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ABSTRACT
The combined use of origami and dynamic geometry software has recently appeared in mathematics education to enrich students' geometric thinking. The objective of this research is to study the roles played by the interaction of two artifacts, paper folding and GeoGebra, in a construction-proving problem as well as its generalization in the Euclidean geometry context. For this, we designed and implemented two mathematical tasks with 52 secondary education students (15–16 years old, 10th grade) during the COVID-19 emergency lockdown period in Italy. The tasks involved four phases: constructing, exploring, conjecturing, and proving. This article presents an epistemic analysis of the tasks and a cognitive analysis of the answers given by one of the students. The theoretical tools of the onto-semiotic approach supported these analyses. Cognitive analysis allows us to confront the intended meanings of the task and the meanings actually employed by a student, thus drawing specific conclusions about the roles of such artifacts in written arguments and give an interpretation of their combined use in mathematics education.

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KEYWORDS
Secondary education; paper folding; GeoGebra; geometric task analysis; onto-semiotic approach

1. Introduction

One of the main difficulties students face in high school is solving geometric problems, especially those that are proof oriented. Research in the field of dynamic geometry over the last 30 years has shown great opportunities for such software tools to offer geometry teaching and learning, thanks to their affordances, such as dragging, measuring, and tracing, among others (Hoyles & Jones, 1998; Laborde, 2000; Lopez-Real & Leung, 2006; Mariotti, 2000; Olivero & Robutti, 2007).

Furthermore, the art of paper folding, also known as origami, has a long history in geometry education, with nineteenth-century origins in Friedrich Fröbel's educational method of folding, '*paperfalten*' as a working tool in his kindergarten (Friedman, 2018). Several other authors have also exploited the power of paper folding because of its manipulative and exploratory nature and the theoretical concepts embedded in each permitted fold. Thus, paper folding is a suitable cognitive tool for teaching geometry and other

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mathematics topics at every level of education (Frigerio, 2002; Golan & Jackson, 2009; Hull, 2013, 2020; Lam & Pope, 2016; Wares, 2011; Wiles, 2013). Regarding geometry teaching and learning through origami, research results show positive effects on students' spatial visualization, achievement in geometry, and geometric reasoning in the 10th grade (Arıcı & Aslan-Tutak, 2015). Other research findings indicate potential positive effects for the visualization and geometric achievement of middle school and college students (Boakes, 2006, 2009) and an improvement in students' geometric thinking in the 11th grade (Gürbüz et al., 2018).

Given these contributions, research in the area has recently studied the potential of the combined use of these two environments, origami and dynamic geometry software, to improve the literacy and mathematics skills of high school students (Budinski et al., 2018; Fenyvesi et al., 2014). In the context of teaching geometry, the book *Developing the Essential Understanding of Geometry in Grades 9–12* (Sinclair et al., 2012) proposes the challenge of working with dynamic diagrams presented both on screen and with physical movements (e.g. with paper folding). Despite this suggestion, to date, there have been no specific studies on the interplay of these representations in a proving process, and our research attempts to shed light on this issue.

The main visual representations used in teaching and learning geometry are static and dynamic geometric diagrams in addition to physical or virtual manipulatives (including geometric object models). Our research focuses on geometric diagrams (or 2D – two dimensional – physical models), widely understood as 'two-dimensional visual representations that accompany problems (e.g. proof problems, find problems, determine problems) in plane geometry' (Dimmel & Herbst, 2015).

This article focuses on the synergy between paper folding and GeoGebra (a dynamic software) applied in two tasks based on a construction-proving problem and its generalization in the Euclidean geometry context.

First, we show the epistemic and cognitive analyses of tasks using onto-semiotic tools (Font et al., 2013; Godino et al., 2007). These analyses will allow us to respond to the following questions: What type of mathematical activity does the student mobilize when solving geometric problems with two different artifacts? Second, what relationship does this have with the built diagrams? Answering these questions will help us to reach the aim to identify the interplay between the chosen artifacts and highlight the interactions between students and diagrams, as reflected in their arguments and actions.

2. Theoretical considerations

This section presents theoretical notions for carrying out the analysis. First, the onto-semiotic approach is used as the main framework of this study. Considering that mathematical activity manifests itself through practices that involve objects and processes, discursive and operational, this approach proposes powerful theoretical tools for the analysis of the complex relationships that are established between them. Second, we point out the main problems and new approaches to the practice of proofs in the geometric context and clarify the meaning of some terms used in the analysis. Third, figural apprehensions are described – a theoretical semiotic-cognitive notion introduced by Duval. These allow a reading of the interactions between an individual and diagrams during geometric problem solving. In the fourth subsection, we briefly present the paper folding practice, while in the fifth subsection, we describe the use of dynamic geometry systems, their peculiarities, and a geometric construction classification. Finally, considering the research aims, we present the interplay that we want to study, taking the Komatsu and Jones (2020) theoretical framework as a reference.

2.1. The onto-semiotic approach

We chose the onto-semiotic approach (OSA) to mathematical education as our study's main lens of observation for the classification of mathematical objects involved in the teaching and learning processes. This perspective stems from the construct of onto-semiotic configuration, 'which generalizes the notion of representation and moves the focus of research towards the system of objects intervening in and emerging from the mathematical activity' (Font et al., 2008, p. 157). From this perspective, a fundamental notion is that of mathematical practice: 'any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems' (Godino et al., 2007, p. 129).

Problems promote and contextualize practices that may be individual (personal) or shared in an institution. Mathematical practice involves and has developed a typology of six primary mathematical objects: 'languages (terms, expressions, notations, graphs), situations (problems, extra or intra – mathematical applications, exercises …), concepts (given by their definitions or descriptions), propositions (properties or attributes), procedures (operations, algorithms, techniques), and arguments (used to validate and explain propositions and procedures)' (Godino et al., 2007, p. 130). Depending on the *language game* in which they are involved, these mathematical objects are classified according to two dimensions: personal (concerning individual subjects) and institutional (shared in a community of practice), ostensive (material, perceptible) and non-ostensive (abstract, ideal, immaterial), extensive (particular) and intensive (general), unitary and -systemic, expression and content.

Practices can be operative, discursive, visual, or non-visual. Visual practices depend on visual perception; non-visual practices (or symbolic/analytical) employ sequential languages such as natural or formal language (Godino et al., 2012). In the formulation of conjectures and in the search for solutions, the visual component plays a key role in understanding a task, while the analytical component is necessary in moments of justification and generalization (Godino et al., 2012). Therefore, it is necessary for visual and analytical thinking to coexist in order to enter into synergy. In conclusion, the configuration of practices, objects, and processes is a powerful tool for investigating the knowledge involved in mathematical tasks.

2.2. Reasoning and proving through exploration

In high-school geometry, students are expected to transition from informal ways of reasoning to more formal ones or, to use the terminology of van Hiele's theory, from the third level of geometric thinking, named *order* (Hoffer, 1981) or *relational-inferential reasoning* (Battista, 2009), to level 4, named *deduction* (Hoffer, 1981) or *formal deductive proof* (Battista, 2009).

The approach to deductive proofs, especially in geometry, is one of the greatest difficulties that students encounter in high school (Mariotti, 2006; Schoenfeld, 1985; Senk, 1985; Usiskin, 1982). This may be due to an incorrect teaching of the proof, presented as a finished product that students must passively learn and reproduce and not as a process of building knowledge (Harel & Sowder, 1998). This awareness has led to a rethinking of the teaching of proofs, conceiving it as a process during which students behave mathematically by means of problems involving them in exploring, conjecturing, explaining, validating, and disproving tentative claims (Zaslavsky et al., 2012), with an experimental approach to theoretical thought (Arzarello et al., 2012; De Villiers, 2003; Fujita & Jones, 2003).

Explorations have become the starting point for genuine mathematical activities. According to Hsieh et al. (2012) the term 'exploration' in a proving process can involve manipulation and interaction with external environments such as hands-on or DGS (dynamic geometry systems) tools. Interaction with external objects acts as a reinforcement of mental exploration and promotes the discovery of properties as well as the discovery of the logical steps required in a proof, thus supporting the related justifications and arguments (Hsieh et al., 2012).

Below, we clarify the meaning of the terms used in the study: argument, argumentation, conjecture, explanation, empirical arguments, deductive arguments, and proof.

