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OVERDETERMINED ELLIPTIC PROBLEMS: BIFURCATION OF SOLUTIONS AND MODICA TYPE ESTIMATES

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Resumen

Esta tesis se ocupa del estudio de problemas elípticos semilineales sobredeterminados de la forma:

$$\begin{cases} \Delta u + f(u) = 0 & \text{en } \Omega, \\ u > 0 & \text{en } \Omega, \\ u = 0 & \text{en } \partial\Omega, \\ \partial_\nu u = c & \text{en } \partial\Omega. \end{cases} \tag{1}$$

Aquí Ω es un dominio regular en $\mathbb{R}^d$, f es una función Lipschitz, c es una constante y $\partial_\nu u$ es la derivada de u en la dirección del vector unitario normal exterior ν en el borde $\partial\Omega$. Estos problemas se denominan "sobredeterminados" debido a las dos condiciones de contorno. Por lo tanto no esperamos, en general, la existencia de soluciones para cada dominio Ω, y aquí la tarea es comprender para cuales dominios Ω podemos tener una solución para este tipo de problema.

El objetivo de esta tesis es construir algunas nuevas soluciones mediante bifurcación local, véase [57, 72, 73], así como establecer un tipo de estimación de tipo Modica, véase [74].

Motivación

Los problemas elípticos sobredeterminados tienen aplicaciones en varios problemas matemáticos y físicos, como la geometría espectral y la hidrodinámica. Comentemos brevemente algunas de las consecuencias del estudio de problemas elípticos sobredeterminados aplicados a problemas físicos:

(1) cuando un fluido viscoso incompresible se mueve en líneas de corriente rectas y paralelas a través de una tubería de una sección transversal dada, la tensión tangencial por unidad de área en la pared de la tubería es la misma en todos los puntos si y solo si la sección transversal es circular;

(2) cuando un líquido sube en un tubo capilar recto, el líquido subirá a la misma altura en la pared del tubo si y sólo si el tubo tiene una sección circular;

(3) una distribución de carga de equilibrio es constante si y solo si hay un solo conductor, que es circular;

(4) cuando ponemos un tubo cilíndrico sólido en un recipiente grande lleno de líquido, la superficie de contacto entre el líquido y la pared del cilindro subirá a la misma altura si y sólo si el tubo cilíndrico sólido tiene una sección circular;

(5) la demostración de la existencia de una solución no trivial a una ecuación elíptica semilineal con condiciones de contorno sobredeterminadas en el plano permite la construcción de una solución débil de las ecuaciones de Euler estacionarias.

Nos referimos a los trabajos [9, 80, 82] para más detalles y otras aplicaciones.

El estudio de los problemas sobredeterminados se remonta al resultado clásico de Serrin [76], donde el autor demostró que si Ω está acotado, f es C^1 y el problema (1) admite una solución, entonces Ω debe ser una bola. El método utilizado por Serrin se conoce universalmente como *método de los planos móviles*, introducido en 1956 por Alexandrov en [3] para demostrar que las únicas hipersuperficies compactas, conexas y embebidas en $\mathbb{R}^d$ con curvatura media constante son las esferas, y también funciona cuando f es solo lipschitziana [66]. Esta demostración mostró una analogía entre los problemas elípticos sobredeterminados y las superficies de curvatura media constante. Desde ese momento, el método de los planos móviles se convirtió en una herramienta muy importante en Análisis para obtener resultados de simetría para soluciones de ecuaciones elípticas semilineales.

Cuando el dominio Ω es ilimitado, Berestycki, Caffarelli y Nirenberg [10] propusieron la siguiente conjetura:

Conjetura BCN. Supongamos que Ω es un dominio suave tal que $\mathbb{R}^d \backslash \Omega$ es conexo. Entonces, la existencia de una solución acotada al problema (1) para alguna función Lipschitz f implica que Ω es una bola, un semiespacio, un cilindro generalizado $B^k \times \mathbb{R}^{d-k}$ (B^k es una bola en $\mathbb{R}^k$), o el complemento de uno de ellos.

La conjetura BCN ha motivado varios trabajos interesantes que daban una respuesta afirmativa para algunas clases de problemas elípticos sobredeterminados. Describamos ahora brevemente algunos de estos resultados. En [37] los autores obtienen una respuesta afirmativa bajo la hipótesis de que Ω es un epigrafo de $\mathbb{R}^2$ o $\mathbb{R}^3$ y la función u satisface algunas hipótesis naturales. En [70] se prueba la conjetura BCN en el plano para algunas clases de no linealidades f. Además, el trabajo [68], prueba la conjetura BCN en dimensión 2 si $\partial\Omega$ es conexo y no acotado.

Sin embargo, resulta que la conjetura es falsa y esto se ha probado con un contraejemplo por Sicbaldi [77]. Se demostró que existen dominios periódicos de revolución tales que el problema (1) admite una solución positiva para $f(u) = \lambda u$ con $\lambda > 0$. Además, estos dominios bifurcan desde el cilindro recto $B \times \mathbb{R} (B \subset \mathbb{R}^{d-1}$ es una bola). Después de este primer resultado, se han dado diferentes construcciones en la literatura, véase por ejemplo [54]. Otro tipo de contraejemplo a la conjetura, de particular interés, ha sido encontrado en [69]. En ese trabajo se muestra que (1) admite una solución para algunos dominios exteriores no radiales (es decir, el complemento de una región compacta en $\mathbb{R}^d$ que no es una bola cerrada), para una función adecuada $f(u)$. En dimensión 2, esto representa la primera construcción de un contraejemplo a la conjetura BCN, que entonces resulta ser falsa en cualquier dimensión.

En esta tesis abordaremos las siguientes cuestiones:

Problemas elípticos sobredeterminados en dominios de tipo onduloide con no linealidades generales

Como mencionamos antes, Sicbaldi en [77] encontró una perturbación periódica del cilindro recto $B^{d-1} \times \mathbb{R}$ que admite una solución periódica al problema (1) con $f(u) = \lambda u, \lambda > 0$. Más precisamente, tales dominios, como se muestra en [75], pertenecen a una familia 1-paramétrica $\{\Omega_s\}_{s \in (-\epsilon, \epsilon)}$ y están dados por

$$\Omega_s = \left\{ (x,t) \in \mathbb{R}^d \times \mathbb{R} \, : \, |x| < 1 + s \cos\left(\frac{2\pi}{T_s} t\right) + O(s^2) \right\}$$

donde ϵ es una constante pequeña, $T_s = T_0 + O(s)$ y T_0 depende solo de la dimensión d. Este resultado refuerza la analogía entre dominios que permiten una solución de (1) y superficies de curvatura media constante, ya que el borde del dominio Ω_s se puede poner en correspondencia con el onduloide (o superficie de Delaunay).

En [32] se demuestra el mismo tipo de resultado en el caso $f \equiv 1$.

Nuestro objetivo principal con respecto al problema (1) es realizar tal construcción bajo supuestos minimales sobre la no linealidad $f(u)$. Es claro que una hipótesis obligatoria es la existencia de una solución del problema de Dirichlet en la bola unitaria B en $\mathbb{R}^d$. Por razones técnicas, necesitamos que la derivada normal en el borde sea distinta de cero, lo cual es una situación típica en problemas elípticos semilineales sobredeterminados. Por lo tanto, asumimos la siguiente hipótesis:

Hipótesis 1: Existe una solución positiva $\phi_1 \in C^{2,\alpha}(\overline{B})$ del problema

$$\begin{cases} \Delta \phi_1 + f(\phi_1) = 0 & \text{en } B, \\ \phi_1 = 0 & \text{en } \partial B, \end{cases} \tag{2}$$

con $\partial_\nu \phi_1(x) \neq 0$ para $x \in \partial B$, donde ν es el vector normal exterior en ∂B.

Observemos que, gracias a [41], cualquier solución ϕ_1 de (2) debe ser una función radialmente simétrica. Por razones técnicas, debemos asumir también que el operador lineal asociado al problema (1) en ϕ_1 no es degenerado. Esta es una suposición bastante natural si se pretende utilizar un argumento de perturbación. Concretamente, nuestra segunda suposición es:

Hipótesis 2: Definimos el operador linealizado $L_D : C^{2,\alpha}_{0,r}(B) \to C^{0,\alpha}_r(B)$ por

$$L_D(\phi) = \Delta \phi + f'(\phi_1)\phi \,,$$

donde $C^{2,\alpha}_{0,r}(B)$ y $C^{0,\alpha}_r(B)$ denotan los espacios de funciones radiales en $C^{2,\alpha}_0(B)$ y $C^{0,\alpha}(B)$ respectivamente. Suponemos que el operador linealizado L_D no es degenerado; en otras palabras, si $L_D(\phi) \equiv 0$ entonces $\phi \equiv 0$.

Ahora estamos en posición de enunciar nuestro primer resultado de la siguiente manera:

Teorema 1. Si $d \geq 1$, $f : [0, +\infty) \to \mathbb{R}$ es $C^{1,\alpha}$ y se cumplen las Hipótesis 1 y 2, entonces existe un número positivo T_* y una curva continua

$$\begin{array}{ccc} (-\epsilon, \epsilon) & \to & C^{2,\alpha}(\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \\ s & \mapsto & (v_s, T_s) \end{array}$$

para ϵ suficientemente pequeño, con $v_s = 0$ si y sólo si $s = 0$. Además $T_0 = T_*$ y el problema sobredeterminado (1) tiene solución en el dominio

$$\Omega_s = \left\{ (x,t) \in \mathbb{R}^d \times \mathbb{R} \, : \, |x| < 1 + v_s \left(\frac{t}{T_s} \right) \right\}.$$

La solución $u = u_s$ del problema (1) es T_s-periódica en la variable t y por lo tanto acotada. Además,

$$\int_0^1 v_s(t)\, dt = 0.$$

Señalemos que las Hipótesis 1 y 2 se cumplen por ejemplo en los siguientes casos (entre muchos otros):

(1) Si $f(0) > 0$ y $f'(u) < \lambda_1$ para cualquier $u \in (0, +\infty)$, donde λ_1 es el primer valor propio del Laplaciano de Dirichlet en la bola unitaria de $\mathbb{R}^d$. En este caso se puede encontrar una solución positiva (por ejemplo, extendiendo $f(u) = f(0)$ si $u < 0$ y minimizando el funcional de Euler-Lagrange correspondiente) y el operador L_D tiene solo valores propios positivos.

(2) Si $f(u) = u^p - u$, $1 < p < \frac{d+2}{d-2}$ si $d > 2$, $p > 1$ si $d = 2$. En este caso es bien conocida la existencia de una solución, y se trata de una solución de paso de montaña. Como consecuencia, L_D tiene un valor propio negativo. Por el análisis de [52], todos los demás valores propios son estrictamente positivos.

(3) Si $f(u) = \lambda e^u$ y $\lambda \in (0, \lambda^*)$ para algún $\lambda^* > 0$ que recibe el nombre de valor extremal. En este caso, ϕ_1 es la llamada solución minimal y L_D solo tiene valores propios positivos (ver, por ejemplo, [26]).

En particular, (1) se cumple cuando $f \equiv 1$, y así recuperamos el resultado en [32]. Por otro lado, cuando $f(u) = \lambda\, u$ para algún $\lambda > 0$, la Hipótesis 1 implica que λ es el primer valor propio del Laplaciano de Dirichlet en la bola unitaria de $\mathbb{R}^d$, pero entonces la Hipótesis 2 claramente no se cumple. Por lo tanto, nuestro teorema es complementario a los resultados en [75, 77].

El teorema 1 es un resultado de bifurcación en el espíritu de [77], ver también [32, 75]. En resumen, se puede reformular la existencia de soluciones para

(1) como los ceros de un operador no lineal de Dirichlet a Neumann, y se usa el Teorema de Crandall-Rabinowitz para concluir la bifurcación local.

Dominios excepcionales en dimensiones superiores

El segundo problema que estudiamos en esta tesis tiene que ver con la existencia de subdominios $\Omega \subset \mathbb{R}^d \times \mathbb{R}$ donde el problema de contorno sobredeterminado

$$\begin{cases} \Delta u = 0 & \text{en} \quad \Omega, \\ u = 1 & \text{en} \quad \partial\Omega, \\ \lim_{|z|\to\infty} u(z,t) = 0 & \text{uniformemente en } t \in \mathbb{R}, \\ \dfrac{\partial u}{\partial \nu} = c & \text{en} \quad \partial\Omega, \end{cases} \tag{3}$$

admite solución. Aquí, hemos elegido las coordenadas $(z,t) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Los dominios Ω que queremos construir son subdominios *excepcionales* no triviales del espacio euclideo $\mathbb{R}^d$, $d \geq 4$.

