Color Comparison in Fuzzy Color Spaces

José Manuel Soto-Hidalgo[a,*], Daniel Sánchez[b], Jesús Chamorro-Martínez[b], Pedro Manuel Martínez-Jiménez[c]

[a]*Department of Electronics and Computer Engineering, University of Córdoba, Campus Universitario de Rabanales, 14071 Córdoba, Spain*
[b]*Department of Computer Science and Artificial Intelligence and CITIC-UGR, University of Granada, C/ Periodista Daniel Saucedo Aranda s/n, 18071 Granada, Spain*
[c]*Department of Automation, Electronics and Computer Architecture and Networks, University of Cádiz, Spain*

Abstract

In this paper we study the problem of color comparison in computers in the setting of fuzzy color spaces. First, we study resemblance relations between precise colors induced by fuzzy colors and fuzzy color spaces. Such resemblance relations are equivalent to fuzzy categorizations of color spaces, that are crucial in order to capture the human's perception of color. Second, we consider the case of color information expressed by means of fuzzy colors used either in a conjunctive or disjunctive way. In order to match pieces of color information we use concepts of resemblance, inclusion, compatibility, and possibility/necessity between the fuzzy colors involved in the definition of different pieces of color information, using well known results from the fuzzy set and possibility theories. Finally, we define inclusion/similarity indexes for comparison of fuzzy colors, including a novel approach to calculate inclusion and overlapping between fuzzy colors on the basis of quantification techniques.

Keywords: Fuzzy colors, fuzzy color spaces, color resemblance, matching color information, conjunctive/disjunctive semantics

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*Corresponding author. José Manuel Soto-Hidalgo. Phone: +34 957 212039
Email addresses: jmsoto@uco.es (José Manuel Soto-Hidalgo), daniel@decsai.ugr.es (Daniel Sánchez), jesus@decsai.ugr.es (Jesús Chamorro-Martínez), pedromartinez@decsai.ugr.es (Pedro Manuel Martínez-Jiménez)

1. Introduction

Fuzzy colors and fuzzy color spaces [1] have been proposed as a suitable way for modelling human color categories. The computational representation of colors by using ordinary color spaces (a triplet represented as a vector in some three-dimensional space), designed to efficiently solve the problem of representing a large amount of very precise colors, is not well suited to capture the human's perception of color categorization and color resemblance, the latter being two sides of the same coin [2, 3]. Whilst, for computers, each vector in a color space is a distinguishable color (a crisp color), the human perception of color is completely different in at least three important respects. First, humans are able to distinguish and name color categories, corresponding to sets of crisp colors that are *resemblant*; hence, color categories can be obtained from a definition of color resemblance (and vice versa) [4, 5]. Second, such categories have fuzzy boundaries, resemblance between precise colors being fuzzy [6, 7]. Third, the categories are subjective and context-dependent [8, 9], that is, the criteria under which we assess color resemblance are not only perceptual and cultural, but also depends on the person and the particular context. In order to cope with these issues, the notion of fuzzy color is introduced as a normal fuzzy subset of colors modeling a distinguishable color category. Also, the notion of fuzzy color space is introduced to be a collection of fuzzy colors representing the relevant color categories in a certain context [1].

In the setting of fuzzy color spaces, while some computational approaches to model color categories by means of fuzzy sets [1, 10] can be found, the present work focuses on the issue of color comparison, which is a key aspect in the computational treatment of color [11, 8, 12, 13]. Color comparison has been widely studied for ordinary (crisp) color spaces, using two main approaches: i) by defining new perceptually uniform color spaces, in the sense that the perceptual dissimilarity between colors is proportional to the Euclidean distance in the space [14, 15, 16], and ii) using an existing color space but modifying the metric employed, usually by introducing weights associated to the color space components [17, 18, 19].

However, to the best of our knowledge, color comparison in terms of fuzzy colors have been mostly neglected in the literature. Some fuzzy color comparison proposals based on histograms in images can be found [20, 21, 22], which are calculated on the basis of the frequency of appearance of certain colors in specific images, with several applications [23, 24, 25, 26, 27, 28, 29, 30, 31]. Nevertheless, in this paper we focus on comparing fuzzy colors not necessarily calculated on the basis of concrete images and/or frequencies. More specifically, we address the following problems:

- Crisp color comparison induced by fuzzy color spaces. Fuzzy colors in fuzzy color spaces define color categories. As in the crisp case, categories and similarity relations are equivalent. Hence, fuzzy colors and fuzzy color spaces induce the corresponding fuzzy resemblance relations, as it is well known in fuzzy set theory. We study these relations and we show how they can be used in crisp color comparison, as well as the conceptual differences with respect to using metrics in crisp color spaces for the same purpose.

- Comparison of color information expressed by means of fuzzy colors. After color categories are defined, they may be employed for many purposes in computer systems. Particularly, color categories (corresponding to fuzzy colors) are frequently used for representing information about color features of objects in information systems. As it is the case in general with crisp and fuzzy sets [32, 33, 34, 35], fuzzy colors can be used either conjunctively or disjunctively. An example of a conjunctive use is to express the information "I like *red* colors". In this case, each crisp color $\mathbf{c}$ in the support of the fuzzy color *red* is a color I like to a degree given by the membership degree of $\mathbf{c}$ to *red*. On the other hand, an example of disjunctive use is to express the information "My car is *red*". In this second case, the actual color of my car is a single crisp color, but we are not pointing out exactly which one. Instead, the (uncertain) information about the car's color is provided by means of a restriction on the set of crisp colors given by the fuzzy color *red*, defining a possibility distribution. In this case, the possibility that a certain crisp color $\mathbf{c}$ is the color of the car is again the membership degree of $\mathbf{c}$ to *red*. Hence, a disjunctively-used fuzzy color is a possibility distribution on the set of crisp colors. We provide measures of compatibility and possibility/necessity between the fuzzy colors involved in the definition of different pieces of color information, using well known results from fuzzy set and possibility theories. From now on we shall denote the conjunctive (resp. disjunctive) use of fuzzy colors as Cu (resp. Du), i.e., we shall say that a fuzzy color is conjunctively used (Cu) or disjunctively used (Du).

- Fuzzy color comparison. The problem here is to compare different fuzzy sets modeling fuzzy colors. A particular case is that of comparing different models of the same color category. These different models may arise because of the subjectivity or context, or even because of the modeling procedure employed. For the general case, we provide fuzzy extensions of the set inclusion and equality predicates following the Sinha-Dougherty axioms [36, 37]. We also provide extensions of inclusion and similarity indexes based on overlapping degrees, particularly an extension of the Jaccard index for crisp sets based on fuzzy quantification.

In order to a better understanding of the different relations between color information, Table 1 contains some illustrative questions involving different types of color information, that we shall address in this paper. Questions 1 and 2 require to determine the resemblance between crisp colors induced by a fuzzy color or a fuzzy color space, respectively. Questions 3 and 4 correspond to relations between crisp colors and fuzzy colors considering the conjunctive and disjunctive use, respectively, while questions 5-11 correspond to relations between fuzzy colors and different uses. In particular, questions 6 and 7 are solved using possibility and necessity, questions 8 and 9 by means of inclusion relations, and questions 10 and 11 by using similarity relations.

Let us remark that the objective of the paper is to highlight the importance of color comparison in the area of fuzzy color modelling and applications, and to show that there are different possible cases of color comparison with different applications (comparison of colors themselves, both crisp and/or fuzzy, and comparison of information expressed in terms of those colors). Our discussion is not aimed to be exhaustive, other questions related to comparison of colors and information expressed using colors can be stated, but we have studied those that we have considered the most important and useful in general. We also provide a solution for every case, based on fuzzy set and possibility theories, in order to show that it is possible to deal with those cases; other solutions might be possible, and some alternatives are also mentioned. However, providing a comparison of different solutions for the same case is out of the scope of this paper; each such comparison would require a whole paper.

The paper is organized as follows. In Section 2 we provide a brief overview about fuzzy colors and fuzzy color spaces. Section 3 is devoted to crisp color comparison induced by fuzzy color spaces (questions like 1 and 2 in Table 1). Section 4 contains our study of matching of color information represented by precise colors, conjunctively-used fuzzy colors, and disjunctively-used fuzzy colors (questions 3 to 7 in Table 1). This section extends our preliminar work in [38] with new formulations of several relations, and also introduces illustrative examples for each case. In Section 5 we show our approaches to comparison of fuzzy colors by means of inclusion and similarity (questions 8 to 11 in Table 1), and we

Question	Color 1	Color 2	Relation
1. How similar are two crisp colors $\mathbf{c}$ and $\mathbf{c}'$ according to the color red?	$\mathbf{c}$ crisp	$\mathbf{c}'$ crisp	Similarity induced by red
2. How similar are two crisp colors $\mathbf{c}$ and $\mathbf{c}'$ according to a given fuzzy color space $\widetilde{\Gamma}$?	$\mathbf{c}$ crisp	$\mathbf{c}'$ crisp	Similarity induced by $\widetilde{\Gamma}$
3. To what degree the color c of the pixel p is red?	$\mathbf{c}$ crisp	red Cu	Compatibility
4. Knowing that my car is red, what is the possibility/necessity that my car is painted in the RGB color $\mathbf{c}$?	$\mathbf{c}$ crisp	red Du	Possibility / Necessity
5. To what extent is it possible to find a car whose color is red and $orange$?	red Cu	$orange$ Cu	Compatibility
6. She told me she liked red cars. My car is $orange$. What is the possibility/necessity that she likes my car?	red Cu	$orange$ Du	Possibility / Necessity
7. Jim saw a red car, and Tim saw an $orange$ car, what is the possibility/necessity that both cars were painted in the same color?	red Du	$orange$ Du	Possibility / Necessity
8. To what extent does this picture use all the $\widetilde{C}$ colors of this $\widetilde{C}'$?	$\widetilde{C}$ Cu	$\widetilde{C}'$ Cu	Inclusion
9. To what extent Jim's knowledge about my car's color $\widetilde{C}$ is more specific than Tim's color $\widetilde{C}'$?	$\widetilde{C}$ Du	$\widetilde{C}'$ Du	Inclusion
10. How similar are fuzzy colors $\widetilde{C}$ and $\widetilde{C}'$ defined by two different users for the same category of color?	$\widetilde{C}$ Cu	$\widetilde{C}'$ Cu	Similarity
11. To what extent Jim's knowledge about my car's color $\widetilde{C}$ is similar to Tim's $\widetilde{C}'$?	$\widetilde{C}$ Du	$\widetilde{C}'$ Du	Similarity

Table 1: Several questions involving different types of color information.

provide an experimental comparison. Finally, Section 6 contains our conclusions and ideas for future research.

