

An Approach to Identify Solutions of Interest from Multi and Many Objective Optimization Problems

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Abstract The result of a multiobjective or a many objective optimization problem is a large set of non-dominated solutions. Once the Pareto Front (or a good approximation of it) has been found, then providing the decision maker with a smaller set of “interesting solutions” is a key step.

Here the focus is on how to select such a set of Solutions of Interest which, in contrast to previous approaches that relied on geometrical features, is carried out considering the Decision Maker’s preferences. The proposed *a posteriori* approach consists in assigning an interval of potential scores to every solution, where such scores depend on the decision maker’s preferences. The solutions are then compared and filtered according to their corresponding intervals, using a recently proposed possibility degree formula. Three examples, with two, three and many objectives are used to show the benefits of the proposal.

Keywords: Multi Objective Optimization, Solutions of Interest, Preferences articulation

1 Introduction

Most of the problems arising in the current social and technological context require the simultaneous optimization of several conflicting objectives. These multiobjective (and many objective) optimization problems (MOPs, in what follows) are approximately solved through specific optimization algorithms that try to obtain

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a set of solutions which hopefully are close to the so-called true Pareto Optimal Set.

Research on these algorithms (mainly those belonging to the class of evolutionary algorithms) has been prolific in the last decades. Despite their type, all of them rely on the fact that “the ultimate goal of multiobjective optimization is to help the decision maker find solutions that meet, at most, his/her preferences” [18].

In this context, the motivation for this work comes from the following observation: *while the Pareto Front of a given MOP is unique, the selected solution will depend on the preferences of the Decision Maker (DM).*

The relation between the DM’s preferences and MOPs has many facets that can, essentially, be grouped into two sets. Firstly, to incorporate the preferences in the solving algorithm (e.g. using goals, weights and reference vectors) thus guiding the searching algorithm to specific regions of the solutions’ space. These methods can be classified as *a priori* or *progressive* approaches [8].

Secondly, to use the preferences *after* the optimization process to help the DM selecting a solution. These approaches are known as *a posteriori* and consist, for example, in applying some techniques to help the DM to select his/her most preferred solutions [6]. In this case, the DM does not need to provide any preference information prior to the search.

A crucial point here is the number of solutions that the DM will have to manage. As stated in [8]:

“The number of elements of the Pareto optimal set that tends to be generated is normally too large to allow an effective analysis from the DM.”

Moreover, in [9], one can read:

“The DM is interested in discovering only the zone of the Pareto front corresponding to his/her preferences, instead of the whole Pareto front. It is essential to provide the DM with a small number of satisfactory alternatives, due to the human cognitive limitations [...]”

And a similar conclusion is given in [21]:

“The empirical investigation revealed dysfunctional effects of information overload if the respondents were provided with ten or more alternatives in the choice set [...]”

Therefore, there is a need to develop approaches to help the DM to find this set of *Solutions of Interest* (SOI, in what follows) that can be vaguely defined as those solutions that are preferred by the DM [4]. Several measures were proposed to identify the SOI (we will comment them later), but to the best of our knowledge, they focus on a more geometric interpretation of the solutions in the Pareto Front, losing the DM’s perspective.

The aim of this contribution is to present an approach to identify a set of SOI from a large set of (already known) non-dominated solutions taking the DM’s preferences into account, which is a clear distinction from existing methods. Thus, in the context of multiobjective optimization, this is an *a posteriori* approach. The DM’s preferences are given as a linear ordering of the objectives, therefore partially

avoiding the issue recognized in [26]: “*in many cases the user does not have a clear preference when little knowledge about the problem is available*”.

While other approaches search for SOI on an n -dimensional objectives space, this proposal relies on comparing solutions in a bi-dimensional space using a possibility distribution function whose values are later used to filter out and reduce the number of SOI.

The paper is organized as follows. Section 2 introduces some preliminary concepts of Multiobjective Optimization; other approaches to detect SOI; also possibility functions to make interval numbers comparison. The proposed approach is detailed in Section 3. Three examples using problems with two, three and many objectives are shown in Section 4 together with an overall discussion of the results. The conclusions are outlined in Section 5.

2 Background and Related Works

We recall here some basic concepts about multiobjective optimization and we comment on other approaches to detect SOI. Then we fully describe Possibility-degree formulae for interval numbers comparison.

2.1 Multiobjective Optimization

Basic definitions associated with MOPs are recalled below. They are adapted from [8], considering maximization problems.

Definition 1 (Multiobjective Optimization Problem, MOP): A MOP is defined as maximizing $F(x) = (f_1(x), \dots, f_n(x))$ subject to $g_i(x) \leq 0$, $i = 1, \dots, p$, and $h_j(x) = 0$, $j = 1, \dots, q$, $x \in \Omega$. A MOP solution maximizes the components of a vector $F(x)$ where x is a k -dimensional decision variable vector $x = (x_1, \dots, x_k)$ from some universe Ω . It is noted that $g_i(x) \leq 0$ and $h_j(x) = 0$ represent constraints that must be fulfilled while maximizing $F(x)$ and Ω contains all possible x that can be used to satisfy an evaluation of $F(x)$.

The term many objective optimization problem is applied when the number of objectives is greater than three.

Definition 2 (Pareto Dominance): A vector $u = (u_1, \dots, u_n)$ is said to dominate another vector $v = (v_1, \dots, v_n)$ (denoted by $u \succeq v$) if, and only if, u is partially greater than v , i.e., $\forall i \in 1, \dots, n, u_i \geq v_i \wedge \exists i \in 1, \dots, n : u_i > v_i$.

