Using the Generalized Collage Theorem for Estimating Unknown Parameters in Perturbed Mixed Variational Equations

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Abstract

In this paper, we study a mixed variational problem subject to perturbations, where the noise term is modelled by means of a bilinear form that has to be understood to be "small" in some sense. Indeed, we consider a family of such problems and provide a result that guarantees existence and uniqueness of the solution. Moreover, a stability condition for the solutions yields a Generalized Collage Theorem, which extends previous results by the same authors. We introduce the corresponding Galerkin method and study its convergence. We also analyze the associated inverse problem and we show how to solve it by means of the mentioned Generalized Collage Theorem and the use of adequate Schauder bases. Numerical examples show how the method works in a practical context.

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1 Introduction: Direct vs Inverse Problem

In applied mathematics there are always two problems associated with a mathematical model of natural phenomena, the so called direct and inverse problem. The direct problem usually refers to the determination and the analysis of the solution to a completely prescribed equation or set of equations. In many contexts, a direct problem assumes the form of differential equations subject to known initial conditions and/or boundary conditions. The inverse problem, instead, describes the model from the parameter estimation point-of-view. Once the model has been created and some empirical solution has been observed, it is of paramount importance to be able to determine a combination of the unknown parameters such that the induced problem admits empirical observation as an approximate solution. One can see the inverse problem as the natural opposite of a direct problem. The study

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of inverse problems has attracted a lot of attention in the literature. Very often, in fact, the inverse problem is ill-posed, while the direct problem is well-posed. When a problem is well-posed, it has the properties of existence, uniqueness, and stability of the solution [25]. On the other hand, an ill-posed problem loses one or more of these desirable properties. This makes the analysis of inverse problems very challenging from a numerical perspective: even when the direct problem is easily solvable, the corresponding inverse problem can be very complex and difficult to solve.

The literature is quite rich in papers proposing ad-hoc methods to address ill-posed inverse problems: These methods usually involve a minimization problem which includes a regularization term that stabilizes the numerical algorithm. One can see [26, 27, 31, 32, 33, 34, 35] and the references therein to get better details about these approaches.

Quite recently other approaches have been introduced to deal with inverse problems when the corresponding direct problem can be viewed as the solution to a fixed point equation and analyzed through the well-known Banach's fixed point theorem. These approaches rely on the so-called Collage Theorem, that it is a simple consequence of the above mentioned Banach's theorem (see [3, 4]). In fractal imaging, these results have been used extensively to approximate a target image by the fixed point (image) of a contractive fractal transform [4, 5, 22, 24, 28, 30, 36]. Over the last few years, the same philosophy has been used to deal with inverse problems for ordinary and partial differential equations. The fact that an ordinary (and even a partial) differential equation can be formulated as a fixed point equation in a specific complete metric space provides the gateway to pursuing analysis based on some of the above results. Indeed, solution frameworks and related results have been established for inverse problems associated with different families of ordinary and partial differential equations (see [6, 10, 15, 16, 17, 18, 19, 20, 21, 23, 29]).

In this paper, we explore systems of mixed variational equations, both from the direct problem and inverse problem point of view. The mixed variational formulation of a linear elliptical boundary problem is obtained from the introduction of a new variable, usually related to any of the derivatives of the original variable, and whose presence is justified in many cases by its applied interest. The theoretical results, known as the *Babuška–Brezzi theory*, and corresponding numerical methods, mixed finite elements, have been successfully developed in the last decades: see, for instance, [2, 7, 9, 13]. What we discuss in this paper, instead, is a modified mixed variational problem that includes a kind of perturbation.

The paper is organized as follows. Section 2 presents a Generalized Collage Theorem for a family of perturbed systems of mixed variational equations. Section 3 analyzes and discusses a Galerkin numerical method for the direct problem. Section 4 presents the formulation of the inverse problem and provides a numerical example. Section 5 concludes the paper.

