A minimax approach for the study of systems of variational equations and related Galerkin schemes

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Abstract. The aim of this work is to analyze the existence of a solution for a quite general variational inequalities system, which includes those appearing in the theory of mixed variational equations, the so-called *Babuška–Brezzi theory*. It is done by means of a minimax inequality, which also provides us with a estimate of the norm of the solution. Then, we derive a numerical method of the Galerkin type to approximate the solution of such a system. Its stability follows from the mentioned control of the norm of the solution.

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1 Introduction

Minimax inequalities are normally associated with the game theory –they arose in this framework– although they have turned out to be a powerful tool in other fields: see, for instance, [3, 4, 10, 11, 13, 14, 15, 16, 17]. One of them, the classical minimax inequality of von Neumann–Fan, will be our starting point for characterizing the existence of a solution for a certain system of variational inequalities.

The class of systems we are going to analyze arises in many situations. To evoke one of them, let us recall that the study of variational equations with constraints emerges naturally, among others, from the context of the elliptic boundary value problem, when their essential boundary conditions are treated as constraints in their standard variational formulation. This leads one to its variational formulation, which coincides with the system of variational equations:

$$\text{find } x_0 \in X \text{ such that } \begin{cases} z \in Z & \Rightarrow f(z) = a(x_0, z) \\ y \in Y & \Rightarrow g(y) = b(x_0, y) \end{cases},$$

for some Banach spaces X and Y, a closed vector subspace Z of X, some continuous bilinear forms $a : X \times X \longrightarrow \mathbb{R}$ and $b : X \times Y \longrightarrow \mathbb{R}$, and $f \in X^*$ and $g \in Y^*$ ("$*$" stands for "topological dual space"): see the details, for instance, in [9, Section 4.6.1]. In a more general way, we deal with the following problem –indeed we will consider this question in a larger class of normed spaces: let X be a real reflexive Banach space, $N \geq 1$, and suppose that for each $j = 1, \ldots, N$, Y_j is a real Banach space, $y_j^* \in Y_j^*$, C_j is a convex subset of Y_j with $0 \in C_j$, and $a_j : X \times Y_j \longrightarrow \mathbb{R}$ is a bilinear form satisfying $y_j \in C_j \Rightarrow a_j(\cdot, y_j) \in X^*$; then

$$\text{find } x_0 \in X \text{ such that } \begin{cases} y_1 \in C_1 \Rightarrow y_1^*(y_1) \leq a_1(x_0, y_1) \\ \cdots \\ y_N \in C_N \Rightarrow y_N^*(y_N) \leq a_N(x_0, y_N) \end{cases}. \tag{1.1}$$

This kind of variational system is so general that it includes certain mixed variational formulations associated with some elliptic problems, those in the so-called *Babuška–Brezzi theory* (see, for instance [2, 8] and some of its generalizations [7]).

The paper is organized as follows. In Section 2 we state, making use of the von Neumann–Fan minimax inequality, a characterization of the existence of a solution for the variational inequalities system under consideration, in terms of that of a positive constant. In particular, such a characterization guarantees the stability of numerical schemes of the Galerkin type for approximating the solution, which is developed in Section 3. The corresponding finite dimensional subspaces are generated from adequate Schauder bases satisfying certain restrictions on their dimensions and depend on the concrete problem. Finally, in Section 4, we illustrate our results with some numerical examples.

2 Variational equations in dual normed spaces

In this section, we focus on deriving an extension of the Lax–Milgram theorem, a characterization of the solvability of a system of variational equations. In the reflexive case, the system under consideration leads us to the system (1.1).

We first evoke the minimax inequality of von Neumann–Fan, a particular case of [5, Theorem 2]:

Theorem 2.1 *Suppose that C and D are convex subsets of two vector spaces in such a way that C is endowed with a topology for which it is compact, and that $\Phi : C \times D \longrightarrow \mathbb{R}$ is concave and* *upper-semicontinuous on C and convex on D. Then*

$$\max_{x\in C}\inf_{y\in D}\Phi(x,y)=\inf_{y\in D}\max_{x\in C}\Phi(x,y).$$

Let us notice that if the system of variational inequalities (1.1) admits a solution $x_0\in X$, then, for all $(y_1,\ldots,y_N)\in\prod_{j=1}^N C_j$ we have that (add the N equations and take $\gamma:=\|x_0\|$)

$$\sum_{j=1}^N y_j^*(y_j)\le\gamma\left\|\sum_{j=1}^N a_j(\cdot,y_j)\right\|.$$

In the following result we prove that this necessary condition is also sufficient.

As usual, $_+$ denotes “positive part”.

