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On the equivariant 2-type of a G-space[1]

M. Bullejos[a,*], J.G. Cabello[a], E. Faro[b]

[a] *Departamento de Álgebra, Universidad de Granada, 18071 Granada, Spain*
[b] *Departamento de Matemática Aplicada, Universidad de Vigo, Vigo, Spain*

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Abstract

A classical theorem of Mac Lane and Whitehead states that the homotopy type of a topological space with trivial homotopy at dimensions 3 and greater can be reconstructed from its π_1 and π_2, and a cohomology class $k_3 \in H^3(\pi_1, \pi_2)$. More recently, Moerdijk and Svensson suggested the possibility of using Bredon cohomology to extend this result to the equivariant case, that is, for spaces X equipped with an action by a fixed group G. In this paper we carry out this suggestion and prove an analogue of the classical result in the equivariant case.

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1. Introduction

Mac Lane and Whitehead proved in [19] that the homotopy type of a pointed space (X, x_0), with trivial homotopy at dimensions 3 and greater, can be reconstructed from its fundamental group $\pi_1(X, x_0)$, together with the $\pi_1(X, x_0)$-module $\pi_2(X, x_0)$ and a cohomology class $k_3 \in H^3(\pi_1(X, x_0), \pi_2(X, x_0))$. The objective of this paper is to prove an analogue of this result (Theorem 7.1) for spaces X equipped with an action by a fixed group G or G-spaces.

From the point of view of its equivariant homotopy type, a G-space X can be regarded as a diagram of spaces, namely, the spaces X^H of points fixed by each subgroup H of G. More concretely: if $\mathcal{O}(G)$ denotes the orbit category of G (see Section 7), then there is a functor $\mathbf{Top}^G \to \mathbf{Top}^{\mathcal{O}(G)^{\mathrm{op}}}$ associating to each G-space X a "fixed points" diagram of spaces

* Corresponding author. E-mail: bullejos@ugr.es.
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$$X^{(-)}: \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Top},$$

(taking each object $G/H \in \mathcal{O}(G)$ to X^H) and it is this diagram what determines the equivariant homotopy type of X. For example, the *equivariant fundamental groupoid* of X is determined by the fundamental groupoids of all the spaces of fixed points X^H, with H a subgroup of G.

The *equivariant fundamental groupoid functor* is obtained from the usual fundamental groupoid functor $\mathbf{Top} \xrightarrow{\pi_1} \mathbf{Gpd}$ as the composite of $\pi_1^* = \pi_1^{\mathcal{O}(G)^{\mathrm{op}}}$ with the functor $\mathbf{Top}^G \to \mathbf{Top}^{\mathcal{O}(G)^{\mathrm{op}}}$ indicated above,

$$\begin{tikzcd} \mathbf{Top}^G \arrow[r] \arrow[rr, bend right, "\pi_1^G"'] & \mathbf{Top}^{\mathcal{O}(G)^{\mathrm{op}}} \arrow[r, "\pi_1^*"] & \mathbf{Gpd}^{\mathcal{O}(G)^{\mathrm{op}}}, \end{tikzcd}$$

so that the equivariant fundamental groupoid of a G-space X is the functor

$$\pi_1^G(X): \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Gpd},$$

which takes any object $G/H \in \mathcal{O}(G)$ to the fundamental groupoid of the space X^H.

Applying the Grothendieck construction (see Section 2.1) to the equivariant fundamental groupoid of a G-space X we obtain a category $\pi^G(X)$ (see Eq. (7.2)) which plays in the equivariant case the same role that the fundamental groupoid plays in the non-equivariant one. So, a local coefficient is just a $\pi^G(X)$-module, i.e. a functor from $\pi^G(X)$ to the category of abelian groups. In particular the equivariant homotopy groups $\pi_n^G(X)$ are local coefficients.

Bredon defined the equivariant cohomology of a G-space in [5], and recently Moerdijk and Svensson [21] have generalized Bredon cohomology to an equivariant cohomology with local coefficients. This generalization is given in terms of the cohomology of small categories. So, in order to generalize the result of Mac Lane–Whitehead we have to substitute cohomology of small categories for group cohomology. To this end, we devote Section 2.4 to introduce and to give some interpretation of the cohomology of a small category. In particular we see that this cohomology is a cotriple cohomology and that we can use Duskin's interpretation theorem to identify the cohomology of a small category with the set of connected components of torsors. Since this interpretation will be essential in the proof of our generalization of the Mac Lane–Whitehead result (Theorem 7.1), we dedicate Section 3 to introduce 2-torsors in the context of the category **Cat** of small categories. We must say that we introduce 2-torsors from an angle that it is different from that appearing in [9] and closer to the ideas of [10]. This new approach (which, in any case, we have learned from Duskin) is based on the simple idea that a 2-torsor over Π is just a groupoid with connected components equal to Π and such that the endomorphism groups are globally determined by a Π-module (see Section 3.1 for details).

The generalization of the Mac Lane–Whitehead result says that the equivariant homotopy type of a G-space X, with trivial equivariant homotopy at dimensions ≥ 3, is completely determined by the $\mathcal{O}(G)^{\mathrm{op}}$-diagram of groupoids $\pi_1^G(X)$, the $\pi^G(X)$-module $\pi_2^G(X)$ and a cohomology element $k_3 \in H^3(\pi^G(X), \pi_2^G(X))$. Here $H^3(\pi^G(X), \pi_2^G(X))$ is the cohomology of the small category $\pi^G(X)$ with coefficients in the $\pi^G(X)$-module $\pi_2^G(X)$.

Our proof of this theorem is based on the adjunction

$$\Sigma : \mathbf{Cat}/\mathscr{C} \rightleftarrows \mathbf{Cat}^{\mathscr{C}} : \int_{\mathscr{C}},$$

(where the right adjoint, $\int_{\mathscr{C}}$, is the Grothendieck semidirect product construction), as well as the extension of this adjunction to the context of 2-categories (see Section 6). Thus, part of this paper is devoted to introduce these functors and to study the way they behave with respect to torsors.

We want to thank I. Moerdijk for having introduced us to this problem. In our work we use most of the techniques that he and Svensson developed for the study of equivariant homotopy problems. On the other hand, we have learned that Joyal and Tierney have developed a more general method of approaching this kind of problems. Their method consists in substituting a topos $\mathscr{E}$ for the topos of sets and using a Quillen model structure in the topos of simplicial $\mathscr{E}$-objects. In particular they have the project of building the whole Postnikov tower of a simplicial $\mathscr{E}$-object so that they will obtain all the k-invariants. The topos of G-sets is a particular but very important case and we all agree that the explicit methods of calculation of the k_3-invariant we presented here are of general interest.

2. Preliminaries

2.1. Grothendieck semidirect product construction

The 2-category **Cat** of small categories has lax colimits, that is, any functor $F : \mathscr{C} \to \mathbf{Cat}$ (with $\mathscr{C}$ small) has a lax colimit (see [4] for the definition of lax colimit). The explicit construction that is used to prove the previous statement is called the Grothendieck construction or Grothendieck integral and can be found, e.g., [14] or [18]. Given a functor $F : \mathscr{C} \to \mathbf{Cat}$ defined on a small category $\mathscr{C}$, the category obtained as the lax colimit of F is called the Grothendieck semidirect product of $\mathscr{C}$ by F and is denoted by $\int_{\mathscr{C}} F$. Its objects are pairs (C,x), where C is an object of $\mathscr{C}$ and x and object of $F(C)$, and an arrow $(C,x) \to (C',x')$ is a pair (f,λ) consisting of an arrow $f : C \to C'$ in $\mathscr{C}$, and an arrow $\lambda : F(f)(x) \to x'$ in $F(C')$, with the obvious composition.

As a lax colimit, the Grothendieck construction has the following property: there is a family of functors, one for each object $C \in \mathscr{C}$,

$$\mathbf{j}^C : F(C) \to \int_{\mathscr{C}} F,$$

and a family of natural transformations

$$\begin{tikzcd} & F(C) \arrow[dr, "\mathbf{j}^C"] & \\ F(C') \arrow[ur, "F(f)"] \arrow[rr, "\mathbf{j}^{C'}"'] \arrow[rr, phantom, shift left=3, "\Uparrow \mathbf{j}^f"] & & \int_{\mathscr{C}} F \end{tikzcd}$$

one for each arrow $f : C' \to C$ in $\mathscr{C}$, satisfying the following properties:

(a) $\mathbf{j}^{1_C} = 1_{\mathbf{j}^C}$,

(b) for any pair of composable arrows $C \xrightarrow{f} C' \xrightarrow{g} C''$ in $\mathscr{C}$ the following diagram of natural transformations commutes

$$\begin{tikzcd} \mathbf{j}^C \circ F(f) \circ F(g) \arrow[rr, "\mathbf{j}^f \circ F(g)"] & & \mathbf{j}^{C'} \circ F(g). \\ & \mathbf{j}^{C''} \arrow[ul, "\mathbf{j}^{gf}"] \arrow[ur, "\mathbf{j}^g"'] & \end{tikzcd}$$

These data are universal in the usual sense for colimits: if there is any category $\mathscr{E}$ (not necessarily small), together with suitable families of functors for each object C, $\mathbf{t}^C$, and natural transformations for each morphism $f : C \to C'$, $\mathbf{t}^f$, satisfying the above properties, then there exists a unique functor $T : \int_{\mathscr{C}} F \to \mathscr{E}$ which determines all the functors $\mathbf{t}^C$ and all the natural transformations $\mathbf{t}^f$ in terms of the $\mathbf{j}^C$ and the $\mathbf{j}^f$ in the usual sense.

There is an obvious projection (forgetful functor) $\int_{\mathscr{C}} F \to \mathscr{C}$ suggesting that the Grothendieck construction $\int_{\mathscr{C}}(-)$ may be seen as a functor from the functor category $\mathbf{Cat}^{\mathscr{C}}$ to the slice category $\mathbf{Cat}/\mathscr{C}$

$$\int_{\mathscr{C}} : \mathbf{Cat}^{\mathscr{C}} \to \mathbf{Cat}/\mathscr{C} .$$

Actually this is even a 2-functor (taking as 2-cells between 1-cells $F, G : (\mathscr{A} \xrightarrow{a} \mathscr{C}) \to (\mathscr{B} \xrightarrow{b} \mathscr{C})$ in $\mathbf{Cat}/\mathscr{C}$ just those natural transformations $\alpha : F \to G$ such that $b\alpha = 1_a$).

Furthermore, if $G : \mathscr{C} \to \mathscr{C}'$ is any functor between small categories, we have a commutative square

$$\begin{array}{ccc} \mathbf{Cat}^{\mathscr{C}'} & \xrightarrow{\int_{\mathscr{C}'}} & \mathbf{Cat}/\mathscr{C}' \\ \downarrow & & \downarrow \\ \mathbf{Cat}^{\mathscr{C}} & \xrightarrow{\int_{\mathscr{C}}} & \mathbf{Cat}/\mathscr{C} \end{array}$$

where the vertical functors are obtained by composition with G and pulling back along G respectively. Thus, the “Grothendieck integral” $\int_{\mathscr{C}} F$ is natural in $\mathscr{C}$, so that one can briefly summarize all the functorial properties of the Grothendieck integral by regarding it as a natural transformation between functors from the 2-category of small categories to the category of 2-dimensional categories,

$$\int : \mathbf{Cat}^{(-)} \to \mathbf{Cat}/(-).$$

Furthermore, if we restrict our attention to groupoids, then the Grothendieck integral restricts to a "natural transformation" between the following functors from the 2-category of small groupoids to the category of 2-dimensional categories,

$$\int : \mathbf{Gpd}^{(-)} \to \mathbf{Gpd}/(-).$$

In other words, if $\mathscr{C}$ is a groupoid, the Grothendieck integral functor $\int_{\mathscr{C}}$ restricts to a functor making the diagram

$$\begin{tikzcd} \mathbf{Gpd}^{\mathscr{C}} \arrow[r, dashed] \arrow[d, hook] & \mathbf{Gpd}/\mathscr{C} \arrow[d, hook] \\ \mathbf{Cat}^{\mathscr{C}} \arrow[r, "\int_{\mathscr{C}}"] & \mathbf{Cat}/\mathscr{C} \end{tikzcd}$$

commutative.

