PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 151, Number 11, November 2023, Pages 4721–4733
https://doi.org/10.1090/proc/16483
Article electronically published on August 4, 2023

DERIVATIONS AND HOMOMORPHISMS IN COMMUTATOR-SIMPLE ALGEBRAS

J. ALAMINOS, M. BREŠAR, J. EXTREMERA, M. L. C. GODOY, AND A. R. VILLENA

(Communicated by Javad Mashreghi)

ABSTRACT. We call an algebra A commutator-simple if $[A, A]$ does not contain nonzero ideals of A. After providing several examples, we show that in these algebras derivations are determined by a condition that is applicable to the study of local derivations. This enables us to prove that every continuous local derivation $D\colon L^1(G) \to L^1(G)$, where G is a unimodular locally compact group, is a derivation. We also give some remarks on homomorphism-like maps in commutator-simple algebras.

1. Introduction

This paper is a continuation of [4] in which derivations and Jordan automorphisms of semisimple finite-dimensional algebras A were determined by certain conditions involving $[A, A]$, the linear span of all commutators in A. As a byproduct, new results on local derivations and local Jordan automorphisms were obtained. We recall that a linear map D from an algebra A to itself is called a *local derivation* if for every $x \in A$ there exists a derivation $D_x\colon A \to A$ such that $D(x) = D_x(x)$. Other "local maps" such as local automorphisms are defined similarly. The obvious problem of whether a local derivation is necessarily a derivation (or a local automorphism is necessarily an automorphism, etc.) has been an active research area since the early 1990s when it was proposed independently by Kadison [13] and Larson and Sourour [14].

We will consider similar conditions in arbitrary, not necessarily finite-dimensional algebras, as well as in Banach algebras. Our main motivation behind the latter has been an open question whether every continuous local derivation D of $L^1(G)$, the group algebra of a locally compact group G, is a derivation.

The following definition introduces a class of algebras to which our methods are applicable.

Definition 1.1. An algebra A is said to be *commutator-simple* if $[A, A]$ does not contain nonzero ideals of A.

Received by the editors October 25, 2022, and, in revised form, March 10, 2023.

2020 *Mathematics Subject Classification.* Primary 43A20, 47L10, 16W20, 16W25.

Key words and phrases. Derivation, automorphism, antiautomorphism, Jordan automorphism, local derivation, local automorphism, group algebra.

The first, third, fourth, and fifth authors were supported by the Grant PID2021-122126NB-C31 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of making Europe" and grant FQM-185 funded by Junta de Andalucía. The second author was supported by the Slovenian Research Agency (ARRS) Grant P1-0288. The fourth author was also supported by grant FPU18/00419 funded by MIU.

Section 2 is devoted to examples of commutator-simple algebras and Banach algebras.

In Section 3 we first show that if A is a commutator-simple semiprime algebra A (over a field of characteristic not 2), then a linear map $D\colon A \to A$ satisfying

$$D(x)x, D(x)x^2 \in [A, A]$$

for every $x \in A$ is a derivation. Using this, we then prove that continuous local derivations on $L^1(G)$ are derivations, provided that G is a unimodular locally compact group. While this does not completely solve the aforementioned question since we still need some conditions on G, it does generalize and unify several known results (see the final part of Section 3 for details). Besides, the approach we take is entirely different from those used earlier.

The final Section 4 discusses the condition that a linear map $T\colon A \to A$, where A is a commutator-simple algebra, satisfies

$$T(x)^3 - x^3 \in [A, A]$$

for every $x \in A$. Under suitable assumptions, T is shown to be a Jordan homomorphism. Some applications to local automorphisms are also given.

2. Commutator-simple algebras

By an algebra we mean an associative algebra which does not necessarily possess a unity. When it does, we call it a unital algebra. We use F to denote a field, and we assume that our algebras are over F. When speaking of Banach algebras, we assume that $F = \mathbb{C}$.

The purpose of this section is to provide examples and constructions of commutator-simple algebras. Our first proposition is trivial, but we record it since one can construct new commutator-simple algebras from old ones (see Propositions 2.4 and 2.8).

Proposition 2.1. *Every commutative algebra is commutator-simple.*

If there exists a linear functional τ on A^2, the linear span of all elements of the form xy with $x, y \in A$, such that for all $x, y \in A$,

$$\tau(xy) = \tau(yx), \tag{2.1}$$

and for all $x \in A$,

$$\tau(xA) = \{0\} \implies x = 0, \tag{2.2}$$

then A is obviously a commutator-simple algebra. The simplest example is the trace on the matrix algebra $M_n(F)$. Slightly more generally, we have the following.