2 Families of Mixed Variational Equations

Unlike the classical system of mixed variational equations corresponding to the mixed variational formulation of a differential problem, we discuss a more general version of it, which includes a certain perturbation. The perturbation term is modelled by means of a new bilinear form, that has to be interpreted to be small in some sense. More specifically, let E and F be real Hilbert spaces, $a : E \times E \longrightarrow \mathbb{R}$, $b : E \times F \longrightarrow \mathbb{R}$ and $c : F \times F \longrightarrow \mathbb{R}$ be continuous bilinear forms, and $x^* : E \longrightarrow \mathbb{R}$ and $y^* : F \longrightarrow \mathbb{R}$ be linear forms. The problem under consideration is given in these terms:

$$\text{find } (x_0, y_0) \in E \times F \text{ such that } \begin{cases} a(x_0, \cdot) + b(\cdot, y_0) = x^* \\ b(x_0, \cdot) + c(y_0, \cdot) = y^* \end{cases} . \tag{2.1}$$

In fact, we state a more general result for a family of problems that include a stability property, (2.3), which will be essential for our purposes since it will allow us to deal with a Galerkin scheme for a specific direct problem as well as with a suitable inverse problem in the next sections. Furthermore, such a stability condition, (2.3), it is a *Generalized Collage Theorem* that extends that in [20] and in [6] in the Hilbertian framework, and that in Section 4 will be useful in order to solve a parameter estimation problem.

In what follows, "$\wedge$" denotes "and".

Theorem 2.1 *Let J be a nonempty set and, for each $j \in J$, let E_j and F_j be real Hilbert spaces, $a_j : E_j \times E_j \longrightarrow \mathbb{R}$, $b_j : E_j \times F_j \longrightarrow \mathbb{R}$ and $c_j : F_j \times F_j \longrightarrow \mathbb{R}$ be continuous and bilinear forms, and let*

$$K_j := \{x \in E_j : \; b_j(x, \cdot) = 0\}.$$

Suppose that

(i) $x \in K_j \;\wedge\; a_j(x, \cdot)_{|K_j} = 0 \Rightarrow \; x = 0,$

and for some $\alpha_j, \beta_j > 0$ the following conditions hold

(ii) $x \in K_j \;\Rightarrow\; \alpha_j \|x\| \leq \|a_j(\cdot, x)_{|K_j}\|,$

(iii) $y \in F \;\Rightarrow\; \beta_j \|y\| \leq \|b_j(\cdot, y)\|.$

Assume in addition that

(iv)

$$\rho := \sup_{j\in J} \max\left\{\frac{1}{\alpha_j}, \frac{1}{\beta_j}\left(1+\frac{\|a_j\|}{\alpha_j}\right), \frac{1}{\beta_j^2}\|a_j\|\left(1+\frac{\|a_j\|}{\alpha_j}\right)\right\} > 0$$

and that for all $j \in J$,

(v) $\|c_j\| < \dfrac{1}{\rho}$.

Then, given $j \in J$ and $(x_j^*, y_j^*) \in E_j^* \times F_j^*$ there exists a unique $(x_j, y_j) \in E_j \times F_j$ such that

$$\left\{\begin{array}{l} a_j(x_j,\cdot) + b_j(\cdot, y_j) = x_j^* \\ b_j(x_j,\cdot) + c_j(y_j,\cdot) = y_j^* \end{array}\right. . \tag{2.2}$$

Moreover, if for each $j \in F$, $(\hat{x}_j, \hat{y}_j) \in E_j \times F_j$, then

$$\inf_{j\in J} \max\{\|x_j - \hat{x}_j\|, \|y_j - \hat{y}_j\|\} \le \inf_{j\in J} \frac{\rho}{1-\rho\|c_j\|}\left(\|x_j^* - a_j(\hat{x}_j,\cdot) - b_j(\cdot,\hat{y}_j)\| + \|y_j^* - b_j(\hat{x}_j,\cdot) - c_j(\hat{y}_j,\cdot)\|\right). \tag{2.3}$$

Proof. Let $j \in J$. The existence and uniqueness of solution for problem (2.2) is a well-known fact (see, for instance [7, Proposition 4.3.2]), but we give a sketch of the proof in order to derive the control of the norms in (2.3) in a precise way. So, let us endow the product space $E_j \times F_j$ with the norm

$$\|(x, y)\| := \max\{\|x\|, \|y\|\}, \qquad (x \in E_j,\ y \in F_j)$$

and its dual space $E_j^* \times F_j^*$ with the corresponding dual norm, that is,

