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The arithmetic of triangular Z-numbers with reduced calculation complexity using an extension of triangular distribution

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ABSTRACT

Information that people rely on is often uncertain and partially reliable. Zadeh introduced the concept of Z-numbers as a more adequate formal construct for describing uncertain and partially reliable information. Most existing applications of Z-numbers involve discrete ones due to the high complexity of calculating continuous ones. However, the continuous form is the most common form of information in the real world. Simplifying continuous Z-number calculations is significant for practical applications. There are two reasons for the complexity of continuous Z-number calculations: the use of normal distributions and the inconsistency between the meaning and definition of Z-numbers. In this paper, we extend the triangular distribution as the hidden probability density function of triangular Z-numbers. We add a new parameter to the triangular distribution to influence its convexity and concavity, and then expand the value's domain of the probability measure. Finally, we implement the basic operations of triangular Z-numbers based on the extended triangular distribution. The suggested method is illustrated with numerical examples, and we compare its computational complexity and the entropy (uncertainty) of the resulting Z-number to the traditional method. The comparison shows that our method has lower computational complexity, higher precision and lower uncertainty in the results.

1. Introduction

In our daily lives, we use information to make sense of the world around us, learn new things, and draw conclusions and predictions. However, because we do not have access to exact knowledge of our environment, uncertainty is a common source of errors. This uncertainty is caused by the methods of measurement used to gain new information, as well as the language we use to communicate on a daily basis. To address this issue, many approaches have been developed to handle uncertainty using soft constraints, such as probability theory [17,41], fuzzy set theory [26], Dempster-Shafer evidence theory [14,13,15], rough sets [31], and others. Among these theories, fuzzy set theory is widely used due to its ability to deal with imprecision and vagueness. In addition to uncertainty, the accuracy or reliability of information is another source of errors. Reliable information can help decision-makers

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make the right decisions; however, they can also be misled by unreliable information. The reliability of information is also imprecise and can be described by fuzzy numbers.

The concept of Z-numbers was proposed by Zadeh as a means of describing both uncertainty and reliability in information [44]. A Z-number is an ordered pair, $Z = (A, B)$, where A is a fuzzy number that constrains the values the variable X can take and B is another fuzzy number measuring the reliability of A. A Z-number is a combination of possibilities and probabilities. To be more exact, it is a summary of probability distributions and the possibilities corresponding to these probability distributions.

Over the last decade or so, mathematical frameworks and applications of Z-numbers have developed rapidly. Due to their powerful ability to process natural language, Z-numbers have proven to be effective in many fields, such as decision-making problems [38,40,35,21], linguistic information processing [20,16], reliability analysis [22,39], risk assessment [30], failure analysis [23,20] and clustering [5]. The mathematical framework of Z-numbers has also become more complete. Aliev et al. presented a general framework for computing with Z-numbers [3,4,2], including some basic arithmetic and some key algebraic operations. The computation process is a combination of probabilistic arithmetic and fuzzy arithmetic. For discrete Z-numbers, this general approach can work well and yield precise Z-numbers. However, for continuous Z-numbers, researchers have had to discretize them to obtain only approximate results rather than exact ones [4]. There are many ranking methods for Z-numbers that have been created, but there is still no universally accepted method. How to rank Z-numbers remains a hot research topic. Chutia designed a ranking method based on the concept of value and ambiguity at levels of decision-making [11]. They calculated the values and ambiguities both for the constraint and reliability components of Z-numbers and combine the scores. Cheng et al. achieved a ranking method based on the developed golden rule representative value [10]. In addition to these basic operations, researchers have also explored other operations on Z-numbers, such as function [6], negation [29], uncertainty measure [24], and cross entropy [36]. A review has concluded that the formalization of Z-number-based mathematical operations and applications of Z-numbers have made significant progress in the decade since its inception in 2011 [9]. Z-number applications are mostly concerned with discrete ones, due to the high complexity of calculating continuous ones [33,29,34,36,12]. However, continuous forms are the most common form of information in the real world. The complexity of calculating continuous Z-numbers limits their wider applications in practice.

There are two reasons for the high complexity of continuous Z-number calculation. The first one is the use of normal distributions. Normally, researchers assume that a continuous Z-number's hidden probability density functions (pdfs) are normal distributions. The normal distribution has a positive property, which is that the addition or subtraction of two independent normal random variables is also a normal distributed. This will facilitate the operations of two random variables in Z-numbers. However, the complex form of the normal distribution makes the process of finding the exact expression of the hidden pdf extremely difficult. By definition, the hidden pdf of a continuous Z-number is a function of the base value of the second fuzzy number and the variable. This difficulty has led researchers to only discretize the second fuzzy number to get an approximate result. Discretization can lead to discretization errors, while approximate estimation can lead to estimation errors, which in turn can result in information loss. Aliev et al. presented the fundamental arithmetic and algebraic operations of continuous Z-numbers in [4]. In the third step, they discretized the second fuzzy number using two approaches for discretizing fuzzy numbers [19]. The same approach was used in computing the Hukuhara difference of continuous Z-numbers [2]. In the numerical examples of Ref. [1], the second fuzzy number was also discretized. In Ref. [38,45], continuous Z-numbers were represented as discrete Z-numbers with key points, providing a more flexible representation of Z-numbers. The hidden pdf becomes the probability distribution on these key points. Moreover, the normal pdf is always convex regardless of the reliability value. In contrast, the hidden probability distribution of discrete Z-numbers is concave when the reliability value is less than 0.5.

Another reason for the high complexity of Z-number calculation is the inconsistency between the actual meaning of a Z-number and its definition. Zadeh stated that the second fuzzy number is a measure of the reliability or certainty of the first fuzzy number [44]. Informally, B can be interpreted as the response to the question: "How sure are you that the variable is A?" [44]. The base value of B ranges from 0 to 1. According to the meaning of a Z-number, reliability is 1, which implies that the first fuzzy number A is completely reliable, and the information provider is absolutely certain that the variable is A. However, according to the definition of a Z-number, reliability is 1 which means the probability measure of A is 1. As a result, the only hidden probability distribution that satisfies this condition is the probability of $1/n$ for each point in the fuzzy number A with a membership degree of 1, where n is the number of points in A with the membership degree of 1. This hidden probability distribution, which represents the real situation, is very different from A. Thus, it is questionable whether we can say that A is completely reliable. This inconsistency also makes it challenging to find some hidden functions of some Z-numbers, especially continuous Z-numbers. Furthermore, due to the complex form of the normal distribution, we have not determined the conditions for the existence of hidden probability density functions. This has led researchers to overlook the role of hidden probabilities in the early days of the Z-number proposal and simply convert Z-numbers to fuzzy numbers [8,27].

In this paper, we extend the triangular distribution to be the form of the hidden pdf of triangular Z-numbers and use it to implement the operations of triangular Z-numbers. First, we add a new parameter to the triangular distribution to influence the convexity and concavity of the triangle distribution. According to the definition of Z-numbers, we find that when the reliability value is in the interval $[1/3, 2/3]$, we can find the corresponding hidden pdf. Then we expand the interval to $[0, 1]$ and keep the ordering relation in order to calculate triangular Z-numbers more conveniently. The hidden pdf is the pdf transformed from A by normalization when the reliability value is 1 [42]. It is the negation of the pdf transformed from A when the reliability value is 0 [43]. Thus, the meaning and definition of the Z-number are consistent. After that, we use extended triangular probability to implement the basic operations of Z-numbers, including addition, subtraction, multiplication, division, minimum, maximum and negation. With the extended triangular probability simple form, we can easily obtain the hidden pdfs and obtain the exact resulting Z-numbers. Some numerical examples are provided to show the validity of the suggested approach, and we compare its computational complexity and

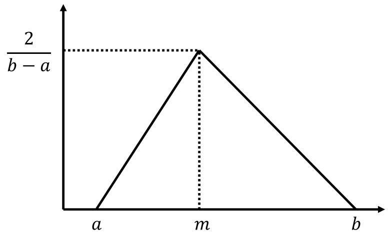


Fig. 1. A triangular density function.

the entropy of the resulting Z-number to the traditional method. The comparison shows that our method has lower computational complexity, higher precision and lower uncertainty in the results.

The main contributions of this paper can be summarized as follows.

- Proposed an extension of triangular distribution, which be added a new parameter to influence the concavity and convexity.
- The extended triangular distribution was used as the form of the hidden probability density function of the triangular Z-numbers to simply the calculation.
- The value domain of probability measure was expanded to uniform the meaning and definition of the Z-number.
- Implemented some basic operations of triangular Z-numbers, including addition, subtraction, multiplication, division, minimum and maximum operations over two triangular Z-numbers and negation of a Z-number.

The paper is structured as follows. Section 2 provides some necessary definitions used in this paper, such as triangular distribution, triangular fuzzy number and Z-number, and some arithmetic of random variables and fuzzy numbers. We analyze the shortcomings of the existing operations in detail and provide the corresponding numerical example in Section 3. In Section 4, we define an extended triangular distribution used in Z-numbers and discuss its properties. Using numerical examples, Section 5 demonstrates how the extended triangular distribution is applied to each of the considered operations and compares its computational complexity and the uncertainty of the resulting Z-number to the traditional method. Finally, several conclusions are drawn in Section 6.

2. Preliminaries

Operation on continuous Z-numbers is a combination of random variables arithmetic and fuzzy numbers arithmetic. In this section, we review some definitions of triangular distribution and the addition, subtraction, multiplication, division, minimum, maximum and complement or negation operations on random variables and fuzzy numbers. We also review other necessary definitions used in this paper.

2.1. Triangular distribution

Definition 1. The pdf of the triangular distribution is a triangle, which has three parameters: the lower limit a, the upper limit b and the mode m, $a \le m \le b \in \mathcal{R}$, as shown in Fig. 1. The pdf of the triangular distribution can be represented as a segmentation function $pdf(x) = T(a, m, b)$ defined as:

$$pdf(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(m-a)}, & a \le x \le m, \\ \dfrac{2(b-x)}{(b-a)(b-m)}, & m \le x \le b, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The expected value of a triangular distribution is $E(X) = (a + m + b)/3$. And its cumulative distribution function (CDF) is:

$$CDF(x) = \begin{cases} \dfrac{(x-a)^2}{(b-a)(c-a)}, & a \le x \le m, \\ 1 - \dfrac{((b-x)^2)}{(b-a)(b-x)}, & m < x \le b, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

2.2. Operations over continuous random variables

Definition 2 *(Addition of random variables [37])*. The pdf of the addition X_3 of two independent continuous random variables X_1 and X_2 is given by $pdf_3(x_3)$, where $pdf_1(x_1)$ and $pdf_2(x_2)$ are the pdfs of X_1 and X_2, respectively.

$$pdf_3(x_3) = \int_{-\infty}^{+\infty} pdf_1(x_1)pdf_2(x_3 - x_1)dx_1 = \int_{-\infty}^{+\infty} pdf_1(x_3 - x_2)pdf_2(x_2)dx_2. \tag{3}$$

Definition 3 *(Subtraction of random variables [37]).* The pdf of the subtraction X_3 of two independent continuous random variables X_1 and X_2 is given by $pdf_3(x_3)$, where $pdf_1(x_1)$ and $pdf_2(x_2)$ are the pdfs of X_1 and X_2, respectively.

$$pdf_3(x_3) = \int_{-\infty}^{+\infty} pdf_1(x_1)pdf_2(x_1 - x_3)dx_1 = \int_{-\infty}^{+\infty} pdf_1(x_3 + x_2)pdf_2(x_2)dx_2. \tag{4}$$

Definition 4 *(Multiplication of random variables [37]).* The pdf of the multiplication X_3 of two independent continuous random variables X_1 and X_2 is given by $pdf_3(x_3)$, where $pdf_1(x_1)$ and $pdf_2(x_2)$ are the pdfs of X_1 and X_2, respectively.

$$pdf_3(x_3) = \int_{-\infty}^{+\infty} \frac{1}{|x_1|} pdf_1(x_1)pdf_2(\frac{x_3}{x_1})dx_1. \tag{5}$$

Definition 5 *(Division of random variables [37]).* The pdf of the division X_3 of two independent continuous random variables X_1 and X_2 is given by $pdf_3(x_3)$, where $pdf_1(x_1)$ and $pdf_2(x_2)$ are the pdfs of X_1 and X_2, respectively.

$$pdf_3(x_3) = \int_{-\infty}^{+\infty} |x_1| pdf_1(x_1)pdf_2(\frac{x_1}{x_3})dx_1. \tag{6}$$

Definition 6 *(Minimum of random variables [37]).* The pdf of the minimum X_3 of two independent continuous random variables X_1 and X_2 is given by $pdf_3(x_3)$, where $pdf_1(x_1)$ and $pdf_2(x_2)$ are the pdfs of X_1 and X_2, respectively.

$$pdf_3(x_3) = pdf_1(x_3) + pdf_2(x_3) - CDF_1(x_3)pdf_2(x_3) - CDF_2(x_3)pdf_1(x_3), \tag{7}$$

where $CDF_1(x_3) = \int_{-\infty}^{x_3} pdf_1(x_1)dx_1$ and $CDF_2(x_3) = \int_{-\infty}^{x_3} pdf_2(x_2)dx_2$.

Definition 7 *(Maximum of random variables [37]).* The pdf of the maximum X_3 of two independent continuous random variables X_1 and X_2 is given by $pdf_3(x_3)$, where $pdf_1(x_1)$ and $pdf_2(x_2)$ are the pdfs of X_1 and X_2, respectively.

$$pdf_3(x_3) = CDF_1(x_3)pdf_2(x_3) + CDF_2(x_3)pdf_1(x_3), \tag{8}$$

where $CDF_1(x_3) = \int_{-\infty}^{x_3} pdf_1(x_1)dx_1$ and $CDF_2(x_3) = \int_{-\infty}^{x_3} pdf_2(x_2)dx_2$.

Definition 8 *(Negation of probability distribution [43]).* Assume there is a probability distribution $P = P(x_i)$ on the frame of reference set $X = \{x_1, x_2, \ldots, x_n\}$, where $0 \le P(x_i) \le 1$ and $\sum_{i=1}^{n} P(x_i) = 1$. Then the negation of P is a probability distribution denoted as $\overline{P}(x_i)$ and defined as:

$$\overline{P}(x_i) = \frac{1 - P(x_i)}{n - 1}. \tag{9}$$

2.3. Operations over continuous fuzzy numbers

Definition 9 *(Triangular fuzzy number (TFN) [26]).* A TFN A is a normal fuzzy subset of the real line $\mathcal{R}$ with a continuous membership function $\mu_A : \mathcal{R} \to [0,1]$.

$$\mu_A(x) = \begin{cases} \dfrac{x-a}{m-a}, & a \le x \le m, \\ \dfrac{b-x}{b-m}, & m < x \le b, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

where $a \le m \le b \in \mathcal{R}$. The TFN can be denoted as (a, m, b).