2. Fuzzy Color Spaces

Color categories (and the corresponding resemblance relations) are fuzzy in nature, membership of a precise color (a triplet represented as a vector in some three-dimensional (color) space) to a color category being a matter of degree. In addition, color categories are subjective and context-dependent. For example, in a general context, two crisp colors could be perceived as totally similar with respect to the "red" color defined by the well-known *ISCC-NBS* system [39], whereas for a winemaker they might be totally different because they belong to two disjoint categories, such as "ruby" and "garnet", in the specific context of her/his work.

Consequently, there are many collections of color categories according to the final user and the context.

In order to account for the fuzziness in color categories, in our previous work [1] we introduced the concept of fuzzy color for representing the correspondence between computational representation of colors as vectors in a color space (crisp colors), and perceptual color categories identified by a color name. Fuzzy colors are formally defined in [1] as normal fuzzy subsets of colors that represent the semantics of a certain human color category. The requirement that the fuzzy color is a normal fuzzy set is because it is requested that at least one crisp color is fully representative of the color category.

We also formalized in [1] the notion of fuzzy color space as the collection of fuzzy colors corresponding to the color categories employed in a certain context/application and/or for a specific user. Also, we introduced different typologies of fuzzy color spaces which are consistent with the characteristics of the collection of colors we want to model for an specific application or context. Let Γ be a crisp color space and $\widetilde{\Gamma} = \{\widetilde{C}_1, \ldots, \widetilde{C}_n\}$ be a fuzzy color space defined on Γ, with $\widetilde{C}_i$ a fuzzy color $\forall 1 \leq i \leq n$. Then:

- $\widetilde{\Gamma}$ is a *covering space* iff for some t-conorm $\bigcup_{\widetilde{C}_i \in \Gamma} \widetilde{C}_i = \Gamma$
- $\widetilde{\Gamma}$ is a *disjoint space* iff $\forall \widetilde{C}_i \in \widetilde{\Gamma}$, $\forall \mathbf{c} \in \Gamma$, $\widetilde{C}_i(\mathbf{c}) = 1$ implies $\widetilde{C}_j(\mathbf{c}) = 0$ $\forall i \neq j$.
- $\widetilde{\Gamma}$ is a *partition space* iff it is a *covering* and *disjoint* space.

Most of the works to represent color terms in the literature attempt to obtain partition spaces containing the *basic* color terms in the sense of Berlin and Kay [40], with small variations in the number of colors, see [1] for a review on the topic. In most of the cases, the idea is to find a representation of each basic color term that can be agreed on by most of people in a given context. The proposal in [1] is more general in that a methodology is provided for representing any color term, either given by an individual person or agreed by a collective in any particular cultural or application context, and not only the abovementioned basic ones.

We consider that it is not mandatory for fuzzy color spaces to be partition spaces. The methodology proposed in [1] complies with this idea since it allows for obtaining both partition and non-partition spaces. It uses the theory of conceptual spaces [41, 42] plus particular fuzzifying techniques introduced in [1] for obtaining the representation of a *single* fuzzy color. The calculation of a whole fuzzy color space $\widetilde{\Gamma} = \{\widetilde{C}_1, \ldots, \widetilde{C}_m\}$, composed by a collection of fuzzy colors, can be done by obtaining each fuzzy color $\widetilde{C}_i$ individually.

Some examples of fuzzy color spaces of different typologies are provided in [1]. First, three partition spaces based on the color prototypes in the ISCC-NBS system, based on the pioneering work of Berlin and Kay [40] about color naming. These spaces correspond to the three color sets (pairs of linguistic term and crisp color) with different levels of color description which are collected in the Universal Language of Color [43]: the *Basic Set* comprised of 13 color terms corresponding to ten basic (pink, red, orange, yellow, brown, olive, green, yellow-green, blue, purple), and 3 achromatic ones (white, gray, and black); the *Extended Set*, comprised of 31 color terms; and the *Complete Set* containing 267 color terms. The fuzzy color spaces for these three color sets are called $\widetilde{\Gamma}_{ISCC-basic}$, $\widetilde{\Gamma}_{ISCC-extended}$, and $\widetilde{\Gamma}_{ISCC-complete}$, respectively in [1], and are designed to be partition spaces since the colors provided by the ISCC-NBS have an exclusive nature (i.e. two colors cannot be fully in two categories, that is, in the core of two fuzzy colors).

Non-covering and non-disjoint spaces are also provided in [1], customized for specific users in the context of a fixed collection of fruit colors categories. On the basis of a collection of fruit color categories and their corresponding representative crisp colors provided by a certain user, fuzzy color spaces are provided. Such spaces are non-covering since not every crisp color is suitable for describing a fruit color category. In addition, they are non-disjoint, since some crisp colors fully match several categories (for instance, *banana* and *lemon*). Several experiments were conducted with 30 users (15 men and 15 women). Users can select colors they consider to be representative of a color term associated with a fruit and they assign it manually a customized color name from the image collection. The image collection and all users data from this experimentation can be downloaded at the website `http://www.jfcssoftware.com`.

In addition to human color categories, fuzzy colors can be defined on the basis of (and employed to represent) information in visual data. For instance, given a particular image, “red colors in the image” is a fuzzy color defined as the collection of crisp colors that match the category “red” in the image. This fuzzy color, which support can be much smaller and even be a crisp singleton in simple images, can be obtained either as a subset of a fuzzy color “red” defined previously, by computing the intersection between the corresponding fuzzy set and the set of crisp colors of the pixels in the image, or even generated specifically for the image by assigning degrees to crisp colors appearing in the image with some other procedure.

In next sections, we use the proposal in [1] for illustrative purposes. However, all the measures and resemblances proposed in the present work can be applied to fuzzy colors and fuzzy color spaces calculated following any other methodology. The only requirement for fuzzy colors, as we have seen at the beginning of this section, is that they must be a normalized fuzzy subset of crisp colors. We have just mentioned as an example the case of fuzzy colors defined by visual data in the previous paragraph. A review of other proposals can be found in [1]. Another recently proposed approach is that of fuzzy colors defined on the basis of several representative prototypes, introduced in [44] and called *granular fuzzy colors*. An important result shown in [44], that we will use later in this work, is that, with an appropriate choice of prototypes, any fuzzy color can be seen in practice as a granular fuzzy color in which its membership function is always decreasing with the distance to some prototype.

3. Resemblance relations between crisp colors

As we have explained in the introduction, color categorization and color resemblance are two sides of the same coin. A color category contains a collection of resemblant precise colors, corresponding to a certain color name. Hence, we can say that a color category *induces* a resemblance relation in the set of precise colors (and vice versa). For instance, if we define a crisp categorization corresponding to a partition of a crisp color space, the corresponding resemblance relation is the crisp equivalence relation having the categorization as quotient set.

An important consequence of the category/resemblance dichotomy is that color resemblance is *relative*, as it depends on the categories employed. Let us recall here the case of the wine colors that may be perceived as not resemblant in the context of red wine tasting because one of them is fully in the category "ruby" and the other one is prototypical of the category "garnet", but they can be perceived as resemblant according to the category "red" in a more general context. On the contrary, crisp color resemblance based on distance in a crisp color space is *absolute*, as it is the case with Euclidean distances in RGB, CIELab and any other space, each of which define a unique comparison framework for crisp colors.

On the other hand, as we have seen in the previous section, color categories coming from humans are not crisp but fuzzy, and can be represented by means of fuzzy sets. As a consequence, the corresponding resemblance relations are not crisp equivalence relations, but fuzzy resemblance relations [45, 46]. We study resemblance relations between two crisp colors induced by a single fuzzy color in Section 3.1. Also, we study resemblance relations induced by a whole fuzzy color space in Section 3.2.

The classical way to study color resemblance in image processing is by means of distances in crisp color spaces. In Section 3.3 we discuss on the relationship between color resemblance, color categories and distances in crisp color spaces.

3.1. Resemblance between crisp colors induced by a fuzzy color

The representation of a fuzzy color is the fuzzy set of crisp colors that are compatible with the fuzzy color. This representation induces a resemblance relation for crisp colors based on the idea that crisp colors appearing to a certain degree in the representation of a fuzzy color are supposed to be resemblant at least to the minimum of those degrees. The semantics of this resemblance relation is the following: the crisp colors $\mathbf{c}$ and $\mathbf{c}'$ are resemblant with respect to the fuzzy color $\widetilde{C}$ to the extent that both $\mathbf{c}$ and $\mathbf{c}'$ match $\widetilde{C}$. This semantics corresponds to the classical notion of *proximity relation*. A proximity relation $r_{\widetilde{C}}$ defined on a fuzzy subset $\widetilde{C}$ of Γ (see [47], page 238) must satisfy the following properties:

$$r_{\widetilde{C}}(\mathbf{c},\mathbf{c}') = r_{\widetilde{C}}(\mathbf{c}',\mathbf{c}) \quad (1)$$

$$r_{\widetilde{C}}(\mathbf{c},\mathbf{c}') \leq \min(r_{\widetilde{C}}(\mathbf{c},\mathbf{c}), r_{\widetilde{C}}(\mathbf{c}',\mathbf{c}')) \quad (2)$$

$$r_{\widetilde{C}}(\mathbf{c},\mathbf{c}) = \widetilde{C}(\mathbf{c}) \quad (3)$$

Hence, a first notion of resemblance between crisp colors is that of proximity induced by a single fuzzy color, that can be defined as follows [47]:

Definition 3.1. *The resemblance (proximity) between crisp colors induced by a single fuzzy color $\widetilde{C}$ is*

$$r_{\widetilde{C}}(\mathbf{c},\mathbf{c}') = min(\widetilde{C}(\mathbf{c}), \widetilde{C}(\mathbf{c}')) \quad (4)$$

Definition 3.1 is useful when answering questions like the first one in Table 1: How similar are two crisp colors $\mathbf{c}$ and $\mathbf{c}'$ according to the color red? Particularly, Eq. (4) can be seen as a type-I comparison measure of two crisp colors in the sense of [48] on the basis of the attribute "matching the fuzzy color $\widetilde{C}$", since we do not consider that two crisp colors are resemblant when none of them match the considered fuzzy color.

