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J. G. CABELLO

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SIMPLICIAL GROUPS AS MODELS FOR *n*-TYPES

by J. G. CABELLO

RESUME. On sait que la catégorie des groupes simpliciaux **Simp(Gp)** fournit des modèles pour tous les types d'espaces connexes. C'est aussi un exemple de catégorie-modèle pour la théorie de l'homotopie. Dans cet article, on définit une nouvelle structure de catégorie-modèle sur **Simp(Gp)** pour chaque $n > 0$ (n-structure), redonnant la version classique lorsque n tend vers l'infini. On démontre que **Simp(Gp)** muni de cette structure est un modèle pour les $(n+1)$-types d'espaces connexes, non seulement au niveau des catégories d'homotopie mais aussi au niveau des théories d'homotopie.

1 Introduction

As proved by Kan, [15], the category of all simplicial groups, **Simp(Gp)**, provides algebraic models for all connected homotopy types. **Simp(Gp)** is also an outstanding example of what a closed model category is (in the sense of Quillen), [18], that is, the weak equivalences -those morphisms which are formally inverted to produce the homotopy category- are accompanied in this case by other two types of relevant morphisms, fibrations and cofibrations, subordinated all of them to a set of axioms. The consequence is that one can reproduce in **Simp(Gp)** -in any closed model category in general- analogues to the classical contructions of homotopy theory of spaces since the homotopy category is extended by the notion of *homotopy theory associated to a closed model category*. This includes the extra structure of loop and suspension functors and fibrations and cofibration sequences.

On the other hand, one has the notion of n-type, first studied by J.H.C. Whitehead, [20], (his "homotopy systems", now generalized by crossed complexes, were a partial solution to the problem of finding an algebraic model for this notion). Since then, several proposals have been given in this direction. One of them is the category $\mathbf{n-HXC(Gp)}$ of n-hypercrossed complexes of groups, a model of n-types of simplicial groups and, consequently, of $(n+1)$-types of connected spaces, [5]. This category consists of certain complexes of non-abelian groups and reduces in the low dimensional levels to the well-known cases of 2-types (crossed modules) and 3-types (2-crossed modules).

The results of this paper are part of the author's Ph.D. thesis, [2], where a standard procedure is introduced to supply any category, suitably related to **Simp(Gp)**, with a closed model structure. It is proved that this procedure particularly gives a closed model structure for the category $\mathbf{n-HXC(Gp)}$. It is possible, then, to regard **Simp(Gp)** as a candidate for modelling $(n+1)$-types of connected spaces as well (or n-types of simplicial groups). In this paper, this is proved by showing that the Quillen's closed model structure on **Simp(Gp)** can be generalized by notions of *n-weak equivalence, n-fibration* and *n-cofibration*, producing several model structures (one for each n, called the *n-structure*). First, one observes that when n runs to infinite, the original Quillen's structure on **Simp(Gp)** is recovered, satisfying our prime purpose. Note that our definition of n-weak equivalences as a truncated version of classical weak equivalences (see 3.6) is the most natural when dealing with n-types. This fact, together with the expression of Kan fibrations in terms of the RLP, suggested the possibility of truncating them as well (level by level) in order to reach the n-fibrations (compare 2.1 with 3.2). This is also the spirit of [14], where something similar was proposed for simplicial sets.

The comparison between both structures (the n-structure and the classical one), is made by means of the pair of adjoint functors $(n+1)-skeleton$ and $(n+1)-coskeleton$, denoted by Sk^{n+1} and $Cosk^{n+1}$. This gives rise to an equivalence of categories

$$Ho_n(Simp(Gp)) \simeq Ho(Simp(Gp)|n-cc)$$

between the localization of **Simp(Gp)** with respect to the n-weak equivalences (on the left) and the full subcategory of the homotopy category $Ho(\mathbf{Simp(Gp)})$ whose objects are those simplicial groups n-coconnected, that is, with vanishing homotopy groups in dimensions greater than or equal to $n+1$, $\Pi_i = 0$ for $i \geq n+1$ (on the right). This shows **Simp(Gp)** as a category of n-types of simplicial groups.

As the category $\mathbf{n - HXC(Gp)}$ has been converted into a closed model category ([2], [3]) it is also reasonable to think of an equivalence between the corresponding homotopy theories. This is achieved by considering in **Simp(Gp)** this n-model structure.

Let us mention that many other categories which, in principle, supported a Quillen's closed model structure (topological spaces, simplicial sets, [18], crossed complexes, [1]) have also been equipped with such an n-structure, generalizing the original one, [9], [10], [14]. In addition, the category of sheaves has been endowed recently with a closed model structure, [6], by a procedure similar to that proposed here.

The plan of this paper is as follows. Next section is devoted to the revision of the set of axioms most frequently used to define a closed model category, together with some other known results of interest for this paper. Section 3 introduces n-weak equivalences, n-fibrations and n-cofibrations for simplicial groups and gives some characterizations of these notions, relevant for the sequel. The detailed proof of the main result, showing that the above definitions lead to the n-closed model structure, is given in section 4. Finally, the last section is left to the conclusions: the comparison of both model structures in **Simp(Gp)**, the classical and the n-structure, is made, obtaining the desired equivalence of homotopy theories.

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2 Preliminaires

Through all this paper **Simp(Gp)** will denote the category of simplicial groups, i.e., the category of functors $Gp^{\Delta^{op}}$ where Δ is the category whose objects are the ordered sets $[0] = \{0\}$, $[1] = \{0, 1\}$, $[2] = \{0, 1, 2\}$, ... , and whose morphisms are the order-preserving functions between them.

Recall that the n-th simplicial kernel of an $(n-1)$-truncated simplicial group $G_{\bullet_{tr}}$, written $\Delta^n(G_{\bullet_{tr}})$, is the subgroup of G_{n-1}^{n+1} whose elements are those $(x_0, x_1, \dots, x_n)$ such that $d_i x_j = d_{j-1} x_i$ for $i < j$. If $d_i : \Delta^n(G_{\bullet_{tr}}) \to G_{n-1}$ denotes the restriction of the canonical projections, there are unique morphisms $s_j : G_{n-1} \to \Delta^n(G_{\bullet_{tr}})$, for $0 \le j \le n-1$ such that

$$\Delta^n(G_{\bullet_{tr}}) \underset{d_n}{\overset{d_0}{\rightrightarrows}} G_{n-1} \underset{d_{n-1}}{\overset{d_0}{\rightrightarrows}} \cdots\cdots G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0$$

is an n-truncated simplicial group. By iterating the simplicial kernel construction one has the functor $cosk^{n-1}$ from the category of $(n-1)$-truncated simplicial groups to **Simp(Gp)**. The resulting functor $cosk^{n-1}$ is right adjoint to the $(n-1)$-truncating functor, tr^{n-1}, that truncates any simplicial group at level $n-1$. By convention, $Cosk^{n-1} = cosk^{n-1} tr^{n-1}$ and $\Delta^n(G_\bullet) = \Delta^n(tr^{n-1}(G_\bullet))$.

