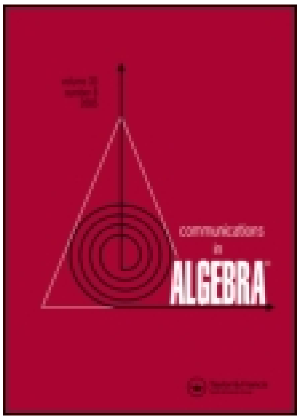

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THE HOMOTOPY TYPE OF SPACES X WITH FUNDAMENTAL GROUPOID $\mathcal{G}$ AND AN UNIQUE NON-TRIVIAL HOMOTOPY $\mathcal{G}$-MODULE $\Pi_n(X)$

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1 Introduction

Bredon cohomology was introduced in [4] to develop an obstruction theory for spaces equipped with an action of a fixed group G. More recently Moerdijk and Svensson have defined Bredon cohomology with local or twisted coefficients and have shown that this cohomology is invariant under weak equivariant homotopy equivalence. They have also constructed a Serre spectral sequence for equivariant fibrations, see [20].

In the non-equivariant case Mac Lane and Whitehead proved, [17], that any pointed connected 2-type (in the classical language, any 3-tuple (G, A, k_3)

consisting of a group G, a G-module A, and a cohomology class $k_3 \in H^3(G, A)$) can be realized by a pointed connected space (X, x_0) with homotopy groups vanishing at dimensions greater or equal to 3, and such that its fundamental group is G and its second homotopy group coincides with A. Our aim was to extend this classical result to the equivariant context. This has been achieved in [5], using results of the present paper which are a good (sufficiently functorial) generalization of the classical non-equivariant results, without assuming connectedness and the choice of a base point. Not only the classical direction - the 2-type allows to find the space- but also the converse have been established in this paper, the latter by means of appropriate arguments of obstruction theory. The argumental line and the results of the present paper lay down the norm for establishing the final generalisation in the equivariant scenario.

We start from a space X with fundamental groupoid $\Pi(X) = \mathcal{G}$ and whose only non-trivial homotopy functor is $A = \Pi_2(X) : \mathcal{G} \to \mathbf{Ab}$. The Eilenberg-Mac Lane cohomology group involved above is then to be replaced by the cohomology group $H^3(\mathcal{G}, A)$ of the small category $\mathcal{G}$ with coefficients on the system of abelian groups A. What we prove, thus, is that the homotopy type of X is *completely determined* by $\mathcal{G}$, A and an element $k_3 \in H^3(\mathcal{G}, A)$ (see Theorem 4.4 below). There is no extra dificulty in suppossing that $A = \Pi_n(X), n \geq 2$, is the only non trivial homotopy group functor of X, so all the results stated have been proved in this general case.

The use of cohomology of categories instead of cohomology of groups seems natural. Even more, it would confirm that the arguments employed in this paper were in the right direction with regard to the equivariant context (as has been proved in [5]) after the identification by Moerdijk-Svenson of Bredon cohomology groups $H^*_G(X, A)$ (of a G-space X with coefficients in an abelian group-valued functor A from the orbit category $\mathcal{O}(G)$, i.e. a $\mathcal{O}(G)$-module) as cohomology groups $H^*(\Delta_G(X), A)$ of the *small category* $\Delta_G(X)$ with coefficients in A.

The plan of the paper is as follows: next section is devoted to cohomology of small categories, reinterpreting it in terms of cotriple cohomology (Theorem 2.3). Section 3 will analyze the notion of n-torsor as a link between the cohomology of small categories and the required n-type. Finally, section 4 is devoted to the conclusions: after the necessary machinery of obstruction theory to set the converse direction, the main result is proved (Theorem 4.4).

2 On cohomology of categories

As we pointed out in the introduction, the passage from the pointed connected case to the equivariant one requires the replacement of Eilenberg- MacLane cohomology of groups by that of small categories. In this section some basic facts concerning cohomology of small categories are reviewed. Some new results will be introduced, with special attention to the identification of this cohomology as a cotriple cohomology.

The category of all small categories and functors between them will be denoted by **Cat**. The standard conventions in any category $\mathcal{C}$, $Arr(\mathcal{C}) = A$ and $Obj(\mathcal{C}) = O$ for sets of arrows and objects, respectively, s for source, t for target and $\circ$ for composition, will be adopted,

$$A \times_O A \xrightarrow{\ \circ\ } A \underset{t}{\overset{s}{\rightrightarrows}} O \quad (Id : O \to A)$$

where $A \times_O A$ contains all pairs of arrows of $\mathcal{C}$ such that their composition makes sense, that is, we will write $(f, g) \in A \times_O A$ or $(a \xrightarrow{f} b \xrightarrow{g} c) \in A \times_O A$ if $g \circ f$ exists.

In order to define cohomology of a category $\mathcal{C}$ we use $\mathcal{C}$-modules as coefficients where, as usual, by a right $\mathcal{C}$-module we mean any functor $\mathcal{A}$ from $\mathcal{C}^{op}$ to the category **Ab** of all abelian groups. Equivalently a right $\mathcal{C}$-module may be viewed as an abelian group object in the topos $\mathcal{S}^{\mathcal{C}^{op}}$ of functors from $\mathcal{C}^{op}$ to the category $\mathcal{S}$ of sets.

For any such $\mathcal{C}$-module $\mathcal{A}$, the cohomology of $\mathcal{C}$ is then defined as the cohomology of the topos $\mathcal{S}^{\mathcal{C}^{op}}$ with coefficients in the abelian group object $\mathcal{A}$, that is,

$$H^n(\mathcal{C},\mathcal{A}) = H^n(\mathcal{S}^{\mathcal{C}^{op}},\mathcal{A}).$$

Here and in the remaining of the paper, $Ner(\mathcal{C})$ stands for the nerve of a small category $\mathcal{C}$. It is the simplicial set having at dimension zero the set of objects O, at dimension one its arrows A, and at dimension n the set of composable n-tuples of arrows in $\mathcal{C}$. The face operators of $Ner(\mathcal{C})$ are $d_0 = t$, $d_1 = s : Ner(\mathcal{C})_1 \longrightarrow Ner(\mathcal{C})_0$, and $d_i : Ner(\mathcal{C})_n \longrightarrow Ner(\mathcal{C})_{n-1}$

$$d_0(\lambda_1,\lambda_2,\dots,\lambda_n) = (\lambda_2,\dots,\lambda_n)$$
$$d_i(\lambda_1,\lambda_2,\dots,\lambda_n) = (\lambda_1,\dots,\lambda_{i-1},\lambda_{i+1}\circ\lambda_i,\lambda_{i+2},\dots,\lambda_n)$$
for $0 < i < n$, and
$$d_n(\lambda_1,\lambda_2,\dots,\lambda_n) = (\lambda_1,\dots,\lambda_{n-1})$$

and the degeneracies $s_j : Ner_n(\mathcal{C}) \to Ner_{n+1}(\mathcal{C}), 0 \le j \le n$. are

$$s_0(\lambda_1,\lambda_2,\dots,\lambda_n) = (Id_{s(\lambda_1)},\lambda_1,\dots,\lambda_n)$$
$$s_j(\lambda_1,\lambda_2,\dots,\lambda_n) = (\lambda_1,\dots,\lambda_{j-1},Id_{s(\lambda_j)},\lambda_j,\dots,\lambda_n)$$
$$s_n(\lambda_1,\lambda_2,\dots,\lambda_n) = (\lambda_1,\lambda_2,\dots,\lambda_n,Id_{s(\lambda_n)}).$$