Definition 10 *(α-cut [26,25]).* The α-cut of a triangular continuous fuzzy number $A = (a, m, b)$ is a closed interval consisting of elements belonging to A to at least the degree α, $\alpha \in [0,1]$:

$$A^\alpha = [\underline{A}^\alpha, \overline{A}^\alpha] = [a + \alpha(m - a), b - \alpha(b - m)]. \tag{11}$$

Definition 11 *(Complement of fuzzy numbers [26,25]).* The complement of fuzzy number A is a fuzzy set denoted as $\overline{A}$ and defined as:

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x). \tag{12}$$

Definition 12 *(Addition of fuzzy numbers [26,25]).* The addition of two continuous fuzzy numbers A_1 and A_2 is a fuzzy number $A_3 = A_1 + A_2$:

$$A_3 = \cup_{\alpha\in[0,1]}\alpha[\underline{A_1}^{\alpha} + \underline{A_2}^{\alpha}, \overline{A_1}^{\alpha} + \overline{A_2^{\alpha}}]. \tag{13}$$

Definition 13 *(Subtraction of fuzzy numbers [26,25]).* The subtraction of two continuous fuzzy numbers A_1 and A_2 is a fuzzy number $A_3 = A_1 - A_2$:

$$A_3 = \cup_{\alpha\in[0,1]}\alpha[\underline{A_1}^{\alpha} - \overline{A_2}^{\alpha}, \overline{A_1}^{\alpha} - \underline{A_2^{\alpha}}]. \tag{14}$$

Definition 14 *(Multiplication of fuzzy numbers [26,25]).* The multiplication of two continuous fuzzy numbers A_1 and A_2 is a fuzzy number $A_3 = A_1 \times A_2$:

$$A_3 = \cup_{\alpha\in[0,1]}\alpha[\min(\underline{A_1}^{\alpha} \times \underline{A_2}^{\alpha}, \underline{A_1}^{\alpha} \times \overline{A_2}^{\alpha}, \overline{A_1}^{\alpha} \times \underline{A_2}^{\alpha}, \overline{A_1}^{\alpha} \times \overline{A_2}^{\alpha}), \max(\underline{A_1}^{\alpha} \times \underline{A_2}^{\alpha}, \underline{A_1}^{\alpha} \times \overline{A_2}^{\alpha}, \overline{A_1}^{\alpha} \times \underline{A_2}^{\alpha}, \overline{A_1}^{\alpha} \times \overline{A_2}^{\alpha})]. \tag{15}$$

Definition 15 *(Division of fuzzy numbers [26,25]).* The division of two continuous fuzzy numbers A_1 and A_2, $0 \notin supp(A_2)$, is a fuzzy number $A_3 = A_1/A_2$:

$$A_3 = \cup_{\alpha\in[0,1]}\alpha[\min(\underline{A_1}^{\alpha} / \underline{A_2}^{\alpha}, \underline{A_1}^{\alpha} / \overline{A_2}^{\alpha}, \overline{A_1}^{\alpha} / \underline{A_2}^{\alpha}, \overline{A_1}^{\alpha} / \overline{A_2}^{\alpha}), \max(\underline{A_1}^{\alpha} / \underline{A_2}^{\alpha}, \underline{A_1}^{\alpha} / \overline{A_2}^{\alpha}, \overline{A_1}^{\alpha} / \underline{A_2}^{\alpha}, \overline{A_1}^{\alpha} / \overline{A_2}^{\alpha})]. \tag{16}$$

Definition 16 *(Minimum of fuzzy numbers [26,25]).* The minimum of two continuous fuzzy numbers A_1 and A_2 is a fuzzy number $A_3 = \min(A_1, A_2)$:

$$A_3 = \cup_{\alpha\in[0,1]}\alpha\{\min(u, v)|u \in A_1^{\alpha}, v \in A_2^{\alpha}\}. \tag{17}$$

Definition 17 *(Maximum of fuzzy numbers [26,25]).* The maximum of continuous two fuzzy numbers A_1 and A_2 is a fuzzy number $A_3 = \max(A_1, A_2)$:

$$A_3 = \cup_{\alpha\in[0,1]}\alpha\{\max(u, v)|u \in A_1^{\alpha}, v \in A_2^{\alpha}\}. \tag{18}$$

2.4. Z-numbers

Definition 18 *(Continuous Z-numbers [44]).* A continuous Z-number is an ordered pair of continuous fuzzy numbers, $Z = (A, B)$, where the first fuzzy number A is a fuzzy constraint on the values that the variable X can take, having a membership function $\mu_A(x) : \mathcal{R} \to [0, 1]$. The second fuzzy number B is a fuzzy constraint on the probability measure of A with the membership function $\mu_B(v) : [0, 1] \to [0, 1]$.

The probability measure of A is defined as $Prob(X \text{ is } A) = \int_{-\infty}^{\infty} \mu_A(x)pdf(x)dx$, where $pdf(x)$ is the hidden pdf of X and is not known. It is known that there is a restriction on $pdf(x)$:

$$\int_{-\infty}^{\infty} \mu_A(x)pdf(x)dx \text{ is } B, \tag{19}$$

$$\mu_B(\int_{-\infty}^{\infty} \mu_A(x)pdf(x)dx = v) = \mu_B(v). \tag{20}$$

Informally, if the centroids of A and $pdf(x)$ are coincident, the pdf is compatible.

$$\int_{-\infty}^{\infty} xpdf(x)dx = \frac{\int_{-\infty}^{\infty} x\mu_A(x)dx}{\int_{-\infty}^{\infty} \mu_A(x)dx}. \tag{21}$$

Definition 19 *(Continuous Z^+-numbers [44]).* Given a Z-number $Z = (A, B)$, a Z^+-number is an ordered pair consisting of a fuzzy number A and a random number R, such that $Z^+ = (A, R)$. Here, A retains its original meaning as in a Z-number, while R represents the probability distribution of a random variable.

R can be viewed as the hidden probability distribution of X in the Z-number $Z = (A, B)$, can be described as a pdf $pdf(x)$, such that $\int_{-\infty}^{\infty} \mu_A(x) pdf(x) du \in supp(B)$.

The traditional arithmetic of continuous Z-numbers is based on the arithmetic of fuzzy numbers and arithmetic of normal pdfs [4]. Here is a brief introduction to the steps involved in the continuous Z-numbers operation. Assume there are two continuous Z-numbers $Z_1(A_1, B_1)$ and $Z_2 = (A_2, B_2)$ describing imperfect information about the values of variables X_1 and X_2. We need to compute $Z'_{12} = Z_1 * Z_2$, $* \in \{+, -, \times, /\}$.

Step 1: Compute $A'_{12} = A_1 * A_2$ by using the operations of continuous fuzzy numbers.

Step 2: Discretize B_i, $i = 1, 2$, and find the hidden pdfs. $supp(B_i)$ will be split into discrete elements v_{il}, $l = 1, \dots, m$, such that the spacing is constant interval. Therefore, B_i can be represented as:

$$B_i = \frac{\mu_{B_i}(v_{i1})}{v_{i1}} + \frac{\mu_{B_i}(v_{i2})}{v_{i2}} + \dots + \frac{\mu_{B_i}(v_{im})}{v_{im}}. \tag{22}$$

Then we have to find such normal pdf $pdf_{il} = \mathcal{N}(\alpha_{il}, \sigma_{il})$ by solving the following nonlinear problem:

$$\int_{-\infty}^{\infty} \mu_{A_i}(x) pdf_{il}(x) dx \longrightarrow v_{il} \tag{23}$$

subject to

$$\begin{cases} \sigma_{il} \geq 0, \\ \int_{-\infty}^{\infty} x pdf_{il}(x) dx = \alpha_{il} = \dfrac{\int_{-\infty}^{\infty} x \mu_{A_i}(x) dx}{\int_{-\infty}^{\infty} \mu_{A_i}(x) dx}. \end{cases} \tag{24}$$

Discretization can lead to discretization errors, which can result in information loss.

Step 3: Compute or (or approximate) $R'_{12} = R_1 * R_2$ as a convolution $pdf_{12s} = pdf_{1l_1} \circ pdf_{2l_2}$ of continuous pdfs. For addition and subtraction, $\alpha_{12s} = \alpha_{1l_1} \pm \alpha_{2l_2}$, $\sigma_{12s} = \sqrt{\sigma^2_{1l_1} + \sigma^2_{2l_2}}$.

Step 4: Compute the discretized base values of B_{12}.

$$v_{12s} = \int_{-\infty}^{\infty} \mu_{A_{12}}(x) pdf_{12s}(x) dx, \tag{25}$$

where $pdf_{12s} = pdf_{1l_1} * pdf_{2l_2}$.

Step 5: Compute the membership function of B'_{12}.

$$\mu_{B_{12}}(v_{12s}) = max_{v_{1l_1}, v_{2l_2}}(\mu_{B_1}(v_{1l_1}) \wedge \mu_{B_2}(v_{2l_2})). \tag{26}$$

As the result, $Z_{12} = Z_1 * Z_2$ is obtained as (A'_{12}, B'_{12}). The second fuzzy number of the resulting Z-numbers is a discrete fuzzy number. The continuous second fuzzy number is approximated by the obtained discrete one. The approximate estimation can lead to estimation errors, which can result in information loss.

Uncertainty (entropy) is a measure of the degree of disorder in the distribution of a random variable. The greater the uncertainty, the more disorder in the distribution of the random variable. Li et al. proposed a method to measure the uncertainty of a continuous Z-number [28]. Assume there is a continuous Z-number $Z = (A, B)$, its uncertainty is calculated by the following steps:

Step 1: Discretize B, by a precision $pre \in [0.1, 0.0001]$ and find the hidden pdfs. Then $supp(B)$ will be split into discrete elements v_l, $l = 1, \dots, m$ such that the spacing is constant interval pre. Therefore, B can be discretized as:

$$B = \frac{\mu_B(v_1)}{v_1} + \frac{\mu_B(v_2)}{v_2} + \dots + \frac{\mu_B(v_m)}{v_m}. \tag{27}$$

Then we have to find such normal pdf $pdf_l = \mathcal{N}(\alpha_l, \sigma_l)$ by solving the following nonlinear problem:

$$\int_{-\infty}^{\infty} \mu_A(x) pdf_l(x) dx \longrightarrow v_l \tag{28}$$

subject to

$$\begin{cases} \sigma \geq 0, \\ \int_{-\infty}^{\infty} x pdf_l(x) dx = \alpha = \dfrac{\int_{-\infty}^{\infty} x \mu_A(x) dx}{\int_{-\infty}^{\infty} \mu_A(x) dx}. \end{cases} \tag{29}$$

Table 1
The normal pdf of Z_1 when reliability is 0.2 to 0.8.

Reliability	$v = 0.2$	$v = 0.3$	$v = 0.4$	$v = 0.5$	$v = 0.6$	$v = 0.7$	$v = 0.8$
PDFs	$N(5, 9.0478)$	$N(5, 6.9272)$	$N(5, 5.0894)$	$N(5, 3.5343)$	$N(5, 2.2619)$	$N(5, 1.2723)$	$N(5, 0.5655)$

Table 2
The probability distribution of Z_1 when reliability is 0.2 to 0.8.

Reliability	$x = 2$	$x = 3$	$x = 4$	$x = 5$	$x = 6$	$x = 7$	$x = 8$
$v = 0.2$	0.2902	0.1338	0.0617	0.0285	0.0617	0.1338	0.2902
$v = 0.3$	0.2181	0.1478	0.1002	0.0679	0.1002	0.1478	0.2181
$v = 0.4$	0.1581	0.1457	0.1343	0.1238	0.1343	0.1457	0.1581
$v = 0.5$	0.1082	0.1321	0.1613	0.1969	0.1613	0.1321	0.1082
$v = 0.6$	0.0675	0.1097	0.1781	0.2894	0.1781	0.1097	0.0675
$v = 0.7$	0.0360	0.0806	0.1808	0.4052	0.1808	0.0806	0.0360
$v = 0.8$	0.0141	0.0478	0.1623	0.5517	0.1623	0.0478	0.0141

Step 2: Compute the uncertainty $H(pdf_l)$ of normal pdf pdf_l by:

$$H(pdf_l) = \frac{1}{2}(\ln(2\pi\sigma_l^2) + 1). \tag{30}$$

Step 3: Compute the uncertainty of Z, $H(Z)$, by:

$$H(Z) = \sum_{i=1}^{m-1} \int_{v_i}^{v_i+pre} \frac{H(pdf_i) + H(pdf_{i+1})}{2} \mu_B(v)dv. \tag{31}$$

3. Motivation

In this section, we provide two examples that illustrate in detail the two main reasons that contribute to the complexity of the current arithmetic operations on continuous Z-numbers. Additionally, we outline the main objectives of this paper.

Normal pdfs are the most widely used pdf as the hidden pdf for continuous Z-numbers. Gardashova applied the operations of Z-numbers to solve decision-making problems by means of normal pdfs [18]. Aliev et al. achieved the arithmetic of Z-numbers by calculating normal pdfs [4,7]. Peng and Wang also explored the multi-criteria group decision-making problems with Z-numbers by using normal pdf as the form of the hidden pdf in continuous Z-numbers [32]. Peng et al. developed a computationally simple and effective Z-number model by using normal pdfs to address game problems [34]. Although the addition and subtraction operations of normal pdfs are simple, it is very difficult to obtain the precise expression of the resulting Z-number due to the complexity of its expression. As a result, researchers have had to discretize continuous Z-numbers [38,4,1,45,7].

The case of using a normal pdf to represent the hidden pdf of a continuous Z-number is not the same as the case of a discrete Z-number. Regardless of reliability, the normal pdf is always convex. In contrast, the hidden probability distribution of a discrete Z-number is convex if the reliability is high and concave when the reliability is low. We will use a numerical example to illustrate this point.