The functor $\int_{\mathscr{C}} : \mathbf{Cat}^{\mathscr{C}} \to \mathbf{Cat}/\mathscr{C}$ has a left adjoint $\Sigma : \mathbf{Cat}/\mathscr{C} \to \mathbf{Cat}^{\mathscr{C}}$, (see, e.g., [14]) and as a consequence we have that the functor $\int_{\mathscr{C}} : \mathbf{Cat}^{\mathscr{C}} \to \mathbf{Cat}/\mathscr{C}$ preserves all limits, and in particular pullback diagrams. Moreover, although the left adjoint Σ does not preserve finite limits in general, it does preserve some. In fact, it is not difficult to see that the functor Σ preserves pullbacks.

We state now a lemma that shows a very interesting property of the adjunction

$$\Sigma : \mathbf{Cat}/\mathscr{C} \rightleftarrows \mathbf{Cat}^{\mathscr{C}} : \int_{\mathscr{C}}.$$

Lemma 2.1. *Given a functor $F : \mathscr{C} \to \mathbf{Cat}$, for any object $C \in \mathscr{C}$, the category $F(C)$ is a co-reflective subcategory of $(\Sigma \int_{\mathscr{C}} F)(C)$. More specifically, there is a canonical inclusion of categories*

$$\mathbf{i}_C : F(C) \hookrightarrow \left(\Sigma \int_{\mathscr{C}} F\right)(C),$$

which embeds $F(C)$ as a full subcategory of $(\Sigma \int_{\mathscr{C}} F)(C)$. Moreover the counit of the adjunction $\Sigma \dashv \int_{\mathscr{C}}$ gives, for any object $C \in \mathscr{C}$, a right adjoint of $\mathbf{i}_C$

$$\varepsilon_C : \left(\Sigma \int_{\mathscr{C}} F\right)(C) \to F(C).$$

2.2. $\mathscr{C}$-modules

Since the functor $\int_{\mathscr{C}} : \mathbf{Cat}^{\mathscr{C}} \to \mathbf{Cat}/\mathscr{C}$ preserves the terminal object, any global point $1 \to F$ of a functor $F \in \mathbf{Cat}^{\mathscr{C}}$ gives rise to a global point of $\int_{\mathscr{C}} F$ in $\mathbf{Cat}/\mathscr{C}$, that is, a section of the projection $\int_{\mathscr{C}} F \to \mathscr{C}$. Thus, if F has exactly one global point, the projection has a canonical section and $\int_{\mathscr{C}} F$ becomes a pointed object in $\mathbf{Cat}/\mathscr{C}$. This occurs, for example, if each of the categories $F(C)$ for $C \in \mathscr{C}$ has exactly one object (making the functor $F : \mathscr{C} \to \mathbf{Cat}$ factor through the full subcategory $\mathbf{Mon} \hookrightarrow \mathbf{Cat}$ of monoids). In such case $\int_{\mathscr{C}} F$ is canonically a pointed object in $\mathbf{Cat}/\mathscr{C}$. Furthermore, in that case, $\int_{\mathscr{C}} F$ has "the same objects" as $\mathscr{C}$, and the projection functor $\int_{\mathscr{C}} F \to \mathscr{C}$ is "the identity on objects" so that $\int_{\mathscr{C}} F$ is canonically a pointed object in the full subcategory of $\mathbf{Cat}/\mathscr{C}$ consisting of those functors $G : \mathscr{T} \to \mathscr{C}$ that are the identity on objects. We shall use the notation $\mathbf{Cat}_0/\mathscr{C}$ to indicate such full subcategory, and we will use an analogous notation in other contexts. For example, if $\mathscr{C}$ is a groupoid we will denote $\mathbf{Gpd}_0/\mathscr{C}$ the full subcategory of $\mathbf{Gpd}/\mathscr{C}$ consisting of those functors $G : \mathscr{T} \to \mathscr{C}$ that are the identity on objects.

The functor $\int_{\mathscr{C}}$ preserves all limits and in particular, cartesian products (even when restricted to $\mathbf{Mon}^{\mathscr{C}}$) and thus it takes every monoid or group object in $\mathbf{Mon}^{\mathscr{C}}$ to a monoid or group object in $\mathbf{Cat}/\mathscr{C}$. But the monoid objects in $\mathbf{Mon}^{\mathscr{C}}$ are commutative and they constitute the full subcategory $\mathbf{ComMon}^{\mathscr{C}} \hookrightarrow \mathbf{Mon}^{\mathscr{C}}$ where $\mathbf{ComMon}$ is the category of commutative monoids. Therefore, if a functor $F : \mathscr{C} \to \mathbf{Cat}$ has the property that all its values are commutative monoids, then $\int_{\mathscr{C}} F$ is canonically a commutative monoid object in $\mathbf{Cat}_0/\mathscr{C}$.

Summing up, the restriction of the Grothendieck construction to the category $\mathbf{ComMon}^{\mathscr{C}}$, factors through the category of commutative monoid objects in $\mathbf{Cat}_0/\mathscr{C}$. The importance that all this has for us lies in the fact that we will be applying the Grothendieck construction to *left $\mathscr{C}$-modules*, i.e. functors $A : \mathscr{C} \to \mathbf{Ab}$ (where the category $\mathbf{Ab}$ of abelian groups is seen as a full subcategory of $\mathbf{Cat}$). In this case the category $\int_{\mathscr{C}} A$ (with its projection to $\mathscr{C}$) is canonically an internal abelian group object in $\mathbf{Cat}_0/\mathscr{C}$

$$\begin{tikzcd}
\mathbf{Ab}^{\mathscr{C}} \arrow[r, dotted, "\int_{\mathscr{C}}"] \arrow[d, hook] & \mathbf{Ab}(\mathbf{Cat}_0/\mathscr{C}) \arrow[d, hook] \\
\mathbf{Cat}^{\mathscr{C}} \arrow[r, "\int_{\mathscr{C}}"] & \mathrm{Cat}/\mathscr{C}.
\end{tikzcd}$$

2.3. Geometric realization and weak equivalences of 2-categories

Our objective now is to define the concept of *weak equivalence of (small) 2-categories* and to establish two lemmas each of which gives sufficient conditions for a 2-functor to be a weak equivalence. These lemmas will be crucial for our main theorem.

As it is done in the case of small categories, the definition of weak equivalence of small 2-categories is reduced to that of spaces by functorially associating to each small 2-category a simplicial set called its *2-nerve* and, then, forming the geometric realization of this simplicial set. There are several possible definitions of the 2-nerve.

One of them is obtained by applying to 2-categories the elegant construction of nerve given by Moerdijk and Svensson in [20]. This, however, we shall not do mainly because we have not been able to prove our required lemmas based on such definition. Instead we give a definition (embodied in diagram 2.3) from which the two Lemmas 2.2 and 2.3 follow easily using well known results. Before we give our definition of 2-nerve we shall review some basic notions mainly to set up our notation.

A 2-dimensional category or 2-category $\mathfrak{C}$ is just a category enriched in the category **Cat** of small categories, that is, for any two objects $A, B \in \mathfrak{C}$, we have a category $\mathfrak{C}(A,B)$ of arrows, whose objects are called the arrows or 1-cells in $\mathfrak{C}$ from A to B and whose arrows are called *deformations* or 2-cells. A deformation α between arrows f and g will be represented as

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet. \end{tikzcd}$$

Thus, in a 2-category $\mathfrak{C}$ there are two different compositions of deformations. One of them, arising from the composition in the categories $\mathfrak{C}(A,B)$, is called the *vertical* composition, while the other, arising from the enrichment composition functors $\mathfrak{C}(A,B) \times \mathfrak{C}(B,C) \to \mathfrak{C}(A,C)$, is called the *horizontal* composition.

A 2-functor between 2-categories, $\mathfrak{f}: \mathfrak{C} \to \mathfrak{C}'$, is just a functor such that for any two objects of $\mathfrak{C}$, A, B, the arrows part of $\mathfrak{f}$,

$$\mathfrak{f}_{A,B}: \mathfrak{C}(A,B) \to \mathfrak{C}'(\mathfrak{f}(A), \mathfrak{f}(B))$$

is a functor. Similarly, a 2-natural transformation between 2-functors is just a natural transformation whose components are functors. Small 2-categories, 2-functors between them and 2-natural transformations constitute a 2-category denoted by **2-Cat**. For useful background information concerning enriched categories and 2-categories, we refer the reader to [4, 6, 15].

We regard any *small* 2-category $\mathfrak{C}$ as an internal category in **Cat** in the following way (of the two possible ways): let $\mathscr{C}_0$ be the underlying category of $\mathfrak{C}$ (we leave out the deformations). Let $\mathscr{C}_1$ be the category whose objects are the same objects of $\mathfrak{C}$ and whose arrows are the deformations of $\mathfrak{C}$, the composition in $\mathscr{C}_1$ being given by the horizontal composition of deformations. Then $\mathfrak{C}$ can be internally represented in **Cat** by the following diagram

$$\mathfrak{C}: \quad \begin{tikzcd} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \arrow[r, "\circ"] & \mathscr{C}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathscr{C}_0 \arrow[l, bend right, "Id"'] \end{tikzcd} \tag{2.1}$$

where the functor Id is the common section of s and t (s for source and t for target) taking any arrow to the identity deformation on it, the pullback is that of s and t, and the composition, $\circ$, is the functor given by the vertical composition. All the categories in the above diagram have the same objects and all the above functors are the identity on objects.

We shall now define the nerve of small 2-categories. Recall that for small categories the nerve functor

$$\mathrm{Ner} : \mathbf{Cat} \to \mathbf{Set}^{\Delta^{\mathrm{op}}} \tag{2.2}$$

is the right adjoint to the categorization functor (see [24]), and therefore it preserves limits, a fact that we will use later.

Now, both of the categories **Cat** and $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ are cartesian closed, thus they are monoidal categories. Since the nerve functor (2.2) preserves cartesian products and terminal object, it is a monoidal functor. Thus, it induces a 2-functor $\mathbf{2\text{-}Cat} \to (\mathbf{Set}^{\Delta^{\mathrm{op}}})\text{-}\mathbf{Cat}$ from the 2-category of small categories enriched in **Cat** to the 2-category of small categories enriched in simplicial sets. Furthermore every small category enriched in simplicial sets can be regarded as a "simplicial category", and one gets a 2-functor $(\mathbf{Set}^{\Delta^{\mathrm{op}}})\text{-}\mathbf{Cat} \to \mathbf{Cat}^{\Delta^{\mathrm{op}}}$. This, composed with the previous one, gives a 2-functor

$$\mathfrak{Ner} : \mathbf{2\text{-}Cat} \to \mathbf{Cat}^{\Delta^{\mathrm{op}}}.$$

On the other hand we have the Artin–Mazur total complex functor $\overline{W}$ from the category of double simplicial sets to the category of simplicial sets,

$$\overline{W} : \mathbf{Set}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \to \mathbf{Set}^{\Delta^{\mathrm{op}}},$$

see [1]. Using (2.2) we can form the composite $\mathbf{N} = \overline{W} \circ \mathrm{Ner}^{\Delta^{\mathrm{op}}} \circ \mathfrak{Ner}$,

$$\begin{tikzcd} \mathbf{2\text{-}Cat} \arrow[r, "\mathfrak{Ner}"] \arrow[rrr, bend right, "\mathbf{N}"'] & \mathbf{Cat}^{\Delta^{\mathrm{op}}} \arrow[r, "\mathrm{Ner}^{\Delta^{\mathrm{op}}}"] & \mathbf{Set}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \arrow[r, "\overline{W}"] & \mathbf{Set}^{\Delta^{\mathrm{op}}}. \end{tikzcd} \tag{2.3}$$

which we define to be the 2-nerve functor.