Theorem 2.2 *Let $N\ge 1$ and let $E,F_1,\ldots,F_N$ be real normed spaces. Assume that for each $j=1,\ldots,N$, $y_j^*\in F_j^*$, C_j is a convex subset of F_j with $0\in C_j$, and that $a_j:E^*\times F_j\longrightarrow\mathbb{R}$ is a bilinear form such that*

$$y_j\in C_j\ \Rightarrow\ a_j(\cdot,y_j)\in E.$$

Then, there exists $x_0^*\in E^*$ satisfying

$$\left\{\begin{array}{c} y_1\in C_1\Rightarrow y_1^*(y_1)\le a_1(x_0^*,y_1)\\ \cdots\\ y_N\in C_N\Rightarrow y_N^*(y_N)\le a_N(x_0^*,y_N)\end{array}\right.$$

if and only if, for some $\gamma\ge 0$, the inequality

$$(y_1,\ldots,y_N)\in\prod_{j=1}^N C_j\ \Rightarrow\ \sum_{j=1}^N y_j^*(y_j)\le\gamma\left\|\sum_{j=1}^N a_j(\cdot,y_j)\right\|$$

is valid.

In addition, if one of these equivalent statements is satisfied, then we can choose a solution $x_0^*\in X^*$ of the system of variational inequalities with

$$\|x_0^*\|=\left(\sup\left\{\frac{\sum_{j=1}^n y_j^*(y_j)}{\left\|\sum_{j=1}^N a_j(\cdot,y_j)\right\|}\ :\ (y_1,\ldots,y_N)\in\prod_{j=1}^N C_j,\ \sum_{j=1}^N a_j(\cdot,y_j)\ne 0\right\}\right)_+. \tag{2.1}$$

Proof. Let $C := \prod_{j=1}^{N} C_j$ and let B_{E^*} stand for the closed unit ball of the Banach space E^*. Then, taking into account that $0 \in \bigcap_{j=1}^{N} C_j$,

$$\text{there exists } x_0^* \in E^* \text{ such that } \left\{ \begin{array}{c} y_1 \in C_1 \Rightarrow y_1^*(y_1) \leq a_1(x_0^*, y_1) \\ \cdots \\ y_N \in C_N \Rightarrow y_N^*(y_N) \leq a_N(x_0^*, y_N) \end{array} \right.$$

$$\Updownarrow$$

$$\text{there exists } \gamma \geq 0,\ x_0^* \in \gamma B_{E^*} :\ (y_1, \ldots, y_N) \in C \ \Rightarrow\ \sum_{j=1}^{N} y_j^*(y_j) \leq \sum_{j=1}^{N} a_j(x_0^*, y_j)$$

that is,

$$\text{there exists } \gamma \geq 0 : \max_{x^* \in \gamma B_{E^*}} \inf_{(y_1,\ldots,y_N) \in C} \left(\sum_{j=1}^{N} a_j(x^*, y_j) - \sum_{j=1}^{N} y_j^*(y_j) \right) \geq 0.$$

But if we apply the minimax theorem, Theorem 2.1, to the function $\Phi : \gamma B_{E^*} \times C \longrightarrow \mathbb{R}$ defined at each $(x^*, (y_1, \ldots, y_N)) \in \gamma B_{E^*} \times C$ as

$$\Phi(x^*, (y_1, \ldots, y_N)) := \sum_{j=1}^{N} a_j(x^*, y_j) - \sum_{j=1}^{N} y_j^*(y_j),$$

(it is linear on its second variable and affine and weak* continuous on its first variable) then, the former assert is equivalent to

$$\text{there exists } \gamma \geq 0 : \inf_{(y_1,\ldots,y_N) \in C} \max_{x^* \in \gamma B_{E^*}} \left(\sum_{j=1}^{N} a_j(x^*, y_j) - \sum_{j=1}^{N} y_j^*(y_j) \right) \geq 0,$$

i.e.,

$$\text{there exists } \gamma \geq 0 \text{ such that } (y_1, \ldots, y_N) \in C \ \Rightarrow\ \sum_{j=1}^{N} y_j^*(y_j) \leq \gamma \left\| \sum_{j=1}^{N} a_j(\cdot, y_j) \right\|,$$

which is the equivalent to the statement that we wanted to prove.