Example 1. There are two Z-numbers: a continuous Z-number $Z_1 = (A_1 = (2, 5, 8), B_1 = (0.2, 0.5, 0.8))$ and the discretized version of Z_1, $Z_2 = (A_2, B_2)$ where A_2 and B_2 are:

$$A_2 = \frac{0.0}{2} + \frac{0.3333}{3} + \frac{0.6667}{4} + \frac{1.0}{5} + \frac{0.6667}{6} + \frac{0.3333}{7} + \frac{0.0}{8};$$
$$B_2 = \frac{0.0}{0.2} + \frac{0.3333}{0.3} + \frac{0.6667}{0.4} + \frac{1.0}{0.5} + \frac{0.6667}{0.6} + \frac{0.3333}{0.7} + \frac{0.0}{0.8}.$$

If we directly use the membership function of B_1 for calculation, we need to calculate the inverse function of the following equation to obtain σ, which is a function of the base value v, $v \in [0, 1]$.

$$\int_2^5 \frac{x-1}{3} \frac{1}{\sqrt{2\pi}\sigma} \exp \frac{-(x-5)^2}{2\sigma^2} dx + \int_5^8 \frac{8-x}{3} \frac{1}{\sqrt{2\pi}\sigma} \exp \frac{-(x-5)^2}{2\sigma^2} dx = v.$$

Deriving the precise expression for σ is extremely complex. Therefore, researchers often discretize B_1 when calculating Z-numbers, to convert the analytical solution of the above equation into a relatively easy-to-obtain numerical solution. However, this approach can result in discretization errors and loss of information. When reliability is 0.2 to 0.8, the hidden normal pdfs of Z_1 are listed in Table 1 and depicted in Fig. 2 and the hidden probability distributions of Z_2 are shown in Table 2 and portrayed in Fig. 3.

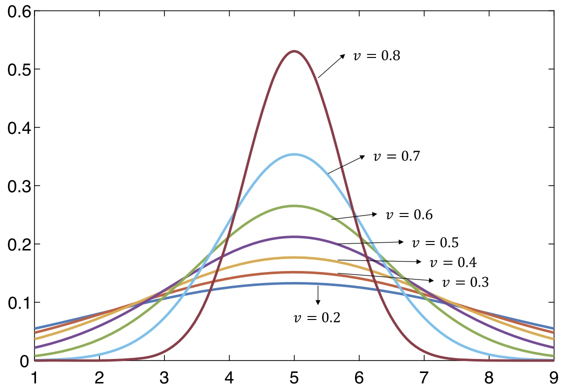


Fig. 2. The normal pdf of Z_1 when reliability is 0.2 to 0.8.

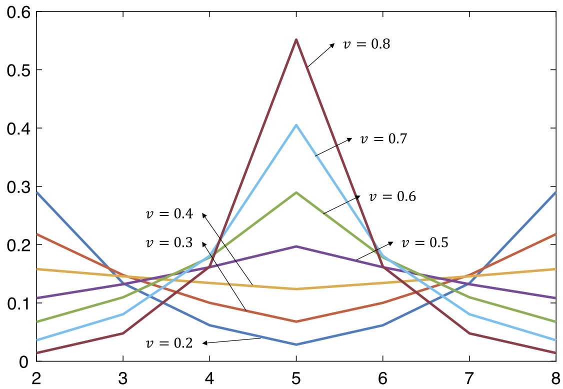


Fig. 3. The probability distribution of Z_1 when reliability is 0.2 to 0.8.

As shown in Figs. 2 and 3, when $v < 0.5$, the probability distributions of Z_2 are concave while the pdfs of Z_1 remain convex irrespective of the value of reliability. Consequently, the normal pdf is not a suitable means for representing the hidden probability density of triangular Z-numbers due to its complex expression and convexity. This paper therefore seeks to identify a suitable pdf that can accurately and easily compute Z-numbers.

According to the meaning of Z-numbers, as the reliability value increases, the first fuzzy number becomes more reliable. However, the hidden probability distribution that satisfies this condition can be very different from the first fuzzy number. We illustrate this issue through an example, using both discrete and continuous Z-numbers.

Example 2. A continuous Z-number $Z_1 = (A_1 = (2, 5, 8), B_1 = (0.9, 1, 1))$ and a discrete Z-number $Z_2 = (A_2, B_2)$ with fuzzy numbers:

$$A_2 = \frac{0.0}{2} + \frac{0.3333}{3} + \frac{0.6667}{4} + \frac{1.0}{5} + \frac{0.6667}{6} + \frac{0.3333}{7} + \frac{0.0}{8};$$

$$B_2 = \frac{0.0}{0.9} + \frac{0.2}{0.92} + \frac{0.4}{0.94} + \frac{0.6}{0.96} + \frac{0.8}{0.98} + \frac{1.0}{1.0}.$$

When the reliability value is between 0.9 and 1.0, the hidden pdfs of Z_1 are depicted in Fig. 4, while the probability distributions of Z_2 are displayed in Fig. 5.

Based on Fig. 4 and Fig. 5, it can be observed that the hidden probability distributions, for both discrete and continuous Z-numbers, can be very different from the first fuzzy number. As the reliability of A increases, the difference between the hidden probability distribution and the first fuzzy number A becomes much larger. That is the inconsistency between the actual meaning of the Z-number and its definition.

The purpose of this paper is to identify a suitable probability density function (pdf) that can accurately represent the hidden function of triangular Z-numbers in a simpler and more computationally efficient manner than the normal distribution, in order to obtain precise results. Additionally, this paper aims to address the inconsistency between the actual meaning of the Z-number and its current definition.

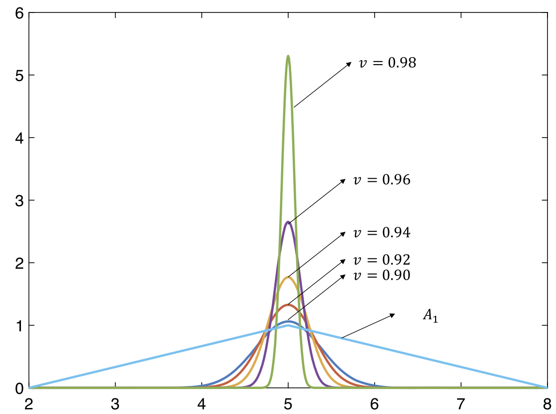


Fig. 4. The normal pdfs of Z_1 when reliability is 0.9 to 1.0.

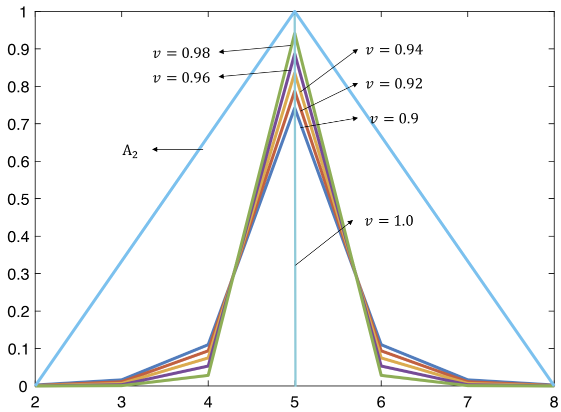


Fig. 5. The probability distributions of Z_1 when reliability is 0.9 to 1.0.

4. Extended triangular distribution

In this section, we design a form of hidden probability distribution for triangular Z-numbers that is based on the triangular distribution.

4.1. Definition and properties

Definition 20. The probability density function pdf of the extended triangular distribution is composed of two trapezoids, and is characterized by four parameters: the lower limit a, the upper limit b, and the mode m, which are parameters of the triangular distribution, as well as a new parameter, the minimum height β, which influences the concavity and convexity of the pdf. The pdf of the extended triangular distribution can be represented as a segmentation function $pdf(x) = \mathcal{T}(a, m, b, \beta)$, where β is in the interval $[0, \frac{2}{b-a}]$, and is defined as follows:

$$pdf(x) = \begin{cases} \dfrac{x-a}{m-a}\left(\dfrac{2}{b-a} - 2\beta\right) + \beta, & a \le x \le m, \\ \dfrac{b-x}{b-m}\left(\dfrac{2}{b-a} - 2\beta\right) + \beta, & m \le x \le b, \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

The extended triangular distribution degrades to the classical triangular distribution when $\beta = 0$. When $0 \le \beta < \frac{1}{b-a}$, the pdf of the extended triangular distribution is convex, as shown in Fig. 6. At this interval, the smaller the value of β, the more convex the distribution will be. If $\beta = \frac{1}{b-a}$, the extended triangular distribution becomes a uniform distribution, as shown in Fig. 7. When $\frac{1}{b-a} < \beta \le \frac{2}{b-a}$, the pdf of the extended triangular distribution is concave, as shown in Fig. 8. At this interval, the bigger the value of β, the more concave the distribution will be.

Theorem 1. $\int_a^b pdf(x)dx = 1$.

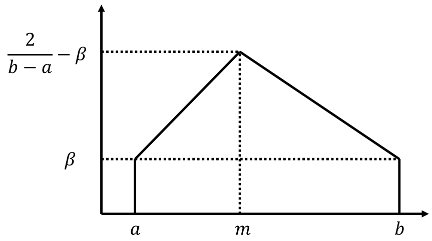


Fig. 6. An extended triangular density function when $0 < \beta < \frac{1}{b-a}$.

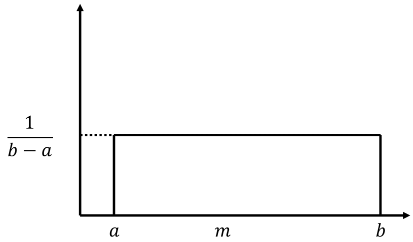


Fig. 7. An extended triangular density function when $\beta = \frac{1}{b-a}$.

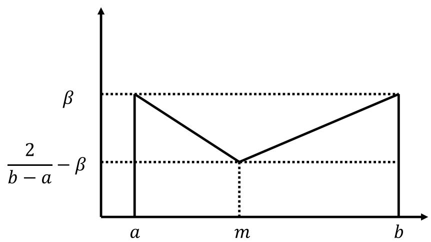


Fig. 8. An extended triangular density function when $\beta > \frac{1}{b-a}$.

Proof 1.

$$\begin{aligned}
\int_a^b pdf(x)dx &= \int_a^m \left(\frac{x-a}{m-a}\left(\frac{2}{b-a}-2\beta\right)+\beta\right)dx + \int_m^b \left(\frac{b-x}{b-m}\left(\frac{2}{b-a}-2\beta\right)+\beta\right)dx \\
&= \frac{1}{m-a}\left(\frac{2}{b-a}-2\beta\right)\frac{1}{2}(m-a)^2+\beta(m-a)+\frac{1}{b-m}\left(\frac{2}{b-a}-2\beta\right)\frac{1}{2}(b-m)^2+\beta(b-m) \\
&= (m-a+b-m)\left(\frac{1}{b-a}-\beta\right)+\beta(m-a+b-m) \\
&= 1-\beta(b-a)+\beta(b-a) \\
&= 1.
\end{aligned}$$

The CDF of the extended triangular distribution is:

$$CDF(x) = \begin{cases} \frac{(x-a)^2}{m-a}\left(\frac{1}{b-a}-\beta\right)+\beta(x-a), & a \le x \le m, \\ \left(\frac{1}{b-a}-\beta\right)(m-a+(m-x)(m-2b+x))+\beta(x-a), & m \le x \le b, \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

Proof 2. According to the definition of CDF, we can obtain,

$$CDF(x) = \int_{-\infty}^{x} pdf(t)dt.$$

When $a \le x \le m$,

$$
\begin{aligned}
CDF(x) &= \int_a^x pdf(t)dt \\
&= \int_a^x \Big(\frac{t-a}{m-a}\big(\frac{2}{b-a} - 2\beta\big) + \beta\Big)dt \\
&= \frac{(x-a)^2}{m-a}\Big(\frac{1}{b-a} - \beta\Big) + \beta(x-a).
\end{aligned}
$$

When $m \le x \le b$,

$$
\begin{aligned}
CDF(x) &= \int_a^x pdf(t)dt \\
&= \int_a^m \Big(\frac{t-a}{m-a}\big(\frac{2}{b-a} - 2\beta\big) + \beta\Big)dt + \int_m^x \Big(\frac{b-t}{b-m}\big(\frac{2}{b-a} - 2\beta\big) + \beta\Big)dt \\
&= \Big(\frac{1}{b-a} - \beta\Big)(m-a+(m-x)(m-2b+x)) + \beta(x-a).
\end{aligned}
$$

Theorem 2. *The expected value of the extended triangular distribution is* $E(X) = (a+m+b)/3 + \frac{\beta(b-a)(a+b-2m)}{6}$.

Proof 3.

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{+\infty} x\,pdf(x)dx \\
&= \int_a^m x\Big(\frac{x-a}{m-a}\big(\frac{2}{b-a} - 2\beta\big) + \beta\Big)dx + \int_m^b x\Big(\frac{b-x}{b-m}\big(\frac{2}{b-a} - 2\beta\big) + \beta\Big)dx \\
&= \frac{1}{6(m-a)}\Big(\frac{2}{b-a} - 2\beta\Big)(m-a)^2(a+2m) + \frac{1}{2}\beta(m^2-a^2) \\
&+ \frac{1}{6(b-m)}\Big(\frac{2}{b-a} - 2\beta\Big)(b-m)^2(b+2m) + \frac{1}{2}\beta(b^2-m^2) \\
&= \frac{1}{3}\Big(\frac{1}{b-a} - \beta\Big)((m-a)(a+2m)+(b-m)(b+2m)) + \frac{1}{2}\beta(b^2-a^2) \\
&= \frac{1}{3}\Big(\frac{1}{b-a} - \beta\Big)(b-a)(a+b+m) + \frac{1}{2}\beta(b^2-a^2) \\
&= \frac{a+m+b}{3} + \frac{\beta(b-a)(a+b-2m)}{6}.
\end{aligned}
$$

From the negation of the probability distribution defined in Definition 8, we can deduce the negation of a pdf.

Definition 21. The negation of a pdf $pdf(x)$ is a pdf denoted as $\overline{pdf}(x)$ and defined as:

$$
\overline{pdf}(x) = \frac{1 - pdf(x)}{\int_{-\infty}^{\infty}(1 - pdf(x))dx}, \tag{34}
$$

where $\int_{-\infty}^{\infty}(1 - pdf(x))dx$ is the normalization coefficient. From the definition, we can see that not all pdfs have the corresponding negation, such as normal distributions.