- *Argument* refers to a reason given to support or disprove something (i.e. a statement, claim, or interpretation) (Pedemonte, 2002).
- *Argumentation* refers to a discursive activity based on arguments (Pedemonte, 2002).
- We adapt from Pedemonte (2007) that *conjecture* is a triplet consisting of a statement, an argumentation, and knowledge (of a person or group of people) that is the result of a system of practices shared within an institution (e.g. the classroom). If the argumentation precedes the statement, it is called constructive, if it follows that, it is called structuring (2007, p. 28).
- *Explanation*, from Latin *explanare* (to make plain or clear), refers to the production of reasons to make a phenomenon understandable.
- We distinguish between *empirical* and *deductive* arguments. Empirical arguments are based on the observation of 'regularities in one or more examples; [Students] use the examples, or relationships observed in them, to justify the truth of their conjecture' (Marrades & Gutiérrez, 2000, p. 91). Deductive arguments are 'characterized by the decontextualization of the arguments used and are based on generic aspects of the problem, mental operations, and logical deductions, all of which aim to validate the conjecture in a general way' (Marrades & Gutiérrez, 2000, p. 93).
- *Proof* refers to a particular argumentation as a sequence of (deductive) arguments that are connected by means of accepted canons of correct inference and are based on accepted truths for or against a mathematical claim (Stylianides, 2008).

2.3. Duval's figural apprehensions

To analyze heuristic figural working, Duval considers four cognitive apprehensions: perceptual, sequential, discursive, and operative (Duval, 1994, 1995). We recognize something (a figure or a sub-figure) by perceptual apprehension and pictorially, by figural organization laws, we can also name what we recognize. Sequential apprehension is required to construct a diagram of a geometrical figure with tools or to describe its construction. Unlike perceptual apprehension, there is a temporal dimension linked to the order in which the instructions for carrying out the construction are executed (Herbst et al., 2017). There are often auxiliary lines that do not belong to the intended figure, and the organization of elementary figural units depends on the technical constraints of the tools used (ruler and compass, paper folding, available primitives in geometrical software) and mathematical properties. In this case, the diagram is a model of a represented mathematical object (Duval, 1995).

Discursive apprehension views a figure in relation to its designation (denomination, capture, primitive commands in a menu, hypothesis) (Duval, 1995). The explicitness of other properties, starting from those indicated, is discursive apprehension. The corresponding cognitive process is deductive reasoning, and its epistemological function is proving (Duval, 1994). We also extend discursive apprehension to cases in which an exploration of a soft construction in a DGS environment gives rise to theoretical discourse by identifying the hypotheses that generate a deductive type of discourse. Finally, operative apprehension consists of all possible modifications, mental or material, of the figure to other figures (reconfigurations, enlargements, deformations, position changes, etc.), which can be created by acting on perceptive sub-figures, separating and recombining them in a new configuration, but also by inserting new elements, such as auxiliary lines, into the starting figure (Duval, 1994).

Challenging construction-demonstration problems involve, in addition to the perceptual, the three apprehensions – discursive, sequential, and operative – and the proof requires complete coordination.

2.4. Paper folding practice

Euclidean geometry is the area in which it is easy to recognize the link between origami (from Japanese *ori* = folding and *kami* = paper) and mathematics. Classic construction tools, straight-edge and compass (SE&C), can be replaced by a piece of paper to fold, assuming all crease lines are straight lines and that every time we fold the paper, we make a single crease (Hull, 2020).

Full characterization of the basic possible operations with origami is harder than with a straight-edge and compass (SE&C), because it is possible to fold paper in many different ways. Seven operations describe the basic admissible folds to construct a new folded line from previously constructed data (points or lines); furthermore, given two lines, we can locate their point of intersection, if it exists (Hull, 2020; Huzita, 1989).

Any SE&C construction can be performed using five of the seven basic operations and vice versa. Any construction using these five operations can be constructed with a SE&C (Geretschlager, 1995). For our purposes, paper folding is a mathematical practice for constructing ostensive representations (models) of geometric figures of the Euclidean plane (non-ostensive). Table 1 shows the five origami operations equivalent to SE&C, their Euclidean meanings, and the number of ways to fold.

2.5. Dynamic geometry environment practices

Dynamic geometry environments have been technological tools in use since the 1980s for teaching and learning geometry. GeoGebra is a commonly used (open-source, free)

Table 1. The five origami operations equivalent to SE&C. Source: authors' own.

Operation	Euclidean meaning of the resulting crease	Number of ways to fold
(O1) Given two distinct points A and B, we can make a unique fold that passes through both of them.	The straight line passing through A and B.	1
(O2) Given two distinct points A and B, we can make a unique fold that places A on B.	The perpendicular bisector of segment AB.	1
(O3) Given two distinct straight-lines r and s, we can make at least one fold that places r on s.	The angle bisector between the lines r and s if r and s are not parallel, the line midway between the two given lines otherwise.	2,1
(O4) Given a point A and a straight-line r, we can make a unique fold perpendicular to r passing through point A.	Perpendicular to r passing through A.	1
(O5) Given two distinct points A and B and a straight-line r, we can, whenever possible, make at least one fold placing A on r and passing through B.	There are more meanings, depending on the relative positions of the line and points. If point A is not on r and the distance from B to line r is less than or equal to the distance from B to A, then the crease is the perpendicular bisector of segment AA′ where A′ is an intersection point of the circle with centre B, and radius AB with line r. In this case, two or just one crease are possible.	0,1,2

software. The environment in GeoGebra allows for the construction and manipulation of geometric figures. Its main features are a set of primitive objects (e.g. points, lines, segments, circles), tools that allow the construction of other objects starting from them (e.g. parallel line, perpendicular line), tools for performing geometric transformations (e.g. axial symmetry, translation), dragging, measuring, animation, hiding objects, and creating procedures. The most important feature is dragging, which allows movement of the free points (those chosen to build new objects) and exploration of how the points and other built objects vary. This dynamic aspect distinguishes a diagram built in a dynamic geometry environment from a paper and pencil diagram or a paper-folding diagram. A figure constructed according to the rules and properties of Euclidean geometry maintains the internal relationships between the elements during dragging. Dragging can perform several functions: providing feedback, mediating between design and geometric figure, testing properties, and searching for new properties. Different types of dragging exist. In this paper, we observe *guided dragging*, that is, moving 'the basic points of a drawing in order to give it a particular shape' (Arzarello et al., 2002, p. 67) and *maintaining dragging,* that is, moving a base point so that the dynamic figure maintains a certain property (Baccaglini-Frank & Mariotti, 2010).

An important distinction is between robust and soft constructions. A robust construction is one that also retains the properties of figures if base points are moved; a soft one is a construction 'in which one of the chosen properties is purposely constructed by eye, allowing the locus of permissible figures to be built up in an empirical manner under the control of the student' (Healy, 2000, p. 142). The two types of construction allow different student experiences of geometric dependence that can be seen as complementary (Laborde, 2005).

2.6. Interplay between physical and digital activity

Analysis of the interplay between physical and digital activity takes the framework proposed by Komatsu and Jones (2020) into account, as shown in Figure 1. In our case, physical

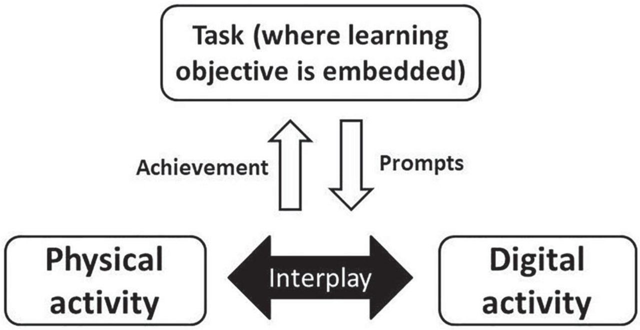


Figure 1. Komatsu and Jones model for the interplay between physical and digital activity. Source: (Komatsu & Jones, 2020, p. 126).

activity concerns constructions and interactions with the paper models, while digital activity refers to the construction (first task) and manipulations (both tasks) of the dynamic model of the folding diagram in the GeoGebra environment.

The interplay that we wish to examine concerns the interactions with diagrams during mathematical activity as interpreted through the discursive and operative, visual, and non-visual practices employed by the student.

3. Materials and methods

3.1. Context and participants

To study the interplay between physical and digital activities, we designed different sequences of tasks as part of a larger project entitled Paper Folding and GeoGebra. In this work, we present a sequence of two tasks designed and administered to a course of 52 10th-grade students.

The easy availability of the material used (a sheet of paper to fold) and use of the free GeoGebra Groups (Tomaschko et al., 2018) combined with the availability of G Suite Apps like Google Meet allowed us to carry out the research during the COVID19 lockdown period in Italy.

In the two-class sessions (90 min each), students faced two connected tasks. In the first session, students tackled the tasks individually, whereas in the second session, students worked in small groups before participating in a classroom discussion (Figure 2).

When they performed the tasks analyzed here, students had basic user knowledge of GeoGebra and had previously worked on the geometric meanings of the first four basic operations of origami geometry, as described in Table 1. Concerning the study of Euclidean geometry, students had little experience with proofs; however, they were familiar with triangle congruence criteria, SAS (side, angle, side), ASA (angle, side, angle), and SSS (side, side, side). The students had a shallow knowledge of the kite.