By the *geometric realization* or *classifying space* $\mathbf{B}(\mathfrak{C})$ of a 2-category $\mathfrak{C}$ we mean the geometric realization of its 2-nerve, $\mathbf{B}(\mathfrak{C}) = |\mathbf{N}(\mathfrak{C})|$. A 2-functor $\mathfrak{f} : \mathfrak{C} \to \mathfrak{C}'$ will be called a *weak equivalence of 2-categories* if the corresponding continuous map $\mathbf{B}(\mathfrak{f}) : \mathbf{B}(\mathfrak{C}) \to \mathbf{B}(\mathfrak{C}')$ is a weak equivalence of spaces or, equivalently, if the simplicial map $\mathbf{N}(\mathfrak{f}) : \mathbf{N}(\mathfrak{C}) \to \mathbf{N}(\mathfrak{C}')$ is a weak equivalence of simplicial sets. It is difficult in general to see when a 2-functor is a weak equivalence. However, as an immediate consequence of the fact that any equivalence of categories induces a homotopy equivalence between the corresponding nerves, and the fact that the "diagonal" functor from $\mathbf{Set}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ to $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ is weak-equivalent to $\overline{W}$, we have that Theorem B.2 of [7] (which can be rephrased as saying: "*Given two functors $F, G : \Delta^{\mathrm{op}} \to \mathbf{Set}^{\Delta^{\mathrm{op}}}$ and a natural transformation $F \xrightarrow{\lambda} G$, if all components of λ are weak equivalences then the diagonal of λ also is a weak equivalence*") implies the following two results (Lemmas 2.2 and 2.3) where $\mathfrak{C}$ and $\mathfrak{C}'$ are small 2-categories regarded as internal categories in **Cat** as indicated above,

Lemma 2.2. *If $\mathfrak{f}:\mathfrak{C}\to\mathfrak{C}'$ is a 2-functor such that:*
1. *it induces a bijection between the corresponding sets of objects, and*
2. *for any two objects $X,Y\in\mathfrak{C}$ the induced functor*

$$\mathfrak{f}_{X,Y}:\mathfrak{C}(X,Y)\to\mathfrak{C}'(\mathfrak{f}(X),\mathfrak{f}(Y))$$

is an equivalence of categories, then $\mathfrak{f}$ is a weak equivalence of 2-categories.

Lemma 2.3. *If $\mathfrak{f}:\mathfrak{C}\to\mathfrak{C}'$ is a 2-functor such that the induced functors $\mathfrak{f}_0:\mathscr{C}_0\to\mathscr{C}'_0$ and $\mathfrak{f}_1:\mathscr{C}_1\to\mathscr{C}'_1$ are equivalences of categories, then $\mathfrak{f}$ is a weak equivalence of 2-categories.*

2.4. Cohomology of small categories

We shall now review some basic facts about cohomology of categories (cf., e.g. [3, 23, 25]).

The cohomology of a small category $\mathscr{C}$ with coefficients in a (left) $\mathscr{C}$-module $A\in\mathbf{Ab}^{\mathscr{C}}$ can be defined (by viewing A as an abelian group object in $\mathbf{Set}^{\mathscr{C}}$) as the cohomology of the topos $\mathbf{Set}^{\mathscr{C}}$ with coefficients in A. This cohomology is calculated as the right derived functors of the global section (or $\lim_{\leftarrow}$) functor

$$\Gamma:\mathbf{Ab}^{\mathscr{C}}\to\mathbf{Ab}.$$

There are other ways to compute this cohomology, for example as the cohomology of a cochain complex associated to the nerve of the category $\mathscr{C}$ and also as the cohomology of certain cochain complex associated to a cosimplicial complex of abelian groups obtained by some kind of bar resolution (see [3]).

A further way to calculate the cohomology of a small category $\mathscr{C}$ is suggested by the fact that, as noted in Section 2.2, for any $\mathscr{C}$-module A, the Grothendieck construction produces an abelian group object $\int_{\mathscr{C}}A\to\mathscr{C}$ in the category $\mathbf{Cat}_0/\mathscr{C}$ and the fact that this category is tripleable over the full subcategory $\mathbf{Gph}_0/U(\mathscr{C})$ of the slice category of graphs over the underlying graph $U(\mathscr{C})$ associated to $\mathscr{C}$. Indeed, the category $\mathbf{Cat}_0/\mathscr{C}$ is tripleable with cotriple associated to the adjunction

$$F:\mathbf{Gph}_0/U(\mathscr{C})\rightleftarrows\mathbf{Cat}_0/\mathscr{C}:U,$$

where F and U are induced by the "*free category*" and the "*underlying graph*" functors respectively.

Let $\mathbb{G}$ denote the corresponding cotriple. It can be proved (by using the resolution in [3] and doing an appropriate translation of what happens in the case of groups – see [8] for the details) that there are natural isomorphisms

$$H^n_{\mathbb{G}}\left(\mathscr{C},\int_{\mathscr{C}}A\right)\cong\begin{cases}Der(\mathscr{C},A) & \text{if } n=0\\ H^{n+1}(\mathscr{C},A) & \text{if } n>0\end{cases}$$

between the cotriple cohomology groups $H^n_{\mathbb{G}}(\mathscr{C},\int_{\mathscr{C}}A)$ of the identity of $\mathscr{C}$ with coefficients in the abelian group object $\int_{\mathscr{C}}A\to\mathscr{C}$ and the cohomology of the small category

$\mathscr{C}$ with coefficients in the $\mathscr{C}$-module A. The set $Der(\mathscr{C},A)$ of derivations of $\mathscr{C}$ is defined by a suitable generalization of the derivations in the case of groups (see, e.g., [13]).

This interpretation of the cohomology of a small category as a cotriple cohomology allows us to use Duskin's interpretation theorem (see [9]) to find natural isomorphisms

$$H^{n+1}(\mathscr{C},A) \cong Tor^n[\mathscr{C},A],$$

where the right-hand side is the set of connected components classes of n-dimensional U-split torsors above $\mathscr{C}$ with coefficients in A. We dedicate Section 3 to study 2-dimensional torsors in this particular context.

We now particularize to the case in which the category $\mathscr{C}$ is a groupoid. In this case we still have that the category $\mathbf{Gpd}_0/\mathscr{C}$ is tripleable over $\mathbf{Gph}_0/U(\mathscr{C})$. Let $\mathbb{G}'$ be the corresponding cotriple. Now, as noted in Section 2.2, for any $\mathscr{C}$-module A, the forgetful functor $\int_{\mathscr{C}} A \to \mathscr{C}$ is also an abelian group object in $\mathbf{Gpd}_0/\mathscr{C}$. Then, we can mimic what happens in groups in order to follow an analogous reasoning with groupoids and get, as above, natural isomorphisms

$$H^n_{\mathbb{G}'}\left(\mathscr{C}, \int_{\mathscr{C}} A\right) \cong H^{n+1}(\mathscr{C},A)$$

for all $n > 0$. So, in the case that $\mathscr{C}$ is a groupoid, the cohomology of $\mathscr{C}$ can be identified as the cohomology of two different cotriples. In fact we have natural isomorphisms

$$H^n_{\mathbb{G}'}\left(\mathscr{C}, \int_{\mathscr{C}} A\right) \cong H^{n+1}(\mathscr{C},A) \cong H^n_{\mathbb{G}}\left(\mathscr{C}, \int_{\mathscr{C}} A\right).$$

Finally we note that the cohomology of a small category can be also interpreted as a singular cohomology when the module of coefficients is a local system of coefficients, that is, when the module of coefficients is a functor $A : \pi_1(\mathscr{C}) \to \mathbf{Ab}$ where $\pi_1(\mathscr{C})$ is the fundamental groupoid (i.e. the groupoid of fractions) of the small category $\mathscr{C}$. Since we have a canonical functor $\mathscr{C} \to \pi_1(\mathscr{C})$ we can consider A as a $\mathscr{C}$-module. Then, a local coefficient is a local coefficient for the geometric realization $\mathbf{B}(\mathscr{C}) = |\mathrm{Ner}(\mathscr{C})|$ of the category $\mathscr{C}$.

Illusie proves in [16] that, for local coefficients, there are natural isomorphisms

$$H^n(\mathscr{C},A) \cong H^n(\mathbf{B}(\mathscr{C}),A),$$

where $H^n(\mathbf{B}(\mathscr{C}),A)$ are the singular cohomology groups of the space $\mathbf{B}(\mathscr{C})$ with coefficients in the local system A.

As an immediate consequence of the above isomorphism we have:

Proposition 2.4. *If a functor $F : \mathscr{C} \to \mathscr{C}'$ is a weak equivalence in the sense that the induced map between the corresponding geometric realizations (or nerves) is a weak equivalence of spaces (or of simplicial sets), then for any local coefficient $A : \pi_1(\mathscr{C}') \to \mathbf{Ab}$ the induced group homomorphisms*

$$F^* : H^n(\mathscr{C}',A) \to H^n(\mathscr{C},F^*(A))$$

are isomorphisms.

3. Two-dimensional torsors

In this section we interpret Duskin's concept of two-dimensional torsor [9] in the case that the base category is $\mathbf{Cat}_0/\mathscr{C}$ and the forgetful functor is $U:\mathbf{Cat}_0/\mathscr{C} \to \mathbf{Gph}_0/U(\mathscr{C})$. We will take this as the definition of 2-torsor (in Section 3.1) leaving to the reader the task of verifying that it coincides with that given by Duskin in [9] when it is particularized to our context. The main reasons we have to use this interpretation of the concept of 2-torsor are:

- to explicitly specify the fiber in the data that define a 2-torsor, and
- to be able to determine the homotopy type of the 2-categories which appear as the fibers of 2-torsors, in some particular contexts (see Theorem 4.3).

We begin by briefly reviewing some known results to help fixing our notation. Given a 2-category $\mathfrak{C}$ regarded as an internal category in **Cat** as we indicated in Section 2.3, diagram (2.1), a (left) action of $\mathfrak{C}$ on a small category, or (left) $\mathfrak{C}$-object in **Cat** is a category over $\mathscr{C}_0$, $P:\mathscr{X} \to \mathscr{C}_0$, equipped with a functor

$$\mu:\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{X} \to \mathscr{X}$$

defined on the pullback of $s:\mathscr{C}_1 \to \mathscr{C}_0$ and P, subject to the usual associative and unit laws, and moreover making the diagram

$$\begin{tikzcd}
\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{X} \arrow[r, "\mu"] \arrow[d, "p_r"'] & \mathscr{X} \arrow[d, "P"] \\
\mathscr{C}_1 \arrow[r, "t"] & \mathscr{C}_0
\end{tikzcd}$$

commutative. There, p_r denotes the canonical projection.

The above required conditions on μ imply that there is an *action* of the set of deformations of $\mathfrak{C}$ on the set of arrows of $\mathscr{X}$ in such a way that the only deformations α that act on an arrow $f:A \to B$ in $\mathscr{X}$ are those whose source is $P(f)$, i.e.

$$\begin{tikzcd}
P(A) \arrow[r, bend left, "P(f)"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & P(B).
\end{tikzcd}$$

Such deformation α acting on f produces an arrow ${}^{\alpha}f = \mu(\alpha, f):A \to B$ in $\mathscr{X}$ such that $P({}^{\alpha}f) = g$, the target of α.

