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The future of Branch Cash Holdings Management is here: New Markov chains

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Highlights

- This paper provides a new methodology: Markov Chains by blocks.
- This would achieve knowledge on the branch cash holdings.
- We study conditions for optimal cash holdings and their steady-states using Ergodicy.
- These findings will also let bank managers know the time validity of the cash holdings
- This incipient mathematical framework may also apply to other contexts.

The future of Branch Cash Holdings Management is here: New Markov chains

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Abstract

Liquidity management is one of the main concerns of the banking sector since it provides control in key areas such as treasury management, working capital financing and business valuation. Under the assumption that branch efficiency makes a fundamental contribution towards the effective performance of the global banking institution, this paper provides a new methodology (Markov Chains by blocks) in order to achieve knowledge on the branch cash holdings: conditions which ensure optimal cash holdings, recurring properties which help to better predict cash holdings shifts and the study of the branch cash holdings steady-states using Ergodic Theory. These findings will let bank managers know *the time validity of the current cash holdings*. This is a crucial advantage to ensure efficient cash management: while helping keep banking institutions on sound financial footing by guaranteeing the compulsory-by-law safety cushion, it also allows bank managers to make sound decisions upon fund investments.

This incipient mathematical framework, based on the re-definition of classical theory on Markov chains, provides an alternative standpoint which may also apply to those dynamical systems which can be categorized into groups of similar features.

Keywords: (D) Economics; Markov chains by blocks; Optimal cash balance; Time validity of the cash holdings; Ergodic Theory

JEL classification: C44; C58; G10; G21

1. Introduction and Literature Review

Corporate/bank cash holdings have always played a crucial role in the development of firms and financial institutions: without cash, they could both become insolvent and at risk of bankruptcy. Thus, efficient cash administration has traditionally focused the attention of managers and shareholders, especially during periods of uncertain market and credit conditions. In this regard, an accurate cash balance *forecast* is critical for successful management while also serving other strategic purposes such as controlling subsidiary groups.

The banking industry has been in search of managerial measures to improve the control of its liquid resources in order to increase efficiency. While efficiency on all fronts (including cash management) has become a primary objective for banking industry over the last decade, there is a body of research which argues that branches have a role to play in helping to improve global bank institution performance. This was firstly suggested in Berger (1997), whose authors stressed the importance of the efficiency of branches as making a fundamental contribution towards the effective performance of the global banking institution. Moreover, the authors of Berger (1997) called attention to the fact that branch efficiency literature is much less complete than banking efficiency literature. As a matter of fact, specific literature to design techniques to improve branching

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performance *as far as cash management is concerned* is quite short[1] apart from those papers which focus on regulatory measures to control under-performing branches. The present paper attempts to help to fill this gap by proposing specific conditions to improve cash (forecasting) management at branch level.

In the area of management of corporate cash holdings, there have been a long series of attempts to determine the optimal investments that organizations should make in cash. Models of cash management or money demand can be categorized into two types: those with demand by households, pioneered by the Baumol-Tobin model, Baumol (1952) and continued by Frenkel and Jovanovic, (1980), Bar-Illan, (1990) and Chang, (1999) and those which concern cash management by firms, pioneered by the paper of Miller and Orr, (1966). Firms differ from households in that firms have daily cash inflow as well as daily expenditures. Also the size of financial transactions differentiates firms from households, as large and instantaneous transactions are more likely. In terms of their mathematical structures, the first describes the money stock between controls by means of Brownian motion with drift whereas Miller and Orr formulated a model under which an organization's cash flow evolves in terms of a stationary random walk. Later proposals feature unified analysis of cash management by combining Brownian motion and compound Poisson processes, as in Bar-Illan (2004). Other authors classify cash management models according to their mathematical fundamentals. Following Melo (2011), these models can be grouped into Inventory Theory models (now both Baumol-Tobin and Miller-Orr belong to the same category), those developed with Linear Programming and those which are based upon Dynamic Programming.

Further papers incorporate stochastic techniques in their patterns of cash management: Baccarin (2009) considers the optimal control of a multidimensional cash management system where the cash balances fluctuate as a homogeneous diffusion process in $\mathbb{R}^n$. Cyert (1962) pioneered research using Markov chains for estimating the allowance of doubtful accounts while Hinderer (2001) analyzed a cash management system in which the distribution of the cash flow depends on a randomly varying environment. Ferstl and Weissensteiner (2008) considers a cash management problem by using a multi-stage stochastic linear program (SLP). In Higson (2010), the authors model the evolution of cash in terms of a square root process modified with a Brownian motion in such a way that the statistical properties of the cash flow process depend on the cash holdings. This functional relationship between cash holdings and cash flow process by means of Hamilton-Jacobi-Bellman equations and other elements of optimal control theory is the core qualitative finding of Anderson (2012). In Bensoussan (2009) the author uses a stochastic maximum principle to obtain an optimal transaction policy. In Sato (2011), a cash management model is built around Brownian motions and Poissson processes while Song (2013) discuss a cash management model for firms based on a stochastic volatility (SV) model. And more recently, Tangsucheeva (2014), where a cash flow forecasting model is developed in terms of Markov chains and bayesian models.

However, there is little current literature about this subject for the *banking industry*, whose specific characteristics differ from firms and other economic organizations. While credit lines are freely available for the banking industry, the private sector must apply for external financing within the framework of those credit channels that are accessible to it. Actually, the relationship between cash holdings and credit risk is present in most former models of cash management: see Acharya (2012), where a dynamic continuous-time model allows for a description of the correlation between credit spreads and cash reserves. Or Anderson (2012), where the authors develop a model of optimal policy toward holding the liquid assets of a firm which faces external financing. Banks differ from firms also in the peculiar dynamics of their cash flow processes. Assorted entries of cash are specific to banks: at the branch level, as daily expected and unexpected deposits and withdrawals, similar to ATM dynamics, while at the aggregate level, large/huge transactions take place as a consequence

[1]"Short" unless the strand of research focused on improving the performance of automatic teller machines, ATMs, would be considered as part of the literature to improve branch cash management.

of movements of money amongst bank entities. Distinct regulation packages are also applied in order to control banking industry versus private sector.

The ultimate aim as far as liquidity management is concerned is to find the optimal level of cash since it would help banking institutions facing short-term obligations at the aggregate and branch level, while minimizing the risk of bankruptcy in long-term projections. This "optimal" level of cash may be also read as "enough" cash. However, finance literature has given very little *precise* guidance on this question: how much money is enough for a banking institution?

Common knowledge suggests that banks that have larger liquid assets should be safer. However, when banking firms keep liquid resources in cash, they renounce a part of their profitability, incurring the opportunity costs of not investing in other alternatives which do generate profits. Thus, the intuition recommends that a balance between minimizing costs and maximizing profits should be kept in order to ensure high levels of efficiency. But, how do the banks identify the right proportion to be held?

This paper attempts to fill this gap by providing answers to the above questions in the context of bank branches. Stated briefly, the contributions of this paper are twofold: a deep study of the branch cash holdings from a dynamic point of view (first contribution) through an approach based on new stochastic financial analytics that we have developed in this paper (second contribution). Actually, this paper provides a new methodology (Markov Chains by blocks) in order to achieve knowledge on the branch cash holdings with proposals to be approached from a variety of perspectives: i)conditions which ensure optimal cash holdings, ii)recurring properties of the branch cash holdings derived from the natural cyclicity exhibited in the branch cash management practice which help to better predict their shifts and iii)the study of the branch cash holdings steady-states using Ergodic Theory, Braido (2013), which let bank managers know *the time validity of the current cash holdings*. In general, we find policies on holding optimal levels of liquid assets aimed at being useful for both bank and branch managers. These conditions are crucial to ensure efficient cash management: while helping keep banking institutions on sound financial footing by guaranteeing the compulsory-by-law safety cushion, it also allows bank managers to make sound decisions upon fund investments. As far as the author knows, this is the first time in the literature that such an analysis on cash holdings at the branch level has been carried out from a dynamic point of view.