Theorem 3. *The negation of the extended triangular distribution is*

$$
\overline{pdf}(x) = \begin{cases} \dfrac{2(x-a)(b\beta - a\beta - 1)}{(b-a)(m-a)(b-a-1)} + \dfrac{1-\beta}{b-a-1}, & a \le x \le m, \\ \dfrac{2(b-x)(b\beta - a\beta - 1)}{(b-a)(b-m)(b-a-1)} + \dfrac{1-\beta}{b-a-1}, & m \le x \le b. \end{cases} \tag{35}
$$

4.2. Extended triangular distributions in Z-numbers

For a Z-number $Z = (A, B)$ with components as TFNs, $A = (a_A, m_A, b_A)$ and $B = (a_B, m_B, b_B)$, we assume that its hidden pdf is an extended triangular distribution $\mathcal{T} = (a, m, b, \beta)$. It is evident that $a = a_A$, $m = m_A$, and $b = b_A$. Our goal is to determine the parameter β. Using the definition of Z-numbers, we can obtain the probability measure of A as follows:

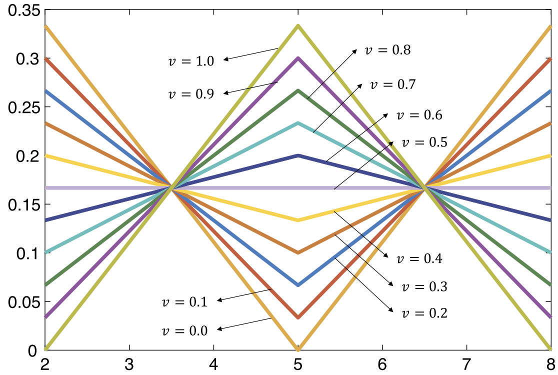


Fig. 9. The hidden extended triangular pdf when reliability is 0.2 to 0.8.

$$
\begin{aligned}
Prob(X \text{ is } A) = v &= \int_{-\infty}^{+\infty} \mu_A(x) pdf(x) dx \\
&= \int_a^m \frac{x-a}{m-a}\left(\frac{x-a}{m-a}\left(\frac{2}{b-a} - 2\beta\right) + \beta\right) dx + \int_m^b \frac{b-x}{b-m}\left(\frac{b-x}{b-m}\left(\frac{2}{b-a} - 2\beta\right) + \beta\right) dx \\
&= \left(\frac{2}{b-a} - 2\beta\right)\frac{m-a}{3} + \frac{\beta(m-a)}{2} + \left(\frac{2}{b-a} - 2\beta\right)\frac{b-m}{3} + \frac{\beta(b-m)}{2} \\
&= \frac{2}{3} - \frac{\beta(b-a)}{6}.
\end{aligned}
$$

Since $\beta \in [0, \frac{2}{b-a}]$, $v \in [\frac{1}{3}, \frac{2}{3}]$. However, $v \in [0,1]$ actually. Hence, when $v \in [0, \frac{1}{3})$ and $v \in (\frac{2}{3}, 1]$, the hidden probability distributions do not exist. But the value of v represents the reliability measure of A. $v = 1$ means A is totally reliable, and $v = 0$ means A is totally unreliable. For the extended triangular distribution, when $\beta = 0$, the pdf is the probability distribution transformed from A by normalization, and the pdf is the negation of the probability distribution transformed from A when $\beta = \frac{2}{b-a}$. Thus, we expand the value domain of $Prob(X \text{ is } A)$ obtained by the extended triangular distribution to $[0,1]$:

$$
\begin{aligned}
\frac{v+1}{3} &= \frac{2}{3} - \frac{\beta(b-a)}{6} \\
\beta &= \frac{2(1-v)}{b-a}.
\end{aligned} \tag{36}
$$

Consequently, for Z-number Z, its hidden pdf is $\mathcal{T}(a_A, m_A, b_A, \frac{2(1-v)}{b_A - a_A})$, $v \in [0,1]$.

Example 3. Let us consider the triangular Z-number in Example 1, $Z_1 = ((2,5,8),(0.2,0.5,0.8))$. Its hidden extended triangular pdf is $\mathcal{T} = (2,5,8,\frac{1}{3} - \frac{v}{3})$. We can obtain an precise hidden pdf without discretizing B_1, which is a function of both x and v. When $v = 0.0$ to 1.0, it is delineated in Fig. 9.

We can see from Fig. 9 that when the reliability of A_1 is 1.0, the hidden pdf is the pdf converted by A_1. This means that the fuzzy number A_1 is totally reliable. When the reliability is 0, the hidden pdf is the negation of the hidden pdf when reliability is 1. Therefore, the fuzzy number A_1 is completely opposite to the real pdf. When the reliability is 0.5, the hidden pdf is a uniform distribution on the interval $[2,8]$. This indicates that the reliability of A_1 is uncertain and there is no information about the value of the variable.

5. Operations on triangular Z-numbers based on the extended triangular distributions

In this section, we list the calculation procedures for addition, subtraction, multiplication, division, minimum and maximum operations over two triangular Z-numbers and negation of a Z-number. We also provide numerical examples to illustrate the calculation procedures and to compare the computational complexity and resulting Z-numbers' uncertainty with the traditional method.

5.1. Addition of triangular Z-numbers

There are two triangular Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ describing imperfect information about the values of variables X_1 and X_2. The addition of X_1 and X_2, $X_{12} = X_1 + X_2$, is a continuous Z-numbers $Z_{12} = Z_1 + Z_2$. The procedures of computation of $Z_{12} = (A_{12}, B_{12})$ are outlined below.

Step 1: Compute the addition of fuzzy numbers A_1 and A_2, $A_{12} = A_1 + A_2$, in accordance with Definition 12.
Step 2: Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2 using Eq. (36), respectively.
Step 3: Compute the addition $pdf_{12} = pdf_1 + pdf_2$ of pdfs using Definition 2.
Step 4: Compute the base value of B_{12}, v_{12}, using the definition of Z-numbers. Note that the addition of two extended triangular distributions is not an extended triangular distribution. However, the aim of computing the addition of pdfs is to calculate the base value of the second fuzzy number. Thus, we can find an extended triangular distribution that can obtain the same base value as the actual addition pdfs.
Step 5: Compute the membership function of B_{12}, $\mu_{B_{12}}(v_{12}) = \max(\mu_{B_1}(v_1) \wedge \mu_{B_2}(v_2))$.

Example 4. Given two triangular Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$, where the components are TFNs: $Z_1 = ((1,2,3),(0.7,0.8,0.9))$, $Z_2 = ((7,8,9),(0.4,0.5,0.6))$. The computation procedures for the addition of Z_1 and Z_2, $Z_{12} = Z_1 + Z_2 = (A_{12}, B_{12})$, are given below.

Step 1: Compute the addition of fuzzy numbers A_1 and A_2, $A_{12} = A_1 + A_2$.

$$A_{12} = (1,2,3) + (7,8,9) = (8,10,12).$$

Step 2: Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2. Assume the pdfs pdf_1 and pdf_2 are two extended triangular distributions, $pdf_1 = \mathcal{T}(1,2,3,\beta_1)$ and $pdf_2 = \mathcal{T}(7,8,9,\beta_2)$. According to Eq. (36), we can get $\beta_1 = 1 - v_1$ and $\beta_2 = 1 - v_2$. Therefore, $pdf_1 = \mathcal{T}(1,2,3,1-v_1)$ and $pdf_2 = \mathcal{T}(7,8,9,1-v_2)$.
Step 3: Compute the addition $pdf_{12} = pdf_1 + pdf_2$ of pdfs.

$$pdf_1(x) = \begin{cases} (x-1)(2v_1-1) + (1-v_1), & 1 \le x \le 2, \\ (3-x)(2v_1-1) + (1-v_1), & 2 \le x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

$$pdf_2(x) = \begin{cases} (x-7)(2v_2-1) + (1-v_2), & 7 \le x \le 8, \\ (9-x)(2v_2-1) + (1-v_2), & 8 \le x \le 9, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} pdf_{12}(z) &= \int_1^2 [(x-1)(2v_1-1) + (1-v_1)] \times pdf_2(z-x)dx + \int_2^3 [(3-x)(2v_2-1) + (1-v_1)] \times pdf_2(z-x)dx \\ &= \int_{z-2}^{z-1} [(z-t-1)(2v_1-1) + (1-v_1)] \times pdf_2(t)dt + \int_{z-3}^{z-2} [(3-z+t)(2v_2-1) + (1-v_2)]dt. \end{aligned}$$

When $8 \le z \le 9$,

$$\begin{aligned} pdf_{12}(z) &= \int_7^{z-1} [(z-t-1)(2v_1-1) + (1-v_1)] \times [(t-7)(2v_2-1) + (1-v_2)]dt \\ &= (2v_1-1)(2v_2-1)\frac{(z-8)^3}{6} + (2v_1-1)(1-v_2)\frac{(z-8)^2}{2} + (2v_2-1)(1-v_1)\frac{(z-8)^2}{2} \\ &\quad + (1-v_1)(1-v_2)(z-8). \end{aligned}$$

When $9 \le z \le 10$,

$$\begin{aligned} pdf_{12}(z) &= \int_{z-2}^{8} [(z-t-1)(2v_1-1) + (1-v_1)] \times [(t-7)(2v_2-1) + (1-v_2)]dt \\ &\quad + \int_8^{z-1} [(z-t-1)(2v_1-1) + (1-v_1)] \times [(9-t)(2v_2-1) + (1-v_2)]dt \\ &\quad + \int_7^{z-2} [(3-z+t)(2v_2-1) + (1-v_2)] \times [(t-7)(2v_2-1) + (1-v_2)]dt \\ &= (2v_1-1)(2v_2-1)(\frac{(10-z)(z^2-14z+46)}{6} + \frac{2(z-9)^2(12-z)}{6}) \end{aligned}$$

$$+ (2v_1 - 1)(1 - v_2)(\frac{(z-8)(10-z)}{2} + \frac{(z-9)^2}{2} + \frac{(z-9)(11-z)}{2})$$
$$+ (2v_2 - 1)(1 - v_1)(\frac{(z-8)(10-z)}{2} + \frac{(z-9)(11-z)}{2} + \frac{(z-9)^2}{2})$$
$$+ (1 - v_1)(1 - v_2)(z-8).$$

When $10 \le z \le 11$,

$$pdf_{12}(z) = \int_{z-2}^{9} [(z-t-1)(2v_1-1) + (1-v_1)] \times [(9-t)(2v_2-1) + (1-v_2)]dt$$
$$+ \int_{z-3}^{8} [(3-z+t)(2v_2-1) + (1-v_2)] \times [(t-7)(2v_2-1) + (1-v_2)]dt$$
$$+ \int_{8}^{z-2} [(3-z+t)(2v_2-1) + (1-v_2)] \times [(9-t)(2v_2-1) + (1-v_2)]dt$$
$$= (2v_1-1)(2v_2-1)(\frac{2(z-8)(z-11)^2}{6} + \frac{(z-10)(z^2-26z+166)}{6})$$
$$+ (2v_1 - 1)(1 - v_2)(\frac{(z-9)(11-z)}{2} + \frac{(z-11)^2}{2} + \frac{(z-10)(12-z)}{2})$$
$$+ (2v_2 - 1)(1 - v_1)(\frac{(z-11)^2}{2} + \frac{(z-9)(11-z)}{2} + \frac{(z-10)(12-z)}{2})$$
$$+ (1 - v_1)(1 - v_2)(12-z).$$

When $11 \le z \le 12$,

$$pdf_{12}(z) = \int_{z-3}^{9} [(3-z+t)(2v_2-1) + (1-v_2)] \times [(9-t)(2v_2-1) + (1-v_2)]dt$$
$$= (2v_1-1)(2v_2-1)\frac{(12-z)^3}{6} + (2v_1-1)(1-v_2)\frac{(z-12)^2}{2} + (2v_2-1)(1-v_1)\frac{(z-12)^2}{2}$$
$$+ (1 - v_1)(1 - v_2)(12-z).$$

Otherwise, $pdf_{12}(z) = 0$.

Step 4: Compute the base value of B_{12}, v_{12}.

$$v_{12} = \int_{-\infty}^{+\infty} \mu_{A_{12}}(x) pdf_{12}(x)dx$$
$$= \int_{8}^{10} \frac{x-8}{2} pdf_{12}(x)dx + \int_{10}^{12} \frac{12-x}{2} pdf_{12}(x)dx$$
$$= (2v_1-1)(2v_2-1)(\frac{1}{60} + \frac{11}{30} + \frac{11}{30} + \frac{1}{60}) + (2v_1-1)(1-v_2)(\frac{1}{16} + \frac{31}{48} + \frac{31}{48} + \frac{1}{16})$$
$$+ (2v_2-1)(1-v_1)(\frac{1}{16} + \frac{31}{48} + \frac{31}{48} + \frac{1}{16}) + (1-v_1)(1-v_2)(\frac{1}{6} + \frac{7}{6} + \frac{7}{6} + \frac{1}{6})$$
$$= \frac{v_1+v_2}{20} + \frac{v_1 v_2}{15} + \frac{3}{5}.$$

We can compute the hidden extended triangular density function $pdf'_{12} = \mathcal{T}(8, 10, 12, \beta_{12})$ of B_{12} that obtain the same base value v_{12}:

$$\frac{v_{12}+1}{3} = \frac{2}{3} - \frac{2}{3}\beta_{12}$$
$$\beta_{12} = \frac{1-v_{12}}{2}$$
$$\beta_{12} = \frac{1}{5} - \frac{v_1+v_2}{40} - \frac{v_1 v_2}{30}.$$

Then, the hidden extended triangular density function of Z_{12} is $pdf'_{12}(x) = \mathcal{T}(8, 10, 12, \frac{1}{5} - \frac{v_1+v_2}{40} - \frac{v_1 v_2}{30})$.

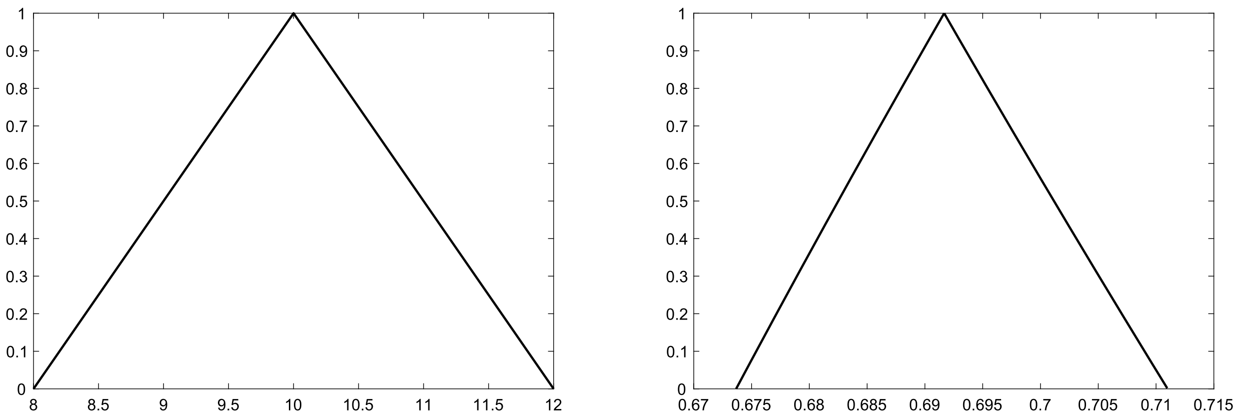


Fig. 10. A_{12} (left) and B_{12} (right).