$$\|(x^*, y^*)\| := \|x^*\| + \|y^*\|, \qquad (x^* \in E_j^*,\ y^* \in F_j^*).$$

According to conditions (i), (ii) and (iii) and from [13, Theorem 2.1], the bounded and linear operator $S_j : E_j \times F_j \longrightarrow E_j^* \times F_j^*$ defined at each $(x, y) \in E_j \times F_j$ as

$$S_j(x, y) := (a_j(x,\cdot) + b_j(\cdot, y), b_j(x,\cdot))$$

is an isomorphism. But, in view of [1, Theorem 2.3.5], in order to state the existence of a unique solution for the perturbed mixed system (2.2) it is enough to show that

$$\|S_j^{-1}\| < \frac{1}{\|c_j\|}. \tag{2.4}$$

The inequality in (2.4) is valid, since in view of [14, Theorem 4.72] or [12, Theorem 3.6], (iv) and (v) we have that

$$\begin{aligned}
\|S_j^{-1}\| &= \sup_{\|x^*\|+\|y^*\|\le 1} \|S_j^{-1}(x^*,y^*)\| \\
&\le \sup_{\|x^*\|+\|y^*\|\le 1} \max\left\{\frac{\|x^*\|}{\alpha_j}+\frac{1}{\beta_j}\left(1+\frac{\|a_j\|}{\alpha_j}\right)\|y^*\|, \frac{1}{\beta_j}\left(1+\frac{\|a_j\|}{\alpha_j}\right)\left(\|x^*\|+\frac{\|a_j\|}{\beta_j}\|y^*\|\right)\right\} \\
&\le \sup_{\|x^*\|+\|y^*\|\le 1} \max\left\{\frac{1}{\alpha_j}, \frac{1}{\beta_j}\left(1+\frac{\|a_j\|}{\alpha_j}\right), \frac{1}{\beta_j}\left(1+\frac{\|a_j\|}{\alpha_j}\right), \frac{\|a_j\|}{\beta_j^2}\left(1+\frac{\|a_j\|}{\alpha_j}\right)\right\}(\|x^*\|+\|y^*\|) \\
&\le \rho \\
&< \frac{1}{\|c_j\|}.
\end{aligned}$$

Furthermore, by making use of (2.4) and of [14, Theorem 4.72] or [12, Theorem 3.6] once again, we arrive at

$$\max\{\|x_j\|, \|y_j\|\} \le \frac{\rho}{1-\rho\|c_j\|}\left(\|x^*\|+\|y^*\|\right), \tag{2.5}$$

where $(x_j, y_j) \in E_j \times F_j$ is the unique solution of (2.2). To conclude, given $(\hat{x}_j, \hat{y}_j) \in E_j \times F_j$, since $(x_j - \hat{x}_j, y_j - \hat{y}_j)$ is the unique solution of the perturbed mixed problem

$$\left\{\begin{array}{l} a_j(x_j-\hat{x}_j,\cdot)+b_j(\cdot,y_j-\hat{y}_j)=x_j^*-a_j(x,\cdot)-b_j(\cdot,\hat{y}_j) \\ b_j(x_j-\hat{x}_j,\cdot)+c_j(y_j-\hat{y}_j,\cdot)=y_j^*-b_j(\hat{x}_j,\cdot)-c_j(\hat{y}_j,\cdot) \end{array}\right.,$$

then, according to inequality (2.5),

$$\max\{\|x_j-\hat{x}_j\|, \|y_j-\hat{y}_j\|\} \le \frac{\rho}{1-\rho\|c_j\|}\left(\|x_j^*-a_j(\hat{x}_j,\cdot)-b_j(\cdot,\hat{y}_j)\|+\|y_j^*-b_j(\hat{x}_j,\cdot)-c_j(\hat{y}_j,\cdot)\|\right).$$

Finally, the arbitrariness of $j \in F$ yields (2.3). □

Example 2.2 Given $\Omega = (0,1)^2$, $\Gamma = \partial\Omega$, $\delta \in \mathbb{R}$ and $f \in H_0^1(\Omega)$, let us consider the boundary value problem:

$$\left\{\begin{array}{rl} \Delta^2\psi+\delta\psi = f & \text{in } \Omega \\ \psi|_\Gamma = 0 & \\ \Delta\psi|_\Gamma = 0 & \end{array}\right.. \tag{2.6}$$