Definition 3 (Pareto Optimal Set): For a given MOP, $F(x)$, the Pareto Optimal Set, \mathcal{P}^* , is defined as:

$$\mathcal{P}^* := \{x \in \Omega \mid \neg \exists x' \in \Omega \mid F(x') \succeq F(x)\}.$$

Definition 4 (Pareto Front): For a given MOP, $F(x)$, and Pareto Optimal Set, \mathcal{P}^* , the Pareto Front \mathcal{PF}^* is defined as:

$$\mathcal{PF}^* := \{u = F(x) \mid x \in \mathcal{P}^*\}.$$

2.2 On Detecting Solutions of Interest

There are two pathways to provide the DM with a reduced set of non-dominated solutions.

The first involves incorporating the DM’s preferences in the solving algorithm thus guiding the searching algorithm to a “Region of Interest” (ROI); and secondly, to use some strategy *after* the optimization process to help the DM in selecting a solution. Methods within the first set require the DM to describe his/her preferences as desired goals, reference points or a pre-ordering on the objectives prior to the search. These *a priori* approaches have been widely used in the past. The interested reader can check some recent reviews on the topic [26, 20, 18]. As has been recently recognized in [18], the definition of ROI is vague: it could be any region of the Pareto Front, controlling its size is far from trivial, and just concentrating on a pre-defined ROI can lead the user to lose some relevant information regarding the boundaries of the Pareto Front. To overcome these problems, the authors propose, in the context of decomposition-based evolutionary multiobjective methods, a systematic way to incorporate the DM’s preference information, either a priori or interactively, based on a nonuniform mapping scheme of the reference points that guide the search. The DM is required to define an expected value on each objective.

Secondly, the reduction of a Pareto Front (obtained by any means) to a smaller set of “diverse solutions” can be attained using some sort of geometric interpretation. Following [5], several approaches exist for detecting SOI in multiobjective problems with more than two objectives.

The first such approach is the maximum convex bulge [11]. It is a purely geometric approach that selects a subset of those located in the “maximum bulge” of the Pareto Optimal Front as SOI. The second one is the hypervolume contribution [32]. The SOI are those that produce the maximum gain in the hypervolume. It is computationally expensive and it is also a purely geometric approach. The third one is the local curvature [4] approach. Some curvature is calculated from a given solution and its neighborhoods, and those with the highest curvature are considered more preferable. The whole approach may fail if the Pareto Front is neither continuous nor symmetric. The Expected Marginal Utility Measure (EMU) [7], the fourth approach, is aimed at identifying the so-called “knee solutions”. The knee is defined as a solution on the tradeoff surface, where significant compromise needs to be made in at least one objective in order to obtain small gains in another. This is a stricter definition than “solution of interest”. Although EMU seems to work well with 2-3 objectives, it is observed that “as the number of objectives increases, the proportion of solutions with nonzero EMUs decreases and there are very few solutions with unique EMU values” [5]. To overcome this limitation, the authors in [5] proposed EMU^r where, using recursive calculations of EMU, they are able to identify K unique sparse SOI.

Some aspects are common in these works: first, as no information regarding the DM’s preferences is required, their authors proposed some metrics to detect solutions that are “interesting” from a geometrical point of view; and second, as these metrics focus on different features, they may end up suggesting a completely different set of solutions.

2.3 Possibility-degree formulae for interval numbers comparison

In our proposal, every solution in the Pareto Front will be assigned to an interval, thus we will compare solutions in terms of their corresponding intervals. Comparing and ranking interval numbers is a widely studied topic. Although reviewing the methods in the literature is out of the scope of this work, some references are given in order to offer a general view of the topic.

Since the initial work from Moore [22] on interval arithmetic, many approaches have been proposed, ranging from establishing order relations between interval numbers (which are summarized in [14]) to ranking methods that depend on probabilistic or fuzzy concepts [16, 23].

In the present work, the focus is on using possibility distributions for interval numbers comparisons, following the approach presented in Liu et al. [19]. The authors propose a formulation of a possibility degree to compare (and thus rank) intervals depending on a function that reflects the attitude, idea and knowledge of the DM. Moreover, under certain conditions, Liu's proposal is able to capture the behaviour of other approaches.

Let $A = [a_l, a_r]$, $B = [b_l, b_r]$ be two nonnegative interval numbers with $a_l, a_r, b_l, b_r \in R_0^+$. The possibility degree of A being greater than B , namely $P(A \geq B)$, proposed in [19] is defined as follows:

1. if $A \cap B = \emptyset$,

$$P(A \geq B) = \begin{cases} 0 & a_r \leq b_l \\ 1 & a_l \geq b_r \end{cases} \quad (1)$$

2. if $A \cap B \neq \emptyset$,

$$P(A \geq B) = \frac{\int_{b_l}^{a_r} f(x) dx}{\int_{b_l}^{a_r} f(x) dx + \int_{a_l}^{b_r} f(x) dx}, \quad (2)$$

where $f(x)$ describes the attitude of the DM.

Three typical functions for modeling the DM's attitudes are suggested: $f(x) = c$ a neutral attitude, $f(x) = 1/x$ a pessimistic attitude¹ and $f(x) = \sqrt{x}$ an optimistic attitude.