En la literatura clásica, un dominio suave Ω del espacio euclideo $\mathbb{R}^d$ se dice que es un dominio *excepcional* si existe una función armónica positiva con dato de borde de Dirichlet igual a cero y dato de borde de Neumann constante distinto de cero. El problema de encontrar dominios excepcionales se remonta al trabajo pionero de Hauswirth, Hélein y Pacard en [46], donde el dominio excepcional no trivial

$$\Omega_0 := \left\{ (x,y) \in \mathbb{R}^2 : |y| < \frac{\pi}{2} + \cosh(x) \right\}$$

fue descubierto en el plano. Notamos que una familia de dominios planos excepcionales infinitamente conectados ya fue descubierta en dinámica de fluidos en [6, 19].

Hasta la fecha, la estructura del conjunto de dominios excepcionales en dimensiones $d \geq 3$ sigue siendo en gran parte desconocida. Con respecto a la existencia de dominios excepcionales *no triviales* en dimensiones superiores, solo conocemos los trabajos recientes [34, 54].

El propósito principal aquí es la construcción de subdominios $\Omega \subset \mathbb{R}^{d-1} \times \mathbb{R}$, $d \geq 4$ de tal manera que el problema sobredeterminado (3) tiene solución. Los dominios bajo consideración son complementos de cilindros perturbados de la forma

$$\Omega_{T,\varphi} := \left\{ (z,t) \in \mathbb{R}^{d-1} \times \mathbb{R} \ : \ |z| > 1 + \varphi\left(\frac{2\pi}{T} t\right) \right\} \subset \mathbb{R}^d, \tag{4}$$

donde $T > 0$, $\varphi : \mathbb{R} \to (0,\infty)$ es una función 2π-periódica de clase $C^{2,\alpha}$, $\alpha \in (0.1)$. El caso $\varphi \equiv 0$ en (4) corresponde al complemento del exterior del cilindro recto $\mathbb{B}_1 \times \mathbb{R}$ y la función $u_1(z) = |z|^{3-d}$ resuelve (3) con $c = d - 3$.

Nuestro resultado principal se puede expresar como sigue:

Teorema 2. Sea $d \geq 4$. Entonces existe un número $T_* > \dfrac{2\pi}{\sqrt{d-2}}$ y una curva suave

$$(-\varepsilon, \varepsilon) \to (0, +\infty) \times C^{2,\alpha}(\mathbb{R}), \qquad s \mapsto (T_s, v_s)$$

con $T_0 = T_*$, $v_0 \equiv 0$ y

$$\int_{-\pi}^{\pi} v_s(t) \cos(t)\, dt = 0,$$

tal que para todo $s \in (-\varepsilon, \varepsilon)$, siendo $\varphi_s(t) = s\cos(t) + sv_s$, existe una única función $u_s \in C^{2,\alpha}(\overline{\Omega_{T_s,\varphi_s}})$ que satisface:

$$\begin{cases} \Delta u_s = 0 & \text{en } \Omega_{T_s,\varphi_s}, \\ u_s = 1 & \text{en } \partial\Omega_{T_s,\varphi_s}, \\ \lim_{|z|\to\infty} u_s(z,t) = 0 & \text{en } \partial\Omega_{T_s,\varphi_s}, \\ \frac{\partial u_s}{\partial \nu} = d-3 & \text{en } \partial\Omega_{T_s,\varphi_s}. \end{cases} \tag{5}$$

Además, u_s es radial en z, así como T_s-periódica y par en t para cada $s \in (-\varepsilon, \varepsilon)$.

Señalamos que, para cada $s \in (-\varepsilon, \varepsilon)$, el dominio Ω_{T_s,φ_s} en el Teorema 2 es excepcional ya que $\tilde{u}_s = 1 - u_s$ es solución de (3) en Ω_{T_s,φ_s}. De hecho, esta función es positiva en Ω_{T_s,φ_s} ya que la función armónica u_s no puede alcanzar un máximo en Ω_{T_s,φ_s} a menos que sea constante, lo cual está excluido por las condiciones de contorno en (5). Se sigue que $0 < u_s < 1$ y por lo tanto $0 < \tilde{u}_s < 1$ en Ω_{T_s,φ_s}.

En realidad, los dominios en el Teorema 2 tienen una forma similar a los encontrados en el trabajo reciente [34] para el caso $d = 3$, pero la construcción subyacente es completamente diferente. De hecho, el enfoque en [34] se basa en propiedades específicas de una representación integral de un operador de Dirichlet a Neumann asociado al problema, que solo está disponible en el caso $d = 3$. Por otro lado, nuestro enfoque depende esencialmente del supuesto $d \geq 4$. La diferencia entre estos dos casos se refleja en la geometría de las funciones de techo asociadas que están acotadas por $d \geq 4$ y tienen un crecimiento logarítmico en la distancia desde el eje del cilindro en el caso $d = 3$. Claramente, estas diferencias están relacionadas con la diferente naturaleza de la solución fundamental de $-\Delta$ en las dimensiones $d = 2$ y $d \geq 3$.

La idea principal de la demostración es de nuevo aplicar el teorema de la bifurcación local de Crandall-Rabinowitz. Para ello es necesario calcular el operador linealizado construido en nuestro trabajo y analizar sus propiedades espectrales.

Problemas elípticos sobredeterminados en dominios contraíbles no triviales de la esfera

El tercer problema que estudiamos es mostrar la existencia de soluciones a problemas elípticos sobredeterminados (1) planteados en variedades riemannianas completas $(\mathcal{M}, g)$ en lugar de la configuración euclídea $(\mathbb{R}^d, g_{eucl})$. En este marco, necesitamos reemplazar en (1) el Laplaciano clásico por el operador de Laplace-Beltrami Δ_g asociado a la métrica g de la variedad $\mathcal{M}$:

$$\begin{cases} \Delta_g u + f(u) = 0 & \text{en } \Omega, \\ u > 0 & \text{en } \Omega, \\ u = 0 & \text{en } \partial\Omega, \\ \partial_\nu u = c & \text{en } \partial\Omega, \end{cases} \tag{6}$$

donde Ω es un dominio de $\mathcal{M}$, y ν es el vector normal unitario exterior sobre $\partial\Omega$ con respecto a g. Aquí estamos interesados en el caso de una esfera con la métrica usual.

Un resultado análogo al de Serrin fue dado por Kumaresan y Prajapat [51], donde se prueba que si Ω está contenido en un hemisferio y (6) admite una solución, entonces el dominio es un disco geodésico. También aquí la prueba se basa en el método de los planos móviles.

Otros dominios naturales de $\mathbb{S}^d$ donde (6) tiene soluciones son los entornos simétricos de cualquier ecuador. Tales anillos simétricos no son contraíbles y su existencia proviene de la geometría de $\mathbb{S}^d$ de la misma manera que existen en un cilindro o en un toro. Además, se ha demostrado la existencia de soluciones de (6) en dominios dados por perturbaciones de un entorn de un ecuador en $\mathbb{S}^d$, véase [33], de la misma manera que se ha hecho para el mismo tipo de dominios en cilindros o en toros [77].

Teniendo en cuenta estos hechos, surge naturalmente la siguiente pregunta: ¿es cierto que si $\Omega \subset \mathbb{S}^d$ es contraíble y (6) tiene solución, entonces Ω debe ser una bola geodésica? En [28], Espinar y Mazet dan una respuesta afirmativa $d = 2$ pero bajo algunas hipótesis sobre el término no lineal $f(u)$. La demostración de tal resultado muestra nuevamente una analogía entre problemas elípticos sobredeterminados y superficies de curvatura media constante, porque está inspirada en el Teorema de Hopf que establece que las únicas superficies de curvatura media constante de género cero inmersas en $\mathbb{R}^3$ son las esferas.

En este apartado mostramos que la respuesta a la pregunta anterior es, en general, negativa: existen dominios contraíbles $\Omega \subset \mathbb{S}^d$, diferentes a las bolas geodésicas, donde (6) admite solución para algunas no linealidades $f(u)$. Esta construcción funciona para cualquier dimensión $d \geq 2$. En vista de [51], dichos dominios no pueden estar contenidos en ningún hemisferio.

Para ser más específicos, probamos el siguiente teorema:

Teorema 3. Sea $d \in \mathbb{N}, d \geq 2$ y $1 < p < \frac{d+2}{d-2}$ ($p > 1$ si $d = 2$). Entonces existen dominios D, que son perturbaciones de una pequeña bola geodésica, tales que el problema

$$\begin{cases} -\varepsilon\Delta_g u + u - u^p = 0 & \text{en } \mathbb{S}^d \setminus D \ , \\ u > 0 & \text{en } \mathbb{S}^d \setminus D \ , \\ u = 0 & \text{en } \partial D \ , \\ \partial_\nu u = c & \text{en } \partial D \ , \end{cases}$$

admite una solución para algún $\varepsilon > 0$.

La idea principal de la demostración es la siguiente. Primero, se usa una dilatación para pasar a un problema planteado en Ω_k el complemento de una bola geodésica de radio 1 en $\mathbb{S}^d(k)$, la esfera de radio $1/k$. Cuando $k \to 0$, el problema converge (en cierto sentido) a un problema límite en $\mathbb{R}^d \setminus B(0,1)$, que se estudia en [69]. Luego usamos el teorema de bifurcación de Krasnoselskii para mostrar la existencia de una rama de soluciones no triviales para (6).

Estimaciones de tipo Modica y resultados de curvatura

El último problema que abordamos en esta tesis consiste en dar una estimación sobre el gradiente de la solución al problema (1) y proporcionar alguna información sobre la curvatura de la frontera.

Dada una F primitiva de f, definimos la función P:

$$P(x) = |\nabla u(x)|^2 + 2F(u(x)). \tag{7}$$

En [58], Modica demostró que si F es una función no positiva y $u : \mathbb{R}^d \to \mathbb{R}$ es una solución acotada C^3 de

$$\Delta u + f(u) = 0,$$

entonces $P \leq 0$. La prueba de este resultado se basa en el hecho de que P satisface $L(P) \geq 0$ para un cierto operador elíptico L, y por lo tanto la conclusión se sigue (no inmediatamente) del principio máximo. Tal resultado se ha extendido a operadores más generales en [13], donde Caffarelli, Garofalo y Segala demuestran la siguiente afirmación:

$$P(x_0) = 0 \text{ para algún } x_0 \in \mathbb{R}^d \Leftrightarrow P(x) = 0 \text{ para todo } x \in \mathbb{R}^d \Leftrightarrow u \text{ es unidimensional.}$$

Esos resultados son muy importantes en el estudio de la conjetura de De Giorgi y problemas relacionados, y tienen consecuencias importantes (por ejemplo, se ha derivado una fórmula de monotonicidad en [59]).

Las estimaciones de tipo Modica para problemas sobredeterminados han sido estudiadas en [89] bajo algunas condiciones sobre la no linealidad f y la constante c. Nuestro objetivo aquí es estudiar la validez de tales estimaciones en general.

Uno de los resultados es exactamente un resultado de rigidez para problemas elípticos sobredeterminados, que en términos generales es el siguiente: si f y c satisfacen una determinada condición, entonces la curvatura media de $\partial\Omega$ es negativa, o Ω es un semi-espacio. Concretamente, si denotamos por $H(p)$ a la curvatura media de $\partial\Omega$ en p, se tiene el siguiente resultado:

Teorema 4. Sea $u \in C^3(\Omega)$ una solución acotada del problema (1), con $f \in C^1$. Si existe una primitiva F no positiva de f tal que

$$c^2 + 2F(0) \geq 0\,, \tag{8}$$

entonces $H(p) < 0$ para cualquier $p \in \partial\Omega$, o bien Ω es un semiespacio y u tiene 1-dimensional, es decir, existe $x_0 \in \mathbb{R}^d$, una función $g : [0, +\infty) \to \mathbb{R}$ y $\vec{a} \in \mathbb{R}^d$, tal que

$$\Omega = \{x \in \mathbb{R}^d : \ \vec{a} \cdot (x - x_0) > 0\}, \quad \text{y } u(x) = g(\vec{a} \cdot (x - x_0)), \quad x \in \Omega.$$

En particular, si se satisface (8), no se puede resolver el problema (1) en una bola, ni en un cilindro, ni en la perturbación de un cilindro. Observe que estamos considerando soluciones acotadas de (1), por lo que f se define en un intervalo cerrado y siempre es posible elegir una primitiva no positiva de f, cambiando f apropiadamente fuera de la imagen de u. Un caso particular del Teorema 4 es el Teorema 2.13 en [70] para dominios periódicos dobles en el plano.