Let us notice that the preceding minimax reasoning guarantees us that for some solution x_0^* of the system, $\|x_0^*\| \leq \gamma$. Since the inequality

$$\sum_{j=1}^{N} y_j^*(z_j) \leq \left(\sup \left\{ \frac{\sum_{j=1}^{n} y_j^*(y_j)}{\left\| \sum_{j=1}^{N} a_j(\cdot, y_j) \right\|} :\ (y_1, \ldots, y_N) \in C,\ \sum_{j=1}^{N} a_j(\cdot, y_j) \neq 0 \right\} \right)_+ \left\| \sum_{j=1}^{N} a_j(\cdot, z_j) \right\|$$

is clearly satisfied for all $z \in C$, then there exists a solution x_0^* of the variational system such that

$$\|x_0^*\| \leq \left(\sup \left\{ \frac{\sum_{j=1}^{n} y_j^*(y_j)}{\left\| \sum_{j=1}^{N} a_j(\cdot, y_j) \right\|} : (y_1, \ldots, y_N) \in C, \ \sum_{j=1}^{N} a_j(\cdot, y_j) \neq 0 \right\} \right)_+ .$$

But for such a solution x_0^*, adding up the inequaities and in view of the continuity of the linear functionals $a_j(\cdot, y_j)$, $j = 1, \ldots, N$, we arrive at

$$\left(\sup \left\{ \frac{\sum_{j=1}^{n} y_j^*(y_j)}{\left\| \sum_{j=1}^{N} a_j(\cdot, y_j) \right\|} : (y_1, \ldots, y_N) \in C, \ \sum_{j=1}^{N} a_j(\cdot, y_j) \neq 0 \right\} \right)_+ \leq \|x_0^*\|,$$

and so we have stated the announced control of the norm (2.1) of one solution. $\square$

If we consider not only some fixed functionals $y_1^*, \ldots, y_N^*$ but also any functionals on $F_1, \ldots, F_N$, we arrive at the following characterization:

Corollary 2.3 *Suppose that E is a real normed space, $N \geq 1$, $F_1, \ldots, F_N$ are real Banach spaces, and that for each $j = 1, \ldots, N$, C_j is a convex subset of F_j with $0 \in C_j$, and $a_j : E^* \times F_j \longrightarrow \mathbb{R}$ is a bilinear form in such a way that*

$$y_j \in C_j \ \Rightarrow \ a_j(\cdot, y_j) \in E.$$

Then, the following assertions are equivalent:

(i) *For all $y_1^* \in F_j^*, \ldots, y_N^* \in F_j^*$ there exists $x_0^* \in E^*$ such that*

$$\left\{ \begin{array}{c} y_1 \in C_1 \Rightarrow y_1^*(y_1) \leq a_1(x_0^*, y_1) \\ \cdots \\ y_N \in C_N \Rightarrow y_N^*(y_N) \leq a_N(x_0^*, y_N) \end{array} \right. . \tag{2.2}$$

(ii) *For some $\rho > 0$*

$$(y_1, \ldots, y_N) \in \prod_{j=1}^{N} C_j \ \Rightarrow \ \rho \sum_{j=1}^{N} \|y_j\| \leq \left\| \sum_{j=1}^{N} a_j(\cdot, y_j) \right\|.$$

Moreover, the validity of one of these equivalent statements guarantees that some solution $x_0^* \in E^*$ of the system of variational inequalities satisfies the control

$$\rho \|x_0^*\| \leq \max_{j=1,\ldots,N} \|y_j^*\|.$$

Proof. According to Theorem 2.2 and the continuity of the linear functionals $y_1^*, \dots, y_N^*$, the implication (ii)⇒(i) is obvious. For the other one, (i)⇒(ii), let us endow the product space $F := \prod_{j=1}^{N} F_j$ with the sum norm, i.e., for $y = (y_1, \dots, y_N) \in F$

$$\|y\| := \sum_{j=1}^{N} \|y_j\|.$$

Let us first notice that, in view of the hypothesis (2.2), we have

$$\sum_{j=1}^{N} a_j(\cdot, y_j) = 0 \ \Rightarrow \ y = 0, \tag{2.3}$$

for any $y = (y_1, \dots, y_N) \in C := \prod_{j=1}^{N} C_j$, since given $y_1^* \in F_1^*, \dots, y_n^* \in F_N^*$, there exists $x_0^* \in E^*$ satisfying (2.2) and then

$$\sum_{j=1}^{N} y_j^*(y_j) \leq 0,$$

therefore, any continuous and linear functional on F is non-positive at y, or in other words, $y = 0$.