If one takes $w := -\Delta\psi$, then this problem is equivalent to

$$\left\{\begin{array}{rl} w+\Delta\psi = 0 & \text{in } \Omega \\ -\Delta w+\delta\psi = f & \text{in } \Omega \\ \psi|_\Gamma = 0 & \\ w|_\Gamma = 0 & \end{array}\right.. \tag{2.7}$$

Then, multiplying its first equation by a test function $v \in H_0^1(\Omega)$, and integrating by parts, we arrive at

$$\int_\Omega wv - \int_\Omega \nabla w \nabla v = 0.$$

On the other hand, when multiplying the second equation of (2.7) by a test function $\phi \in H_0^1(\Omega)$, and, proceeding as above, we write it as

$$-\int_\Omega \nabla w \nabla \phi - \delta \int_\Omega \psi\phi = -\int_\Omega f\phi.$$

Therefore, if we take the real Hilbert spaces $E = F := H_0^1(\Omega)$, the continuous bilinear forms $a : E \times E \longrightarrow \mathbb{R}$, $b : E \times F \longrightarrow \mathbb{R}$ and $c : F \times F \longrightarrow \mathbb{R}$ defined for each $w, v \in E$, and $\phi, \psi \in F$, as

$$a(w, v) := \int_\Omega wv,$$

$$b(v, \psi) := -\int_\Omega \nabla v \nabla \psi,$$

and

$$c(\psi, \phi) := -\delta \int_\Omega \psi\phi,$$

and the continuous linear forms $x^* \in E^*$ and $y^* \in F^*$ given by

$$x^*(v) := 0 \qquad (v \in E)$$

and

$$y^*(\phi) := -\int_\Omega f\phi, \qquad (\phi \in F),$$

then we have derived this variational formulation of the problem (2.6): find $(w, \psi) \in E \times F$ such that

$$\left\{ \begin{array}{rcll} v \in E & \Rightarrow & a(w, v) + b(v, \psi) & = x^*(v) \\ w \in W & \Rightarrow & b(w, \phi) + c(\psi, \phi) & = y^*(\phi) \end{array} \right. ,$$

which adopts the form of (2.2) with $\operatorname{card}(J) = 1$. Then, taking into account that the operator $\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ is an isomorphism, it is very easy to check that, when $\delta < 1$, Theorem 2.1 applies and this problems admits a unique solution (w, ψ) such that, for any $(\hat{w}, \hat{\psi}) \in E \times F$,

$$\max\{\|w - \hat{w}\|, \|\psi - \hat{\psi}\|\} \leq \frac{1}{1-\delta} \left(\|a(x, \cdot) + b(\cdot, y)\| + \|y^* - b(x, \cdot) - c(y, \cdot)\|\right)$$

and, in particular,

$$\max\{\|w\|, \|\psi\|\} \leq \frac{\|f\|}{1-\delta}.$$

$\square$

3 The Galerkin Algorithm

Now we focus our effort on developing the Galerkin method for the perturbed mixed problem (2.2) when $\mathrm{card}(F) = 1$.

Theorem 3.1 *Let E and F be real Hilbert spaces and that $a : E \times E \longrightarrow \mathbb{R}$, $b : E \times F \longrightarrow \mathbb{R}$ and $c : F \times F \longrightarrow \mathbb{R}$ are continuous bilinear forms. Given $n \in \mathbb{N}$, let E_n and F_n be finite dimensional vector subspaces of E and F, respectively, and let*

$$K_n := \{x \in E_n : \ b(x, \cdot)_{|F_n} = 0\}.$$

Let us also suppose that

(i) $x \in K_n \ \wedge \ a(x, \cdot)_{|K_n} = 0 \Rightarrow \ x = 0$

and there exist $\alpha_n, \beta_n > 0$ such that

(ii) $x \in K_n \ \Rightarrow \ \alpha_n \|x\| \leq \|a(\cdot, x)_{|K_n}\|$,

(iii) $y \in F_n \ \Rightarrow \ \beta_n \|y\| \leq \|b(\cdot, y)_{|E_n}\|$

and for

(iv)

$$\rho_n := \max \left\{ \frac{1}{\alpha_n}, \frac{1}{\beta_n} \left(1 + \frac{\|a\|}{\alpha_n}\right), \frac{1}{\beta_n^2} \|a\| \left(1 + \frac{\|a\|}{\alpha_n}\right) \right\} > 0,$$

there holds

(v) $\|c_{|F_n \times F_n}\| < \dfrac{1}{\rho_n}$.