Then, the cohomology of $\mathcal{C}$ with coefficients in a $\mathcal{C}$-module $\mathcal{A}$ can also be computed (see [1]) as the cohomology of the abelian cochain complex coming from the cosimplicial abelian group defined by

$$C^n(\mathcal{C},\mathcal{A}) = \prod_{x_0\to\dots\to x_n\in Ner(\mathcal{C})_n} \mathcal{A}(x_0)$$

with differentials $\delta : C^n(\mathcal{C},\mathcal{A}) \longrightarrow C^{n-1}(\mathcal{C},\mathcal{A})$ obtained from the face maps of the nerve of $\mathcal{C}$ by taking alternating sums, in the standard fashion. Then

$$H^n(\mathcal{C}, \mathcal{A}) = H^n(C^*(\mathcal{C}, \mathcal{A})).$$

We can also use some kind of *bar resolution*, to compute this cohomology. In fact we define the simplicial co-complex $B(\mathcal{C}, \mathcal{A})$ of abelian groups by

$$B^0(\mathcal{C}, \mathcal{A}) = \{f : O \longrightarrow \sum_{x \in O} \mathcal{A}(x) / f(x_0) \in \mathcal{A}(x_0) \text{ for } x_0 \in O\}$$

and for $n \geq 1$, $B^n(\mathcal{C}, \mathcal{A})$ is defined as

$$\{f : Ner(\mathcal{C})_n \longrightarrow \sum_{x \to y \in G} \mathcal{A}(x) / f(x_0 \xrightarrow{\lambda_1} x_1 \dots x_{n-1} \xrightarrow{\lambda_n} x_n) \in \mathcal{A}(x_0)\}$$

where $\sum$ denotes the coproduct (disjoint union) in the category of sets. Note that each $B^n(\mathcal{C}, \mathcal{A})$ is converted into an abelian group by defining $f+g$ for $f, g \in B^n(\mathcal{C}, \mathcal{A})$, to be the sum into $\mathcal{A}(x_0)$. The face operators of this cosimplicial complex are

$$d_i : B^n(\mathcal{C}, \mathcal{A}) \to B^{n+1}(\mathcal{C}, \mathcal{A}).\ 0 \leq i \leq n$$

$$d_0 f(\lambda_1, \lambda_2, \dots, \lambda_{n+1}) = \mathcal{A}_C(\lambda_1) f(\lambda_2, \dots, \lambda_{n+1})$$

$$d_i f(\lambda_1, \lambda_2, \dots, \lambda_{n+1}) = f(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1} \circ \lambda_i, \lambda_{i+2}, \dots, \lambda_{n+1}),$$

for $0 < i < n$, and

$$d_n f(\lambda_1, \lambda_2, \dots, \lambda_{n+1}) = f(\lambda_1, \lambda_2, \dots, \lambda_n)$$

and the degeneracy operators are

$$s_j : B^{n+1}(\mathcal{C}, \mathcal{A}) \to B^n(\mathcal{C}, \mathcal{A}),\ 0 \leq j \leq n-1$$

$$s_0 g((\lambda_1, \lambda_2, \dots, \lambda_n) = g(Id_{x_0}, \lambda_1, \lambda_2, \dots, \lambda_n)$$

$$s_j g(\lambda_1, \lambda_2, \dots, \lambda_n) = g(\lambda_1, \dots, \lambda_{j-1}, Id_{s(\lambda_j)}, \lambda_j, \dots, \lambda_n)$$

for $0 < j < n-1$, and

$$s_{n-1} g(\lambda_1, \lambda_2, \dots, \lambda_n) = g(\lambda_1, \lambda_2, \dots, \lambda_n, Id).$$

From this cosimplicial complex of abelian groups, a cochain complex is produced by defining the coboundary operators in a bar resolution fashion:

$$\delta_{B^n} : B^n(\mathcal{C},\mathcal{A}) \to B^{n+1}(\mathcal{C},\mathcal{A})$$

$$\begin{aligned}\delta_{B^n}(f)(\lambda_1,\lambda_2,\ldots,\lambda_{n+1}) = \; & \mathcal{A}(\lambda_1)[f(\lambda_2,\ldots,\lambda_{n+1})] \\ & + \textstyle\sum_{i=1}^{n}(-1)^i f(\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1}\circ\lambda_i,\ldots,\lambda_{n+1}) \\ & +(-1)^{n+1} f(\lambda_1,\ldots.\lambda_n)\end{aligned}$$

for all $n \geq 1$, and

$$\delta_{B^0} : B^0(\mathcal{C}.\mathcal{A}) \to B^1(\mathcal{C}.\mathcal{A})$$

$$\delta_{B^0}(f)(\lambda : a \to b) = \mathcal{A}(\lambda)(f(b)) - f(a)$$

Then we have

$$H^n(\mathcal{C},\mathcal{A}) = H^n(B^*(\mathcal{C}.\mathcal{A})),$$

since both complexes $C^*(\mathcal{C},\mathcal{A})$ and $B^*(\mathcal{C},\mathcal{A})$ are isomorphic. The isomorphism is given at each dimension n as follows:

$$C^n(\mathcal{C},\mathcal{A}) \underset{p_n}{\overset{q_n}{\rightleftarrows}} B^n(\mathcal{C},\mathcal{A})$$

for each tupla $(\alpha_{x_0})_{(x_0 \xrightarrow{\lambda_1} x_1 \ldots x_{n-1} \xrightarrow{\lambda_n} x_n)\in Ner(\mathcal{C})_n}$ to be considered in the product $\prod_{x_0\to\ldots\to x_n\in Ner(\mathcal{C})_n} \mathcal{A}(x_0)$, we define

$$p_n((\alpha_{x_0})_{(x_0 \xrightarrow{\lambda_1} x_1 \ldots x_{n-1} \xrightarrow{\lambda_n} x_n)\in Ner(\mathcal{C})_n})$$

as the following element of $B^n(\mathcal{C})$:

$$p_n((\alpha_{x_0})_{(x_0 \xrightarrow{\lambda_1} x_1 \ldots x_{n-1} \xrightarrow{\lambda_n} x_n)\in Ner(\mathcal{C})_n})(y_0 \xrightarrow{\mu_1} y_1 \ldots y_{n-1} \xrightarrow{\mu_n} y_n) = \alpha_{y_0}$$

Conversely, given any $f \in B^n(\mathcal{C},\mathcal{A})$, $q_n(f)$ is defined as

$$q_n(f) = (f(x_0 \xrightarrow{\lambda_1} x_1 \ldots x_{n-1} \xrightarrow{\lambda_n} x_n))_{(x_0 \xrightarrow{\lambda_1} x_1 \ldots x_{n-1} \xrightarrow{\lambda_n} x_n)\in Ner(\mathcal{C})_n}$$

Both p_n and q_n describe the desired biyection.

In the particular case of the category $\mathcal{C}$ being a group G, viewed as an only-one-object category the topos $\mathcal{S}^{G^{op}}$ coincides with the topos of right G-sets and the above cohomology equals the usual cohomology of groups.

For any category $\mathcal{C}$, a right $\mathcal{C}$-set can also be regarded as a set (over the set O of objects of $\mathcal{C}$) equipped with a right $\mathcal{C}$-action. In other words, a right $\mathcal{C}$-set is equivalent to giving a set $X \xrightarrow{p} O$ over O together with a $\mathcal{C}$-action on the right

$$X \times_O A \xrightarrow{\mu} X$$

of $\mathcal{C}$ on X, where A is considered as object over O via t, i.e., the elements of $X \times_O A$ are pairs $(x, g) \in X \times A$ such that $p(x) = t(g)$.

The conditions of a $\mathcal{C}$-action on the right are expresed as usual by

1. $p(x \cdot g) = s(g)$.
2. $x \cdot Id_{p(x)} = x$.
3. $x \cdot (h \circ g) = (x \cdot h) \cdot g$,

where we write $\mu(x, g) = x \cdot g$.

We have then a forgetful functor

$$U : \mathcal{S}^{\mathcal{C}^{op}} \longrightarrow \mathcal{S}/O$$

from the category of right $\mathcal{C}$-sets to the slice category of sets over O, which in fact has a left adjoint. The functor U preserves products and so it induces a functor M between the corresponding categories of internal abelian group objects. This situation is illustrated in the following commutative diagram,

$$\begin{array}{ccc} \mathcal{S}^{\mathcal{C}^{op}} & \xrightarrow{U} & \mathcal{S}/O \\ \uparrow & & \uparrow \\ \mathbf{Ab}^{\mathcal{C}^{op}} & \xrightarrow{M} & \mathbf{Ab}(\mathcal{S}/O) \end{array}$$

where the vertical arrows are the corresponding forgetful functors from the categories of internal abelian groups objects. If $\mathcal{A} \in \mathbf{Ab}^{\mathcal{C}^{op}}$, $M(\mathcal{A})$ is defined by

$$M(\mathcal{A}) = \sum_{a \in O} \mathcal{A}(a) \xrightarrow{pr} O,$$

with pr the canonical projection.