This incipient mathematical framework, based on the re-definition of classical theory on Markov chains, provides an alternative standpoint which may also apply to those dynamical systems which can be categorized into groups of similar features.

The remainder of the paper is organized as follows. In Section 2, an overview of *clustering* methods (*identification of similarities*) is presented since the cyclicity that exists in the branch cash management practices (which is at the heart of our study) relies on the idea of grouping the weeks into *blocks of weeks with similar features*. Section 3 presents the general framework of the liquids funds of a branch. Section 4 is aimed at setting conditions to ensure optimal cash holdings. Recurring properties on branch cash holdings are presented in section 5, while time validity on cash holdings is analyzed in depth in section 6. Section 7 contains a numerical example, based on real data for a branched-bank. Finally, Section 8 concludes the paper.

2. Clustering: related approaches

As mentioned before, one of the key insights of the paper is the natural cyclicity exhibited in the branch cash management practice: in detail, that refers to the usual branch managers' partition of the year into *blocks of weeks with similar features* in order to require the same amount of case for all weeks inside the same block[2]. Around this idea, a new methodology -called Markov Chains by blocks- is developed in this paper: specifically, the whole temporal sequence of branch cash holdings,

[2]For instance, a block of weeks is "first weeks of each month".

which is previously shown to be a Markov chain, is partitioned into blocks such that this partition is well correlated with the partition of the year by branch managers[3]. This new methodology is aimed at providing knowledge on branch cash holdings.

The practice of grouping weeks is close to *clustering*. As a matter of fact, following Manning and Schütze (2000), clustering is the process of partitioning a set of objects into groups or clusters. In the financial sector, where data are generated on large scale, the clustering can be improved by pattern recognition and data mining techniques. Although this paper' approach is heading in a different direction (Markov Chains by blocks and the application of Ergodic Theory), a review on distance based methods for clustering might be of interest.

Since clustering is the grouping of similar objects, some kind of measure which may conclude whether two objects are similar or not, is required. In global terms, two kinds of measures may be used to estimate this relationship: *distance measures* (which result in distance-based methods) and *similarity measures* (giving rise to similarity functions like the well-known Pearson correlation measure). Distance-based methods run on the basis of the shorter the distance between objects, the more similar to one another they are. These methods may be categorized according to the distance taken, which would vary depending on the *type of attributes* of data: *numeric attributes* (the similarity between two data instances may be calculated using the Minkowski metric, with the well-known Euclidean distance as a particular case), *binary attributes* (the distance between objects may be calculated through contingency tables), or other types as *nominal, ordinal or mixed-type attributes* for which specific definitions of distance are required.

Together with a vast literature on clustering methods, there are also many criteria upon which they could be categorized. Mainly, they may be divided into hierarchical and partitional clustering, based on the way they produce the results, see Fraley and Raftery (1998): specifically, hierarchical methods construct the clusters by recursively partitioning the instances while partitional ones relocate instances by moving them from one cluster to another, starting from an initial partitioning. A more comprehensive classification, see Han and Kamber (2011), categorizes them into hierarchical', partitional', density-based methods (which assume that the points that belong to each cluster are drawn from a specific probability distribution, see Banfield and Raftery (1993)), model-based methods (which attempt to optimize the fit between the given data and some mathematical models), grid-based methods (which partition the space into a finite number of cells that form a grid structure on which all of the operations for clustering are performed, see Han and Kamber, (2011)) and finally soft-computing methods (with fuzzy clustering as main exponent). One example of model-based methods are Markov Mixture Models (MMM), which was firstly analyzed by Chib (1996) as models of a class of mixture distributions in which the component populations, from one observation to the next, were selected according to an unobserved Markov process. In the context of clustering, this is an approach that uses a Markov model to represent the data in each of the clusters: e.g, *if there are two clusters, they should be represented by two different Markov models.*

The approach suggested in this paper is different from its conception to its implementation, as proved in next sections: firstly, the whole sequence of branch cash holdings, denoted as $\{CH^n\}_{n\in\mathbb{N}}$, is shown to constitute a discrete-time Markov chain (Theorem 4.4). While it is shown not to be irreducible (hence, Ergodic Theory does not apply), a suitable definition of a new equivalent relation over the set of states of the Markov chain $\{CH^n\}_{n\in\mathbb{N}}$ is proposed (equivalent Definitions 5.1 and 5.2) aimed at achieving irreducible chains. Subsequently, Theorem 5.6 establishes the correlation between the partition on the whole sequence of branch cash holdings originated by the definitions cited above, and the partition of the year as a result of the branch managers' practices. Now, Ergodic Theory applies to these equivalent classes of cash holdings (called *blocks*) providing knowledge on them.

[3]This new methodology also applies to any dynamical system $\{X^n\}_{n\in\mathbb{N}}$ which holds the Markov property, see Conclusion section for further details.

3. Dynamics of the liquids funds of a branch

This section is devoted to stating the dynamics of the liquids funds of any branch. To this regard, we start with a few words on the functioning of the branches. Every day, branches perform numerous transactions which cause cash inflows and outflows. Each branch should keep its total liquid assets (known as **cash holdings**) at an optimal level, without generating either a surplus or a shortage of money. It should not be too low, in order to refrain from insolvency and to adhere to minimum capital regulations. But it should not either be too high so the branch would not have security problems due to heavy cash load or the bank would not suffer from opportunity cost (i.e., the opportunity loss of not investing in other alternatives which do generate profit).

Hence, periodically, the branch adjusts its cash levels to its necessities -deposits and withdrawals- avoiding generating dormant money. To accomplish this task, the branch requires help from its cash central. Thus, an armoured van either evacuates the surplus or provides the deficit of cash up to reach a confident level of case. As for this "confident level of cash", it should be mentioned that every branch has a *cash upper bound* fixed by the bank company as an internal control mechanism. This cash upper bound is assigned according to the branch size and it will be denoted by C_{max} (i.e., maximum cash allowed to be held by the branch).

The *cash entries* are the following: the own branch requests of cash to the cash central and the deposits made by individual users/companies. As far as the branch *cash expenses* are concerned, these include withdrawals and other costs which mainly consist of logistic costs for transport and handling as well as opportunity costs[4]. The branch movements of cash are reflected in Figure 1.