When $A \cap B \neq \emptyset$ the possibility degree for each attitude is calculated as follows:

$$\text{Neutral attitude} \rightarrow P(A \geq B) = \frac{a_r - b_l}{a_r - a_l + b_r - b_l} \quad (3)$$

$$\text{Pessimistic attitude} \rightarrow P(A \geq B) = \frac{\ln a_r - \ln b_l}{\ln a_r - \ln a_l + \ln b_r - \ln b_l} \quad (4)$$

$$\text{Optimistic attitude} \rightarrow P(A \geq B) = \frac{a_r \sqrt{a_r} - b_l \sqrt{b_l}}{a_r \sqrt{a_r} - a_l \sqrt{a_l} + b_r \sqrt{b_r} - b_l \sqrt{b_l}} \quad (5)$$

The previous formulation with a neutral attitude (Eq. (3)) is able to capture the behaviour of other approaches [10, 12, 25, 27, 31]. Also, the formulations proposed

¹ For certain attitude functions like $f(x) = 1/x$ the intervals must be defined in R^+ .

	$P(A \geq B = [0.3, 0.7])$		
	Neutral	Optim.	Pessim.
$A_0 = [0.1, 0.2]$	0	0	0
$A_1 = [0.1, 0.6]$	0.33	0.35	0.26
$A_2 = [0.3, 0.6]$	0.43	0.42	0.45
$A_3 = [0.3, 0.7]$	0.50	0.50	0.50
$A_4 = [0.4, 0.8]$	0.63	0.62	0.64
$A_5 = [0.5, 0.7]$	0.67	0.64	0.72
$A_6 = [0.8, 0.9]$	1	1	1

Table 1 Comparison of seven intervals A_i against a reference one $B = [b_l, b_r] = [0.3, 0.7]$. The evaluations show the value of $P(A \geq B)$ with Neutral, Optimistic and Pessimistic DM's attitudes.

in [30,33] are equivalent to some of the above [17]. That means that a proper use of the formulation proposed by Liu et al. allows the same results as other methods to be obtained while, at the same time, the interval comparison is easily modified to consider different DM's attitudes.

Table 1 shows seven cases of comparison between intervals according to the previous definition of possibility degree and three different attitudes. For each interval $A = \{A_0, A_1, A_2, A_3, A_4, A_5, A_6\}$ the possibility degree of A being greater than B , $P(A \geq B = [0.3, 0.7])$ is displayed. An interval A_i has a possibility degree greater than 0.5 if the interval lies "above" B , and less than 0.5 if it lies "below". Note that, with all attitudes, the interval $A_0 = [0.1, 0.2]$ has zero possibility of being greater than interval $B = [0.3, 0.7]$ (it has no overlapping and is always lower than B) and the interval $A_6 = [0.8, 0.9]$ has a possibility of 1 (it has no overlapping and is always greater than B). All attitudes also lead to $P(A_3 \geq B) = 0.5$ because an interval always has a possibility degree of 0.5 of being greater than itself.

In what follows, and for the sake of clarity and simplicity, the neutral attitude detailed in Eq. (3) will be used.

3 An approach to identify Solutions of Interest

In this section we describe our proposal: an approach to identify a set of "Solutions of Interest" (SOI) from a possibly large Pareto Front.

The general idea is to assign an interval to every solution. The intervals represent the range of potential scores that a solution can attain after an aggregation process. Then, using the possibility degree formulae, a comparison between those intervals is made which is later used to filter them obtaining the set of SOI. The overall approach is shown with the workflow in Fig. 1. There are four steps, namely, 1) intervals calculation, 2) reference interval identification, 3) intervals comparison and 4) set of SOI calculation. These steps are explained below.

3.1 Step 1: Intervals Calculation

After solving a given MOP by any means, we obtain a set of solutions $s_j \in \mathcal{P}^*$, $j = 1, \dots, m$ that can be organized as a table where every row is a solution $\{s_1, s_2, \dots, s_m\}$ and every column represents each objective function $\{f_1, f_2, \dots, f_n\}$.

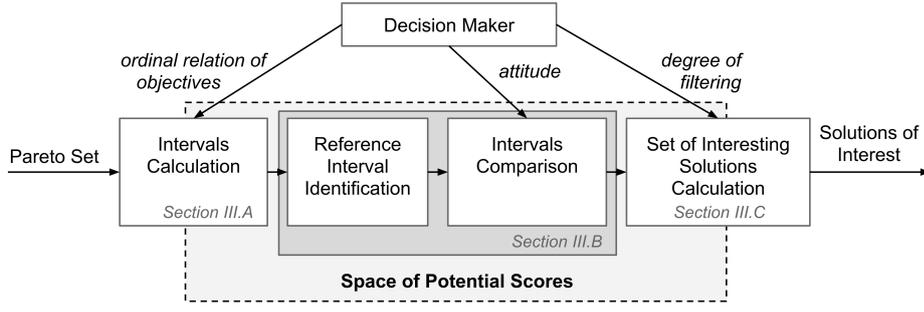


Fig. 1 Workflow diagram. The DM indicates the ordinal relation of objectives to calculate the intervals. Then, a reference interval is identified and the comparison between intervals is made according to the DM's attitude, obtaining an interval's evaluation. The last step is to use these evaluations to calculate the SOI.