3.2. Data collection and analysis techniques

In this article, we describe and analyze the work of one student (F) while solving the two tasks described below. We selected Student F because his detailed production allows for

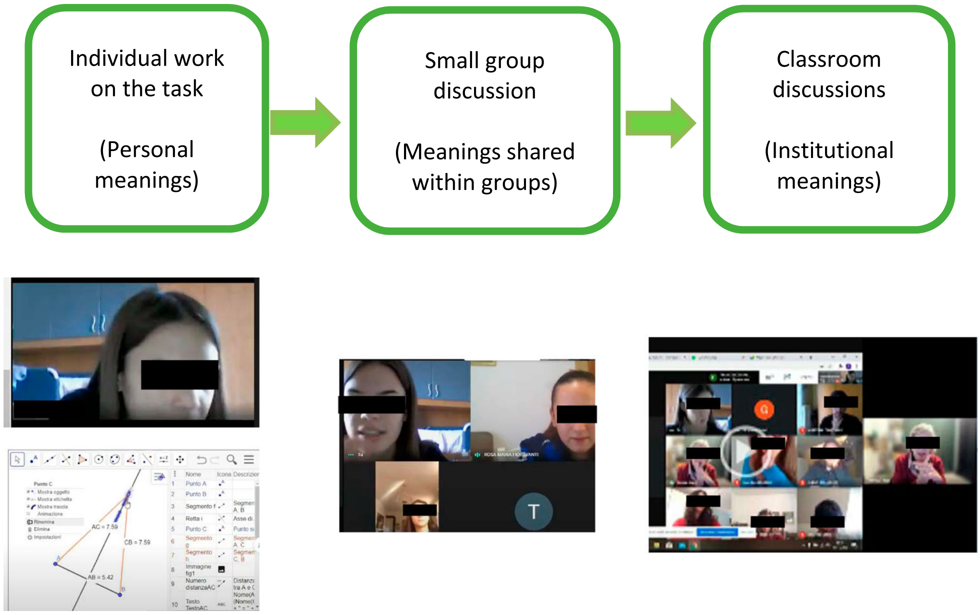


Figure 2. Didactic methodology. Source: authors' own.

the analysis of the role of the chosen artifacts in the development of his arguments. The data analyzed were the student's written and digital (e.g. digital diagrams) production, his interventions in the classroom discussion, and the folded physical models, allowing us to triangulate the information during analysis.

The following paragraph first provides a description of the two tasks, then gives an epistemic analysis of the OSA object configurations involved in the mathematical practices to solve the tasks, highlighting the duality of the generated processes. Finally, we analyze the cognitive OSA configuration of objects that emerged in student F's mathematical practice, distinguishing the typology of arguments and the type of apprehension and pointing out the role of physical and digital diagrams and artifacts in each step. Our elaboration of object configurations is inspired by the model of Font et al. (Font et al., 2010) described above.

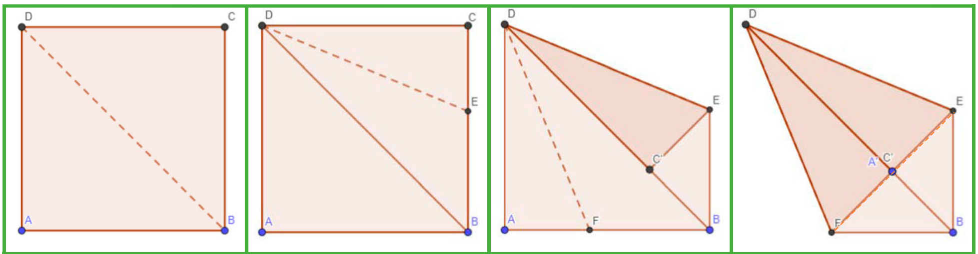
4. Task analysis and discussion

4.1. Tasks description

Students were given two connected tasks for which an online worksheet had been provided (GeoGebra activity). Both tasks involved physical and digital activities with the possibility of interference, as students could work with both representations.

In this section, we describe how students were expected to work, and in the following sections of epistemic analysis, we clarify, for each step of the two tasks, the previous and emerging objects during the resolution of the task.

In task 1, instructions were given to fold a square sheet of paper (Figure 3). allowing students to construct a two-dimensional origami model.



1) Label the vertices of your square as shown. Fold diagonal BD and unfold.	2) Place side DC on diagonal BD to make crease DE.	3) Place side AD on diagonal BD to make crease DF.	4) Fold along FE and unfold. You are done!

Figure 3. Paper-folding instructions. Source: authors' own.

Students were asked to analyze the paper model, with visual and hands-on explorations, formulating observations on congruencies between the various subfigures, and trying to justify them. A direct question asked them to formulate a conjecture about the nature of the folded quadrilateral (a kite): '*What kind of polygon is FBED?*'

Students were then asked: (1) to model the paper folding operations in the GeoGebra environment, starting from a pre-assigned square, explaining the mathematical meanings embedded in each operation, (2) to test the construction by dragging, and (3) to make any new observations and/or justifications in support of the conjecture already made.

Since a mathematical result obtained in a particular case (in our case, that of a folded square) has greater value and meaning if it can inspire a sense of generality that can be acquired by the students if they are given the opportunity to vary, extend, and recontextualize the previous experience, this opportunity was offered by the second task.

Task 2 aims to generalize the result obtained in the case of the square sheet to a wider class of quadrilaterals, with generalization being the process/product of 'passing from consideration of a given set of objects to that of a larger set containing the given one' (Polya, 1954, vol. I, p. 12).

The students were initially prompted to fold, as in task 1, non-square quadrangular sheets (general quadrilateral, rectangle, trapezoid, rhombus, kites, etc.) whose templates had been provided, identifying similarities and differences (see epistemic analysis for more detail) with the square sheet case. Students were then asked to find the conditions on the initial quadrilateral ABCD under which the folding instructions would produce a quadrilateral FBED of the *same kind* as that obtained in task 1, formulating a conjecture (ABCD kite) and providing a justification for it. We emphasize that the ambiguity of the expression *of the same kind* is intentional, but apparent; in fact, it was used in order not to reveal the nature of the quadrilateral right away, but by the time the second task was solved, all the students had determined it was a kite (during the classroom session, the students' work was constantly monitored), as expected. Finally, the students were asked to decide which further conditions on the initial quadrilateral ABCD would make it so that the triangles folded in steps (2) and (3) of the instruction would fit perfectly into triangle FED, as in the case of the folded square, trying to give a justification (see epistemic analysis for more detail). A GeoGebra pre-build construction (described later), showing the folding output, was also embedded in the worksheet that students could explore and work with.

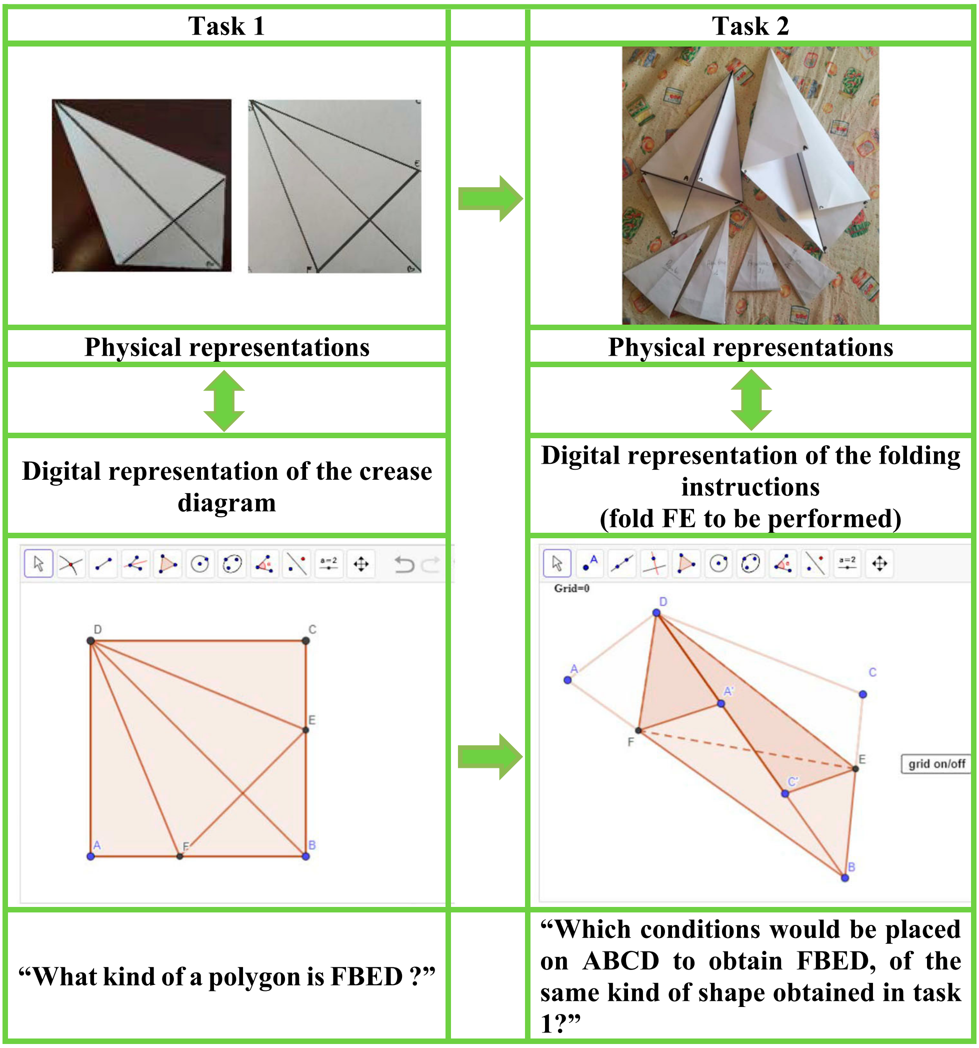


Figure 4. Synthetic illustration of the tasks. Source: authors’ own.