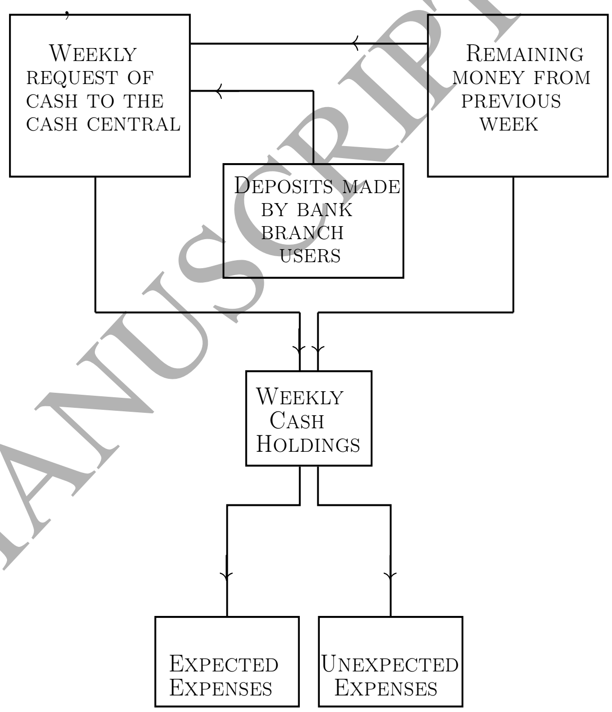

Figure 1 The dynamic of the branch cash holdings

Let C stands for the total amount of money that the branch requests from its cash central. This quantity C is adjusted weekly[5] to the cash necessities. One of the aims as far as liquidity management at the branch level is concerned is to find the optimal amount C which covers all branch expenses without generating either surplus or shortage of money. Habitual bank branches cash management routines as far as this computation is concerned consist of historical data handling. That means that the branch registers the cash quantity on some particular day (workable, weekend, holidays, etc) and the result obtained at the end of the journey (exceed or shortage of case) and copies the successful amounts. In the process of the decision-making, often the staff in charge reaches a decision with only partial information. The author of the present paper developed in García Cabello (2013)[6] a mathematical procedure in order to compute C such that C covers the

[4]Some of these costs can be anticipated and others are of a random nature. The same classification can be established for deposits.

[5]This unit of time -one week- may be changed without loss of generality.

[6]A patent has been requested for the paper García Cabello (2013) by the University of Granada, "Method for managing liquidity in bank branches", number ES201431094, United States

branch demand of cash as accurate as possible -i.e., without generating either a surplus or a shortage of money. This procedure is summarized below. Let N be the number of branch costumers during the considered period of time. Such arrival processes are described by a Poisson process $N(t)$ or N_t. The counter tells the number of arrivals in the interval $(0, t)$. The withdrawals and deposits movements are described as $N_t = N_t^w + N_t^d + O_t$, where N_t^w stands for the number of withdrawals in the interval $(0, t)$, N_t^d is the number of deposits made in the interval $(0, t)$ and O_t gathers the rest of operations. Let W_i represent the withdrawn amount made by i-branch user. The withdrawal process, parameterized by certain rate λ, is defined as the compound Poisson process $X_t := \sum_{i=1}^{N_t^w} W_i$, as independent and identically distributed (i.i.d.) random variables. Thus, the total amount of money which has been withdrawn for $t = 1$ is $X_1 = \sum_{i=1}^{N^w} W_i$. Then, in García Cabello (2013), some formula to compute the branch total expected withdrawals for a unit of time, E_{X_1}, and the branch total expected deposits for a unit of time, E_{Y_1}, were given. Moreover, the key result of García Cabello (2013) is the following:

Theorem 3.1 (García Cabello (2013)). *Let K[7] be both branch expected expenses/deposits for a unit of time. The total amount of cash C which will cover the branch demand of cash, C, may be computed as*

$$\begin{array}{rcl} C & = & E_{X_1} - E_{Y_1} + K \\ C & \leq & C_{max}, \end{array} \tag{1}$$

since no amount of cash should exceed the branch cash upper bound C_{max}.

Let CH stands for the branch **cash holdings** at some moment, that is, the total liquid assets to be held by the branch at some moment. Hence CH includes C. *The main objective of this paper is to study CH with proposals to be approached from a variety of perspectives.* To carry out this analysis, we shall use superscripts to point out a certain period of time (discrete-time). Hence, CH^n are the branch cash holdings at week n; similary, C^n stands for the total amount of money that the branch requests from its cash central at week n. To start this study, let us first formalize the concept of *size of a branch*. The notion of branch size[8] is intuitively identified with the volume of its turnover. However, there are many criteria which quantify the size of a branch amongst bank managers. The most accepted is to consider size of a branch as increasing in function with the total branch cash needs: the bigger branch sizes correspond to the bigger branch cash needs, related to mayor larger moves -entries and exits- of liquid resources. Thus, we will identify branch size with the maximum value of branch cash holdings during some period of time:

Definition 3.2. BS = branch size $= max\{CH^n / n \in \mathbb{N}\}$.

Note that the foregoing C_{max} always performs under the size of the branch: $C_{max} \leq BS$.

4. Optimal branch cash holdings

This section is devoted to determining conditions which ensure optimal cash holdings for each branch. This involves a suitable definition of optimal cash holdings in terms of technical efficiency. Firstly, we specifically define branch weekly cash holdings following former Figure 1.

Definition 4.1. For any bank branch, we define its cash holdings at the n-th week as

$$CH^n = C^n + R^{n-1} \tag{2}$$

where R^{n-1} stands for the remaining money from previous week. We shall write $CH = C + R$ when the week no longer needs to be emphasized.

[7] K may be considered as part of the *security cash level* (settlement accounts) that the banking institutions holds for precautionary reasons.

[8] Bank managers apply this term as benchmark to position the branches with respect to each other for several different purposes.

Remark 4.2. Since C is computed taking into account all branch cash entries and expenditures (see García Cabello (2013) for further details), if the computation of C^n would be completely accurate, the remaining money from previous week R^{n-1} would be 0. That is, R is the error committed in the computation of the cash requested to the branch cash central C. In consequence, $R^n = se[C^n] := \sqrt{Var_{C^n}}$ could be considered as function of C. Hence, equation (2) $CH^n = C^n + R^{n-1}$ could perform like a dynamical system provided initial and/or boundary conditions are given.

Furthermore, according to the notion of technical efficiency (the maximum output produced from the minimum quantity of inputs), the optimal cash holdings should be those which cover all branch expenses without (or producing the minimum of) remaining money, as follows:

Definition 4.3. For any bank branch, we define its optimal cash holdings CH^* as that which holds $CH = C + R$ with almost no committed error (negligible) in the calculation of C. That is to say, $CH^* \approx C$.

4.1. Main features of the cash holdings CH

Before studying the main features of the branch cash holdings $\{CH^n\}_{n\in\mathbb{N}}$, let us point out a few properties of the branch cash requests $\{C^n\}_{n\in\mathbb{N}}$. Firstly, $\{C^n\}_{n\in\mathbb{N}}$ constitute a random walk. Moreover, as the set of states of C^n represents the set of cash amounts which may be required by branch managers, while C^n would in principle be allowed to take values in $\mathbb{R}$, in practice the set of states of C^n may be considered finite, for every n. Indeed, branch managers envision *only a few* quantities to be required to central hubs when weekly adjusting branch cash needs, which vary depending on the characteristics of each week. *This is the result of branch managers' practice of categorizing the weeks according to their specific features in order to simplify the cash requirements.* Let $\{C^{nk}, k = 1, 2 \ldots, m\}_{n\in\mathbb{N}}$ be the set of all quantities of cash which could be required to the central hub at the nth-week. These are illustrated in Figure 2.

We proceed now with the study of $\{CH^n\}_{n\in\mathbb{N}}$. The main result is the following:

Theorem 4.4. *The branch cash holdings constitute a discrete-time Markov chain $\{CH^n\}_{n\in\mathbb{N}}$.*

PROOF. $CH^n = C^n + R^{n-1}$, where C^n is computed by Theorem 3.1 and the committed error in the computation is given by $R^n = se[C^n] := \sqrt{Var_{C^n}}$. Substituting $R^{n-1} = se[C^{n-1}]$ into equation $CH^n = C^n + R^{n-1}$ would give $CH^n = C^n + se[C^{n-1}]$ which shows that every outcome CH^n depends on the outcome of previous step. In consequence, the Markov property holds.