It is easy to show that $r_{\widetilde{C}}$ satisfies Eqs. (1)-(3), particularly it is a symmetric fuzzy relation. It is also easy to show that $r_{\widetilde{C}}$ is locally reflexive, that is,

$$r_{\widetilde{C}}(\mathbf{c},\mathbf{c}) \geq \max_{c' \in \Gamma}(r_{\widetilde{C}}(\mathbf{c},\mathbf{c}'), r_{\widetilde{C}}(\mathbf{c}',\mathbf{c})) \tag{5}$$

and max-min transitive, i.e.,

$$r_{\widetilde{C}}(\mathbf{c},\mathbf{c}'') \geq \max_{c' \in \Gamma} \min(r_{\widetilde{C}}(\mathbf{c},\mathbf{c}'), r_{\widetilde{C}}(\mathbf{c}',\mathbf{c}'')) \tag{6}$$

The local character of reflexivity just requires that no color is more similar to a certain color $\mathbf{c}$ than $\mathbf{c}$ itself. It is reasonable since, as a proximity relation based on a certain fuzzy color $\widetilde{C}$, we are restricting the semantics of our resemblance to the case of matching $\widetilde{C}$. As stated in Eq. (3), the proximity of a color $\mathbf{c}$ to itself in the context of a fuzzy color $\widetilde{C}$ is $\widetilde{C}(\mathbf{c})$, being the largest resemblance value to $\mathbf{c}$.

Let us illustrate the usefulness of this proposal with an example: suppose that in a medical image, pixels corresponding to a tumour are known to be *red* (a fuzzy color category) and topologically connected, and one wanted to perform a fuzzy segmentation of the tumour on the basis of color similarity between neighbour pixels, like in [49]. Then, one might be interested in computing the resemblance between the crisp colors of neighbour pixels in the image on the basis of the *red* category only, so that similarity with crisp colors out of the support of *red* yields 0, even if two neighbour pixels are assigned the same such crisp color.

Other proposals can be employed for resemblance induced by a fuzzy color. For instance, when reflexivity is required for a particular application, that is, when we require

$$r_{\widetilde{C}}(\mathbf{c},\mathbf{c}) = 1 \tag{7}$$

we can use as an alternative the technique proposed in page 267 of [47], that generates a decomposable fuzzy equivalence relation as:

$$r_{\widetilde{C}}(\mathbf{c},\mathbf{c}') = \begin{cases} 1 & c = c' \\ min(\widetilde{C}(\mathbf{c}), \widetilde{C}(\mathbf{c}')) & \end{cases} \tag{8}$$

Note that Eq. (8) is a slight modification of Eq. (4) that guarantees reflexivity and, at the same time, keeps symmetry and max-min transitivity, as shown in [47]. It also keeps Eq. (2), but not Eq. (3), which is replaced by Eq. (7).

In order to illustrate resemblance induced by a single fuzzy color as defined by Eq. (4), we consider the precise colors $c_1 - c_8$ in Table 2. Note that the alternative in Eq. (8) yields the same values, except for the case of comparison of each color with itself, in which Eq. (8) yields 1 in every case (main diagonal). Tables 3, 4, 5 and 6 show the resemblance degrees between the crisp colors $c_1 - c_8$ according to the fuzzy colors *Blue*, *Red*, *Orange* and *Yellow*, respectively, from the fuzzy color space $\widetilde{\Gamma}_{ISCC-basic}$ based on the ISCC-NBS *Basic Set* mentioned in Section 2. In Table 3 it can be observed that the resemblance between the crisp colors c_1 and c_2 induced by the fuzzy color *Blue* is greater than 0 since both colors are perceived as similar with respect to the color *Blue*, while it is 0 for the rest of the colors (c_3-c_8). However, resemblance of c_1 and c_2 with respect to the fuzzy color *Red* (Table 4) is 0, showing how the resemblance is relative to the reference fuzzy color employed. The same happens for color c_1 and c_2 with respect to the fuzzy colors *Orange* and *Yellow* (tables 5 and 6). It can be seen that colors c_3, c_4 and c_5 are resemblant to some degree with respect to the fuzzy color *Red* (Table 4), colors c_4, c_5, c_6 and c_7 with respect to the fuzzy color *Orange* (Table 5), and colors c_6, c_7 and c_8 with respect to the fuzzy color *Yellow* (Table 6).

On the issue of reflexivity that we have discussed above, note that $r_{\widetilde{C}}(c1, c1) \neq 1$ in Table 3 because the property of reflexivity in Definition 3.1 is not imposed. The same happens to colors c_4, c_5 with respect to the fuzzy colors *Red*, *Orange*, and c_6 and c_7 with respect to *Orange* and *Yellow*. As we have seen, when reflexivity is required, we can use alternatively Eq. (8).

Color	RGB Value
c_1	$[5, 209, 233]$
c_2	$[1, 126, 170]$
c_3	$[200, 26, 51]$
c_4	$[255, 51, 53]$
c_5	$[255, 75, 37]$
c_6	$[250, 152, 40]$
c_7	$[225, 171, 8]$
c_8	$[254, 253, 21]$

Table 2: A collection of crisp colors $c_1 - c_8$ and their corresponding RGB values.

3.2. Resemblance between crisp colors induced by a fuzzy color space

The notion of resemblance induced by a fuzzy color can be extended to resemblance induced by a fuzzy color space just taking the maximum degree among all the fuzzy colors of the fuzzy color space. Note that we only consider the case of finite fuzzy color spaces, since the amount of color categories humans are able to

c_1	0.439							
c_2	0.439	1						
c_3	0	0	0					
c_4	0	0	0	0				
c_5	0	0	0	0	0			
c_6	0	0	0	0	0	0		
c_7	0	0	0	0	0	0	0	
c_8	0	0	0	0	0	0	0	0
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8

Table 3: Resemblance degree induced by the fuzzy color *Blue* of the fuzzy color space $\widetilde{\Gamma}_{ISCC-basic}$ defined in [1] between the crisp colors c_1-c_8 of Table 2.

c_1	0							
c_2	0	0						
c_3	0	0	1					
c_4	0	0	0.658	0.658				
c_5	0	0	0.298	0.298	0.298			
c_6	0	0	0	0	0	0		
c_7	0	0	0	0	0	0	0	
c_8	0	0	0	0	0	0	0	0
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8

Table 4: Resemblance degree induced by the fuzzy color *Red* of the fuzzy color space $\widetilde{\Gamma}_{ISCC-basic}$ defined in [1] between the crisp colors c_1-c_8 of Table 2.

c_1	0							
c_2	0	0						
c_3	0	0	0					
c_4	0	0	0	0.342				
c_5	0	0	0	0.342	0.702			
c_6	0	0	0	0.342	0.702	0.865		
c_7	0	0	0	0.262	0.262	0.262	0.262	
c_8	0	0	0	0	0	0	0	0
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8

Table 5: Resemblance degree induced by the fuzzy color *Orange* of the fuzzy color space $\widetilde{\Gamma}_{ISCC-basic}$ defined in [1] between the crisp colors c_1-c_8 of Table 2.

c_1	0							
c_2	0	0						
c_3	0	0	0					
c_4	0	0	0	0				
c_5	0	0	0	0	0			
c_6	0	0	0	0	0	0.135		
c_7	0	0	0	0	0	0.135	0.738	
c_8	0	0	0	0	0	0.135	0.738	1
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8

Table 6: Resemblance degree induced by the fuzzy color *Yellow* of the fuzzy color space $\widetilde{\Gamma}_{ISCC-basic}$ defined in [1] between the crisp colors c_1-c_8 of Table 2.

manage is finite. The semantics of this resemblance is the following: two crisp colors $\mathbf{c}$ and $\mathbf{c}'$ are resemblant according to a fuzzy color space $\widetilde{\Gamma}$ to the extent that $\exists \widetilde{C} \in \widetilde{\Gamma}$ such that $\mathbf{c}$ and $\mathbf{c}'$ are resemblant according to $\widetilde{C}$ as defined in the previous section, that is, when both crisp colors pertain to $\widetilde{C}$. On the basis of this idea, we introduce the following definition:

Definition 3.2. *The resemblance between crisp colors* $\mathbf{c}$ *and* $\mathbf{c}'$ *induced by a fuzzy color space* $\widetilde{\Gamma}$ *is*

$$R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}') = \max_{\widetilde{C} \in \widetilde{\Gamma}} r_{\widetilde{C}}(\mathbf{c}, \mathbf{c}') \tag{9}$$

Definition 3.2 is useful in practice when answering questions like the second one in Table 1: How similar are two crisp colors $\mathbf{c}$ and $\mathbf{c}'$ according to a given fuzzy color space? And related questions such as *What RGB colors are resemblant to color* c*?*, or *Give me images containing colors that resemble that of pixel* p, etc.