Given two $\mathfrak{C}$-objects, $(\mathscr{X} \xrightarrow{P} \mathscr{C}_0, \mu)$, and $(\mathscr{Y} \xrightarrow{R} \mathscr{C}_0, \nu)$, an equivariant functor or morphism of $\mathfrak{C}$-objects from one to the other is a morphism $(\mathscr{X} \xrightarrow{P} \mathscr{C}_0) \xrightarrow{F} (\mathscr{Y} \xrightarrow{R} \mathscr{C}_0)$ in $\mathbf{Cat}/\mathscr{C}_0$, compatible with the actions, in the obvious way. Thus, F is such that if α acts on $(A \xrightarrow{f} B) \in \mathscr{X}$ then necessarily α acts also on $F(f)$ and $F({}^{\alpha}f) = {}^{\alpha}F(f)$.

We will let $\mathfrak{C}$-**Cat** to denote the category of $\mathfrak{C}$-objects and equivariant functors. This category is also known as the category of internal functors from $\mathfrak{C}$ to **Cat**, which is often denoted by $\mathbf{Cat}^{\mathfrak{C}}$, because there is a forgetful functor $\mathfrak{C}\text{-}\mathbf{Cat} \to \mathbf{Cat}/\mathscr{C}_0$ that has a left adjoint and the monad on $\mathbf{Cat}/\mathscr{C}_0$ induced by this adjoint pair has $\mathfrak{C}$-**Cat** as category of algebras.

Evidently any 2-functor $\mathfrak{f}:\mathfrak{C} \to \mathfrak{C}'$ induces in a functorial way (via pullback) a functor

$$\mathfrak{f}^* : \mathfrak{C}'\text{-}\mathbf{Cat} \to \mathfrak{C}\text{-}\mathbf{Cat}, \tag{3.1}$$

which preserves in particular finite products, a fact that will be used in Section 3.1 in the definition of 2-torsors.

In a second place, some considerations about internally connected groupoids in $\mathbf{Cat}_0/\Pi$ have to be made. Let us consider the functor

$$\Pi_0 : \mathbf{2\text{-}Cat} \to \mathbf{Cat} \tag{3.2}$$

which sends any 2-category $\mathfrak{C}$ to the category $\Pi_0(\mathfrak{C})$ of connected components of $\mathfrak{C}$, that is, to the coequalizer of the functors s and t, in the diagram (2.1),

$$\mathscr{C}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathscr{C}_0 \longrightarrow \Pi_0(\mathfrak{C}).$$

For a given 2-category $\mathfrak{C}$ let $\Pi = \Pi_0(\mathfrak{C})$. Clearly $\mathfrak{C}$ can be considered as an internal category in $\mathbf{Cat}_0/\Pi$ and as such it is *connected* since Π "is" just the terminal object of $\mathbf{Cat}_0/\Pi$.

If the 2-category $\mathfrak{C}$ is actually a category enriched in the category **Gpd** of small groupoids (so that every deformation has an inverse with respect to the vertical composition) then $\mathfrak{C}$ is also a groupoid as internal category in $\mathbf{Cat}_0/\Pi$ and we can say that $\mathfrak{C}$ is an *internally connected groupoid* in $\mathbf{Cat}_0/\Pi$.

Conversely, let us suppose that a small category Π is given and $\mathfrak{C}$ is a 2-category which, when regarded as an internal category in **Cat**, it is an internally connected groupoid in $\mathbf{Cat}_0/\Pi$. Then evidently $\Pi = \Pi_0(\mathfrak{C})$ and $\mathfrak{C}$ is actually a category enriched in groupoids.

In what follows our interest will be focused on those 2-categories which are internally connected groupoids in $\mathbf{Cat}_0/\Pi$ for a given small category Π.

Let $\mathfrak{C}$ be a (small) 2-category which is enriched in the category of groupoids. We shall now build a special $\mathfrak{C}$-object which has a canonical structure of abelian group object in $\mathfrak{C}$-**Cat**. Consider the functor $\mathscr{E} : \mathbf{2\text{-}Cat} \to \mathbf{Cat}$ which associates to each small 2-category $\mathfrak{C}$ the equalizer of (s,t) (see (2.1)),

$$\mathscr{E}(\mathfrak{C}) \longrightarrow \mathscr{C}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathscr{C}_0,$$

that is, $\mathscr{E}(\mathfrak{C})$ has the same objects as $\mathfrak{C}$ and its arrows $\alpha : A \to B$ are just deformations α in $\mathfrak{C}$ from any arrow $f : A \to B$ in $\mathfrak{C}$ to itself. The composition in $\mathscr{E}(\mathfrak{C})$ is given by the horizontal composition of deformations and the obvious functor $\mathscr{E}(\mathfrak{C}) \to \mathscr{C}_0$ is the identity on objects and sends any arrow α, as above, to the corresponding f (its source and target).

Now, if $\mathfrak{C}$ is enriched in the category of groupoids, we can specify an action

$$\mu : \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{E}(\mathfrak{C}) \to \mathscr{E}(\mathfrak{C})$$

given by conjugation (with respect to the vertical composition), i.e. a deformation β acts on an endodeformation α only in the case that the vertical composition $\beta\alpha$ exists and then the result of the action is ${}^{\beta}\alpha = \beta\alpha\beta^{-1}$. In this way we have that $\mathscr{E}(\mathfrak{C})$ is canonically a $\mathfrak{C}$-object in **Cat**, i.e. an object in $\mathfrak{C}$-**Cat**. Moreover, the projection $\mathscr{E}(\mathfrak{C}) \to \mathscr{C}_0$ has a canonical section $\mathscr{C}_0 \to \mathscr{E}(\mathfrak{C})$, which sends any arrow f in $\mathscr{C}_0$ to its own identity deformation and it is clear that the vertical composition of deformations makes of $\mathscr{E}(\mathfrak{C})$ a group object in $\mathfrak{C}$-**Cat**. This group plays an important role in the definition of 2-dimensional torsors.

3.1. *Definition and properties of 2-torsors*

Let Π be any small category. If we consider it as a discrete 2-category then Π-$\mathbf{Cat} = \mathbf{Cat}/\Pi$. Thus, if $\mathfrak{C}$ is a 2-category which is an internally connected groupoid in $\mathbf{Cat}_0/\Pi$, then the canonical projection $\mathbf{P} : \mathfrak{C} \to \Pi$ (as a morphism of 2-categories) induces, as in (3.1), a finite-product preserving functor $\mathbf{Cat}/\Pi \to \mathfrak{C}$-**Cat** (pulling back along **P**), which in turn (since it preserves finite products) induces a functor

$$\mathbf{P}^* : \mathbf{Ab}(\mathbf{Cat}/\Pi) \to \mathbf{Ab}(\mathfrak{C}\text{-}\mathbf{Cat}),$$

between the corresponding categories of internal abelian group objects. Thus, we have:

Definition 3.1. Let Π be any small category and $A : \Pi \to \mathbf{Ab}$ a left Π-module. A two-dimensional torsor above Π with coefficients in A (or just a Π-A 2-torsor or an A-2-torsor above Π) is a pair $(\mathfrak{C}, \lambda)$ consisting of an internally connected groupoid $\mathfrak{C}$ in $\mathbf{Cat}_0/\Pi$ (called the *fiber* of the torsor), together with a group isomorphism

$$\lambda : \mathscr{E}(\mathfrak{C}) \xrightarrow{\cong} \mathbf{P}^*\left(\int_{\Pi} A\right).$$

Note that for any 2-torsor $(\mathfrak{C}, \lambda)$, since the group object $\int_{\Pi} A \to \Pi$ is abelian and λ is an isomorphism, $\mathscr{E}(\mathfrak{C})$ is abelian. This means that for any arrow $f \in \mathfrak{C}$ the group of deformations of f to itself is an abelian group, and this implies that the action of $\mathfrak{C}$ on $\mathscr{E}(\mathfrak{C})$, given by conjugation, is trivial (meaning that the action of any two deformations with the same source and target is the same). So, the isomorphism λ can be seen just as an isomorphism of group objects in $\mathbf{Cat}/\mathscr{C}_0$ (forgetting the action). Hence the following is an equivalent definition of 2-torsor, which we will prefer.

Definition 3.2. A two-dimensional torsor above Π with coefficients in A is a pair $(\mathfrak{C}, \lambda)$ consisting of an internally connected groupoid $\mathfrak{C}$ in $\mathbf{Cat}_0/\Pi$ and a functor $\lambda : \mathscr{E}(\mathfrak{C}) \to \int_{\Pi} A$, compatible with the corresponding group structures and such that it makes the following diagram a pullback in **Cat**

$$\begin{tikzcd} \mathscr{E}(\mathfrak{C}) \arrow[r, "\lambda"] \arrow[d] & \int_\Pi A \arrow[d] \\ \mathscr{C}_0 \arrow[r] & \Pi \end{tikzcd}.$$

A morphism of 2-torsors, $\mathfrak{f}:(\mathfrak{C},\lambda)\to(\mathfrak{C}',\lambda')$, is a 2-functor $\mathfrak{f}:\mathfrak{C}\to\mathfrak{C}'$, making commutative in $\mathbf{Cat}/\Pi$ the following diagram

$$\begin{tikzcd} \mathscr{E}(\mathfrak{C}) \arrow[dr, "\lambda"] \arrow[dd, "\mathscr{E}(f)"'] & \\ & \int_\Pi A \\ \mathscr{E}(\mathfrak{C}') \arrow[ur, "\lambda'"'] & \end{tikzcd} \tag{3.3}$$

where $\mathscr{E}(\mathfrak{f})$ is the obvious functor induced by $\mathfrak{f}$, which is clearly compatible with the group structures.

The category of Π-A 2-torsors and morphisms between them will be denoted by $\mathrm{Tor}^2(\Pi,A)$ and the corresponding set of connected components by $\mathrm{Tor}^2[\Pi,A]$. Then, as we indicated in Section 2.4 we have natural isomorphisms

$$H^3(\Pi,A)\cong \mathrm{Tor}^2[\Pi,A].$$

Note now that any 2-category which is the fiber of a 2-torsor above Π has the same set of objects as Π and that any morphism $\mathfrak{f}:(\mathfrak{C},\lambda)\to(\mathfrak{C}',\lambda')$ of 2-torsors above Π is the identity on objects. Moreover, for any two objects $X,Y\in\Pi$ the induced functor

$$\mathfrak{f}_{X,Y}:\mathfrak{C}(X,Y)\to\mathfrak{C}'(X,Y)$$

is an equivalence of groupoids. In fact, the groupoids $\mathfrak{C}(X,Y)$ and $\mathfrak{C}'(X,Y)$ have the same set of connected components, namely $\Pi(X,Y)$, and, on the other hand, the commutativity of the diagram (3.3) and the conditions of torsor imply that the square

$$\begin{tikzcd} \mathscr{E}(\mathfrak{C}) \arrow[r, "\mathscr{E}(f)"] \arrow[d] & \mathscr{E}(\mathfrak{C}') \arrow[d] \\ \mathscr{C}_0 \arrow[r, "\mathfrak{f}_0"'] & \mathscr{C}'_0 \end{tikzcd}$$

is a pullback. So, the functor $\mathfrak{f}$ induces, for any arrow $g:X\to Y$ in $\mathfrak{C}$, an isomorphism between the group of deformations in $\mathfrak{C}$ from g to itself and the group of deformations in $\mathfrak{C}'$ from $\mathfrak{f}(g)$ to itself.

We can then apply Lemma 2.2 which immediately implies the following:

Theorem 3.3. *Any morphism $\mathfrak{f}:(\mathfrak{C},\lambda)\to(\mathfrak{C}',\lambda')$ of 2-torsors induces a weak equivalence $\mathbf{B}(\mathfrak{f}):\mathbf{B}(\mathfrak{C})\to\mathbf{B}(\mathfrak{C}')$ between the corresponding geometric realizations. In particular the fiber groupoids of any two 2-torsors in the same connected component have the same homotopy type.*

4. Two-dimensional torsors above a groupoid

In this section we are going to study 2-torsors above a groupoid instead of merely above an arbitrary category. In particular we will prove that any 2-groupoid is the fiber of a 2-torsor above its groupoid of connected components.