We denote as $f_{ij} = f_i(s_j)$, the value of the solution s_j under the objective function f_i . Without loss of generality, it is assumed that $f_{ij} \in [0, 1]$, $i = 1, \dots, n$, $j = 1 \dots m$.

One simple way to detect SOI is to assign a score to every solution and then rank them, keeping the top ones. This can be readily done using aggregation functions. An aggregation function $ag(F(s_j), W)$, $ag : [0, 1]^n \rightarrow [0, 1]$ combines the inputs (f_{ij}) into a single value (score) that can be later used to sort the solutions. There is a great diversity of aggregation functions. The reader may be referred to [3] for a complete introduction to the topic. One well known aggregation function is the weighted sum:

$$ag(F(s_j), W) = \sum_{i=1}^n w_i f_{ij}, \quad (6)$$

where $W = \{w_1, w_2, \dots, w_n\}$ with $\sum w_i = 1$ and $w_i \in [0, 1]$.

Regarding W , it should be noted that a) weights' determination is far from trivial and studies have shown that the use of aggregation affects the quality of the decision [15], and b) any specific selection of the set of weights W leads to a different score for every solution. Possible sets of weights could be obtained using *Surrogate weighting methods* [1] with the most common ones being the Rank Sum (RS) weights, Rank Reciprocal (RR) weights [24], Rank Order Centroid (ROC) weights and Equal (EW) weights [2] or more sophisticated methods [28, 29] in the event of uncertain linguistic environments.

In this proposal, it is assumed that the DM's preferences are given using an ordinal relation among the objectives (thus avoiding that the DM provides a specific set of weights). We shall assume, without loss of generality, that this relation is $f_1 \succeq_p f_2 \succeq_p \dots \succeq_p f_n$. Symbol \succeq_p should be read as "is at least as preferred to". In terms of the weights, this implies $w_1 \geq w_2 \geq \dots \geq w_n$.

The Potential Scores space is then defined as a bi-dimensional space that represents all the scores a solution s_j can obtain under the assumption of $w_1 \geq w_2 \geq \dots \geq w_n$. It is built combining the inputs f_{ij} , $i = 1, \dots, n$, into two values $I_j = [L_j, U_j]$ (the minimum and maximum scores that a solution s_j can attain) with $L_j \leq U_j$ and $L_j, U_j \in [0, 1]$. It holds that $ag(F(s_j), W) \in [L_j, U_j]$, $\forall W$ satisfying $w_1 \geq w_2 \geq \dots \geq w_n$, $\sum w_i = 1$ and $w_i \in [0, 1]$.

Now the question is how such interval bounds L_j and U_j could be obtained. The solution of the two following independent Linear Programming (LP) problems, where $w_i, i = 1, \dots, n$ are the decision variables, is the answer.

$$\begin{aligned}
 \text{MIN} \quad & L_j = \sum_{i=1}^n w_i f_{ij}, \\
 \text{s.t.} \quad & \\
 & w_1 \geq w_2 \geq \dots \geq w_n, \\
 & \sum_{i=1}^n w_i = 1, \\
 & w_i \in [0, 1].
 \end{aligned} \tag{7}$$

$$\begin{aligned}
 \text{MAX} \quad & U_j = \sum_{i=1}^n w_i f_{ij}, \\
 \text{s.t.} \quad & \\
 & w_1 \geq w_2 \geq \dots \geq w_n, \\
 & \sum_{i=1}^n w_i = 1, \\
 & w_i \in [0, 1].
 \end{aligned} \tag{8}$$

Nowadays, solving these linear programs is easy and very fast from a computational point of view. However, as stated in [1], there is no need to run an algorithm like Simplex to solve these specific LP problems, because “*by the well-known properties of a linear program, only the extreme points of the ranked weights need to be considered to effect the desired optimum and they are readily available*”. This implies that, instead of solving the LP problems, solutions need to be scored only at the extreme points (three sets of weights). Then, the minimum and maximum are selected as L_j and U_j and the process is finished. Let n be the number of objectives, thus, the sets of weights to consider are:

1. $W = (1, 0, \dots, 0)$: all the weight is assigned to the most preferred objective.
2. $W = (1/(n-1), 1/(n-1), \dots, 1/(n-1))$: the least preferred objective is assigned $w_n = 0$ while the others get $w_i = 1/(n-1), \forall i \neq n$.
3. $W = (1/n, 1/n, \dots, 1/n)$: all the objectives are equally important.

For example, having a decision problem with three objectives, $n = 3$, and $f_1 \succeq_p f_2 \succeq_p f_3$, every solution (out of m) needs to be evaluated on just these three sets of weights: $(1, 0, 0)$, $(1/2, 1/2, 0)$ and $(1/3, 1/3, 1/3)$. Then the maximum and minimum value attained are selected.

Such process is linear in the number of objectives, thus the interval calculation is $O(n)$. Having m solutions, the computational complexity of this step is $O(mn)$. However, as the value of n is expected to be small², the interval calculation can be considered as constant. As a consequence, the whole process for this step can be considered linear in the number of solutions ($O(m)$).

It should be noted that this space is always bi-dimensional regardless of the number of objectives the initial MOP has. In other words, the approach can be readily used in a Many objective Optimization Problem.