On the other hand, given any truncated simplicial group,

$$G_{\bullet_{tr}} : \qquad G_{n-1} \underset{d_{n-1}}{\overset{d_0}{\rightrightarrows}} \cdots\cdots G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0$$

its nth-simplicial cokernel $\nabla(G_{\bullet_{tr}})$ is defined as the quotient of the coproduct $\sqcup_{i=0}^{n-1} G_{n-1}$, i.e., free product in **Gp**, under the congruence generated by the elements $(s_i s_j x, s_{j-1} s_i x), x \in G_{n-2}$ for $0 \le i < j \le n-2$, where $s_j : G_{n-1} \to \nabla(G_{\bullet_{tr}})$ are the morphisms induced by the canonical inclusions. The $(n-1)$-skeleton of the $(n-1)$-truncated simplicial group $G_{\bullet_{tr}}$, denoted by $sk^{n-1}(G_{\bullet_{tr}})$ may be defined by iteration of successive simplicial cokernels. Writing $Sk^{n-1} = sk^{n-1} tr^{n-1}$, one has the following adjunction

$$Sk^{n-1} : \mathbf{Simp(Gp)} \rightleftarrows \mathbf{Simp(Gp)} : Cosk^{n-1}, \qquad Sk^{n-1} \dashv Cosk^{n-1}$$

The same adjunction is known to exist for simplicial sets, which allows us to identify, for any simplicial set $X_\bullet$, simplicial maps $\dot{\Delta}[n] = Sk^{n-1}\Delta[n] \to X_\bullet$ with simplicial maps $\Delta[n] \to Cosk^{n-1}(X_\bullet)$ and so with elements of $\Delta^n(X_\bullet)$ (see [7], prop. 1.5).

Given any $G_\bullet \in$ **Simp(Gp)**, $\Lambda_i^n(G_\bullet)$ will stand for the universal group having morphisms $d_j : \Lambda_i^n(G_\bullet) \to G_{n-1}$, for $0 \leq j \leq n, j \neq i$ satisfying $d_j d_k = d_{k-1} d_j, j < k, k \neq i$, which is called the object of open i-horns at dimension n. If $\Delta[n,k], 0 \leq k \leq n > 0$, denotes the simplicial subset of $\Delta[n]$ generated by $d_i(i_n), 0 \leq i \leq n, i \neq k$, (where $i_n = (0,1,\dots,n) \in \Delta[n]_n$, see [7]), note the equivalence between giving an element of $\Lambda_k^n(G_\bullet)$ and giving a simplicial map $\Delta[n,k] \to G_\bullet$. A morphism in **Simp(Gp)**, $f_\bullet : E_\bullet \to B_\bullet$, is said to be a Kan fibration iff the canonical morphism

$$E_s \to B_s \times_{\Lambda_k^s(B_\bullet)} \Lambda_k^s(E_\bullet)$$

is surjective, for $0 \leq k \leq s, s > 0$.

Let us recall as well that the Moore complex $N(G_\bullet)$ associated to any simplicial group, $G_\bullet$, [18], is the following

$$N(G_\bullet) : \dots N_q(G_\bullet) \xrightarrow{\delta_q} N_{q-1}(G_\bullet) \longrightarrow \cdots \longrightarrow N_1(G_\bullet) \xrightarrow{\delta_1} N_0(G_\bullet)$$

where $N_0(G_\bullet) = G_0$ and $N_q(G_\bullet) = \cap_{i=0}^{q-1} Ker d_i \subseteq G_q$ and δ_q is the restriction of $d_q : G_q \to G_{q-1}$ to $N_q(G_\bullet)$. For later use, let us recall that the Moore complex of the simplicial group $Cosk^{n-1}(G_\bullet)$ is the following:

$$N_q(Cosk^{n-1}(G_\bullet)) = \begin{cases} N_q(G_\bullet) & \text{if } 0 \leq q \leq n-1 \\ Ker(\delta_{n-1} : N_{n-1}(G_\bullet) \to N_{n-2}(G_\bullet)) & \text{if } q = n \\ 0 & \text{otherwise} \end{cases}$$

The (Moore) homotopy groups of any simplicial group $G_\bullet$ are defined then as the corresponding homology groups of its Moore complex, that is, for $n \geq 0$

$$\Pi_n(G_\bullet) = \frac{\cap_{i=0}^{n} Ker(d_i : G_n \to G_{n-1})}{d_{n+1}(\cap_{i=0}^{n} Ker(d_i : G_{n+1} \to G_n))}$$

A closed model category in the sense of Quillen, [19], is a category $\mathcal{C}$ with three distinguished classes of morphisms called fibrations, cofibrations and weak equivalences, satisfying the usual axioms,

CM1. The category $\mathcal{C}$ has finite limits and colimits.

CM2. For any composable pair of arrows f, g in $\mathcal{C}$, if any two of f, g, fg are weak equivalences so is the third.

CM3. If f is a retract of g and g is a fibration, cofibration or weak equivalence, then f is also such, where one says that a morphism $f : X \to Y$ is a retract of $g : W \to Z$ if there are morphisms $i : X \to W, r : W \to X, j : Y \to Z$ and $s : Z \to Y$ such that $ri = Id_X, sj = Id_Y, gi = jf$ and $fr = sg$.

CM4. Lifting axiom. Given a solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} \qquad (0)$$

where i is a cofibration, p is a fibration and either i or p is also a weak equivalence, then the dotted arrow exists.

CM5. Factorization axiom. Any morphism f in $\mathcal{C}$ can be factored both as $f = pi$ and $f = qj$, where p and q are fibrations, i, j are cofibrations and p, j are weak equivalences.

This set of axioms is, of course, equivalent to the original (see [18]). A morphism in $\mathcal{C}$, $i : A \to B$, is said to have the *left lifting property*, LLP, with respect to a morphism $p : X \to Y$ (and p is said to have the *right lifting property*, RLP, with respect to i) if the dotted arrow exists in any diagram of the form (0).

Finally, let us recall that the Quillen's closed model structure over **Simp(Gp)** described in [18], has the Kan fibrations as fibrations, those morphisms which induce isomorphisms between the homotopy groups as weak equivalences and the cofibrations are defined by the LLP with respect to trivial fibrations (fibrations and weak equivalences). By means of the free group functor F : **Simp(Sets)** $\to$ **Simp(Gp)** from the category of all simplicial sets, (trivial) fibrations in **Simp(Gp)** can be characterized as follows:

Proposition 2.1 (**[18], Prop. 1.2**). *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of simplicial groups. Then,*

1. *$f_\bullet$ is a fibration iff it has the RLP with respect to the morphisms $F\Delta[n,k] \to F\Delta[n]$, induced by the inclusions $\Delta[n,k] \hookrightarrow \Delta[n]$, for $0 \leq k \leq n, n > 0$.*

2. *$f_\bullet$ is a trivial fibration iff it has the RLP with respect to the family of morphisms $F\dot{\Delta}[n] \to F\Delta[n]$, induced by the inclusions $\dot{\Delta}[n] \hookrightarrow \Delta[n]$, for all $n \geq 0$.*

3 Truncating the closed model structure in Simp(Gp)

Definition 3.1. *A morphism $p_\bullet : E_\bullet \to B_\bullet$ in* **Simp(Gp)** *is said to be a truncated Kan fibration at dimension n if it satisfies the Kan condition up to dimension n, that is, the canonical morphism*

$$E_s \to B_s \times_{\Lambda^s_k(B_\bullet)} \Lambda^s_k(E_\bullet)$$

is surjective for $0 \leq k \leq s$ and $0 < s \leq n$. A morphism $p_\bullet : E_\bullet \to B_\bullet$ in **Simp(Gp)** *is said to be an n-fibration, $n \geq 1$, if $p_\bullet$ is a truncated Kan fibration at dimension $n+2$ and the induced morphism on the $(n+1)$-th homotopy groups, $\overline{p}_{n+1} : \Pi_{n+1}(E_\bullet) \to \Pi_{n+1}(B_\bullet)$, is surjective.*

Hence, one observes that an ∞-fibration in **Simp(Gp)** is exactly a Kan fibration, retrieving so the classical definition. This can also be observed from the following characterization of n-fibrations in terms of the RLP:

Proposition 3.2. *For any morphism $p_\bullet : E_\bullet \to B_\bullet$ in* **Simp(Gp)** *the following are equivalent:*