Proposition 2.2. *The algebra of all finite rank linear operators on a (finite or infinite dimensional) vector space X is commutator-simple.*

Proof. This algebra has a trace τ. If x is a finite rank linear operator on X, then

$$\tau\big(x(\xi \otimes f)\big) = f(x(\xi))$$

for each $\xi \in X$ and each linear functional f on X. This clearly implies that τ satisfies (2.2). □

Proposition 2.3. *For any (not necessarily finite) group G, the group algebra $F[G]$ is commutator-simple.*

Proof. For any element $x = \sum_{g\in G} \lambda_g g \in F[G]$, we define $\tau(x)$, the trace of x, as λ_e, the coefficient of the identity $e \in G$. Taking another element $y = \sum_{g\in G} \mu_g g \in F[G]$, we have

$$\tau(xy) = \sum_{g\in G} \lambda_g \mu_{g^{-1}} = \sum_{g\in G} \mu_g \lambda_{g^{-1}} = \tau(yx).$$

Thus, τ satisfies (2.1). Since $\tau(xg^{-1}) = \lambda_g$, it also satisfies (2.2). □

Proving that the ordinary group algebra $F[G]$ is commutator-simple is thus very easy. However, we are more interested in group algebras $L^1(G)$ where G is a locally compact group. We will discuss this at the end of the section.

The following proposition needs no proof.

Proposition 2.4. *If $(A_i)_{i\in I}$ is a family of commutator-simple algebras, then the direct product $\Pi_{i\in I} A_i$ is also commutator-simple.*

We remark that a simple algebra A is obviously commutator-simple if and only if $A \neq [A, A]$.

Proposition 2.5. *Every finite-dimensional semisimple algebra A is commutator-simple.*

Proof. In light of Proposition 2.4, we may assume that A is simple. The property that a unital F-algebra A is commutator-simple does not depend on the field F, so there is no loss of generality in assuming that A is a central F-algebra (i.e., the center of A is F). Let $\overline{F}$ be the algebraic closure of F. It is easy to see that $A = [A, A]$ implies that $\overline{A} = \overline{F} \otimes_F A$, the scalar extension of A to $\overline{F}$, also satisfies $\overline{A} = [\overline{A}, \overline{A}]$. However, $\overline{A}$ is isomorphic to the matrix algebra $M_n(\overline{F})$ which, by Proposition 2.2, does not have this property. Therefore, $A \neq [A, A]$, meaning that A is commutator-simple. □

Proposition 2.5 does not hold for infinite-dimensional simple algebras, in fact not even for division algebras – see, for example, [10]. One may therefore ask what are examples of infinite-dimensional algebras that are both simple and commutator-simple.

The classical Weyl algebra A_1 is simple, but not commutator-simple (as it is equal to $[A_1, A_1]$). However, many *generalized Weyl algebras* $A(F[z], a, \sigma)$ are commutator-simple. These algebras were introduced by V. V. Bavula [2] to whom the authors are thankful for providing the relevant information. We recall the definition. Let a be an element of the polynomial algebra $F[z]$, where F is a field of characteristic 0, and let σ be an automorphism of $F[z]$. The generalized Weyl algebra $A = A(F[z], a, \sigma)$ is the algebra obtained by adjoining to $F[z]$ the new variables x and y subject to the relations

$$yx = a, \ xy = \sigma(a),$$

and

$$xf = \sigma(f)x, \ fy = y\sigma(f)$$

for all $f \in F[z]$. This algebra is simple if and only if the difference of two roots of the polynomial a is not an integer [2]. Further, if σ is defined by $\sigma(f) = f - \lambda$ with $0 \neq \lambda \in F$ and the degree of a is greater than 1, then $A \neq [A, A]$ by [7]. Therefore, the following holds.

Proposition 2.6. *Let $a \in F[z]$ be a polynomial of degree at least 2 such that the difference of two roots of a is not an integer, and let σ be the automorphism of $F[z]$ defined by $\sigma(f) = f - \lambda$ with $0 \neq \lambda \in F$. Then the generalized Weyl algebra $A(F[z], a, \sigma)$ is commutator-simple (and simple).*

Algebras from the next example are not simple.

Proposition 2.7. *Let F be with $\operatorname{char}(F) = 0$ and let $A = F\langle X\rangle$ be a free algebra on at least two indeterminates. Then $[A, A]$ does not contain nonzero subalgebras. In particular, A is commutator-simple.*

Proof. Suppose there exists a nonzero $f \in F\langle X\rangle$ such that $f^j \in [A, A]$ for every $j \geq 1$. By [9, Theorem 1.7.2], we can pick a positive integer n such that $f = f(x_1, \ldots, x_m)$ is not a polynomial identity of $M_n(F)$. For any $A_1, \ldots, A_m \in M_n(F)$,

$$f(A_1, \ldots, A_m)^j \in [M_n(F), M_n(F)]$$

and therefore $f(A_1, \ldots, A_m)^j$ has trace 0. It is well known that this implies that the matrix $f(A_1, \ldots, A_m)$ is nilpotent. This means that f^n is a polynomial identity. However, this contradicts Amitsur's theorem [9, Theorem 1.12.4]. □

Besides the direct product, we can also use the tensor product to obtain new examples of commutator-simple algebras.