Let us suppose that $\mathfrak{C}$ is a 2-groupoid and let $\mathscr{C}_0$ be its underlying groupoid. As we have seen in Section 3, in this case the group of deformations of any arrow is always an abelian group and so $\mathscr{E}(\mathfrak{C})$ is an abelian group object in $\mathbf{Cat}/\mathscr{C}_0$. We shall now prove that this abelian group is canonically the Grothendieck semidirect product of a $\mathscr{C}_0$-module.

To this end, we define a functor

$$E_{\mathfrak{C}} : \mathscr{C}_0 \to \mathbf{Ab} \tag{4.1}$$

which sends every object $C \in \mathscr{C}_0$ to the abelian group $E_{\mathfrak{C}}(C)$ of endodeformations of the identity arrow of C and sends every arrow $f : C \to C'$ to the group homomorphism

$$E_{\mathfrak{C}}(f) : E_{\mathfrak{C}}(C) \to E_{\mathfrak{C}}(C')$$

given by conjugation (with respect to the horizontal composition) with the identity endodeformation of f, that is, $E_{\mathfrak{C}}(f)(\beta) = f\beta f^{-1}$.

Note that for every f the group homomorphism $E_{\mathfrak{C}}(f)$ is an isomorphism. Moreover, for any deformation α in $\mathfrak{C}$ the equality between the two possible ways of composing the deformations in the following diagram

$$\begin{tikzcd}[column sep=huge]
\mathscr{C}' \arrow[r, bend left=70, "f^{-1}"] \arrow[r, "f^{-1}" description] \arrow[r, bend right=70, "g^{-1}"'] \arrow[r, phantom, shift left=4ex, "\| Id_f"] \arrow[r, phantom, shift right=4ex, "\Downarrow\alpha^{-1}"] & \mathscr{C} \arrow[r, bend left=70, "1_C"] \arrow[r, "1_C" description] \arrow[r, bend right=70, "1_C"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\beta"] \arrow[r, phantom, shift right=4ex, "\| Id_{1_C}"] & \mathscr{C} \arrow[r, bend left=70, "f"] \arrow[r, "f" description] \arrow[r, bend right=70, "g"'] \arrow[r, phantom, shift left=4ex, "\| Id_f"] \arrow[r, phantom, shift right=4ex, "\Downarrow\alpha"] & \mathscr{C}',
\end{tikzcd}$$

that is, the equality $f\beta f^{-1} = \alpha\beta\alpha^{-1}$, gives that $E_{\mathfrak{C}}(f)$ can be obtained by conjugation (with respect to the horizontal composition) with any deformation α with source f. Analogously, $E_{\mathfrak{C}}(f)$ can be obtained by conjugation with any deformation α' with *target* f. From this we deduce that if there is a deformation between two arrows f and g in $\mathfrak{C}$ then $E_{\mathfrak{C}}(f) = E_{\mathfrak{C}}(g)$. On the other hand, for any arrow $f : C \to C'$, any deformation $\alpha : f \Rightarrow f$ is completely determined by f and a deformation of the identity arrow of C, so we conclude:

Lemma 4.1. *Given a 2-groupoid $\mathfrak{C}$, the functor over $\mathscr{C}_0$*

$$\begin{tikzcd}
\int_{\mathscr{C}_0} E_{\mathfrak{C}} \arrow[rr] \arrow[dr] & & \mathscr{E}(\mathfrak{C}) \arrow[dl] \\
& \mathscr{C}_0 &
\end{tikzcd},$$

which takes any arrow $(f,\alpha): C \to C'$ *of* $\int_{\mathscr{C}_0} E_{\mathfrak{C}}$ *to the horizontal composition*

$$\begin{tikzcd} C \arrow[r, "f"] & C' \arrow[r, bend left, "1'_C"] \arrow[r, bend right, "1'_C"'] \arrow[r, phantom, "\Downarrow\alpha"] & C', \end{tikzcd}$$

is an isomorphism of abelian group objects in $\mathfrak{C}$**-Cat**.

As an immediate consequence of the above Lemma 4.1 we have that to give a 2-torsor above a groupoid Π with Π-module of coefficients $A: \Pi \to \mathbf{Ab}$ and fiber a 2-groupoid $\mathfrak{C}$ is equivalent to give a natural isomorphism λ,

$$\begin{tikzcd} & \Pi \arrow[dr, "A"] \arrow[d, phantom, "\Uparrow\lambda"] & \\ \mathscr{C}_0 \arrow[ur] \arrow[rr, "E_{\mathfrak{C}}"'] & & \mathbf{Ab} \end{tikzcd} \tag{4.2}$$

of $\mathscr{C}_0$-modules.

Let us suppose that Π is the groupoid of connected components of a 2-groupoid $\mathfrak{C}$ and let us choose a section σ (at the underlying graph level) of the canonical projection $\mathscr{C}_0 \to \Pi$. Since for any two arrows f and g of $\mathfrak{C}$ in the same connected component we have $E_{\mathfrak{C}}(f) = E_{\mathfrak{C}}(g)$, we can define a Π-module $A: \Pi \to \mathbf{Ab}$ by $A(C) = E_{\mathfrak{C}}(C)$ and $A(\bar{f}) = E_{\mathfrak{C}}(\sigma(\bar{f}))$, for any arrow $\bar{f} \in \Pi$. This Π-module does not depend on the section σ we have chosen, and, on the other hand, there is a natural equivalence λ as in diagram (4.2) above, which gives rise to a 2-torsor above Π with coefficients in A. This torsor (which is not canonical) is usually called the *obstruction of* $\mathfrak{C}$.

Summing up, we have:

Theorem 4.2. *If* $\mathfrak{C}$ *is a 2-groupoid with* Π *as groupoid of connected components, and* $A: \Pi \to \mathbf{Ab}$ *is the* Π*-module defined above, then* $\mathfrak{C}$ *is the fiber of a 2-torsor above* Π *with coefficients on* A*, called its obstruction.*

Let us remark that in general, if a 2-groupoid $\mathfrak{C}$ is the fiber of a 2-torsor above a groupoid Π with coefficients in a Π-module A, a direct use of the results of Dwyer–Kan in [11], when they are applied to the simplicial groupoid $\mathfrak{Ner}(\mathfrak{C})$, gives that the classifying space $\mathbf{B}(\mathfrak{C})$ is a 2-type with fundamental groupoid Π and A as second homotopy group functor. The next theorem extends this result to any 2-category which is the fiber of a 2-torsor above a groupoid.

Theorem 4.3. *If a 2-category $\mathfrak{C}$ is the fiber of a 2-torsor above a groupoid Π with coefficients on a Π-module A, then $\mathbf{B}(\mathfrak{C})$ is a 2-type with fundamental groupoid Π and A as second homotopy group functor.*

Proof. If Π is a groupoid and A is a Π-module, in Section 2.4 we have given isomorphisms

$$H^n_{\mathbb{G}}\left(\Pi, \int_\Pi A\right) \cong H^{n+1}(\Pi, A) \cong H^n_{\mathbb{G}'}\left(\Pi, \int_\Pi A\right),$$

where $\mathbb{G}$ and $\mathbb{G}'$ are the cotriples associated to the adjunctions

$$F : \mathbf{Gph}_0/U(\Pi) \rightleftarrows \mathbf{Cat}_0/\Pi : U \quad \text{and} \quad F' : \mathbf{Gph}_0/U(\Pi) \rightleftarrows \mathbf{Gpd}_0/\Pi : U$$

respectively. In particular we have isomorphisms

$$H^2_{\mathbb{G}}\left(\Pi, \int_\Pi A\right) \cong H^3(\Pi, A) \cong H^2_{\mathbb{G}'}\left(\Pi, \int_\Pi A\right), \tag{4.3}$$

which show that any cohomology class can be interpreted either in terms of 2-torsors defined internally in the category $\mathbf{Cat}_0/\Pi$ (which are just those we are using until now) or else in terms of 2-torsors defined internally in the category $\mathbf{Gpd}_0/\Pi$. The only difference between 2-torsors defined internally in $\mathbf{Cat}_0/\Pi$ and 2-torsors defined in $\mathbf{Gpd}_0/\Pi$ is that the fiber in the second case is necessarily a 2-groupoid.

So the isomorphisms (4.3) imply that in the connected component class of any 2-torsor above a groupoid Π, there is one whose fiber is a 2-groupoid. Thus, by Theorem 3.3 and the above remark the proof is complete. □

5. Whitehead 2-groupoid

The following construction can be found in [20]: Let X be a topological space, let $Y \subseteq X$ be any subspace and let $S \subseteq Y$ be a set of ("base")-points. The Whitehead 2-groupoid $W(X,Y,S)$ of (X,Y,S) is defined as follows:

- the underlying groupoid of $W(X,Y,S)$ is the fundamental groupoid $\pi_1(Y,S)$,
- the deformations of $W(X,Y,S)$ are homotopy classes of maps from the square $I \times I$ into X, which are constant along the vertical edges with values in S, and map the horizontal edges into Y,
- the domain and codomain of one such deformation are given by restriction to $I \times 0$ and $I \times 1$ respectively, and the composition of deformations is defined in the usual way.

As in [20], we next apply Whitehead's construction to the following particular case: let T be a simplicial set, with $T^{(k)}$ its k-skeleton, and let $|\cdot|$ denote geometric realization. We define

$$W(T) = W(|T|, |T^{(1)}|, |T^{(0)}|).$$

This 2-groupoid has the following properties (see [20]):

(i) the underlying groupoid $W(T)_0$ of $W(T)$ is the fundamental groupoid of $T^{(1)}$, i.e.

$$W(T)_0 = \pi_1(T^{(1)}),$$

and therefore $W(T)_0$ is just the free groupoid on the graph

$$\begin{tikzcd} T_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & T_0 \arrow[l, bend right, "s_0"'] \end{tikzcd},$$

given by the 1-truncation of T,

(ii) the groupoid of connected components of $W(T)$ is the fundamental groupoid $\pi_1(T)$ of T, and the projection

$$W(T)_0 \to \pi_1(W(T))$$

is the functor induced by the inclusion $T^{(1)} \to T$,

(iii) the functor $E_{W(T)} : W(T)_0 \to \mathbf{Ab}$ (as in diagram (4.1)) takes any point x_0 to the homotopy group $\pi_2(T, T^{(1)}, x_0)$, so that we identify

$$E_{W(T)} = \pi_2(T, T^{(1)}) : \pi_1(T^{(1)}) \to \mathbf{Ab}$$

and also we identify

$$\mathscr{E}(W(T)) = \int_{\pi_1(T^{(1)})} \pi_2(T, T^{(1)}) \to W(T)_0.$$

Let us note that there is a natural equivalence λ_T, also natural in T,

$$\begin{tikzcd} & \pi_1(T) \arrow[dr, "\pi_2(T)"] \arrow[d, phantom, "\Uparrow \lambda_T"] & \\ \pi_1(T^{(1)}) \arrow[ur] \arrow[rr, "E_{W(T)} = \pi_2(T, T^{(1)})"'] & {} & \mathbf{Ab}. \end{tikzcd}$$

So the pair $(W(T), \lambda_T)$ is a $\pi_2(T)$ 2-torsor above $\pi_1(T)$, which corresponds to the obstruction of the 2-groupoid $W(T)$ as in Theorem 4.2.