1. *$p_\bullet$ is an n-fibration.*

2. *$p_\bullet$ has the RLP with respect to the following two families of morphisms in* **Simp(Sets)***:*

$$* \to S^{n+1} = \frac{\Delta[n+1]}{\dot{\Delta}[n+1]}$$

and

$$\Delta[s,k] \hookrightarrow \Delta[s],\ 0 \le k \le s, 0 < s \le n+2,$$

or equivalently with respect to the induced families of morphisms of simplicial groups:

$$F* \to FS^{n+1} \text{ and } F\Delta[s,k] \to F\Delta[s], 0 \le k \le s, 0 < s \le n+2,$$

Proof. In order to show the existence of lifting $D_\bullet$ in any given commutative square

$$\begin{array}{ccc} \Delta[s,k] & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow & \overset{D_\bullet}{\nearrow} & \downarrow p_\bullet \\ \Delta[s] & \xrightarrow{b_\bullet} & B_\bullet \end{array} \qquad (1)$$

with $0 \le k \le s, 0 < s \le n+2$, first one applies the condition of Kan fibration at dimension s to the element $(y,(x_0,\dots,x_{k-1},x_{k+1},\dots,x_s)) \in B_s \times_{\Lambda_k^s(B_\bullet)} \Lambda_k^s(E_\bullet)$, where $(x_0,\dots,x_{k-1},x_{k+1},\dots,x_s)) \in \Lambda_k^s(E_\bullet)$ is determined by the simplicial map $a_\bullet$, and $y \in B_s$ is the s-simplex represented by $b_\bullet$. The analogy in properties between simplicial maps coming from $\Delta[s]$ and s-simplices of the target -as remarked in section 2- is again present since the element $(y,(x_0,\dots,x_{k-1},x_{k+1},\dots,x_s))$ belongs to the fiber product $B_s \times_{\Lambda_k^s(B_\bullet)} \Lambda_k^s(E_\bullet)$, because of the commutativity of (1). As $p_\bullet$ is a truncated Kan fibration at dimension $n+2$ there exists $x \in E_s$ such that $d_i x = x_i$, for $i \neq k$ and $p_s(x) = y$. The required lifting for (1), $D_\bullet$, is then the representing map of $x \in E_s$. As for the RLP with respect to $* \to S^{n+1}$, consider any commutative square of type

$$\begin{array}{ccc} * & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow & \overset{d_\bullet}{\nearrow} & \downarrow p_\bullet \\ S^{n+1} & \xrightarrow{b_\bullet} & B_\bullet \end{array} \qquad (2)$$

and take $y \in B_{n+1}$ represented by the simplicial map $b_\bullet$. As it happened above, $d_i y = e$, $0 \le i \le n+1$ because of the commutativity of the diagram (2). For the surjectivity of $\overline{p}_{n+1}$ there is then $\overline{x} \in \Pi_{n+1}(E_\bullet)$ such that $\overline{p}_{n+1}(\overline{x}) = \overline{y}$. Let $z \in B_{n+2}$ the homotopy between $p_{n+1}(x)$ and y, i.e., z holds $d_{n+2}z = y, d_{n+1}z = p_{n+1}(x)$ and $d_i z = s_n d_i p_{n+1}(x) = s_n d_i y$, for $i = 1, \ldots, n$. Using then the condition of $p_\bullet$ being a Kan fibration at dimension $n+2$ for the element $(z, (x_0, \ldots, x_k, x_{k+1}, \ldots, x_{n+1})) \in B_{n+2} \times_{\Lambda^{n+2}_{n+2}(B_\bullet)} \Lambda^{n+2}_{n+2}(E_\bullet)$ where $x_j = s_n d_j x, j = 0, \ldots, n, x_{n+1} = x$, and taking into consideration that z verifies that $d_i z = p_{n+1}(x_i)$ for $i \neq n+2$, there exists $x' \in E_{n+2}$ such that $d_i x' = x_i, i \neq n+2$ and $p_{n+2}(x') = z$.

Note that $d_{n+2}x' \in E_{n+1}$ verifies $d_i d_{n+2} x' = e, i = 0, \ldots, n+1$ and $p_{n+1}(d_{n+2}x') = y$; hence, the required lifting for (2), $d_\bullet$, is defined as the representing map of $d_{n+2}x'$. That proves the first half.

As for the opposite direction, it should be noted that if $p_\bullet$ has the RLP with respect to $\Delta[s,k] \hookrightarrow \Delta[s]$ for $0 \leq k \leq s, 0 < s \leq n+2$, then it is a truncated Kan fibration at dimension $n+2$ just by reversing the same argument given for the first implication. Then, it only remains to show that $\overline{p}_{n+1} : \Pi_{n+1}(E_\bullet) \to \Pi_{n+1}(B_\bullet)$ is surjective. To this end, take $y \in B_{n+1}$ with $d_i y = e, i = 0, \ldots, n+1$ a representative of any $\overline{y} \in \Pi_{n+1}(B_\bullet)$ and consider the following square:

$$\begin{array}{ccc} * & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow & \overset{D_\bullet}{\nearrow} & \downarrow p_\bullet \\ S^{n+1} & \xrightarrow{b_\bullet} & B_\bullet \end{array}$$

where $a_\bullet$ is the unique simplicial map from $*$ and $b_\bullet$ is the representing map of y. This diagram commutes owing to $d_i y = e, i = 0, \ldots, n+1$ and so there is a lifting $D_\bullet : S^{n+1} \to E_\bullet$. If $d_\bullet$ is the composition $\Delta[n+1] \longrightarrow S^{n+1} \xrightarrow{D_\bullet} E_\bullet$, it represents a simplex $x \in E_{n+1}$ verifying $d_i x = e, i = 0, \ldots, n+1$ and $p_{n+1}(x) = y$. Then, $\overline{x} \in \Pi_{n+1}(E_\bullet)$ is the desired preimage which shows $\overline{p}_{n+1}$ to be surjective. □

The following lemma summarizes truncated versions of some results due to Quillen ([18]):

Lemma 3.3. *Let $f_\bullet : G_\bullet \to H_\bullet$ be a morphism in* **Simp(Gp)**. *Then,*

1. *$f_q : G_q \to H_q$ is surjective (injective) for $0 \leq q \leq n$ iff $N_q(f) : N_q(G_\bullet) \to N_q(H_\bullet)$ is surjective (injective) for $0 \leq q \leq n$.*

2. *$f_\bullet$ is a truncated Kan fibration at dimension n iff $N_q(f) : N_q(G_\bullet) \to N_q(H_\bullet)$ is surjective for $0 < q \leq n$.*

3. *$f_q : G_q \to H_q$ is surjective for $0 \leq q \leq n$ iff $f_\bullet$ is a truncated Kan fibration at dimension n and the induced morphism on the 0-th homotopy group, $\overline{f}_0 : \Pi_0(G_\bullet) \to \Pi_0(H_\bullet)$ is surjective.*

After the previous result, the n-fibrations can be also characterized as follows:

Corollary 3.4. *For any morphism $p_\bullet : E_\bullet \to B_\bullet$ in* **Simp(Gp)** *the following are equivalent:*

1. *$p_\bullet$ is an n-fibration.*

2. *$N_q(p_\bullet)$ is surjective, $0 < q \leq n+2$ and $\overline{p}_{n+1} : \Pi_{n+1}(E_\bullet) \to \Pi_{n+1}(B_\bullet)$ is surjective.*

As a direct consequence of this, the n-fibrant simplicial groups are identified:

Corollary 3.5. *Every object $G_\bullet$ in* **Simp(Gp)** *is n-fibrant, $n > 0$, i.e., the morphism $G_\bullet \to *$ is an n-fibration.*

Next, other kind of relevant morphisms for the future closed model n-structure.

Definition 3.6. *A morphism $f_\bullet : X_\bullet \to Y_\bullet$ in* **Simp(Gp)** *is said to be an n-weak equivalence, $n > 0$, if the induced morphism on the homotopy groups, $\overline{f}_q : \Pi_q(X_\bullet) \to \Pi_q(Y_\bullet), 0 \leq q \leq n$, is an isomorphism.*

The morphism $f_\bullet$ will be called an n-trivial fibration if it is an n-fibration and an n-weak equivalence.