As for the state space, the Markov chain $\{CH^n\}_{n\in\mathbb{N}}$ takes values in $\mathbb{R}$. Nevertheless, while the set of states of $\{C^n\}_{n\in\mathbb{N}}$ could be considered finite for the branch managers's practice, the set of states of the Markov chain $\{CH^n\}_{n\in\mathbb{N}}$ is not finite. Let $\{CH^{nk}, k \in \mathbb{N}\}$ be the set of feasible states of the Markov chain $\{CH^n\}_{n\in\mathbb{N}}$ at week n. This is shown in Figure 3:

$$
\begin{array}{ccccc}
\multicolumn{5}{c}{BS} \\ \hline
\multicolumn{5}{c}{C_{max}} \\ \hline
C^{1m} & C^{2m} & C^{3m} & \ldots & C^{nm}\ldots \\
\vdots & \vdots & \vdots & & \vdots \\
C^{12} & C^{22} & C^{32} & \ldots & C^{n2}\ldots \\
C^{11} & C^{21} & C^{31} & \ldots & C^{n1}\ldots \\
\| & \| & \| & & \| \\
C^{1} & C^{2} & C^{3} & \ldots & C^{n}\ldots
\end{array}
$$

Figure 2 The sequential structure of requests of cash C

$$
\begin{array}{cccc}
\multicolumn{4}{c}{BS} \\ \hline
\vdots & \vdots & & \vdots \\
CH^{1m} & CH^{2m} & \ldots & CH^{nm} \\
\vdots & \vdots & & \vdots \\
CH^{12} & CH^{22} & \ldots & CH^{n2} \\
CH^{11} & CH^{21} & \ldots & CH^{n1} \\
\| & \| & & \| \\
CH^{1} & CH^{2} & \ldots & CH^{n}
\end{array}
$$

Figure 3 The sequential structure of branch cash holdings CH

4.2. Conditions for the cash holdings CH to be optimal

We focus here on analyzing the properties of the sequence of states of "remaining money" R^n in order to study when R^n becomes minimal. As a result, conditions for the sequence of cash holdings to be optimal are proved in the following theorem:

Theorem 4.5. *If $E[C^{n1},\dots,C^{nm}]$ stands for the mean of variables $C^{n1},\dots,C^{nm}$, we denote $E[C^n]$ to the set $E[C^n]=\{E[C^{n1},\dots,C^{nm}],k=1,\dots,m\}$. Thus*

$$(CH^n)\to(CH^n)^* \quad \textit{if and only if} \quad C^n\to E[C^n].$$

Proof. Recall that, from $CH^n=C^n+\sqrt{Var_{C^{n-1}}}$ where to minimize $\sqrt{Var_{C^n}}$ is equivalent to minimize Var_{C^n}. Since the set of states of C^n is finite, variance becomes sample variance V with variables $C^{n1},\dots,C^{nm}$, $V(C^{n1},\dots,C^{nm})$. From the definition of sample variance, it follows that $V(C^{n1},\dots,C^{nm})=\dfrac{1}{m}\sum_{k=1}^{m}(C^{nk}-E[C^{n1},\dots,C^{nm}])^2$. From the first-order conditions, the stationary points of function V might be a solution to the system of partial derivatives

$$\frac{\partial V}{\partial C^{nk}}=\frac{2}{m}(C^{nk}-E[C^{n1},\dots,C^{nm}])=0\Rightarrow C^{nk}=E[C^{n1},\dots,C^{nm}],\quad for\ k=1,\dots,m.$$

As for the second-order conditions for minimum of function V, the Hessian of the function, $HessV$, is a diagonal matrix of order n with the only (strictly positive) eigenvalue equal to $2/m$ (V is a strictly convex function in consequence). Hence, the conditions for $\{C^{n1},\dots,C^{nm}\}$ to be a global minimum are $C^{nk}=E[C^{n1},\dots,C^{nm}], for\ k=1,\dots,m$.

Finally, following theorem establishes conditions for cash holdings to be optimal:

Theorem 4.6. *The branch cash holdings amount $(CH^n)=C^n+R^{n-1}$ is the optimum $(CH^n)^*$ whenever the quantities C^n (cash requirements to its central hub) are computed as outlined in Theorem 3.1.*

Proof. According to Theorem 3.1, all quantities $\{C^{nm}\}_m$ are equal to $C^n=E^n_{X_1}-E^n_{Y_1}+K^n$ for each m. The properties of means bring to the desired result:

$$C^n=E^n_{X_1}-E^n_{Y_1}+K^n=E[X^n_1]-E[Y^n_1]+K^n=E[X^n_1-Y^n_1+K^n]\Rightarrow E[C^n]=E[E[X^n_1-Y^n_1+K^n]].$$

5. Recurring properties of the branch cash holdings: Markov Chains by blocks

We develop here the general properties of the Markov chains as far as recurrence is concerned, thereby formalizing current branch managers' practices with regard to branch forecasting procedures. An important contribution has been the re-definition of communication amongst Markov chain states versus the classical definition, which will provide an alternative point of view for Markov chains.

Before proceeding, let it be observed that this theoretical framework on recurrence is considered to be due to the natural cyclicity properties which are exhibited by the branch cash requirements *in the practice of branch cash management*. Actually, in the practical context, the staff imitate the successful amounts to be required according the branch's historical records, where the already required cash quantities on some particular week are registered depending on that week's particular features -workable/holidays, beginning/end of month...- (see subsection 3.1 for further details). That means that the problem of computing the optimal quantity of case to be required is solved in practice by partitioning the year into blocks of weeks with similar features *in order to require the same amount of case for all weeks inside the same block*: for instance, *all* beginning-of-month weeks are considered to belong to the beginning-of-month block whereas *all* ending-of-month weeks belong to the ending-of-month block. Thus, the required quantities of case might be the same for all weeks inside the same block. This partition into blocks is done in each branch depending on its features (size or geographical location). The result of this intuitive method is a set of classes, in a similar way that defining an equivalence relation within the set of all weeks of the year partitions this set into cells.

We proceed by exposing some general concepts and properties on recurrency. Let $\{S^n\}$ a Markov chain with set of states $\{1, 2 \ldots, i, \ldots, j, \ldots\}$. A state j is accessible from state i , $i \to j$, if there is a possibility of reaching j from i in some number of steps, that is, if the nth-step transition probability from state i to state j, denoted by $p_{ij}^{(t)} = P_i[S^n = j \text{ for some } n]$ is strictly positive, $p_{ij}^{(t)} > 0$. We simply denote $p_{ij}^{(1)}$ as p_{ij}. If j is not accessible from i ($p_{ij} = 0$), thus the chain started form i never visits the state j. We say that i communicates with j if $i \to j$ and $j \to i$. This equivalent relation divides states into classes. Within each class, all states communicate to each other, but no pair of states in different classes communicates. The chain is irreducible if there is only one class. A state i is recurrent if $P_i[S^n = i \text{ for infinitely many steps } n] = 1$,, that is to say, if the probability of returning to i in a finite set of steps is equal to 1. It all applies to the Markov chain $\{CH^n\}_{n\in\mathbb{N}}$ with set of states all cash amounts: $\{CH^{lk}, l, k \in \mathbb{N}\}$ where the first superscript l is the counter of weeks. With the (classical) equivalence relation, in despite of the transition probabilities are almost zero for states corresponding to weeks with no similar features, all states CH^{lk} communicate, hence the chain is irreducible. Nevertheless, the set of states might prove to be uncontrollable.

We define a new equivalent relation over the set of states of the Markov chain $\{CH^n\}_{n\in\mathbb{N}}$ in order to continue advance in applying the branch cash managers' intuitive practices. This new equivalent relation shall partition the set of states of the branch cash holdings $\{CH^n\}_{n\in\mathbb{N}}$ into **blocks**, as we will shortly be proposing. Besides, this new equivalence relation has advantage comparing to the classical one in reducing the number of cash holdings (states) to control them better. Actually, the marginal benefit of using the blocks approach should make the problem more granular and therefore closer to desired optimality. For this, state $i, j, \ldots$... shall be written as $CH^{n_i}, CH^{n_j}, \ldots$ or $CH^{n_i k_i}$ when it becomes necessary to point out a concrete cash holding amount corresponding to state CH^{n_i}.