Let us note that this functor M, in the case of groups, factors through the category of internal abelian group objects of a slice category of groups, by a construction of semidirect products of groups and kernels. All the mentioned structures can be reproduced in the case of categories by mean of the so-called *Grothendieck construction.* Given any contravariant functor $F : \mathcal{C}^{op} \to \mathbf{Cat}$, the Grothendieck semidirect construction is the category $\int_{\mathcal{C}} F$ whose objects are pairs (c, x) with $c \in O$ and x an object in $F(c)$. An arrow $(f, \lambda) : (c, x) \to (c', y)$ consists of an arrow $f : c \to c' \in A$ and an arrow in $F(c)$, $\lambda : x \to F(f)(y)$. The composition is defined in the obvious way.

The Grothendieck construction, when considered on $\mathcal{C}$-modules $\mathcal{A}$ (regarding each abelian group as a category with just one object), produces a category whose set of objects is the same of $\mathcal{C}$, and whose arrows are pairs $(f, a) : c \to c'$ with $c \xrightarrow{f} c'$ an arrow in $\mathcal{C}$ and $a \in \mathcal{A}(c)$, the source and target of (f, a) are the source c and the target c' of f, respectively. Particulary, the composition in $\int_{\mathcal{C}} \mathcal{A}$ is given by

$$c \xrightarrow{(g,b)} c' \xrightarrow{(f,a)} c''. \qquad (f,a)\circ(g,b) = (f \circ g, \mathcal{A}(g)(a) + b)$$

The natural projection $\int_{\mathcal{C}} F \xrightarrow{pr} \mathcal{C}$ is the identity on objects. This justifies the restriction of our setting to the full subcategory $\mathbf{Cat}_O/\mathcal{C}$ (of the slice category $\mathbf{Cat}/\mathcal{C}$) formed by those categories over $\mathcal{C}$ with same set of objects O as $\mathcal{C}$. $\int_{\mathcal{C}} \mathcal{A}$ will be always assumed to be supplied with the natural projection to $\mathcal{C}$, pr.

This produces an abelian group object which will be an adequate coefficient for certain cohomology, as follows:

Lemma 2.1. *For any category $\mathcal{C}$ and rigth $\mathcal{C}$-module $\mathcal{A}$, the category $\int_{\mathcal{C}} \mathcal{A}$ is an abelian group object internal in the slice category $\mathbf{Cat}_O/\mathcal{C}$.*

Proof. The group operation $+ : \int_{\mathcal{C}} \mathcal{A} \times \int_{\mathcal{C}} \mathcal{A} \longrightarrow \int_{\mathcal{C}} \mathcal{A}$ is given by $(f, a) + (f, a') = (f, a + a')$ □

And even more:

Lemma 2.2. *When the category considered is a groupoid $\mathcal{G}$, so it is $\int_{\mathcal{G}} \mathcal{A}$, for any $\mathcal{G}$-module $\mathcal{A}$. If $\mathbf{Gpd}_O/\mathcal{G}$ stands for the corresponding slice category of groupoids (over $\mathcal{G}$) with fixed set of objects, then $\int_{\mathcal{C}} \mathcal{A}$ is abelian group object in $\mathbf{Gpd}_O/\mathcal{G}$.* □

The Grothendieck construction gives rise to a functor

$$\int_{\mathcal{C}} - : \mathbf{Ab}^{\mathcal{C}^{op}} \longrightarrow \mathbf{Ab}(\mathbf{Cat}_O/\mathcal{C}),$$

which in the particular case of groupoids is indeed an equivalence of categories, $\mathbf{Ab}^{\mathcal{G}^{op}} \cong \mathbf{Ab}(\mathbf{Gpd}_O/\mathcal{G})$, for any fixed groupoid $\mathcal{G}$. The close relationship between abelian group objects in $\mathcal{S}^{\mathcal{C}^{op}}$ and abelian group objects in $\mathbf{Cat}_O/\mathcal{C}$ (that in case of groupoids turns out to be an identification), together with the fact that this last category $\mathbf{Cat}_O/\mathcal{C}$ is tripleable, suggest interpreting the cohomology $H^n(\mathcal{C}, \mathcal{A})$ as a cotriple cohomology. In order to articulate this idea, let us observe, on the one hand, that the set

$$Hom_{\mathbf{Cat}_O/\mathcal{C}} \begin{pmatrix} \mathcal{C} & \int_{\mathcal{C}} \mathcal{A} \\ \downarrow & , \quad \downarrow \\ \mathcal{C} & \mathcal{C} \end{pmatrix}$$

of functors (over $\mathcal{C}$) from $\mathcal{C}$ to $\int_{\mathcal{C}} \mathcal{A}$ can be parametrized by the set $Der(\mathcal{C}, \mathcal{A})$ of "*derivations*" from $\mathcal{C}$ to $\mathcal{A}$, where an element $d \in Der(\mathcal{C}, \mathcal{A})$ is a correspondence which maps any arrow $a \xrightarrow{f} b$ of $\mathcal{C}$ to an element $d(h) \in \mathcal{A}(\alpha(a))$ such that for any pair

$$a \xrightarrow{f_2} b \xrightarrow{f_1} c$$

of composable arrows of $\mathcal{C}$, the identity

$$d(f_1 \circ f_2) = \mathcal{A}(\alpha(f_2))(d(f_1)) + d(f_2),$$

or, in terms of actions,

$$d(f_1 \circ f_2) = d(f_1)^{\alpha(f_2)} + d(f_2),$$

holds. On the other hand, recall that the category **Cat** of small categories is tripleable over the category **Grph** of small (directed and reflexive) graphs, with cotriple associated to the adjunction

$$F : \mathbf{Grph} \rightleftarrows \mathbf{Cat} : U \,. \quad F \dashv U \qquad (1)$$

where F and U are the "*free category*" and the "*underlying graph*" functors respectively. Now, the above adjunction *(1)* induces, for any groupoid $\mathcal{C}$, another one between the corresponding slice categories

$$F : \mathbf{Grph}/U(\mathcal{C}) \rightleftarrows \mathbf{Cat}/\mathcal{C} : U \,, \qquad (2)$$

in such a way that $\mathbf{Cat}/\mathcal{C}$ is tripleable over $\mathbf{Grph}/U(\mathcal{C})$, where the cotriple comes from the adjunction *(2)*. Besides. this last adjunction *(2)* could be restricted to another one

$$F : \mathbf{Grph}_O/U(\mathcal{C}) \rightleftarrows \mathbf{Cat}_O/\mathcal{C} : U \,, \qquad (3)$$

where $\mathbf{Grph}_O/\mathcal{C}$ is the full subcategory of $\mathbf{Grph}/U(\mathcal{C})$ having as objects those graphs $\mathcal{T} \to U(\mathcal{C})$ over $U(\mathcal{C})$ with the identity on vertices. The category $\mathbf{Cat}_O/\mathcal{C}$ is then tripleable over $\mathbf{Grph}_O/\mathcal{C}$ by means of the adjunction *(3)*.

With the obvious abuse of language, $\mathcal{C}$ for the identity $Id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, we may consider then the cotriple cohomology $H^n_{\mathbb{G}}(\mathcal{C}, \int_{\mathcal{C}} \mathcal{A})$, $n \geq 0$, of $Id_{\mathcal{C}}$ with coefficients in the abelian group objects $\int_{\mathcal{C}} \mathcal{A}$. Thus we prove

Theorem 2.3. *For any small category $\mathcal{C}$ and any $\mathcal{C}$-module $\mathcal{A}$, there are natural isomorphisms*

$$H^n_{\mathbb{G}}(\mathcal{C}, \int_{\mathcal{C}} \mathcal{A}) \cong \begin{cases} Der(\mathcal{C}, \mathcal{A}) & if \quad n = 0 \\ H^{n+1}(\mathcal{C}, \mathcal{A}) & if \quad n > 0 \end{cases}$$

between the cotriple cohomology of $\mathcal{C}$ with coefficients in $\int_{\mathcal{C}} \mathcal{A}$ and the cohomology of the small category $\mathcal{C}$ with coefficients in the $\mathcal{C}$-module $\mathcal{A}$.

Proof. An standard cochain complex K consists of contravariant functors to the category of abelian groups, $K^n : \mathcal{C}^{op} \to \mathbf{Ab}$, together with natural transformations $d^{n+1} : K^n \to K^{n+1}$ such that $d^{n+1}d^n = 0$, $\forall n \geq 0$, as usual. Thus, we shall prove the theorem as an application of the result given in [2] (Corollary 3.2.):

"If K y L are standard cochain complexes, both $\mathbb{G}$- acyclic and $\mathbb{G}$-representables, and $K^{-1} \cong L^{-1}$, then K and L are (homotopy) equivalents. In particular, they produce the same cohomology".