Formulation and proof of the corresponding characterization in terms of the RLP can be given now for n-trivial fibrations.

Proposition 3.7. *For any morphism $p_\bullet : E_\bullet \to B_\bullet$ in* **Simp(Gp)** *the following are equivalent:*

1. *$p_\bullet$ is an n-trivial fibration.*

2. *$p_\bullet$ has the RLP with respect to the following two families of maps in* **Simp(Sets)***:*

$$\dot{\Delta}[s] \hookrightarrow \Delta[s],\ 0 \leq s \leq n+1$$

and

$$\Delta[n+2,k] \hookrightarrow \Delta[n+2],\ 0 \leq k \leq n+2,$$

or equivalently with respect to the induced families of morphisms of simplicial groups:

$$F\dot{\Delta}[s] \to F\Delta[s],\ 0 \leq s \leq n+1$$

and

$$F\Delta[n+2,k] \to F\Delta[n+2],\ 0 \leq k \leq n+2,$$

Proof. Let us prove the first direction. Note that, by 3.2, $p_\bullet$ has the RLP with respect to $\Delta[n+2,k] \hookrightarrow \Delta[n+2]$ particularly. Hence, it remains to show the existence of lifting in commutative squares such as

$$\begin{array}{ccc} \dot{\Delta}[s] & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow & \nearrow & \downarrow{\scriptstyle p_\bullet} \\ \Delta[s] & \xrightarrow{b_\bullet} & B_\bullet \end{array} \qquad (3)$$

For $s = 0$, it can be observed that the existence of lifting in (3) is equivalent to p_0 being surjective. To this end, take any $y \in B_0$ and consider its homotopy class into $\Pi_0(B_\bullet)$, $\overline{y}$. As $\overline{p}_0 : \Pi_0(E_\bullet) \to \Pi_0(B_\bullet)$ is surjective (even more, it is an isomorphism), there exists $\overline{x} \in \Pi_0(E_\bullet)$ such that $\overline{p}_0(\overline{x}) = \overline{y}$. Let z be the corresponding homotopy between them, i.e., $d_0 z = p_0(x)$ and $d_1 z = y$ and then, since $p_\bullet$ is an 1-truncated Kan fibration, there is $x' \in E_1$ such that $d_0 x' = x$ and $p_1(x') = z$. The element $d_1 x'$ is then the required preimage for y.

In order to prove that there exists a lifting in (3), $0 < s \leq n+1$, let $y \in B_s$ be the simplex with representing morphism $b_\bullet$ and $(x_0, \dots, x_{s-1}, x_s)$ the element of $\Delta^n(E_\bullet)$ determined by $a_\bullet$.

As usual, the commutativity of (3) establishes $(y, (x_0, \dots, x_{s-1}))$ as an element of $B_s \times_{\Lambda^s_s(B_\bullet)} \Lambda^s_s(E_\bullet)$. Now, $p_\bullet$ is a truncated Kan fibration at dimension $n+2$ so there exists $\alpha \in E_s$ such that $d_i\alpha = x_i, i \neq s$ and $p_s(\alpha) = y$. It is not difficult to confirm that $\overline{x_s - d_s\alpha} \in Ker\overline{p}_{s-1}$, which is particularly injective, so $\overline{x_s - d_s\alpha} = \overline{e}$. Let us call α' to the homotopy between $x_s - d_s\alpha$ and e, i.e., $d_s\alpha' = x_s - d_s\alpha$ and $d_i\alpha' = e, i = 0, \dots, s-1$, and consider the s-simplex of $E_\bullet$, $\alpha'' = \alpha' + \alpha$, which verifies $d_i\alpha'' = x_i$ and $d_i(p_s(\alpha'') - y) = e, i = 0, \dots, s$. That allows one to consider $\overline{p_s(\alpha'') - y} \in \Pi_s(B_\bullet)$ and, since $\overline{p}_s$ is an epimorphism for $0 < s \leq n+1$ (an isomorphism up to dimension n, indeed), there exists $\overline{z} \in \Pi_s(E_\bullet)$ such that $\overline{p}_s(\overline{z}) = \overline{p_s(\alpha'') - y}$.

Let $w \in B_{s+1}$ be the homotopy between $p_s(\alpha'') - y$ and $p_s(z)$ and apply again the condition of Kan fibration at dimension $s+1$ to the element $(w, (e, \overset{s)}{\dots}, e, z)) \in B_s \times_{\Lambda^{s+1}_{s+1}(B_\bullet)} \Lambda^{s+1}_{s+1}(E_\bullet)$ to find $\lambda \in E_{s+1}$ which holds $d_i\lambda = e, i = 0, \dots, s-1$, $d_s\lambda = z$ and $p_{s+1}(\lambda) = w$.

Finally, the required lifting for (3) is defined as the representing morphism of $x = -d_{s+1}\lambda + \alpha''$.

The converse will be proved in two steps:

First, it will be showed that if $p_\bullet$ has the RLP with respect to $\dot{\Delta}[s] \hookrightarrow \Delta[s]$, for $0 \leq s \leq n+1$ then $\overline{p}_s$ is surjective and $\overline{p}_{s-1}$ is injective. That leads to $\overline{p}_{n+1}$ surjective and $\overline{p}_i$ isomorphism for $0 \leq i \leq n$ (particularly, $p_\bullet$ will be an n-weak equivalence).

In order to prove that $\overline{p}_s$ is surjective, take any $\overline{y} \in \Pi_s(B_\bullet)$ and consider the following square

$$\begin{array}{ccc} \dot{\Delta}[s] & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow & \overset{D_\bullet}{\nearrow} & \downarrow p_\bullet \\ \Delta[s] & \xrightarrow{b_\bullet} & B_\bullet \end{array}$$

constructed by the representing map $b_\bullet$ of any representative element y of $\overline{y}$, and the simplicial map $a_\bullet$ defined by the element of the simplicial kernel $(e, \overset{s+1)}{\dots}, e)$. It is clear that this square commutes so there exists a lifting $D_\bullet : \Delta[s] \to E_\bullet$. The homotopy class of the s-simplex of $E_\bullet$ represented by $D_\bullet$ proves then that $\overline{p}_s$ is surjective, $0 \leq s \leq n+1$.

To prove now that $\overline{p}_{s-1}$ is injective, take any $\overline{x} \in Ker\overline{p}_{s-1}$ and let $w \in B_s$ be the homotopy between $p_{s-1}(x)$ and e (so $d_sw = p_{s-1}(x)$ and $d_iw = e$ for $i = 0, \dots, s-1$). With the representing map of w, say $b'_\bullet$, and that of the element of the simplicial kernel $(e, \overset{s)}{\ldots}, e, x)$, written $a'_\bullet$, the following commutative square can be constructed for $0 \leq s \leq n+1$,

$$\begin{array}{ccc} \dot{\Delta}[s] & \xrightarrow{a'_\bullet} & E_\bullet \\ \downarrow & D'_\bullet \nearrow & \downarrow p_\bullet \\ \Delta[s] & \xrightarrow{b'_\bullet} & B_\bullet \end{array}$$

where the lifting $D'_\bullet$, that there exists by applying the hypothesis, determines the homotopy which shows that $\overline{x} = \overline{e}$. Hence, $\overline{p}_{s-1}$ is injective, $0 \leq s \leq n+1$.