Proposition 2.8. *Let A and B be commutator-simple unital algebras. If A is simple and central, then the algebra $A \otimes_F B$ is commutator-simple.*

Proof. Set $T = A \otimes_F B$. Suppose $[T, T]$ contains a nonzero ideal I. Take a nonzero

$$w = u_1 \otimes v_1 + \cdots + u_m \otimes v_m \in I.$$

Without loss of generality, we may assume that $u_1, \ldots, u_m$ are linearly independent and $v_1 \neq 0$. Pick $a \in A \setminus [A, A]$. By the Artin-Whaples Theorem (see [3, Corollary 5.24]), there exist $s_i, t_i \in A$ such that

$$\sum_i s_i u_1 t_i = a \quad \text{and} \quad \sum_i s_i u_j t_i = 0,\ j = 2, \ldots, m.$$

This gives

$$a \otimes v_1 = \sum_i s_i \otimes 1 \cdot w \cdot t_i \otimes 1 \in I,$$

which readily implies that $a \otimes J \subseteq I$ where J is the ideal of B generated by v_1. By assumption, there exists a $b \in J \setminus [B, B]$. Since $a \otimes b \in I \subseteq [T, T]$, there exist $x_k, y_k \in A$ and $z_k, w_k \in B$ such that

$$a \otimes b = \sum_k [x_k \otimes z_k, y_k \otimes w_k] = \sum_k x_k y_k \otimes z_k w_k - y_k x_k \otimes w_k z_k,$$

which can be written as

$$a \otimes b + \sum_k [y_k, x_k] \otimes z_k w_k = \sum_k y_k x_k \otimes [z_k, w_k].$$

Considering $\sum_k [y_k, x_k] \otimes z_k w_k$ as an element of the vector space $[A, A] \otimes B$, we can rewrite it as $\sum_\ell c_l \otimes d_l$ for some linearly independent $c_\ell \in [A, A]$ and some $d_\ell \in B$. Since $a \notin [A, A]$, a does not lie in the linear span of the elements c_ℓ. Therefore,

$$a \otimes b + \sum_\ell c_l \otimes d_l = \sum_k y_k x_k \otimes [z_k, w_k]$$

implies that b lies in the linear span of the elements $[z_k, w_k]$ (see [3, Lemma 4.9]). However, this is a contradiction since $b \notin [B, B]$. $\square$

Taking A to be the matrix algebra $M_n(F)$, we obtain the following.

Corollary 2.9. *If B is a commutator-simple unital algebra, then so is $M_n(B)$.*

We now turn our attention to Banach algebras. We start with some of those in which the argument involving the trace is applicable.

Proposition 2.10. *Let X be a normed space. Then the algebra $F(X)$ of all continuous finite rank linear operators on X is commutator-simple.*

Proof. As in Proposition 2.2 this algebra has a trace τ, and for any continuous linear operator x on X, we have $\tau\big(x(\xi \otimes f)\big) = f(x(\xi))$ for each $\xi \in X$ and each continuous linear functional f on X. It follows from the Hahn-Banach theorem that τ satisfies (2.2). $\square$

Proposition 2.11. *Let H be a complex Hilbert space. Then each of the following operator algebras is commutator-simple:*

(i) *The algebra $S^1(H)$ of all trace class operators on H;*
(ii) *The algebra $S^2(H)$ of all Hilbert-Schmidt operators on H.*

Proof. Let τ be the natural trace on the trace class operators (see [17, Definition 9.1.34]). By [17, Theorem 9.1.35], $F(H) \subset S^1(H) \subset S^2(H)$ and $\tau(xy) = \tau(yx)$ for all x, $y \in S^2(H)$. This shows that τ satisfies (2.1) in both cases (i) and (ii). Further,

$$\tau\big(x(\xi \otimes \eta^*)\big) = \tau\big(\xi(x) \otimes \eta^*\big) = \langle \xi(x) | \eta \rangle$$

for each continuous linear operator x and all $\xi, \eta \in H$, which gives (2.2). $\square$

Let G be a locally compact group. We write $L^1(G)$ for the usual Banach L^1-space with respect to the left Haar measure on G. It becomes a Banach algebra with respect to the convolution product defined by

$$(f * g)(t) = \int_G f(s) g(s^{-1}t)\, ds.$$

If G is a discrete group, then the Haar measure is the counting measure, and the corresponding group algebra is usually written as $\ell^1(G)$. This algebra coincides with the purely algebraic group algebra $\mathbb{C}[G]$ in the case where G is a finite group. The group G is called *unimodular* if the left Haar measure is also a right Haar measure. This is a very important and wide class of groups, which includes Abelian groups, compact groups, discrete groups, and many others (we refer the reader to [17, Chapter12] for examples and a thorough discussion of this class of groups). We confine our attention to unimodular locally compact groups. In this case we have

$$\int_G f(t^{-1})\, dt = \int_G f(t)\, dt$$

for each $f \in L^1(G)$, and the group algebra $L^1(G)$ turns into a Banach $\star$-algebra by defining

$$f^\star(t) = \overline{f(t^{-1})}$$

for all $f \in L^1(G)$ and $t \in G$.

Proposition 2.12. *Let G be a unimodular locally compact group. Then each of the following subalgebras of the group algebra $L^1(G)$ is commutator-simple:*

(i) *The algebra $C_{00}(G)$ of continuous functions on G of compact support.*
(ii) *The algebra $L^1(G) \cap L^2(G)$.*

Proof. Let e be the identity of the group G.