As a consequence

$$\left\{ \frac{(y_1, \dots, y_N)}{\left\| \sum_{j=1}^{N} a_j(\cdot, y_j) \right\|} : \ (y_1, \dots, y_N) \in C, \ \sum_{j=1}^{N} a_j(\cdot, y_j) \neq 0 \right\}$$

$$\|$$

$$\left\{ \frac{(y_1, \dots, y_N)}{\left\| \sum_{j=1}^{N} a_j(\cdot, y_j) \right\|} : \ (y_1, \dots, y_N) \in C, \ (y_1, \dots, y_N) \neq 0 \right\},$$

and, thanks to (2.2), Theorem 2.2 and the uniform boundedness principle, we conclude that this subset of F is bounded, that is, there exists $\rho > 0$ such that

$$(y_1, \dots, y_N) \in C \ \Rightarrow \ \rho \sum_{j=1}^{N} \|y_j\| \leq \left\| \sum_{j=1}^{N} a_j(\cdot, y_j) \right\|. \tag{2.4}$$

Finally, Theorem 2.2 implies the existence of a solution $x_0^* \in E^*$ of the system of variational inequalities in such a way that

$$\|x_0^*\| = \left(\sup \left\{ \frac{\sum_{j=1}^{n} y_j^*(y_j)}{\left\| \sum_{j=1}^{N} a_j(\cdot, y_j) \right\|} : (y_1, \dots, y_N) \in C, \ \sum_{j=1}^{N} a_j(\cdot, y_j) \neq 0 \right\} \right)_+ ,$$

and so, from this, (2.3) and (2.4), we deduce that

$$\begin{aligned} \|x_0^*\| &\leq \sup \left\{ \frac{\sum_{j=1}^{N} \|y_j\|}{\left\| \sum_{j=1}^{N} a_j(\cdot, y_j) \right\|} : (y_1, \dots, y_N) \in C, \ \sum_{j=1}^{N} \|y_j\| \neq 0 \right\} \max_{j=1,\dots,N} \|y_j^*\| \\ &\leq \frac{1}{\rho} \max_{j=1,\dots,N} \|y_j^*\|. \end{aligned}$$

$\square$

The uniqueness of solution for system (2.2) is an easy algebraical question when the convex sets C_j are balanced ($C_j = -C_j$): if that system admits a solution, it is unique if, and only if, each $x^* \in E^*$ is null, whenever

$$(y_1, \dots, y_N) \in \prod_{j=1}^{N} C_j \ \Rightarrow \ \sum_{j=1}^{N} a_j(x^*, y_j) = 0.$$

Let us emphasize that when the dual space E^* is reflexive, i.e., E is also it, then the weak* continuity condition

$$j = 1, \dots, N, \ y_j \in F_j \ \Rightarrow \ a_j(\cdot, y_j) \in E$$

becomes a norm (or weak) continuity assumption. Moreover, if the convex sets coincide with the whole spaces, then they contain 0 and the inequalities are, in fact, equalities; that is, the system adopts the form:

$$\text{find } x_0^* \in E^* \text{ such that } \left\{ \begin{array}{c} y_1^* = a_1(x_0^*, \cdot) \\ \cdots \\ y_N^* = a_N(x_0^*, \cdot) \end{array} \right. .$$

Example 2.4 Given $a, b, \mu \in \mathbb{R}$ and $f \in L^p(0,1)$ with $1 < p < \infty$, let us consider the boundary value problem:

$$\left\{ \begin{array}{r} -z'' + \mu z = f \quad \text{on } (0,1) \\ z(0) = a, \quad z(1) = b \end{array} \right. . \tag{2.5}$$

If one takes $x := z'$, then this problem is equivalent to

$$\left\{\begin{array}{rl} x = z' & \text{on } (0,1) \\ -x' + \mu z = f & \text{on } (0,1) \\ z(0) = a, & z(1) = b \end{array}\right. . \tag{2.6}$$

Then, multiplying its first equation by a test function $y \in W^{1,q}(0,1)$, where q is the conjugate exponent of p, and integrating by parts, we arrive at

$$\int_0^1 z'y = \langle \mathrm{tr}(y), (b,-a)\rangle - \int_0^1 zy',$$

$\langle \cdot, \cdot \rangle$ being the usual inner product in $\mathbb{R}^2$ and $\mathrm{tr} : W^{1,q}(0,1) \longrightarrow \mathbb{R}^2$ the trace operator in $W^{1,q}(0,1)$. On the other hand, when multiplying the second equation of (2.6) by a test function $w \in L^q(0,1)$, we write it as

$$\int_0^1 x'w - \mu \int_0^1 zw = -\int_0^1 fw.$$

Therefore, if we take the reflexive Banach spaces

$$X := W^{1,p}(0,1),\ Y := W^{1,q}(0,1),\ Z := L^p(0,1),\ W := L^q(0,1),$$

the continuous bilinear forms $a : X \times Y \longrightarrow \mathbb{R}$, $b : Y \times Z \longrightarrow \mathbb{R}$, $c : X \times W \longrightarrow \mathbb{R}$ and $d : Z \times W \longrightarrow \mathbb{R}$ defined for each $x \in E, y \in F,\ z \in G$ and $w \in H$ as