² The benchmark provided in the Competition on Many-Objective Optimization at the 2018 IEEE Congress on Evolutionary Computation, included functions with 5, 10 and 15 objectives https://www.cs.bham.ac.uk/~chengr/CEC_Comp_on_Ma00/2018/webpage.html

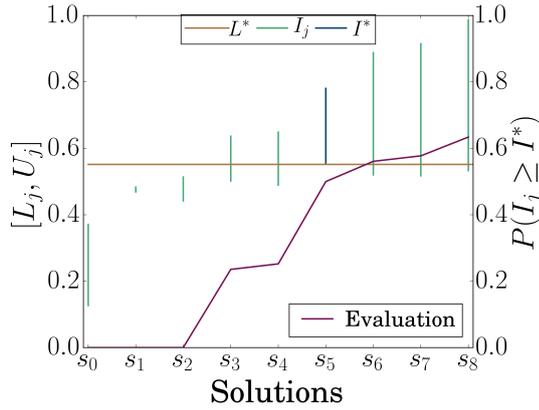


Fig. 2 Example of interval representation $I_j = [L_j, U_j]$ for each solution s_j . The reference interval is $I^* = I_5$. The evaluation line represents the value $P(I_j \geq I^*)$.

The following step consists in comparing solutions using the corresponding intervals in the Potential Scores space.

3.2 Steps 2 and 3: Reference Interval Identification and Intervals Comparison

The next step is to compare the solutions s_j using their corresponding intervals. Instead of using a full pairwise comparison, which would be computationally expensive, a reference solution s^* will be identified, and then the intervals $I_j = [L_j, U_j]$ will be compared against $I^* = [L^*, U^*]$ (the interval corresponding to s^*). It is here that the intervals comparison detailed in Sec. 2.3 appears.

The reference solution s^* is the one having the greatest lower bound value among all intervals in the Potential Scores space³, i.e., the one which has $I^* = [L^*, U^*] \mid L^* \geq L_j, \forall j$.

Then, for every solution s_j , the possibility degree $P(I_j \geq I^*) \in [0, 1]$ is calculated following Eq. (2) (considering a neutral attitude, $f(x) = c$). As s^* is the solution with the greatest lower bound there is no interval that verifies $P(I_j \geq I^*) = 1$. This means that there is no solution that *always* scores better than the chosen reference solution. The solutions s_j such that their intervals I_j verify $P(I_j \geq I^*) = 0$ will always attain a lower score than s^* : there is no set of weights that can make s_j better than s^* . Also, the reference solution s^* will always score 0.5 as $P(I^* \geq I^*) = 0.5$.

The reference interval identification is $O(m)$. The comparison of two intervals using the possibility distribution with a neutral attitude is $O(1)$, thus the whole process is $O(m)$.

Figure 2 shows the evaluation of $P(I_j \geq I^*)$ where $I^* = I_5$ (the reference solution is identified as $s^* = s_5$). This should be understood as follows: the possibility degree of interval $I_8 = [0.53, 0.99]$ being greater than the reference interval $I^* = [0.55, 0.78]$ is calculated as $P(I_8 \geq I^*) = \frac{0.99 - 0.55}{0.99 - 0.53 + 0.78 - 0.55} = 0.64$.

³ In the special case where this interval is not unique, the reference interval is the one that also has the greatest upper bound.

In this example, the solutions s_0, s_1, s_2 will never score better than the reference solution $s^* = s_5$ because their intervals on the Potential Scores space have a possibility degree of zero of being greater than the reference interval, that is, $P(\{I_0, I_1, I_2\} \geq I^*) = 0$. In general, every interval that lies completely below the horizontal line (which represents L^*) has always zero possibilities of being greater than the reference interval and, because of that, the corresponding solution is always worse than the reference solution.

3.3 Step 4: Set of Solutions of Interest Calculation

Up to this point, the reference solution s^* has been identified, every solution $s_j \in \mathcal{P}^*$ has an associated interval $I_j = [L_j, U_j]$ and the possibility degrees $P(I_j \geq I^*)$ were calculated. The idea now is to filter out solutions to reduce the amount of information the DM has to evaluate.

The set of SOI is defined as:

$$S_\lambda = \{s_j \mid P(I_j \geq I^*) > \lambda\}, \quad (9)$$

where $\lambda \in [0, 1]$ is a value selected by the DM and stands for the minimum value of P that a solution needs to have in order to be considered as ‘‘interesting’’. It holds that if $\lambda_1 < \lambda_2$ then $S_{\lambda_2} \subseteq S_{\lambda_1}$, the higher the value of λ , the smaller the size of S_λ . Also, $s^* \in S_\lambda \iff \lambda < 0.5$.

Note that if the solutions are sorted in terms of their possibility degrees (these can be done in $O(m \log(m))$), obtaining the different S_λ sets can be done in constant time.

We shall see some examples using the solutions from Fig. 2. If $\lambda = 0$, the set of SOI consists in the solutions s_j with $P(I_j \geq I^*) > 0$. Then, $S_{\lambda=0} = \{s_3, s_4, s_5, s_6, s_7, s_8\}$, 6 solutions have more than 0 possibility of being greater than the reference. If $\lambda = 0.3$, the set of SOI is $S_{\lambda=0.3} = \{s_5, s_6, s_7, s_8\}$.

Once the set of SOI is defined, the DM has a much smaller set of relevant solutions to work with.

4 Illustrative Examples

In this section, using examples from optimization problems with two, three and many objectives the behaviour of the proposal is illustrated.

Firstly, the DM needs to indicate his/her preferences among the objectives. Without loss of generality, it is assumed $f_1 \succeq_p f_2 \succeq_p \dots \succeq_p f_n$ and $f_i \in [0, 1]$, $\forall i$. Regarding the possibility distribution, a neutral attitude is considered (Eq. 3).