(i) It is easy to check that $C_{00}(G)$ is a $\star$-subalgebra of $L^1(G)$. We define a linear functional $\tau\colon C_{00}(G) \to \mathbb{C}$ by

$$\tau(f) = f(e).$$

Using the unimodularity of G, for each $f, g \in C_{00}(G)$, we have

$$\begin{aligned}\tau(f * g) = (f * g)(e) &= \int_G f(t)g(t^{-1}e)\,dt = \int_G f(t)g(t^{-1})\,dt \\ &= \int_G f(t^{-1})g(t)\,dt = \int_G g(t)f(t^{-1}e)\,dt = (g * f)(e) = \tau(g * f),\end{aligned}$$

which shows that τ is a trace on $C_{00}(G)$. Further, suppose that $f \in C_{00}(G)$ is such that $\tau(f * g) = 0$ for each $g \in C_{00}(G)$. We have

$$0 = \tau(f * f^\star) = (f * f^\star)(e) = \int_G f(t)f^\star(t^{-1}e)\,dt = \int_G |f(t)|^2\,dt,$$

whence $f = 0$.

(ii) Write $A = L^1(G) \cap L^2(G)$. Since G is unimodular, [8, Proposition 2.40] shows that A is a subalgebra of $L^1(G)$, and it is easily seen that it is $\star$-invariant. Further, if $f, g \in A$, then [8, Proposition 2.41] shows that $f * g$ is defined everywhere on G and it is a continuous function vanishing at infinity. This implies that we can define a linear functional $\tau\colon A^2 \to \mathbb{C}$ by $\tau(f) = f(e)$ for each $f \in A^2$. We can check that both (2.1) and (2.2) hold for τ as in (i). ☐

Proposition 2.13. *Let G be a locally compact group. Then the group algebra $L^1(G)$ is commutator-simple in each of the following cases:*

(i) *The group G has* small invariant neighborhoods*, i.e., every neighborhood of the identity contains a compact neighborhood of the identity which is invariant under all inner automorphisms;*
(ii) *The group G is* maximally almost periodic*, i.e., the are enough continuous finite-dimensional unitary representations to separate the points of G.*

Proof. (i) Let $(e_\lambda)_{\lambda\in\Lambda}$ be a bounded approximate identity for $L^1(G)$ consisting of central elements of $L^1(G)$ and such that $e_\lambda \in L^1(G) \cap L^2(G)$ for each $\lambda \in \Lambda$ (see [16, Page 530]).

Suppose that I is an ideal of $L^1(G)$ with $I \subseteq [L^1(G), L^1(G)]$. Then, for each $\lambda \in \Lambda$,

$$L^1(G) * e_\lambda \subseteq L^1(G) \cap L^2(G)$$

and so

$$I * e_\lambda * e_\lambda \subseteq [L^1(G) * e_\lambda, L^1(G) * e_\lambda] \subseteq [L^1(G) \cap L^2(G), L^1(G) \cap L^2(G)].$$

Further, $I * e_\lambda * e_\lambda$ is an ideal of $L^1(G) \cap L^2(G)$. From Proposition 2.12 we see that $I * e_\lambda * e_\lambda = \{0\}$. Finally, since $(e_\lambda)_{\lambda\in\Lambda}$ is a bounded approximate identity for $L^1(G)$, for each $f \in I$ we have

$$f = \lim_{\lambda\in\Lambda} \underbrace{f * e_\lambda * e_\lambda}_{=0} = 0.$$

(ii) Suppose that I is an ideal of $L^1(G)$ such that $I \subseteq [L^1(G), L^1(G)]$, and take $f \in I$.

Let π be a continuous, unitary representation of G on a finite-dimensional Hilbert space H. Then π gives rise to a representation of $L^1(G)$ on the algebra $B(H)$ of all continuous linear operators on H, still denoted by π, defined by

$$\pi(f) = \int_G f(t)\pi(t)\,dt$$

for all $f \in L^1(G)$. Since H is finite-dimensional we can define a linear functional $\tau_\pi \colon L^1(G) \to \mathbb{C}$ by

$$\tau_\pi(g) = \operatorname{trace}(\pi(g))$$

for all $g \in L^1(G)$. It is immediate to check that τ_π is a trace. Consequently, $\tau_\pi(I) = \{0\}$ and hence

$$0 = \tau_\pi(f * f^\star) = \operatorname{trace}(\pi(f)\pi(f)^*),$$

which implies that $\pi(f) = 0$.

We thus get $\pi(f) = 0$ for each finite-dimensional, continuous, unitary representation π of G, and since G is an MAP-group, it may be concluded that $f = 0$. $\square$

For a comprehensive treatment of the classes of groups, [SIN] and [MAP], appearing in Proposition 2.13, we refer the reader to [17, Chapter 12]. It should be pointed out that all of these groups are unimodular. Further, each discrete group G has small invariant neighborhoods and therefore the algebra $\ell^1(G)$ is commutator-simple.

3. Derivations

Let A be an algebra. A linear map $D \colon A \to A$ is called a *Jordan derivation* if it satisfies

$$D(x^2) = D(x)x + xD(x)$$

for all $x \in A$. If A is a *semiprime* algebra, i.e., A has no nonzero nilpotent ideals, and the characteristic of the underlying field is not 2, every Jordan derivation is automatically a *derivation* [6], meaning that it satisfies

$$D(xy) = D(x)y + xD(y)$$

for all $x, y \in A$.