Let us define an standard cochain complex, K, by

$$\{K^n(\mathcal{C})\}_{n \geq -1} = \begin{cases} Hom_{\mathbf{Cat}_O/\mathcal{C}}(\mathcal{C}, \int_{\mathcal{C}} \mathcal{A}) & if \quad n = -1 \\ Hom_{\mathbf{Cat}_O/\mathcal{C}}(\mathbb{G}^{n+1}(\mathcal{C}), \int_{\mathcal{C}} \mathcal{A}) & if \quad n \geq 0 \end{cases}$$

It is well known that the cohomology produced by this complex is the cotriple cohomology of $\mathcal{C}$ with coefficients in $\int_{\mathcal{C}} \mathcal{A}$.

This cochain complex is $\mathbb{G}$-acyclic, that is, there is a functorial contracting homotopy in the "composite" complex $\{K^n(\mathbb{G}(\mathcal{C}))\}_{n\geq -1}$. This comes from the fact that the cosimplicial complex $\{K^n(\mathbb{G}(\mathcal{C}))\}_{n\geq -1}$ could be regarded as

$$Hom_{(\mathbf{Cat}_O/\mathcal{C})}(Dec^1\mathbb{G}^\bullet(\mathcal{C}), \int_{\mathcal{C}} \mathcal{A})$$

where Dec^1 denotes the shift functor which acts by forgetting the last face operator at each level and renumbering the levels (see [14]). Then, just take into account that a simplicial contraction $\{c_n : X_{n-1} \to X_n\}_{n\geq -1}$ in an augmented simplicial complex $\{X_n\}_{n\geq -1}$ (i.e., an extra degeneracy operator) produces a contracting homotopy in the cochain complex obtained by "*Hom-ing*" and taking alternating sum of faces. This is defined by

$$s_n = (-1)^{n+1} Hom_{(\mathbf{Cat}_O/\mathcal{C})}(c_{n+1}, -)$$

So, the required contracting homotopy for $\{K^n(\mathbb{G}(\mathcal{C}))\}_{n\geq -1}$ comes from the simplicial contraction $Dec^1(\mathbb{G}^\bullet(\mathcal{C}))$ has (which is the forgotten degeneracy).

Besides, the complex $\{K^n(\mathcal{C})\}_{n\geq -1}$ is $\mathbb{G}$-representable , which means in this case that there are morphisms $K^n(\mathbb{G}(\mathcal{C})) \xrightarrow{\alpha^n} K^n(\mathcal{C})$ such that $\alpha^n\mathbb{G}^{n+1}(\epsilon_{\mathcal{C}})^* = Id_{K^n(\mathcal{C})}$. This is achived by defining the morphisms α^n to be $\mathbb{G}^n(\delta_{\mathcal{C}})$. Then, the required equality is satisfied for the counitary law of the cotriple.

In order to have a complete information about $\{K^n(\mathbb{G}(\mathcal{C}))\}_{n\geq -1}$, note now that the -1 level of this complex, $Hom_{\mathbf{Cat}_O/\mathcal{C}}(\mathcal{C}, \int_{\mathcal{C}} \mathcal{A})$, can be described as being isomorphic to $Der(\mathcal{C}, \mathcal{A})$, as remarked before.

On the other hand, let us consider the complex $B^n(\mathcal{C}, \mathcal{A})$ introduced previously and, particulary, consider the coboundary map

$$\delta_{B^1} : B^1(\mathcal{C}, \mathcal{A}) \to B^2(\mathcal{C}, \mathcal{A})$$

which is defined by

$$\delta_{B^1}(f)(\lambda_1, \lambda_2) = \mathcal{A}(\lambda_1)[f(\lambda_2)] - f(\lambda_2 \circ \lambda_1) + f(\lambda_1)$$

for each $x_0 \xrightarrow{\lambda_1} x_1 \xrightarrow{\lambda_2} x_2$, whose kernel equals $Der(\mathcal{C},\mathcal{A})$. This allows us to consider the cochain complex $B^n(\mathcal{C},\mathcal{A})$ with a shift in dimension, that is, starting in $B^1(\mathcal{C},\mathcal{A})$, as cochain complex coaugmented on $Der(\mathcal{C},\mathcal{A})$. Let us give the following name to this

$$\{L^n(\mathcal{C})\}_{n\geq -1} = \begin{cases} Der(\mathcal{C},\mathcal{A}) & if \quad n=-1 \\ B^{n+1}(\mathcal{C},\mathcal{A}) & if \quad n\geq 0 \end{cases}$$

and let us observe that this complex is the one of those taken to define cohomology of the category $\mathcal{C}$ with coefficients in a $\mathcal{C}$-module $\mathcal{A}$, as well as $\{K^n(\mathcal{C})\}_{n\geq -1}$ defines the cotriple cohomology.

We show now that the cochain complex $\{L^n(\mathcal{C})\}_{n\geq -1}$ is $\mathbb{G}$-acyclic and $\mathbb{G}$-representable. In order to prove $\mathbb{G}$-acyclicness for L, we shall find the contracting homotopy $s_n : L^{n+1}\mathbb{G}\mathcal{C} \to L^n\mathbb{G}\mathcal{C}$ as follows:

For $f \in L^{n+1}\mathbb{G}\mathcal{C}$ and $(p_1,p_2,\dots,p_{n+1}) \in Ner(\mathbb{G}\mathcal{C})_{n+1}$, we define s_n by induction on the length of the word p_1, considering the following cases:

1. If $p_1 = (w)g, w \in G$ and $g \in FG$ define

$$\begin{aligned} s_nf(p_1,\dots,p_{n+1}) = \; & \mathcal{A}(w)[s_nf(g,p_2,\dots,p_{n+1})] \\ & -f((w),g,p_2,\dots,p_{n+1}) \end{aligned}$$

2. If $p_1 = Id$, define

$$s_nf(p_1,\dots,p_{n+1}) = f(Id,Id,p_2,\dots,p_{n+1})$$

where it is not indicated explicitely where the identity Id is, since this is forced by the belonging of $(Id,Id,p_2,\dots,p_{n+1})$ to the nerve. As terminology, (w) stands for the word consisting of an unique letter w, according to the construction of the free groupoid functor F.

Finally, let us show that $\{L^n(\mathcal{C})\}_{n\geq -1}$ is $\mathbb{G}$-representable as well. To this end, define the morphisms $\alpha^n : L^n(\mathbb{G}\mathcal{C}) \to L^n(\mathcal{C})$ as follows:

$$\alpha^n h(g_1,g_2,\dots,g_{n+1}) = h((g_1),(g_2),\dots,(g_{n+1}))$$

where $h \in L^n(\mathbb{G}C)$ and $(g_1, g_2, \dots, g_{n+1}) \in Ner(C)_{n+1}$. These maps must verify $\alpha^n L^n(\epsilon_C) = Id_{L^n(C)}$, where the map $L^n(\epsilon_C)$ is defined as follows:

Let f be in L^nC and $(w_1, w_2, \dots, w_{n+1}) \in Ner(\mathbb{G}(C))_{n+1}$, that is, each of the words w_i consists of letters,

$$w_i = l^i_{s_i} l^i_{s_i-1} \cdots l^i_2 l^i_1 \qquad \text{for} \qquad i = 1, \dots, n+1.$$

Then one has

$$L^n(\epsilon_C)(f)(w_1, w_2, \dots, w_{n+1}) =$$
$$f(l^1_{s_1} \circ l^1_{s_1-1} \cdots l^1_1, l^2_{s_2} \circ l^2_{s_2-1} \cdots l^2_1, \dots, l^{n+1}_{s_{n+1}} \circ l^{n+1}_{s_{n+1}-1} \cdots l^{n+1}_1)$$

where these compositions exist since $(w_1, w_2, \dots, w_{n+1})$ is an element of $Ner(\mathbf{G}(C))_{n+1}$. With such definitions, the required property holds. □

Remark 2.4. *The adjunction (3) can be replaced by*

$$F : \mathrm{Grph}_O/U(\mathcal{G}) \rightleftarrows \mathrm{Gpd}_O/\mathcal{G} : U\,, \qquad (4)$$

for a fixed groupoid $\mathcal{G}$, with set of objects O and arrows A. For any $\mathcal{G}$-module $\mathcal{A}$, $\int_{\mathcal{G}} \mathcal{A}$ is an adequate coefficient, by 2.2 , for the cotriple cohomology $H^*_{\mathbb{G}'}(\mathcal{G}, \int_{\mathcal{G}} \mathcal{A})$, with cotriple $\mathbb{G}'$ associatted to adjunction (4); hence, an anologous proof to that given by 2.3, shows the correponding result for $\mathrm{Gpd}_O/\mathcal{G}$ *to be true. This will be fully used in next sections.*

3 Relationship with torsors

The identification provided in theorem 2.3 gives a new interpretation of the cohomology groups of any category C. This section is devoted to analizing the advantages of such interpretation, as a link to the required homotopy type.