When $r_{\widetilde{C}}$ is the proximity relation defined by Eq. (4), it is easy to show that $R_{\widetilde{\Gamma}}$ satisfies the following properties:

$$\begin{aligned}
R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}') &= R_{\widetilde{\Gamma}}(\mathbf{c}', \mathbf{c}) && (10)\\
R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}') &\leq \min(R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}), R_{\widetilde{\Gamma}}(\mathbf{c}', \mathbf{c}')) && (11)\\
R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}) &= \max_{\widetilde{C} \in \widetilde{\Gamma}} \widetilde{C}(\mathbf{c}) && (12)\\
R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}) &\geq \max_{c' \in \Gamma}(R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}'), R_{\widetilde{\Gamma}}(\mathbf{c}', \mathbf{c})) && (13)\\
R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}'') &\geq \max_{c' \in \Gamma} \min(R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}'), R_{\widetilde{\Gamma}}(\mathbf{c}', \mathbf{c}'')) && (14)
\end{aligned}$$

Particularly, we have that $R_{\widetilde{\Gamma}}$ is symmetric (Eq. (10)), max-min transitive (Eq. (14)) and locally reflexive (Eq. (13)), as it is the case for $r_{\widetilde{C}}$ defined by Eq. (4).

As in the previous section, when reflexivity is required, we can use in Eq. (9) the resemblance $r_{\widetilde{C}}$ defined by Eq. (8). In such case, the values yield by $R_{\widetilde{\Gamma}}$ are the same except with the single exception that $R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}) = 1$ for all $c \in \Gamma$. This alternative satisfies the same properties as the previous one, except for Eq. (12), and $R_{\widetilde{\Gamma}}$ is a fuzzy equivalence relation.

Note also that, in the first alternative, reflexivity can also be achieved in the particular case that the cores of the fuzzy colors in $\widetilde{\Gamma}$ form a crisp covering of Γ, since in that case it is

$$\max_{\widetilde{C} \in \widetilde{\Gamma}} \widetilde{C}(\mathbf{c}) = 1 \tag{15}$$

However, such kind of fuzzy color space is not usual in practical applications.

In order to illustrate our proposal, Table 7 shows the resemblance induced by the fuzzy colors *Blue*, *Red*, *Orange*, and *Yellow* of the fuzzy color space $\widetilde{\Gamma}_{ISCC-basic}$ between the crisp colors c_1-c_8 of Table 2, using the first alternative in this section (the second alternative yields the same values except for the main diagonals of the matrices, where all values are 1 as reflexivity is imposed).

3.3. Color resemblance, color categories and distances in crisp color spaces

The problem of how to compare precise colors in terms of resemblance has been dealt with in the image processing area by means of distances in color spaces, typically Euclidean distance [14, 15, 16]. An important question that arises is, what is the relation between distances and categories/resemblance? We can discuss this issue from the point of view of both categories and resemblance relations.

Categories: With respect to categories, it is important to remark that distances are on the basis of most of the techniques for building fuzzy color spaces. This idea is natural for humans since we are used to employ color prototypes for defining basic color categories; as an example, color naming techniques [50] provide prototypes for defining correspondences between color categories and computational representations of color on the basis of the distance. Roughly, membership to a fuzzy color decreases with distance to some of the crisp representatives of the color (with a rate that can depend on the direction along which we move in the color space domain [44]), so membership can be seen as inversely proportional to distance. In particular, considering a maximum distance M in a color space on the basis of a distance d, a membership function for a color category $\widetilde{C}_{\mathbf{r}}$ with respect to a representative prototype $\mathbf{r}$ could be defined as

$$\widetilde{C}_{\mathbf{r}}(\mathbf{c}) = 1 - \frac{d(\mathbf{r}, \mathbf{c})}{M} \tag{16}$$

c_1	0.439							
c_2	0.439	1						
c_3	0	0	1					
c_4	0	0	0.658	0.658				
c_5	0	0	0.298	0.342	0.702			
c_6	0	0	0	0.342	0.702	0.865		
c_7	0	0	0	0.262	0.262	0.262	0.738	
c_8	0	0	0	0	0	0.135	0.738	1
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8

Table 7: Resemblance degree induced by the fuzzy colors *Blue*, *Red*, *Orange*, and *Yellow* of the fuzzy color space $\widetilde{\Gamma}_{ISCC-basic}$ between the crisp colors c_1-c_8 of Table 2.

c_1	0.744							
c_2	0.744	0.925						
c_3	0.418	0.418	0.918					
c_4	0.332	0.375	0.820	0.820				
c_5	0.321	0.375	0.789	0.789	0.845			
c_6	0.328	0.375	0.717	0.780	0.845	0.900		
c_7	0.323	0.375	0.717	0.780	0.845	0.900	0.930	
c_8	0.247	0.375	0.717	0.721	0.721	0.859	0.859	0.859
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8

Table 8: Resemblance degree between the crisp colors c_1-c_8 of Table 2 induced by the set of four fuzzy colors of the form $\widetilde{C}_{\mathbf{r}}$ of Eq. (16) in RGB, with r being the representatives in the ISCC-Basic system of the colors *Blue*, *Red*, *Orange*, and *Yellow*.

c_1	0.597							
c_2	0.597	0.816						
c_3	0.487	0.487	0.917					
c_4	0.458	0.424	0.806	0.806				
c_5	0.458	0.424	0.789	0.789	0.796			
c_6	0.458	0.424	0.676	0.742	0.796	0.927		
c_7	0.458	0.424	0.676	0.742	0.796	0.854	0.916	
c_8	0.458	0.403	0.618	0.618	0.618	0.787	0.839	0.839
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8

Table 9: Resemblance degree between the crisp colors c_1-c_8 of Table 2 induced by the set of four fuzzy colors of the form $\widetilde{C}_{\mathbf{r}}$ of Eq. (16) in CIEDE2000, with r being the representatives in the ISCC-Basic system of the colors *Blue*, *Red*, *Orange*, and *Yellow*.

Let us remark that there are many approaches to transform a distance measure to a resemblance one [51, 52, 48], and the linear case in Eq. (16) is a particular case with well-known limitations. Regardless the transformation applied, what is usual in fuzzy color literature is to define membership functions on the basis of parametric functions of the distance [53, 1]. Most recent works face this issue by taking into account that precise colors can be roughly divided into three categories in terms of their distance to the representative of a fuzzy color: those that are close enough for their membership to be 1; those that are far enough for their membership to be 0; and those having an intermediate distance to the representative, that are assigned membership degrees in $(0, 1)$. The first two groups can be defined in terms of bounds; the third one is defined by those same bounds together with a way to compute membership in a proportional (and inverse) way to the distance. Hence, in every direction, membership to fuzzy categories can be defined as a piecewise function which is constantly 1 in the vicinity of the crisp representative of the category as defined by a first bound; constantly 0 beyond a certain distance given by a second bound; and decreasing from 1 to 0 with distance between both bounds. This is not the case for fuzzy colors defined following Eq. (16). In fact, every precise color being at a distance less than M from $\mathbf{r}$ is in the support of $\widetilde{C}_{\mathbf{r}}$. Nevertheless, Eq. (3) can be seen as the simplest transformation that interprets distance as a membership degree, allowing us to compare more complex approaches with the use of distance as membership degree. In this sense, let us remark that our goal is to discuss the relationship between distances in crisp color spaces and color categories, but not to analyze or compare different approaches to transform a distance measure to a resemblance.

Table 8 shows the use of Eq. (16) for calculating resemblance between the colors c_1-c_8 of our example in the previous section. Fuzzy colors are calculated using Eq. (16) with the Euclidean distance in the RGB color space with $M = 255\sqrt{3}$ and the representatives in the ISCC-NBS *Basic Set* for colors *Blue* ($[1, 161, 194]$), *Red* ($[190, 1, 50]$), *Orange* ($[243, 132, 1]$), and *Yellow* ($[243, 195, 1]$). Resemblance of precise colors is computed using Eqs. (4) and (9) for that collection of fuzzy colors. The results are shown in Table 8. The same computation is performed with the CIEDE2000 metric[1] [18] with $M = 100$ as maximum distance[2], and the

[1]The CIEDE2000 metric based on the CIE Lab system is specifically defined so that distances are perceptually more significant for humans, i.e., so that the corresponding space is perceptually uniform.

[2]Note that $M = 100$ is rather conservative and does not correspond to the maximum possible value of the CIEDE2000 metric, which is larger, with a larger value of M increasing even more

results are shown in Table 9.

It is easy to see the difference between the resemblance as calculated in tables 7 and those of tables 8 and 9. Zero resemblances in Table 7 are replaced by high resemblances in tables 8 and 9, see for instance resemblance between colors c_3 and c_6-c_8, due to the fact that only distance is used in the definition of fuzzy colors, without taking into consideration human-like semantics in the form of bounds, as discussed above. Let us remark that this is going to be the case for any color space. It can be also appreciated in that the resemblances evolve in a softer way. For instance, consider the columns corresponding to color c_3. Resemblance decreases abruptly from 1 to 0 as we go from rows c_3 to c_6 in Table 7, whilst in Table 8 it decreases from 0.918 to 0.717, and in Table 9 it decreases from 0.917 to 0.676. In addition, these results are less in accordance with the human semantics regarding comparison in terms of colors *Blue*, *Red*, *Orange* and *Yellow*. This can be seen in the resemblance between c_1 and c_3, which are clearly very different according to these colors (and hence in Table 7 their resemblance is 0), whilst in Tables 8 and 9, resemblances of 0.418 and 0.487, respectively, are provided.

Resemblance: With respect to resemblance, a fuzzy resemblance relation can be defined directly between precise colors as

$$r_d(\mathbf{c}, \mathbf{c}') = 1 - \frac{d(\mathbf{c}, \mathbf{c}')}{M} \tag{17}$$

Implicitly, this relation is associated to the collection of *all* fuzzy colors $\widetilde{C}_{\mathbf{r}}$ calculated using Eq. (16) for all $\mathbf{r}$ in the crisp color space, since $r_d(\mathbf{c}, \mathbf{c}')$ corresponds to the membership of c' to $\widetilde{C}_{\mathbf{c}}$ (and equivalently to the membership of c to $\widetilde{C}_{\mathbf{c}'}$). It is clear that not all of these fuzzy colors will correspond to human color categories. Also, such collection of categories is always the same, independent of the context, and huge, and hence is not suited to deal with the human perception of color categories, from which categories are subjective and context/application dependent.