$$a(x,y) := \int_0^1 xy,$$

$$b(y,z) := \int_0^1 y'z,$$

$$c(x,w) := \int_0^1 x'w$$

and

$$d(z,w) := -\mu \int_0^1 zw,$$

and the continuous linear forms $y_0^* \in Y^*$ and $w_0^* \in W^*$ given by

$$y_0^*(y) := \langle \mathrm{tr}(y), (b,-a)\rangle, \qquad (y \in Y)$$

and

$$w_0^*(w) := -\int_0^1 fw, \qquad (w \in W),$$

then we have derived this variational formulation of the problem (3.2): find $(x_0, z_0) \in X \times Z$ such that

$$\left\{\begin{array}{rll} y \in Y \ \Rightarrow & a(x_0,y) + b(y,z_0) & = y_0^*(y) \\ w \in W \ \Rightarrow & c(x_0,w) + d(z_0,w) & = w_0^*(w) \end{array}\right. .$$

But this system adopts the form of (2.2) with $N = 2$, the dual (reflexive) real space $E := (X \times Z)^*$, the Banach spaces $F_1 := Y$, $F_2 := W$, the convex sets $C_1 := F_1$, $C_2 := F_2$, the continuous bilinear forms $a_1 : E^* \times F_1 \longrightarrow \mathbb{R}$ and $a_2 : E^* \times F_2 \longrightarrow \mathbb{R}$ defined at each $(x, z) \in E^*$, $y \in F_1$ and $w \in F_2$ as

$$a_1((x, z), y) := a(x, y) + b(y, z)$$

and

$$a_2((x, z), w) := c(x, w) + d(z, w),$$

and the continuous linear forms $y_1^* := y_0^*$ and $y_2^* := w_0^*$.

We are going to prove that this variational formulation admits a unique solution $(x, z) \in E^* = X \times Z$, provided that $|\mu| < 0.5$, or in other words, we will show that when endowing, for instance, the linear spaces $X \times Z$ and $Y^* \times W^*$ with the complete product norms

$$\|(x, z)\| := \max\{\|x\|, \|z\|\}, \qquad (x \in X,\ z \in Z)$$

and

$$\|(y^*, w^*)\| := \|y^*\| + \|z^*\|, \qquad (y^* \in Y^*,\ w^* \in W^*),$$

then the continuous and linear operator $T : X \times Z \longrightarrow Y^* \times W^*$ defined at each $(x, z) \in X \times Z$ as

$$T(x, z) := (a(x, \cdot) + b(\cdot, z), c(x, \cdot) + d(z, \cdot))$$

is a one-to-one and onto operator. The fact that it is injective (uniqueness of solution) is an easy and straightforward issue, so let us deal with the surjectivity of T (existence of solution), applying Corollary 2.3. To conclude that the assumptions in such a result are valid, let us define the auxiliary continuous (the same product norms) and linear operator $S : X \times Z \longrightarrow Y^* \times W^*$ by

$$S(x, z) := (a(x, \cdot) + b(\cdot, z), c(x, \cdot)), \qquad ((x, z) \in X \times Z).$$

This operator is bijective, as was stated in [7, Example 3.8] via a particular case of Corollary 2.3, and it follows from that reasoning that

$$\|S^{-1}\| \leq 2. \tag{2.7}$$

Let us notice that

$$\begin{aligned} \|T - S\| &= \sup\{\|T(x, z)\| : \ \|(x, z)\| = 1\} \\ &= \sup\{\|d(z, \cdot)\| : \ \|(x, z)\| = 1\} \\ &= \sup\{|\mu|\|z\| : \ \|z\| = 1\} \\ &= |\mu|, \end{aligned}$$

therefore, acoording to [1, Theorem 2.3.5], the inequality (2.7) and the fact that $|\mu| < 0.5$, we deduce the surjectivity of T, i.e., our variational problem admits a unique solution. In particular, our continuous bilinear forms satisfy the hypothesis of Corollary 2.3. □

It is worth mentioning that the preceding example does not fall into the scope of the Babuška–Brezzi theory, or even the more general one of [7], where the analysis of Corollary 2.3 is done by means of independent conditions of the involved bilinear forms.