The second step consists of showing that to have the RPL with respect to $\dot{\Delta}[s] \hookrightarrow \Delta[s]$, for $0 \leq s \leq n+1$ implies that $p_\bullet$ is a truncated Kan fibration at dimension $n+1$. To this end, take any element $(y, (x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_s)) \in B_s \times_{\Lambda^s_k(B_\bullet)} \Lambda^s_k(E_\bullet)$, $, 0 \leq k \leq s, 0 \leq s \leq n+1$, and define the following collection of s $(s-2)$-simplices of $E_\bullet$:

$$z_i = \begin{cases} d_{k-1}x_i & if \quad i \leq k-1 \\ d_k x_{i+1} & if \quad i > k-1 \end{cases}$$

It appears clear that $(z_0, \ldots, z_{k-1}, z_{k+1}, \ldots, z_s) \in \Delta^{s-1}(E_\bullet)$. Then, considering the simplicial map $c_\bullet : \dot{\Delta}[s-1] \to E_\bullet$ associated to this, and the representing map of $d_k y \in B_{s-1}, d_\bullet : \Delta[s-1] \to B_\bullet$, one may form the following commutative square

$$\begin{array}{ccc} \dot{\Delta}[s-1] & \xrightarrow{c_\bullet} & E_\bullet \\ \downarrow & H_\bullet \nearrow & \downarrow p_\bullet \\ \Delta[s-1] & \xrightarrow{d_\bullet} & B_\bullet \end{array}$$

for which there is a lifting $H_\bullet : \Delta[s-1] \to E_\bullet$. Let x_k be the $(s-1)$-simplex of $E_\bullet$ with representing map $H_\bullet$ and note that the element $(x_0, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_s) \in \Delta^s(E_\bullet)$. Since $d_i y = p_{s-1}(x_i)$ for $i = 0, \ldots, k, \ldots, s$, one may find a lifting $G_\bullet$ on the square

$$\begin{array}{ccc} \dot{\Delta}[s] & \xrightarrow{c'_\bullet} & E_\bullet \\ \downarrow & G_\bullet \nearrow & \downarrow p_\bullet \\ \Delta[s] & \xrightarrow{d'_\bullet} & B_\bullet \end{array}$$

where $c'_\bullet$ is associated to $(x_0, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_s)$ and $d'_\bullet$ is the representing map of $y \in B_s$. The s-simplex x' of $E_\bullet$ with representing map $G_\bullet$, verifies that $d_i x' = x_i, i \neq k$ and $p_s(x') = y$. Consequently, $p_\bullet$ is a truncated Kan fibration at dimension $n+1$. This brings the proof to an end since the condition of being a Kan fibration at dimension $n+2$ is equivalent (see proof of 3.2) to the RLP with respect to $\Delta[n+2,k] \hookrightarrow \Delta[n+2]$, for $0 \leq k \leq n+2$. □

Another characterization of n-trivial fibrations can be given as a consequence of the following lemma, which is a truncated version of that given by Quillen (see [18], chapter II, §3, Prop. 2).

Lemma 3.8. *For any morphism $p_\bullet : E_\bullet \to B_\bullet$ in* **Simp(Gp)** *the following are equivalent:*

1. *$p_\bullet$ is a truncated Kan fibration at dimension n, $\overline{p}_q : \Pi_q(E_\bullet) \to \Pi_q(B_\bullet)$ is an isomorphism for $0 \leq q \leq n$ and $\overline{p}_{n+1}$ is surjective.*
2. *p_q is surjective and $\Pi_q(Kerp_\bullet) = 0$, for $0 \leq q \leq n$.*

Then, the announced result is the following:

Corollary 3.9. *For any morphism $p_\bullet : E_\bullet \to B_\bullet$ in* **Simp(Gp)**, *$p_\bullet$ is an n-trivial fibration iff p_q is surjective, $0 \leq q \leq n+2$ and $\Pi_q(Kerp_\bullet) = 0$ for $0 \leq q \leq n$.*

The third distinguished class of morphisms in a closed model category is defined as follows:

Definition 3.10. *A morphism $f_\bullet : X_\bullet \to Y_\bullet$ in* **Simp(Gp)** *is said to be an n-cofibration, $n > 0$, if it has the LLP with respect to the n-trivial fibrations.*

At this point, it is interesting to remark the relationship between all these truncated definitions. Note that one has the following sequence of inclusions for n-trivial fibrations:

$$(\infty - triv.fib.) \subseteq \dots \subseteq ((n+1) - triv.fib.) \subseteq (n - triv.fib.) \subseteq \dots$$

Consequently, the definition of n-cofibration sets the corresponding sequence for them:

$$\dots \subseteq (n - cofib.) \subseteq ((n+1) - cofib.) \subseteq \dots \subseteq (\infty - cofib.)$$

4 The n-structure in Simp(Gp)

The above definitions of n-fibration, n-cofibration and n-weak equivalence will be used now to determine the n-structure in **Simp(Gp)**, i.e., a closed model structure in the sense of Quillen for each n, that generalizes the classical one.

The following lemma will be needed in the proof of the main theorem of this section.

Lemma 4.1. *1. If $\{f^i_\bullet : A^i_\bullet \to B^i_\bullet\}_{i\in I}$ is a family of n-cofibrations in* **Simp(Gp)**, *then the induced morphism on the coproducts, $\sqcup_I A^i_\bullet \to \sqcup_I B^i_\bullet$ is an n-cofibration.*

2. If $f_\bullet : A_\bullet \to B_\bullet$ is an n-cofibration in **Simp(Gp)** *and $g_\bullet : A_\bullet \to C_\bullet$ is arbitrary, the induced morphism into the pushout, $C_\bullet \to B_\bullet \sqcup_{A_\bullet} C_\bullet$ is an n-cofibration.*

3. If $A_0 \to A_1 \to \cdots \to A_m \to \cdots$ is a sequence of n-cofibrations (resp. n-weak equivalences) in **Simp(Gp)**, *the canonical morphism $A_0 \to A_\infty = \varinjlim A_m$ is an n-cofibration (resp. n-weak equivalence).*

Proof. The first two properties, together with the half of the third one concerning n-cofibrations, can be easily proved by the universal property of coproducts, pushouts and direct limits, in each case. As for the n-weak equivalences, just recall that in **Simp(Gp)**, one has that $\Pi_i(\varinjlim X_m) = \varinjlim \Pi_i(X_m)$. □

The main theorem can be enounced and showed now:

Theorem 4.2. *With the given definitions of n-fibration, n-cofibration and n-weak equivalence, $n > 0$, the category* **Simp(Gp)** *is, in each case, a closed model category.*

Proof. The first axiom is well-known and the second can be proved without any difficulty. As for the third related to n-fibrations, let $p_\bullet : E_\bullet \to B_\bullet$ be an n-fibration and $f_\bullet : X_\bullet \to Y_\bullet$ a retract of $p_\bullet$ with

$r_\bullet : E_\bullet \to X_\bullet$ as retraction. The required liftings in the commutative diagrams

$$\begin{array}{ccc} \Delta[s,k] & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow f_\bullet \\ \Delta[s] & \longrightarrow & Y_\bullet \end{array} \quad \text{and} \quad \begin{array}{ccc} * & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow f_\bullet \\ S^{n+1} & \longrightarrow & Y_\bullet \end{array}$$

are given by the composition with $r_\bullet$ of the corresponding liftings for $p_\bullet$. The cases involving n-cofibrations and n-weak equivalences are analogues.