As a final conclusion of this section, human compliant color resemblance is relative to the categories defining the context for the comparison, and hence can be computed from a suitable fuzzy color space for such context. Fuzzy colors capturing human color semantics cannot be defined in terms of distance alone, but as piecewise functions whose support do not cover the whole color space.

the resemblance values in Table 9. The maximum distance among colors c_1-c_8 in our example is between 85 and 86.

Distance has a different role: it is necessary in order to define categories, since the semantics of the latter are defined implicitly by piecewise functions defined on the former. As a consequence, there is no necessity of having a perceptually uniform distance when defining fuzzy colors, since we are not using distance, but similarity, for comparing crisp colors, and the uniformity of perception of similarity comes from the fuzzy sets defining the fuzzy colors. Finally, distance is also useful when the color comparison is not requested to be compliant with human semantics, but for other different purposes in the setting of image processing.

4. Relations between two pieces of color information

Crisp and fuzzy colors can be used in different ways to represent color information in a computer, filling the semantic gap with human's perception of color. However, as we explained in the introduction, fuzzy colors may represent information with different semantics depending on its use. In order to consider matching of colors according to different semantics, in this section we consider color information expressed by means of:

1. A crisp color $\mathbf{c} \in \Gamma$, e.g., *this pixel has color [255,0,0] in RGB*.
2. A conjunctively-used fuzzy color, e.g., *I like red cars*; *the grass is green*. It defines the matching (compatibility) between a linguistic term and the crisp colors in a crisp color space.
3. A disjunctively-used fuzzy color, corresponding to a description by means of a flexible restriction of our knowledge about the actual value of a variable whose possible values are crisp colors. For example, *my car is red*.

In the following we describe different situations in which pairs of color informations involving at least one fuzzy color are compared or matched, and the corresponding relations and measures that may be used for determining the degree of resemblance, compatibility, or possibility/necessity for each case. Particularly, we analyze the case of a crisp and a fuzzy color in Section 4.1, and the case of two fuzzy colors in Section 4.2.

It is important to note that in this work we do not work with the physical magnitudes of the color (wavelengths) in order to represent the crisp color spaces on which we build our fuzzy color spaces, but with a discrete crisp color space Γ resulting from the light digitalization process through a suitable device (a camera, in this case). We assume a discrete and finite subset of crisp colors in the color space Γ, as usual in the computer representation of crisp colors, although theoretically it works on an infinite vector space. This restriction is necessary when calculating the different expressions for the proposed relations on the basis of fuzzy colors, and does not imply loss of generality, since humans can only distinguish a finite set of crisp colors, usually lower than the computers can represent. It is also finite the set of color categories that a human is able to manage, each category being defined as a fuzzy subset in the crisp color space, as we have pointed out in previous sections.

4.1. Relations between a crisp color and a fuzzy color

In this section we deal with the case in which only one of the color informations to be compared is expressed via a fuzzy color. We propose a compatibility relation between a crisp color and a fuzzy color Cu (conjunctively-used) in Section 4.1.1. Possibility/necessity relations between a crisp color and a fuzzy color Du (disjunctively-used) are proposed in Section 4.1.2.

4.1.1. Compatibility between a crisp color and a conjunctively-used fuzzy color

Compatibility between a crisp color and a fuzzy color Cu depends on the representation of the fuzzy color, so a crisp color will be compatible with a fuzzy color Cu to a certain degree given by its membership function. This semantics is captured by the following definition:

Definition 4.1. *The compatibility between a crisp color* $\mathbf{c}$ *and a fuzzy color Cu* $\widetilde{C} \in \widetilde{\Gamma}$ *is*

$$K(\mathbf{c}, \widetilde{C}) = \widetilde{C}(\mathbf{c}) \tag{18}$$

This definition is useful for answering questions such as the question 3 of Table 1, *to what degree pixel p is* red*?*, *give me images containing* red *pixels*, etc. As an example, Table 10 shows the compatibility of the crisp colors c_1-c_8 of Table 2 with the fuzzy colors *Blue*, *Red*, *Orange* and *Yellow* defined in $\widetilde{\Gamma}_{ISCC-basic}$. As can be observed in the table, the color c_2 is 100% compatible with the fuzzy color *Blue* since it belongs to the core of *Blue*. This is also the case of the crisp color c_3 with respect to the fuzzy color *Red*, as well as c_8 with respect to the *Yellow*. The rest of crisp colors (c_1, c_4, c_5, c_6 and c_7) are also compatible with the fuzzy colors to the degree they belong to each fuzzy color.

4.1.2. Possibility and necessity of a crisp color compatible with a disjunctively-used fuzzy color

A fuzzy color used in a disjunctive way is a possibility distribution on the set of crisp colors. Hence for answering questions using disjunctive colors, we need

Crisp color	*Blue*	*Red*	*Orange*	*Yellow*
c_1	0.439	0	0	0
c_2	1	0	0	0
c_3	0	1	0	0
c_4	0	0.658	0.342	0
c_5	0	0.298	0.702	0
c_6	0	0	0.865	0.135
c_7	0	0	0.262	0.738
c_8	0	0	0	1

Table 10: Compatibility of the crisp colors c_1-c_8 of Table 2 with the fuzzy colors *Blue*, *Red*, *Orange* and *Yellow* defined in $\widetilde{\Gamma}_{ISCC-basic}$.

to measure possibility/necessity of the compatibility between a crisp color and a fuzzy color Du as follows:

Definition 4.2. *The possibility and the necessity that a certain color variable V takes value* $\mathbf{c}$ *knowing that its value is in a Du fuzzy color $\widetilde{C} \in \widetilde{\Gamma}$ are, respectively*

$$pos(V = \mathbf{c}|V \textit{ is } \widetilde{C}) = \widetilde{C}(\mathbf{c}) \tag{19}$$

and

$$nec(V = \mathbf{c}|V \textit{ is } \widetilde{C}) = 1 - \max_{\mathbf{c}' \in \Gamma \ | \ \mathbf{c}' \neq \mathbf{c}} \{\widetilde{C}(\mathbf{c}')\} \tag{20}$$

Note that the *possibility* that a color variable V takes the value $\mathbf{c}$ knowing that its value is a known fuzzy color $\widetilde{C}$, is equal to the compatibility of the crisp color $\mathbf{c}$ with the fuzzy color $\widetilde{C} \in \widetilde{\Gamma}$. In addition, note that for any fuzzy color $\widetilde{C} \in \widetilde{\Gamma}$ such that its core is not a singleton, $Nec(V = \mathbf{c}|V \text{ is } \widetilde{C}) = 0$. As a final remark, if $\widetilde{C}$ is a singleton, it is easy to show that

$$pos(V = \mathbf{c}|V \text{ is } \{\mathbf{c}'\}) = Nec(V = \mathbf{c}|V \text{ is } \{\mathbf{c}'\}) =$$
$$= \begin{cases} 1 & \mathbf{c} = \mathbf{c}' \\ 0 & \text{otherwise} \end{cases}$$

This definition is useful for answering questions such as the question 4 of the Table 1, *knowing that my car is red, what is the possibility/necessity that the color of my car is* $\mathbf{c}$*?*. For example, as shown in Table 10, the *possibility* that a car painted in the red color of Table 2 is painted with the crisp color c_3 is 1, whereas for the color c_4 is 0.658. In the case of *necessity* it is always 0 since the cores of the fuzzy colors in $\widetilde{\Gamma}_{ISCC-basic}$ are not singletons. Likewise, if a fuzzy color space is defined by fuzzy colors having a singleton as core, the *necessity* of a crisp color $\mathbf{c}$ and a fuzzy color $\widetilde{C}$ will be 1 in the case of $\widetilde{C}(c') = 1$ and $c = c'$. Intermediate values can be obtained for other fuzzy colors. For instance, if the fuzzy color representing the *red* colors in a certain image is $\widetilde{C} = 1/c + \alpha/c'$, with $\alpha \in [0, 1]$, then it is $nec(V = \mathbf{c}|V \text{ is } \widetilde{C}) = 1 - \alpha$.

4.2. Relations between two fuzzy colors

In this section we consider the case of matching two pieces of color informations expressed by fuzzy colors. We consider relations between fuzzy colors used in a conjunctive and/or disjunctive way. Our objective is to show that it is possible to solve such cases using existing tools, in our case possibility theory, by making specific proposals; however, other approaches to solve these cases are possible. A compatibility relation between fuzzy colors Cu is proposed in Section 4.2.1. Possibility/necessity relations for the case Cu - Du and Du - Du are proposed in sections 4.2.2 and 4.2.3, respectively.

4.2.1. Compatibility between two conjunctively-used fuzzy colors

Two fuzzy colors Cu will be compatible to the extent that there exists a crisp color belonging to both fuzzy colors. Therefore, the compatibility of a fuzzy color Cu with itself will be 1. In general, compatibility is defined as follows:

Definition 4.3. *The compatibility between two fuzzy colors Cu can be defined as*

$$K(\widetilde{C}, \widetilde{C}') = \max_{\mathbf{c} \in \Gamma} \min\{\widetilde{C}(\mathbf{c}), \widetilde{C}'(\mathbf{c})\} \tag{21}$$

This definition is useful when answering questions such as the question 5 of the Table 1, like *to what extent it is possible to find a crisp color that is* red *and* $orange$*?* As an example, table 11 shows the compatibility between the fuzzy colors *Blue*, *Red*, *Orange* and *Yellow* defined in $\widetilde{\Gamma}_{ISCC-basic}$. Since $\widetilde{\Gamma}_{ISCC-basic}$ is a partition space, the compatibility of fuzzy colors is not expected to be higher than 0.5. For example, the higher compatibility between different fuzzy colors in our example is that of the fuzzy colors *Red* and *Yellow* (0.5). The compatibility between *Orange* and *Yellow* is also rather high for a partition space (0.484). Finally, the compatibility between the fuzzy color *Blue* and the rest is 0 since there is no crisp color belonging to both colors, as expected. Also expected is the fact that the compatibility of each fuzzy color with itself is 1.

Blue	1			
Red	0	1		
Orange	0	0.5	1	
Yellow	0	0	0.484	1
	Blue	*Red*	*Orange*	*Yellow*

Table 11: Compatibility between the fuzzy colors *Blue*, *Red*, *Orange* and *Yellow* defined in $\widetilde{\Gamma}_{ISCC-basic}$.