3 The Galerkin method

Now we deal with stating a Galerkin scheme for the system of inequalities under study. More specifically, we consider the case in which the convex sets C_j coincide with the space F_j and the bilinear forms are continuous; in that way, as in Example 2.4, the inequalities become continuous equalities. To this end, we discretize Corollary 2.3 instead of Theorem 2.2, since the abstract uniformity in that result means numerical stability:

Corollary 3.1 *Let E be a real normed space, $N \geq 1$, $F_1, \ldots, F_N$ real Banach spaces, and for each $j = 1, \ldots, N$, let $a_j : E^* \times F_j \longrightarrow \mathbb{R}$ be a continuous bilinear form. Assume that for all $k \geq 1$, ${E^*}^k$, $F_1^k, \ldots, F_N^k$ are finite-dimensional vector subspaces of E^*, $F_1, \ldots, F_N$, respectively. Then, given $k \geq 1$, for any $y_1^* \in F_1^*, \ldots, y_n^* \in F_N^*$, the existence of a unique $x_k^* \in {E^*}^k$ such that*

$$\left\{ \begin{array}{c} y_1 \in F_1^k \ \Rightarrow \ y_1^*(y_1) = a_1(x_k^*, y_1) \\ \cdots \\ y_N \in F_N^k \ \Rightarrow \ y_N^*(y_N) = a_1(x_k^*, y_N) \end{array} \right. \tag{3.1}$$

is equivalent to that of $\rho_k > 0$ with

$$(y_1, \ldots, y_N) \in \prod_{j=1}^N F_j^k \ \Rightarrow \ \rho_k \sum_{j=1}^N \|y_j\| \leq \left\| \sum_{j=1}^N a_j(\cdot, y_j)_{|{E^*}^k} \right\|,$$

and the fact that if $x^* \in {E^*}^k$ satisfies

$$(y_1, \ldots, y_N) \in \prod_{j=1}^N F_j^k \ \Rightarrow \ \sum_{j=1}^N a_j(x^*, y_j) = 0,$$

then $x^* = 0$.

In the case that these statements are satisfied, then

$$\min\{\|x_k^*\| : \ x_k^* \in {E^*}^k \text{ is a solution of } (3.1)\} \leq \frac{1}{\rho_k} \max_{j=1,\ldots,N} \|y_j^*\|.$$

Proof. It is a straightforward consequence of Corollary (2.3). □

By assuming a certain uniformity in the infsup conditions in the previous result, one arrives at the main result regarding the numerical method for solving the system of variational equations under consideration:

Corollary 3.2 *Let E be a real normed space, $N \geq 1$, $F_1, \ldots, F_N$ real Banach spaces, and for each $j = 1, \ldots, N$, let $y_j^* \in F_j^*$ and $a_j : E^* \times F_j \longrightarrow \mathbb{R}$ be a continuous bilinear form. Let us also assume that for all $k \geq 1$, ${E^*}^k$, $F_1^k, \ldots, F_N^k$ are finite-dimensional vector subspaces of E^*, $F_1, \ldots, F_N$, respectively, and $\rho_k > 0$, such that the corresponding system of variational equations*

$$\begin{cases} y_1 \in F_1^k \;\Rightarrow\; y_1^*(y_1) = a_1(x_0^*, y_1) \\ \cdots \\ y_N \in F_N^k \;\Rightarrow\; y_N^*(y_N) = a_1(x_0^*, y_N) \end{cases}$$

admits a unique solution $x_0^* \in E^*$, the inequality

$$\rho_k \sum_{j=1}^N \|y_j\| \leq \left\| \sum_{j=1}^N a_j(\cdot, y_j)_{|{E^*}^k} \right\|$$

is valid for any $(y_1, \ldots, y_N) \in \prod_{j=1}^N F_j^k$, and

$$\rho := \inf_{k \geq 1} \rho_k > 0.$$

Suppose that if $x^* \in {E^*}^k$

$$(y_1, \ldots, y_N) \in \prod_{j=1}^N F_j^k \;\Rightarrow\; \sum_{j=1}^N a_j(x^*, y_j) = 0,$$

then $x^* = 0$.

If for all $k \geq 1$, $x_k^* \in {E^*}^k$ is the unique solution of the discrete system

$$\begin{cases} y_1 \in F_1^k \;\Rightarrow\; y_1^*(y_1) = a_1(x_k^*, y_1) \\ \cdots \\ y_N \in F_N^k \;\Rightarrow\; y_N^*(y_N) = a_1(x_k^*, y_N) \end{cases},$$

then, for some $\delta > 0$, there holds

$$k \geq 1 \;\Rightarrow\; \|x_0^* - x_k^*\| \leq \delta \operatorname{dist}(x_0^*, {E^*}^k).$$