We prove now **CM5**, concerning the factorization of any morphism $f_\bullet : X_\bullet \to Y_\bullet$ into n-cofibration followed by n-trivial fibration, by means of the so-called "small object argument". Consider the following diagram

$$\begin{array}{ccccccc} X_\bullet & \xrightarrow{i_{\bullet_0}} & X_{\bullet_0} & \xrightarrow{i_{\bullet_1}} & X_{\bullet_1} & \xrightarrow{i_{\bullet_2}} & \cdots \\ f_\bullet \downarrow & \swarrow q_{\bullet_0} & & \swarrow q_{\bullet_1} & & & \\ Y_\bullet & & & & & & \end{array}$$

where, having obtained $X_{\bullet_m}$ and considering all commutative diagrams of the form

$$\begin{array}{ccc} F\dot{\Delta}[s] & \longrightarrow & X_{\bullet_m} \\ \downarrow & \overset{\lambda}{\nearrow} & \downarrow q_{\bullet m} \\ F\Delta[s] & \longrightarrow & Y_\bullet \end{array} \quad \text{and} \quad \begin{array}{ccc} F\Delta[n+2,k] & \longrightarrow & X_{\bullet_m} \\ \downarrow & \overset{\mu}{\nearrow} & \downarrow q_{\bullet m} \\ F\Delta[n+2] & \longrightarrow & Y_\bullet \end{array}$$

$0 \leq s \leq n+1$ and $0 \leq k \leq n+2$, distinguished by indexes λ and μ, then the morphism $i_{\bullet_{m+1}} : X_{\bullet_m} \to X_{\bullet_{m+1}}$ is defined by the pushout diagram

$$\begin{array}{ccc} (\sqcup_\lambda F\dot{\Delta}[s]) \sqcup (\sqcup_\mu F\Delta[n+2,k]) & \longrightarrow & X_{\bullet_m} \\ \downarrow & & \downarrow i_{\bullet_{m+1}} \\ (\sqcup_\lambda F\Delta[s]) \sqcup (\sqcup_\mu F\Delta[n+2]) & \longrightarrow & X_{\bullet_{m+1}} \end{array}$$

Let us define now $X_{\bullet_\infty} = \varinjlim X_m$ and $q_\bullet = \varinjlim q_m$ to have a factorization

$$\begin{array}{ccc} X_\bullet & \xrightarrow{f_\bullet} & Y_\bullet \\ & {\scriptstyle i_\bullet}\searrow \quad \nearrow{\scriptstyle q_\bullet} & \\ & X_{\bullet_\infty} & \end{array}$$

where $q_\bullet$ is an n-trivial fibration by 3.7. In order to show now that $i_\bullet$ is an n-cofibration, note that $F\dot{\Delta}[s] \to F\Delta[s], 0 \leq s \leq n+1$ and $F\Delta[n+2, k] \to F\Delta[n+2], 0 \leq k \leq n+2$ are n-cofibrations and so, by 4.1, the coproducts of them are also such, so every $i_{\bullet_m}$ is. Finally $i_\bullet$ is an n-cofibration, again by 4.1.

We shall be concerned now about the factorization of any morphism $f_\bullet$ into an n-trivial cofibration followed by an n-fibration. To this end, let us repeat the same argument used in the factorization already proved, taking in this case all diagrams of the form

$$\begin{array}{ccc} F* & \longrightarrow & X_{\bullet_m} \\ \downarrow & \overset{\lambda}{\nearrow} & \downarrow {\scriptstyle p_{\bullet m}} \\ FS^{n+1} & \longrightarrow & Y_\bullet \end{array} \quad \text{and} \quad \begin{array}{ccc} F\Delta[s,k] & \longrightarrow & X_{\bullet_m} \\ \downarrow & \overset{\mu}{\nearrow} & \downarrow {\scriptstyle p_{\bullet m}} \\ F\Delta[s] & \longrightarrow & Y_\bullet \end{array}$$

for $0 \leq k \leq s, 0 < s \leq n+2$. Let us define then $j_{\bullet_m} : X_{\bullet_m} \to X_{\bullet_{m+1}}$ by the pushout diagram

$$\begin{array}{ccc} (\sqcup_\lambda F*) \sqcup (\sqcup_\mu F\Delta[s,k]) & \longrightarrow & X_{\bullet_m} \\ \downarrow & & \downarrow {\scriptstyle j_{\bullet_{m+1}}} \\ (\sqcup_\lambda FS^{n+1}) \sqcup (\sqcup_\mu F\Delta[s]) & \longrightarrow & X_{\bullet_{m+1}} \end{array}$$

Let $X_{\bullet_\infty}$ be $\varinjlim X_m$ and $p_\bullet = \varinjlim p_m$ one obtains, as above, a factorization $f_\bullet = p_\bullet j_\bullet$, where $p_\bullet$ is an n-fibration by 3.2 and $j_\bullet$ is an n-cofibration by 4.1, since $F* \to FS^{n+1}$ and $F\Delta[s,k] \to F\Delta[s], 0 \leq k \leq s, 0 < s \leq n+2$ are n-cofibrations.

It only remains to prove that $j_\bullet$ is an n-weak equivalence, which will be achieved by showing that every $j_{\bullet_m}$ is, applying 4.1 again. For this, it should be noted that each $j_{\bullet_m} : X_{\bullet_{m-1}} \to X_{\bullet_m}$ is the composition

$X_{\bullet_{m-1}} \xrightarrow{\alpha_\bullet} T_\bullet \xrightarrow{\beta_\bullet} X_{\bullet_m}$ of simplical maps $\alpha_\bullet$ and $\beta_\bullet$, both obtained from the following pushouts diagrams:

$$\begin{array}{ccc} \sqcup_\lambda F\Delta[s,k] & \longrightarrow & X_{\bullet_{m-1}} \\ \downarrow & & \downarrow \alpha_\bullet \\ \sqcup_\lambda F\Delta[s] & \longrightarrow & T_\bullet \end{array} \quad \text{and} \quad \begin{array}{ccc} \sqcup_\mu F* & \longrightarrow & T_\bullet \\ \downarrow & & \downarrow \beta_\bullet \\ \sqcup_\mu FS^{n+1} & \longrightarrow & X_{\bullet_m} \end{array}$$

It is clear that $\alpha_\bullet$ is a weak equivalence -with the classical model structure in **Simp(Gp)**- so it is an n-weak equivalence, for all $n > 0$. In order to prove that $\beta_\bullet$ is an n-weak equivalence as well, note that the free simplicial groups $F*$ and FS^{n+1} are equal to $\mathbb{Z}$ up to dimension n ($F*$ is the constant simplicial group having $\mathbb{Z}$ at each dimension whereas FS^{n+1} has $\mathbb{Z}$ up to dimension n and the free product $\mathbb{Z} \sqcup \mathbb{Z}$ at level $n+1$, indeed). Hence, $\beta_\bullet : T_\bullet \to X_{\bullet_m}$ is an isomorphism up to dimension n, inducing then isomorphisms on the corresponding homotopy groups up (and including) dimension $n-1$.

Let us analyze now the case of the nth-homotopy groups. First, note the existence of a morphism $v_\bullet : X_{\bullet_m} \to T_\bullet$ such that $v_\bullet \beta_\bullet = Id_{T_\bullet}$ so $\beta_\bullet$ is a section. This comes from the fact that $*$ is the terminal object in **Simp(Sets)** and hence, there is a morphism $S^{n+1} \to *$ such that, for the uniqueness of such maps, verifies that the composition $* \to S^{n+1} \to *$ is equal to the identity on $*$. As for the rest, it comes from the properties of the free group functor, F, coproducts and pushouts diagrams, respectively.

In order to compute the nth-homotopy groups, let us consider them as the nth-homology groups of the corresponding Moore complex. This situation can be pictured in the following diagram,

$$\begin{array}{ccccccc} N_{n+1}(T_\bullet) & \longrightarrow & N_n(T_\bullet) & \longrightarrow & N_{n-1}(T_\bullet) & \longrightarrow & \cdots \\ \uparrow\downarrow & & \downarrow \cong & & \downarrow \cong & & \\ N_{n+1}(X_{\bullet_m}) & \longrightarrow & N_n(X_{\bullet_m}) & \longrightarrow & N_{n-1}(X_{\bullet_m}) & \longrightarrow & \cdots \end{array}$$

which cleary shows that $\Pi_n(T_\bullet) \cong \Pi_n(X_{\bullet_m})$. Then, $j_\bullet$ is an n-trivial cofibration. In addition, $j_\bullet$ has by construction the LLP with respect to any n-fibration.