Then, given $(x^*, y^*) \in E^* \times F^*$, there exists a unique $(x_n, y_n) \in E_n \times F_n$ such that

$$\left\{ \begin{array}{l} a(x_n, \cdot)_{|E_n} + b(\cdot, y_n)_{|E_n} = x^*_{|E_n} \\ b(x_n, \cdot)_{|F_n} + c(y_n, \cdot)_{|F_n} = y^*_{|F_n} \end{array} \right. . \tag{3.1}$$

Furthermore, for all $(x, y) \in E \times F$ we have that

$$\max\{\|x_n - x\|, \|y_n - y\|\} \leq \frac{\rho_n}{1 - \rho_n \|c\|} \left(\|x^*_{|E_n} - a(x, \cdot)_{|E_n} - b(\cdot, y)_{|E_n}\| + \|y^*_{|F_n} - b(x, \cdot)_{|E_n} - c(y, \cdot)_{|F_n}\| \right).$$

PROOF. It follows from Theorem 2.1, by means of standard arguments. □

We conclude the section by illustrating these results with the discretization of Example 2.2.

Example 3.2 Let us consider the boundary value problem in Example 2.3

$$\left\{\begin{array}{r l} \Delta^2\psi+\delta\psi=f & \text{in } \Omega \\ \psi|_\Gamma=0 & \\ \Delta\psi|_\Gamma=0 & \end{array}\right. , \tag{3.2}$$

with $\delta \in \mathbb{R}$ and $f \in H_0^1(\Omega)$. We take $\delta = 1/15$, and the function $f \in H_0^1(\Omega)$ defined for $(x, y) \in (0, 1)^2$ in order to have the solution $\psi_0(x, y) := 10^3(x(x-1)y(y-1))^4$.

Now let us consider the Haar system $\{h_k\}_{k\geq 1}$ in $L^2(0, 1)$, which is a Schauder basis for such real Hilbert space. We construct a basis for $H_0^1(0, 1)$ using $h_{k k\geq\geq 1}$ by definting $g_1 := 1$ and for all $k > 1$,

$$g_k(t) = \int_0^t h_{k-1}(s)\, ds.$$

It is easy to prove (see [11]) that the collection of functions $\{g_k\}_{k\geq 1}$ is a Schauder basis for the real Hilbert space $H^1(0, 1)$ and, as a consequence, $\{g0_k\}_{k\geq 1}$, where $g0_k = g_{k+2}$, is a basis for $H_0^1(0, 1)$. We now use the following bijective mapping from $\mathbb{N}$ onto $\mathbb{N}\times\mathbb{N}$ to define a bivariate basis for $H_0^1((0, 1)^2)$: let $[\ ]$ stand for "floor function" and let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}\times\mathbb{N}$ be the mapping given by

$$\sigma(n) := \left\{\begin{array}{l l} (\sqrt{n}, \sqrt{n}) & if \quad [\sqrt{n}] = \sqrt{n} \\ (n-[\sqrt{n}]^2, [\sqrt{n}]+1) & if \quad 0 < n-[\sqrt{n}]^2 \leq [\sqrt{n}] \\ ([\sqrt{n}]+1, n-[\sqrt{n}]^2-[\sqrt{n}]) & if \quad [\sqrt{n}] < n-[\sqrt{n}]^2 \end{array}\right. . \tag{3.3}$$

Then, the sequence $\{G0_k\}_{k\geq 1}$ defined as

$$G0_n(s, t) = g_p0(s)g_q0(t), \qquad (s, t \in (0, 1))$$

where $\sigma(n) = (p, q)$, is a Schauder basis for the real Hilbert space $H_0^1((0, 1)^2)$.