Obtendremos el Teorema 4 como corolario de resultados más generales, que representan estimaciones tipo Modica para problemas sobredeterminados.

Teorema 5. Sea $\Omega \subset \mathbb{R}^d, d \geq 1$, un dominio regular, $F \in C^2(\Omega)$ una función no positiva, $F' = f$, $u \in C^3(\Omega)$ una solución acotada del problema (1) y P dado por (7). Entonces

$$P(x) \leq \max\{0, c^2 + 2F(0)\} \ \text{ para cada } x \in \Omega.$$

Además, si existe un punto $x_0 \in \Omega$ tal que

$$P(x_0) = \max\{0, c^2 + 2F(0)\},$$

entonces P es constante, u es 1-dimensional y Ω es un semi-espacio. Es decir, existe $x_0 \in \mathbb{R}^d$, una función $g : [0, +\infty) \to \mathbb{R}$ y $\vec{a} \in \mathbb{R}^d$, tal que

$$\Omega = \{x \in \mathbb{R}^d : \ \vec{a} \cdot (x - x_0) > 0\}, \quad \text{y } u(x) = g(\vec{a} \cdot (x - x_0)), \quad x \in \Omega.$$

La demostración utiliza el principio del máximo junto con argumentos de escala y un cuidadoso paso al límite en los argumentos por contradicción.

Si P está acotado superiormente por $c^2 + 2F(0)$ podemos dar información extra sobre la curvatura media de $\partial\Omega$.

Teorema 6. Sea $\Omega \subset \mathbb{R}^d, d \geq 1$, un dominio regular que admite una solución acotada $u \in C^3(\Omega)$ al problema (1) con $c \neq 0$. Asumimos que:

$$P(x) \leq c^2 + 2F(0) \;\text{ para cada } x \in \Omega.$$

Entonces, $H(p) \leq 0$ para cualquier $p \in \partial\Omega$. Además, si existe $p \in \partial\Omega$ tal que $H(p) = 0$, entonces P es constante, u es 1-dimensional y Ω es un semi-espacio o el dominio entre dos hiperplanos paralelos.

Señalemos que el Teorema 6 no requiere que F sea no positiva. Por el contrario, requiere que la derivada normal sobre $\partial\Omega$ no sea cero. Como hemos comentado anteriormente, existen soluciones del problema (1) en bolas, cilindros u onduloides generalizados (ver por ejemplo [77]). El teorema 6 implica que en todos estos casos,

$$\sup_{x \in \Omega} P(x) > c^2 + 2F(0)\,.$$

Contents

Chapter 1

Introduction. Objectives. Results.

This thesis is concerned with the study of semilinear overdetermined elliptic problems in the form

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a regular domain in $\mathbb{R}^d$, f is a Lipschitz function, c is a constant, and $\partial_\nu u$ is the derivative of u in the direction of the outward normal unit vector ν on the boundary $\partial\Omega$. These problems are called "overdetermined" because of the two boundary conditions. Hence we do not expect, in general, existence of solutions for every domain Ω, and here the task is to understand for which domains Ω we may have a solution to this kind of problem.

Being specific, the objective in this thesis is to construct some new bifurcating solutions, see [57, 72, 73], and establish a kind of Modica type estimate, see [74].

1.1 Motivation

Overdetermined elliptic problems have applications in various mathematical and physical problems, that we briefly describe below. We refer to the works [9, 80, 82] for more details and further applications.

1.1.1 Fluid moving in a straight pipe

We start with a viscous incompressible fluid moving through a straight pipe of a given cross section. Fixing Cartesian coordinates (x, y, z) in space with the z axis directed along the pipe, the section of the pipe is therefore a domain Ω in the (x, y)-plane. The velocity of the fluid u can be seen as a function of (x, y) defined in Ω.

With the adherence condition, the function u satisfies

$$\begin{cases} \Delta u + \frac{\delta}{\eta l} = 0 & \text{in } \Omega\,, \\ u > 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.2}$$

where δ is the change of pressure between the two ends of the pipe, l is the length and η is the viscosity. It turns out that the tangential stress at a point (x, y, z) of the pipe wall is given by $\left|\eta \frac{\partial u}{\partial \nu}\right|$.

An interesting question arises: when is such tangential stress the same at all points? This leads to solutions to (1.2) that satisfy the Neumann type boundary condition

$$\frac{\partial u}{\partial \nu} = c \text{ on } \partial\Omega. \tag{1.3}$$

If Ω is a ball with radius R, the unique solution to (1.2) is given by

$$u(x, y) = \frac{\delta}{4\eta l}(R^2 - x^2 - y^2),$$

which satisfies (1.3). Actually, if Ω is not a ball, the solution to (1.2) does not satisfy (1.3). This is a consequence of the classical result by Serrin [76].

1.1.2 The torsion problem.

We now consider the torsion of a solid cylindrical bar. Suppose we have a cylinder of arbitrary cross-section Ω, made of an isotropic material, with the z-axis along the axis of the cylinder as in the previous example. Generally, its cross-section bends instead of remaining a plane after twisting. After rotating the bar, any point $P(x, y, z)$ on the bar will occupy a new point $P'(x', y', z')$. If the rotation angle is small, it is physically natural to modelize the displacements as

$$x' = x - \alpha z y, \quad y' = y + \alpha z x, z' = z + \alpha\varphi(x, y),$$

where φ is called the torsion function and α is a constant (the twist per unit length).

Using the generalized Hooke's Law, it is possible to show that φ is harmonic, see [82, Chapter 4, equations (29.1)]. Define ψ the complex harmonic conjugate of φ.

Let us now consider the "stress function" introduced by L. Prandtl as

$$u(x,y) = \psi(x,y) - \frac{1}{2}(x^2 + y^2),$$

which solves the following problem

$$\begin{cases} \Delta u + 2 = 0 & \text{in } \Omega\,, \\ u = c & \text{on } \partial\Omega\,. \end{cases}$$

It turns out that the magnitude of the tangential stress is given by $\tau(x,y) = -\frac{\partial u}{\partial \nu}\mu\alpha$, where μ is the modulus of the stiffness of the bar, see [82, Chapter 4, Sections 34, 35].

A natural question appears: for which shapes is the magnitude of the resulting stress on the lateral surface of the bar constant?

1.1.3 The interior capillarity problem

Overdetermined problem can be also considered for more general elliptic operators instead of Laplacian one. The next example involves a homogeneous and incompressible liquid rising in a straight capillary tube. Taking the same setting as in the previous examples, we define the height of liquid by $u(x,y)$ with respect to the level of Ω.

In this situation, the Euler-Laplace condition (reduced by Euler's condition and Laplace's condition) and Dupré-Young condition for hydrostatic equilibrium can be written as

$$div \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} - bu - c = 0 \ \text{ in } \Omega\,,$$

and

$$\frac{\partial u}{\partial \nu} = \cos\alpha \sqrt{1 + |\nabla u|^2} \ \text{ on } \partial\Omega\,,$$

where $b = \frac{\rho g}{\sigma}$, ρ is the density of the fluid, σ is the surface tension, g is the gravity and α is the contact angle between the liquid surface and the wall of the tube, which is constant. The natural question that arises here is to study shapes Ω such that the liquid reaches the same height on the tube wall.

Therefore, if $u = a$ on $\partial\Omega$, where a is a constant, then $-\frac{\partial u}{\partial \nu} = |\nabla u|$. Hence

$$\frac{\partial u}{\partial \nu} = \cos\alpha \sqrt{1 + \left|\frac{\partial u}{\partial \nu}\right|^2},$$

which implies that $\frac{\partial u}{\partial \nu} = \cot\alpha$. Therefore, u is the solution of the problem:

$$\begin{cases} div\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - bu - c = 0 & \text{in } \Omega\,, \\ u = a & \text{on } \partial\Omega\,, \\ \frac{\partial u}{\partial \nu} = \cot\alpha & \text{on } \partial\Omega\,. \end{cases}$$

Of course, a simple translation allows to change the Dirichlet boundary condition by $u = 0$.

1.1.4 The electrostatics problem

We consider a smooth conducting body G in $\mathbb{R}^3$, which has a charge distributed on the boundary, $\alpha \in C(\partial G)$. The potential generated by α is

$$u(x) = \int_{\partial G} \alpha(y)\gamma(|x-y|)d\sigma_y$$

in G, where the function γ is given by

$$\gamma(t) = -\frac{1}{4\pi t}.$$

We consider the case in which α is constant. We say that the charge α is at equilibrium if u is constant on the boundary. In this case, u solves the following equation

$$\begin{cases} \Delta u = 0 & \text{in } G\,,\text{ and } \Omega = \mathbb{R}^3 \backslash G\ , \\ u = 0 & \text{on } \partial G\,, \\ \frac{\partial u}{\partial \nu} = \alpha & \text{on } \partial G. \end{cases}$$

We can also consider this problem in dimension $d \neq 3$. Notice that if G is a ball, the potential u, which is rotationally invariant and harmonic, is constant on ∂G. A natural question arises: are there other domains G with potential at equilibrium? This question is answered negatively for $d = 2$ and $d = 3$ by Martensen [55] and Reichel [67] respectively.

1.1.5 The exterior capillarity problem

Here, we consider an infinite reservoir filled with a homogeneous and incompressible liquid, in which we submerge a solid right cylinder. We study the contact surface between the liquid and the cylinder's wall. This problem is dual to the interior capillarity problem and leads to the same equation, but on the outside of the cylinder.

More generally, we consider m solid cylindrical bodies of arbitrary (smooth) cross sections $G_i, i = 1, ..., m$ submerged in a reservoir, without touching each other. This causes the liquid to rise around their walls to a certain level, greater than the reference level of the liquid. The question here is: given such a set of cylinders, can we add more cylinders to create an equilibrium system, with each contact surface at a constant height?

Mathematically, as in the previous problem, assume that $G_i, i = 1, ..., m$ are mutually disjoint and bounded domains such that $\mathbb{R}^d \backslash G$ is connected, being G the union of all. We are interested in the solvability of the problem:

$$\begin{cases} div \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - bu - c = 0 & \text{in } \mathbb{R}^d \backslash G\,, \\ u \geq u_\infty & \text{in } \mathbb{R}^d \backslash G\,, \\ u \to u_\infty & \text{as } |x| \to \infty\,, \\ u = a_i > 0 & \text{on } \partial G_i, i = 1, ..., m\,, \\ \frac{\partial u}{\partial \nu} = \cot \alpha_i & \text{on } \partial G_i, i = 1, ..., m\,. \end{cases}$$

Note that the above overdetermined problem does not admit solutions, unless $m = 1$ and G is a ball, see [79].

1.1.6 The Euler equations

The incompressible Euler equations are given by

$$\begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p & \text{in } \Omega, \\ div\ v = 0 & \text{in } \Omega, \\ v \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \in \mathbb{R}^2$ is a regular domain, $v : \mathbb{R} \times \Omega \to \mathbb{R}^2$ is the velocity vector field, $p : \mathbb{R} \times \Omega \to \mathbb{R}$ is the pressure, and the variable t stands for time. The stationary Euler equations for an incompressible and inviscid flow v are:

$$\begin{cases} (v \cdot \nabla) v = -\nabla p & \text{in } \Omega, \\ div\ v = 0 & \text{in } \Omega, \\ v \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

To see the relation between the Euler equations and semilinear elliptic equation, one needs to talk about the streamfunction. Taking $\phi = v^\perp = (-v_2, v_1)$, it follows that there exists $u : \bar{\Omega} \to \mathbb{R}$ such that $\phi = \nabla u$ since

$$\frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_1}{\partial y} = 0,$$

and

$$\int_{\partial\Omega} \phi \cdot \tau = \int_{\partial\Omega} \phi^{\perp} \cdot \nu = \int_{\partial\Omega} v \cdot \nu = 0,$$

where τ is the tangent vector to $\partial\Omega$, since $div\ v = 0$ and $v \cdot \nu = 0$ on $\partial\Omega$.

If $\omega = \nabla \times v$ is the vorticity of the fluid, then $\Delta u = \omega$. Applying $\nabla\times$ to the first equation of (1.4), we get

$$\nabla\omega \cdot v = 0.$$

Also the above equation can reduce to

$$\nabla(\Delta u) \parallel \nabla u,$$

which is satisfied for any radially symmetric function u, for instance. One type of solutions to this is any solutions of a semilinear equation $\Delta u + f(u) = 0$ since

$$-\nabla(\Delta u) = f'(u)\nabla u \Rightarrow \nabla(\Delta u) \parallel \nabla u.$$

Moreover, the boundary condition $v{\cdot}\nu = 0$ is equivalent to $u = c_i$ in each connected component of the boundary $\Gamma_i = (\partial\Omega)_i, i \geq 1$.