Step 5: Compute the membership function of B_{12}, $\mu_{B_{12}}(v_{12})$.

$$\mu_{B_{12}}(\frac{v_1+v_2}{20}+\frac{v_1v_2}{15}+\frac{3}{5}) = \max(\mu_{B_1}(v_1)\wedge\mu_{B_2}(v_2)),$$

$$\mu_{B_{12}}(v_{12}) = \begin{cases} \frac{\sqrt{6(1000v_{12}-561)}}{2}-13, & \frac{417}{619}\leq v_{12}\leq\frac{83}{120}; \\ 15-\frac{\sqrt{6(1000v_{12}-561)}}{2}, & \frac{83}{120}\leq v_{12}\leq\frac{711}{1000}. \end{cases}$$

The addition of Z-numbers Z_1 and Z_2 is a Z-number $Z_{12}=(A_{12},B_{12})$, which is shown in Fig. 10.

Example 5. Let us consider the addition of Z_1 and Z_2 in Example 4 using the traditional approach, $Z'_{12}=Z_1+Z_2$. The procedures of computation of $Z'_{12}=(A'_{12},B'_{12})$ are listed below.

Step 1: The addition of fuzzy number A_1 and A_2, $A'_{12}=A_1+A_2$.

$$A'_{12}=(1,2,3)+(7,8,9)=(8,10,12).$$

Step 2: Discretize B_1 and B_2 and find the hidden pdfs. $supp(B_1)$ and $supp(B_2)$ are split into five discrete elements:

$$B_1=\frac{0}{0.7}+\frac{0.5}{0.75}+\frac{1}{0.8}+\frac{0.5}{0.85}+\frac{0}{0.9},$$
$$B_2=\frac{0}{0.4}+\frac{0.5}{0.45}+\frac{1}{0.5}+\frac{0.5}{0.55}+\frac{0}{0.6}.$$

In the traditional approach, the hidden pdf is a normal pdf. Then we can find pdf_{il}, $i=1,2$, $l=1,2,\dots,5$, by using the following nonlinear optimization problem:

$$\int_{-\infty}^{\infty}\mu_{A_i}(x)\frac{1}{\sqrt{2\pi}\sigma_{il}}\exp(-\frac{(x-\alpha_i)^2}{2\sigma_{il}^2})dx\longrightarrow v_{il}$$

subject to

$$\alpha_i=\frac{\int_{-\infty}^{\infty}x\mu_{A_i}(x)dx}{\int_R\mu_{A_i}(x)dx},$$
$$\sigma_{il}>0.$$

$\alpha_1=2$ and $\alpha_2=8$. For different values of v_{il}, the resulting values of σ_{il} are different. Therefore, it is necessary to solve the above optimization problem ten times. The results are shown in Table 3.

Step 3: Compute the hidden pdf pdf'_{12s} of Z_{12}. $\alpha_{12}=\alpha_1+\alpha_2=10$ and $\sigma_{12s}=\sqrt{\sigma_{1l}^2+\sigma_{2l}^2}$. Then the discretized hidden pdf pdf'_{12s} is shown in Table 4.

Step 4: The base values v'_{12s} of B'_{12} is computed by

$$v'_{12s}=\int_8^{10}\frac{x-8}{2}\frac{1}{\sqrt{2\pi}\sigma_{12s}}\exp(-\frac{(x-\alpha_{12})^2}{2\sigma_{12s}^2})dx+\int_{10}^{12}\frac{12-x}{2}\frac{1}{\sqrt{2\pi}\sigma_{12s}}\exp(-\frac{(x-\alpha_{12})^2}{2\sigma_{12s}^2})dx.$$

Table 3
The obtained solutions σ_{il} for v_{il}.

v_{1l}	0.7	0.75	0.8	0.85	0.9
σ_{1l}	0.3772	0.3135	0.2507	0.1880	0.1253
v_{2l}	0.4	0.45	0.5	0.55	0.6
σ_{2l}	0.9075	0.7832	0.6801	0.5918	0.5140

Table 4
The values of σ_{12s} for different σ_{1l} and σ_{2l}.

σ_{2l} \ σ_{1l}	0.3772	0.3135	0.2507	0.1880	0.1253
0.9075	0.9828	0.9601	0.9415	0.9268	0.9161
0.7832	0.8693	0.8436	0.8223	0.8054	0.7932
0.6801	0.7777	0.7489	0.7248	0.7056	0.6915
0.5918	0.7018	0.6697	0.6427	0.6209	0.6049
0.5140	0.6376	0.6021	0.5719	0.5473	0.5291

Table 5
The base values of B'_{12} for different v_{1l} and v_{2l}.

v_{2l} \ v_{1l}	0.7	0.75	0.8	0.85	0.9
0.4	0.6155	0.6235	0.6301	0.6354	0.6392
0.45	0.6564	0.6660	0.6740	0.6804	0.6851
0.5	0.6910	0.7021	0.7115	0.7190	0.7245
0.55	0.7205	0.7331	0.7438	0.7524	0.7587
0.6	0.7458	0.7599	0.7719	0.7817	0.7889

The results are shown in Table 5.

Step 5: The membership function $\mu_{B'_{12}}$ of B'_{12} is computed by

$$\mu_{B'_{12}}(v'_{12s}) = \max(\mu_{B_1}(v_{1l_1}) \wedge \mu_{B_2}(v_{2l_2})).$$

Therefore, there are eight base values with membership degree 0.5:

$$\mu_{B'_{12}}(0.6660) = \mu_{B'_{12}}(0.6740) = \mu_{B'_{12}}(0.6804) = \mu_{B'_{12}}(0.7021)$$
$$= \mu_{B'_{12}}(0.7190) = \mu_{B'_{12}}(0.7331) = \mu_{B_{12}}(0.7438) = \mu_{B'_{12}}(0.7524) = 0.5.$$

There is one base values with membership degree 1: $\mu_{B'_{12}}(0.7115) = 1$. The membership degree of other base values is zero. The B'_{12} is approximated as a TFN $B'_{12} = (0.6155, 0.7115, 0.7889)$.

Therefore, $Z'_{12} = Z_1 + Z_2 \approx ((8, 10, 12), (0.6155, 0.7115, 0.7889))$.

By comparing the calculation procedures of Example 4 and Example 5, it can be concluded that their first steps are identical. However, the second step of the traditional method involves discretizing B_i and solving several nonlinear optimization problems to determine the hidden pdfs. This discretization process can lead to discretization errors. The number of nonlinear optimization problems required to be solved depends on the level of discretization precision, with higher precision requiring more optimization problems to be solved. In contrast, our proposed method only requires a simple calculation to obtain all possible hidden psds. Due to the additive property of normal probability distributions, the third step of the traditional method is relatively straightforward. Conversely, our method requires computing the convolution of two segmentation functions. However, the computational complexity of the traditional method still depends on the level of discretization precision. In the fourth step, both methods require computing integrals. However, the number of integrals that need to be computed in the traditional method increases with the level of precision, while our method only requires computing one integral. In the fifth step, the traditional method estimates the results, which introduces estimation errors.

The traditional method is prone to errors due to discretization and estimation, and requires solving numerous nonlinear optimization problems to find the hidden probability function. On the other hand, our method is error-free and involves simpler calculations.

According to Eqs. (27) to (31), we compute the entropy of Z-numbers Z_1, Z_2 and their addition computed by two approaches Z'_{12} and Z_{12}, with *pre* set to 0.01. $H(Z_1) = 0.0013$, $H(Z_2) = 0.1034$, $H(Z'_{12}) = 0.0966$, $H(Z_{12}) = 0.0209$. From the results, it can be observed that the addition computed by the traditional method yields more uncertainty.

5.2. Subtraction of triangular Z-numbers

There are two triangular Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ describing imperfect information about the values of variables X_1 and X_2. The subtraction of X_1 and X_2, $X_{12} = X_1 - X_2$, is a continuous Z-numbers $Z_{12} = Z_1 - Z_2$. The procedures of computation of $Z_{12} = (A_{12}, B_{12})$ are given below.

Step 1: Compute the subtraction of fuzzy numbers A_1 and A_2, $A_{12} = A_1 - A_2$, based on Definition 13.
Step 2: Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2 by using Eq. (36), respectively.
Step 3: Compute the subtraction $pdf_{12} = pdf_1 - pdf_2$ of pdfs with Definition 2.
Step 4: Compute the based value of B_{12}, v_{12}, by using the definition of Z-numbers. The subtraction of two extended triangular distributions is not an extended triangular distribution. However, we can find an extended triangular distribution that can obtain the same base value as the actual subtraction pdfs.
Step 5: Compute the membership function of B_{12}, $\mu_{B_{12}}(v_{12}) = \max(\mu_{B_1}(v_1) \wedge \mu_{B_2}(v_2))$.

Example 6. The subtraction of X_1 and X_2 in Example 4, $X_{12} = X_2 - X_1$ is a continuous Z-numbers $Z_{12} = Z_2 - Z_1$. The procedures of computation of the subtraction of Z_1 and Z_2, $Z_{12} = Z_2 - Z_1 = (A_{12}, B_{12})$, are given below.

Step 1: Compute the subtraction of fuzzy numbers A_1 and A_2, $A_{12} = A_2 - A_1$.

$$A_{12} = A_2 - A_1 = (4, 6, 8).$$

Step 2: Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2.

$$pdf_1 = \mathcal{T}(1, 2, 3, 1 - v_1), pdf_2 = \mathcal{T}(7, 8, 9, 1 - v_2).$$

Step 3: Compute the subtraction $pdf_{12} = pdf_2 - pdf_1$ of pdfs.

$$\begin{aligned} pdf_{12}(z) &= \int_1^2 [(x-1)(2v_1-1)+(1-v_1)] \times pdf_2(x+z)dx + \int_2^3 [(3-x)(2v_1-1)+(1-v_1)] \times pdf_2(x+z)dx \\ &= \int_{z+1}^{z+2} [(t-z-1)(2v_1-1)+(1-v-1)]pdf_2(t)dt + \int_{z+2}^{z+3} [(3+z-t)(2v_1-1)+(1-v_1)]pdf_2(t)dt. \end{aligned}$$

When $4 \le z \le 5$,

$$\begin{aligned} pdf_{12}(z) &= \int_7^{z+3} [(3+z-t)(2v_1-1)+(1-v_1)][(t-7)(2v_2-1)+(1-v_2)]dt \\ &= (2v_1-1)(2v_2-1)\frac{(z-4)^3}{6} + (2v_1-1)(1-v_2)\frac{(z-4)^2}{2} \\ &+ (2v_2-1)(1-v_1)\frac{(z-4)^2}{2} + (1-v_1)(1-v_2)(z-4). \end{aligned}$$

When $5 \le z \le 6$,

$$\begin{aligned} pdf_{12}(z) &= \int_7^{z+2} [(t-z-1)(2v_1-1)+(1-v_1)][(t-7)(2v_2-1)+(1-v_2)]dt \\ &+ \int_{z+2}^{8} [(3+z-t)(2v_2-1)+(1-v_1)][(t-7)(2v_2-1)+(1-v_2)]dt \\ &+ \int_8^{z+3} [(3+z-t)(2v_2-1)+(1-v_1)][(9-t)(2v_2-1)+(1-v_2)]dt \\ &= (2v_1-1)(2v_2-1)(\frac{2(z-5)^2(8-z)}{6} + \frac{(6-z)(z^2-6z+6)}{6}) \\ &+ (2v_1-1)(1-v_2)(\frac{(z-5)(7-z)}{2} + \frac{(z-4)(6-z)}{2} + \frac{(z-5)^2}{2}) \\ &+ (1-v_1)(2v_2-1)(\frac{(z-5)^2}{2} + \frac{(z-4)(6-z)}{2} + \frac{(z-5)(7-z)}{2}) \\ &+ (1-v_1)(1-v_2)(z-4). \end{aligned}$$

When $6 \le z \le 7$,

$$pdf_{12}(z) = \int_{z+1}^{8} [(t-z-1)(2v_1-1)+(1-v_1)][(t-7)(2v_2-1)+(1-v_2)]dt$$
$$+ \int_{8}^{z+2} [(t-z-1)(2v_1-1)+(1-v_1)][(9-t)(2v_2-1)+(1-v_2)]dt$$
$$+ \int_{z+2}^{9} [(3+z-t)(2v_2-1)+(1-v_1)][(9-t)(2v_2-1)+(1-v_2)]$$
$$= (2v_1-1)(2v-2-1)(\frac{2((z-4)(z-7)^2)}{6} + \frac{(z-6)(z^2-18z+78)}{6})$$
$$+ (2v_1-1)(1-v_2)(\frac{(z-7)^2}{2} + \frac{(z-6)(8-z)}{2} + \frac{(z-5)(7-z)}{2})$$
$$+ (2v_2-1)(1-v_1)(\frac{(z-5)(7-z)}{2} + \frac{(z-6)(8-z)}{2} + \frac{(z-7)^2}{2})$$
$$+ (1-v_1)(1-v_2)(8-z).$$

When $7 \le z \le 8$,

$$pdf_{12}(z) = \int_{z+1}^{9} [(t-z-1)(2v_1-1)+(1-v-1)][(9-t)(2v_2-1)+(1-v-2)]dt$$
$$= (2v_1-1)(2v_2-1)\frac{(8-z)^3}{6} + (2v_1-1)(1-v_2)\frac{(8-z)^2}{2}$$
$$+ (2v_2-1)(1-v_1)\frac{(8-z)^2}{+}(1-v_1)(1-v_2)(8-z).$$

Otherwise, $pdf_{12}(z) = 0$.

Step 4: Compute the based value of B_{12}, v_{12}.

$$v_{12} = \int_{-\infty}^{+\infty} \mu_{A_{12}}(x) pdf_{12}(x)dx$$
$$= \int_{4}^{6} \frac{x-4}{2} pdf_{12}(x)dx + \int_{6}^{8} \frac{8-x}{2} pdf_{12}(x)dx$$
$$= (2v-1)(2v_2)(\frac{1}{60} + \frac{11}{30} + \frac{11}{30} + \frac{1}{60}) + (2v_1-1)(1-v_2)(\frac{1}{16} + \frac{31}{48} + \frac{31}{48} + \frac{1}{16})$$
$$+ (2v_2-1)(1-v-1)(\frac{1}{16} + \frac{31}{48} + \frac{31}{48} + \frac{1}{16}) + (1-v_1)(1-v_2)(\frac{1}{6} + \frac{7}{6} + + \frac{7}{6} + \frac{1}{6})$$
$$= \frac{v_1+v_2}{20} + \frac{v_1 v_2}{15} + \frac{3}{5}.$$

Therefore, the hidden extended triangular density function of Z_{12} is $pdf'_{12}(x) = \mathcal{T}(4, 6, 8, \frac{1}{5} - \frac{v_1+v_2}{40} - \frac{v_1 v_2}{30})$.