We can now use this basis to construct finite dimensional subspaces of the real Hilbert spaces above: For each $m \geq 1$, let us consider the finite-dimensional subspaces of E and F

$$E_m := F_m := \text{span}\{G0_1, G0_2, \ldots, G0_m\}.$$

Then, the corresponding discrete problem satisfies the assumptions in Theorem 3.1. More precisely: since $E_m = F_m$, then $K_m = \{0\}$, so (i) is trivially true, as well as (ii), for which any constant α_m can be chosen, then we fix $\alpha_m = \alpha$, for a certain $\alpha > 0$. Moreover, (iii) is a straightforward consequence of Poincaré's inequality ([8, Proposition 8.13]), with a uniform constant $\beta_m = \beta > 0$. In particular,

ρ_m also is constant. Finally, since $\|c\| \leq \delta$, it suffices to choose α in order that $\delta < 1/\rho_m$ to guarantee the validity of (v). Therefore, conditions in Theorem 3.1 are uniformly satisfied.

Now, the linearity of the problem implies that the discrete problem can be equivalently reformulated as: Find $(w_m, \psi_m) \in E_m \times F_m$, the unique solution of the discrete perturbed system

$$\begin{cases} a(w_m, G0_i) + b(G0_i, \psi_m) = x^*(G0_i) & i = 1, \ldots, m, \\ b(w_m, G0_{i-m}) + c(\psi_m, G0_{i-m}) = y^*(G0_{i-m}) & i = m+1, \ldots, 2m. \end{cases}$$

We show, in the following tables, the numerical results obtained for various values of m. The value (w_0, ψ_0) denotes the exact solution of the continuous problem with δ given above.

	$m = 9$	$m = 25$	$m = 81$
$\|\psi_m - \psi_0\|_{L^2(\Omega)}$	1.33×10^{-3}	9.53×10^{-4}	4.33×10^{-4}
$\|\psi_m - \psi_0\|_{H_0^1(\Omega)}$	1.46×10^{-2}	1.16×10^{-2}	7.11×10^{-3}
$\|w_m - w_0\|_{L^2(\Omega)}$	9.41×10^{-2}	7.09×10^{-2}	2.56×10^{-2}
$\|w_m - w_0\|_{H_0^1(\Omega)}$	1.48	1.22	7.8×10^{-1}

4 The Inverse Problem

In this section we discuss the general formulation of the inverse problem for the system of mixed variational equations (2.1). Suppose that $(\hat{x}_j, \hat{y}_j) \in E_j \times F_j$ is a pair of observed/interpolated functions. Suppose, in addition, that $a_j : E_j \times E_j \longrightarrow \mathbb{R}$, $b_j : E_j \times F_j \longrightarrow \mathbb{R}$ and $c_j : F_j \times F_j \longrightarrow \mathbb{R}$ are families of bilinear forms, and $x_j^* : E_j \longrightarrow \mathbb{R}$ and $y_j^* : F_j \longrightarrow \mathbb{R}$ are families of linear forms, all them fulfilling hypotheses (i) and (ii) in Theorem 2.1. The inverse problem can be formulated as follows: Find $\hat{j} \in J$, where J is a compact subset of $\mathbb{R}^p$, such that $(\hat{x}_j, \hat{y}_j)$ is an approximate solution to the perturbed mixed variational system (2.2). Assuming that

$$\alpha := \inf_{j \in J} \alpha_j > 0, \quad \beta := \inf_{j \in J} \beta_j > 0, \quad \delta := \sup_{j \in J} \|a_j\|, \quad \gamma := \inf_{j \in J} \|c_j\| > 0,$$

then conditions (iii) and (iv) in Theorem 2.1 are valid as soon as $\rho\gamma < 1$, and so, such a result applies.

Then, in view of the collage estimation (2.3), the inverse problem can be solved by minimizing the following objective function

$$\xi(j) := \|x_j^* - a_j(\hat{w}, \cdot) - b_j(\cdot, \hat{\psi})\| + \|y_j^* - b_j(\hat{w}, \cdot) - c_j(\hat{\psi}, \cdot)\| \tag{4.1}$$

over $j \in J$. This objective function measures the distance between the left and the right hand-side of Eq. (2.2). If the optimal value is closer to zero the better the approximation will be as the distance

noise	C_1	C_2	C_3	Collage Distance
0%	1.000107013477	0.9999842592864	0.46822577278	0.00054485531439904
0.5%	1.000059971746	1.0000823273167	0.30727953435	0.00056207875045237
1.0%	1.000009640383	1.0001771911339	0.15302707869	0.00059892386287046
1.5%	0.999956019932	1.0002688495507	0.00547416104	0.00065538437959271
2%	0.999899110999	1.0003573014412	-0.13537360557	0.00073145365856316

Table 1: Results of the Numerical Simulation. True values are $(C_1, C_2, C_3) = (1, 1, \frac{1}{4})$.

between the target solution $(\hat{x}_j, \hat{y}_j)$ and the theoretical one (x_j, y_j) gets very small. The optimization problem can be discretized by means of Schauder bases in the real Hilbert spaces involved, along the lines of [6, Section 3] and [20, Section 4]. The minimization algorithm has been implemented using the MAPLE 2018 optimization toolbox. The optimal solution provides the estimation of the unknown parameters of the model.