4.1 Example A: two objectives with random Pareto Front

The objective of this first example is to provide a clear view of the sets of SOI and their sizes using the Pareto Front, shown in Fig. 3(a). It contains 300 solutions from a two objective problem, generated using a randomized greedy algorithm available at <https://github.com/TorresM/DataSets>.

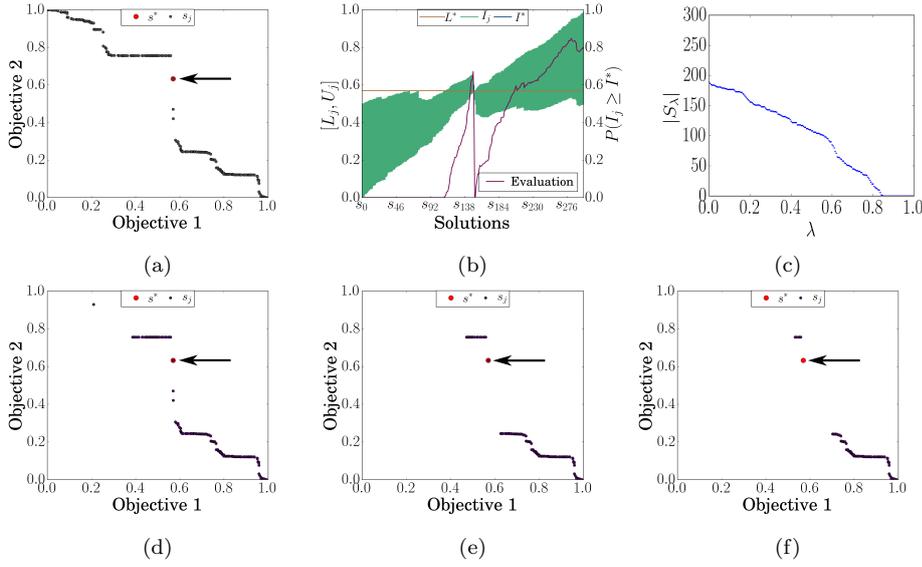


Fig. 3 Example A: (a) the initial Pareto Front with 300 solutions. The solutions are evaluated on two objectives with the objective 1 being preferred to the objective 2. Both of them are normalized. In (b), the interval representation of I_j , $I^* = [L^*, U^*]$ and the evaluations $P(I_j \geq I^*)$ for every solution s_j are depicted. Solutions follow the same order in both plots. The value of $|S_\lambda|$ as a function of λ is shown in (c). Three sets of SOI are shown below, (d) $S_{\lambda=0}$, (e) $S_{\lambda=0.25}$, and (f) $S_{\lambda=0.5}$, where the reference solution s^* is identified with an arrow.

Initially, the intervals on the Potential Scores space are calculated and the reference interval I^* is identified. In Fig. 3(a) the corresponding reference solution is marked with an arrow. Next, the possibility degree values $P(I_j \geq I^*)$ are calculated. Figure 3(b) shows both the scores' intervals I_j and the possibility degree $P(I_j \geq I^*)$ for every solution, s_j . The reference interval corresponds to the solution s_{151} , with $I^* = I_{151} = [0.57, 0.60]$.

Figure 3(c) shows the variation of the number of SOI ($|S_\lambda|$) as a function of λ . When $\lambda = 0$, $|S_{\lambda=0}| = 189$, which, in turn, means that 111 solutions are not in the set of SOI $S_{\lambda=0}$. In other words, 1/3 of the initial solutions are readily discarded: such solutions will never score better than the reference solution. As λ increases, the size of S_λ decreases. For example, $|S_{\lambda=0.25}| = 150$, $|S_{\lambda=0.5}| = 108$, $|S_{\lambda=0.6}| = 85$ and $|S_{\lambda=0.8}| = 20$. In this particular example, there is a clear linear relation between λ and $|S_\lambda|$.

Now the question is where these SOI are located in the Pareto Front. It should be recalled that $f_1 \succeq_p f_2$. Figure 3 shows the sets (d) $S_{\lambda=0}$, (e) $S_{\lambda=0.25}$ and (f) $S_{\lambda=0.5}$ together with the location of the reference solution. Although $s^* \notin S_{\lambda=0.5}$, it is included in the plot for visualization purposes.