We also consider the Bernoulli function $B = \frac{|v|^2}{2} + p$.

Claim: B is constant on Γ_i.
We now compute the derivative of B and can get

$$\nabla B = \frac{1}{2} Dv \cdot v^{\perp} + \frac{1}{2} v \cdot Dv^{\perp} + \nabla p.$$

Let us discuss two cases:
Case 1: $v \neq 0$.

$$\nabla B \cdot v = \frac{1}{2} Dv \cdot v^{\perp} \cdot v + \frac{1}{2} v \cdot Dv^{\perp} \cdot v + \nabla p \cdot v = 0.$$

Therefore, $\nabla B \perp v$. As a result, B is constant on streamlines, that is, $\frac{|v|^2}{2} + p$ is constant on Γ_i.
Case 2: $v = 0$.

$$\nabla B = \nabla p = -(v \cdot \nabla) v = 0.$$

Again, $\frac{|v|^2}{2} + p$ is constant on Γ_i.

Let us first give a definition of the weak solution v to the Euler equation in $\mathbb{R}^2$ by

$$\int_{\mathbb{R}^2} (v \otimes v) : D\psi = 0, \quad \int_{\mathbb{R}^2} v \cdot \nabla\varphi = 0$$

for any vector field $\psi \in C_0^1(\mathbb{R}^2)^2$ with $div\ \psi = 0$ and any scalar function $\varphi \in C_0^1(\mathbb{R}^2)$, see [15]. Here $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^{2} A_{ij} B_{ij}$ for $\mathbf{A} = \{A_{ij}\}$ and $\mathbf{B} = \{B_{ij}\}$.

Lemma. Let $\Omega \subset \mathbb{R}^2$ be a regular domain, p be of class C^1 and v be a C^1 solution to (1.4), then

$$\tilde{v} = \begin{cases} v, & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases}$$

is a weak solution of the Euler equation in $\mathbb{R}^2$ if and only if $|v|$ is constant on $\Gamma_i, i \geq 1$.

Proof. Taking an arbitrary vector field $\psi \in C_0^1(\mathbb{R}^2)^2$ with $div\ \psi = 0$, what we need to verify is that

$$\int_{\mathbb{R}^2} (\tilde{v} \otimes \tilde{v}) : D\psi = 0.$$

It suffices to prove

$$\begin{aligned} \int_\Omega (v \otimes v) : D\psi &= -\int_\Omega div(v \otimes v) \cdot \psi + \int_{\partial\Omega} |v|^2 \psi \cdot \nu \\ &= \int_\Omega div(p\psi) + \int_{\partial\Omega} |v|^2 \psi \cdot \nu \\ &= \int_{\partial\Omega} (p + |v|^2) \psi \cdot \nu. \end{aligned}$$

If $p + |v|^2$ is a constant on Γ_i, then $\int_{\partial\Omega} (p + |v|^2) \psi \cdot \nu = \sum_i c_i \int_{\Gamma_i} \psi \cdot \nu = 0$ since $div\ \psi = 0$. It follows that $\tilde{v}$ is a weak solution of the Euler equation in $\mathbb{R}^2$.

Now we assume that $p + |v|^2$ is not a constant on Γ_1 to get a contradiction. Let $g = p + |v|^2 \neq constant$ on Γ_1 with $\int_{\Gamma_1} (g - \bar{g}) = 0$, where $\bar{g} = \frac{1}{|\Gamma_1|} \int_{\Gamma_1} g$. Take G such that $G_\tau = g - \bar{g}$, where τ is the tangent vector. We extend G outside Γ_1 with compact support so that $supp\ G \cap \Gamma_i = \emptyset, i \geq 2$. Taking $\psi = (\nabla G)^\perp \in C^1$ then $div \psi = 0$ and ψ has compact support. It follows

$$\int_{\partial\Omega} (p + |v|^2) \psi \cdot \nu = \int_{\Gamma_1} g \cdot \psi \cdot \nu = \int_{\Gamma_1} g \cdot (\nabla G)^\perp \cdot \nu = \int_{\Gamma_1} g \cdot G_\tau = \int_{\Gamma_1} (g - \bar{g})^2 \neq 0,$$

which is a contradiction.

Then $p + |v|^2$ is a constant on Γ_i if $\tilde{v}$ is a weak solution of the Euler equation in $\mathbb{R}^2$. Together with the claim that B is a constant on Γ_i, we have therefore get that p and $|v|$ are constant on Γ_i.

Finally, for all $\varphi \in C_0^1(\mathbb{R}^2)$,

$$0 = \int_{\mathbb{R}^2} \tilde{v} \cdot \nabla\varphi = \int_\Omega v \cdot \nabla\varphi = -\int_\Omega \varphi div\ v + \int_{\partial\Omega} \varphi v \cdot \nu$$

since $div\ v = 0$ in Ω and $v \cdot \nu = 0$ on $\partial\Omega$. □

One can refer to [25, Lemmas 1.1, 2.1] for other relations between overdetermined elliptic problem and fluid equations.

From the above result, it is clear that the existence of a solution to a semilinear elliptic equation with overdetermined boundary values in dimension 2 gives rise to a weak solution of the steady Euler equations, which is 0 outside Ω.

1.2 Antecedents

The study of overdetermined boundary value problems can be traced back to Serrin's classical result [76], where the author proved that if Ω is bounded, f is C^1 and problem (1.1) admits a solution, then Ω must be a ball. The method used by Serrin is universally known as the *moving plane method*, introduced in 1956 by Alexandrov in [3] to prove that the only compact, connected, embedded hypersurfaces in $\mathbb{R}^d$ with constant mean curvature are the spheres, and also works when f is only Lipschitz continuous [66]. This proof showed an analogy between overdetermined elliptic problems and constant mean curvature surfaces. From that moment, the moving plane method has become a very important tool in Analysis to obtain symmetry results for solutions of semilinear elliptic equations.

When the domain Ω is unbounded, Berestycki, Caffarelli and Nirenberg [10] proposed the following conjecture:

BCN Conjecture: Assume that Ω is a smooth domain such that $\mathbb{R}^d \backslash \Omega$ is connected. Then the existence of a bounded solution to the problem (1.1) for some Lipschitz function f implies that Ω is either a ball, a half-space, a generalized cylinder $B^k \times \mathbb{R}^{d-k}$ (B^k is a ball in $\mathbb{R}^k$), or the complement of one of them.

Such conjecture was motivated first by the result of Serrin for bounded domains, and for unbounded domains by some rigidity results obtained in epigraphs ([10]) and exterior domains ([2, 67]). The BCN Conjecture has motivated various interesting works giving an affirmative answer for some classes of overdetermined elliptic problems. Let us now briefly describe some of such results. In [37] the authors get an affirmative answer under the hypothesis that Ω is an epigraph of $\mathbb{R}^2$ or $\mathbb{R}^3$ and the function u satisfies some natural assumptions. In addition, the BCN conjecture is proved in the plane for some classes of nonlinearities f in [70]. Moreover, the work [68] proves the validity of the BCN conjecture in dimension 2 if $\partial\Omega$ is connected and unbounded.

It turns out, however, that the conjecture has been answered negatively with a counterexample by Sicbaldi [77]. It is shown that there exist periodic domains of revolution such that the problem (1.1) admits a positive solution for $f(u) = \lambda u$ where $\lambda > 0$. Moreover, these domains bifurcate from a straight cylinder $B \times \mathbb{R}$, where $B \subset \mathbb{R}^{d-1}$ is a ball. The construction in [77] relies on topological degree theory and therefore does not give rise to a smooth branch of domains. After such first result, different constructions have been given in the literature, see for instance [54].

The first problem that we address in this thesis is the validity of this perturbation result for general nonlinearities $f(u)$ (Chapter 3).

In the spirit of the corresponding results of Alexandrof and Serrin, a parallelism between overdetermined elliptic problems and constant mean curvature surfaces has attracted a lot of attention. Indeed, the boundary of the domain built in [77] has a shape that looks like an unbounded constant mean curvature surface, showing again an important analogy with the onduloid (or Delaunay surfaces). Moreover, in [22] similar solutions are found for the Allen-Cahn nonlinearity, but in domains that are perturbations of a dilated straight cylinder, i.e. perturbations of $(\epsilon^{-1} B^d) \times \mathbb{R}$ for ϵ small, or more in general domains that are perturbations of a dilation of the region contained in an onduloid.

If Ω is an epigraph, the problem is also related to the famous De Giorgi's conjecture (see [20]):
De Giorgi conjecture: Let $u : \mathbb{R}^d \to \mathbb{R}$ be an entire solution of the Allen-Cahn problem:

$$\Delta u + u - u^3 = 0.$$

Assume that for $x \in \mathbb{R}^d$,

$$\frac{\partial u}{\partial x_d} > 0.$$

Then the level sets of u are hyperplanes, that is, u is one-dimensional, at least if $d \leq 8$.

The relationship between De Giorgi's conjecture and overdetermined problems is not surprising if we recall that this conjecture is the counterpart of Bernstein's conjecture on minimal surfaces (1914), which stated that all entire minimal graphs in $\mathbb{R}^d$ should be hyperplanes, and has been disproved by Bombieri, De Giorgi and Giusti for $d \geq 9$ ([12]). Starting from the Bombieri-De Giorgi-Giusti entire minimal graph, the entire nontrivial monotone solutions to the Allen-Cahn equation if $d \geq 9$ are built by Del Pino, Kowalczyk and Wei in [21]. In other words, they show with an example that in the De Giorgi conjecture, "at least if $d \leq 8$" is indeed necessary. In this spirit, Del Pino, Pacard and Wei have shown that there exist nontrivial domains Ω that support solutions to (1.1) when $f(u)$ is of Allen-Cahn type (i.e. $f(u) = u - u^3$ or other nonlinearities with similar behavior) if $d \geq 9$, see [22].

For the study of the De Giorgi's conjecture, Modica estimates are useful. In [58], Modica proved that if F is a nonpositive function and $u : \mathbb{R}^d \to \mathbb{R}$ is a bounded C^3 solution of

$$\Delta u + F'(u) = 0,$$

then $P(x) = |\nabla u(x)|^2 + 2F(u(x)) \leq 0$. The proof of this result is based on the fact that P satisfies $L(P) \geq 0$ for a certain elliptic operator L, and hence the conclusion follows (not immediately) from the maximum principle.

One of the questions that we deal with in this thesis is to study to which point this kind of inequalities hold also for overdetermined elliptic problems, and what is its exact form, in general, (Chapter 6).

The harmonic case $f = 0$ has some intrinsic features and deserves a separate discussion. In the classical literature, a smooth domain Ω of the Euclidean space $\mathbb{R}^d$ is said to be an *exceptional* domain if there exists a positive harmonic function with vanishing Dirichlet boundary data and constant nonzero Neumann boundary data. Such a function is referred to as the *roof function.*

The problem of finding exceptional domains goes back to the pioneer work [46], where the nontrivial exceptional domain

$$\Omega := \left\{(x, y) \in \mathbb{R}^2 : |y| < \frac{\pi}{2} + \cosh(x)\right\}$$

was discovered in the plane. Moreover, Traizet [88] was able to prove that the only examples up to rotation and translation of planar exceptional domains having finitely boundary components are the exterior of a disk, a halfplane and the nontrivial domain Ω. In [88, Example 7.3], he further proved the existence of a nontrial periodic exceptional domain corresponding to Scherk's simply periodic minimal bigraphs. In higher dimension, a catenoid-type solution has been recently found, see [54].

In this thesis, we will show the existence of exceptional domains of $\mathbb{R}^d, d \geq 4$, where such overdetermined problem admits a solution (Chapter 4).

For an exterior domain $\Omega = \mathbb{R}^d \setminus \overline{D}$, the first result was obtained by Reichel [67], who proved that if there exists $u \in C^2(\overline{\Omega})$ with

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega, \\ u = 1 & \text{on} \quad \partial\Omega, \\ \lim_{|z|\to\infty} u(z) = 0, & \\ \dfrac{\partial u}{\partial \nu} = c & \text{on} \quad \partial\Omega, \\ 0 < u \leq 1 & \text{in} \quad \Omega, \end{cases} \tag{1.5}$$

then D is ball and u is radially symmetric and radially decreasing with respect to the center of D. Here $t \mapsto f(t)$ is a locally Lipschitz function, non-increasing for non negative and small values of t. In the special case $f \equiv 0$, this result therefore characterizes the ball as the only electric conductor such that the intensity of the corresponding electric field is constant on the boundary. To prove his result, Reichel used the *moving plane method.*

Closely related to [67] is the work [2], where the authors studied (1.5) without the decay at infinity and under different assumptions on f including the interesting case $f(t) = t^p$ with $\frac{d}{d-2} < p \leq \frac{d+2}{d-2}$. Moreover, Sirakov [79] proved that the result in [67] holds without the assumption $u < 1$ in Ω and for possibly multi-connected sets $D_i, i = 1, ..., m, m \in \mathbb{N}$.