Both conditions fulfilling the task requirements are exclusive, that is, sufficient as well as necessary (if and only if).

In this second task, we expect the exploration of the GeoGebra construction not only to confirm the situations already explored on paper but also to enrich students’ observations (e.g. to note that the condition ABCD kite is not only sufficient but also necessary) and arguments.

Figure 4 below briefly illustrates the central questions in the two tasks.

4.2. Task 1 epistemic analysis

The paper-folding instructions were given in diagrammatic language (Figure 3), with the label and text indicating that the diagram was supposed to be seen as a geometric figure by discursive apprehension. This involves an interpretation by means of non-ostensive objects (the diagonal, the angle bisector …) of the ostensive objects (the square sheet, the folds …).

Fold BD (Figure 3. Step 1) corresponds to the first origami operation and represents the diagonal of the square ABCD. Folds DE and DF correspond to the third origami operation and represent the segments of the angle bisectors of ∠BDC and ∠ADB inside the square (Figure 3. Steps 2 and 3). The last fold, FE, corresponds to the first origami operation and represents the segment joining points F and E (Figure 3. Step 4). The folding procedure requires sequential apprehension.

The construction of the digital model of the fold diagram requires a change in the representational system, forcing students to reason about the meanings of the folds in non-ostensive terms. Discursive and sequential apprehensions intervene in this practice. The conjecture on FBED can be considered as a 'fact', provided without any argumentation, directly from diagrams or the model, through perceptual apprehension, and the related arguments will be structuring arguments.

The digital diagram students had to build is a robust construction, where the dragging function tests the correctness of the construction in a 'large' set of squares, showing the generality of the construction process. The digital counterparts of the folds and the square sheet determine the construction invariants (premise of conditional statements), whereas congruencies such as those between DF and DE or between BF and DE are the invariants derived from the construction (conclusion of conditional statements). The link between invariants can also emerge from the epistemic potential of the folding actions, assuming the square sheet and the folds as a *before* and the results as an *after* of a conditional statement expressed in a diachronic mode.

It is possible to prove that FBED is a kite in many ways, informal or more formal, also involving symmetries. Assuming the *folds* as additional hypotheses to the square ABCD, the Euclidean proof requires the application of congruence triangle criteria and congruence properties.

Figure 5 shows the epistemic configuration related to the expected practices in the resolution of the task. The question marks refer to possible emergent objects.

From an onto-semiotic approach, the epistemic configuration reflected in Figure 5 is complemented with an analysis of the dual processes employed.

- Ostensive-non-ostensive: The starting point is reasoning on a material representation model of an ideal object (a kite) intertwining the dual processes of materialization and idealization of the concepts and operations to achieve a mathematical proof (Figure 6).
- Extensive-intensive: This duality allows the complexity of a generic element, that is, the dual relationship between the particular and the general. In this sense, the use of rules, criteria, and formulas is applied to specific cases of the problem. For example, congruent triangle criteria are applied in the case of a folded shape.
- Unitary-systemic: This duality is linked to the processes of reification (constitution of objects as a whole) and decomposition (inverse). In this case, it is an ostensive geometric figure that intervenes as a unitary whole that must be decomposed into different elements: triangles, sides, congruent angles, and so on.

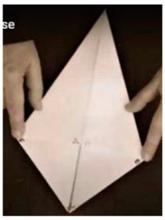
Languages

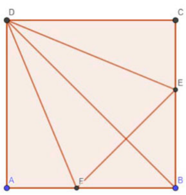
Natural

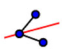
In the task text and its resolution

Diagrammatic

In the task assignment;





Iconic (GeoGebra icons:  , …)

Analytic/symbolic

BD, ABCD, …

Express and support

Regulate the use of

Problem-situations

Purpose: Development of deductive arguments to justify observed facts

Construction of an origami model and the digital diagram of its crease diagram; explorations of the model and the diagrams (crease and digital diagrams) to discover properties; justification of these properties.

Motivate **Solve**

Concepts

Previous: angle, segment, polygon, side of a polygon, interior angle of a polygon, triangle, quadrilateral as a polygon with four sides, square (defined as quadrilateral with four congruent sides and four congruent angles), kite (deltoid) as a quadrilateral with two distinct pairs of congruent adjacent sides, diagonal, right angle, angle bisector, isosceles triangle, right triangle, congruence of plane figures, axial symmetry.

Emerging: kite as a quadrilateral symmetric to a diagonal (?); main diagonal of a kite…

Propositions

Previous: congruent triangles criteria (side-side-side; side-angle-side; angle-side-angle), the halves of congruent angles are congruent, congruence properties: reflexive, …, (isosceles triangle properties)

Emerging (main propositions): triangles BCD and DAB are congruent (either the diagonal BD of square ABCD bisects the angles with the vertices at its ends, or square ABCD is symmetrical to the diagonal BD); $\angle ADF \cong \angle FDB \cong \angle BDE \cong \angle EDC$; triangles AFD and DEC are congruent; FED and FBE are isosceles triangles; **quadrilateral FBED has two distinct pairs of congruent adjacent sides;** quadrilateral FBED is symmetrical to diagonal BD;…

Procedures

Figure 5. Epistemic configuration related to task 1. Source: authors' own.

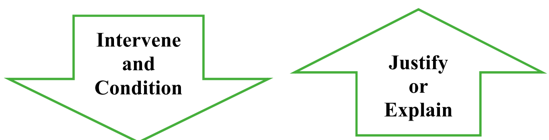
		Previous **With tools:** perform the first four origami operations, use the main, general construction, movement, measurement, and transformation tools in the GeoGebra graphic view to represent geometrical objects and interact with these representations. **Mathematics:** classify triangles, apply triangle congruence criteria, develop simple deductive chain. **Emerging** **With tools:** perform the paper-folding instructions related to the task, perform hand-on manipulations of the paper model (e.g., folding along BD), construct the digital diagram of the crease diagram, use the movement tool to drag free objects, use the measurement tools to empirically verify or discover congruencies, and use the axial symmetry tool to empirically verify congruencies. **Mathematics:** apply propositions (previous and emergent in the practice) and concepts (apply congruent triangles criteria, apply propositions on halves of congruent angles…), develop written arguments.
		
		Arguments/Explanations
		Empirical/visual in explorative activities (perceptual), deductive (formal or informal) in justifications.

Figure 5. Continued.

4.3. Task 2 epistemic analysis

Task 2 aims to extend the domain of the problem to quadrangular non-square sheets and create conflict situations. The task asks students two questions. In the first question, students are asked to determine under which conditions the initial quadrilateral in the paper-folding instructions would produce a quadrilateral of the *same kind* as the one obtained in task 1 (a kite). They then must formulate a conjecture and justify it. Students first worked with quadrangular paper sheets of different shapes (rectangle, trapezoid, rhombus, kite, right kite…) and then with a pre-built GeoGebra construction, as shown in Figure 7.