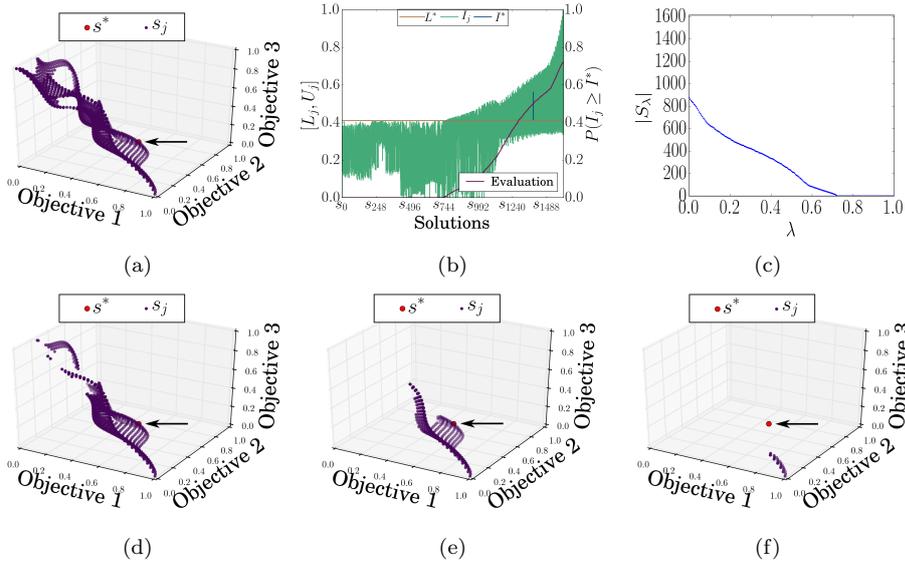


Fig. 4 Example B: (a) the initial Pareto Front with 1612 solutions from a three objective problem. In (b), the interval representation I_j , $I^* = [L^*, U^*]$ and $P(I_j \geq I^*)$ for every solution s_j . In (c), the value of $|S_\lambda|$ as a function of λ . Three sets of SOI are shown below, (d) $S_{\lambda=0}$, (e) $S_{\lambda=0.35}$, and (f) $S_{\lambda=0.7}$. The reference solution s^* is identified with an arrow.

4.2 Example B: three objectives problem

In this second example, taken from [7], the approach is applied to a larger Pareto Front corresponding to a three objectives optimization problem. The Pareto Front has 1612 solutions and is depicted in Fig. 4(a). Here we have $f_1 \succeq_p f_2 \succeq_p f_3$.

Firstly, the intervals on the Potential Scores space are calculated and the reference interval is identified: $I^* = I_{1391} = [0.41, 0.56]$. The corresponding reference solution is marked with an arrow. Then, the possibility degree of each interval $P(I_j \geq I^*)$ is calculated.

The intervals and their evaluations are shown in Fig. 4(b). In order to facilitate visualization, the solutions are ordered by their possibility degree value.

Figure 4(c) shows the size of SOI for different λ values. When considering $\lambda = 0$, then $|S_{\lambda=0}| = 879$ which means that the number of “interesting solutions” can be halved. If λ is further increased, smaller sets of SOI can be achieved. For example, $|S_{\lambda=0.35}| = 382$ and $|S_{\lambda=0.7}| = 23$. At this point, the DM will have a number of affordable solutions to consider.

It is also relevant to know the location of the SOI in the original Pareto Front. The sets (d) $S_{\lambda=0}$, (e) $S_{\lambda=0.35}$ and (f) $S_{\lambda=0.7}$ are depicted in Fig. 4. Although $s^* \notin S_{\lambda=0.7}$, it is included in the plot for visualization purposes.

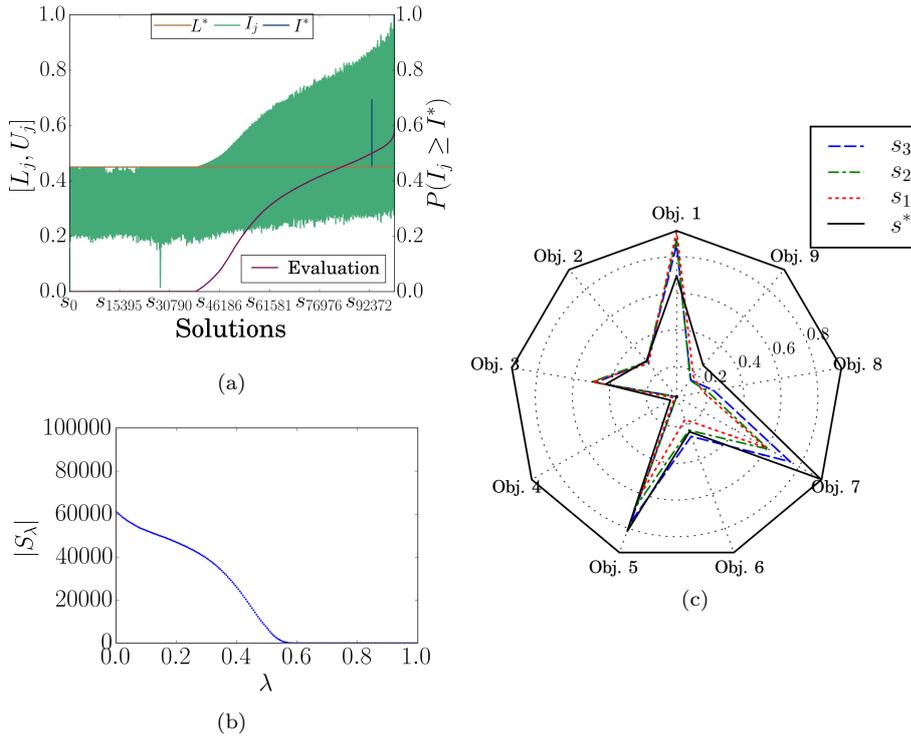


Fig. 5 Example C: (a) interval representation and evaluation of $P(I_j \geq I^*)$ for every solution s_j . In (b), the value of $|S_\lambda|$ as a function of λ is shown. In (c), the three solutions from set $S_{\lambda=0.6}$ and the reference solution s^* are depicted.

4.3 Example C: many objectives problem

The proposed approach is next applied to a real data case taken from [5]. The Pareto Front consists of 100070 non-dominated solutions from a 9 objectives radar waveform design problem originally published in [13].