In [69], it is shown that (1.1) admits a solution for some nonradial exterior domains (i.e. the complement of a compact region in $\mathbb{R}^d$ that is not a closed ball), for a suitable function $f(u)$, which gives another type of counterexample to the conjecture. In dimension 2, this represents this first construction of a counterexample to the BCN conjecture, that turns to be false in any dimension. It is worth pointing out that such result breaks the analogy with the theory of constant mean curvature surfaces.

A case of interest is that in cones. Let D be a smooth domain on the unit sphere $\mathbb{S}^{d-1}, d \geq 2$ and let Σ be the cone spanned by D, namely

$$\Sigma := \{x \in \mathbb{R}^d : x = st, t \in D, s \in (0, +\infty)\}.$$

For a domain $\Omega \subset \Sigma$ we set:

$$\Gamma := \partial\Omega \cap \Sigma, \ \Gamma_1 := \partial\Omega \cap \partial\Sigma.$$

Then the overdetermined problem in the form

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma, \\ \dfrac{\partial u}{\partial \nu} = c & \text{on} \quad \Gamma, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \Gamma_1 \setminus \{0\}, \end{cases} \tag{1.6}$$

has been treated.

In [64] (see also [65]), the authors gave a rigidity result of Serrin type by showing that the existence of a solution to the problem (1.6) for $f(u) = 1$ implies that Ω is a spherical sector, under the assumption that the cone Σ is convex. Such result has been extended to more general operators in [17]. The case of the nonconvex cones is also considered in [16, 47]. In [47], the authors proved that there exist nonradial domains inside a nonconvex cone, in which the problem (1.6) admits a solution for $f(u) = 1$. In [16], the authors presented that there is a nonradial positive solution to the problem (1.6) for some locally Lipschitz continuous functions f.

Another natural field of research that has attracted considerable attention is to study the overdetermined problem (1.1) in the case of domains of a Riemannian manifold $(\mathcal{M}, g)$ instead of the Euclidean setting. In this framework, we need to replace in (1.1) the classical Laplacian by the Laplace-Beltrami operator Δ_g

associated to the metric g of the manifold $\mathcal{M}$:

$$\begin{cases} \Delta_g u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where Ω is a domain of $\mathcal{M}$.

For general Riemannian manifolds, solutions of overdetermined elliptic problems of the form (1.7) are obtained in [23, 24, 36, 61, 63, 78].

Here we are interested in the case of subdomains of the sphere $\mathbb{S}^d$, where the problem (1.7) can be solved (Chapter 5).

1.3 Onduloid type domains in $\mathbb{R}^d$

The first study subject of this thesis is to understand the existence of the solution to the problem (1.1) in $\mathbb{R}^{d+1}, d \geq 1$ for a very general class of functions f, where the domain Ω we would like to construct is a perturbation of the straight cylinder. This result is complementary to the results with linear source and gives another counterexample to the BCN conjecture.

As we mentioned before, Sicbaldi in [77] found a periodic perturbation of the straight cylinder $B^{d-1} \times \mathbb{R}$ that supports a periodic solution to the problem (1.1) with $f(u) = \lambda u, \lambda > 0$. More precisely, such domains, as shown in [75], belong to a 1-parameter family $\{\Omega_s\}_{s \in (-\epsilon, \epsilon)}$ and are given by

$$\Omega_s = \left\{ (x,t) \in \mathbb{R}^d \times \mathbb{R} \,:\, |x| < 1 + s \cos\left(\frac{2\pi}{T_s} t\right) + O(s^2) \right\}$$

where ϵ is a small constant, $T_s = T_0 + O(s)$ and T_0 depends only on the dimension d. This result reinforces the boundary of domains that allow a solution of (1.1) and CMC surfaces, as the domain Ω_s can be put in correspondence to the onduloid (or Delaunay surface). In [32] the same kind of result is proved in the case $f \equiv 1$.

1.3.1 Objectives and strategies of the proofs

Our principle objective with respect to the problem (1.1) is to perform such a construction under somewhat minimal assumptions on the nonlinearity $f(u)$. It is clear that a mandatory assumption is the existence of a solution of the Dirichlet problem in the unit ball B in $\mathbb{R}^d$. For technical reasons we need the normal derivative at the boundary to be nonzero, which is a typical situation in overdetermined semilinear elliptic problems. Hence, we will consider the following hypothesis:

Assumption 1: There exists a positive solution $\phi_1 \in C^{2,\alpha}(\overline{B})$ of the problem

$$\begin{cases} \Delta\phi_1 + f(\phi_1) = 0 & \text{in } B, \\ \phi_1 = 0 & \text{on } \partial B, \end{cases} \tag{1.8}$$

with $\partial_\nu \phi_1(x) \neq 0$ for $x \in \partial B$, where ν denotes the exterior unit vector normal to ∂B.

Observe that by [41], any solution ϕ_1 of (1.8) needs to be a radially symmetric function. For technical reasons, we need to assume also that the linearized operator associated to problem (1.1) at ϕ_1 is non-degenerate (in a radially symmetric setting). This is a rather natural assumption if one intends to use a perturbation argument. Precisely, our second assumption is:

Assumption 2: Define the linearized operator $L_D : C^{2,\alpha}_{0,r}(B) \to C^{0,\alpha}_r(B)$ by

$$L_D(\phi) = \Delta\phi + f'(\phi_1)\phi\,,$$

where $C^{2,\alpha}_{0,r}(B)$ and $C^{0,\alpha}_r(B)$ denote the spaces of radial functions in $C^{2,\alpha}_0(B)$ and $C^{0,\alpha}(B)$ respectively. We assume that the linearized operator L_D is non-degenerate; in other words, if $L_D(\phi) \equiv 0$ then $\phi \equiv 0$.

We are now in position to state our main result, joint work with David Ruiz and Pieralberto Sicbaldi ([73]).

Theorem 1.1. *If $d \geq 1$, $f : [0, +\infty) \to \mathbb{R}$ is $C^{1,\alpha}$ and Assumptions 1 and 2 hold, then there exist a positive number T_* and a continuous curve*

$$\begin{array}{ccc} (-\epsilon, \epsilon) & \to & C^{2,\alpha}(\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \\ s & \mapsto & (v_s, T_s) \end{array}$$

for some ϵ small, with $v_s = 0$ if and only if $s = 0$. Moreover $T_0 = T_*$ and the overdetermined problem (1.1) has a solution in the domain

$$\Omega_s = \left\{ (x,t) \in \mathbb{R}^d \times \mathbb{R} \,:\, |x| < 1 + v_s\left(\frac{t}{T_s}\right) \right\}.$$

The solution $u = u_s$ of problem (1.1) is T_s-periodic in the variable t and hence bounded. Moreover,

$$\int_0^1 v_s(t)\, dt = 0.$$

Let us point out that Assumptions 1 and 2 hold for example in the following cases (among many others):

(1) If $f(0) > 0$ and $f'(u) < \lambda_1$ for any $u \in (0, +\infty)$, where λ_1 is the first eigenvalue of the Dirichlet Laplacian in the unit ball of $\mathbb{R}^d$. In this case a positive solution can be found (for instance, extending $f(u) = f(0)$ if $u < 0$ and minimizing the corresponding Euler-Lagrange functional) and the operator L_D has only positive eigenvalues.

(2) If $f(u) = u^p - u$, $1 < p < \frac{d+2}{d-2}$ if $d > 2$, $p > 1$ if $d = 2$. In this case the existence of a solution is well known, and it is a mountain-pass solution. As a consequence, L_D has a negative eigenvalue. By the analysis of [52], all other eigenvalues are strictly positive.

(3) If $f(u) = \lambda e^u$ and $\lambda \in (0, \lambda^*)$ for some $\lambda^* > 0$ that receives the name of extremal value. In this case ϕ_1 is the so-called minimal solution and L_D has only positive eigenvalues (see for instance [26]).

In particular, (1) holds when $f \equiv 1$, and we recover in this way the result in [32]. On the other hand, when $f(u) = \lambda u$ for some $\lambda > 0$, Assumption 1 implies that λ is the first eigenvalue of the Dirichlet Laplacian in the unit ball of $\mathbb{R}^d$, but then Assumption 2 is clearly not satisfied. Hence, our theorem is complementary to the results in [75, 77].

Theorem 1.1 is a bifurcation result in the spirit of [77], see also [32, 75]. In sum, one can reformulate the existence of solutions to (1.1) as the zeroes of a nonlinear Dirichlet-to-Neumann operator, and the Crandall-Rabinowitz Theorem is used to conclude local bifurcation. But here the situation is more involved because of the general term $f(u)$. In fact, the operator L_D may have negative eigenvalues, and the bifurcation argument requires a finer spectral analysis. In particular, the Dirichlet-to-Neumann operator can be built only for certain values of T, which are related to the nondegeneracy of the Dirichlet problem in the cylinder. We are able to show the existence of a bifurcation branch by taking advantage of the ideas of [69], where the linearized problem has a negative eigenvalue.

1.4 Exceptional domains in $\mathbb{R}^d$

The second problem that we study in this thesis is to deal with the equation (1.1) with $f(u) = 0$, nonzero Dirichlet condition and additional decay condition at infinity. More specifically, we are concerned with the existence of subdomains $\Omega \subset \mathbb{R}^d \times \mathbb{R}$ where the overdetermined boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = 1 & \text{on} \quad \partial\Omega, \\ \lim_{|z|\to\infty} u(z,t) = 0 & \text{uniformly in } t \in \mathbb{R}, \\ \dfrac{\partial u}{\partial \nu} = c & \text{on} \quad \partial\Omega, \end{cases} \tag{1.9}$$

is solvable. Here, we have chosen the coordinates $(z, t) \in \mathbb{R}^{d-1} \times \mathbb{R}$. The domains Ω we would like to construct are actually *nontrivial exceptional* subdomains of the Euclidean space $\mathbb{R}^d$, $d \geq 4$.

The classification of planar exceptional domains was done by Traizet [88]. We note that a family of infinitely connected planar exceptional domains was already discovered in fluid dynamics in [6, 19].

Up to date the structure of the set of exceptional domains in dimensions $d \geq 3$ remains largely unknown. We are only aware of the recent work [34, 54].

1.4.1 Objectives and strategies of the proofs

The main purpose here is the construction of subdomains $\Omega \subset \mathbb{R}^{d-1} \times \mathbb{R}$, $d \geq 4$ such that the overdetermined boundary value problem (1.9) is solvable. The domains under consideration are complements of perturbed cylinders of the form

$$\Omega_{T,\varphi} := \left\{(z,t) \in \mathbb{R}^{d-1} \times \mathbb{R} \ : \ |z| > 1 + \varphi\left(\frac{2\pi}{T}t\right)\right\} \subset \mathbb{R}^d, \tag{1.10}$$

where $T > 0$, $\varphi : \mathbb{R} \to (0, \infty)$ is a 2π-periodic function of class $C^{2,\alpha}$, for some $\alpha \in (0.1)$. The case $\varphi \equiv 0$ in (1.10) corresponds to the complement of the exterior of the straight cylinder $B_1 \times \mathbb{R}$ and the function $u_1(z) = |z|^{3-d}$ solves (1.9) with $c = -(3-d)$.

Our main result can be then stated as follows. This is joint work with Tobias Weth and Ignace Aristide Minlend ([57]).

Theorem 1.2. *Let $d \geq 4$. Then there exist a number $T_* > \dfrac{2\pi}{\sqrt{d-2}}$ and a smooth curve*

$$(-\varepsilon, \varepsilon) \to (0, +\infty) \times C^{2,\alpha}(\mathbb{R}), \qquad s \mapsto (T_s, v_s),$$

where $T_0 = T_*$ and $v_0 \equiv 0$ with

$$\int_{-\pi}^{\pi} v_s(t)\cos(t)\,dt = 0,$$

such that for all $s \in (-\varepsilon, \varepsilon)$, letting $\varphi_s(t) = s\cos(t) + sv_s$, there exists a unique function $u_s \in C^{2,\alpha}(\overline{\Omega_{T_s,\varphi_s}})$ satisfying

$$\begin{cases} \Delta u_s = 0 & \text{in } \Omega_{T_s,\varphi_s}, \\ u_s = 1 & \text{on } \partial\Omega_{T_s,\varphi_s}, \\ \lim_{|z|\to\infty} u_s(z,t) = 0, & \\ \frac{\partial u_s}{\partial \nu} = d-3 & \text{on } \partial\Omega_{T_s,\varphi_s}. \end{cases} \tag{1.11}$$

Moreover, u_s is radial in z and T_s-periodic and even in t for every $s \in (-\varepsilon, \varepsilon)$.