Step 5: Compute the membership function of B_{12}, $\mu_{B_{12}}(v_{12})$.

$$\mu_{B_{12}}(\frac{v_1+v_2}{20} + \frac{v_1 v_2}{15} + \frac{3}{5}) = \max(\mu_{B_1}(v_1) \wedge \mu_{B_2}(v_2))$$
$$\mu_{B_{12}}(v_{12}) = \begin{cases} \frac{\sqrt{6(1000v_{12}-561)}}{2} - 13, & \frac{417}{619} \le v_{12} \le \frac{83}{120}; \\ 15 - \frac{\sqrt{6(1000v_{12}-561)}}{2}, & \frac{83}{120} \le v_{12} \le \frac{711}{1000}. \end{cases}$$

The subtraction of Z-numbers Z_1 and Z_2 is a Z-number $Z_{12} = (A_{12}, B_{12})$, visualized in Fig. 11.

The subtraction of Z_1 and Z_2, $Z'_{12} = Z_2 - Z_1$, calculated by traditional method is $Z'_{12} = (A'_{12}, B'_{12}) \approx ((4,6,8), (0.6155, 0.7115, 0.7889))$. The comparison of subtraction operation is similar to that of addition. According to Eqs. (27) to (31), we compute the entropy of Z-numbers Z'_{12} and Z_{12}, with *pre* set to 0.01. $H(Z'_{12}) = 0.0966$, $H(Z_{12}) = 0.0209$. From the results, it can be observed that the subtraction computed by the traditional method yields more uncertainty.

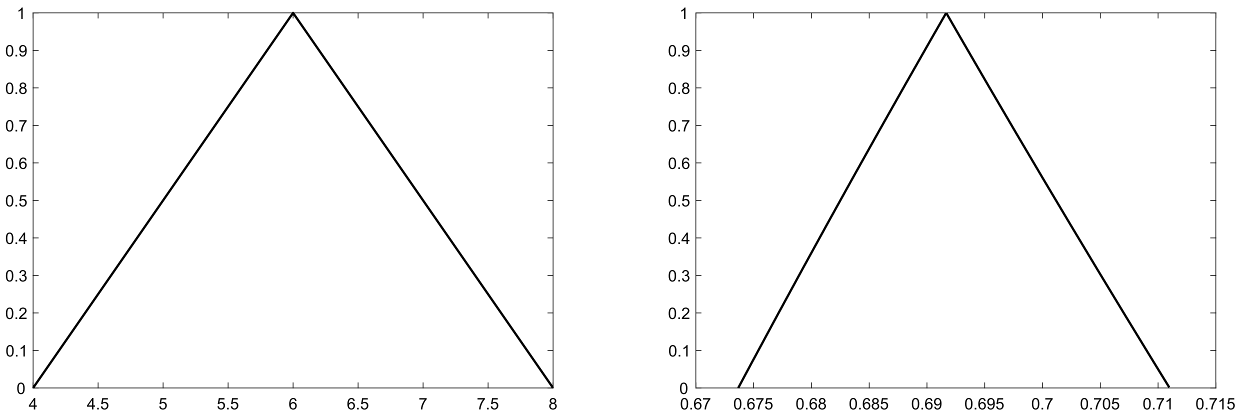


Fig. 11. A_{12} (left) and B_{12} (right).

5.3. Multiplication of triangular Z-numbers

There are two triangular Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ describing imperfect information about the values of variables X_1 and X_2. The multiplication of X_1 and X_2, $X_{12} = X_1 \times X_2$, is a continuous Z-numbers $Z_{12} = Z_1 \times Z_2$. The procedures of computation of $Z_{12} = (A_{12}, B_{12})$ can be found below.

Step 1: Compute the multiplication of fuzzy numbers A_1 and A_2, $A_{12} = A_1 \times A_2$, in accordance with Definition 14.
Step 2: Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2 by using Eq. (36), respectively.
Step 3: Compute the multiplication $pdf_{12} = pdf_1 \times pdf_2$ of pdfs with Definition 4.
Step 4: Compute the based value of B_{12}, v_{12}, by using the definition of Z-numbers. The multiplication of two extended triangular distributions is not an extended triangular distribution. However, we can find an extended triangular distribution that can obtain the same base value as the actual multiplication pdfs.
Step 5: Compute the membership function of B_{12}, $\mu_{B_{12}}(v_{12}) = \max(\mu_{B_1}(v_1) \wedge \mu_{B_2}(v_2))$.

Example 7. The multiplication of X_1 and X_2 in Example 4, $X_{12} = X_1 \times X_2$ is a continuous Z-numbers $Z_{12} = Z_1 \times Z_2$. The procedures of computation of the multiplication of Z_1 and Z_2, $Z_{12} = Z_1 \times Z_2 = (A_{12}, B_{12})$, are given below.

Step 1: Compute multiplication of fuzzy numbers A_1 and A_2, $A_{12} = A_1 \times A_2$.

$$A_{12} = A_1 \times A_2 = (7, 16, 27).$$

Step 2: Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2.

$$pdf_1 = \mathcal{T}(1, 2, 3, 1 - v_1), pdf_2 = \mathcal{T}(7, 8, 9, 1 - v_2).$$

Step 3: Compute the multiplication $pdf_{12} = pdf_1 \times pdf_2$ of pdfs.

$$pdf_{12}(z) = \int_1^2 \frac{1}{x}[(x-1)(2v_1 - 1) + (1 - v_1)] \times pdf_2(\frac{z}{x})dx + \int_2^3 \frac{1}{x}[(3-x)(2v_1 - 1) + (1 - v_1)] \times pdf_2(\frac{z}{x})dx.$$

When $7 \le z \le 8$,

$$pdf_{12}(z) = \int_7^z \frac{1}{t}[(\frac{z}{t} - 1)(2v_1 - 1) + (1 - v_1)] \times [(t-7)(2v_2 - 1) + (1 - v_1)]dt.$$

When $8 \le z \le 9$,

$$pdf_{12}(z) = \int_7^8 \frac{1}{t}[(\frac{z}{t} - 1)(2v_1 - 1) + (1 - v_1)] \times [(t-7)(2v_2 - 1) + (1 - v_1)]dt$$

$$+ \int_8^z \frac{1}{t}[(\frac{z}{t} - 1)(2v_1 - 1) + (1 - v_1)] \times [(9-t)(2v_2 - 1) + (1 - v_1)]dt.$$

When $9 \le z \le 14$,

$$pdf_{12}(z) = \int_{7}^{8} \frac{1}{t}[(\frac{z}{t} - 1)(2v_1 - 1) + (1 - v_1)] \times [(t - 7)(2v_2 - 1) + (1 - v_1)]dt$$

$$+ \int_{8}^{9} \frac{1}{t}[(\frac{z}{t} - 1)(2v_1 - 1) + (1 - v_1)] \times [(9 - t)(2v_2 - 1) + (1 - v_1)]dt.$$

When $14 \le z \le 16$,

$$pdf_{12}(z) = \int_{z/2}^{8} \frac{1}{t}[(\frac{z}{t} - 1)(2v_1 - 1) + (1 - v_1)] \times [(t - 7)(2v_2 - 1) + (1 - v_1)]dt$$

$$+ \int_{8}^{9} \frac{1}{t}[(\frac{z}{t} - 1)(2v_1 - 1) + (1 - v_1)] \times [(9 - t)(2v_2 - 1) + (1 - v_1)]dt$$

$$+ \int_{7}^{z/2} \frac{1}{t}[(3 - \frac{z}{t})(2v_1 - 1) + (1 - v_1)] \times [(t - 7)(2v_2 - 1) + (1 - v_1)]dt.$$

When $16 \le z \le 18$,

$$pdf_{12}(z) = \int_{z/2}^{9} \frac{1}{t}[(\frac{z}{t} - 1)(2v_1 - 1) + (1 - v_1)] \times [(9 - t)(2v_2 - 1) + (1 - v_1)]dt$$

$$+ \int_{7}^{8} \frac{1}{t}[(3 - \frac{z}{t})(2v_1 - 1) + (1 - v_1)] \times [(t - 7)(2v_2 - 1) + (1 - v_1)]dt$$

$$+ \int_{8}^{z/2} \frac{1}{t}[(3 - \frac{z}{t})(2v_1 - 1) + (1 - v_1)] \times [(9 - t)(2v_2 - 1) + (1 - v_1)]dt.$$

When $18 \le z \le 21$,

$$pdf_{12}(z) = \int_{7}^{8} \frac{1}{t}[(3 - \frac{z}{t})(2v_1 - 1) + (1 - v_1)] \times [(t - 7)(2v_2 - 1) + (1 - v_1)]dt$$

$$+ \int_{8}^{9} \frac{1}{t}[(3 - \frac{z}{t})(2v_1 - 1) + (1 - v_1)] \times [(9 - t)(2v_2 - 1) + (1 - v_1)]dt.$$

When $21 \le z \le 24$,

$$pdf_{12}(z) = \int_{z/3}^{8} \frac{1}{t}[(3 - \frac{z}{t})(2v_1 - 1) + (1 - v_1)] \times [(t - 7)(2v_2 - 1) + (1 - v_1)]dt$$

$$+ \int_{8}^{9} \frac{1}{t}[(3 - \frac{z}{t})(2v_1 - 1) + (1 - v_1)] \times [(9 - t)(2v_2 - 1) + (1 - v_1)]dt.$$

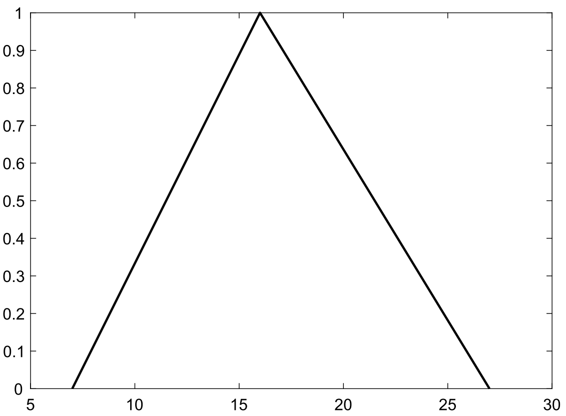
When $24 \le z \le 27$,

$$pdf_{12}(z) = \int_{z/3}^{9} \frac{1}{t}[(3 - \frac{z}{t})(2v_1 - 1) + (1 - v_1)] \times [(9 - t)(2v_2 - 1) + (1 - v_1)]dt.$$

Otherwise, $pdf_{12}(z) = 0$.

Step 4: Compute the based value of B_{12}, v_{12}.

$$v_{12} = \int_{-\infty}^{+\infty} \mu_{A_{12}}(x) pdf_{12}(x)dx$$



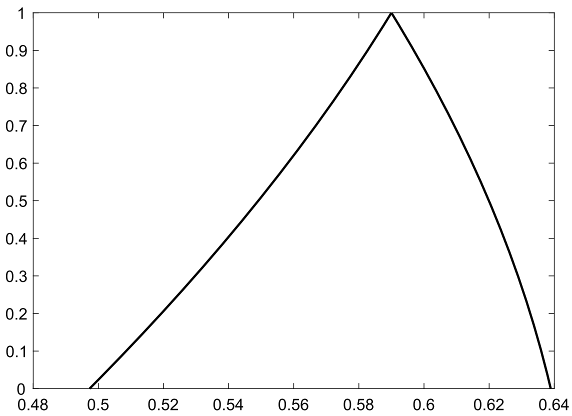


Fig. 12. A_{12} (left) and B_{12} (right).

$$
\begin{aligned}
&= \int_{7}^{16} \frac{x-7}{9} pdf_{12}(x)dx + \int_{16}^{27} \frac{27-x}{11} pdf_{12}(x)dx \\
&= (2v_1-1)(2v_2-1)(\frac{220160}{297}\ln 3 - \frac{202240}{297}\ln 7 + \frac{10240}{11}\ln 2 - \frac{119269}{891}) \\
&+ (2v_1-1)(1-v_2)(\frac{5120}{99}\ln 3 + \frac{1703}{22}\ln 7 - \frac{10240}{33}\ln 2 + \frac{380525}{37422}) \\
&+ (2v_2-1)(1-v_1)(\frac{40960}{33}\ln 2 - \frac{17920}{99}\ln 7 - \frac{5120}{11}\ln 3 + \frac{146}{33}) \\
&+ (1-v_1)(1-v_2)(\frac{2560}{11}\ln 7 - \frac{5120}{99}\ln 3 + \frac{292}{33}) \\
&= -2.1888v_1v_2 + 2.4506v_1 + 1.1029v_2 - 0.6421.
\end{aligned}
$$

Therefore, the hidden extended triangular density function is $pdf'_{12}(x) = (7, 16, 27, 0.2189v_1v_2 - 0.2451v_1 - 0.1103v_2 + 0.1642)$.

Step 5: Compute the membership function of B_{12}, $\mu_{B_{12}}(v_{12})$.

$$\mu_{B_{12}}(v_{12}) = \max(\mu_{B_1}(v_1) \wedge \mu_{B_2}(v_2))$$

$$
\mu_{B_{12}}(v_{12}) = \begin{cases} \dfrac{19097}{7296} - \dfrac{\sqrt{1574143601 - 2432000000v_{12}}}{7296}, & 0.497306 \le v_{12} < 0.59, \\ \dfrac{\sqrt{1574143601 - 2432000000v_{12}}}{7296} - \dfrac{4505}{7296}, & 0.59 \le v_{12} \le 0.638918. \end{cases}
$$

The multiplication of Z-numbers Z_1 and Z_2 is a Z-number $Z_{12} = (A_{12}, B_{12})$, visualized in Fig. 12.

The multiplication of Z_1 and Z_2, $Z_{12} = Z_1 \times Z_2$, calculated by traditional method is $Z_{12} = (A_{12}, B_{12}) \approx ((7, 16, 27), (0.35, 0.55, 0.75))$ [4]. Due to the fact that normal probability distributions only possess the additive and subtractive properties, for multiplication operations, the traditional method again performs an approximate estimation in the third step, $\alpha_{12} \approx \alpha_1 \times \alpha_2$ and $\sigma^2_{12s} \approx \alpha_2^2\sigma^2_{1l_1} + \alpha_1^2\sigma^2_{2l_2} + \sigma^2_{1l_1}\sigma^2_{2l_2}$. According to Eqs. (27) to (31), we compute the entropy of Z-numbers Z'_{12} and Z_{12}, with *pre* set to 0.01. $H(Z'_{12}) = 0.6385$, $H(Z_{12}) = 0.1381$. From the results, it can be observed that the multiplication computed by the traditional method yields more uncertainty.

5.4. Division of triangular Z-numbers

There are two triangular Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ describing imperfect information about the values of variables X_1 and X_2. The division of X_1 and X_2, $X_{12} = X_1/X_2$, is a continuous Z-numbers $Z_{12} = Z_1/Z_2$. The procedures of computation of $Z_{12} = (A_{12}, B_{12})$ are listed below.