Following the same steps as before, the intervals, the reference interval $I^* = I_{93085} = [0.45, 0.69]$ and the possibility degrees $P(I_j \geq I^*)$ are calculated (elements depicted in Fig. 5(a)). In order to facilitate the visualization, solutions are ordered by their possibility degree value.

The relation between $|S_\lambda|$ and λ is shown in Fig. 5(b). Departing from a set of 100070 solutions, and considering the following sets of SOI: $|S_{\lambda=0}| = 61271$, $|S_{\lambda=0.2}| = 46949$, $|S_{\lambda=0.4}| = 25983$ and $|S_{\lambda=0.6}| = 3$, reductions of 39%, 54%, 74% and 99.9% are obtained, respectively.

The solutions in $S_{\lambda=0.6}$ are displayed in Fig. 5(c). Although $s^* \notin S_{\lambda=0.6}$, it is included in the plot for visualization purposes. It can be observed how such solutions are better in terms of objective 1 (the most preferred one) and worse in objectives 7,8 and 9 (the lesser preferred).

4.4 Discussion

The key aspect of the proposal is to assign to every solution an interval representing the potential scores it can achieve if a weighted aggregation of the objectives is made. So interval construction and interval comparison are essential features.

Intervals construction relies on how the DM' preferences are managed and how the solution's information is aggregated. Here preferences are given as a linear ordering of the objectives. In this situation, upper and lower bounds for the solution's interval can be efficiently calculated when the potential scores are obtained through a weighted aggregation. This decision allows for a fast exploration of different objectives' ordering, thus allowing the DM to explore different areas of the Pareto Front.

Of course other aggregation functions may be used, but at the cost of potentially increasing the computational complexity of the interval calculation step. In other words, upper and lower bounds for the potential scores of the solutions could be hard to calculate under other aggregation functions.

The intervals comparison is made using a possibility distribution. Instead of doing a fully pairwise comparison among the solutions, we select a reference solution (the one with the highest lower bound) and just $m - 1$ comparisons are then needed to rank the solutions in terms of the reference.

The use of such possibility distributions allow to include some DM's characteristics in the process. Considering a neutral attitude as an initial step allows for a fast computation of the results. Again, other ways are possible to compare intervals and may deserve further research.

The previous examples showed the role of the parameter λ as a filtering degree. The ability to reduce the set of solutions in S_λ can be further analyzed exploring the relation $|S_\lambda|/|S|$ where $|S|$ is the number of initial solutions on the Pareto Front. The first element to highlight is that, by setting $\lambda = 0$, the SOI sets exclude only those solutions that would never be interesting for the DM because they will never attain a score better than that of the reference solution. Such solutions can be readily discarded with no "risk". For the examples A, B and C, $S_{\lambda=0}$ sets are considerably reduced: $|S_{\lambda=0}|/|S| = 0.63$, $|S_{\lambda=0}|/|S| = 0.55$ and $|S_{\lambda=0}|/|S| = 0.61$, respectively. This means that even when just those solutions s_j with $P(I_j \geq I^*) = 0$ are eliminated, the reductions are around 40% of the initial Pareto Front sets. Smaller sets of SOI can be obtained with higher λ values, like $|S_{\lambda=0.8}|/|S| = 0.07$, $|S_{\lambda=0.7}|/|S| = 0.01$ and $|S_{\lambda=0.6}|/|S| = 0.01$ for examples A, B and C, respectively.

So, a DM may start inspecting just a few solutions corresponding to some λ values (like $S_{\lambda=0.6}$ in example C) and progressively reducing such λ for obtaining more solutions. If DM wants a different kind of solutions, then a new preference order should be defined and the calculations should be repeated. As we stated in the proposal description, the computational complexity can be considered linear in the number of solutions. For the biggest example (with 100070 solutions), the whole process takes much less than a minute.

It is important to highlight that the number of objectives of the MOP is somehow irrelevant as the comparison of solutions is done over the corresponding intervals and, once the reference solution is identified, just $m - 1$ (with m being the size of the Pareto Front) intervals comparisons are needed in order to obtain the initial set of SOI.

5 Conclusions

In the context of multi or many objective optimization problems (MOP), an *a posteriori* approach is proposed to help the DM to obtain a reduced set of Solutions of Interest (SOI) after the MOP solving process is completed.

This approach, in contrast with others based on geometrical features, associates the concept of “interest” in terms of the DM’s preferences. Moreover, the DM’s attitude can be incorporated into the possibility distribution function used to compare intervals and a filtering degree (parameter λ) can be adjusted to control the size of SOI’s set.

Overall, the proposal is a) simple to understand and implement, b) it has a low computational complexity, and c) the examples clearly illustrate how a reduced set of SOI can be obtained for multi and many objectives optimization problems.

Several lines of research arise at this point.

Once a smaller set of SOI is obtained, it could be interesting to apply a multi-criteria decision making methods to rank the solutions using a more sophisticated modeling of the DM’s preferences (like using pairwise comparison among solutions). Also, the impact of using different aggregation functions for the intervals’ calculation needs to be assessed. Finally, the implementation of a visualization tool, linking the solutions at the Potential Scores space with the Pareto Front and the set of SOI may also help to better understand the relation among these spaces.

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Author contributions

All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by M. Torres and D. Pelta. All authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Conflict of Interest

The authors declare that they have no conflict of interest.

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