4.2.2. Possibility and necessity of a conjunctively-used fuzzy color compatible with a disjunctively-used fuzzy color

Since one of the colors is used in a disjunctive way, we have relations of possibility and necessity. We introduce the following definitions:

Definition 4.4. *The possibility and the necessity that a certain color variable V takes a value compatible with a Cu fuzzy color $\widetilde{C}$ knowing that the value of V is in a Du fuzzy color $\widetilde{C'}$ are, respectively*

$$pos(V \in \widetilde{C} | V \textit{ is } \widetilde{C}') = \max_{\mathbf{c} \in \Gamma} \min\{\widetilde{C}'(\mathbf{c}), \widetilde{C}(\mathbf{c})\} \tag{22}$$

$$nec(V \in \widetilde{C} | V \textit{ is } \widetilde{C}') = 1 - \max_{\mathbf{c} \in \Gamma} \min\{\widetilde{C}'(\mathbf{c}), 1 - \widetilde{C}(\mathbf{c})\} \tag{23}$$

This definition is useful when answering questions such as question 6 of Table 1: *She told me she likes red cars. My car is $orange$. What is the possibility/necessity that she likes my car?*

4.2.3. Possibility and necessity of a disjunctively-used fuzzy color compatible with a disjunctively-used fuzzy color

As in the previous section, since both colors are used in a disjunctive way, we have relations of possibility and necessity. We introduce the following definitions:

Definition 4.5. *The possibility and the necessity that a certain color variable V takes a value compatible with a Du fuzzy color $\widetilde{C}$ knowing that the value of V is in a Du fuzzy color $\widetilde{C'}$ are, respectively*

$$pos(V = V' | V \textit{ is } \widetilde{C} \textit{ and } V' \textit{ is } \widetilde{C}') = \max_{\mathbf{c} \in \Gamma} \min\{\widetilde{C}'(\mathbf{c}), \widetilde{C}(\mathbf{c})\} \tag{24}$$

$$nec(V = V' | V \textit{ is } \widetilde{C} \textit{ and } V' \textit{ is } \widetilde{C}') = 1 - \max_{\mathbf{c}, \mathbf{c}' \in \Gamma \ | \ \mathbf{c}' \neq \mathbf{c}} \min\{\widetilde{C}(\mathbf{c}), \widetilde{C}'(\mathbf{c}')\} \tag{25}$$

This definition is useful when answering questions such as question 7 of Table 1: *Jim saw a red car, and Tim saw an $orange$ car, what's the possibility/necessity that both cars were painted in the same color?*.

In the previous relations, a classic equality in the vectorial space is used as equality criteria. Another alternative is to consider the resemblance induced by a fuzzy color space $\widetilde{\Gamma}$ as equality criteria, as follows:

Definition 4.6. *The possibility and necessity that the value of a certain color variable V which is in a fuzzy color Du $\widetilde{C}$ is similar according to the fuzzy color space $\widetilde{\Gamma}$ to the value of a variable V' which is in a fuzzy color Du $\widetilde{C'}$, are, respectively*

$$pos(V = V'|V \text{ is } \widetilde{C} \text{ and } V' \text{ is } \widetilde{C}') = \max_{\mathbf{c},\mathbf{c}' \in \Gamma} \min\{\widetilde{C}(\mathbf{c}), \widetilde{C}'(\mathbf{c}'), R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}')\} \quad (26)$$

$$nec(V = V'|V \text{ is } \widetilde{C} \text{ and } V' \text{ is } \widetilde{C}') = 1 - \max_{\mathbf{c},\mathbf{c}' \in \Gamma} \min\{\widetilde{C}(\mathbf{c}), \widetilde{C}'(\mathbf{c}'), 1 - R_{\widetilde{\Gamma}}(\mathbf{c}, \mathbf{c}')\} \quad (27)$$

This last definition illustrates that there are many possible ways to perform comparison for each of the cases we discuss in the paper, even with different semantics. Note that results provided by Def. (4.6) can be very different from those provided by Def. (4.5). Consider for instance the case of a covering but non-disjoint space as introduced in Section 2. Let us assume that the supports of $\widetilde{C}$ and $\widetilde{C}'$ are disjoint. Hence, Eq. (24) yields 0. Now, let us assume that there exists a third fuzzy color $\widetilde{C}'' \in \widetilde{\Gamma}$ and two crisp colors $\mathbf{c}, \mathbf{c}' \in \Gamma$ such that $\widetilde{C}''(\mathbf{c}) = \widetilde{C}''(\mathbf{c}') = 1$, $\widetilde{C}(\mathbf{c}) = 1$, and $\widetilde{C}'(\mathbf{c}') = 1$. Then Eq. (26) yields 1.

This same alternative could have been applied in the relations defined previously in this section by using the similarity between crisp colors instead of equality in the color space Γ, among other alternative formulations.

5. Inclusion and similarity between fuzzy colors

Inclusion and similarity indexes for fuzzy colors are useful for different purposes, and can be applied to fuzzy colors under different uses (but the same for both colors). For instance, the degree of inclusion between fuzzy colors is useful for determining whether a fuzzy color is a particularization or restriction of another fuzzy color. The degree of similarity allows us to compare fuzzy colors defined for different contexts and see to which extent they are the same.

Fuzzy extensions of the usual predicates of inclusion and equality are provided in sections 5.1 and 5.2. Extensions of similarity and inclusion indexes for fuzzy sets are proposed in Section 5.4 on the basis of a new notion of *degree of overlapping* of fuzzy sets, introduced in Section 5.3. Some experiments for comparing both approaches are shown in Section 5.5.

As in previous sections, let us remark that there are many possible ways to compute degrees of inclusion and similarity, with different semantics and properties. Different studies and proposals can be found in [36, 51, 37, 52, 48, 54, 55]. The amount of such indexes that can be applied is virtually infinite. Our proposals here are intended to show that it is possible to perform these kind of comparisons using fuzzy set theory.

5.1. Inclusion of a fuzzy color in another

In the fuzzy set theory the usual definition of inclusion is crisp: a set A is included in B if $A(x) \leq B(x)\ \forall x$. This notion can be extended by introducing inclusion degrees using different functions. One approach to this problem is to define *inclusion indicators* satisfying a set of axioms known as the *Sinha-Dougherty axioms* [36]. We can use the following definitions from [37]:

Definition 5.1 ([37]). *Let A, B be two fuzzy sets defined on a crisp set X. Let I be a contrapositive fuzzy implication. Then, the family of functions given by*

$$Sub(A, B) = \min_{x \in X} I(A(x), B(x)) \tag{28}$$

is a family of inclusion indicators *that satisfy all the Sinha-Dougherty axioms [37].*

On this basis, it is possible to define an inclusion relation between fuzzy colors as follows:

Definition 5.2. *The degree of inclusion of a fuzzy color $\widetilde{C}$ in another fuzzy color $\widetilde{C'}$ is calculated by means of a suitable inclusion indicator in the family given by Eq. 28, i.e.,*

$$Inc(\widetilde{C}, \widetilde{C}') = Sub(\widetilde{C}, \widetilde{C}') = \min_{\mathbf{c} \in \Gamma} I(\widetilde{C}(\mathbf{c}), \widetilde{C}'(\mathbf{c})) \tag{29}$$

An example of contrapositive fuzzy implication I satisfying the conditions above is the Lukasiewicz's implicator I_L, given by

$$I_L(x, y) = \min(1, 1 - x + y) \tag{30}$$

In particular, the degree of inclusion of a fuzzy color $\widetilde{C}$ in another fuzzy color $\widetilde{C'}$ based on Lukasiewicz's implicator is

$$Inc(\widetilde{C}, \widetilde{C}') = \min_{\mathbf{c}\in\Gamma} \min(1, 1 - \widetilde{C}(\mathbf{c}) + \widetilde{C}'(\mathbf{c})) \tag{31}$$

The inclusion indicator between pieces of color information is useful in the case of fuzzy colors Du when answering questions such as question 8 of the Table 1: *to what extent Jim's knowledge about my car's color is more specific than Tim's?* In the context of Du fuzzy sets, specificity refers to the difficulty of determining the value of the variable when the available information is that its value is within the fuzzy set. Specificity can also be seen roughly as how close is the fuzzy set to be a crisp singleton [56]. Specificity and inclusion are clearly related in the case of normal fuzzy sets: if $F \subset G$, then F is more specific than G. A study of different specificity measures and their properties can be found for instance in [57, 56]. The work in [56] is also somehow related to the present work in that specificity measures are defined on the basis of measures of similarity for fuzzy sets.

In the case of Cu fuzzy colors, inclusion indicators are useful for questions such as question 9 of Table 1, *to what extent this picture uses all the colors of this one?* In Section 5.5, some examples of inclusions in fuzzy color spaces are shown.

5.2. Similarity between two fuzzy colors

The inclusion relation between fuzzy colors can be extended to a similarity relation naturally as the degree to which the double inclusion between sets holds. In this way a family of indicators of similarity between fuzzy sets can be defined as follows [54]:

Definition 5.3. *The degree of similarity between two fuzzy colors $\widetilde{C}$ and $\widetilde{C'}$ is calculated by means of one function of the family Sub as*

$$Sim(\widetilde{C}, \widetilde{C}') = \min\{Sub(\widetilde{C}, \widetilde{C}'), Sub(\widetilde{C}', \widetilde{C})\} \tag{32}$$

The similarity between fuzzy colors, which is a reflexive and symmetric fuzzy resemblance relation, is useful for answering questions such as questions 10 and 11 of the Table 1: *how similar are fuzzy colors defined by two different users for the same category of color?*, and *to what extent Jim's knowledge about my car's color is similar to Tim's?* Again, we will see examples of similarity in fuzzy color spaces in Section 5.5.

As we mentioned at the beginning of this section, there are many other possible ways to compute similarity, see for instance [51, 52, 48, 54, 55].