We remark that there are many algebras in which every derivation D is of the form

$$D(x) = [x, m]$$

where m is a fixed element from A, or, sometimes, from a larger algebra M that contains A as an ideal. These are the derivations that are of particular interest to us. Observe that they satisfy

$$D(x)x^n = [x, m]x^n = [x, mx^n] \in [A, A] \tag{3.1}$$

for all $x \in A$ and all positive integers n. We will actually need this only for $n = 1$ and $n = 2$.

We now state our basic result on derivations in commutator-simple algebras. Its proof is short and uses an idea from [4].

Theorem 3.1. *Let A be a commutator-simple semiprime algebra over a field F with* $\operatorname{char}(F) \neq 2$. *If a linear map $D\colon A \to A$ satisfies*

$$D(x)x, D(x)x^2 \in [A, A]$$

for every $x \in A$, then D is a derivation.

Proof. For any $x, y \in A$, we write $x \equiv y$ if $x - y \in [A, A]$. Our assumption can thus be read as $D(x)x \equiv 0$ and $D(x)x^2 \equiv 0$ for every $x \in A$. Writing $x + y$ for x in the first relation we obtain

$$D(x)y \equiv -D(y)x \tag{3.2}$$

for all $x, y \in A$, and writing $x \pm y$ for x in the second relation, we obtain

$$D(x)(xy + yx) + D(y)x^2 \equiv 0 \tag{3.3}$$

for all $x, y \in A$ (here we used the assumption that $\operatorname{char}(F) \neq 2$). In light of (3.2), (3.3) can be written as

$$D(x)(xy + yx) - D(x^2)y \equiv 0. \tag{3.4}$$

Since

$$D(x)yx - xD(x)y = [D(x)y, x] \equiv 0,$$

(3.4) further gives

$$\big(D(x)x + xD(x) - D(x^2)\big)y \equiv 0. \tag{3.5}$$

Suppose D is not a derivation. By the aforementioned result, D is neither a Jordan derivation, so

$$a = D(x)x + xD(x) - D(x^2) \neq 0$$

for some $x \in A$. By (3.5), $ay \equiv 0$ for all $y \in A$. Since $zay = [z, ay] + a(yz)$ it follows that $zay \equiv 0$ for all $y, z \in A$. This means that $[A, A]$ contains the ideal AaA, which is nonzero since A is semiprime. This contradicts our assumption that A is commutator-simple. □

Theorem 3.1 is of course applicable to most algebras mentioned in Section 2, but we will not list them here and rather focus on local derivations of group algebras $L^1(G)$. For this purpose, we need the following technical corollary to Theorem 3.1.

Corollary 3.2. *Let A be a semiprime algebra over a field F with* $\operatorname{char}(F) \neq 2$. *Suppose A is an ideal of an algebra M such that every derivation from A to A is of the form $x \mapsto [x, m]$ for some $m \in M$. Then every local derivation $D\colon A \to A$ is a derivation on each commutator-simple ideal I of M contained in A.*

Proof. Take $x \in I$. Since $x \in A$, by our assumption there exists an $m_x \in M$ such that $D(x) = [x, m_x] \in I$. Therefore, D maps I into itself, and just as in (3.1) we see that

$$D(x)x^n = [x, m_x]x^n = [x, m_x x^n] \in [I, I]$$

for every positive integer n. As an ideal of a semiprime algebra, I itself is semiprime. Applying Theorem 3.1 to the algebra I it follows that D is a derivation on I. □

Theorem 3.3. *Let G be a unimodular locally compact group. Then every continuous local derivation $D\colon L^1(G) \to L^1(G)$ is a derivation.*

Proof. We will check that Corollary 3.2 applies with $A = L^1(G)$ and $M = M(G)$, the Banach algebra of complex Radon measures on G. Of course, A is a closed ideal of M.

A remarkable property of the group algebra $L^1(G)$ is that each derivation from $L^1(G)$ to itself is of the form $f \mapsto f * \mu - \mu * f$ for some $\mu \in M(G)$ (see [1, 15]). We take $I = L^1(G) \cap L^2(G)$, which is an ideal of A and, by Proposition 2.12, I is a commutator-simple algebra. From Corollary 3.2 we see that D is a derivation on I. Since I is dense in $L^1(G)$ and D is continuous on $L^1(G)$, we conclude that D is a derivation on $L^1(G)$. □

In [18, 19], the author shows that every continuous (approximately) local derivation from $L^1(G)$ to any Banach $L^1(G)$-bimodule X is a derivation for a large class of groups including PG-groups, IN-groups, MAP-groups, and totally disconnected groups. In order to relate this result to Theorem 3.3, we mention that the class of unimodular groups strictly contains the classes [PG], [IN], and [MAP] (we refer the reader to [17, Diagram 1, page 1486] for this fact), and, on the other hand, unlike [18, 19], we are confined to maps from $L^1(G)$ to itself. To the best of our knowledge, Theorem 3.3 gives a new information.