Let us determine here some simplicial notation. Having previously restricted our attention to groupoids with a fixed set of objects, the adequate simplicial setting now is the category $Simp(\mathbf{Gpd}_O)$ of simplicial groupoids with constant object of objects, that is, simplicial groupoids in the sense of Dwyer and Kan, [11], whose category is usually denoted by **Gd**. The suitable relationship of **Gd** with the category of topological spaces by means of a sequence of adjunctions, will lead us to the desired homotopy type in the next section.

For the fixed groupoid $\mathcal{G}$, we consider the slice category $\mathbf{Gd}/\mathcal{G}$ by regarding $\mathcal{G}$ as constant simplicial groupoid with all faces and degeneracies the identity. Also, any object of $\mathbf{Gd}/\mathcal{G}$ can be viewed as an augmented simplicial groupoid $E_\bullet \to \mathcal{G}$, in the obvious way.

An augmented simplicial groupoid $E_\bullet \to \mathcal{G}$ is called *U-split* if the augmented simplicial graph $U(E_\bullet) \to U(\mathcal{G})$ has a simplicial contraction, that is, an extra degeneracy operator, called the U-splitting.

The *n-dimensional simplicial kernel* $\Delta_n(E_\bullet)$ is the groupoid over $\mathcal{G}$ with same set of objects as $\mathcal{G}$ and whose arrows are $(n+1)$-tuples $(e_0, \dots, e_k, \dots, e_n)$ of arrows $e_i \in E_{n-1}$ such that $d_i e_j = d_{j-1} e_i$, for all $0 \le i < j \le n$. The source and target of an arrow $(e_0, \dots, e_k, \dots, e_n)$ in $\Delta_n(E_\bullet)$ are the source and target, respectively, of any of its components e_i and the composition in $\Delta_n(E_\bullet)$ is given componentwise. We will write

$$D_n = < d_0, \dots, d_n >: E_n \to \Delta_n(E_\bullet)$$

for the map which is the identity on objects and takes an arrow e of E_n to $D_n(e) = (d_0(e), \dots, d_n(e))$. A simplicial groupoid is said to be *aspherical at dimension n* if the canonical map D_n is surjective and *aspherical* if it is aspherical at dimension n, for all $n \ge 0$. As convention, $D_1 = < d_0, d_1 >: E_1 \to \Delta_1(E_\bullet) = E_0 \times_{\mathcal{G}} E_0$, while at dimension 0, $d_0 : E_0 \to \Delta_0(E_\bullet) = \mathcal{G}$ is the augmentation map.

In a similar way, for any $0 \le k \le n$, one may define the *n-groupoid of open k-horns*, $\Lambda_n^k(E_\bullet)$. The set of objects of this groupoid is O and its arrows are n-tuples $(e_0, \dots, e_{k-1}, -, e_{k+1}, \dots, e_n)$ of arrows e_i in E_{n-1} (where the k-th component is missing), satisfying

$$d_i e_j = d_{j-1} e_i, \text{ for all } 0 \leq i < j \leq n \text{ and } i, j \neq k.$$

There are also canonical maps at each dimension n,

$$< d_i >_{i \neq k} = < d_0, \dots, d_{k-1}, -, d_k, \dots, d_n >: E_n \to \Lambda_n^k(E_\bullet)$$

defined as the identity on objects, and on arrows e of E_n to be $(d_0(e), \dots, d_{k-1}(e), -, d_k(e), \dots, d_n(e))$.

Note now that by iterating the simplicial kernel construction we obtain the *coskeleton* functor.

$$cosk^n : Tr_n(\mathbf{Gd}/\mathcal{G}) \to \mathbf{Gd}/\mathcal{G}$$

from the category of truncated, at dimension n, simplicial groupoids over $\mathcal{G}$. This functor $cosk^n$ is right adjoint to the functor Tr_n which takes a simplicial groupoid over $\mathcal{G}$ to its truncation at dimension n. Then, $cosk^n$ of an n-dimensional truncated simplicial groupoid over $\mathcal{G}$ has simplicial kernels from dimension $n+1$ on. We write the composition functor $cosk^n \circ Tr^n$ as $Cosk^n$; then, the unit of the above adjunction $D_\bullet : E_\bullet \to Cosk^n(E)$ is the identity until dimension n and the map $D_m : E_m \to \Delta_m(E_\bullet))$ at dimensions $m \geq n+1$.

Let $\mathcal{A}$ be any $\mathcal{G}$-module. We may describe now the Eilenberg-Mac Lane complexes $K(\int_\mathcal{G} \mathcal{A}, n)$, asociated with the corresponding abelian group object $\int_\mathcal{G} \mathcal{A}$ in $\mathbf{Gpd}_O/\mathcal{G}$. These simplicial complexes are defined to be the $cosk^{n+1}$ of the truncated complex

$$(\textstyle\int_\mathcal{G} \mathcal{A})^{n+1} \underset{d_{n+1}}{\overset{d_0}{\rightrightarrows}} \textstyle\int_\mathcal{G} \mathcal{A} \underset{d_n}{\overset{d_0}{\rightrightarrows}} \mathcal{G} = \mathcal{G} \cdots \mathcal{G} = \mathcal{G}$$

which is constant until dimension $n-1$ (and equal to the identity on $\mathcal{G}$), it has $\int_\mathcal{G} \mathcal{A}$ at dimension n (with all faces the projection pr) and the product in

$\mathbf{Gpd}_O/\mathcal{G}$ of $n+1$ copies of $\int_{\mathcal{G}}\mathcal{A}$ at dimension $n+1$. The faces $d_i : (\int_{\mathcal{G}}\mathcal{A})^{n+1} \to \int_{\mathcal{G}}\mathcal{A}$ are the canonical projections for $0 \leq i \leq n$, while d_{n+1} is given by the identity on objets and

$$d_{n+1} = (-1)^n \sum_{i=0}^{n} (-1)^i d_i,$$

on arrows.

Note that $K(\int_{\mathcal{G}}\mathcal{A}, n)_m \cong \Lambda^k_m(K(\int_{\mathcal{G}}\mathcal{A}, n))$, for $0 \leq k \leq m$ and $m \geq n+1$. Simplicial objects satisfying this last condition (of having open horns at dimensions greater than or equal to $n+1$, in the sense that the canonical map $< d_i >_{i \neq k}: E_m \to \Lambda^k_m(E_\bullet)$ is an isomorphism, for all $0 \leq k \leq m, m \geq n+1$) are called *n-hypergroupoids*. An n-hypergroupoid is said to be *exact* if it is aspherical in dimensions q, $0 \leq q \leq n$.

We explore now the notion of n-torsor (see [9], [13]) since it plays a central rôle in the determination of the required homotopy type. The importance of such devices is that they give a complete interpretation for the cotriple cohomology groups in all dimensions that now can be applied to the cohomology of the small groupoid $\mathcal{G}$ with coefficients in a $\mathcal{G}$-module $\mathcal{A}$, by 2.3.

Definition 3.1. *A $K(\int_{\mathcal{G}}\mathcal{A}, n)$-torsor relative to the forgetful functor U in (3) above $\mathcal{G}$, denoted by $(E_\bullet, \xi_\bullet, \int_{\mathcal{G}}\mathcal{A})$, consists of an augmented simplicial complex $E_\bullet \to \mathcal{G} \in \mathbf{Gd}/\mathcal{G}$, supplied with a simplicial map $\xi_\bullet : E_\bullet \to K(\int_{\mathcal{G}}\mathcal{A}, n)$ such that*

1. *the canonical map $D_\bullet : E_\bullet \to Cosk^{n-1}(E_\bullet)$ is an isomorphism.*

2. *the commutative square of canonical maps*

$$\begin{array}{ccc} E_n & \xrightarrow{\xi_n} & K(\int_{\mathcal{G}}\mathcal{A}, n)_n \\ {\scriptstyle <d_i>_{i\neq k}}\downarrow & & \downarrow{\scriptstyle <d_i>_{i\neq k}} \\ \Lambda^k_n(E_\bullet) & \longrightarrow & \Lambda^k_n(K(\int_{\mathcal{G}}\mathcal{A}, n)) = \mathcal{G} \end{array}$$

is cartesian, for $0 \leq k \leq n$, *and*

3. $E_\bullet \to \mathcal{G}$ *is an augmentated* U*-split complex.*

Note that condition 1 together with condition 2, implies that $\Delta_n(E_\bullet) \cong \Lambda_n^k(E_\bullet) \times_{\mathcal{G}} \int_{\mathcal{G}} \mathcal{A}$ (which intuitively means that $\int_{\mathcal{G}} \mathcal{A}$ acts on $\Lambda_n^k(E_\bullet)$ to fill its missing k-face in a fashion which is compatible with the degeneracy operators).