We point out that, for every $s \in (-\varepsilon, \varepsilon)$, the domain Ω_{T_s,φ_s} in Theorem 1.2 is exceptional with *roof function* given by $\tilde{u}_s = 1 - u_s$ in Ω_{T_s,φ_s}. Indeed this function is positive in Ω_{T_s,φ_s} since the harmonic function u_s cannot attain a maximum in Ω_{T_s,φ_s} unless it is constant, which is excluded by the boundary conditions in (1.11). It follows that $0 < u_s < 1$ and therefore $0 < \tilde{u}_s < 1$ in Ω_{T_s,φ_s}.

Actually, the domains in Theorem 1.2 have a similar shape as those found in the recent work [34] for the case $d = 3$, but the underlying construction is completely different. In fact, the approach in [34] relies on specific properties of an integral representation of an associated Dirichlet-to-Neumann operator which is only available in the case $d = 3$. On the other hand, our approach depends essentially on the assumption $d \geq 4$ (see e.g. Theorem 4.4 and Proposition 4.11). The difference between these two cases is reflected by the geometry of associated roof functions which are bounded for $d \geq 4$ and have a logarithmic growth in the distance from the cylinder axis in the case $d = 3$. Clearly, these differences are related to the different nature of the fundamental solution of $-\Delta$ in dimensions $d = 2$ and $d \geq 3$.

Related to Theorem 1.2 are some recent results in [31, 60]. In [60], Morabito obtained a family of bifurcation branches of domains which are small deformation of the complement of a solid cylinder in $\mathbb{R}^3$. We note that in contrast to (1.11), [60] considers a non-constant Neumann condition involving the mean curvature of the boundary. Furthermore, it is shown in [31] the existence of a foliation by perturbations of a large coordinate sphere whose enclosures solve (1.11) in an asymptomatically flat manifold.

We now explain the construction of the domains in (1.10). First, we rephrase the main problem (1.9) to an equivalent one on the fixed domain $\Omega_1 = B_1^c \times \mathbb{R}$. We emphasize that our analysis strongly relies on the decay assumption in (1.9) which motivates the functional setting in weighted Hölder spaces. Furthermore, we need the pull back operator in Lemma 4.1 to map between these spaces. We do this by parametrizing the set $\Omega_{T,\varphi}$ as in (4.3) with a suitably chosen diffeomorphism which minimizes the effect of the perturbation away from the boundary. In this functional analytic setting we are able to reformulate our problem as a nonlinear operator equation of the form $F(T, \varphi) \equiv 0$, to which we then apply the Crandall-Rabinowitz bifurcation theorem. For this it is necessary to compute the linearised operator $D_\varphi F(T, 0)$ and analyze its spectral properties.

1.5 Nontrivial contractible domains in $\mathbb{S}^d$

The third problem we study is to show that there exist solutions to overdetermined elliptic problems (1.7) in subdomains of the sphere.

It is clear that any symmetry result on the solutions of (1.7) tightly depends on the symmetry of the ambient manifold. In a given arbitrary manifold, geodesic balls are not domains where (1.7) can be solved. As shown in [23], for small volumes it is possible to construct solutions of (1.7) with $f(u) = \lambda\, u$ in perturbations of geodesic balls centered at specific points of the manifold, but such domains in general are not geodesic balls. A similar approach was followed by Fall and Minlend [30] to show the existence of solutions to (1.7) with $f(u) = 1$. In fact, one can expect to obtain a Serrin-type result only for manifolds that are symmetric in a suitable sense. More precisely (see also the introduction of [24]), the key ingredient for the moving plane method is the use of the reflexion principle in any point and any direction. For this we need that for any $p \in \mathcal{M}$ and any two vectors $v, w \in T_p\mathcal{M}$ there exists an isometry of $\mathcal{M}$ leaving p fixed and transporting v into w (i.e. $\mathcal{M}$ is isotropic) and that such isometry is induced by the reflection with respect to a hypersurface. Such last hypersurface must be totally geodesic, being the set of fixed points of an isometry. But an isotropic manifold admitting totally geodesic hypersurfaces must have constant sectional curvature (see for example [11], p. 295) and the only isotropic Riemannian manifolds of constant sectional curvature are the Euclidean space $\mathbb{R}^d$, the round sphere $\mathbb{S}^d$, the hyperbolic space $\mathbb{H}^d$ and the real projective space $\mathbb{RP}^d$ (see [86]). Now, domains in $\mathbb{RP}^d$ arise naturally to domains in its universal covering $\mathbb{S}^d$, so we are left to consider our problem in $\mathbb{R}^d$, $\mathbb{S}^d$ and $\mathbb{H}^d$.

Being the problem in $\mathbb{R}^d$ completely understood by Serrin, the framework of the other two space form manifolds has been treated in 1998 in the paper by Kumaresan and Prajapat [51]. In the case of $\mathbb{H}^d$ they obtained a complete counterpart of the Serrin's theorem: namely, by using the moving plane method, they show that if Ω is a bounded domain of $\mathbb{H}^d$ and (1.7) admits a solution, then Ω must be a geodesic ball. The case of $\mathbb{S}^d$ is different. In fact, even if the reflexion principle is valid in any point, one needs to have a totally geodesic hypersurface that does not intersect the domain in order to start the moving plane method. This is not a problem in $\mathbb{R}^d$ or $\mathbb{H}^d$, but it is in $\mathbb{S}^d$. Since the totally geodesic hypersurfaces are the equators, one can start the moving plane method if and only if the domain is contained on a hemisphere. And this is exactly the case considered in [51]: if Ω is contained in a hemisphere and (1.7) admits a solution, then Ω must be a geodesic ball.

Other natural domains of $\mathbb{S}^d$ where (1.7) has solutions are symmetric neighborhoods of any equator. Such symmetric annuli are not contractible and their existence comes from the geometry of $\mathbb{S}^d$ in the same way as they exist in a cylinder or in a torus. Moreover, perturbations of neighborhoods of an equator in $\mathbb{S}^d$ where (1.7) still admits a solution have been built in [33] in the same way as this has been done for the same kind of domains in cylinders or in tori [77].

Taking these facts in account, the following question arises naturally: is it true that if $\Omega \subset \mathbb{S}^d$ is contractible and (1.7) can be solved, then Ω must be a geodesic ball? In [28], Espinar and Mazet give an affirmative answer if $d = 2$ but under some extra assumptions on the nonlinear term $f(u)$. The proof of such result shows again an analogy between overdetermined elliptic problems and constant mean curvature surfaces, because it is highly inspired by the proof of the Hopf's

Theorem that states that the only immersed constant mean curvature surfaces of genus zero in $\mathbb{R}^3$ are the spheres.

In this thesis we show that the answer to the previous question is negative, in general: there exist contractible domains $\Omega \subset \mathbb{S}^d$, different from geodesic balls, where (1.7) can be solved for some nonlinearities f. This construction works for any dimension $d \geq 2$. In view of [51], such domains cannot be contained in any hemisphere.

1.5.1 Objectives and strategies of the proofs

To be more specific, we are going to prove the following theorem, joint work with David Ruiz and Pieralberto Sicbaldi ([72]) .

Theorem 1.3. *Let $d \in \mathbb{N}, d \geq 2$ and $1 < p < \frac{d+2}{d-2}$ ($p > 1$ if $d = 2$). Then there exist domains D, which are perturbations of a small geodesic ball, such that the problem*

$$\begin{cases} -\varepsilon\Delta_g u + u - u^p = 0 & \text{in } \mathbb{S}^d \setminus D \ , \\ u > 0 & \text{in } \mathbb{S}^d \setminus D \ , \\ u = 0 & \text{on } \partial D \ , \\ \partial_\nu u = c & \text{on } \partial D \ , \end{cases}$$

admits a solution for some $\varepsilon > 0$.

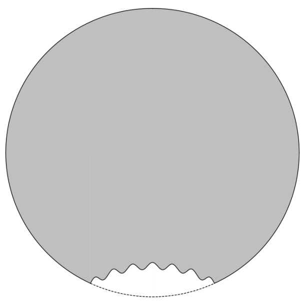

Figure 1.1: The domain $\mathbb{S}^d \setminus D$

The main idea of the proof is the following. First, one uses a dilation to pass to a problem posed in $\mathbb{S}^d(k)$, the sphere of radius $1/k$, where k will be a small parameter. We take a geodesic ball of radius 1 in $\mathbb{S}^d(k)$, and consider now Ω_k the complement of such ball in $\mathbb{S}^d(k)$. The main idea is that, as $k \to 0$, the domain Ω_k converges (in a certain sense) to the exterior domain $\mathbb{R}^d \setminus B(0,1)$. Thanks to the result in [69], we have the existence of nontrivial solutions of (1.1) for a suitable choice of f, bifurcating from a family of regular solutions u_λ of the problem:

$$\begin{cases} -\lambda\Delta u + u - u^p = 0 & \textit{in } \mathbb{R}^d \setminus B(0,1)\ , \\ u > 0 & \textit{in } \mathbb{R}^d \setminus B(0,1)\ , \\ u = 0 & \textit{on } \partial B(0,1)\ , \\ \partial_\nu u = c & \textit{on } \partial B(0,1). \end{cases}$$

Firstly, we show that for k sufficiently small there exists a similar family of solutions posed in Ω_k. This is accomplished by making use of a (quantitative) Implicit Function Theorem. Then we study the behavior of the linearized operator by using a perturbation argument, and taking into account the case studied in [69]. In such way, we can use the Krasnoselskii bifurcation theorem to show the existence of a branch of nontrivial solutions to (1.7) .

In our arguments, we rely on the study of the linearized operator given in [69]. At a certain point the assumption $p < \frac{d+2}{d-2}$ is needed in [69], and hence our result is also restricted to that case. Moreover, such assumption is needed also in order to get L^∞ uniform estimates on the solutions.

1.6 Modica type estimates and curvature results

Our final subject in this thesis is to build an estimate on the gradient of the solution to the problem (1.1) and provide some information about the curvature of the boundary.

Let u be a bounded C^3 solution of $\Delta u + f(u) = 0$ in $\mathbb{R}^d$. Given a primitive F of f, we define the P-function:

$$P(x) = |\nabla u(x)|^2 + 2F(u(x)). \tag{1.12}$$

Modica in [58], as mentioned in the Antecedents, proved that if F is nonpositive, then $P \leq 0$. Such result has been extended to more general operators in [13], where Caffarelli, Garofalo and Segala prove the following result:

$$P(x_0) = 0 \text{ for some } x_0 \in \mathbb{R}^d \Leftrightarrow P(x) = 0 \text{ for all } x \in \mathbb{R}^d \Leftrightarrow u \text{ is 1-dimensional.}$$

These results have many consequences (e.g. a monotonicity formula has been derived in [59]), and are very important in the study of a famous conjecture of De Giorgi and related questions.

Modica type gradient estimates has been extended to domains Ω, which are epigraphs with nonnegative mean curvature on the boundary in [38], and to compact manifolds with nonnegative Ricci tensor, see [39] and the references therein.

1.6.1 Objectives and strategies of the proofs

In this objective, we want to study to which point a Modica type estimate holds also for overdetermined elliptic problems, and what its exact form is, in general. A first result in this direction has been given in [89] under some assumptions on the nonlinearity and also on the normal derivative at the boundary. The result of [89] implies, in particular, that the mean curvature of $\partial\Omega$ is everywhere negative. Let us point out that this cannot hold in general since there are examples of solutions to (1.1) in spheres or onduloids, for instance. Here we plan to clarify this point and give a general form of a Modica type estimate that generalizes that of [89]. It is to be expected that the presence of the boundary will play a decisive role in the description of the question. Of course, we are also interested in the equality case.