4. Jordan homomorphisms

A linear map T from an algebra A to itself is called a *Jordan homomorphism* if

$$T(x^2) = T(x)^2$$

for every $x \in A$. Basic examples are homomorphisms and antihomomorphisms and, under suitable conditions, these are also the only examples. However, we shall not go into this here.

The idea of the proof of the following theorem is also taken from [4]. However, there are important differences in details.

Theorem 4.1. *Let A be a commutator-simple unital algebra over a field F with $\operatorname{char}(F) \neq 2, 3$. If a surjective linear map $T\colon A \to A$ satisfies $T(1) = 1$ and $T(x)^3 - x^3 \in [A, A]$ for every $x \in A$, then T is a Jordan homomorphism.*

Proof. As above, we write $x \equiv y$ for $x - y \in [A, A]$. Since $\operatorname{char}(F) \neq 2$, replacing x by $x \pm y$ in $T(x)^3 \equiv x^3$ gives

$$T(x)^2T(y) + T(x)T(y)T(x) + T(y)T(x)^2 \equiv x^2y + xyx + yx^2.$$

Observing that

$$T(x)^2T(y) \equiv T(x)T(y)T(x) \equiv T(y)T(x)^2,$$
$$x^2y \equiv xyx \equiv yx^2,$$

and using $\operatorname{char}(F) \neq 3$ it follows that

$$T(x)^2T(y) \equiv x^2y$$

for all $x, y \in A$. Hence,

$$T(x)^2T(y^2) \equiv x^2y^2 \equiv y^2x^2 \equiv T(y)^2T(x^2) \equiv T(x^2)T(y)^2.$$

Replacing y by $y + 1$ in $T(x)^2T(y^2) = T(x)^2T(y)^2$ and using $T(1) = 1$ we arrive at

$$T(x)^2T(y) \equiv T(x^2)T(y)$$

for all $x, y \in A$. Since T is surjective, this means that

$$\big(T(x^2) - T(x)^2\big)A \subseteq [A, A],$$

which, by $zay = [z, ay] + a(yz)$, implies that

$$A\big(T(x^2) - T(x)^2\big)A \subseteq [A, A].$$

Since A is commutator-simple, it follows that $T(x^2) - T(x)^2 = 0$. □

Let A be a unital algebra. By a *local inner automorphism* of A we, of course, mean a linear map $T\colon A \to A$ such that, for every $x \in A$, there exists an inner automorphism T_x of A such that $T(x) = T_x(x)$; that is, $T(x)$ is always of the form $u_x x u_x^{-1}$ for some invertible $u_x \in A$. Observe that such a map is automatically injective and sends 1 to 1.

Corollary 4.2. *If A is a commutator-simple unital algebra over a field F with* $\mathrm{char}(F) \neq 2, 3$*, then every surjective local inner automorphism of A is a Jordan automorphism.*

Proof. Observe that

$$uxu^{-1} - x = [u, xu^{-1}] \equiv 0$$

for every invertible $u \in A$ and every $x \in A$. Hence

$$T(x)^3 = (u_x x u_x^{-1})^3 = u_x x^3 u_x^{-1} \equiv x^3$$

for every $x \in A$, and so Theorem 4.1 applies. □

Every matrix x is similar to its transpose x^t, so $x \mapsto x^t$ is an example of a surjective local inner automorphism of the matrix algebra $M_n(F)$ that is a Jordan automorphism but not an automorphism. We also remark that since all automorphisms of this algebra are automatically inner, the notion of a local inner automorphism here coincide with the notion of a local automorphism.

We also need a technical refinement of this corollary that concerns algebras that are not necessarily unital.

Corollary 4.3. *Let A be a commutator-simple algebra over a field F with* $\mathrm{char}(F) \neq 2, 3$*. Assume further that $y \in Ay$ for each $y \in A$ and that A is an ideal of a unital algebra M. If $T\colon A \to A$ is a surjective linear map such that for each $y \in A$ there exists an invertible element $u_y \in M$ satisfying $T(y) = u_y y u_y^{-1}$, then T is a Jordan automorphism.*

Proof. Write $A_1 = F1 + A$. We claim that A_1 is a commutator-simple algebra. Indeed, let I be an ideal of A_1 such that $I \subseteq [A_1, A_1]$. Since $[A_1, A_1] = [A, A] \subseteq A$ we see that I is an ideal of A contained in $[A, A]$, and so $I = \{0\}$.

Define $T_1\colon A_1 \to A_1$ by

$$T_1(\alpha 1 + y) = \alpha 1 + T(y)$$

for all $\alpha \in F$ and $y \in A$. Observe that T_1 is a well-defined linear map. Moreover, it is surjective and $T_1(1) = 1$. We claim that $T_1(x)^3 \equiv x^3$ for each $x \in A_1$. Set $x = \alpha 1 + y \in A_1$. Take $e \in A$ such that $y = ey$ and write $u = u_y \in M$. Then

$$\begin{aligned} T_1(x)^3 - x^3 &= (\alpha 1 + uyu^{-1})^3 - (\alpha 1 + y)^3 \\ &= 3\alpha^2(uyu^{-1} - y) + 3\alpha(uy^2u^{-1} - y^2) + (uy^3u^{-1} - y^3) \\ &= 3\alpha^2[ue, yu^{-1}] + 3\alpha^2[y, e] + 3\alpha[uy, yu^{-1}] + [uy, y^2u^{-1}] \in [A, A], \end{aligned}$$

as claimed. By Theorem 4.1 it follows that T_1 is a Jordan automorphism and hence T is a Jordan automorphism. $\square$

Let A be an algebra of linear operators on a vector space that contains all finite rank operator. As a local automorphism of A preserves the rank of any finite rank operator, it seems likely that its form can be determined by using well known results on rank preservers. What we will establish in the next lines could therefore be obtained by other means, or may even be already known. However, our method of proof is certainly new.