Let us denote by $Tors_U^n(\mathcal{G}, \int_{\mathcal{G}} \mathcal{A})$ and $Tors_U^n[\mathcal{G}, \int_{\mathcal{G}} \mathcal{A}]$ the category of $K(\int_{\mathcal{G}} \mathcal{A}, n)$- torsors relative to U above $\mathcal{G}$ (where a morphism of n-torsors is an equivariant simplicial map) and the set of their connected components, respectively.

Thus, $K(\int_{\mathcal{G}} \mathcal{A}, n)$-torsors interpret all cotriple cohomology groups, as follows:

Theorem 3.2 (Theorem 5.2, [10]). *Consider the functor* U *given in (4) and the cohomology groups obtained from the corresponding cotriple. Then, there are natural isomorphisms*

$$Tors_U^n[\mathcal{G}, \int_{\mathcal{G}} \mathcal{A}] \cong H_{\mathbb{G}}^n(\mathcal{G}, \int_{\mathcal{G}} \mathcal{A})$$

for each $\int_{\mathcal{G}} \mathcal{A} \in Ab(\mathbf{Gpd}_O/\mathcal{G})$. □

Once we have a new view of the cohomology of categories, as a direct consequence of the former result, there are isomorphisms $H^{n+1}(\mathcal{G}, \mathcal{A}) \cong Tor_U^n[\mathcal{G}, \int_{\mathcal{G}} \mathcal{A}], n > 0$. So each cohomology class $k_{n+1} \in H^{n+1}(\mathcal{G}, \mathcal{A})$ appears to be associated to a class of connected componentes of $K(\int_{\mathcal{G}} \mathcal{A}, n)$ -torsors, $\xi_\bullet : E_\bullet \to K(\int_{\mathcal{G}} \mathcal{A}, n)$. Consequentely,

Theorem 3.3. *There exists a one-to-one correspondence between cohomology classes* $k_{n+1} \in H^{n+1}(\mathcal{G}, \mathcal{A})$ *and classes of connected components of* $K(\int_{\mathcal{G}} \mathcal{A}, n)$*-torsors relative to* U *above* $\mathcal{G}$, $\xi_\bullet : E_\bullet \to K(\int_{\mathcal{G}} \mathcal{A}, n)$, *for any abelian group object* $\int_{\mathcal{G}} \mathcal{A} \in Ab(\mathbf{Gpd}_O/\mathcal{G})$. □

The search of required homotopy type passes through n-torsors, as we have indicated. More precisely, it passes through a particular substructure that any $K(\int_{\mathcal{G}} \mathcal{A}, n)$-torsor has, called its *fibre*. This is the following:

Definition 3.4. *Let $(E_\bullet, \xi_\bullet, \int_{\mathcal{G}} \mathcal{A})$ be a $K(\int_{\mathcal{G}} \mathcal{A}, n)$-torsor. Its fiber $Fib(E_\bullet)$ is defined to be the pullback simplicial groupoid*

$$\begin{array}{ccc} Fib(E_\bullet) & \hookrightarrow & E_\bullet \\ \downarrow & & \downarrow{\scriptstyle \xi_\bullet} \\ \mathcal{G} & \underset{j_\bullet}{\hookrightarrow} & K(\int_{\mathcal{G}} \mathcal{A}, n) \end{array}$$

provided $\mathcal{G}$ is regarded as constant simplicial groupoid in $\mathrm{Gd}/\mathcal{G}$, with $j_\bullet$ the inclusion.

From this definition, the fibre of an n-torsor is an $(n-1)$-hypergroupoid which, in addition, is exact. The relevant fact is that

Lemma 3.5. *Every exact $(n-1)$-hypergroupoid may be considered as the fibre of an U-split n-torsor.*

Proof. To this end, let $M_\bullet \to \mathcal{G}$ be an exact $(n-1)$-hypergroupoid,

$$\ldots \Lambda_n^k(E_\bullet) \cong M_n \underset{d_n}{\overset{d_0}{\rightrightarrows}} M_{n-1} \underset{d_{n-1}}{\overset{d_0}{\rightrightarrows}} M_{n-2} \rightrightarrows \ldots M_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} M_0 \xrightarrow{d_0} \mathcal{G}$$

for each $0 \leq k \leq n$. The n-torsor above $\mathcal{G}$, $E_\bullet^{M_\bullet}$ must be $Cosk^{n-1}(M_\bullet)$. Then, the abelian group object is defined as follows. First, let us remind that the kernel of a functor α between groupoids in $\mathbf{Gpd}_O/\mathcal{G}$, which is the identity on objects, is the group over O whose fiber at any point $x \in O$ is the group of automorphisms of x in the domain that goes to the identity on x by α. Let us define then $A'_{M_\bullet} = Kerd_0 \cap Kerd_1 \cap \ldots \cap Kerd_{n-1} \subseteq M_{n-1}$, which is equipped with an action of M_0, carrying each $a \in A'_{M_\bullet}$ to M_0 by sucessive degeneracies. For the exactness, this one defines an action of $\mathcal{G}$ on $A'_{M_\bullet}$, as $g \cdot a = s_0^{n-1}(m)a(s_0^{n-1}(m))^{-1}$, $g \in \mathcal{G}, a \in A'_{M_\bullet}$, for all $m \in M_0$

such that $d_0(m) = g$. Using now the equivalence between abelian groups objects of $\mathcal{G}$-sets and that of $\mathbf{S}^{\mathcal{G}^{op}}$, we denote also by $A'_{M_\bullet}$ the corresponding functor $\mathcal{G}^{op} \to \mathbf{Ab}$. Thus, the abelian group object in $\mathbf{Gpd}_O/\mathcal{G}$ is obtained as $\int_{\mathcal{G}} A'_{M_\bullet}$, written $A_{M_\bullet}$ for simplicity. Finally, the functor $\xi_n : E_n \to A_{M_\bullet}$ (which is the identity on objects), is defined on maps as $\xi_n(x_0, x_1, \ldots, x_n) = (d_0^n(x_0), d_n(x_0, x_1, \ldots, x_{n-1})x_n^{-1}) \in \int_{\mathcal{G}} A'_{M_\bullet}$ (see [8] for groups). Moreover, this n-torsor is U-split. In fact, the U-splitting $s_q : U(M_{q-1}) \to U(M_q)$ is defined as:

- $s_1 : U(M_0) \to U(M_1)$ is given by $s_1(x) = h_1(h_0 d_0 x, x)$, for each $x \in U(M_0)$, where h_0 and h_1 are the corresponding splittings in the underlying maps of sets $U(D_0) : U(M_0) \to U(\Delta_0(M_\bullet))$ and $U(D_1) : U(M_1) \to U(\Delta_1(M_\bullet))$, and
- $s_q(x) = h_q(s_{q-1}d_0x, s_{q-1}d_1x, \ldots, s_{q-1}d_{q-1}x, x)$, for each $x \in U(M_{q-1})$, where h_n is the corresponding U-splitting for $U(D_n)$.

□

The key property of $Fib(E_\bullet)$ in our develloppement is the following:

Proposition 3.6. *For any right $\mathcal{G}$-module $\mathcal{A}$ and any $K(\int_{\mathcal{G}} \mathcal{A}, n)$-torsor relative to U above $\mathcal{G}$, $(E_\bullet, \xi_\bullet, \int_{\mathcal{G}} \mathcal{A})$, then*

$$\Pi_0(Fib(E_\bullet)) = \mathcal{G}$$

and there is an isomorphism

$$\Pi_{n-1}(Fib(E_\bullet)) \cong \mathcal{A}$$

as right G-modules. Besides. the remaining of homotopy functors are trivial.