The naturality of the above constructions assures that for any simplicial morphism $f: T \to K$ we have a commutative cube

$$\begin{tikzcd}
& \mathscr{E}(W(K)) \arrow[rr, "\lambda_K"] \arrow[dd] & & \int_{\pi_1(K)}\pi_2(K) \arrow[dd] \\
\mathscr{E}(W(T)) \arrow[ur] \arrow[rr, "\lambda_T"] \arrow[dd] & & \int_{\pi_1(T)}\pi_2(T) \arrow[ur] \arrow[dd] & \\
& W(K)_0 \arrow[rr] & & \pi_1(K) \\
W(T)_0 \arrow[ur] \arrow[rr] & & \pi_1(T) \arrow[ur] &
\end{tikzcd}$$

in which the front and back faces are pullbacks.

6. An extension of Grothendieck construction

We are going to use the following extension of the Grothendieck semidirect product construction. Note that this extension is different from the one given by Moerdijk and Svensson in [20].

Let $\mathscr{C}$ be a category and let $\mathfrak{F}: \mathscr{C} \to \mathbf{2\text{-}Cat}$ be a functor from $\mathscr{C}$ to the category of small 2-categories. By composing $\mathfrak{F}$ with the "domain" forgetful functor $(-)_0: \mathbf{2\text{-}Cat} \to \mathbf{Cat}$ (see diagram (6.1) below) we obtain a functor $F_0: \mathscr{C} \to \mathbf{Cat}$ which sends any object $C \in \mathscr{C}$ to the underlying category of the 2-category $\mathfrak{F}(C)$. Our extension of the Grothendieck construction will give us a 2-category $\int\!\!\int_{\mathscr{C}} \mathfrak{F}$ with underlying category

$$\left(\int\!\!\int_{\mathscr{C}} \mathfrak{F}\right)_0 = \int_{\mathscr{C}} F_0,$$

and a canonical projection 2-functor

$$\int\!\!\int_{\mathscr{C}} \mathfrak{F} \to \mathscr{C}$$

(where $\mathscr{C}$ is regarded as a 2-category with no deformations except the identities), in such a way that the following square will be commutative:

$$\begin{tikzcd}
\mathbf{2\text{-}Cat}^{\mathscr{C}} \arrow[r, dotted, "\int\!\int_{\mathscr{C}}"] \arrow[d, "(-)_0^*"'] & \mathbf{2\text{-}Cat}/\mathscr{C} \arrow[d, "(-)_0^*"] \\
\mathbf{Cat}^{\mathscr{C}} \arrow[r, "\int_{\mathscr{C}}"'] & \mathbf{Cat}/\mathscr{C}.
\end{tikzcd}$$

A deformation

$$\begin{tikzcd}(C,x) \arrow[r, bend left, "{(f,\lambda)}"] \arrow[r, bend right, "{(f',\lambda')}"'] \arrow[r, phantom, "\Downarrow"] & (C',x')\end{tikzcd}$$

in $\iint_{\mathscr{C}} \mathfrak{F}$ exists only if $f' = f$ and in that case it is defined as a deformation

$$\begin{tikzcd}\mathfrak{F}(f)(x) \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\lambda'"'] \arrow[r, phantom, "\Downarrow\alpha"] & x'\end{tikzcd}$$

in $\mathfrak{F}(C')$. The vertical and horizontal compositions of deformations are defined in the obvious way.

This construction determines a functor

$$\iint_{\mathscr{C}} : \mathbf{2\text{-}Cat}^{\mathscr{C}} \to \mathbf{2\text{-}Cat}/\mathscr{C}$$

which extends the usual Grothendieck construction as indicated above. Moreover this functor commutes with the calculus of connected components in the following sense: let $\Pi_0 : \mathbf{2\text{-}Cat} \to \mathbf{Cat}$ be the functor (3.2) which sends any 2-category $\mathfrak{C}$ to its category of connected components. Then we have the following lemma, whose proof is of a rather technical character but not difficult.

Lemma 6.1. *For any small category $\mathscr{C}$ the following diagram commutes*

$$\begin{tikzcd}\mathbf{2\text{-}Cat}^{\mathscr{C}} \arrow[r, "\iint_{\mathscr{C}}"] \arrow[d, "\Pi_0^*"'] & \mathbf{2\text{-}Cat}/\mathscr{C} \arrow[d, "\Pi_0^*"] \\ \mathbf{Cat}^{\mathscr{C}} \arrow[r, "\int_{\mathscr{C}}"'] & \mathbf{Cat}/\mathscr{C}\end{tikzcd},$$

where Π_0^* denotes in both cases the functors induced by $\Pi_0 : \mathbf{2\text{-}Cat} \to \mathbf{Cat}$.

On the other hand, if for every object $C \in \mathscr{C}$ the 2-category $\mathfrak{F}(C)$ is a locally connected groupoid in the sense of Section 3, then the 2-category $\iint_{\mathscr{C}} \mathfrak{F}$ is again a locally connected groupoid. Moreover, in this case it is also easy to prove that this extension of the Grothendieck semidirect product construction commutes with the functor $\mathscr{E} : \mathbf{2\text{-}Cat} \to \mathbf{Cat}$, defined in Section 3. In fact we have:

Lemma 6.2. *For any functor $\mathfrak{F} : \mathscr{C} \to \mathbf{2\text{-}Cat}$ such that for every $C \in \mathscr{C}$ the 2-category $\mathfrak{F}(C)$ is a locally connected groupoid, there is a natural isomorphism $\theta : \mathscr{E}(\iint_{\mathscr{C}} \mathfrak{F}) \cong$*

$\int_{\mathscr{C}} \mathscr{E}\mathfrak{F}$, *which makes commutative the diagram*

$$\begin{tikzcd} \mathscr{E}(\int\int_{\mathscr{C}}\mathfrak{J}) \arrow[rr, "\theta", "\cong"'] \arrow[dr] & & \int_{\mathscr{C}}\mathscr{E}\mathfrak{J} \arrow[dl] \\ & \int_{\mathscr{C}} F_0 & \end{tikzcd} .$$

On the other hand, the adjunction

$$\Sigma : \mathbf{Cat}/\mathscr{C} \rightleftarrows \mathbf{Cat}^{\mathscr{C}} : \int_{\mathscr{C}},$$

given in Section 2.1, can be extended to the context of 2-categories, so that the functor $\int\int_{\mathscr{C}}$ also has a left adjoint

$$\Sigma : \mathbf{2\text{-}Cat}/\mathscr{C} \to \mathbf{2\text{-}Cat}^{\mathscr{C}}.$$

Let us note that if we consider the functors

$$\mathbf{2\text{-}Cat} \underset{(-)_1}{\overset{(-)_0}{\rightrightarrows}} \mathbf{Cat} \tag{6.1}$$

which send a 2-category $\mathfrak{C}$ to its underlying category $\mathscr{C}_0$ and to the category $\mathscr{C}_1$ of deformations of $\mathfrak{C}$ respectively, then for any 2-category above $\mathscr{C}$, $\mathfrak{C} \xrightarrow{P} \mathscr{C}$, the 2-category $\Sigma(P)$ satisfies that

$$(\Sigma(P))_0 = \Sigma(\mathscr{C}_0 \xrightarrow{P} C)$$

and

$$(\Sigma(P))_1 = \Sigma(\mathscr{C}_1 \xrightarrow{P\cdot s} C) = \Sigma(\mathscr{C}_1 \xrightarrow{P\cdot t} C).$$

Therefore, since the functor Σ preserves colimits (it has a right adjoint) we have:

Lemma 6.3. *For any small category $\mathscr{C}$ the following diagram commutes*

$$\begin{tikzcd} \mathbf{2\text{-}Cat}/\mathscr{C} \arrow[r, "\Sigma"] \arrow[d, "\Pi_0^*"'] & \mathbf{2\text{-}Cat}^{\mathscr{C}} \arrow[d, "\Pi_0^*"] \\ \mathbf{Cat}/\mathscr{C} \arrow[r, "\Sigma"'] & \mathbf{Cat}^{\mathscr{C}} \end{tikzcd},$$

where Π_0^* denotes in both cases the functors induced by $\Pi_0 : \mathbf{2\text{-}Cat} \to \mathbf{Cat}$.

The way the functor Σ is defined assures that if $\mathfrak{C} \xrightarrow{P} \mathscr{C}$ is a 2-category above $\mathscr{C}$ which is enriched in groupoids (i.e. any deformation has an inverse with respect to the horizontal composition), then $\Sigma(P)$ is a functor with the property that for any object $C \in \mathscr{C}$ the 2-category $\Sigma(P)(C)$ is enriched in groupoids and in this case $\mathscr{E}(\Sigma(P)(X)) = \Sigma(\mathscr{E}(\mathfrak{C}))(X)$. In this sense we will say that the functors Σ and $\mathscr{E}$ commute.

7. The theorem

We come now to the main part of this paper. A group G will be regarded as a one-object category, so that the category of G-spaces is the functor category $\mathbf{Top}^G$ of functors from G to the category $\mathbf{Top}$ of topological spaces. The orbit category of the group G is the category $\mathcal{O}(G)$ whose objects are left G-sets of the form G/H with $H \subseteq G$ a subgroup and whose arrows are all the equivariant maps.

The equivariant homotopy type of a G-space X is determined by the homotopy types of the spaces X^H of points fixed by each subgroup H of G. More concretely, any G-space X determines a functor

$$\mathrm{Map}_G(-,X) = X^{(-)} : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Top} \tag{7.1}$$

which sends any object G/H of $\mathcal{O}(G)$ to the space

$$X^H = \mathrm{Map}_G(G/H,X)$$

of points fixed by H, so that we have a functor $\mathbf{Top}^G \to \mathbf{Top}^{\mathcal{O}(G)^{\mathrm{op}}}$. The homotopy invariants of the G-space X are the homotopy invariants of the above diagram (7.1) of spaces.

The equivariant fundamental groupoid of X is the functor

$$\pi_1^G(X) : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Gpd},$$

which takes any object $G/H \in \mathcal{O}(G)$ to the fundamental groupoid of the space X^H. If we apply the Grothendieck semidirect product construction (see Section 2.1) to this contravariant functor we obtain a category above $\mathcal{O}(G)^{\mathrm{op}}$, which we simply denote $\pi^G(X)$:

$$\pi^G(X) = \left(\int_{\mathcal{O}(G)^{\mathrm{op}}} \pi_1^G(X) \to \mathcal{O}(G)^{\mathrm{op}} \right). \tag{7.2}$$

Now, for any integer $n \geq 2$ and any object G/H of $\mathcal{O}(G)$, we can consider the homotopy groups $\pi_n(X^H)$ as functors, or $\pi_1(X^H)$-modules,

$$\pi_1(X^H) \xrightarrow{\pi_n(X^H)} \mathbf{Ab},$$

which send an object of $\pi_1(X^H)$ (a point $x \in X^H$) to the n-th homotopy group $\pi_n(X^H, x)$. Moreover for any map $f : G/H \to G/H'$ we have a continuous map $X^f : X^{H'} \to X^H$ and so a natural transformation

$$\begin{tikzcd} & \pi_1(X^H) \arrow[dr, "\pi_n(X^H)"] \arrow[d, phantom, "\Uparrow \pi_n(f)"] & \\ \pi_1(X^{H'}) \arrow[ur, "\pi_1(X^f)"] \arrow[rr, "\pi_n(X^{H'})"'] & {} & \mathbf{Ab}\,. \end{tikzcd}$$

By the universal property of $\pi^G(X)$ (of being a lax colimit), the above data is equivalent to a functor or $\pi^G(X)$-module

$$\pi^G(X) \xrightarrow{\pi_n^G(X)} \mathbf{Ab}$$

which is a local coefficient for the G-space X, called the n-th dimensional G-equivariant homotopy group of X.

After having developed all the required machinery in the preceding sections, we are finally ready to state and prove the following theorem, which is the equivariant version of the classical theorem proved by Mac Lane–Whitehead [19] for usual spaces.