It only remains to prove the lifting axiom **CM4**. In fact, the only half to be shown is that dealing with the existence of lifting in any commutative square of the form

$$\begin{array}{ccc} A_\bullet & \xrightarrow{a_\bullet} & X_\bullet \\ \downarrow{\scriptstyle k_\bullet} & & \downarrow{\scriptstyle q_\bullet} \\ B_\bullet & \xrightarrow{b_\bullet} & Y_\bullet \end{array} \qquad (4)$$

with $q_\bullet$ n-fibration and $k_\bullet$ n-trivial cofibration, because the other half comes straightforward from 3.10. To this end, factor the morphism $k_\bullet$ by **CM5** into an n-fibration $p_\bullet$ followed by an n-trivial cofibration $i_\bullet$ and note that, since both $k_\bullet$ and $i_\bullet$ are n-weak equivalences so it is $p_\bullet$, by **CM2**. On the other hand, there exists a lifting $h_\bullet$ in the following commutative square

$$\begin{array}{ccc} A_\bullet & \xrightarrow{a_\bullet} & X_\bullet \\ \downarrow{\scriptstyle i_\bullet} & \nearrow{\scriptstyle h_\bullet} & \downarrow{\scriptstyle q_\bullet} \\ A_\infty & \xrightarrow{b_\bullet p_\bullet} & Y_\bullet \end{array}$$

since $i_\bullet$ has the LLP with respect to all n-fibrations. Finally, the required lifting for (4) is given by composing $h_\bullet$ with the lifting $s_\bullet$ obtained from the diagram

$$\begin{array}{ccc} A_\bullet & \xrightarrow{i_\bullet} & A_\infty \\ \downarrow{\scriptstyle k_\bullet} & \nearrow{\scriptstyle s_\bullet} & \downarrow{\scriptstyle p_\bullet} \\ B_\bullet & = & B_\bullet \end{array}$$

which exists because $k_\bullet$ is n-cofibration (n-trivial cofibration really) and $p_\bullet$ is an n-trivial fibration, as remarked before. □

Throughout the above proof the key to characterize the n-cofibrations has also been given and so, the n-cofibrant simplicial groups are identified.

Proposition 4.3. *A morphism $f_\bullet : X_\bullet \to Y_\bullet$ in* **Simp(Gp)** *is an n-cofibration iff it is a retract of the morphism $i_\bullet : X_\bullet \to X_\infty$ obtained* *from the factorization of $f_\bullet$ into a n-cofibration followed by an n- trivial fibration.*

Proof. Supposing $f_\bullet$ is an n-cofibration, let us factor it by the first half of **CM5** into n-cofibration, $i_\bullet$, and n-trivial fibration, $q_\bullet$. The following diagram shows, then, that $f_\bullet$ is a retract of $i_\bullet$:

$$\begin{array}{ccc} X_\bullet & = & X_\bullet \\ \downarrow{\scriptstyle f_\bullet} & & \downarrow{\scriptstyle i_\bullet} \\ Y_\bullet & \underset{q_\bullet}{\overset{d_\bullet}{\rightleftarrows}} & X_\infty \end{array}$$

where $d_\bullet$ is the lifting found by **CM4** in the commutative square

$$\begin{array}{ccc} X_\bullet & \xrightarrow{i_\bullet} & X_\infty \\ \downarrow{\scriptstyle f_\bullet} & \nearrow{\scriptstyle d_\bullet} & \downarrow{\scriptstyle q_\bullet} \\ Y_\bullet & = & Y_\bullet \end{array}$$

Conversely, if $f_\bullet$ is a retract of the n-cofibration $i_\bullet$, $f_\bullet$ is, by **CM3**, an n-cofibration. □

5 Comparing model structures in Simp(Gp)

In previous sections, the n-structure in **Simp(Gp)** has been showed to be a generalization of the classical Quillen's closed model structure. In this one, it will be appear as the tool to present the category **Simp(Gp)** as a model for connected $(n+1)$-types of spaces.

To this end, the machinery used will be the skeletal-coskeletal adjunction. Let us point out then that $\Pi_i(Cosk^{n+1}G_\bullet) = 0$ for $i > n$ and so, the image of $Cosk^{n+1}$ could be identified with the full subcategory of **Simp(Gp)** formed by the n-coconnected simplicial groups, that is, those simplicial groups $H_\bullet$ such that $\Pi_i(H_\bullet) = 0$ for $i > n$.

Theorem 5.1. *For the adjunction*

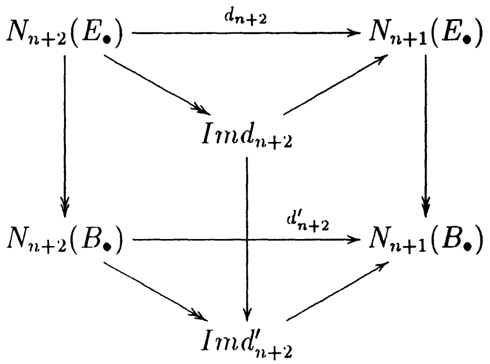
$$Sk^{n+1} : \mathbf{Simp(Gp)} \rightleftarrows \mathbf{Simp(Gp)} : Cosk^{n+1} \qquad (5)$$

it holds:

1. *If $p_\bullet : E_\bullet \to B_\bullet$ is an n-fibration (resp. n-weak equivalence), then $Cosk^{n+1}p_\bullet : Cosk^{n+1}E_\bullet \to Cosk^{n+1}B_\bullet$ is a Kan fibration (resp. weak equivalence).*

2. *If $i_\bullet : A_\bullet \to C_\bullet$ is a cofibration (resp. weak equivalence) in* **Simp(Gp)**, *then $Sk^{n+1}i_\bullet$ is an n-cofibration (resp. n-weak equivalence).*

Proof. Suppose $p_\bullet : E_\bullet \to B_\bullet$ is a n-fibration. By 3.4, this implies that $N_q(p_\bullet) : N_q(E_\bullet) \to N_q(B_\bullet)$ is surjective for $0 < q \leq n+2$. In order to prove that $Cosk^{n+1}p_\bullet$ is a Kan fibration, $N_q(Cosk^{n+1}p_\bullet)$ to be surjective, for $q > 0$, will be proved: for the special feature of $N_q(Cosk^{n+1}G_\bullet)$ for any simplicial group $G_\bullet$, the surjectivity is clear for $0 < q \leq n+1$. In the next level, the situation is as pictured:



From this, $Imd_{n+2} \to Imd'_{n+2}$ is surjective and then, since $\overline{p}_{n+1} : \Pi_{n+1}(E_\bullet) \to \Pi_{n+1}(B_\bullet)$ is onto for being $p_\bullet$ an n-fibration, the following diagram of short exact sequences gives the surjectivity of $N_{n+2}(Cosk^{n+1}p_\bullet) : Ker d_{n+1} \to Ker d'_{n+1}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Imd_{n+2} & \longrightarrow & Kerd_{n+1} & \longrightarrow & \Pi_{n+1}(E_\bullet) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Imd'_{n+2} & \longrightarrow & Kerd'_{n+1} & \longrightarrow & \Pi_{n+1}(B_\bullet) & \longrightarrow & 0 \end{array}$$

The case involving n-weak equivalences follows straightforward: for $i > n$ because $\Pi_i(Cosk^{n+1}E_\bullet) = 0$ and for $0 \leq i \leq n$, because $p_\bullet$ is an n-weak equivalence. This ends the first part of the theorem.