Proof. Let $k \geq 1$ and let $\overline{x}_k^* \in {E^*}^k$. Since $x_k^* - \overline{x}_k^*$ is the unique solution of the problem: find $\tilde{x}_k^* \in {E^*}^k$ such that

$$\begin{cases} y_1 \in F_1^k \;\Rightarrow\; {y_1^*}^k(y_1) = a_1(\tilde{x}_k^*, y_1) \\ \cdots \\ y_N \in F_N^k \;\Rightarrow\; {y_N^*}^k(y_N) = a_1(\tilde{x}_k^*, y_N) \end{cases},$$

where for all $j = 1, \ldots, N$,

$${y_j^*}^k := a_j(x_0^* - \overline{x}_k^*, \cdot)_{|{F_j}^k},$$

we conclude, in view of Corollary 3.1 and the assumption on ρ_k, that

$$\begin{aligned} \|\overline{x}_k^* - x_k^*\| &\leq \frac{1}{\rho_k} \max_{j=1,\ldots,N} \|y_j^{*k}\| \\ &\leq \frac{1}{\rho} \max_{j=1,\ldots,N} \|a_j\| \|x_0^* - \overline{x}_k^*\|. \end{aligned}$$

The announced inequality follows from the triangle inequality and the arbitrariness of $\overline{x}_k^* \in E^{*k}$, with

$$\delta = 1 + \frac{1}{\rho} \max_{j=1,\ldots,N} \|a_j\|.$$

□

We conclude the section by illustrating these results with the discretization of Example 2.4 in two cases:

Example 3.3 Let us consider the boundary value problem in Example 2.4

$$\left\{ \begin{array}{rl} -z'' + \mu z = f & \text{on } (0,1) \\ z(0) = a, & z(1) = b \end{array} \right. , \tag{3.2}$$

with $a, b, \mu \in \mathbb{R}$ and $f \in L^p(0,1)$ $(1 < p < \infty)$ in the two following cases:

1. We take $a := 0$, $b := 0$, $\mu := \frac{1}{3}$ and the function $f \in L^{5/4}(0,1)$ defined for $t \in (0,1)$ as

$$f(t) := \frac{1}{3}\left(\frac{25}{14} - \frac{25}{14}(1-t)^{7/5} - \frac{25}{14}t\right) + \frac{1}{(1-t)^{3/5}}.$$

   Let us notice that $f \notin L^2(0,1)$. The Banach spaces in Example 2.4 considered in this case are $X = W^{1,5/4}(0,1)$, $Z = L^{5/4}(0,1)$, $E = (X \times Z)^*$, $F_1 = W^{1,5}(0,1)$ and $F_2 = L^5(0,1)$.

2. Now $a := 0$, $b := \frac{225}{238}$, $\mu := \frac{1}{5}$ and take the function $f \in L^{3/2}(0,1)\backslash L^2(0,1)$ defined for $t \in (0,1)$ as

$$f(t) := \frac{-113 - 125(1-\sqrt{t})^{3/5} - 75\sqrt{t} + 5(-37 + 34(1-\sqrt{t})^{3/5})t + 95t^{3/2} + 40t^2}{238(1-\sqrt{t})^{3/5}},$$

   so the corresponding Banach spaces are $X = W^{1,3/2}(0,1)$,$Z = L^{3/2}(0,1)$, $E = (X \times Z)^*$, $F_1 = W^{1,3}(0,1)$ and $F_2 = L^3(0,1)$.

In order to generate the finite dimensional subspaces of the real Banach spaces above, let us recall the well-known fact that for any $1 \leq p < \infty$, the Haar system $\{h_k\}_{k\geq 1}$ in $L^p(0,1)$ is a basis for this Banach space that satisfies the orthogonality property

$$\int_0^1 h_i h_j = \delta_{ij},$$

where δ_{ij}, the Kronecker symbol, takes the value 1 when $i = j$ and 0 elsewhere. Let $\{g_k\}_{k\geq 1}$ be the sequence of real-valued functions defined for each $t \in [0,1]$ as

$$g_1(t) := 1$$

and for all $i > 1$,

$$g_i(t) = \int_0^1 h_{i-1}(s)\, ds.$$

As proven in [6, Proposition 4.8], the sequence $\{g_k\}_{k\geq 1}$ is a basis for the Banach space $W^{1,p}(0,1)$ with $1 \leq p < \infty$. We now introduce, for each $k \geq 1$, the finite-dimensional subspaces of E^*, F_1 and F_2, respectively:

$$X^k := \text{span}\{g_1, g_2, \ldots, g_{k+1}\}, \quad F_1^k := X^k,$$

$$Z^k := \text{span}\{h_1, h_2, \ldots, h_k\}, \quad F_2^k := Z^k,$$

and

$$E^{*k} := X^k \times Z^k.$$

Then, the corresponding discrete problem is: find $(x_k, z_k) \in E^{*k}$, the unique solution of the discrete system

$$\begin{cases} y_{1k} \in F_1^k & \Rightarrow \quad a_1((x_k, z_k), y_{1k}) = y_1^*(y_{1k}) \\ y_{2k} \in F_2^k & \Rightarrow \quad a_2((x_k, z_k), y_{2k}) = y_2^*(y_{1k}) \end{cases}.$$