Now we illustrate a numerical implementation of the algorithm. We start with the system in the Example 2.2, setting $\delta = \frac{1}{4}$ and choosing $f(x, y)$ such that the solution $u(x, y)$ to the problem is $10^3[x(1-x)y(1-y)]^4$. We solve the system in COMSOL. Isotherms and surface contour plots are shown in Figure 1. Then we sample the numerical solution on a uniform grid of 9×9 interior points of $[0, 1]^2$. We interpolate each set of 81 points, with low-amplitude relative noise added, to build two target functions $\hat{u}$ and $\hat{w}$. We feed these representations into our Generalized Collage Theorem machinery. We finite dimensionalize Eq. (4.1) by working with a uniform, piecewise-linear, finite-element basis on [0,1] with 81 interior nodes, representing each of the two target functions in this basis. Finally, knowing $f(x, y)$, we recover C_1, C_2, C_3 so that $\hat{u}$ and $\hat{w}$ are approximate solutions to the system

$$\begin{cases} C_1 \Delta u + C_2 w = 0, \\ -C_1 \Delta w + C_3 u = f(x, y), \end{cases}$$

The true values are $C_1 = 1$, $C_2 = 1$,$C_3 = \frac{1}{4}$. The results are presented in Table 1. The number in the final column of the table is the value of the generalized collage distance. We say that for low relative noise values, the method does reasonably well.

One can easily notice that the estimation of the coefficient C_3 is not so good: this is depending on the numerical approximation of the Δu rather than the method itself. When solving an inverse problem, in fact, empirical data and observations for u are used to estimate the unknown parameters. In this model, however, the empirical data is used to get a numerical approximation of Δu which turns out to add more noise to the inverse problem implementation. In addition, the use of a piecewise-linear

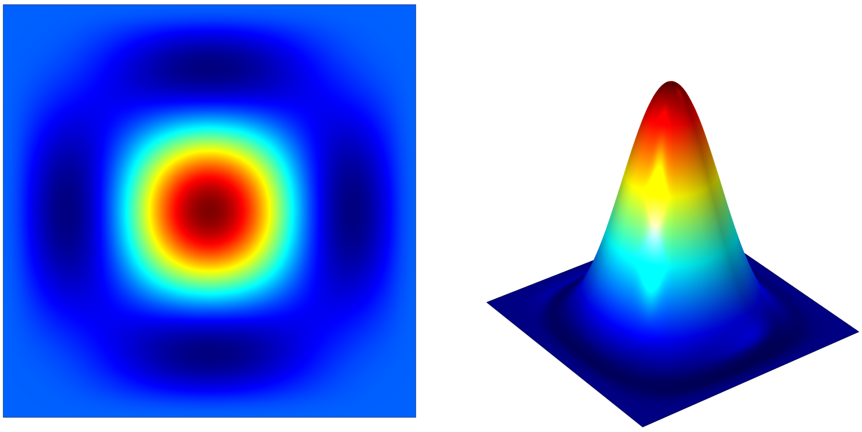

Figure 1: Isotherms and surface contour plot for the target solution in the example.

basis means that derivatives of the basis representations are piecewise constant and imprecise, also generating numerical error; given this observation, we view the results in Table 1 for low-amplitude relative noise quite positively.

5 Conclusion

In this paper we have studied the direct problem and the inverse problem for perturbed mixed variational equations. We have shown conditions that guarantee the existence and uniqueness of the solution to the direct problem and formulated the inverse problem as an optimization problem using an extension of the Collage Theorem. We have also provided a numerical Galerkin scheme to approximate the solution to this model. A potential application to a fourth-order PDE example is also illustrated: by substitution one can reduce this example to a perturbed mixed variational problem and then use the theory and the numerical treatment presented in this work to solve it.

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