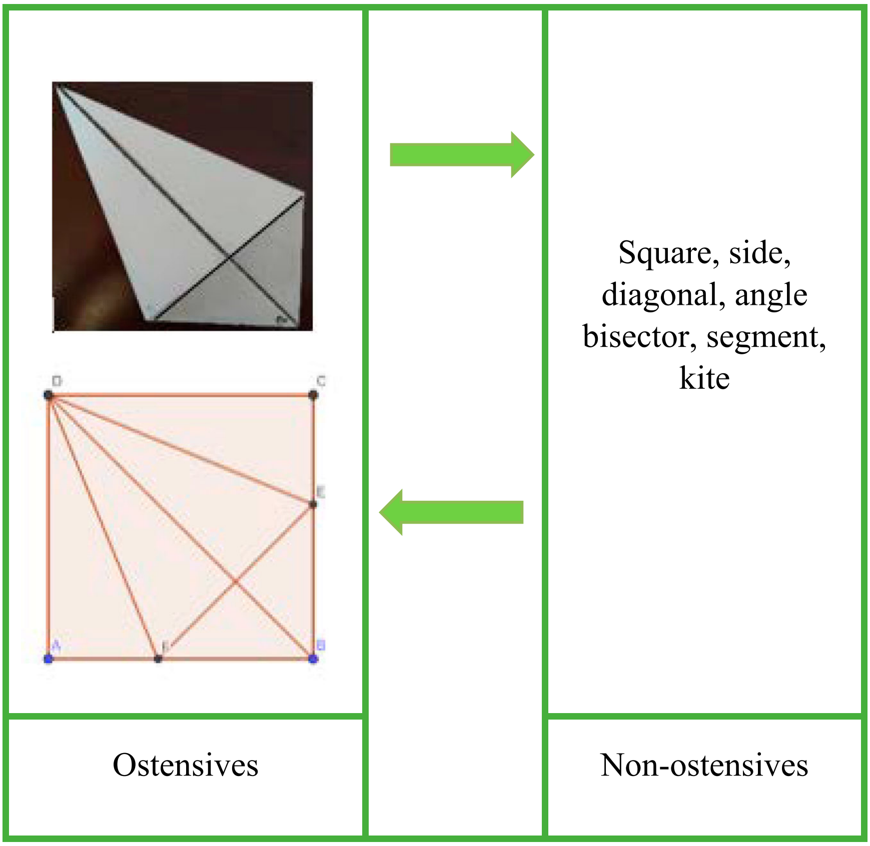


Figure 6. Example of the dialectic between the ostensive and non-ostensive facets of mathematical activity. Source: authors' own.

Our intention was that the digital construction should have made it possible to visualize a greater variety of configurations than those seen on paper and with greater precision but also to perform different actions due to the perceived affordances. These actions could be integrated with those performed on paper as complementary and supplementary to formulate conjectures and related arguments. Of course, we did not exclude the possibility that students could solve the task by working exclusively with the templates provided.

Here, dragging can be used with different focuses and intentions. It can be used to induce, as a soft (Healy, 2000; Laborde, 2005), invariant, FBED kite (conclusion) to identify a hypothesis (premise) and to formulate a conditional statement for a sufficient condition. This does not exclude the fact that the same dragging could be used for a dynamic exploration beyond the conditions set by the hypothesis, this time focusing on ABCD, also in light of the experience made with the paper sheets, to arrive at a statement that also expresses the necessity of the condition. Guided and maintaining dragging strategies were expected.

Square and rhombuses as kites, concave kites (called darts), or degenerate kites could emerge, favoring the generalization process and enriching the students' concept image of

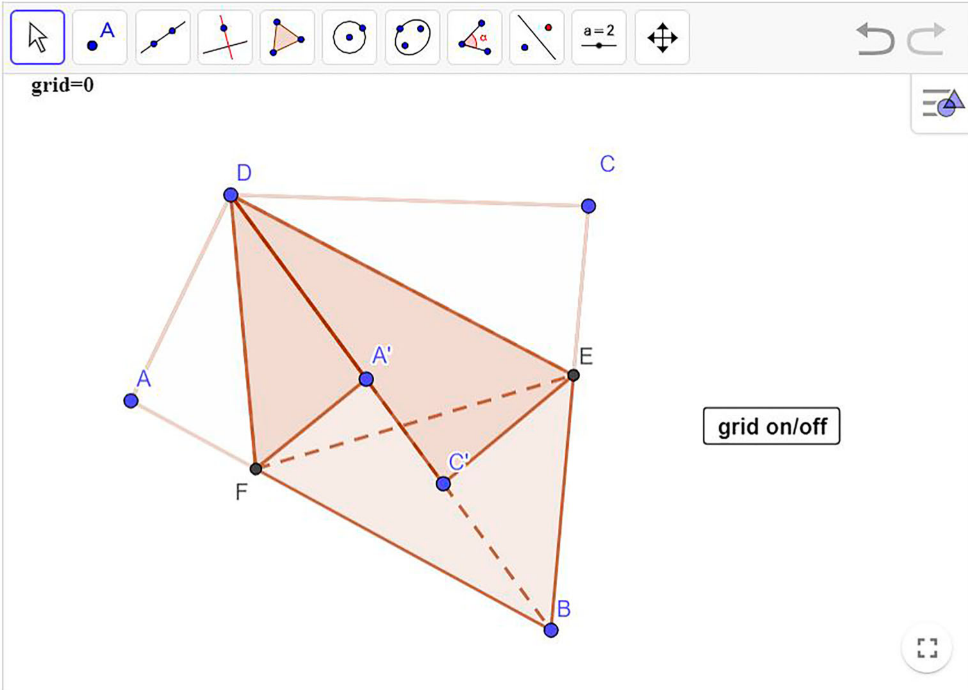


Figure 7. The digital representation for task 2. Source: authors' own.

the kite. In this task, we expected constructive arguments. One possible correctly formulated statement for the conjecture is as follows: 'In a kite (and only in a kite), if the main diagonal is drawn and, from one of its ends, we draw the bisectors of the angles that it forms with each side sharing the end, the bisectors cut the other two sides at two points which, together with the ends of the diagonal, are the vertices of the other kite'. The following statement expresses the same information in simpler terms: 'if (and only if) ABCD is a kite, then FBED is a kite' (Figure 8a).

The sufficient condition (ABCD kite) could be verified using a robust construction, starting for example, constructing a kite, ABCD, by fixing three vertices that determine a triangle, such as BCD, and applying an axial symmetry with a BD axis, (the key idea of which is the symmetry of quadrilateral ABCD with respect to diagonal BD) and then applying the same sequence of instructions as in task 1. Similarly (with the awareness of the equivalence of the statements 'if ABCD is not a kite, then FBED is not a kite' and 'if FBED is a kite, then ABCD is a kite'), starting from a robust FBED kite, the necessity of the condition ABCD kite could be verified. In this case, students could refer to either the material or virtual folding. Starting from the kite FBED, one can unfold the two folded triangles AFD and DEC and obtain the original quadrilateral ABCD. Point A is the intersection of the segment BF extended beyond B and the segment resulting from the reflection of the diagonal DB with respect to the side DF (see Figure 8a). Similarly, point C is the intersection of the segment BE extended beyond E and the segment resulting from the reflection of the diagonal DB with respect to the side DE. Due to the symmetry of FBED with respect to BD, the quadrilateral ABCD, constructed in this way, is a kite.

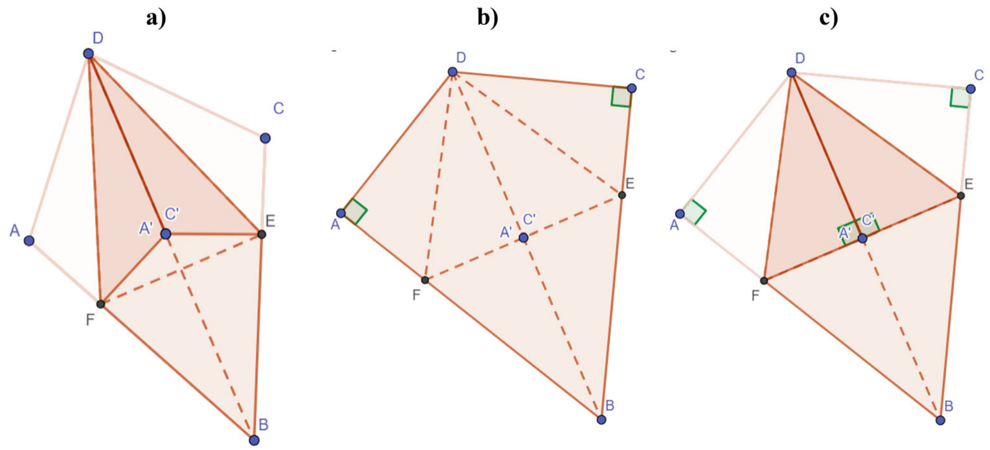


Figure 8. Different situations where if ABCD is a kite, then FBED is a kite. Source: authors' own.

In the second question, students were expected to conjecture and justify that if (and only if) ABCD is a right kite (Figure 8b), then triangles ADF and CDE, once folded (Figure 8c), would fit perfectly into triangle FED, as in the case of the square sheet.

All the conjectures can be proven using congruence criteria for triangles and congruence properties.

Figure 9 shows the epistemic configuration of task 2. The question marks refer to objects that could emerge from practice.

4.4. Task 1 cognitive analysis

Figure 10 shows the cognitive configurations of the objects that emerged from Student F's practice. Within square brackets are references to words implicit in the student's discourse. We categorize each argument according to the artifact used for its formulation and its typology (empirical or deductive).