Step 1: Compute the division of fuzzy numbers A_1 and A_2, $A_{12} = A_1/A_2$, based on Definition 15.
Step 2: Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2 by using Eq. (36).
Step 3: Compute the division $pdf_{12} = pdf_1/pdf_2$ of pdfs with Definition 5.
Step 4: Compute the based value of B_{12}, v_{12}, by using the definition of Z-numbers. The division of two extended triangular distributions is not an extended triangular distribution. However, we can find an extended triangular distribution that can obtain the same base value as the actual division pdfs.

Step 5: Compute the membership function of B_{12}, $\mu_{B_{12}}(v_{12}) = \max(\mu_{B_1}(v_1) \wedge \mu_{B_2}(v_2))$.

Example 8. The division of X_1 and X_2 in Example 4, $X_{12} = X_2/X_1$ is a continuous Z-number $Z_{12} = Z_2/Z_1$. The procedures of computation of the division of Z_1 and Z_2, $Z_{12} = Z_2/Z_1 = (A_{12}, B_{12})$, are given below.

1. Compute the division of fuzzy numbers A_1 and A_2, $A_{12} = A_2/A_1$.

$$A_{12} = A_2/A_1 = (\frac{7}{3}, 4, 9).$$

2. Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2.

$$pdf_1 = \mathcal{T}(1, 2, 3, 1 - v_1), pdf_2 = \mathcal{T}(7, 8, 9, 1 - v_2).$$

3. Compute the division $pdf_{12} = pdf_1/pdf_2$ of pdfs.

$$\begin{aligned} pdf_{12}(z) &= \int_{-\infty}^{+\infty} |x| pdf_1(x) pdf_2(xz) dx \\ &= \int_1^2 x[(x-1)(2v_1-1) + (1-v_1)] pdf_2(xz) dx + \int_2^3 x[(3-x)(2v_1-1) + (1-v_1)] pdf_2(xz) dx \\ &= \int_z^{2z} \frac{t}{z^2}[(\frac{t}{z}-1)(2v_1-1) + (1-v_1)] pdf_2(t) dt + \int_{2z}^{3z} \frac{t}{z^2}[(3-\frac{t}{z})(2v_1-1) + (1-v_1)] pdf_2(t) dt. \end{aligned}$$

When $\frac{7}{3} \le z \le \frac{8}{3}$,

$$pdf_{12}(z) = \int_7^{3z} \frac{t}{z^2}[(3-\frac{t}{z})(2v_1-1) + (1-v_1)][(t-7)(2v_2-1) + (1-v_2)] dt.$$

When $\frac{8}{3} \le z \le 3$,

$$\begin{aligned} pdf_{12}(z) &= \int_7^{8} \frac{t}{z^2}[(3-\frac{t}{z})(2v_1-1) + (1-v_1)][(t-7)(2v_2-1) + (1-v_2)] dt \\ &+ \int_8^{3z} \frac{t}{z^2}[(3-\frac{t}{z})(2v_1-1) + (1-v_1)][(9-t)(2v_2-1) + (1-v_2)] dt. \end{aligned}$$

When $3 \le z \le \frac{7}{2}$,

$$\begin{aligned} pdf_{12}(z) &= \int_7^{8} \frac{t}{z^2}[(3-\frac{t}{z})(2v_1-1) + (1-v_1)][(t-7)(2v_2-1) + (1-v_2)] dt \\ &+ \int_8^{9} \frac{t}{z^2}[(3-\frac{t}{z})(2v_1-1) + (1-v_1)][(9-t)(2v_2-1) + (1-v_2)] dt. \end{aligned}$$

When $\frac{7}{2} \le z \le 4$,

$$\begin{aligned} pdf_{12}(z) &= \int_{2z}^{8} \frac{t}{z^2}[(3-\frac{t}{z})(2v_1-1) + (1-v_1)][(t-7)(2v_2-1) + (1-v_2)] dt \\ &+ \int_8^{9} \frac{t}{z^2}[(3-\frac{t}{z})(2v_1-1) + (1-v_1)][(9-t)(2v_2-1) + (1-v_2)] dt \\ &+ \int_7^{2z} \frac{t}{z^2}[(\frac{t}{z}-1)(2v_1-1) + (1-v_1)][(t-7)(2v_2-1) + (1-v_2)] dt. \end{aligned}$$

When $4 \le z \le \frac{9}{2}$,

$$pdf_{12}(z) = \int_{2z}^{9} \frac{t}{z^2}[(3-\frac{t}{z})(2v_1-1)+(1-v_1)][(9-t)(2v_2-1)+(1-v_2)]dt$$
$$+\int_{7}^{8} \frac{t}{z^2}[(\frac{t}{z}-1)(2v_1-1)+(1-v_1)][(t-7)(2v_2-1)+(1-v_2)]dt$$
$$+\int_{8}^{2z} \frac{t}{z^2}[(\frac{t}{z}-1)(2v_1-1)+(1-v_1)][(9-t)(2v_2-1)+(1-v_2)]dt.$$

When $\frac{9}{2} \le z \le 7$,

$$pdf_{12}(z) = \int_{7}^{8} \frac{t}{z^2}[(\frac{t}{z}-1)(2v_1-1)+(1-v_1)][(t-7)(2v_2-1)+(1-v_2)]dt$$
$$+\int_{8}^{9} \frac{t}{z^2}[(\frac{t}{z}-1)(2v_1-1)+(1-v_1)][(9-t)(2v_2-1)+(1-v_2)]dt.$$

When $7 \le z \le 8$,

$$pdf_{12}(z) = \int_{z}^{8} \frac{t}{z^2}[(\frac{t}{z}-1)(2v_1-1)+(1-v_1)][(t-7)(2v_2-1)+(1-v_2)]dt$$
$$+\int_{8}^{9} \frac{t}{z^2}[(\frac{t}{z}-1)(2v_1-1)+(1-v_1)][(9-t)(2v_2-1)+(1-v_2)]dt.$$

When $8 \le z \le 9$,

$$pdf_{12}(z) = \int_{z}^{9} \frac{t}{z^2}[(\frac{t}{z}-1)(2v_1-1)+(1-v_1)][(9-t)(2v_2-1)+(1-v_2)]dt.$$

Otherwise, $pdf_{12}(z) = 0$.

4. Compute the based value of B_{12}, v_{12}.

$$\begin{aligned} v_{12} &= \int_{-\infty}^{+\infty} \mu_{A_{12}}(x) pdf_{12}(x)dx \\ &= \int_{7/3}^{4} \frac{3x-7}{5} pdf_{12}(x)dx + \int_{4}^{9} \frac{9-x}{5} pdf_{12}(x)dx \\ &= 0.7398(2v_1-1)(2v_2-1) + 1.4642(2v_1-1)(1-v_2) + 1.2289(2v_2-1)(1-v_1) + 2.4410(1-v_1)(1-v_2) \\ &= 0.0140v_1v_2 + 0.2367v_1 + 0.0014v_2 + 0.4877. \end{aligned}$$

Therefore, the hidden extended triangular density function $pdf'_{12}(x) = (\frac{7}{3}, 4, 9, 0.1537 - 0.0042v_1v_2 - 0.0710v_1 - 0.0004v_2)$.

5. Compute the membership function of B_{12}, $\mu_{B_{12}}(v_{12})$.

$$\mu_{B_{12}}(v_{12}) = \max(\mu_{B_1}(v_1) \wedge \mu_{B_2}(v_2))$$

$$\mu_{B_{12}}(v_{12}) = \begin{cases} \dfrac{\sqrt{5600000v_{12}+3137457}}{28} - \dfrac{2535}{28}, & 0.58728 \le v_{12} < 0.61277, \\ \dfrac{2591}{28} - \dfrac{\sqrt{5600000v_{12}+3137457}}{28}, & 0.61277 \le v_{12} \le 0.63854. \end{cases}$$

The division of Z-numbers Z_1 and Z_2 is a Z-number $Z_{12} = (A_{12}, B_{12})$, visualized in Fig. 13.

The division of Z_1 and Z_2, $Z_{12} = Z_2 \times Z_1$, calculated by traditional method is $Z_{12} = (A_{12}, B_{12}) \approx ((\frac{7}{3}, 4, 9), (0.35, 0.55, 0.75))$ [4]. Due to the fact that normal probability distributions only possess the additive and subtractive properties, for division operations, the

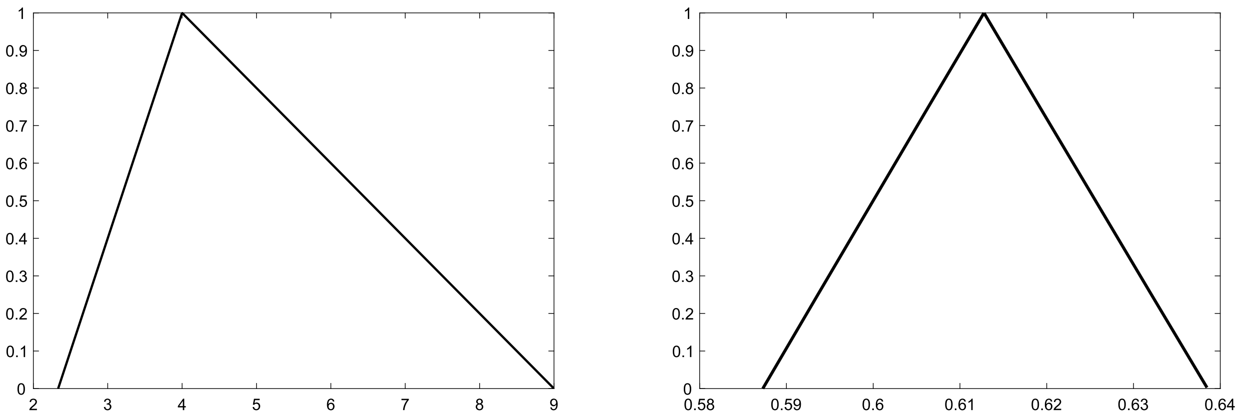


Fig. 13. A_{12} (left) and B_{12} (right).

traditional method again performs an approximate estimation in the third step, $\alpha_{12s} \approx \frac{\alpha_2}{\alpha_1} + \frac{\alpha_2 \sigma^2_{1l_1}}{4\alpha_1^3}$, $\sigma^2_{12s} \approx \frac{\sigma^2_{2l_2}}{\alpha_1^2} + \frac{\alpha_2^2 \sigma^2_{1l_1}}{\alpha_1^4} - \frac{\alpha_2^2 \alpha_1^3}{2\alpha_1^5}$. According to Eqs. (27) to (31), we compute the entropy of Z-numbers Z'_{12} and Z_{12}, with *pre* set to 0.01. $H(Z'_{12}) = 0.4125$, $H(Z_{12}) = 0.0455$. From the results, it can be observed that the division computed by the traditional method yields more uncertainty.

5.5. Minimum of triangular Z-numbers

There are two triangular Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ describing imperfect information about the values of variables X_1 and X_2. The minimum of X_1 and X_2, $X_{12} = \min(X_1, X_2)$, is a continuous Z-numbers $Z_{12} = \min(Z_1, Z_2)$. The procedures of computation of $Z_{12} = (A_{12}, B_{12})$ are listed below.

Step 1: Compute the minimum of fuzzy numbers A_1 and A_2, $A_{12} = \min(A_1, A_2)$, based on Definition 16.
Step 2: Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2 by using Eq. (36).
Step 3: Compute the minimum $pdf_{12} = \min(pdf_1, pdf_2)$ of pdfs with Definition 6.
Step 4: Compute the based value of B_{12}, v_{12}, by using the definition of Z-numbers. The minimum of two extended triangular distributions is not an extended triangular distribution. However, we can find an extended triangular distribution that can obtain the same base value as the actual minimum pdfs.
Step 5: Compute the membership function of B_{12}, $\mu_{B_{12}}(v_{12}) = \max(\mu_{B_1}(v_1) \wedge \mu_{B_2}(v_2))$.

Example 9. The minimum of X_1 and X_2 in Example 4, $X_{12} = \min(X_2, X_1)$ is a continuous Z-number $Z_{12} = \min(Z_1, Z_2)$. The procedures of computation of the minimum of Z_1 and Z_2, $Z_{12} = \min(Z_1, Z_2) = (A_{12}, B_{12})$, are given below.

Step 1: Compute the minimum of fuzzy numbers A_1 and A_2, $A_{12} = \min(A_1, A_2)$.

$$A_{12} = \min(A_1, A_2) = A_1 = (1, 2, 3).$$

Step 2: Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2.

$$pdf_1 = \mathcal{T}(1, 2, 3, 1 - v_1), pdf_2 = \mathcal{T}(7, 8, 9, 1 - v_2).$$

Step 3: Compute the minimum $pdf_{12} = \min(pdf_1, pdf_2)$ of pdfs.

$$pdf_{12}(z) = pdf_1(z) + pdf_2(z) - F_1(z)pdf_2(z) - F_2(z)pdf_1(z).$$

When $1 \le z \le 2$,

$$pdf_{12}(z) = (z - 1)(2v_1 - 1) + (1 - v_1).$$

When $2 \le z \le 3$,

$$pdf_{12}(z) = (3 - z)(2v_1 - 1) + (1 - v_1).$$

Otherwise, $pdf_{12}(z) = 0$.

Step 4: Compute the based value of B_{12}, v_{12}.

$$v_{12} = \frac{1}{3} + \frac{v_1}{3}.$$

Step 5: Compute the based value of B_{12}, v_{12}.

$$B_{12} = B_1 = (0.7, 0.8, 0.9).$$

The minimum of Z-numbers Z_1 and Z_2 is $Z_{12} = \min(Z_1, Z_2) = Z_1$.

The minimum of Z_1 and Z_2, $Z_{12} = \min(Z_1, Z_2)$, calculated by traditional method is $Z_{12} = (A_{12}, B_{12}) \approx ((1, 2, 3), (0.55, 0.65, 0.75))$. It can be seen that the minimum value of the two Z-numbers calculated by traditional method is not any of them, which is counterintuitive. Our method not only yields more reasonable results, but is also computationally simpler.

5.6. Maximum of triangular Z-numbers

There are two triangular Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ describing imperfect information about the values of variables X_1 and X_2. The maximum of X_1 and X_2, $X_{12} = \max(X_1, X_2)$, is a continuous Z-numbers $Z_{12} = \max(Z_1, Z_2)$. The procedures of computation of $Z_{12} = (A_{12}, B_{12})$ are listed below.

Step 1: Compute the maximum of fuzzy numbers A_1 and A_2, $A_{12} = \max(A_1, A_2)$, based on Definition 17.
Step 2: Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2 by using Eq. (36).
Step 3: Compute the maximum $pdf_{12} = \max(pdf_1, pdf_2)$ of pdfs with Definition 7.
Step 4: Compute the based value of B_{12}, v_{12}, by using the definition of Z-numbers. The maximum of two extended triangular distributions is not an extended triangular distribution. However, we can find an extended triangular distribution that can obtain the same base value as the actual maximum pdfs.
Step 5: Compute the membership function of B_{12}, $\mu_{B_{12}}(v_{12}) = \max(\mu_{B_1}(v_1) \wedge \mu_{B_2}(v_2))$.