Corollary 4.4. *Let A be the algebra of all finite rank linear operators on a vector space X over a field F with* $\operatorname{char}(F) \neq 2, 3$. *Then every surjective local automorphism of A is a Jordan automorphism.*

Proof. Let M be the algebra of all linear operators on X, and we will check that all requirements in Corollary 4.3 are satisfied. By Proposition 2.2, A is commutator-simple. For each $x \in A$, we can choose an idempotent operator $P \colon X \to X$ mapping onto the range of x, which implies that $P \in A$ and $x = Px$, so that $x \in Ax$. Further, M is a unital algebra and A is an ideal of M. For each $x \in A$ there exists an automorphism $\Phi_x \colon A \to A$ such that $T(x) = \Phi_x(x)$, and [12, Section IV.11] shows that there exists an invertible linear operator $u_x \colon X \to X$ such that $\Phi_x(y) = u_x y u_x^{-1}$ for each $y \in A$. Proposition 4.3 gives the desired conclusion. $\square$

Corollary 4.5. *Let X be a normed space. Then every surjective local automorphism of $F(X)$ is a Jordan automorphism.*

Proof. Set $A = F(X)$, and let M be the algebra of all continuous linear operators on X. Then Proposition 2.10 shows that A is a commutator-simple algebra. If $x \in A$, then the range Y of x is finite-dimensional and therefore there exists a continuous linear projection P of X onto Y. Consequently, $P \in A$ and $x = Px$, which gives $x \in Ax$. Moreover, M is a unital algebra and A is an ideal of M. Take $x \in A$. Then there exists an automorphism $\Phi_x \colon F(X) \to F(X)$ such that $T(x) = \Phi_x(x)$. By [5, Theorem 3.1], there exists a continuous invertible linear operator $u_x \colon X \to X$ such that $\Phi_x(y) = u_x y u_x^{-1}$ for each $y \in F(X)$. From Proposition 4.3 we obtain that T is a Jordan automorphism. $\square$

Corollary 4.6. *Let H be a Hilbert space, let $1 \leq p \leq \infty$, and let $S^p(H)$ be the algebra of pth Schatten class operators on H. Then every continuous surjective local automorphism T of $S^p(H)$ is a Jordan automorphism. Further, if H is an infinite-dimensional separable Hilbert space, then T is an automorphism.*

Proof. Our method consists in considering the restriction of T to the ideal $F(H)$ of $S^p(H)$. Take $x \in F(H)$. Then there exists an automorphism $\Phi_x \colon S^p(H) \to S^p(H)$ such that $T(x) = \Phi_x(x)$. By [5, Corollary 3.2], there exists a continuous invertible linear operator $u_x \colon H \to H$ such that $\Phi_x(y) = u_x y u_x^{-1}$ for each $y \in S^p(H)$. Consequently, $T(x) = u_x x u_x^{-1} \in F(H)$ and hence T maps $F(H)$ into $F(H)$ and is a local automorphism of $F(H)$. Further, if $y \in F(H)$, then there exists an $x \in S^p(H)$ with $T(x) = y$. Since $T(x) = u_x x u_x^{-1}$, it follows that $x = u_x^{-1} y u_x \in F(H)$. This shows that T is a surjective local automorphism of $F(H)$. Corollary 4.5 tells us that T is a Jordan automorphism of $F(H)$. Since $F(H)$ is dense in $S^p(H)$ and T is continuous on $S^p(H)$, it may be concluded that T is a Jordan automorphism of $S^p(H)$. From [11] we see that T is either an automorphism or an antiautomorphism.

We now suppose that H is an infinite-dimensional separable Hilbert space, let (ξ_n) be an orthonormal basis of H, let $s \in S^p(H)$ be the weighted shift operator defined through

$$s(\xi_n) = 2^{-n}\xi_{n+1}$$

for all $n \in \mathbb{N}$, so that

$$s^*(\xi_1) = 0, \quad s^*(\xi_n) = 2^{-n}\xi_{n-1} \quad n \geq 2,$$

and let $u_s\colon H \to H$ be a continuous invertible linear operator such that $T(s) = u_s s u_s^{-1}$. Assume towards a contradiction that T is an antiautomorphism. Take a conjugation c on H and define Φ on $S^p(H)$ by $\Phi(x) = T(cx^*c)$ for each $x \in S^p(H)$. Then Φ is an automorphism of $S^p(H)$ and [5, Corollary 3.2] shows that there exists a continuous invertible linear operator $v\colon H \to H$ such that $\Phi(x) = vxv^{-1}$ for each $x \in S^p(H)$. We thus get