Definition 5.1. Consider two states CH^{n_i} (shorter state i) and CH^{n_j} (state j), corresponding to weeks n_i and n_j. We say that i communicates with j, $i \leftrightarrow j$, if both weeks n_i and n_j exhibit the same set of features as defined by the branch staff.

Following the branch cash managers intuitive practice, this is equivalent to the following:

Definition 5.2. We say that states CH^{n_i} and CH^{n_j} communicate if the required amount of cash to the central hub is the same for both weeks n_i and n_j: $CH^{n_i} = C^{n_i} + R^{n_i - 1}, CH^{n_j} = C^{n_j} + R^{n_j - 1}$, with $C^{n_i} = C^{n_j}$.

Remark 5.3. From Definitions 5.1 and 5.2, not every state of the Markov chain $\{CH^n\}_{n\mathbb{N}}$ communicates each other, only those which correspond to weeks with similar characteristics. We call **block** the set of weeks with similar features. Now, irreducible subchains in $\{CH^n\}_{n\in\mathbb{N}}$ are those formed by cash holdings corresponding to weeks inside the same block.

Let us go on by formalizing the branch cash managers intuitive practice. For this, recall that the period of a state i, $d(i)$ defined as $d(i) = gcd\{n \in \mathbb{N}, p_{ii}(t) > 0\}$ (gcd denotes "greatest common divisor"), is a class property. Besides, recall that two integers n, m are congruent modulo d if $n - m = \alpha d$ for some $\alpha \in \mathbb{Z}$. Let $[\quad]_d$ stand for the equivalence class modulo d. In order to require the same amount of cash from the central hub, branch managers usually group the weeks whose main characteristics coincide with peaks and low cash-demand periods. The following example is intended for illustrative purposes, according to this branch cash managers intuitive routine:

Example 5.4 (First weeks of each month). "First weeks of each month" is amongst the customary criteria applied by branch managers to group the weeks. Next Figure displays the first weeks of each month, for whom the same quantity of cash is required (C^{fm}, fm=first month) making visible the period of this block:

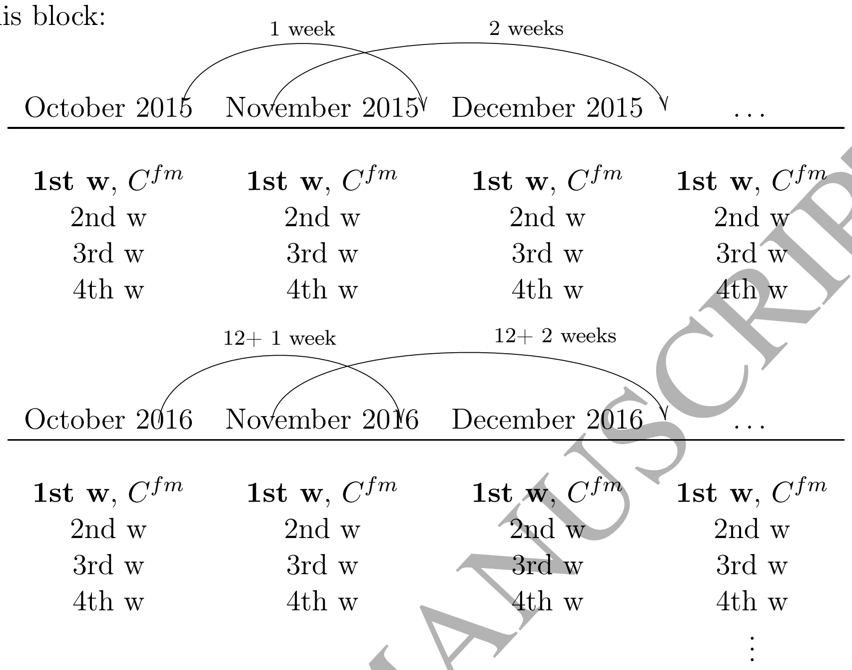

Figure 4 "First weeks of each month" criterion

According to the "first weeks of each month" criterion, Figure 4 lets visible the period: this is $d(i) = 4$ and the quotient set is $\mathbb{Z}_4 = \{[0], [1], [2], [3]\}$, where each block $[i]$ is as follows:

$$\begin{aligned} [0] &= \{0, 4, 8\ldots\} = \text{last weeks of month} \\ [1] &= \{1, 5, 9\ldots\} = \text{first weeks of month} \\ [2] &= \{2, 6, 10\ldots\} \\ [3] &= \{3, 7, 11\ldots\} \end{aligned}$$

Let us now reinterpret the parameter μ which measures the cash flows fluctuations. The definition of μ is somehow present already at the classical issue of the Transaction Demand for the Cash, Baumol (1952), Miller (1966), Tobin (1956) when defining the variance of daily changes in the cash balance, A, as $A = \mu^2 t$, where $\frac{1}{t}$ represents some small fraction of a working day. Recently, it has been considered in García Cabello (2013).

Definition 5.5 (Cash flow fluctuations index). For each branch, μ denotes the amount of euros that the branch cash balance increases or decreases in some small fraction of a working day. We refer to μ as branch cash flow fluctuations index (see Figure 5).

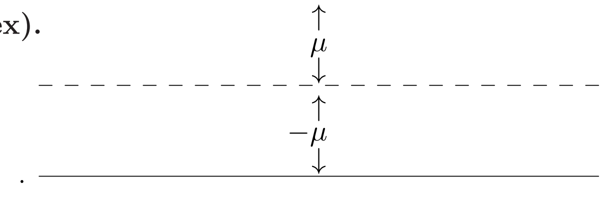

Figure 5 Cash flow fluctuations index

In our context, μ may be reinterpreted as the measure of similarity between two cash holdings whose corresponding weeks are in the same block: the smaller μ, the more similar the quantities.

As defined, the parameter μ depends mainly on the geographic location of the branch. To simplify the wide variety on branch locations, with different objectives and/or consumers financial habits, branches are here categorized into two main classes: city center branches, with rural site branches as a particular case, and business center branches. Since μ reflects the fluctuations of the branch cash holdings, it should be lower for branches located at city centers or rural locations while major fluctuations indexes should correspond to branches located near cash business centers.

Next Theorem summarizes the above underlying ideas:

Theorem 5.6. *For a branch with cash flow fluctuations index μ, the following statement are equivalent:*

1. $CH^{n_i} \leftrightarrow CH^{n_j}$ *(shorter, $i \leftrightarrow j$).*
2. $C^{n_i} = C^{n_j}$.
3. n_i, n_j *are into the same block.*
4. n_i, n_j *are congruent modulo $d(i)$.*
5. $[n_i]_{d(i)} = [n_j]_{d(i)}$ *or* $[n_i] = [n_j]$.
6. $CH^{n_i}, CH^{n_j} \in \left[C^{n_i k_i} - \mu, C^{n_i k_i} + \mu\right]$.
7. $|CH^{n_i} - CH^{n_j}| \leq \mu$.

Proof. Consider two states which communicate, $CH^{n_i} \leftrightarrow CH^{n_j}$ (shorter, $i \leftrightarrow j$). Hence the corresponding C-amounts are equal, $C^{n_i} = C^{n_j}$. Moreover, the former statement is equivalent to weeks n_i, n_j to be inside the same block. On the other hand, simple algebraic calculations show that weeks inside the same block follow a pattern of congruence modulo the period of the corresponding state. If $[n_i]_{d(i)} = [n_j]_{d(i)}$ stands for the congruence equivalence class modulo $d(i)$ (when no misunderstanding would arise, we will simply denote $[\quad]_{d(i)}$ as $[\quad]$) this statement allows us to identify $[n_i]$ with the block of weeks with similar features to that of n_i.