Theorem 7.1. *The equivariant homotopy type of a G-space X, with trivial equivariant homotopy at dimensions greater or equal than* 3, *is completely determined by a diagram of groupoids*

$$\Pi : \mathcal{O}(G)^{op} \to \mathbf{Gpd},$$

a $\int_{\mathcal{O}(G)^{op}} \Pi$*-module A and an element*

$$k_3 \in H^3\left(\int_{\mathcal{O}(G)^{op}} \Pi, A\right).$$

In order to prove this theorem we shall first rephrase it in terms of the homotopy category of $\mathcal{O}(G)^{\mathrm{op}}$-diagrams of simplicial sets. Recall that the homotopy category of G-spaces is equivalent to the homotopy category of $\mathcal{O}(G)^{\mathrm{op}}$-diagrams of simplicial sets, see, e.g., [12] or [20]. This equivalence is induced by the singular complex functor $\mathbf{Top} \xrightarrow{S} \mathbf{Set}^{\Delta^{\mathrm{op}}}$ through the functor $\mathbf{Top}^G \xrightarrow{S^G} (\mathbf{Set}^{\Delta^{\mathrm{op}}})^{\mathcal{O}(G)^{\mathrm{op}}}$ which sends any G-space X to the diagram of simplicial sets

$$S^G(X) : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Set}^{\Delta^{\mathrm{op}}}$$

which takes an object $G/H \in \mathcal{O}(G)$ to $S(X^H)$, the singular complex of the space X^H. Here the homotopy invariants of a diagram of simplicial sets are defined pointwise. So, to prove the above Theorem 7.1 is equivalent to proving the following.

Theorem 7.2. *The equivariant homotopy type of a $\mathcal{O}(G)^{op}$-diagram of simplicial sets*

$$\mathbf{T} : \mathcal{O}(G)^{op} \to \mathbf{Set}^{\Delta^{op}},$$

with trivial equivariant homotopy at dimensions greater or equal than 3, *is completely determined by a diagram of groupoids*

$$\Pi : \mathcal{O}(G)^{op} \to \mathbf{Gpd},$$

a $\int_{\mathcal{O}(G)^{op}} \Pi$*-module* A *and an element*

$$k_3 \in H^3\left(\int_{\mathcal{O}(G)^{op}} \Pi, A\right).$$

Proof. Given the diagram of simplicial sets

$$\mathbf{T} : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Set}^{\Delta^{\mathrm{op}}},$$

let T^H be the image of an object $G/H \in \mathcal{O}(G)$ and let $T^f : T^{H'} \to T^H$ be the image of a G-map $f : G/H \to G/H'$. We then have:

(i) a functor $\Pi = \pi_1(\mathbf{T}) : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Gpd}$, taking an object G/H to the fundamental groupoid of T^H (where T^H is seen as a space via the usual functor $\mathbf{Set}^{\Delta^{\mathrm{op}}} \to \mathbf{Top}$), so $\Pi(G/H) = \pi_1(T^H)$,

(ii) for each object $G/H \in \mathcal{O}(G)$, a functor $\pi_2(T^H) : \pi_1(T^H) \to \mathbf{Ab}$, and

(iii) for each arrow $f : G/H \to G/H'$ in $\mathcal{O}(G)$, a natural transformation

$$\begin{tikzcd}
 & \pi_1(T^H) \arrow[dr, "\pi_2(T^H)"] \arrow[d, phantom, "\Uparrow \pi_2(T^f)"] & \\
\pi_1(T^{H'}) \arrow[ur, "\pi_1(T^f)"] \arrow[rr, "\pi_2(T^{H'})"'] & {} & \mathbf{Ab}.
\end{tikzcd}$$

Now, as the data in (ii) and (iii) satisfy the analogous of conditions (a) and (b) in Section 2.1, we obtain a $\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi$-module

$$A : \int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi \to \mathbf{Ab}$$

which sends an object $(G/H, x) \in \int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi$ to the abelian group $\pi_2(T^H, x)$. On the other hand we can also build a functor

$$\int_{\pi_1(T^{(-)})} \pi_2(T^{(-)}) : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Gpd}$$

which sends any object $G/H \in \mathcal{O}(G)$ to the groupoid $\int_{\pi_1(T^H)} \pi_2(T^H)$ and any arrow $f : G/H \to G/H'$ to the functor

$$\int_{\pi_1(T^{H'})} \pi_2(T^{H'}) \to \int_{\pi_1(T^H)} \pi_2(T^H)$$

induced by the natural transformation $\pi_2(T^f)$ as in (iii). Then we can apply the Grothendieck construction to get a category

$$\int_{\mathcal{O}(G)^{\mathrm{op}}}\left(\int_{\pi_1(T^{(-)})}\pi_2(T^{(-)})\right).$$

Moreover, the canonical functors $\int_{\pi_1(T^H)}\pi_2(T^H) \to \pi_1(T^H)$ induce a functor

$$\int_{\mathcal{O}(G)^{\mathrm{op}}}\int_{\pi_1(T^{(-)})}\pi_2(T^{(-)}) \to \int_{\mathcal{O}(G)^{\mathrm{op}}}\Pi,$$

with the property that for any object $G/H \in \mathcal{O}(G)$ the square (where the horizontal arrows are the canonical inclusions)

$$\begin{tikzcd}
\int_{\pi_1(T^H)}\pi_2(T^H) \arrow[r] \arrow[d] & \int_{\mathcal{O}(G)^{\mathrm{op}}}\int_{\pi_1(T^{(-)})}\pi_2(T^{(-)}) \arrow[d] \\
\pi_1(T^H) \arrow[r] & \int_{\mathcal{O}(G)^{\mathrm{op}}}\Pi
\end{tikzcd}$$

is a pullback.

It is also straightforward to see that there is a canonical isomorphism of objects above $\int_{\mathcal{O}(G)^{\mathrm{op}}}\Pi$

$$\begin{tikzcd}
\int_{\int_{\mathcal{O}(G)^{\mathrm{op}}\Pi}} A \arrow[rr, "\cong"] \arrow[dr] & & \int_{\mathcal{O}(G)^{\mathrm{op}}}\int_{\pi_1(T^{(-)})}\pi_2(T^{(-)}) \arrow[dl] \\
& \int_{\mathcal{O}(G)^{\mathrm{op}}}\Pi &
\end{tikzcd}$$

On the other hand, for every H the simplicial set T^H has associated a 2-torsor above $\pi_1(T^H)$ with coefficients in $\pi_2(T^H)$

$$(W(T^H), \lambda_{T^H}),$$

as in Section 5, where λ_{T^H} is a functor which makes the square

$$\begin{tikzcd}
\mathscr{E}(W(T^H)) \arrow[r, "\lambda_{T^H}"] \arrow[d] & \int_{\pi_1(T^H)}\pi_2(T^H) \arrow[d] \\
W(T^H)_0 \arrow[r] & \pi_1(T^H)
\end{tikzcd} \tag{7.3}$$

to be a pullback (see Section 5). This construction is natural in T^H. Moreover the Whitehead 2-groupoid construction is functorial so, if we only take care of the fibers

$W(T^H)$ of the above torsors, we have a functor

$$W\mathbf{T} : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{2\text{-}Gpd}.$$

Let

$$\mathbf{W}(\mathbf{T}) = \iint_{\mathcal{O}(G)^{\mathrm{op}}} W\mathbf{T}$$

be the corresponding 2-category. Now, since any $W(T^H)$ is a 2-groupoid with $\pi_1(T^H)$ as groupoid of connected components, we can see $W(T^H)$ as a locally connected groupoid in $\mathbf{Cat}_0/\pi_1(T^H)$, for any object G/H. Then we have (by Lemma 6.1) that $\mathbf{W}(\mathbf{T})$ is a locally connected groupoid in $\mathbf{Cat}/(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi)$, in particular the category of connected components of $\mathbf{W}(\mathbf{T})$ is $\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi$. If we now apply the Grothendieck construction to the pullback squares (7.3), we get squares

$$\begin{array}{ccc}
\int_{\mathcal{O}(G)^{\mathrm{op}}} \mathscr{E}(W(T^{(-)})) & \xrightarrow{\int_{\mathcal{O}(G)^{\mathrm{op}}} \lambda_{T^{(-)}}} & \int_{\mathcal{O}(G)^{\mathrm{op}}} \int_{\pi_1(T^{(-)})} \pi_2(T^{(-)}) \\
\big\downarrow & & \big\downarrow \\
\int_{\mathcal{O}(G)^{\mathrm{op}}} W(T^{(-)})_0 & \longrightarrow & \int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi
\end{array}$$

which are also pullbacks, by the results stated in 2.1. But we can identify:
- $\int_{\mathcal{O}(G)^{\mathrm{op}}} \mathscr{E}(W(T^{(-)}))$ with $\mathscr{E}(\mathbf{W}(\mathbf{T}))$ (by Lemma 6.2),
- $\int_{\mathcal{O}(G)^{\mathrm{op}}} \int_{\pi_1(T^{(-)})} \pi_2(T^{(-)})$ with $\int_{\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi} A$ and
- $\int_{\mathcal{O}(G)^{\mathrm{op}}} W(T^{(-)})_0$ with $\mathbf{W}(\mathbf{T})_0$.

Therefore we have a diagram

$$\begin{array}{ccc}
\mathscr{E}(\mathbf{W}(\mathbf{T})) & \xrightarrow{\lambda_{\mathbf{T}}} & \int_{\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi} A \\
\big\downarrow & & \big\downarrow \\
\mathbf{W}(\mathbf{T})_0 & \longrightarrow & \int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi
\end{array}$$

that is a pullback, so the pair

$$(\mathbf{W}(\mathbf{T}), \lambda_{\mathbf{T}})$$

is a 2-torsor above $\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi$ with coefficients in A. This torsor determines an element

$$k_3^{\mathbf{T}} \in H^3\left(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi, A\right).$$

Besides, from the $\mathcal{O}(G)^{\mathrm{op}}$-diagram of simplicial sets $\mathbf{T}$, we have obtained:
- a $\mathcal{O}(G)^{\mathrm{op}}$-diagram of groupoids $\Pi : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Gpd}$,
- a $\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi$-module A, and
- an element $k_3^{\mathbf{T}} \in H^3(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi, A)$.