5.3. *Quantifier-based overlapping indexes for comparison of fuzzy colors*

The measures of inclusion and similarity between fuzzy colors introduced in the previous sections are gradual in that they provide degrees of inclusion and similarity. However, such graduality is due to membership being in $[0, 1]$ only. More specifically, the proposed measures are natural extensions of the crisp notions of inclusion and similarity since, for every finite and arbitrarily large set $X \neq \emptyset$ and fuzzy subsets A and B of X, having a single object $x_0 \in X$ such that $A(x_0) = 1$ and $B(x_0) = 0$ yields $Inc(A, B) = Sim(A, B) = 0$ even when $A(x) = B(x)$ $\forall x \in X \backslash \{x_0\}$. Hence, being completely different in one single element yields a value 0 for inclusion (of the set having the element with respect to the other) and similarity, despite the agreement on the rest of elements for an arbitrarily large set X.

In many cases, similarity and inclusion indexes with a different semantics are needed. For instance, in the crisp case there are similarity indexes that take into account the amount of elements in which A and B agree among all elements in A and B (type-I comparison measure in the sense of [48]), or even including common dissagreements as well (type-II in [48])). An example of type-I measure is the well-known Jaccard index, defined for crisp sets A and B as

$$J(A, B) = \frac{|A \cap B|}{|A \cup B|} \tag{33}$$

The difference between this kind of index and the measures introduced in the previous section can be easily illustrated by a crisp example: let $|X| = 100$, $x \in X$, and $A = X \backslash \{x\}$. Then $Sim(A, X) = 0$ (since there is one element in X that is not in A) whilst $J(A, X) = 0.99$ (since A and X agree in 99% of the elements). It is easy to show that $Sim(A, B) \leq J(A, B)$ for all crisp $A, B \subset X$ with $X \neq \emptyset$ a finite set.

In this section we introduce similarity and inclusion indexes between fuzzy colors that take into account both i) the graduality in the membership to the fuzzy colors, and ii) the amount of elements in which the fuzzy sets agree. They can be seen as fuzzy extensions of similarity and inclusion indexes. Our proposal will be based on the notion of Q-overlapping of fuzzy sets, that we introduce in the following section. In Section 5.4 we introduce our proposals, whilst in Section 5.5 we shall provide an experimental comparison between both kinds of similarity and inclusion measures.

5.3.1. Q-overlapping index

The notion of Q-overlapping index is based on fuzzy quantification[3] as follows:

Definition 5.4. *Let Q be a fuzzy relative quantifier defined by a membership function $Q : [0,1] \to [0,1]$ such that $Q(v) = 0$ iff $v = 0$, $Q(v) = 1$ iff $v = 1$, and Q is non-decreasing. Let A, B be two fuzzy sets defined on a finite set $X \neq \emptyset$ with $A \cup B \neq \emptyset$. The **Q-overlapping** of A and B, denoted $O_Q(A, B)$, is given by the evaluation of the quantified sentence*

Q of $(A \cup B)$ are $(A \cap B)$

using a suitable evaluation method, where union and intersection are performed using a t-conorm and a t-norm, respectively.

Definition 5.4 provides a wide family of indexes parametrized by the t-norm and t-conorm and the quantifier employed, and the quantifier sentence evaluation method. A recent study about quantification methods and their properties can be found in [59]. Many quantifiers can be employed. In particular, we can use the following result:

Proposition 5.1. *Let $N : [0,1] \to [0,1]$ be a strict fuzzy negation. Then, $Q^N : [0,1] \to [0,1]$ defined as*

$$Q^N(v) = 1 - N(v) \tag{34}$$

is a quantifier satisfying the conditions required in Definition 5.4.

Proof: Since N is a strict fuzzy negation then it is continuous and strictly decreasing. Hence, Q^N is strictly increasing and thus, Q^N is non-decreasing. In addition, $N(0) = 1$, and $N(1) = 0$, and hence $Q^N(0) = 1 - N(0) = 0$ and $Q^N(1) = 1 - N(1) = 1$; moreover, since Q^N is strictly increasing, it is $Q(v) = 0$ iff $v = 0$ and $Q(v) = 1$ iff $v = 1$.

A particular case is the quantifier QM defined as

$$QM(v) = v \quad \forall v \in [0,1] \tag{35}$$

[3]Fuzzy quantification has been employed before in the setting of fuzzy color spaces, particularly for defining *linguistic histograms* [58].

which can be obtained using Equation (34) from the standard negation $N(v) = 1 - v$. Let us remark that it is possible to use other quantifiers that cannot be obtained using Proposition 5.1, particularly non-continuous quantifiers.

As a particular case, when using QM as quantifier, a t-norm t and a t-conorm s, and Zadeh's evaluation method for quantified sentences (see [59]), we have:

$$O_{QM}(A, B) = \frac{\sum_{x \in X} t(A(x), B(x))}{\sum_{x \in X} s(A(x), B(x))} \tag{36}$$

which is the usual fuzzy extension of the Jaccard index using as fuzzy cardinality the sigma-count [52, 48].

A suitable method for evaluation of quantified sentences, that we shall employ in the rest of the paper, is the method GD introduced in [60]. Given F, G fuzzy subsets of a finite set $X \neq \emptyset$ with $G \neq \emptyset$ and a fuzzy quantifier Q, the evaluation of the quantified sentence *Q of G are F*, denoted $GD_Q(F/G)$, is

$$GD_Q(F/G) = \sum_{\alpha_i \in \Lambda(F/G)} (\alpha_i - \alpha_{i+1})\, Q\left(\frac{|(F \cap G)_{\alpha_i}|}{|G_{\alpha_i}|}\right) \tag{37}$$

where the intersection is performed via the minimum, F_{α_i} is the $\alpha_i - cut$ of F (same for G), and $\Lambda(F/G) = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is the union of the level sets of F and G, that is, the finite subset containing all the values $F(x) > 0$ and $G(x) > 0$ with $x \in X$, with $1 = \alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha_{m+1} = 0$. In addition, if G is not a normal fuzzy set, G is normalized and the same normalization factor is applied to $F \cap G$ before the evaluation.

5.4. Similarity and inclusion based on the Q-overlapping index

Using the Q-overlapping index we can define similarity and inclusion indexes for fuzzy colors with finite support as follows:

Definition 5.5. *The degree of similarity between two fuzzy colors $\widetilde{C}$ and $\widetilde{C'}$ with finite support, based on a suitable quantified sentence evaluation method, and a non-decreasing quantifier Q with $Q(v) = 0$ iff $v = 0$ and $Q(v) = 1$ iff $v = 1$, is defined as*

$$QSim(\widetilde{C}, \widetilde{C}') = O_Q(\widetilde{C}, \widetilde{C}') \tag{38}$$

Definition 5.5 introduces a family of fuzzy resemblance relations (reflexive and symmetric) depending on the quantifier, fuzzy operators and evaluation method considered. In this paper we shall employ the quantifier QM and the evaluation method GD because of their suitable properties shown in the previous section.

Definition 5.6. *The degree of inclusion of a fuzzy color $\widetilde{C}$ with respect to another fuzzy color $\widetilde{C'}$, both with finite support, based on a suitable quantified sentence evaluation method and a non-decreasing quantifier Q with $Q(v) = 0$ iff $v = 0$ and $Q(v) = 1$ iff $v = 1$, is defined as*

$$QInc(\widetilde{C}, \widetilde{C}') = QSim(\widetilde{C} \cap \widetilde{C}', \widetilde{C}) = O_Q(\widetilde{C} \cap \widetilde{C}', \widetilde{C}). \tag{39}$$

where the intersection is performed using the minimum.

The rationale behind Definition 5.6 is that F is included in G to the degree that $F \cap G = G$. Note that by definition, $QInc(A, B)$ is the evaluation of the quantified sentence ***Q of A ∪ (A ∩ B) are A ∩ (A ∩ B)***. Since maximum and minimum are employed for union and intersection, this is equivalent to ***Q of A are A ∩ B*** which, using the minimum as t-norm and GD as the evaluation method, is the same as the evaluation of ***Q of A are B*** [60], with a clear semantics of inclusion of A in B. Hence, apart from the membership degrees, $QInc$ takes into account the amount of elements in A that are in B, whilst $QSim$ takes into account the amount of elements in $A \cup B$ that are in $A \cap B$. Moreover, as we have shown in the previous section, in the crisp case with QM and using GD, we have

$$QSim(A, B) = O_{QM}(A \cap B / A \cup B) = \frac{|A \cap B|}{|A \cup B|} = J(A, B) \tag{40}$$

In the same case, by the properties of method GD, it is

$$QInc(A, B) = O_{QM}(A \cap B / A) = \frac{|A \cap B|}{|A|} \tag{41}$$

5.5. Experiments and discussion

In order to illustrate the inclusion relation, Table 12 shows some examples of the inclusion degree between fuzzy colors. The column Inc represents the inclusion degree given by Eq. (31), whilst column $QInc$ corresponds to the inclusion as given by Eq. (39) using the quantifier QM and method GD. Specifically, this table shows the inclusion degree between fuzzy colors from the fuzzy color space $\widetilde{\Gamma}_{ISCC-complete}$ based on the ISCC-NBS *Complete Set* mentioned in Section 2 (see [1]) with respect to their related fuzzy colors *Blue*, *Red*, *Orange* and *Yellow* defined in the corresponding space $\widetilde{\Gamma}_{ISCC-basic}$ based on the ISCC-NBS *Basic Set*.

In this case, as explained in [1], the fuzzy colors defined in the fuzzy color space $\widetilde{\Gamma}_{ISCC-complete}$ are more specific than the fuzzy colors defined in $\widetilde{\Gamma}_{ISCC-basic}$ because $\widetilde{\Gamma}_{ISCC-complete}$ is based on the complete set of colors of the *ISCC-NBS*.