Student F folded the sheet correctly, highlighting the understanding of the instructions in the text and representing them ostensibly. The student correctly reconstructed the crease diagram in the GeoGebra environment, associating the square and folds with their corresponding non-ostensive objects and then choosing the appropriate GeoGebra tools. The student used dragging to test the correctness of the construction with perceptual apprehension.

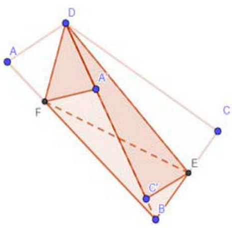
The student *looked* (perceptual apprehension) at the quadrilateral FBED (artifact L_2-a) as a quadrilateral composed of two isosceles triangles on the same base (arguments A_4, A_5, and A_6), suggesting that it was a kite. The folds made in steps 2 and 3 of the instructions that highlight an isosceles triangle of double thickness may have influenced the student, or this way of looking may also reflect the student's personal concept image of the kite. Manipulating the physical model, he consequently folded triangle FED along the symmetry axis BD (argument A_1 with artifact L_2-b) with an operative apprehension that affected the choice of sub-configurations to which discursive apprehension could be attached: triangles ADF and ECD (argument A_2 with artifact L_2) superimposed after folding. It seems to us that the student identified triangles ADF and ECD with DFG and DGE, respectively (which

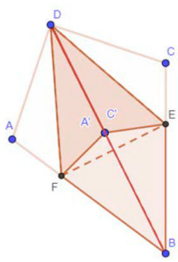
Languages

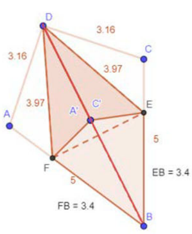
Natural

For example, in the task assignment

Diagrammatic







Express and support

Regulate the use of

Problem-situations

Purpose:

a) To start a generalization process (extension of a result to a larger class) by folding prepared sheets of paper with possible conjectures formulation;

b) To support the inquiries by exploring a dynamic soft app with some robust elements. To formulate conjectures with the form: If...then… To formulate arguments to justify conjectures. To conceptualize a class of quadrilaterals.

Explorations of paper models and diagrams (crease and digital diagrams) to find the quadrilaterals that are "folded" according to task 1 instructions and produced a quadrilateral that "looks like"…

Motivate **Solve**

Concepts

Previous: angle, segment, polygon, side of a polygon, interior angle of a polygon, quadrilateral as a polygon with four sides, square, triangle, diagonal, right angle, angle bisector, isosceles triangle, right triangle, congruence of plane figures, axial symmetry, rectangle, rhombus, kite (as in task 1)

Emerging

A kite as a class of quadrilateral symmetrical to a diagonal, a kite as…, a right kite (as a kite with the angles whose vertices are not ends of the main diagonal right angles) (?)

Propositions

Previous: congruent triangles criteria, the halves/differences of congruent angles are congruent, congruence properties: reflexive, …, (isosceles triangle properties), if ABCD is a square then FBED is a kite (emerged in task 1)

Figure 9. Epistemic configuration of task 2. Source: authors' own.

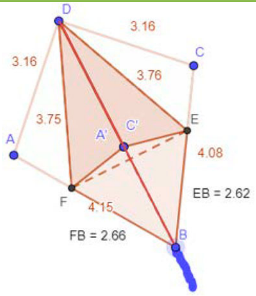
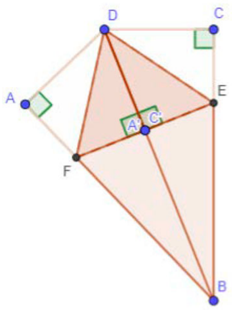
  **Iconic** (GeoGebra icons: ,…) **Analytic/symbolic** FBED, …		**Emerging (principal propositions)** If ABCD is a rectangle (not square), then FBED is not a kite If ABCD is a rhombus, then FBED is a kite… If ABCD is a kite, then FBED is a kite If ABCD is not a kite, then FBED is not a kite A kite is symmetrical to a diagonal, kite properties If ABCD is a right kite, then FBED is a kite and the diagonal BD divides triangle FED into two triangles, both congruent to AFD and ECD. If ABCD is a kite but not a right kite, then FBED is a kite, but the diagonal BD does not divide triangle FED into two triangles, both congruent to AFD and ECD.
		Procedures
		Previous **With tools:** a) perform the first four origami operations; b) use of the main general, construction, movement, measurement, and transformation tools in the GeoGebra graphic view to represent geometrical objects and interact with these representations, guided dragging or drag-to-fit, maintaining dragging procedures **Mathematics:** classify triangles, apply triangle congruence criteria, develop simple deductive chain **Emerging** **With tools:** Perform the paper-folding instructions related to task 1 with some quadrangular sheets. Explore the paper models and the crease diagram by hand manipulation. Explore the GeoGebra diagram. Use the movement tool to drag free objects; use the measurement tools, and to empirically verify congruences. Use dragging modality procedures. Use guided dragging to find kite FBED. Move only B or only D in a fixed direction (AC bisector) to maintain FBED as a kite (maintaining dragging). Build a

Figure 9. Continued.

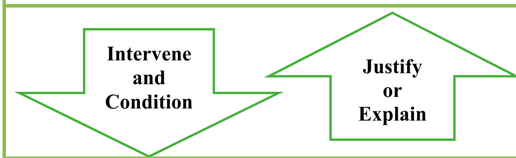
		robust kite, ABCD, from a triangle (e.g., BCD) with the axial symmetry tool or with different strategies. **Mathematics:** Apply propositions and concepts, previous and emergent, in practice (apply congruent triangles criteria, apply propositions on halves of congruent angles…), to develop written arguments.
		
		Arguments/Explanations
		Empirical/visual in the explorative activities (perceptual), deductive (formal or informal) in the justifications, analogical.

Figure 9. Continued.

he successively represented with the L_3 artifact), based on the empirical evidence of the multilayer overlap after folding (steps 3–4 of the folding instructions).

A deductive argument was formulated by working with the material diagram (argument A_2 with artifact L_2). The deductive argument A_2 presents an implicit premise: ∠ADB congruent to ∠BDC. This implicit premise was later made explicit and proved (deductive argument A_3 with artifact L_3) after the use of the angle-measuring tool in the digital environment. Therefore, we believe that, in this instance, the use of *measure* (here with a discursive apprehension) in the digital environment allowed the student to recall the meanings of the folds and to reorganize their previous knowledge in order to develop a deductive argument.

The student used the angle measuring tools (artifact L_3) as a marker to visualize elements (as he would have done in a traditional diagram) and also as a tool to verify the congruence of some angles and of the segments AF and CE (the latter used in argument A_4) during the proving process. With argument A_4 (artifact L_3), the student justified the FDE isosceles proposition (operative and discursive apprehensions). In this sense, its purpose is evident: to prove that FBED is a quadrilateral consisting of two isosceles triangles with a common base.

Claims about P_{P6}–P_{P11} emerging propositions are all justified by deductive arguments (basically correct, in a semi-paragraph format style), using other well-known theorems or propositions (P_{P1}–P_{P5}), with the exception of P_{P12}. This highlights the fact that the student only observed the symmetry of the kite with respect to the BD diagonal at the end of his work, but as we will see in the analysis of Task 2, this is the key idea during the resolution process.

Languages

Natural

L1. "...[FED] *It's an isosceles triangle. If I fold it in half starting from the vertex at the top, I get two congruent right triangles*"

...

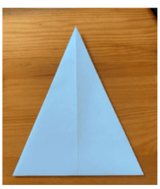
...

Diagrammatic

L2. Material

a)

b)



L3. Digital

Express and support

Regulate the use of

Problem-situations

Purpose: Development of deductive arguments to justify observed facts

Construction of an origami model and the digital diagram of its crease diagram; explorations of the model and the diagrams (crease and digital diagrams) to discover properties; justification of these properties.

Motivate **Solve**

Concepts

Previous: square, diagonal, angle, bisector of an angle, right angle, amplitude and measure of an angle, measure of lengths, side of a polygon, congruence of figures, triangle, isosceles triangle, right triangle, vertex of a polygon, kite (deltoid) as a quadrilateral with two distinct pairs of congruent adjacent sides.

Emerging: kite as a quadrilateral symmetric to a diagonal.

Propositions

Previous accepted propositions:

Pp1. Side-side-side congruence triangles criterion;

Pp2. Angle-side-angle congruence triangles criterion;

Pp3. Halves of congruent angles are congruent;

Pp4. Differences of respectively congruent segments are congruent;

Pp5. Isosceles triangle properties: …

Emerging propositions:

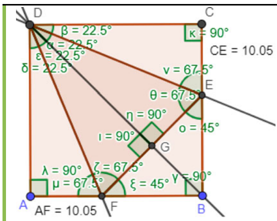
Pp6. *Triangles ADF and ECD are congruent;*

Pp7. *FED is an isosceles triangle*;

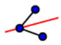
Pp8. *Triangles BCD and DAB are congruent;*

Pp9. *The diagonal [BD] of a square [ABCD] is also the angle*

Figure 10. Cognitive configuration of the resolution of task 1 by Student F. Source: authors' own.