For weeks inside the same block, say n_i, n_j, the corresponding cash holdings CH^{n_i}, CH^{n_j} might be similar cash amounts, since the bulk of the cash holdings (that is, C^{n_i} and C^{n_j}) are equal. That is to say, for weeks n_i, n_j inside the same block, it holds that $CH^{n_i}, CH^{n_j} \in [C^{n_i} - \mu, C^{n_i} + \mu]$ or, equivalently, $|CH^{n_i} - CH^{n_j}| \leq \mu$.

As a consequence, the set of spaces of cash holdings can be considered as shown in next figure, upper bounded by the branch size BS:

$$
\begin{array}{cccc}
\multicolumn{4}{c}{BS} \\
\hline
\left[C^{1m} - \mu, C^{1m} + \mu\right] & \left[C^{2m} - \mu, C^{2m} + \mu\right] & \ldots & \left[C^{nm} - \mu, C^{nm} + \mu\right] \\
\vdots & \vdots & & \vdots \\
\left[C^{12} - \mu, C^{12} + \mu\right] & \left[C^{22} - \mu, C^{22} + \mu\right] & \ldots & \left[C^{n2} - \mu, C^{n2} + \mu\right] \\
\left[C^{11} - \mu, C^{11} + \mu\right] & \left[C^{21} - \mu, C^{21} + \mu\right] & \ldots & \left[C^{n1} - \mu, C^{n1} + \mu\right] \\
\| & \| & & \| \\
CH^1 & CH^2 & \ldots & CH^n
\end{array}
$$

Figure 6 The sequential structure of the branch cash holdings by blocks

In global terms, Definitions 5.1 and 5.2 give a new angle on the Markov chains fundamentals. In particular, let it be noticed that they enable central equation $CH^n = C^n + R^{n-1}$ (2) to only perform within each block of weeks:

Corollary 5.7. *Theorem 5.6 enables equation $CH^n = C^n + R^{n-1}$ to only perform within each block of weeks: $CH^{[n]} = C^{[n]} + R^{[n-1]}$. It is also equivalent to regarding the former equation as a dynamical system which considers the previous step with similar features instead of the prior step.*

Then, those dynamical systems for whom Markov property holds and which may be categorized into groups of similar features are a potential application of the defined blocks approach.

6. Time validity of the current cash holdings

This section aims to outline conditions under which the branch cash holdings remain constant, that is, conditions which let bank managers know the time validity of the current cash holdings. Conditions of this type are the key to guaranteeing efficient branch management for several reasons. They help keep branches on sound financial footing while providing the compulsory-by-law financial safety cushion. They guarantee an optimal level of cash inside the branch without generating either surplus or shortage of money. They allow banking entity management to make sound decisions upon fund investments. And in general, they help bank managers avoid not-desirable fluctuations of the branch cash flow. Such conditions will be studied separately as long-term projections, useful for the managers responsible for bank entities, and as short-term rules, suitable for branch managers. Previously, we derived some results as functional branch rules in order to further fine-tune the internal branch decision processes with regard to daily branch cash management.

Let us start with a few words about the hitting time. Let τ_i be the hitting time of the first visit to state i, $\tau_i = min\{n \in \mathbb{N} /\ S^n = i\}$. Therefore, m_i is defined as the expectation of τ_i: $m_i = E[\tau_i] = \sum_{i=n}^{\infty} n f_i(t)$ where $f_i(t)$ denotes de probability of a first visit to i after n steps. Hence, in our particular context, m_i represents the expected number of steps in return process to a concrete cash holdings, CH^{n_i}. The following result holds:

Theorem 6.1. *Consider the Markov chain $\{CH^n\}_{n\in\mathbb{N}}$ with n as counter of weeks, Thus, the discrete random variable τ_i which measures the hitting time to first visit to state i is a multiple of the period of the state,*

$$\tau_i = \tau \cdot d(i). \tag{3}$$

Proof. This follows from the definition of hitting time as minimum of steps in returning to each state together with the new angle provided by Definitions 5.1 and 5.2 since "visiting a state" now means to reach any of the states belonging to the same equivalence class.

Let us continue examining the period $d(i)$ of each state i. In branch reality, each period $d(i)$ comes from a branch managers' choice of a criterion which groups the weeks in order to require from the central hub the same cash amount for all of them. Once the criterion is picked up, the year becomes partitioned into blocks of weeks, as shown in Figure 7:

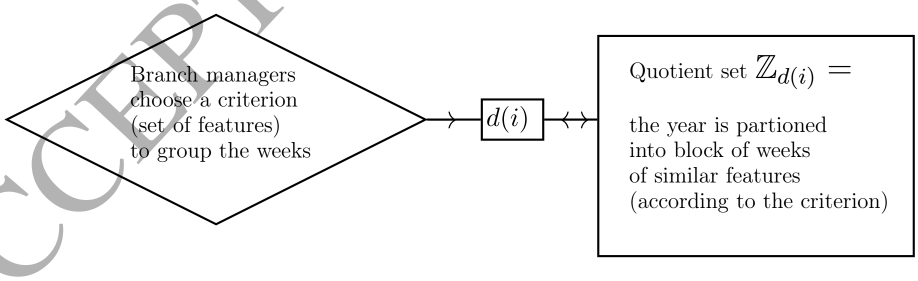

Figure 7 The year partitioned into block of weeks

The number of suitable criteria which would apply to a branch depends on branch features as size or geographic location. We refer to these criteria as feasible. The following corollary details the dependence of m_i upon feasible criteria:

Corollary 6.2. *Let $d(i_1), \ldots, d(i_r)$ be the feasible criteria for a branch. Thus, for each state i, m_i depends on the number of its feasible periods. Actually, m_i is a linear combination of feasible periods $d(i_1), \ldots, d(i_r)$:*

$$m_i = \tau \left[(f(d(i_1))d(i_1) + \ldots + f(d(i_r)d(i_r) \right]. \tag{4}$$

PROOF. Note that the number of feasible periods $d(i)$ is finite, $\{d(i_1), \ldots, d(i_r)\}$, since the set of states of $\{C^n\}_{n\in\mathbb{N}}$ could be considered finite for the branch managers's practice. Hence,

$$\begin{aligned} m_i &= E[\tau_i] = \\ &= E[\tau \cdot d(i)] = \\ &= \tau E[d(i)] = \\ &= \tau \left[f(d(i_1))d(i_1) + \ldots + f(d(i_r)d(i_r) \right] \end{aligned}$$

where $f(d(i_k))$ denotes the probability of occurrence of period $d(i_k)$.

Thus, we claim that

Proposition 6.3. *For each state i, m_i increasingly depends on the fluctuations on the branch cash holdings, μ. That is, $m_i = m_i(\mu)$, with* $\dfrac{\partial m_i}{\partial \mu} > 0$.

PROOF. This is consequence of the assumption from the empirical evidence that the number of criteria to be applied at each branch increasingly depends on the kind of branch as classified before: this shall be higher for branches with high fluctuations on cash holdings while lower for those branches with low fluctuations on cash holdings.

Next Theorem computes the expected number of steps in return process to state i, m_i, inside a block $\{CH^i\}$, $i \equiv 0 \; mod \; d(i))$.