One of the results is exactly a rigidity result for overdetermined elliptic problems, that roughly speaking is the following: if f and c satisfy a certain condition then either the mean curvature of $\partial\Omega$ is negative, or Ω is a half-space. More precisely, denoting by $H(p)$ the mean curvature of $\partial\Omega$ at p, we will prove the following rigidity result. This is joint work with David Ruiz and Pieralberto Sicbaldi ([74]):

Theorem 1.4. *Let $u \in C^3(\Omega)$ be a bounded solution to the problem (1.1), with $f \in C^1$. If there exists a non positive primitive F of f such that*

$$c^2 + 2F(0) \geq 0\,, \tag{1.13}$$

then either $H(p) < 0$ for any $p \in \partial\Omega$, or Ω is a half-space and u is 1-dimensional, i.e., there exists $x_0 \in \mathbb{R}^d$, a function $g : [0, +\infty) \to \mathbb{R}$ and $\vec{a} \in \mathbb{R}^d$, such that

$$\Omega = \{x \in \mathbb{R}^d : \ \vec{a} \cdot (x - x_0) > 0\}, \quad \text{and } u(x) = g(\vec{a} \cdot (x - x_0)), \quad x \in \Omega.$$

In particular, if (1.13) is satisfied, one cannot solve the problem (1.1) in a ball, nor in a cylinder, nor in the perturbation of a cylinder. Notice that we are considering bounded solutions of (1.1), so f is defined in a closed interval and it is always possible to choose a non positive primitive of f, by changing f appropriately outside the image of u. A particular case of Theorem 1.4 is Theorem 2.13 in [70] for double periodic domains in the plane.

We will obtain Theorem 1.4 as a corollary of more general results, that represent Modica type estimates for overdetermined problems.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^d, d \geq 1$, be a regular domain, $F \in C^2(\Omega)$ be a non-positive function, $F' = f$, $u \in C^3(\Omega)$ be a bounded solution to the problem (1.1) and P be given by (1.12). Then*

$$P(x) \leq \max\{0, c^2 + 2F(0)\} \quad \text{for all } x \in \Omega.$$

Moreover, if there exists a point $x_0 \in \Omega$ such that

$$P(x_0) = \max\{0, c^2 + 2F(0)\},$$

then P is constant, u is 1-dimensional and Ω is a half-space. Namely, there exists $x_0 \in \mathbb{R}^d$, a function $g : [0, +\infty) \to \mathbb{R}$ and $\vec{a} \in \mathbb{R}^d$, such that

$$\Omega = \{x \in \mathbb{R}^d : \ \vec{a} \cdot (x - x_0) > 0\}, \quad \text{and } u(x) = g(\vec{a} \cdot (x - x_0)), \quad x \in \Omega.$$

As a first ingredient for Theorem 1.5, we prove a uniform gradient bound on u. This is based on a rescaling argument and a Harnack's inequality, together with regularity estimates. Once this is obtained, we make use of the fact that the function P is a subsolution for a certain elliptic operator outside the critical points of u. For the study of the supremum of P we need to study several cases, depending on the behavior of the maximizing sequences. The proof concludes again by a scaling argument and passing to a limit, in a certain sense. This part of the proof is inspired by [48].

If P is bounded above by $c^2+2F(0)$ we can give information on the mean curvature of $\partial\Omega$.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^d, d \geq 1$, be a regular domain that supports a bounded solution $u \in C^3(\Omega)$ to the problem (1.1) with $c \neq 0$. Assume that*

$$P(x) \leq c^2 + 2F(0) \quad \text{for all } x \in \Omega.$$

Then, $H(p) \leq 0$ for any $p \in \partial\Omega$. Moreover, if there exists $p \in \partial\Omega$ such that $H(p) = 0$, then P is constant, u is 1-dimensional and Ω is either a half-space or the domain between two parallel hyperplanes.

Let us point out that Theorem 1.6 does not require that F is nonpositive. On the contrary, it requires that the normal derivative on $\partial\Omega$ does not vanish. As has been commented before, there exist solutions of problem (1.1) in balls, cylinders, or generalized onduloids. Theorem 1.6 implies that in all such cases,

$$\sup_{x \in \Omega} P(x) > c^2 + 2F(0)\,.$$

1.7 Local bifurcation theorems

Since our first three results are all proved by a local bifurcation argument and for the sake of completeness, we end this chapter by recalling the bifurcation theorems that will be used.

Let us first consider the the case of bifurcation from a simple eigenvalue. For the proof and for many other applications we refer to [49, 81] and to the original exposition [18, Theorem 1.7]

Theorem 1.7. ***(Crandall-Rabinowitz Bifurcation Theorem)*** *Let X and Y be Banach spaces, and let $U \subset X$ and $\Gamma \subset \mathbb{R}$ be open sets, where we assume $0 \in U$. Denote the elements of U by v and the elements of Γ by λ. Let $G : U \times \Gamma \to Y$ be a C^1 operator such that*

i) $G(0, \lambda) = 0$ *for all* $\lambda \in \Gamma$;

ii) $\mathrm{Ker}\ D_v G(0, \lambda_*) = \mathbb{R}\, w$ *for some* $\lambda_* \in \Gamma$ *and some* $w \in X \setminus \{0\}$;

iii) $\mathrm{codim\ Im}\ D_v G(0, \lambda_*) = 1$;

iv) The cross derivative $D_\lambda D_v G$ *exists and is continuous, and* $D_\lambda D_v G(0, \lambda_*)(w) \notin \mathrm{Im}\ D_v G(0, \lambda_*)$.

Then there is a nontrivial continuous curve

$$s \to (v(s), \lambda(s)) \in X \times \Gamma\,, \tag{1.14}$$

$s \in (-\delta, +\delta)$ *for some* $\delta > 0$*, such that* $(v(0), \lambda(0)) = (0, \lambda_*)$*,* $v(s) \neq 0$ *if* $s \neq 0$ *and*

$$G\big(v(s), s\big) = 0 \quad for \quad s \in (-\delta, +\delta)\,.$$

Moreover there exists a neighborhood $\mathcal{N}$ *of* $(0, \lambda_*)$ *in* $X \times \Gamma$ *such that all solutions of the equation* $G(v, \lambda) = 0$ *in* $\mathcal{N}$ *belong to the trivial solution line* $\{(0, \lambda)\}$ *or to the curve* (1.14)*. The intersection* $(0, \lambda_*)$ *is called a bifurcation point.*

Now, we shall present another theorem for the case of bifurcation from an eigenvalue with odd multiplicity due to Krasnoselskii, one can refer to [49, 81] for more details.

Theorem 1.8. ***(Krasnoselskii Bifurcation Theorem)*** *Let Y be a Banach space, and let $U \subset Y$ and $\Gamma \subset \mathbb{R}$ be open sets, where we assume $0 \in U$. Denote the elements of U by w and the elements of Γ by λ. Let $G : U \times \Gamma \to Y$ be a C^1 operator such that*

i) $G(0, \lambda) = 0$ *for all* $\lambda \in \Gamma$;

ii) $G(w, \lambda) = w - K(w, \lambda)$*, where* $K(w, \lambda)$ *is a compact map;*

iii) We denote by $i(\lambda)$ *the index of* $D_w G(0, \lambda)$*, i.e., the sum of the multiplicities of all negative eigenvalues of* $D_w G(0, \lambda)$*. We assume that there exist* $\bar{\lambda} < \hat{\lambda}$ *such that:*

   1. $D_w G(0, \bar{\lambda})$*,* $D_w G(0, \hat{\lambda})$ *are non degenerate.*

   2. $i(\bar{\lambda})$ *and* $i(\hat{\lambda})$ *have different parity.*

Then there exists $\lambda_* \in (\bar{\lambda}, \hat{\lambda})$ a bifurcation point for $G(w, \lambda) = 0$ in the following sense: $(0, \lambda_*)$ is a cluster point of nontrivial solutions $(w, \lambda) \in Y \times \mathbb{R}, w \neq 0,$ of $G(w, \lambda) = 0$.

Remark 1.9. *Let us point out that the above version of the Krasnoselskii theorem is not the standard one, as usually one imposes the existence of an isolated point where $D_w G(0, \lambda)$ is degenerate. However, the proof of the theorem works equally well for the version stated above. See [69, Remark 6.3] on this regard.*

Chapter 2

Methodology

These objectives are discussed from different perspectives and require different methods in multiple mathematical areas, such as Analysis, Partial Differential Equations and Geometry. The methodology for the development of this thesis can be divided into several items:

i. *Human-being Support.* Under the joint supervision of David Ruiz and Pieralberto Sicbaldi at University of Granada, the basic research ideas of this thesis are conceived by frequent working sessions. Many of them were made online at the beginning of the pandemic. Pieralberto Sicbaldi is an expert in the field, particularly in its geometric aspects, whereas David Ruiz is a researcher specialized in PDEs.

   There were also some short stays in Frankfurt, including a one-week visit from 10/07/2022 to 15/07/2022 and a three-month stay from mid-March to mid-June 2023. During the one-week stay, there were some talks about the overdetermined elliptic problem for the harmonic function in the higher dimension with Tobias Weth and Ignace Aristide Minlend. During the second stay in Frankfurt, we first discussed the previous topic to complete it so that we could submit it to a scientific journal. At the same time, we also had some working sessions on a new project related to the Schiffer conjecture, an important open question in this framework. So far, the exchange with all these professors are of great significance for completing the thesis.

ii. *Framework.* The Ph.D. program "Fisica y Matematicas" at University of Granada had offered a wide range of high-level doctoral courses, conferences and seminars by distinguished experts in different fields. These academic training activities either had a direct and positive impact on the preparations of this thesis or enhanced education in other disciplines.

   Moreover, the libraries of the Faculty of Science and the Department of Mathematical Analysis are well equipped with academic resources and the

Institute of Mathematics of the University of Granada (IMAG) is staffed by a sufficient number of outstanding mathematicians from Granada. IMAG was awarded with the Excellence Seal "María de Maeztu" in 2021, and became an associated center of the Banff International Research Station (BIRS) in 2022. As a consequence, many activities such as conferences, workshops, seminars, advanced courses, etc. have been carried out. These have created an ideal framework for the formation of a P.h.D. student. It also hosts post-docs and visiting researchers who collaborate with local or national researchers.

iii. *Communication of the results obtained in this thesis.* In addition to the publications in several high-level mathematical journals, we take it seriously to present our research findings in different seminars of scientific events, such as:

(1) XXVI Congress of Differential Equations and Applications XVI Congress of Applied Mathematics, June 2021, Gijón, Spain;

(2) Seminar of Young Researchers at the University of Granada, February 9, 2022, Granada, Spain;

(3) Applied Analysis Seminar at the University of Frankfurt, July 14, 2022, Frankfurt, Germany;

(4) Seminar of Differential Equations at the University of Granada, December 2, 2022, Granada, Spain;

(5) VI Congress of Young Researchers of the Royal Spanish Mathematical Society, February 2023, León, Spain.

I also participated in the following conferences:

(1) Workshop on PDEs: Modelling, Analysis and Numerical Simulation, January 2020, Granada, Spain;

(2) Geometric PDEs@Caserta, September 2021, Caserta, Italy;

(3) Encuentro REAG, March 2022, Granada, Spain.

Due to the breakout of the pandemic, the realization of the conferences and workshops was stopped. In addition, some planned research stays and visits have been put on hold, which has had a major impact on our research.

iv. *Use of online resources.* The database of the electronic resources provided by the institution, as well as some digital publishing portals, have been provided free of charge, which are necessary for the completion of the project. In addition, online seminars and conferences also play an important role in my formation as a PhD student, such as:

(1) One World PDE seminar;

(2) Online Analysis and PDE seminar;

(3) International Conference on Nonlinear Analysis and Nonlinear Partial Differential Equations, August 2022, Xi'an, China;

(4) ICMC Summer Meeting on Differential Equations, February 2021, Sao Carlos, Brazil.

As is well known, financial support is always of great importance for researchers to conduct their academic activities and make contributions to their research field. I have been awarded a scholarship from the State Scholarship Fund by the China Scholarship Council, which is a non-profit organization formed by the Chinese Ministry of Education and provides a primary vehicle through which the Chinese government awards excellent students with international academic exchanges both at home and abroad. This provides us with an excellent framework for producing a high-quality doctoral dissertation.

Last but not least, the doctoral thesis at the University of Granada is also funded by the Andalusian Commission Investigation Team (FQM-116), which has allowed me to attend different international conferences and seminars to get in contact with researchers in the field and exchange ideas in a natural research environment.

Chapter 3

Onduloid type domains in $\mathbb{R}^d$

This chapter is devoted to the study of the problem (1.1). Firstly, we give some basic preliminary results on the Dirichlet problem in a ball and a cylinder respectively such that we can construct the nonlinear Dirichlet-to-Neumann operator. Once this is done, we can compute the linearization of this operator and study the properties of the linearized operator. With all those ingredients, a local bifurcation argument can be used to prove Theorem 1.1.