In this example it is remarkable that there are not big differences between both inclusion degrees though, as expected, $QInc$ provides larger values. We can observe to what extent some colors are more specific than others. For instance, colors like *Vivid Orange* and *Vivid Yellow* in the *Complete Set* are completely included in *Orange* and *Yellow* of the *Basic Set*, respectively. High values are also obtained by *Vivid Red* and, to a lesser extent, *Vivid Blue*. On the contrary, the lower values are obtained for colors like *Very Light Blue*, *Very Deep Red*, *Dark Red*, *Light Orange* and *Deep Yellow*. These results show that, for some of the adjectives employed in the construction of the *Complete Set* from the terms of the *Basic Set*, the modification of the color they reflect yields colors that have little relation to the original ones. This is the case for instance of *Deep*, *Very Deep*, and *Dark*, corresponding to low intensity. On the contrary, some others like *Vivid*, corresponds to colors that are closer to the pure representative of the basic color term.

For our last example, we have employed a non-partition typology of fuzzy color space for the definition of color terms related to fruits, presented in [1], comprising among others the color terms *Banana* and *Lemon*. In [1], an experiment was conducted for determining a fuzzy color space for fruits for each of 30 users, in order to check for differences in the definitions of the spaces, as well as for compliance with each user's subjectivity[4].

For these fuzzy color spaces, in Table 13, we have calculated the degree of inclusion and similarity between the fuzzy colors corresponding to the color terms *Banana* and *Lemon* defined by each user, using the two approaches to inclusion and similarity we have seen previously. The non-partition typology can be seen in that, for users 2 and 9, both fuzzy colors are equal. It can be also observed that, following the inclusion from Eq. (31) (Inc column) there are users who consider colors that clearly belong to the concept *Banana* and do not belong to the concept *Lemon*, and vice versa, since the corresponding degree of inclusion between fuzzy colors is 0 (users 5, 6, 10, 13, 14, 15, 18, 19, 20, 24, 25, 26 and 30) while others do not consider that possibility (users 2, 9, 11, 16, 17, 22, 27 and 29) since the degree of inclusion is greater than 0. On the other hand, we

[4]A detailed description of the experiment, as well as the dataset and software employed, can be found in `http://www.jfcssoftware.com`.

	Blue Inc	Blue QInc
Vivid Blue	0.611	0.832
Brilliant Blue	0.427	0.706
Strong Blue	0.906	0.997
Deep Blue	0.207	0.473
Very Light Blue	0	0.053
Light Blue	0	0.168
Moderate Blue	0.25	0.499
Dark Blue	0	0.089

(a)

	Red Inc	Red QInc
Vivid Red	0.76	0.959
Strong Red	0.04	0.535
Deep Red	0	0.099
Very Deep Red	0	0
Moderate Red	0	0.01
Dark Red	0	0

(b)

	Orange Inc	Orange QInc
Vivid Orange	1	1
Brilliant Orange	0.525	0.696
Strong Orange	0.9	0.989
Deep Orange	0.424	0.576
Light Orange	0	0.085
Moderate Orange	0.399	0.515

(c)

	Yellow Inc	Yellow QInc
Vivid Yellow	1	1
Brilliant Yellow	0.425	0.492
Strong Yellow	0.435	0.7
Deep Yellow	0	0.077
Light Yellow	0.062	0.127
Moderate Yellow	0.312	0.415

(d)

Table 12: Inclusion degree between fuzzy colors defined in $\widetilde{\Gamma}_{ISCC-complete}$ with respect to their related fuzzy colors defined in $\widetilde{\Gamma}_{ISCC-basic}$. (a) *Blue*, (b) *Red*, (c) *Orange* and (d) *Yellow*.

can see that significantly larger values are obtained for some users following the inclusion from Eq. (39) (column $QInc$). This is the case in the *Banana* $\subseteq$ *Lemon* column, which raises from the value 0 provided by Inc to values as high as 0.716 (user 24), with other five users raising from 0 to values higher than 0.5. The same happens in the *Lemon* $\subseteq$ *Banana* column for four users. This example illustrates how one single object in A not being in B makes Inc yields a value 0 even if a high percentage of overlapping exists between both fuzzy sets, and how $QInc$ is able to discard such cases by measuring the overlapping. Finally, in the last column, the similarity between *Banana* and *Lemon* can be observed from two different points of view: as an extension of the crisp notion of similarity where a value 0 is yielded if there is at least one element in A that is not in B (Sim), and taking into account the amount of elements in which A and B agree among all elements in A and B ($QSim$). It can be observed that, following the similarity from Eq. (32) (Sim

column) there are users who consider that the color concepts *Banana* and Lemon are completely different (similarity degree = 0), whereas following the similarity from Eq. (38) ($QSim$ column) values greater than 0.5 can be obtained (users 8, 16, 20, 23 and 24).

6. Conclusions

Color comparison is a key issue and one of the most important problems in computational systems dealing with color categories and color information. In

	Banana $\subseteq$ Lemon		Lemon $\subseteq$ Banana		Lemon $=$ Banana	
	Inc	QInc	Inc	QInc	Sim	QSim
user1	0	0.504	0	0.381	0	0.287
user2	1	1	1	1	1	1
user3	0	0.551	0.32	0.641	0	0.424
user4	0	0.33	0	0.381	0	0.217
user5	0	0.049	0	0.011	0	0.009
user6	0	0.37	0.18	0.33	0	0.222
user7	0.674	0.971	0.475	0.804	0.475	0.785
user8	0	0.65	0.809	0.981	0	0.642
user9	1	1	1	1	1	1
user10	0	0.417	0	0.209	0	0.169
user11	0.451	0.72	0.422	0.889	0.422	0.662
user12	0	0.694	0.092	0.581	0	0.469
user13	0.082	0.603	0	0.461	0	0.363
user14	0	0.164	0	0.089	0	0.067
user15	0	0.442	0	0.537	0	0.323
user16	0.776	0.913	0	0.53	0	0.504
user17	0.192	0.584	0	0.625	0	0.457
user18	0	0.021	0	0.037	0	0.014
user19	0	0.018	0	0.008	0	0.006
user20	0.507	0.881	0	0.635	0	0.586
user21	0.769	0.882	0.746	0.906	0.746	0.81
user22	0.263	0.789	0.942	0.965	0.263	0.767
user23	0	0.702	0.571	0.851	0	0.626
user24	0	0.716	0.534	0.798	0	0.609
user25	0	0.182	0	0.084	0	0.064
user26	0	0.053	0	0.027	0	0.019
user27	0.742	0.962	0.824	0.913	0.742	0.882
user28	0	0.38	0	0.201	0	0.168
user29	0.787	0.959	0.639	0.813	0.639	0.787
user30	0	0.259	0	0.187	0	0.136

Table 13: Inclusion and similarity degrees between the fuzzy colors *Banana* and *Lemon* as defined by 30 users.

	$\mathbf{c}'$	Cu $\widetilde{C}'$	Du $\widetilde{C}'$
$\mathbf{c}$	$r_{\widetilde{C}}(\mathbf{c}, \mathbf{c}')$ (4) $R(\mathbf{c}, \mathbf{c}')$ (9)	K($\mathbf{c}$,$\widetilde{C}'$) (18)	Pos(V=$\mathbf{c}$ — V is $\widetilde{C}'$) (19) Nec(V=$\mathbf{c}$ — V is $\widetilde{C}'$) (20)
Cu $\widetilde{C}$		K($\widetilde{C}$,$\widetilde{C}'$) (21) Inc($\widetilde{C}$,$\widetilde{C}'$) (29) Sim($\widetilde{C}$,$\widetilde{C}'$) (32) QInc($\widetilde{C}$,$\widetilde{C}'$) (39) QSim($\widetilde{C}$,$\widetilde{C}'$) (38)	Pos($V \in \widetilde{C}$ — V is $\widetilde{C}'$) (22) Nec($V \in \widetilde{C}$ — V is $\widetilde{C}'$) (23)
Du $\widetilde{C}$			Pos(V is $\widetilde{C}$ — V is $\widetilde{C}'$) (24) Nec(V is $\widetilde{C}$ — V is $\widetilde{C}'$) (25) Inc($\widetilde{C}$,$\widetilde{C}'$) (29) Sim($\widetilde{C}$,$\widetilde{C}'$) (32) QInc($\widetilde{C}$,$\widetilde{C}'$) (39) QSim($\widetilde{C}$,$\widetilde{C}'$) (38)

Table 14: Relations between color information and the corresponding equation numbers. Table is symmetric, only elements in the diagonal or above are shown.

this paper we have shown that fuzzy set and possibility theories provide suitable tools for solving two main, widely present color comparison problems: fuzzy color resemblance (isomorphic to fuzzy color categorization), and matching of color information under different uses of fuzzy colors. Although we use [1] for illustrative purposes to define colors which we use to compare, our proposal here is not dependent on the way in which colors are defined. In the same way, the particular formulations of resemblance, compatibility, inclusion and similarity measures in this paper are just particular ways to perform such comparisons, but there are many other tools in the literature that can be used for the same purposes, as we have discussed and referenced all through the paper.

This work integrates and updates with new formulations some of our previous, partial results for some of these problems, with additional contributions regarding inclusion and similarity on the basis of the concepts of quantifier-based overlapping indexes, that are here introduced for the first time, as well as illustrative examples and an experimental comparison. Table 14 summarizes our contributions and contains the most common relations that, in our opinion, can be considered when matching color information, where $\mathbf{c}$ and $\mathbf{c}' \in \Gamma$ are crisp colors, and $\widetilde{C}$, $\widetilde{C}'$ $\in \widetilde{\Gamma}$ are fuzzy colors. To the best of our knowledge, this is the first work to offer a global and integrated view of the problem of color comparison based on fuzzy colors.

As future work, we shall address the problem of matching color information defined by level 2 fuzzy sets considering conjunctive and disjunctive use, like for instance *the colors I like are 0.5/red + 1/grey + 1/blue* (conjunctively-used) and *she told me to paint the whole wall in 0.8/red + 0.7/green + 1/grey* (disjunctively-used). We shall also study the effect of the evaluation method and quantifier employed in the definition of Q-overlapping measures, in order to determine the most suitable combination for particular purposes. Finally, our results will be used in real-world problems, particularly fuzzy image segmentation based on color and texture, image information retrieval, and linguistic description of images including color terms in the setting of data-to-text systems.

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