Proof. Any augmented simplicial groupoid. $\mathcal{K}_\bullet \to \mathcal{G}$ in $\mathbf{Gd}/\mathcal{G}$, may be considered as ordinary simplicial groupoid by simply forgetting the augmentation.

On the other hand, every ordinary simplicial groupoid $\mathcal{X}_\bullet \in \mathbf{Gd}$ may be seen as augmented by considering the coequalizer of the maps $\mathcal{X}_1 \rightrightarrows \mathcal{X}_0 \to \Pi_0(\mathcal{X}_\bullet)$, which is called the connected component groupoid of $\mathcal{X}_\bullet$. This construction gives rise to a pair of adjoint functors

$$\mathbf{Gd} \underset{C_0}{\overset{Fg}{\rightleftarrows}} \mathbf{Gd}_{aug}$$

between $\mathbf{Gd}$ and the category of augmented simplicial groupoids, with Fg for *forgetting* functor and $C_0(\mathcal{K}_\bullet) = \mathcal{K}_\bullet \to \Pi_0(\mathcal{K})$. It is clear that $Fg \circ C_0 = Id$ and note that the other composition $C_0 \circ Fg$ is not equal to the identity for all simplicial complexes but for those which are aspherical (dimensions 0 and 1).

In order to show that $C_0 \circ Fg = Id$, let us prove that U-splitness implies asphericity in our particular context. First, one observes that the existence of a simplicial contraction in an augmented simplicial groupoid makes itself aspherical. Consider then any $K(\int_{\mathcal{G}} \mathcal{A}, n)$-torsor in $\mathbf{Gd}/\mathcal{G}$, $\xi_\bullet : E_\bullet \to K(\int_{\mathcal{G}} \mathcal{A}, n)$, which is U-split. Hence, the canonical map

$$U(D_n) : U(E_n) \to U(\Delta_n(E_\bullet))$$

is surjective. for all $n \geq 0$. As the free groupoid functor F has right adjoint, it preserves colimits, particulary epimorphisms. Finally, the commutative diagram

$$\begin{array}{ccc} FU(E_n) & \xrightarrow{FU(D_n)} & FU(\Delta_n(E)) \\ {\scriptstyle \epsilon_{E_n}}\downarrow & & \downarrow{\scriptstyle \epsilon_{\Delta_n(E)}} \\ E_n & \xrightarrow[D_n]{} & \Delta_n(E) \end{array}$$

shows that $E_\bullet$ is aspherical since the counit of the adjunction, ϵ, is a surjective map of simplicial groupoids when evaluated at each of them. Then, for the $K(\int_{\mathcal{G}} \mathcal{A}, n)$-torsor $\xi_\bullet : E_\bullet \to K(\int_{\mathcal{G}} \mathcal{A}, n)$, it is clear that $\Pi_0(E_\bullet \to \mathcal{G}) = \mathcal{G}$. As the fibre $Fib(E_\bullet)$ coincides with $E_\bullet$ until dimension $n-1$, as we pointed out above, $\Pi_0(Fib(E_\bullet)) = \mathcal{G}$ as well.

Let us study the homotopy groups of the fibre. In general, for any simplicial groupoid $\mathcal{K}_\bullet \to \mathcal{G}$ in $\mathbf{Gd}/\mathcal{G}$, its homotopy groups at each vertex $a \in O$ are defined to be the homotopy groups (always at the identity on a) of the simplicial group formed by the endomorphisms of a at each dimension, called $End_{\mathcal{K}_\bullet}(a)$. These groups can be computed as the homology groups of the corresponding Moore complex (see [18]), so

$$\Pi_q(\mathcal{K}_\bullet, a) = \{k + Imd_{q+1} \in End_{\mathcal{K}_q}(a); d_i k = Id_a, i = 0, \dots, q\}.$$

However, the $(n-1)$-th homotopy group is particulary simple when dealing with a $(n-1)$-hypergroupoid. In fact, as the Moore complex vanishes at dimensions greater than $n-1$, it is

$$\Pi_{n-1}(\mathcal{K}_\bullet. a) = \{k \in End_{\mathcal{K}_{n-1}}(a); d_i k = Id_a, i = 0, \dots, n-1\}.$$

One observes, then, that $Fib(E_\bullet)$ has homotopy at dimension $n-1$ since the $K(\int_{\mathcal{G}} \mathcal{A}, n)$-torsor is aspherical. In fact we prove that $\Pi_{n-1}(Fib(E_\bullet), a) = \mathcal{A}(a)$, for each vertex $a \in O$. Indeed, take any $x \in \Pi_{n-1}(Fib(E_\bullet), a)$ and note that $(Id_a, \stackrel{n)}{\dots}, Id_a, x) \in \Delta_n(E_\bullet)$. By the isomorphism $\gamma : \Delta_n(E_\bullet) \stackrel{\cong}{\to} \Lambda_n^n(E_\bullet) \times_{\mathcal{G}} \int_{\mathcal{G}} \mathcal{A}$ coming from the definition of n-torsor, one has associated to the given $x \in \Pi_{n-1}(Fib(E_\bullet). a)$ one and only one $(f, r) \in \int_{\mathcal{G}} \mathcal{A}$ such that

$$\gamma(Id_a, \stackrel{n)}{\dots}, Id_a, x) = ((Id_a, \dots, Id_a, -), (f, r))$$

Note that $r \in \mathcal{A}(a)$ since the belonging of $((Id_a, \dots, Id_a, -), (f, r))$ to $\Lambda_n^n(E_\bullet) \times_{\mathcal{G}} \int_{\mathcal{G}} \mathcal{A}$ determines that f is an arrow with source a. This determines the desired biyective correspondence $(x \mapsto r)$ so the result comes. $\square$

4 Generalizing Mac Lane and Whitehead's result

All the necessary work has been done in preceding sections, so we are ready to formulate and prove the announced generalization of Mac Lane and Whitehead's result.

Theorem 4.1. *For each choice of a groupoid $\mathcal{G}$, a right $\mathcal{G}$-module $\mathcal{A}$ and a cohomology class $k_{n+1} \in H^{n+1}(\mathcal{G}, \mathcal{A})$ there exists a non necesarily connected topological space X such that $\Pi_1(X) = \mathcal{G}$ and $\Pi_n(X) = \mathcal{A}$.*

Proof. In order to determine the required topological space from the given data, let us consider $\xi_\bullet : E_\bullet \to K(\int_{\mathcal{G}} \mathcal{A}, n)$, the $K(\int_{\mathcal{G}} \mathcal{A}, n)$-torsor associated to k_{n+1}, according to 3.3, which is in $\mathbf{Gd}/\mathcal{G}$.

The category **Gd** is related to that of simplicial sets, $Simp(\mathcal{S})$, by means of the classifying groupoid functor $\overline{W} : \mathbf{Gd} \to Simp(\mathcal{S})$ and its left adjoint, the loop groupoid functor, $\overline{G} : Simp(\mathcal{S}) \to \mathbf{Gd}$. We consider the corresponding adjunction between slice categories,

$$\overline{G} : Simp(\mathcal{S})/\overline{W}(\mathcal{G}) \rightleftarrows \mathbf{Gd}/\mathcal{G} : \overline{W} ,$$

It is well known that $\Pi_{i+1}(\overline{W}(\mathcal{K}_\bullet)) \cong \Pi_i(\mathcal{K}_\bullet)$, as homotopy functors, for all $i \geq 0$ and $\mathcal{K}_\bullet \in \mathbf{Gd}$. Then, the required homotopy type X is defined by

$$X = |\overline{W}(Fib(E_\bullet))|$$

since, by 3.6, one has

$$\begin{aligned} \Pi_1(X) &= \Pi_1(|\overline{W}(Fib(E_\bullet))|) \\ &\cong \Pi_1(\overline{W}(Fib(E_\bullet))) \\ &\cong \Pi_0(Fib(E_\bullet)) \\ &\cong \mathcal{G}, \end{aligned}$$

and, in the same way,

$$\begin{aligned} \Pi_n(X) &= \Pi_n(|\overline{W}(Fib(E_\bullet))|) \\ &\cong \Pi_n(\overline{W}(Fib(E_\bullet))) \\ &\cong \Pi_{n-1}(Fib(E_\bullet)) \\ &\cong \mathcal{A}, \end{aligned}$$

which shows that $\Pi_n(X)$ and $\mathcal{A}$ are isomorphic as $\mathcal{G}$-modules. □

The generalization of the converse problem, solved in the connected context by means of Postnikov invariants, leads us to recall some classical terminology in obstruction theory (see [16]). First, a plain generalization of the Moore complex functor for any simplicial groupoid in $\mathbf{Gd}/\mathcal{G}$, as $N_q(\mathcal{K}_\bullet) = \{f \in \mathcal{K}_q / d_i f = Id, 0 \leq i \leq q-1\}$, with $d_q : N_q(\mathcal{K}_\bullet) \to N_{q-1}(\mathcal{K}_\bullet)$ as differential operator.