Theorem 6.4 (Number of weeks in return process). *Let $\{CH^i\}$, $i \equiv 0 \; mod \; d(i))$ be a Markov chain of cash holdings corresponding to weeks inside a block. Thus, the expected number of steps in return process to some state CH^i (shortly i), m_i, is*

$$m_i = \frac{d(i)}{gcd\{i, d(i)\}}. \tag{5}$$

PROOF. Recall that the hitting time of the first visit to state i, τ_i, is $\tau_i = min\{n \in \mathbb{N} \,/\, S^n = i\}$ whereas m_i is defined as the expectation of τ_i, $m_i = E[\tau_i]$. From Theorem 6.1, τ_i is a multiple of $d(i)$, $\tau_i = \tau \cdot d(i)$. The same property holds for i since $i \equiv i \, mod \, d(i)$.
Let lcm/gcd be least common multiple and greatest common divisor, respectively. As a consequence of the product property for quotient sets modulo ($a \equiv b \, mod \, m, c \equiv d \, mod \, m \; \Rightarrow \; ac \equiv bd \, mod \, m$) $\tau_i \cdot i$ is multiple of $d(i)$ as well as multiple of i. For the definition of τ_i, thus $\tau_i \cdot i = \text{lcm}\,\{i, d(i)\}$.
Hence

$$i \cdot d(i) = \text{lcm}\,\{i, d(i)\} \cdot \text{gcd}\{i, d(i)\},$$

from Number Theory. Then,

$$i \cdot d(i) = \underbrace{\text{lcm}\,\{i, d(i)\}}_{\tau_i \cdot i} \cdot \text{gcd}\{i, d(i)\} \Rightarrow \tau_i = \frac{d(i)}{\text{gcd}\{i, d(i)\}},$$

which proves the desired result.

Once m_i has been computed, we may apply Ergodic Theory. This is the study of long-term behavior in dynamical systems from a statistical point of view, intimately connected with the time evolution of systems modeled by measure-preserving actions, where the action representing the passage of time. Particularly, the Ergodic theorem establishes conditions to determine the steady state behavior of a Markov chain as an application of the Strong Law of Large Numbers, provided that the Markov chain is irreducible. Ergodic Theorem will enable us to state conditions under which the branch cash holdings remain constant. It should be applied to chains of cash holdings corresponding to weeks into the same block (irreducible).

We adopt here the notation $V_i(n)$ to represent the number of visits to state i before step n. This theorem states that the long-run value of the ratio $V_i(n)/n$, which is the proportion of time spent in state i before step n, equals to the inverse of the expected return time to state i.

Theorem 6.5 (Ergodic Theorem). *For any irreducible Markov chain* $\{S^n\}$*, then*

$$lim_{n\to\infty}\frac{V_i(n)}{n} = \frac{1}{m_i}. \tag{6}$$

The dominant interpretation of the Ergodic Theorem is that $1/m_i$ is the average time of permanence at state i. In our context, Ergodic Theorem provides thus conditions under which the cash holdings remain constant although this should be applied to each block since the Markov chain ${CH^n}_{n\in\mathbb{N}}$ is not irreducible under the classical definitions. Thereby, the average amount of time that the Markov chain stays on state i equals to the inverse of the expected number of steps in return process to state i. That is to say,

Theorem 6.6 (Ergodic Theorem for branches). *For those weeks into the same block, the average time validity of the current cash holdings* CH^i *is equal to*

$$\frac{gcd\{i, d(i)\}}{d(i)}. \tag{7}$$

PROOF. It is sufficies to apply previous Ergodic Theorem, 6.5, toghether with the result achieved at Theorem 6.4.

Next result can also be demonstrated:

Theorem 6.7 (The average time validity). *The average time validity of the current cash holdings is a class property.*

PROOF. Let i be a week and consider any other j such that $[i] = [j]$, i.e., $j = i+\alpha\cdot d(i),$ for some $\alpha \in \mathbb{Z}$. Besides, the period of a state i, $d(i)$, is a class property: this implies in turn that $d(i) = d(j)$. From basic Number Theory, $\gcd\{b + \alpha \cdot a, a\} = \gcd\{b, a\}$. Hence,

$$\gcd\{j, d(i)\} = \gcd\{i + \alpha \cdot d(i), d(i)\} = \gcd\{i, d(i)\} \Rightarrow \frac{\gcd\{j, d(i)\}}{d(i)} = \frac{\gcd\{i, d(i)\}}{d(i)}.$$

7. A numerical example

This section is aimed at developing a case in point -intended for illustrative purposes- which should show how the previous theorems may be run. This numerical example, supported by real banking records based on data transactions, should highlight the real gain for entities who adopt the proposed method as a complement for IT technologies[9] since it may be easily (and at no cost) converted into an algorithm[10].

[9]The main features of this proposal -it is precise and very simple to be implemented at daily branch practices, assuring costs reductions- would allow it to co-exist with IT technologies, providing extra-support for branch managers' decisions.

[10]This result shall be addressed in a forthcoming paper.

In order to carry out this task, real banking information has been processed. The dataset is based upon excel files that contain all daily branch operations from June to December 2012 of some representative Spanish branch of a well known Spanish bank[11]. Despite our initial database was originally written using the entity's specific code, significant external operations have been extracted/separated from those internal organizational orders (accounting entries) as part of the database processing.

The general procedure is based on the following steps:

1. *Branch managers select a criteria for grouping the weeks.* Once again (see former example 5.4), we select "first weeks of each month", since it is amongst the customary criteria applied by branch managers.
2. *Once the criterion is picked up, the year becomes partitioned into blocks of weeks.* According to the "first weeks of each month" criterion, the period of a state i is $d(i) = 4$ and the quotient set is $\mathbb{Z}_4 = \{[0], [1], [2], [3]\}$, where each block $[i]$ is as follows (see former Figure 4):

$$\begin{aligned} [0] &= \{0, 4, 8 \ldots\} = \text{last weeks of month} \\ [1] &= \{1, 5, 9 \ldots\} = \text{first weeks of month} \\ [2] &= \{2, 6, 10 \ldots\} \\ [3] &= \{3, 7, 11 \ldots\} \end{aligned}$$

3. For those weeks inside the block of "first weeks of each month", we will compute the average time validity of the current cash holdings CH^i, according to Theorem 6.6. Next table shows how time only moves along the block of "first weeks of each month", with a week as time unit:

2012	Jan	Feb	March	April	May	June	July	Aug	Sep	Oct	Nov	Dec
First weeks						✓	✓	✓	✓	✓	✓	✓
Second weeks												
Third weeks												
Last weeks												

Table 1: Average time validity of the cash holdings.

As the time validity should be the same for those weeks inside the same block, according to Theorem 6.7, we only compute this for June. This is as follows:

$$\text{June 2012, i=1:} \quad \frac{\gcd\{1, d(1)\}}{d(1)} = \frac{\gcd\{1, 4\}}{4} = \frac{1}{4} \text{ week (1,75 days).}$$

It also applies for the average time validity of the cash holdings at states $5, 9, \ldots$, (first week of July is state $5 = 1 + 1 \cdot 4$, first week of August is state $9 = 1 + 2 \cdot 4 \ldots$)

4. The Theorem 6.6 should be applied now. Recall that this result provides conditions under which the cash holdings CH^i remain constant. This sentence may be interpreted as CH^i *do not deviate from the expected value stated for* i *(standard)*. These *standards* for each state i are an inherent feature for each branch. They may be easily computed by bank entities through their own huge amounts of data. Furthermore, for each specific branch, the expertise' eye of the office director is certainly familiar with them.

[11]In order to comply with legislation, the name of the bank must be kept private.