Iconic

L4. GeoGebra icons:  ,

…

Analytic/symbolic

L5. BD, ABCD, …

bisector;

Pp10. *FBE is an isosceles right triangle;*

Pp11. *FBED has two distinct pairs of congruent adjacent sides;*

Pp12. *The diagonal BD divides FBED into two congruent triangles.*

Procedures

Previous

With tools: perform the first four origami operations, use of the main general, construction, movement, measurement, and transformation tools in the GeoGebra graphic view to represent geometrical objects and interact with these representations

Mathematics: classify triangles, apply triangle congruence criteria, develop simple deductive chain (meta-procedure)

Emerging

With tools: the student performed the paper-folding instructions related to the task, manipulated the paper model by folding along the diagonal BD, constructed the digital diagram of the crease diagram, used the movement tool to drag free objects, used the measurement tools to empirically verify congruencies.

Mathematics: the student identified sub-configurations in the diagram to apply propositions (previous and emergent in the practice) and concepts, (apply congruent triangles criteria, apply congruence properties…), developed written arguments.

Intervene and Condition | **Justify or Explain**

Arguments/Explanations

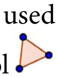
A_i = Argument, E = Empirical, D = Deductive

A1. "[FED] *It's an isosceles triangle. If I fold it in half starting from the vertex at the top, I get two*	**E**

Figure 10. Continued.

		congruent right triangles" (Artifact **L**2)	
		A2. "[FED] *is an isosceles triangle* [because] *folds DE and DF are angle bisectors of the angle [whose vertex is] D in triangles ADB and CDB. AD is congruent to CD, angle ADF is congruent to angle CDE because [they are] halves of congruent angles and angle DAF is congruent to angle DCE because [they are] 90° angles, so the two triangles ADF and ECD are congruent for the second congruency criterion, consequently FD is congruent to DE and FA is congruent to CE."* (Artifact **L**2)	**D**
		A3. *"Because the diagonal* [BD] *of a square* [ABCD] *is also the angle bisector* [bisects the angles ∠B and ∠D], *in fact triangles DAB and DCB are congruent for the third criterion. AB is congruent to BC; AD is congruent to DC (sides of a square) and DB is in common."* (Artifact **L**3)	**D**
		A4. *"FED is an isosceles* [proved above], *FBE is an isosceles right triangle.* [∠] *B is 90 degrees (because it is on the vertex of a square).* [In fact] *FB is congruent to EB by difference of congruent segments."* (Artifact **L**3)	**D**
		A5. "[Since FED and FBE are isosceles triangles] *In FBED, FB is congruent to EB and DE is congruent to FD (pairs of consecutive congruent sides)"* (Artifact **L**3)	**D**

Figure 10. Continued.

Moreover, the student used the intersection tool  to add a point (G) to the digital diagram and the triangle tool  to highlight sub-configurations FGD and EGD and used the angle measurement tool without any written statements. During the classroom discussion, the student said he was trying to prove the congruence of the four right triangles ADF, FDG, DGE, and DEC, which explains why, once folded, the ADF and DEG triangles overlap the isosceles triangle FED perfectly. Here, a less formal approach based on symmetries would be acceptable.

All of the student's conjectures were observed facts based on perceptual apprehension of the physical diagram, and his arguments are all structuring type, supported by a coordination of sequential, operative, and discursive apprehensions.

4.5. Task 2 cognitive analysis

Figure 11 shows the cognitive configurations of the objects that emerged from Student F's practice. Within square brackets, we include the words implicit in the student's discourse and within curly brackets, some of our specifications. Each argument is categorized according to the artifact used for its formulation and its typology (empirical or deductive).

The student apprehended the unlabeled paper shapes by discursive (a rectangle, a rhombus, ...) apprehension and labeled them. He then performed the paper-folding instructions with sequential and discursive apprehensions, even by varying the labeling. The request that quadrilateral ABCD should be symmetrical with respect to diagonal BD (first fold) to achieve a kite emerged from hands-on explorations (language L_2, arguments A_1–A_3) by perceptual apprehension that allowed the student to formulate a kind of pre-conjecture but not explained through a general statement. It seems to us that in this phase, the student collected examples and non-examples, identifying the condition that, besides being sufficient, seemed to his eyes also necessary, that is, the symmetry of the quadrilateral with respect to a diagonal (along which the first fold must be made). Although the arguments were empirical and not detached from the physical context, they provided the basis for subsequent deeper exploration with GeoGebra.

In the GeoGebra environment, the student first looked for a confirmation of what had already been seen on paper, in the logic of yes (Soldano et al., 2019), and with an inductive approach and perceptual apprehension, starting from squares and rhombuses and ending with kites (artifact L_3, arguments A_4–A_5). The student used the grid tool only at the beginning of the exploration, as an aid, then performed a guided dragging (and guided measuring of the sides) focusing on ABCD to reconfirm what he had already explored through paper folding. The student then performed a maintaining dragging (to maintain FBED, a kite moving point D or point B) without tracing; this is an operative apprehension because he changed the initial geometrical figure (the kite obtained) into another one, 'while keeping the properties of the initial figure' (Duval, 1999, p. 18). The student confirmed this cognitive process during classroom discussion (Figure 12).

With argument A_6: '*The ABCD quadrilateral always has two* [distinct?] *pairs of congruent consecutive sides*', the student explained the symmetry condition hypothesized by explicitly referring to the sides of quadrilateral ABCD (we believe that dragging while measuring facilitated this logical step), coordinating the operative apprehension with a discursive apprehension, establishing the theoretical starting assumptions for the Euclidean deductive reasoning set out below (arguments A_7–A_8). Argument A_6 also shows awareness that squares and rhombuses belong to the class of kites (this shows a conceptual control of the figure given by the fusion of image and concept into a single mental object favored by dragging). It is evident that, unlike in Task 1, the student viewed the kite as a quadrilateral that was symmetrical with respect to a diagonal. Although the data available to us are

Languages

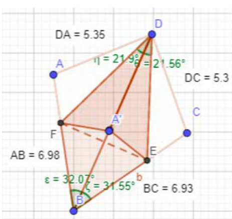
Natural

L1. *"Always in squares and diamonds because the two diagonals of ABCD represent the symmetry axes and also the bisectors of the quadrilateral"*

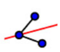
…

L2. Diagrammatic material

L3. Digital



Iconic

L4. GeoGebra icons:  , …

Analytic/symbolic

L5. $\angle ADB \cong \angle BDC$, …

Express and support

Regulate the use of

Problem-situations

Purpose:

a) To start a generalization process (extension of a result to a larger class) by folding prepared sheets of paper with possible conjecture formulation

b) To support the inquiry by exploring a dynamic soft app with some robust elements. To formulate a conjecture using the form: If...then…; to formulate arguments that justify a conjecture

Explorations of paper models and diagrams (crease and digital diagrams) to find the quadrilaterals that "folded" according to task 1 instructions to produce a quadrilateral that "looks like" something else

Motivate **Solve**

Concepts

Previous: angle, segment, polygon, side of a polygon, interior angle of a polygon, quadrilateral as a polygon with four sides, square, triangle, diagonal, right angle, angle bisector, isosceles triangle, right triangle, congruence of plane figures, axial symmetry, rectangle, rhombus, kite (as in task 1).

Emerging

A kite as a quadrilateral symmetrical to a diagonal!

Right kite (a kite with the angles whose vertices are not ends of the main diagonal right angles).

Propositions

Previous accepted propositions:

Pp1. Side-side-side congruent triangles criterion;

Pp2. Angle-side-angle congruent triangles criterion;

Pp3. Halves of congruent angles are congruent;

Pp4. Congruence properties: reflexive, …,

Figure 11. Cognitive configuration of Student F's resolution of task 2. Source: authors' own.

Pp5. If ABCD is a square, then FBED is a kite;

Pp6. In a square, the diagonals are axes of symmetry;

Emerging (principal propositions)

Pp7. If ABCD is a rectangle {not square}, then FBED is not a kite;

{similar propositions for ABCD non-kite}

Pp8. If ABCD is a rhombus, then FBED is a kite;

Pp9. If ABCD is a kite, then FBED is a kite;

Pp10. A kite is symmetrical to a diagonal;

Pp11. Squares and rhombi are kites;

Pp12. If ABCD is a kite and with two right angles…, then FBED is a kite and the triangles folded in steps 3 and 4 of the paper-folding instructions fit perfectly into the FED triangle.