$$u_s s u_s^{-1} = T(s) = \Phi(cs^*c) = vcs^*cv^{-1}.$$

Then the operator $u_s^{-1}vc$ is invertible, so that $\xi = u_s^{-1}vc\xi_1 \neq 0$ and, on the other hand,

$$s\xi = u_s^{-1}vcs^*cv^{-1}u_s\xi = u_s^{-1}vcs^*\xi_1 = 0.$$

This is a contradiction as s is injective. □

References

[1] U. Bader, T. Gelander, and N. Monod, *A fixed point theorem for L^1 spaces*, Invent. Math. **189** (2012), no. 1, 143–148, DOI 10.1007/s00222-011-0363-2. MR2929085

[2] V. V. Bavula, *Generalized Weyl algebras and their representations* (Russian), Algebra i Analiz **4** (1992), no. 1, 75–97; English transl., St. Petersburg Math. J. **4** (1993), no. 1, 71–92. MR1171955

[3] Matej Brešar, *Introduction to noncommutative algebra*, Universitext, Springer, Cham, 2014, DOI 10.1007/978-3-319-08693-4. MR3308118

[4] Matej Brešar, *Automorphisms and derivations of finite-dimensional algebras*, J. Algebra **599** (2022), 104–121, DOI 10.1016/j.jalgebra.2022.02.010. MR4390457

[5] Paul R. Chernoff, *Representations, automorphisms, and derivations of some operator algebras*, J. Functional Analysis **12** (1973), 275–289, DOI 10.1016/0022-1236(73)90080-3. MR0350442

[6] J. M. Cusack, *Jordan derivations on rings*, Proc. Amer. Math. Soc. **53** (1975), no. 2, 321–324, DOI 10.2307/2040004. MR399182

[7] M. A. Farinati, A. Solotar, and M. Suárez-Álvarez, *Hochschild homology and cohomology of generalized Weyl algebras* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **53** (2003), no. 2, 465–488. MR1990004

[8] Gerald B. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR1397028

[9] Antonio Giambruno and Mikhail Zaicev, *Polynomial identities and asymptotic methods*, Mathematical Surveys and Monographs, vol. 122, American Mathematical Society, Providence, RI, 2005, DOI 10.1090/surv/122. MR2176105

[10] Bruno Harris, *Commutators in division rings*, Proc. Amer. Math. Soc. **9** (1958), 628–630, DOI 10.2307/2033220. MR96697

[11] I. N. Herstein, *Jordan homomorphisms*, Trans. Amer. Math. Soc. **81** (1956), 331–341, DOI 10.2307/1992920. MR76751

[12] Nathan Jacobson, *Structure of rings*, American Mathematical Society Colloquium Publications, Vol. 37, American Mathematical Society, 190 Hope Street, Providence, R.I., 1956. MR0081264

[13] Richard V. Kadison, *Local derivations*, J. Algebra **130** (1990), no. 2, 494–509, DOI 10.1016/0021-8693(90)90095-6. MR1051316

[14] David R. Larson and Ahmed R. Sourour, *Local derivations and local automorphisms of* $\mathcal{B}(X)$, Operator theory: operator algebras and applications, Part 2 (Durham, NH, 1988), Proc. Sympos. Pure Math., vol. 51, Amer. Math. Soc., Providence, RI, 1990, pp. 187–194, DOI 10.1090/pspum/051.2/1077437. MR1077437

[15] Viktor Losert, *The derivation problem for group algebras*, Ann. of Math. (2) **168** (2008), no. 1, 221–246, DOI 10.4007/annals.2008.168.221. MR2415402

[16] Theodore W. Palmer, *Banach algebras and the general theory of* **-algebras. Vol. I*, Encyclopedia of Mathematics and its Applications, vol. 49, Cambridge University Press, Cambridge, 1994. Algebras and Banach algebras, DOI 10.1017/CBO9781107325777. MR1270014

[17] Theodore W. Palmer, *Banach algebras and the general theory of* $*$*-algebras. Vol. 2*, Encyclopedia of Mathematics and its Applications, vol. 79, Cambridge University Press, Cambridge, 2001. $*$-algebras, DOI 10.1017/CBO9780511574757.003. MR1819503

[18] Ebrahim Samei, *Approximately local derivations*, J. London Math. Soc. (2) **71** (2005), no. 3, 759–778, DOI 10.1112/S0024610705006496. MR2132382

[19] Ebrahim Samei, *Reflexivity and hyperreflexivity of bounded* n*-cocycles from group algebras*, Proc. Amer. Math. Soc. **139** (2011), no. 1, 163–176, DOI 10.1090/S0002-9939-2010-10454-9. MR2729080

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, Granada, Spain

Email address: `alaminos@ugr.es`

Faculty of Mathematics and Physics, University of Ljubljana, and Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

Email address: `matej.bresar@fmf.uni-lj.si`

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, Granada, Spain

Email address: `jlizana@ugr.es`

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, Granada, Spain

Email address: `mgodoy@ugr.es`

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, Granada, Spain

Email address: `avillena@ugr.es`