Thus, from previous point, cash holdings remain constant $\frac{1}{4}$ week for those weeks inside the block $[1] = \{1, 5, 9 \ldots\}$ (=first weeks of month). This can be seen from following figures, which display the comparison between current cash holdings and the corresponding standard for the first week of months from June to December 2012. Grey bars represent real cash holdings and black ones the standards. The x axis shows days of the corresponding first week (blanks represent weekends or holidays) while cash amounts in Euros appear on the y axis.

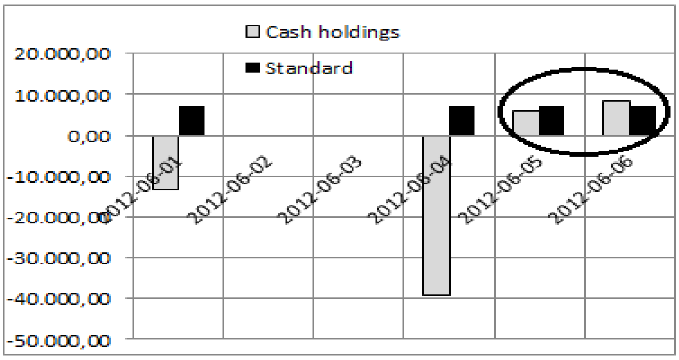

Figure 8 First week of June

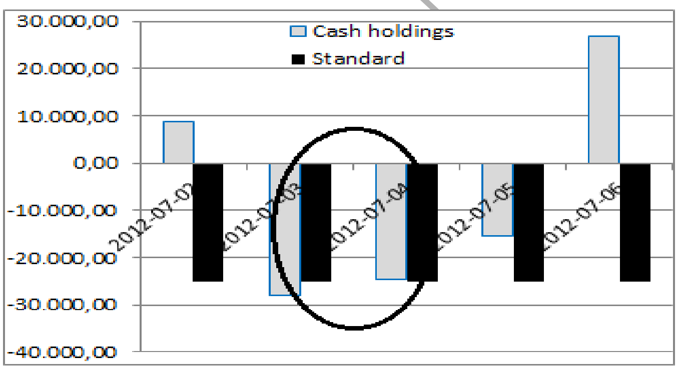

Figure 9 First week of July

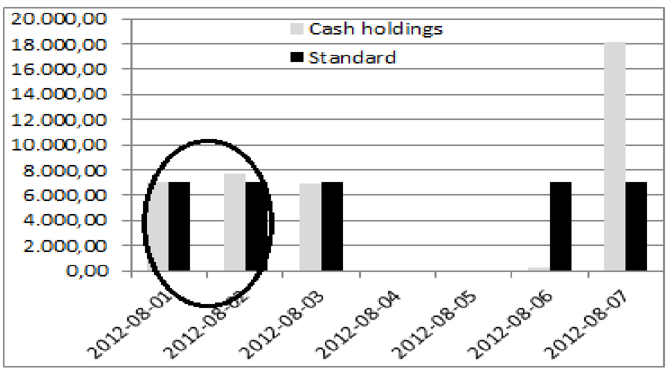

Figure 10 First week of August

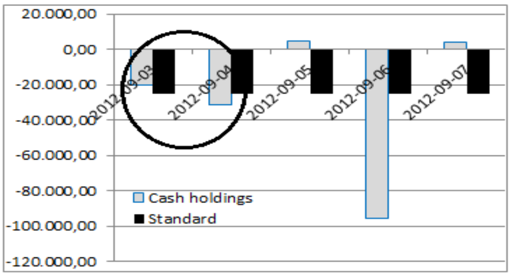

Figure 11 First week of September

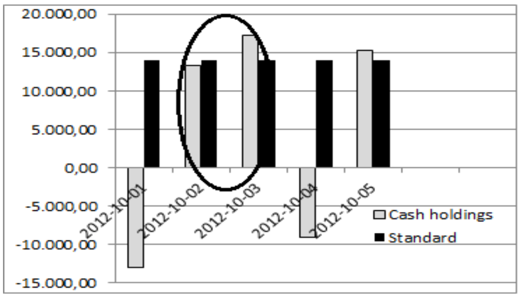

Figure 12 First week of October

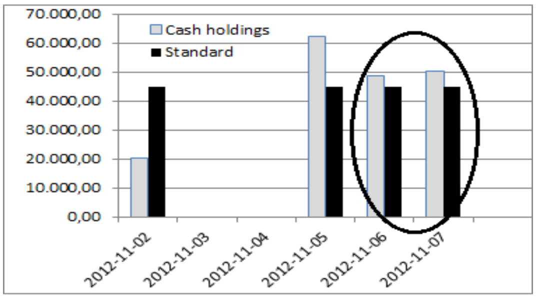

Figure 13 First week of November

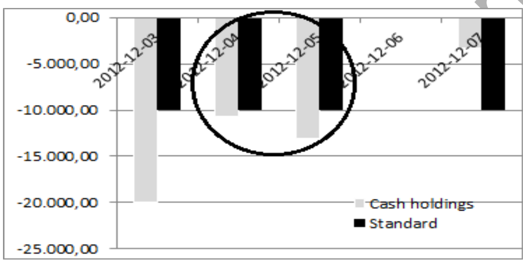

Figure 14 First week of December

8. Conclusions and direction for future research

This paper attempts to provide a new methodology by re-defining the classical fundamentals of Markov chains, thereby providing an alternative point of view to the classical. This new theoretical setting is applied for analyzing bank branching cash holdings: conditions which ensure optimality, recurring properties to better predict cash holdings shifts and knowledge about their steady-states in order to envision *the time validity of the current cash holdings.*

The marginal benefit of using the blocks approach -instead of the classical approach- is based on the fact that having blocks makes the problem more granular and therefore closer to the desired optimality. Once the real scenario has been mathematically modeled, this new framework offers many chances to continue exploring other possibilities, apart from those employed in the present paper. This incipient perspective may be as fruitful as the classical theory in applying a wide variety of properties to the cause of modeling banking branch cash holdings.

The results that we achieved in this paper enjoyed such broad support since they attempt to be suitable for all kind of branches, regardless their size or geographic location. Moreover, this set of theorems may be easily (and at no cost) converted into an algorithm which would co-exist with IT technologies, providing extra-support for branch managers' decisions: this is a future research project within a foreseeable period of time.

These suitability criteria also allowed for its application for different contexts apart from the banking scenario. This is the case for currency exchange offices as well as other settings where liquid provisions have to be made by adjusting monetary exits and entries. The breadth of our mathematical groundwork is an advantage: this new methodology may also apply to contexts where cash is not the star product. Actually, any dynamical system with the Markov property *which may be categorized into groups of similar features* is a potential application of the defined framework.

Specifically, let $\{X^n\}$ be a dynamical system where the Markov property holds with time variable n to be incremented discretely corresponding to the integers $\{0, 1, 2, 3, 4, ...\}$ (discrete dynamical system holding the Markov property). Besides by hypothesis, any of the states X^n and/or the corresponding time unit n may be categorized into groups of similar features. Thus, we let definition

5.1 remain the same: two states X^{n_i} (shorter state i) and X^{n_j} (state j), corresponding to time units n_i and n_j communicate, $i \leftrightarrow j$, if both time units n_i and n_j exhibit the same set of features. Next, we set definition 5.2 to be adequate to the required context: we say that

X^{n_i} and X^{n_j} communicate if `we put here the desired definition`.

As a result, not every state X^n communicates each other, only those which correspond to time units with similar characteristics. Therefore, the set of these constitues a block. Moreover, irreducible subchains in $\{X^n\}_{n\in\mathbb{N}}$ are those formed by states corresponding to time units inside the same block. Next step is to apply Ergodic Theory to the blocks in order to determine the dynamical system steady-states as well as exploring other possibilities in the desired context.

9. Acknowledgements

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