Procedures

Previous

With tools: a) perform the first four origami operations; b) use the main general, construction, movement, measurement, and transformation tools in the GeoGebra graphic view to represent geometrical objects and interact with these representations, guided dragging, drag to-fit

Mathematics: classify triangles, apply triangle congruence criteria, develop a simple deductive chain

Emerging

With tools: The student performed the paper-folding instructions related to task 1 with some quadrangular sheets, he explored the paper models with hands-on manipulation, he explored the GeoGebra diagram using the movement tool to drag free objects (guided and maintaining dragging strategies) and the measurement tools to empirically verify congruences. He used the grid option.

Mathematics: The student applied propositions and concepts (previous and emergent in the practice: apply congruent triangles criteria, apply propositions on halves of congruent

Figure 11. Continued.

		angles…), and developed written arguments.	
		Intervene and Condition **Justify or Explain**	
		Arguments/Explanations	
		A_i = Argument, E = Empirical, D = Deductive	
		A1. "[With the rectangle] *No I can't* [have the same situation] *because I can't make the last fold* [?]. *The diagonal is not a symmetry axis or the angle bisector of angle ∠B and angle ∠D."* (Artifact **L_2**) {the sheet of paper given to the student was a rectangle, not square}	**E**
		A2. "[With the rhombus] *Yes, I can reproduce it* [the kite], *obviously [I have to] change the values of the angles because the rhombus does not have right angles. I can do it whether I first fold the major diagonal or the minor one; the diagonals are symmetry axes and angle bisectors."* (Artifact **L_2**)	**E**
		A3. "[With kites, to obtain a kite] *I can only do the DB fold on the diagonal that represents the symmetry axis of the kites and the bisector of angles ∠D and ∠B. In kite n.2 because angle ∠B is very large, making the last fold, I overlap vertices A and C."* (Artifact **L_2**)	**E**
		{similar arguments to A_1 with the paper sheets not representing a special or more general kite} (Artifact **L_2**)	**E**
		A4. "[FBED is a kite] *Always in squares and rhombuses because the two diagonals of ABCD represent the symmetry axes and [are] also the bisectors of the quadrilateral."* (Artifact **L_3**)	**E**

Figure 11. Continued.

		A5. *"Whereas if ABCD is a kite* [to obtain a kite], *it is necessary to first make the DB fold on the diagonal that represents the only axis of symmetry and that is also the bisector of angles ∠D and ∠B."* (Artifact **L**$_3$)	**E**
		A6. "[To obtain a kite] *The ABCD quadrilateral always has two* [distinct?] *pairs of congruent consecutive sides."* (Artifact **L**$_3$)	**E**
		A7. "[if AB≅*BC* and *AD≅DC* then] *DB is an angle bisector of ABCD* [in fact] *ABD≅DCB according to [the] III congruence criterion;* [then] *∠ABD≅∠CBD, ∠DCB≅∠DAB, ∠ADB≅∠BDC."* (Artifact **L**$_3$)	**D**
		A8. *"I consider triangles BDF and BED,* [they have] *∠FBD≅∠EBD, proved above; ∠FDB≅∠EDB, by hypothesis, halves of congruent angles; DB [is] in common; therefore, they are congruent for the II congruence criterion and [so] FB≅EB, ED≅FD as result."* (Artifact **L**$_3$)	**D**

Figure 11. Continued.

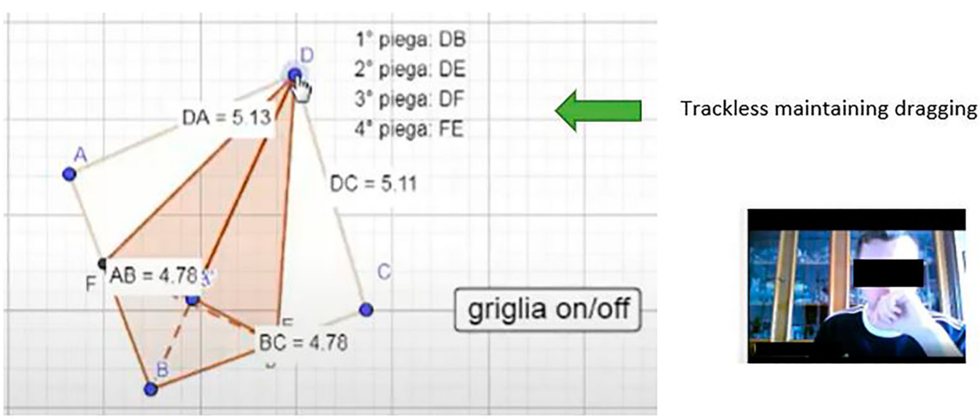


Figure 12. Student F's intervention during the classroom discussion. Source: data collected by the authors.

insufficient to make a firm claim, from analysis of the student intervention in class discussion, we believe that the student's explorations went beyond the conditions of the identified hypothesis, in the logic of not (Soldano et al., 2019), and that, in the absence of counter-examples, led him to think the general statement 'if ABCD is not a kite, then FBED is not a kite', expressed during the discussion by student utterances such as *ABCD must be a kite*. Perhaps additional prompts in the worksheet could have stimulated arguments about this aspect, but in our task design, we preferred the spontaneous emergence and evolution of arguments (also to see the possible contribution of artefacts in this respect).

The student finally observed that if the kite is a right kite, then the folding procedure will give a very similar result to that obtained with the square sheet but provided no discursive argument for this claim (proposition P_{p12}). It seems to us that this statement comes only from folding the paper right kite chosen from the templates and was not explored further in the digital diagram.

No student succeeded in providing arguments for this question, which was addressed in the classroom discussion, showing how arguments based on symmetries (or on a mix of symmetries and congruences) instead of rigorous arguments based on the congruence of triangles sometimes make it easier to solve a task (De Villiers, 2011).

For this task, it is possible to clearly distinguish different argumentative phases for the first question: Some arguments contribute to the construction of the conjecture for the sufficient condition (arguments A_1–A_6) while others structure it to realize an acceptable proof (arguments A_7–A_8). Student F did not seem to have difficulty developing deductive arguments (A_7–A_8) after coordinating the various apprehensions, starting from the sequential one that contains some 'premises' and the 'conclusion', the operative one as well as the discursive one.

5. Conclusions

In this article, we analyzed the mathematical practices of a 10th-grade student in the resolution of mathematical tasks undertaken in online learning class sessions during the COVID-19 lockdown in Italy (May 2020).

The aim of this study was to examine the roles played by the representational artifacts used (paper folding and dynamic geometry) in a construction-proving problem (task 1) and its generalization (task 2) in the field of Euclidean geometry. Task 1 consisted of conjecturing and proving a certain statement that was valid in a particular case (a square) in the kite category. Task 2 generalized the statement through explorative activities. We first characterized the expected mathematical activity (epistemic analysis) and then the one that effectively emerged (cognitive analysis) in the work of a student, analyzing the mathematical objects involved (language, situations, concepts, propositions, procedures, arguments) based on the semiotic approach. We complemented the analysis with other theoretical tools (Duval apprehensions, dragging modalities, typing of arguments) to better highlight the interplay of the artifacts arising in this mathematical activity. The interplay and interactions with diagrams during the mathematical activity were interpreted through the discursive, operative, visual, and non-visual practices employed by a student. The artifacts used established two different visual scenarios governed by their production rules and the affordances offered by the environments. The physical and digital artifacts sometimes played complementary roles, whereas at other times they reinforced each other and allowed the student to make inferences or rework the actions that the student carried out.

In the first task, there was no privileged artifact to start a deductive discourse. In the second task, it seems to us that the digital one favored the discourse by acting as a bridge between configural reasoning and proof. However, we cannot confirm the futility of the digital diagram in the first task for at least three reasons. First, the reconstruction of folding operations in the digital environment strengthened the student's awareness of the hypotheses. Second, because the student did not work with a paper and pencil diagram, because the digital diagram took this role too, and finally, because the use of measuring tools (which also replaced the usual marks that the student affixes on his drawings) allowed the student to make inferences by reasoning with the diagram, some previous arguments were allowed to be completed. The synergy between visual practices and analytical practices mediated by the developed discourse arises from the completeness and richness of the objects that emerged from the student's practice, although he did not tackle the task in its full generality.

It is clear that this study has strong limitations (the analysis of only one student and a limited amount of data); however, it shows how multiple visual representations such as folded models and crease diagrams and their digital counterparts allow for better coordination of operative and discursive apprehensions during the problem-solving process, especially during the process from conjecture generation to proof construction. Even if more studies are required (with more challenging problems, different promptings, individual or group data) our results seem to suggest that it is desirable for these practices to enter into teaching. Many different possibilities of proof-oriented task design are possible, even drawing on origami literature (Frigerio, 2002; Haga, 2008) for substantial problems that can be modeled in dynamic geometry environments. The choice of the problem and the task design obviously depend on many factors, including the previous knowledge of students and the degree of confidence in paper folding and dynamic geometry practices.

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