We show, in the following tables, the numerical results obtained in both cases for $k = 16, 32, 64$. The value (x_0, z_0) denotes the exact solution of the continuous problem (2.4) with a, b, μ given above. For the first case we have

	$k = 16$	$k = 32$	$k = 64$
$\|x_k - x_0\|_{L^{5/4}(0,1)}$	2.53×10^{-2}	1.10×10^{-2}	4.76×10^{-3}
$\|z_k - z_0\|_{L^{5/4}(0,1)}$	9.56×10^{-3}	4.46×10^{-3}	2.11×10^{-3}

while, for the second one,

	$k = 16$	$k = 32$	$k = 64$
$\|x_k - x_0\|_{L^{3/2}(0,1)}$	4.81×10^{-2}	2.29×10^{-2}	1.09×10^{-2}
$\|z_k - z_0\|_{L^{3/2}(0,1)}$	2.10×10^{-2}	9.71×10^{-3}	4.46×10^{-3}

□

Let us emphasize to conclude that, in the reflexive case, the numerical treatment of some inverse problems related to the systems of variational equalities under consideration has been developed in [12].

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References

[1] K. Atkinson and W. Han, *Theoretical numerical analysis. A functional functional analysis framework*, Third Edition, Texts in Applied Mathematics **39**, Springer, 2009.

[2] D. Boffi et al., *Mixed finite elements, compatibility conditions and applications*, Lecture Notes in Mathematics **1939**, Springer–Verlag, Berlin, 2008.

[3] J.M. Borwein and O. Giladi, *Some remarks on convex analysis in topological groups*, J. Convex. Anal. **23** (2016), 313–332.

[4] X.T. Deng, Z.F. Li and S.Y. Wang, *A minimax portfolio selection strategy with equilibrium*, European J. Oper. Res. **166** (2005), 278–292.

[5] K. Fan, *Minimax theorems*, Proceedings of the National Academy of Sciences of the United States of America **39** (1953), 42–47.

[6] S. Fučik, *Fredholm alternative for nonlinear operators in Banach spaces and its applications to differential and integral equations*, Časopis Pěst. Mat. **96** (1971), 371–390.

[7] A.I. Garralda Guillem and M. Ruiz Galán, *Mixed variational formulations in locally convex spaces*, J. Math. Anal. Appl. **414** (2014), 825–849.

[8] G.N. Gatica, *A simple introduction to the mixed finite element method. Theory and applications*, SpringerBriefs in Mathematics, Springer, Cham, 2014.

[9] C. Grossmann, H.G. Roos and M. Stynes, *Numerical treatment of partial differential equations*, Universitext, Springer–Verlag, Berlin, 2007.

[10] P.Q. Khanh and N.H. Quan, *General existence theorems, alternative theorems and applications to minimax problems*, Nonlinear Anal. **72** (2010), 2706–2715.

[11] N. Kenmochi, *Monotonicity and compactness methods for nonlinear variational inequalities, Handbook of differential equations: stationary partial differential equations, IV*, 203–298, Elsevier/North-Holland, Amsterdam, 2007.

[12] H. Kunze, D. La Torre, K. Levere and M. Ruiz Galán, *Inverse problems via the "generalized collage theorem" for vector-valued Lax-Milgram-based variational problems*, Math. Probl. Eng. 2015, Art. ID 764643, 8 pp.

[13] Y. Polyanskiy, *Saddle point in the minimax converse for channel coding*, IEEE Trans. Inform. Theory **59** (2013), 2576–2595.

[14] B. Ricceri, *On a minimax theorem: an improvement, a new proof and an overview of its applications*, Minimax Theory Appl. **2** (2017), 99–152.

[15] M. Ruiz Galán, *The Gordan theorem and its implications for minimax theory*, J. Nonlinear Convex Anal. **17** (2016), 2385–2405.

[16] J. Saint Raymond, *A new minimax theorem for linear operators*, Minimax Theory Appl. **3** (2018), 131–160.

[17] S. Simons, *Minimax and monotonicity*, Lecture Notes in Mathematics **1693**, Springer–Verlag, Berlin, 1998.