Example 10. The maximum of X_1 and X_2 in Example 4, $X_{12} = \max(X_2, X_1)$ is a continuous Z-number $Z_{12} = \max(Z_1, Z_2)$. The procedures of computation of the maximum of Z_1 and Z_2, $Z_{12} = \max(Z_1, Z_2) = (A_{12}, B_{12})$, are given below.

Step 1: Compute the maximum of fuzzy numbers A_1 and A_2, $A_{12} = \max(A_1, A_2)$.

$$A_{12} = \max(A_1, A_2) = A_2 = (7, 8, 9).$$

Step 2: Compute the hidden pdfs pdf_1 and pdf_2 of Z_1 and Z_2.

$$pdf_1 = \mathcal{T}(1, 2, 3, 1 - v_1), pdf_2 = \mathcal{T}(7, 8, 9, 1 - v_2).$$

Step 3: Compute the maximum $pdf_{12} = \max(pdf_1, pdf_2)$ of pdfs.

$$pdf_{12}(z) = F_1(z)pdf_2(z) + F_2(z)pdf_1(z).$$

When $7 \le z \le 8$,

$$pdf_{12}(z) = (z - 7)(2v_2 - 1) + (1 - v_2).$$

When $8 \le z \le 9$,

$$pdf_{12}(z) = (9 - z)(2v_2 - 1) + (1 - v_2).$$

Otherwise, $pdf_{12}(z) = 0$.
Step 4: Compute the based value of B_{12}, v_{12}.

$$v_{12} = \frac{1}{3} + \frac{v_2}{3}.$$

Step 5: Compute the based value of B_{12}, v_{12}.

$$B_{12} = B_2 = (0.4, 0.5, 0.6).$$

The maximum of Z-numbers Z_1 and Z_2 is $Z_{12} = \max(Z_1, Z_2) = Z_2$.

The maximum of Z_1 and Z_2, $Z_{12} = \max(Z_1, Z_2)$, calculated by traditional method is $Z_{12} = (A_{12}, B_{12}) \approx ((7, 8, 9), (0.55, 0.65, 0.75))$. It can be seen that the maximum of the two Z-numbers calculated by traditional method is not any of them, which is counterintuitive. Our method not only yields more reasonable results, but is also computationally simpler.

5.7. Negation of a triangular Z-number

There are a triangular Z-number $Z = (A, B)$. The negation of Z, $\overline{Z} = (\overline{A}, \overline{B})$, is a continuous Z-numbers. The procedures of computation of $\overline{Z}$ are listed as follows.

Step 1: Compute the complement of fuzzy number A, $\overline{A}$, based on the Definition 11.
Step 2: Compute the hidden pdf pdf of Z by using Eq. (36).
Step 3: Compute the negation of pdf with Eq. (35).
Step 4: Compute the base value of $\overline{B}$, $\overline{v}$ by using the definition of Z-numbers.
Step 5: Compute the membership function of $\overline{B}$, $\mu_{\overline{B}}(\overline{v}) = \mu_B(v)$.

Example 11. The procedures of computation of the negation of Z_1 described in Example 4, $\overline{Z_1} = (\overline{A_1}, \overline{B_1})$, are given below.

Step 1: Compute the complement of fuzzy number A, $\overline{A}$, based on the Definition 11.

$$\mu_{\overline{A_1}}(x) = \begin{cases} 2 - x, & 1 \le x \le 2, \\ x - 2, & 2 \le x \le 3. \end{cases}$$

Step 2: Compute the hidden pdf pdf of Z by using Eq. (36).

$$pdf_1 = \mathcal{T}(1, 2, 3, 1 - v_1).$$

Step 3: Compute the negation of pdf with Eq. (35).

$$\overline{pdf_1}(x) = \begin{cases} v_1 - (2v_1 - 1)(x - 1), & 1 \le x \le 2, \\ v_1 + (2v_1 - 1)(x - 3), & 2 \le x \le 3. \end{cases}$$

Step 4: Compute the base value of $\overline{B}$, $\overline{v}$ by using the definition of Z-numbers.

$$\begin{aligned} \overline{v_1} &= \int_1^3 pdf_1(x)\mu_{\overline{A_1}}(x)dx \\ &= \int_1^2 (2 - x)(v_1 - (2v_1 - 1)(x - 1))dx + \int_2^3 (x - 2)(v_1 + (2v_1 - 1)(x - 3)) \\ &= \frac{1}{3} + \frac{v_1}{3}. \end{aligned}$$

Step 5: Compute the membership function of $\overline{B}$, $\mu_{\overline{B}}(\overline{v}) = \mu_B(v)$.

$$\overline{B_1} = B_1 = (0.7, 0.8, 0.9).$$

The negation of a Z-number $Z = (A, B)$ is $\overline{Z} = (\overline{A}, B)$.

The traditional method does not provide any calculation for the negation of Z-numbers.

6. Conclusions

The concept of Z-number is introduced as a more adequate formal construct for describing real-world information. The continuous Z-number is the most common information in the real world. However, the complexity and inaccuracy of continuous Z-number calculations limit its application. Thus, simplifying the calculation of continuous Z-numbers is significant. We analyzed in detail two reasons for the complexity of continuous Z-number calculations. The first one is that existing arithmetic always assumes that the hidden pdf is normal pdf. Its complex expressions make the process of finding the expression of the hidden pdf extremely difficult. The second one is the inconsistency between the meaning and definition of the Z-number. After that, we designed an extended triangular distribution for the triangular Z-number as its hidden pdf and expand the value domain of probability measure to unify the meaning and definition of Z-number. Finally, we implemented the basic operations of Z-numbers based on the extended triangular distribution, including addition, subtraction, multiplication, division, minimum and maximum over two triangular Z-numbers and the negation of a Z-number. Compared to the traditional method, our method had lower computational complexity, higher precision and lower uncertainty in the results. The corresponding numerical examples were provided to show the validity and advantages of the suggested approach.

It should be noted that the variable range used in the proposed method is $[a, b]$, rather than the entire real number line.

In the future, we will explore other important algebraic operations of continuous Z-numbers, such as ranking and distance, and their practical applications.

CRediT authorship contribution statement

Y. Li: Conceptualization, Methodology, Formal analysis, Software, Validation, Writing – original draft, Investigation, Data curation, Visualization, Revision.

E. Herrera-Viedma: Writing – review & editing, Supervision, Funding acquisition.

Ignacio Javier Pérez: Writing – review & editing, Visualization, Supervision.

W. Xing: Writing – review & editing, Visualization, Supervision.

J.A. Morente-Molinera: Writing – review & editing, Supervision, Funding acquisition, Revision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] R. Aliev, A. Alizadeh, O. Huseynov, An introduction to the arithmetic of Z-numbers by using horizontal membership functions, Proc. Comput. Sci. 120 (2017) 349–356.
[2] R. Aliev, W. Pedrycz, O. Huseynov, Hukuhara difference of Z-numbers, Inf. Sci. 466 (2018) 13–24.
[3] R.A. Aliev, A.V. Alizadeh, O.H. Huseynov, The arithmetic of discrete Z-numbers, Inf. Sci. 290 (2015) 134–155.
[4] R.A. Aliev, O.H. Huseynov, L.M. Zeinalova, The arithmetic of continuous Z-numbers, Inf. Sci. 373 (2016) 441–460.
[5] R.A. Aliev, W. Pedrycz, B. Guirimov, O.H. Huseynov, Clustering method for production of Z-number based if-then rules, Inf. Sci. 520 (2020) 155–176.
[6] R.A. Aliev, W. Pedrycz, O.H. Huseynov, Functions defined on a set of Z-numbers, Inf. Sci. 423 (2018) 353–375.
[7] R.A. Aliev, W. Pedrycz, O.H. Huseynov, Hukuhara difference of Z-numbers, Inf. Sci. 466 (2018) 13–24.
[8] A.S.A. Bakar, A. Gegov, Multi-layer decision methodology for ranking Z-numbers, Int. J. Comput. Intell. Syst. 8 (2015) 395–406.
[9] R. Banerjee, S.K. Pal, J.K. Pal, A decade of the Z-numbers, IEEE Trans. Fuzzy Syst. 30 (2021) 2800–2812.
[10] R. Cheng, J. Zhang, B. Kang, Ranking of Z-numbers based on the developed golden rule representative value, IEEE Trans. Fuzzy Syst. (2022), https://doi.org/10.1109/TFUZZ.2022.3170208.
[11] R. Chutia, Ranking of Z-numbers based on value and ambiguity at levels of decision making, Int. J. Intell. Syst. 36 (2021) 313–331.
[12] S. Das, A. Garg, S.K. Pal, J. Maiti, A weighted similarity measure between Z-numbers and bow-tie quantification, IEEE Trans. Fuzzy Syst. 28 (2019) 2131–2142.
[13] Y. Deng, Information volume of mass function, Int. J. Comput. Commun. Control 15 (2020) 3983, https://doi.org/10.15837/ijccc.2020.6.3983.
[14] Y. Deng, Uncertainty measure in evidence theory, Sci. China Inf. Sci. 63 (2020) 210201.
[15] Y. Deng, Random permutation set, Int. J. Comput. Commun. Control 17 (2022) 4542, https://doi.org/10.15837/ijccc.2022.1.4542.
[16] C.Y. Duan, H.C. Liu, L.J. Zhang, H. Shi, An extended alternative queuing method with linguistic Z-numbers and its application for green supplier selection and order allocation, Int. J. Fuzzy Syst. 21 (2019) 2510–2523.
[17] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 2, John Wiley & Sons, 2008.
[18] L.A. Gardashova, Application of operational approaches to solving decision making problem using Z-numbers, Appl. Math. 2014 (2014).
[19] M. Hanss, Applied Fuzzy Arithmetic, Springer, 2005.
[20] J. Huang, D.H. Xu, H.C. Liu, M.S. Song, A new model for failure mode and effect analysis integrating linguistic Z-numbers and projection method, IEEE Trans. Fuzzy Syst. 29 (2019) 530–538.
[21] Q. Jia, J. Hu, Q. He, W. Zhang, E. Safwat, A multicriteria group decision-making method based on AIVIFSs, Z-numbers, and trapezium clouds, Inf. Sci. 566 (2021) 38–56.
[22] W. Jiang, Y. Cao, X. Deng, A novel Z-network model based on Bayesian network and Z-number, IEEE Trans. Fuzzy Syst. 28 (2020) 1585–1599.
[23] W. Jiang, C. Xie, B. Wei, Y. Tang, Failure mode and effects analysis based on Z-numbers, Intell. Autom. Soft Comput. (2017) 1–8.
[24] B. Kang, Y. Deng, K. Hewage, R. Sadiq, A method of measuring uncertainty for Z-number, IEEE Trans. Fuzzy Syst. 27 (2018) 731–738.
[25] A. Kaufmann, M.M. Gupta, Introduction to Fuzzy Arithmetic: Theory and Applications, VanNostrand Reinhold, New York, 1991.
[26] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Possibility Theory versus Probab. Theory, vol. 32, 1996, pp. 207–208.
[27] R.A. Krohling, A.G. Pacheco, G.A. dos Santos, TODIM and TOPSIS with Z-numbers, in: Frontiers of Information Technology & Electronic Engineering, vol. 20, 2019, pp. 283–291.
[28] Y. Li, F.J. Cabrerizo, E. Herrera-Viedma, J.A. Morente-Molinera, A modified uncertainty measure of Z-numbers, Int. J. Comput. Commun. Control 17 (2022).
[29] Q. Liu, H. Cui, Y. Tian, B. Kang, On the negation of discrete Z-numbers, Inf. Sci. 537 (2020) 18–29.
[30] M.A. Onari, S. Yousefi, M.J. Rezaee, Risk assessment in discrete production processes considering uncertainty and reliability: Z-number multi-stage fuzzy cognitive map with fuzzy learning algorithm, Artif. Intell. Rev. (2020) 1–35.
[31] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (1982) 341–356.
[32] H.G. Peng, J.Q. Wang, A multicriteria group decision-making method based on the normal cloud model with Zadeh's Z-numbers, IEEE Trans. Fuzzy Syst. 26 (2018) 3246–3260.
[33] H.G. Peng, X.K. Wang, J.Q. Wang, New MULTIMOORA and pairwise evaluation-based MCDM methods for hotel selection based on the projection measure of Z-numbers, Int. J. Fuzzy Syst. 24 (2022) 371–390.

[34] H.G. Peng, X.K. Wang, T.L. Wang, J.Q. Wang, Multi-criteria game model based on the pairwise comparisons of strategies with Z-numbers, Appl. Soft Comput. 74 (2019) 451–465.

[35] D. Qiao, K.W. Shen, J.Q. Wang, T.L. Wang, Multi-criteria PROMETHEE method based on possibility degree with Z-numbers under uncertain linguistic environment, J. Ambient Intell. Humaniz. Comput. 11 (2020) 2187–2201.

[36] D. Qiao, X.K. Wang, J.Q. Wang, K. Chen, Cross entropy for discrete Z-numbers and its application in multi-criteria decision-making, Int. J. Fuzzy Syst. 21 (2019) 1786–1800.

[37] M.D. Springer, The algebra of random variables, Technical Report, 1979.

[38] Y. Tian, L. Liu, X. Mi, B. Kang, ZSLF: a new soft likelihood function based on Z-numbers and its application in expert decision system, IEEE Trans. Fuzzy Syst. 29 (2021) 2283–2295.

[39] Y. Tian, X. Mi, H. Cui, P. Zhang, B. Kang, Using Z-number to measure the reliability of new information fusion method and its application in pattern recognition, Appl. Soft Comput. 111 (2021) 107658.

[40] T. Wang, H. Li, X. Zhou, D. Liu, B. Huang, Three-way decision based on third-generation prospect theory with Z-numbers, Inf. Sci. 569 (2021) 13–38.

[41] Q. Wu, Y. Deng, N. Xiong, Exponential negation of a probability distribution, Soft Comput. 26 (2022) 2147–2156.

[42] R.R. Yager, Element selection from a fuzzy subset using the fuzzy integral, IEEE Trans. Syst. Man Cybern. 23 (1993) 467–477.

[43] R.R. Yager, On the maximum entropy negation of a probability distribution, IEEE Trans. Fuzzy Syst. 23 (2014) 1899–1902.

[44] L.A. Zadeh, A note on Z-numbers, Inf. Sci. 181 (2011) 2923–2932.

[45] R. Zhu, Q. Liu, C. Huang, B. Kang, Z-ACM: an approximate calculation method of Z-numbers for large data sets based on kernel density estimation and its application in decision-making, Inf. Sci. 610 (2022) 440–471.