Proposition 4.2. *Let $\mathcal{K}_\bullet \to \mathcal{G}$ be an augmented simplicial groupoid in* $\mathbf{Gd}/\mathcal{G}$. *Then the following statements are equivalent:*

1. *$\mathcal{K}_\bullet \to \mathcal{G}$ is an exact n-hypergroupoid.*
2. *It satisfies both*
   (a) $N_q(\mathcal{K}_\bullet) = 0$, for $q \geq n+1$, and
   (b) $\Pi_i(\mathcal{K}_\bullet, a) = 0$ for $0 < i < n$, at each vertex $a \in O$.

Proof. Using the generalization of the Moore complex functor for simplicial groupoids, it is straightforward to see the equivalence between being an n-hypergroupoid and having trivial Moore complex at dimensions greater than n. Hence, an exact n-hypergroupoid has trivial homotopy Π_0-functors at dimensions greater or equal than $n+1$.

Let us prove now the equivalence between exactness and $\Pi_i(\mathcal{K}_\bullet, a) = 0$ for $0 < i < n$ and $a \in O$, making use of the fact that each of the homotopy functors $\Pi_i(\mathcal{K}_\bullet) : \Pi_0(\mathcal{K}_\bullet) \to \mathbf{Ab}$ are defined at $a \in O$ to be the homotopy (at dimension q) of $End_{\mathcal{K}_\bullet}(a)$.

Then, for $k \in End_{\mathcal{K}_q}(a)$ such that $d_i k = Id_a$, $0 \leq i \leq q$ (i.e., $k + Imd_{q+1} \in \Pi_q(\mathcal{K}_\bullet, a)$) one may find $y \in \mathcal{K}_{q+1}$ such that $d_i y = k$, for $0 \leq i \leq q+1$, by applying asphericity to $(k, \ldots^{q+2)}, k) \in \Delta_{q+1}(\mathcal{K}_\bullet)$. Note that, as source and target maps commute with the face operators, y is an endomorphism of a as k is. Particulary $d_{q+1} y = k$, so that $k \in Imd_{q+1}$ and hence $\Pi_q(\mathcal{K}_\bullet)$ is trivial for $0 < q < n$ and all a.

Conversely, suppose $\mathcal{K}_\bullet \to \mathcal{G} \in \mathbf{Gd}$ such that $\Pi_q(\mathcal{K}_\bullet) = 0$ (at each vertex) for $0 < q < n$ and let us prove that $D_q = < d_0, \ldots, d_q >: \mathcal{K}_q \to \Delta_q(\mathcal{K}_\bullet)$ is surjective, $0 \le q \le n$. First, we find a preimage for those elements of $\Delta_q(\mathcal{K}_\bullet)$ which are of the form $(Id_a, Id_a, \ldots, Id_a, k)$: as one of these is equivalent to giving $k + Imd_q \in \Pi_{q-1}(\mathcal{K}_\bullet, a) = 0$, there exists $y \in N_q(\mathcal{K}_\bullet)$ such that $d_q y = k$ and so, y is the required preimage. Let us consider then any $(k_0, \ldots, k_{q-1}, k_q) \in \Delta_q(\mathcal{K}_\bullet)$ and take the corresponding element in the qth-groupoid of open horns, $(k_0, \ldots, k_{q-1}, -) \in \Lambda^q_q(\mathcal{K}_\bullet)$.

Since each simplicial groupoid is a Kan complex, there exists $y \in \mathcal{K}_q$ such that $d_i y = k_i$ for $i = 0, \ldots, q-1$. Now it is a matter of plain calculations to check out that $(Id, Id, \ldots, Id, d_q y^{-1} k_q)$ belongs to $\Delta_q(\mathcal{K}_\bullet)$ and so there is $y' \in \mathcal{K}_q$ such that $d_i y' = Id$ for $i = 0, \ldots, q-1$ and $d_q y' = d_q y^{-1} k_q$. Then, the required preimage for $(k_0, \ldots, k_{q-1}, k_q)$ is yy'. □

Beyond the former characterization, the non-trivial homotopy Π_0-functors of an exact n-hypergroupoid just amount to Π_n, called its *center*. Also note that $\Pi_0(\mathcal{K}_\bullet) = \mathcal{G}$, for any exact n-hypergroupoid in $\mathbf{Gd}/\mathcal{G}$, by the exactness.

The assignement of a cohomology class in $H^n(\Pi_1(X), \Pi_n(X))$ for any given topological space X can be performed in two parts. First, in a similar fashion to that used by Eilenberg and Mac Lane in defining the classical obstruction mapping, there is a map

$$Obs : \{\text{exact}\,(n-1) - \text{hypergroupoid with center}\,\mathcal{A}\} \to H^{n+1}(\mathcal{G}, \mathcal{A})$$

defined as follows: let us take any exact $(n-1)$-hypergroupoid with center $\mathcal{A}$, say $M_\bullet$. By appliying 3.5, consider the $K(A_{M_\bullet}, n)$-torsor (U-split) which $M_\bullet$ is the fibre of. Then, by 3.3, the correspondence to $H^{n+1}(\mathcal{G}, \mathcal{A})$ comes.

In order to complete the assignement of an $(n-1)$-dimensional abstract kernel with center $\mathcal{A}$ for every topological space X with $\Pi_n(X)$ as only non-trivial $\Pi_1(X)$-module, let us consider the following sequence of adjunctions:

$$Cat^{n-1}(\mathbf{Gpd}_1) \underset{\mathbb{N}}{\overset{\mathbb{P}}{\rightleftarrows}} Gd^{n-1} \underset{\mathbb{D}}{\overset{\mathbb{T}_{n-1}}{\rightleftarrows}} \mathbf{Gd} \underset{\overline{W}}{\overset{\mathcal{G}}{\rightleftarrows}} Simp(\mathcal{S})$$

where $Cat^n(\mathbf{Gpd}_1)$ stands for the category of cat^n-groupoids, that is, a suitable generalization of that of cat^n-groups defined by Loday, [15], and $\mathbf{Gd}^n$ is the category formed by simplicial n-groupoids, i.e., those objects having n independent simplicial structures in each of the coordinate directions. These two categories are related by a multinerve functor, $\mathbb{N}$, and its left adjoint $\mathbb{P}$, which reproduces the fundamental groupoid construction at each direction.

The second adjunction is set by the diagonal functor and its left adjoint, $\mathbb{T}_n$, defined as a generalization of the Illusie's functor Total Dec, (for further details, see [6]).

Moving onto the corresponding slice categories, the second part of the required assignement is given in the following

Proposition 4.3. *Given a topological space X with $\Pi_n(X)$ as unique non-trivial homotopy $\Pi_1(X)$-functor, then $M_\bullet(X) = \mathbb{DNPT}_{n-1}\mathcal{G}Sing(X)$ is an exact (n-1)-hypergroupoid.*

Proof. To this end, we have to check out the conditions given in 4.2. In effect, it has vanishing Moore complex at dimensions greater than or equal to n ([7], Theorem 1.3). On the other hand, let us prove that $\Pi_i(M_\bullet(X)) = 0$ for each vertex and $0 < i < n$. In fact,

$$\begin{aligned} \Pi_i(X) &= \Pi_i(|\overline{W}(M_\bullet(X))|) \\ &\cong \Pi_i(\overline{W}(M_\bullet(X)) \\ &\cong \Pi_{i-1}(M_\bullet(X)) \end{aligned}$$

□

The result anounced at the beginning of the paper has been finally proved:

Theorem 4.4. *The homotopy type of a space X with fundamental groupoid $\Pi_1(X)$ and a unique non-trivial homotopy functor $\Pi_n(X) : \Pi_1(X) \to \mathbf{Ab}$ is completely determined by $\Pi_1(X)$, $\Pi_n(X)$ and a cohomology class $k_{n+1} \in H^{n+1}(\Pi_1(X), \Pi_n(X))$.* $\square$

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