As for the second half, let $i_\bullet : A_\bullet \to C_\bullet$ be any cofibration in **Simp(Gp)**. Consequently, to prove that $Sk^{n+1}i_\bullet$ is an n-cofibration consists of finding lifting in any commutative square of the form

$$\begin{array}{ccc} Sk^{n+1}A_\bullet & \longrightarrow & E_\bullet \\ {\scriptstyle Sk^{n+1}i_\bullet}\downarrow & & \downarrow{\scriptstyle p_\bullet} \\ Sk^{n+1}C_\bullet & \longrightarrow & B_\bullet \end{array}$$

with $p_\bullet$ any n-trivial fibration. This is equivalent, by the adjunction (5), to doing it through the diagram

$$\begin{array}{ccc} A_\bullet & \longrightarrow & Cosk^{n+1}E_\bullet \\ {\scriptstyle i_\bullet}\downarrow & & \downarrow{\scriptstyle Cosk^{n+1}p_\bullet} \\ C_\bullet & \longrightarrow & Cosk^{n+1}B_\bullet \end{array}$$

where the desired lifting exists for being $Cosk^{n+1}p_\bullet$ a trivial fibration (by the first half, already proved). Hence, $Sk^{n+1}i_\bullet$ is an n-cofibration.

As for the weak equivalences, let $i_\bullet : A_\bullet \to C_\bullet$ be a weak equivalence, note that, for any simplicial group $G_\bullet$, one has $\Pi_i(Sk^{n+1}G_\bullet) = \Pi_i(G_\bullet), 0 \leq i \leq n$. Then, $Sk^{n+1}i_\bullet$ is an n-weak equivalence. □

If Ho_n(**Simp(Gp)**) stands for the homotopy category of **Simp(Gp)** associated to the n-structure and Ho(**Simp(Gp)**$|n -$ **coconnected**) denotes the full subcategory of Ho(**Simp(Gp)**) consisting of n-coconnected simplicial groups (i.e., with trivial homotopy groups for $i \geq n+1$), one has

Corollary 5.2. *The functors Sk^{n+1} and $Cosk^{n+1}$ induce an equivalence of categories*

$$Ho_n(\mathbf{Simp(Gp)}) \simeq Ho(\mathbf{Simp(Gp)}|n - \mathit{coconnected})$$

Theorem 5.1 also produces a new approach to the n-cofibrant simplicial groups. To this end, a simplicial group $G_\bullet$ will be said to be

n – *squeletal* if $Sk^n G_\bullet = G_\bullet$. Then, a large number of n-cofibrant simplicial groups are determined by the following result.

Corollary 5.3. *Any retract of a free and* $(n+1)$*-squeletal simplicial group is* n*-cofibrant.*

Proof. Let $G_\bullet$ be a retract of a free and $(n+1)$-squeletal simplicial group $Z_\bullet$, i.e., there are simplicial morphisms

$$G_\bullet \underset{j_\bullet}{\overset{i_\bullet}{\rightleftarrows}} Z_\bullet$$

such that $j_\bullet i_\bullet = Id_{G_\bullet}$. Note then that the unique morphism $g_\bullet : * \to G_\bullet$ is a retract of $z_\bullet : * \to Z_\bullet$. In order to show that $G_\bullet$ is n-cofibrant, let us find lifting on any commutative square

$$\begin{array}{ccc} * & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow{\scriptstyle g_\bullet} & \nearrow{\scriptstyle D_\bullet} & \downarrow{\scriptstyle p_\bullet} \\ G_\bullet & \xrightarrow{b_\bullet} & B_\bullet \end{array}$$

with $p_\bullet$ any n-trivial fibration. In fact, $D_\bullet$ exists, defined as the composition $d_\bullet i_\bullet$, where $d_\bullet$ is the lifting corresponding to the diagram

$$\begin{array}{ccccc} Sk^{n+1}* & = & * & \xrightarrow{a_\bullet} & E_\bullet \\ \downarrow{\scriptstyle Sk^{n+1}z_\bullet} & & \downarrow{\scriptstyle z_\bullet} & \nearrow{\scriptstyle d_\bullet} & \downarrow{\scriptstyle p_\bullet} \\ Sk^{n+1}Z_\bullet & = & Z_\bullet & \xrightarrow{b_\bullet j_\bullet} & B_\bullet \end{array}$$

whose existence is justified by 5.1, because $Z_\bullet$ is a free simplicial group and so $z_\bullet$ is a cofibration. Particulary, $Sk^{n+1}z_\bullet = z_\bullet$ is an n-cofibration and the result follows. □

At this point, let us recall that the non-abelian version of the classical Dold-Kan's theorem, given in [5], allowed to find, by a canonical process of truncation, a category, **n – HXC(Gp)**, of n-hypercrossed complexes of groups, which provides algebraic models for n-simplicial groups and so, for connected $(n+1)$-types of spaces. This category is equivalent to the reflexive subcategory of **Simp(Gp)** whose objects are those simplicial groups with vanishing Moore complex in dimensions greater than

n. This last category is denoted $\mathbf{n-Hypgd(Gp)}$ because it is just the category of n-hypergroupoids of groups in the sense of Duskin-Glenn, [12].

Now, the category $\mathbf{n-Hypgd(Gp)}$ inherits a closed model structure from the structure of **Simp(Gp)**, via the pair of adjoint functors J and $\mathbb{P}$, the inclusion functor and the reflector one,

$$\mathbb{P}:\mathbf{Simp(Gp)} \rightleftarrows \mathbf{n-Hypgd(Gp)}:J\ ,\qquad \mathbb{P}\dashv J$$

by defining fibrations (weak equivalences) as those morphisms $f_\bullet$ such that $J(f_\bullet)$ is a fibration (weak equivalence) of simplicial groups, and defining cofibrations by the LLP with respect to trivial fibrations (see [3], or [4]). The category $\mathbf{n-HXC(Gp)}$ supports then a closed model structure as well.

Following Quillen, [18], for any closed model category $\mathcal{C}$ the notion of homotopy category, $Ho(\mathcal{C})$, will be considered as extended by the notion of homotopy theory associated to $\mathcal{C}$. The Quillen's criterion to determine an equivalence of homotopy theories (see [18], Chap.I, §4, Theorem 3) will be used for the following theorem:

Theorem 5.4. *The homotopy theory of* **Simp(Gp)** *(with the closed model n-structure) and that of* $\mathbf{n-Hypgd(Gp)}$ *(with the closed model structure given in [3]) are equivalent.*

Proof. Let us recall that the reflector functor $\mathbb{P}$ is given by

$$\mathbb{P}(G_\bullet)=cosk^{n+1}\Big(\frac{G_{n+1}}{H_{n+1}} \underset{d_{n+1}}{\overset{d_0}{\rightrightarrows}} \frac{G_n}{d_{n+1}(N_{n+1}G_\bullet)} \underset{d_n}{\overset{d_0}{\rightrightarrows}} G_{n-1} \underset{d_{n-1}}{\overset{d_0}{\rightrightarrows}} \cdots\cdots G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0\Big)$$

where H_{n+1} is the normal subgroup of G_{n+1} formed by those $x\in G_{n+1}$ such that $d_ix\in d_{n+1}(N_{n+1}G_\bullet)$, for $0\le i\le n+1$ (compare with that given in [12]).

Since $\mathbb{P}$ preserves both cofibrations and weak equivalences, so does J with fibrations and weak equivalences. In addition, the unit of the adjunction is a weak equivalence (i.e., an n-weak equivalence of simplicial groups). Consequently, the adjoint situation

$$\mathbb{P}:Ho_n(\mathbf{Simp(Gp)}) \rightleftarrows Ho(\mathbf{n-Hypgd(Gp)}):J\ ,\qquad \mathbb{P}\dashv J$$

provides an equivalence of categories.

Since both **Simp(Gp)** and $\mathbf{n} - \mathbf{Hypgd(Gp)}$ are pointed, this equivalence turns out to be an equivalence of homotopy theories. □

It is clear now that

Corollary 5.5. *The closed model categories* **Simp(Gp)** *with the n-structure, and* $\mathbf{n} - \mathbf{HXC(Gp)}$*, with the structure given in [3], have equivalent homotopy theories.*

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J.G.Cabello,
Dpto. Algebra, Facultad de Ciencias, Granada, 18071, España.
cabello@goliat.ugr.es