Note that if we have a natural transformation $\eta : \mathbf{T} \to \mathbf{T}'$ between $\mathcal{O}(G)^{\mathrm{op}}$-diagrams of simplicial sets such that each of its components is a weak equivalence, then by applying the above constructions we get the same diagram of groupoids Π associated to $\mathbf{T}$ and $\mathbf{T}'$, the same $\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi$-module A and (by the naturality of the Whitehead 2-groupoid construction) 2-torsors $(\mathbf{W}(\mathbf{T}), \lambda_{\mathbf{T}})$ and $(\mathbf{W}(\mathbf{T}'), \lambda_{\mathbf{T}'})$ which are in the same connected component, so they determine the same element $k_3^{\mathbf{T}} = k_3^{\mathbf{T}'} \in H^3(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi, A)$. This completes the first part of the proof. Conversely, let us suppose that we start with a $\mathcal{O}(G)^{\mathrm{op}}$-diagram of groupoids

$$\Pi : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Gpd},$$

a $\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi$-module A and an element

$$k_3 \in H^3\left(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi, A\right).$$

For any object $G/H \in \mathcal{O}(G)$, let us write A^H the $\Pi(G/H)$-module obtained from the $\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi$-module A by restriction through the canonical inclusion

$$\Pi(G/H) \hookrightarrow \int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi.$$

As before we can define a functor

$$\int_{\Pi(-)} A^{(-)} : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Gpd}$$

which takes any object $G/H \in \mathcal{O}(G)$ to $\int_{\Pi(G/H)} A^H$, in such a way that there is a canonical isomorphism of objects above $\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi$,

$$\begin{tikzcd}
\int\int_{\mathcal{O}(G)^{\mathrm{op}}\Pi} A \arrow[rr, "\cong"] \arrow[dr] & & \int_{\mathcal{O}(G)^{\mathrm{op}}} \int_{\Pi(-)} A^{(-)} \arrow[dl] \\
& \int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi &
\end{tikzcd}.$$

Let us choose a 2-torsor $(\mathfrak{C}, \lambda)$ which represents the cohomology class k_3. By Theorem 3.3, all the fiber groupoids of 2-torsors which represent the same cohomology class have the same homotopy type so it does not matter which 2-torsor we have chosen. As usual, let us consider the diagram

$$\begin{tikzcd} & \mathscr{E}(\mathfrak{C}) \arrow[r, "\lambda"] \arrow[d] & \iint_{\mathcal{O}(G)^{\mathrm{op}}\Pi} A \arrow[d] \\ \mathscr{C}_1 \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & \mathscr{C}_0 \arrow[r] & \int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi \end{tikzcd}, \tag{7.4}$$

which can be thought of as a diagram above $\mathcal{O}(G)^{\mathrm{op}}$. Let

$\Sigma(\mathfrak{C}) : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{2\text{-}Cat},$

be the functor obtained by applying the functor

$\Sigma : \mathbf{2\text{-}Cat}/\mathcal{O}(G)^{\mathrm{op}} \to \mathbf{2\text{-}Cat}^{\mathcal{O}(G)^{\mathrm{op}}}$

to the canonical projection

$$\mathfrak{C} \to \int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi \to \mathcal{O}(G)^{\mathrm{op}}$$

as 2-category above $\mathcal{O}(G)^{\mathrm{op}}$. Then, by Lemma 6.3, for any object $G/H \in \mathcal{O}(G)$ the category of connected components of $\Sigma(\mathfrak{C})(G/H)$ is $\Sigma(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi)(G/H)$, hence (by the note after Lemma 6.3) $\Sigma(\mathfrak{C})(G/H)$ is a locally connected groupoid in the category $\mathbf{Cat}_0/\Sigma(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi)(G/H)$.

Moreover, since the square in diagram (7.4) is a pullback and Σ preserves pullbacks (see 2.1), we see that for any object $G/H \in \mathcal{O}(G)$ the square

$$\begin{tikzcd} \Sigma(\mathscr{E}(\mathfrak{C}))(G/H) \cong \mathscr{E}(\Sigma(\mathfrak{C}(G/H))) \arrow[r, "\lambda^H"] \arrow[d] & \Sigma(\iint_{\mathcal{O}(G)^{\mathrm{op}}\Pi} A)(G/H) \arrow[d] \\ \Sigma(\mathscr{C}_0)(G/H) \arrow[r] & \Sigma(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi)(G/H) \end{tikzcd}$$

(where λ^H is the functor induced by λ) is also a pullback. Consider now the diagram

$$\begin{tikzcd} & \mathscr{E}(\Sigma(\mathfrak{C}(G/H))) \arrow[r, "\lambda^H"] \arrow[d] & \Sigma(\iint_{\mathcal{O}(G)^{\mathrm{op}}\Pi} A)(G/H) \arrow[d] \\ \Sigma(\mathscr{C}_1)(G/H) \arrow[r, shift left] \arrow[r, shift right] & \Sigma(\mathscr{C}_0)(G/H) \arrow[r] & \Sigma(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi)(G/H) \end{tikzcd}$$

and let $\mathfrak{C}^H$ be the 2-category obtained by lifting the 2-category $\Sigma(\mathfrak{C})(G/H)$ by pullback via the canonical inclusion

$$\Pi(G/H) \hookrightarrow \Sigma\left(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi\right)(G/H).$$

Then $\mathfrak{C}^H$ is a locally connected groupoid in $\mathbf{Cat}_0/\Pi(G/H)$ and $\mathscr{E}(\mathfrak{C}^H)$ is the pullback

$$\begin{tikzcd}
\mathscr{E}(\mathfrak{C}^H) \arrow[r] \arrow[d] & \mathscr{E}(\Sigma(\mathfrak{C}(G/H))) \arrow[d] \\
(\mathfrak{C}^H)_0 \arrow[r] & (\Sigma(\mathfrak{C}(G/H)))_0
\end{tikzcd}$$

Moreover the functor λ^H induces a functor λ^{H^*}, as in the diagram

$$\begin{tikzcd}
& & \mathscr{E}(\mathfrak{C}^H) \arrow[dl] \arrow[dd] \arrow[rr, "\lambda^{H^*}"] & & \int_{\Pi(G/H)} A^H \arrow[dl] \arrow[dd] \\
\mathscr{C}_1^H \arrow[r, shift left] \arrow[r, shift right] \arrow[dd] & \mathscr{C}_0^H \arrow[rr] \arrow[dd] & & \Pi(G/H) \arrow[dd] & \\
& & \mathscr{E}(\Sigma(\mathfrak{C}(G/H))) \arrow[dl] \arrow[rr, "\lambda^H"] & & \Sigma\left(\int_{\mathcal{O}(G)^{\mathrm{op}}} \int_{\Pi(-)} A^{(-)}\right)(G/H), \arrow[dl] \\
\Sigma(\mathscr{C}_1)(G/H) \arrow[r, shift left] \arrow[r, shift right] & \Sigma(\mathscr{C}_0)(G/H) \arrow[rr] & & \Sigma\left(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi\right)(G/H) &
\end{tikzcd}$$

in which the top face is obtained by lifting the bottom one by pulling back along the canonical inclusion $\Pi(G/H) \hookrightarrow \Sigma(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi)(G/H)$ (note that all squares in the cube are pullbacks). Thus, the pair $(\mathfrak{C}^H, \lambda^{H^*})$ is a 2-torsor above $\Pi(G/H)$ with coefficients in A^H. Moreover, since $\Pi(G/H)$ is a groupoid we can apply Theorem 4.3 to deduce that the geometric realization $\mathbf{B}(\mathfrak{C}^H)$ is a 2-type with

$$\pi_1(\mathbf{B}(\mathfrak{C}^H)) = \Pi(G/H) \quad \text{and} \quad \pi_2(\mathbf{B}(\mathfrak{C}^H)) = A^H.$$

Summing up: We start with a 2-torsor $(\mathfrak{C}, \lambda)$ above $\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi$, then we consider this data above the category $\mathcal{O}(G)^{\mathrm{op}}$ (via the canonical projections) and we apply the corresponding functor Σ to obtain a $\mathcal{O}(G)^{\mathrm{op}}$-diagram of functors, such that for any object $G/H \in \mathcal{O}(G)$ the pair $(\Sigma(\mathfrak{C})(G/H), \lambda^H)$ is a 2-torsor above $\Sigma(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi)(G/H)$. Now, the groupoid $\Pi(G/H)$ is a co-reflexive subcategory of the groupoid $\Sigma(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi)(G/H)$ via the canonical inclusion

$$\Pi(G/H) \hookrightarrow \Sigma\left(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi\right)(G/H)$$

(see Lemma 2.1) which therefore is a weak equivalence of categories, so the set of connected components of torsors above $\Pi(G/H)$ is in bijective correspondence with the set of connected components of torsors above $\Sigma(\int_{\mathcal{O}(G)^{\mathrm{op}}} \Pi)(G/H)$, when the coefficients are appropriated (see the end of Section 2.4). Finally we lift the torsors $(\Sigma(\mathfrak{C})(G/H), \lambda^H)$ via the above inclusions of categories to get torsors $(\mathfrak{C}^H, \lambda^{H^*})$ above $\Pi(G/H)$ with coefficients the $\Pi(G/H)$-modules A^H.

Let us now note that the construction of the torsors $(\mathfrak{C}^H, \lambda^{H^*})$ is natural in G/H in the following sense: for any arrow $G/H \to G/H'$ in $\mathcal{O}(G)$, we have a commutative diagram

$$\begin{tikzcd}
 & & & \Pi(G/H') \arrow[dl] \arrow[dd] \\
\Sigma(\mathscr{C}_1)(G/H') \arrow[r, shift left] \arrow[r, shift right] \arrow[dd] & \Sigma(\mathscr{C}_0)(G/H') \arrow[r] \arrow[dd] & \Sigma\left(\int_{\mathcal{O}(G)^{\mathrm{op}}}\Pi\right)(G/H') \arrow[dd] & \\
 & & & \Pi(G/H) \arrow[dl] \\
\Sigma(\mathscr{C}_1)(G/H) \arrow[r, shift left] \arrow[r, shift right] & \Sigma(\mathscr{C}_0)(G/H) \arrow[r] & \Sigma\left(\int_{\mathcal{O}(G)^{\mathrm{op}}}\Pi\right)(G/H) &
\end{tikzcd}$$

which induces a commutative diagram

$$\begin{tikzcd}
 & & \mathscr{S}(\mathfrak{C}^{H'}) \arrow[rr, "\lambda^{H'^*}"] \arrow[dl] \arrow[dd] & & \int_{\Pi(G/H')}\pi_2(T^{H'}) \arrow[dl] \arrow[dd] \\
\mathscr{C}_1^{H'} \arrow[r, shift left] \arrow[r, shift right] \arrow[dd] & \mathscr{C}_0^{H'} \arrow[rr] \arrow[dd] & & \Pi(G/H') \arrow[dd] & \\
 & & \mathscr{S}(\mathfrak{C}^{H}) \arrow[rr, "\lambda^{H^*}"] \arrow[dl] & & \int_{\Pi(G/H)}\pi_2(T^{H}) \arrow[dl] \\
\mathscr{C}_1^{H} \arrow[r, shift left] \arrow[r, shift right] & \mathscr{C}_0^{H} \arrow[rr] & & \Pi(G/H) &
\end{tikzcd}$$

and so a 2-functor

$$\mathfrak{C}^f : \mathfrak{C}^{H'} \to \mathfrak{C}^H.$$

By this naturality, if we only take care of the fibers $\mathfrak{C}^H$ and we use the *2-dimensional nerve*, we get a simplicial map

$$\mathbf{N}(\mathfrak{C}^f) : \mathbf{N}(\mathfrak{C}^{H'}) \to \mathbf{N}(\mathfrak{C}^H),$$

and so a $\mathcal{O}(G)^{\mathrm{op}}$-diagram of simplicial sets

$$\mathbf{T}_{k_3} = \mathbf{N}\mathfrak{C}^{(-)} : \mathcal{O}(G)^{\mathrm{op}} \to \mathbf{Set}^{\Delta^{\mathrm{op}}}$$

such that, for any object $G/H \in \mathcal{O}(G)$, the simplicial set $\mathbf{T}_{k_3}(G/H)$ is a 2-type with

$$\pi_1(\mathbf{T}_{k_3}(G/H)) = \Pi(G/H) \quad \text{and} \quad \pi_2(\mathbf{T}_{k_3}(G/H)) = A^H.$$

To finish the proof of this theorem we only need to note that the unity and counity of the adjunction

$$\Sigma : \mathbf{2\text{-}Cat}/\mathcal{O}(G)^{\mathrm{op}} \rightleftarrows \mathbf{2\text{-}Cat}^{\mathcal{O}(G)^{\mathrm{op}}} : \iint_{\mathcal{O}(G)^{\mathrm{op}}}$$

allows us to prove that if we start from a diagram of simplicial sets $\mathbf{T}$ as in the statement of the theorem, and we build the k_3-invariant $k_3^{\mathbf{T}}$ and the simplicial diagram $\mathbf{T}_{k_3^{\mathbf{T}}}$, then we get a diagram which is weak equivalent to $\mathbf{T}$. And conversely, if start with an element $k_3 \in H^3(\int_{\mathcal{O}(G)^{op}} \Pi, A)$ we build the diagram of simplicial sets $\mathbf{T}_{k_3}$, then the k_3-invariant $k_3^{\mathbf{T}_{k_3}}$ is again k_3. $\square$

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