3.1 Some preliminaries about related linear problems in the ball

Let B be the unit ball in $\mathbb{R}^d$ centered at the origin. It will be useful to define the following Hölder spaces:

$$C_r^{k,\alpha}(B) = \left\{\phi \in C^{k,\alpha}(B) : \phi(x) = \phi(|x|),\ x \in B\right\},$$

$$C_{0,r}^{k,\alpha}(B) = \left\{\phi \in C_0^{k,\alpha}(B) : \phi(x) = \phi(|x|),\ x \in B\right\}.$$

We also define the following Sobolev spaces:

$$H_r^1(B) = \left\{\phi \in H^1(B) : \phi(x) = \phi(|x|),\ x \in B\right\},$$

$$H_{0,r}^1(B) = \left\{\phi \in H_0^1(B) : \phi(x) = \phi(|x|),\ x \in B\right\}.$$

We will write $r = |x|$, and for functions ϕ in such spaces we will use both notations $\phi(x)$ and $\phi(r)$ according to the computations. We recall that ϕ_1 is a radial solution of (1.8), that is,

$$\begin{cases} \phi_1''(r) + \frac{d-1}{r}\phi_1'(r) + f(\phi_1(r)) = 0 & \text{in } (0,1]\ , \\ \phi_1(1) = 0, \quad \phi_1'(0) = 0. \end{cases} \tag{3.1}$$

The operator L_D defined in Assumption 2 has a diverging sequence of eigenvalues γ_{D_j}, hence there are only a finite number l of them which are negative:

$$\gamma_{D_1} < \gamma_{D_2} < \cdots < \gamma_{D_l} < 0,\ \gamma_{D_{l+1}} > 0.$$

Next result is rather standard, we include it here for the sake of completeness.

Lemma 3.1. *The eigenvalues γ_{D_j} are all simple.*

Proof. Assume that ψ_1, ψ_2 are two nontrivial eigenfunctions corresponding to γ_{D_j}, i.e.

$$L_D(\psi_i) + \gamma_{D_j}\psi_i = 0\,,\ i = 1, 2\,.$$

Let us choose $k_1, k_2 \in \mathbb{R}$, $(k_1, k_2) \neq (0, 0)$, such that

$$k_1\psi_1'(1) + k_2\psi_2'(1) = 0\,.$$

If $\psi = k_1\psi_1 + k_2\psi_2$, we have

$$L_D(\psi) + \gamma_{D_j}\psi = 0,\ \psi(1) = 0,\ \psi'(1) = 0\,.$$

By the uniqueness of the solution of the Cauchy problem for ODE, $\psi \equiv 0$. That is, ψ_1, ψ_2 are linearly dependent. $\square$

We denote by $z_j \in C^{2;\alpha}_{0,r}(B)$ the eigenfunction with eigenvalue γ_{D_j}, i.e.

$$\begin{cases} \Delta z_j + f'(\phi_1)z_j + \gamma_{D_j}z_j = 0 & \text{in } B\,, \\ z_j = 0 & \text{on } \partial B\,, \end{cases} \tag{3.2}$$

normalized by $\|z_j\|_{L^2} = 1$.

As is well known, the operator L_D is related to the quadratic form

$$Q_D : H^1_{0,r}(B) \to \mathbb{R},\quad Q_D(\phi) := \int_B \left(|\nabla\phi|^2 - f'(\phi_1)\phi^2\right).$$

For instance, the first eigenvalue of L_D is given by

$$\gamma_{D_1} = \inf\left\{Q_D(\phi) : \|\phi\|_{L^2(B)} = 1\right\}. \tag{3.3}$$

Later, our computations will involve another quadratic form Q defined as

$$Q : H^1_r(B) \to \mathbb{R},\quad Q(\psi) := \int_B \left(|\nabla\psi|^2 - f'(\phi_1)\psi^2\right) + \bar{c}\,\omega_d\psi(1)^2\,,$$

where ω_d is the area of $\mathbb{S}^{d-1}$ and

$$\bar{c} = -\frac{\phi_1''(1)}{\phi_1'(1)} = d - 1 + \frac{f(0)}{\phi_1'(1)}\,. \tag{3.4}$$

To get the last equality we used (3.1). Observe that

$$Q|_{H^1_{0,r}(B)} = Q_D\,. \tag{3.5}$$

Analogously we can define

$$\gamma_1 = \inf\left\{Q(\phi) : \|\phi\|_{L^2(B)} = 1\right\}\,. \tag{3.6}$$

It is rather standard to show that γ_1 is achieved by a solution ψ_1 of the problem:

$$\begin{cases} \Delta\psi_1 + f'(\phi_1)\psi_1 + \gamma_1\psi_1 = 0 & \text{in } B,\\ \partial_\nu\psi_1(x) + \bar{c}\,\psi_1(x) = 0 & \text{on } \partial B\,. \end{cases} \tag{3.7}$$

As in Lemma 3.1 one can show that γ_1 is simple, so ψ_1 is uniquely determined up to a sign.

We finish this section with an estimate of the eigenvalue γ_1.

Lemma 3.2. *There holds:* $\gamma_1 < \min\{0, \gamma_{D_1}\}$.

Proof. We first show that $\gamma_1 < 0$; for this it suffices to find $\psi \in H^1_r(B)$ such that $Q(\psi) < 0$. Since Q is considered among radially symmetric functions, we can write the quadratic form as

$$\begin{aligned} Q(\psi) &= \int_B \left[|\nabla\psi|^2 - f'(\phi_1)\psi^2\right] + \bar{c}\,\omega_d\psi(1)^2\\ &= \omega_d \int_0^1 r^{d-1}\left[\psi'(r)^2 - f'(\phi_1)\psi(r)^2\right]dr + \bar{c}\,\omega_d\psi(1)^2. \end{aligned}$$

Now we compute the derivative in (3.1) to obtain:

$$\phi_1'''(r) + \frac{d-1}{r}\phi_1''(r) - \frac{d-1}{r^2}\phi_1'(r) + f'(\phi_1)\phi_1'(r) = 0. \tag{3.8}$$

If we multiply the equation (3.8) by $r^{d-1}\phi_1'(r)$ and integrate, we obtain

$$\int_0^1 r^{d-1}\left[\phi_1''(r)^2 - f'(\phi_1)\phi_1'(r)^2\right]dr = \phi_1'(1)\phi_1''(1) - (d-1)\int_0^1 r^{d-3}\phi_1'(r)^2 dr.$$

This last equality comes from the computation:

$$\begin{aligned} &\int_0^1 r^{d-1}\phi_1'''(r)\phi_1'(r)dr = \int_0^1 r^{d-1}\phi_1'(r)d\phi_1''(r)\\ &= r^{d-1}\phi_1'(r)\phi_1''(r)\Big|_0^1 - \int_0^1 \phi_1''(r)d\big(r^{d-1}\phi_1'(r)\big)\\ &= \phi_1'(1)\phi_1''(1) - \int_0^1 r^{d-1}\phi_1''(r)^2 dr - (d-1)\int_0^1 r^{d-2}\phi_1'(r)\phi_1''(r)dr. \end{aligned}$$

We can take the test function $\phi_1'(r) \in H_r(B)$ obtaining:

$$
\begin{aligned}
Q(\phi_1'(r)) &= \omega_d \int_0^1 r^{d-1} \left[\phi_1''(r)^2 - f'(\phi_1)\phi_1'(r)^2\right] dr + \bar{c}\,\omega_d \phi_1'(1)^2 \\
&= -(d-1)\omega_d \int_0^1 r^{d-3} \phi_1'(r)^2 dr.
\end{aligned}
$$

If $d > 1$, we have already found a radial function ψ such that $Q(\psi) < 0$. In the case $d = 1$, $Q(\phi_1') = 0$, and indeed ϕ_1' is a solution of the linearized problem:

$$
\begin{cases}
\psi'' + f'(\phi_1)\psi = 0 & \text{in } [-1, 1], \\
\psi'(1) + \bar{c}\,\psi(1) = 0, & -\psi'(-1) + \bar{c}\,\psi(-1) = 0.
\end{cases}
$$

However this solution cannot correspond to the first eigenvalue γ_1 since ϕ_1' changes sign in $[-1, 1]$. As a consequence, γ_1 is negative.

We now show that $\gamma_1 < \gamma_{D1}$. From (3.3), (3.6) and (3.5), we have immediately $\gamma_1 \leq \gamma_{D_1}$. Assume, reasoning by contradiction, that $\gamma_1 = \gamma_{D_1}$. Hence the minimizer $z_1 \in H^1_{0,r}(B)$ works also for the minimizing problem defining γ_1. In particular, z_1 solves (3.7), and its boundary condition implies that $\partial_\nu z_1(x) = 0$ for $x \in \partial B$. Summing up, z_1 solves:

$$
\begin{cases}
\Delta z_1 + f'(\phi_1) z_1 + \gamma_1 z_1 = 0 & \text{in } B, \\
z_1(x) = 0 & \text{on } \partial B, \\
\partial_\nu z_1(x) = 0 & \text{on } \partial B.
\end{cases}
$$

But, by the uniqueness of the Cauchy problem for ODE we conclude that $z_1 = 0$, a contradiction. □

3.2 Eigenvalue estimates for related linear problems in the cylinder

As commented in the introduction, the construction of the Neumann-to-Dirichlet operator (which will be made in next section) can be performed only if the Dirichlet problem in the cylinder is not degenerate. The main purpose of this section is to study this question. We will show that we have nondegeneracy for all $T \in (0, \overline{T})$, for some specific value of $\overline{T}$. Hence the rest of the computations of the next sections will always require $T \in (0, \overline{T})$.

Let us consider the Dirichlet problem for the linearized equation in a straight cylinder for periodic functions, namely,

$$\begin{cases} \Delta\psi + f'(\phi_1)\psi = 0 & \text{in } B \times \mathbb{R}, \\ \psi(x) = 0 & \text{on } (\partial B) \times \mathbb{R}, \end{cases} \tag{3.9}$$

where $\psi(x,t)$ is T-periodic in the variable t. Define:

$$C_1^T = B \times \mathbb{R}/T\mathbb{Z}.$$

Hence (3.9) is just the linearization of the problem:

$$\begin{cases} \Delta\phi + f(\phi) = 0 & \text{in } C_1^T, \\ \phi = 0 & \text{on } \partial C_1^T. \end{cases} \tag{3.10}$$

We define the following Hölder spaces of radial functions:

$$C_r^{k,\alpha}(C_1^T) = \left\{\phi \in C^{k,\alpha}(C_1^T) : \phi(x,t) = \phi(|x|,t),\ (x,t) \in C_1^T\right\},$$

$$C_{0,r}^{k,\alpha}(C_1^T) = \left\{\phi \in C_0^{k,\alpha}(C_1^T) : \phi(x,t) = \phi(|x|,t),\ (x,t) \in C_1^T\right\}.$$

We also define the following Sobolev spaces:

$$H_r^1(C_1^T) = \left\{\phi \in H^1(C_1^T) : \phi(x,t) = \phi(|x|,t),\ (x,t) \in C_1^T\right\},$$

$$H_{0,r}^1(C_1^T) = \left\{\phi \in H_0^1(C_1^T) : \phi(x,t) = \phi(|x|,t),\ (x,t) \in C_1^T\right\}.$$

For functions in such spaces sometimes we will write $\phi(r)$ and $\phi(r,t)$ instead of respectively $\phi(x)$ and $\phi(x,t)$, with $r = |x|$. The reader will understand in each case if we refer to the variable x or r.

If ϕ_1 is the solution of the problem (1.8), then the function $\phi_1(x,t) = \phi_1(x)$ (we use a natural abuse of notation) solves (3.10). Define the linearized operator $L_D^T : C_{0,r}^{2,\alpha}(C_1^T) \to C_r^{\alpha}(C_1^T)$ (associated to the problem (3.10)) by

$$L_D^T(\phi) = \Delta\phi + f'(\phi_1)\phi,$$

and consider the eigenvalue problem

$$L_D^T(\phi) + \tau\phi = 0.$$

Then the functions $z_j(x,t) = z_j(x)$ from (3.2) solve the problem

$$\begin{cases} \Delta z_j + f'(\phi_1)z_j + \tau_j z_j = 0 & \text{in } C_1^T, \\ z_j = 0 & \text{on } \partial C_1^T. \end{cases} \tag{3.11}$$

Let us define the quadratic form $Q_D^T : H_{0,r}^1(C_1^T) \to \mathbb{R}$ related to L_D^T,

$$Q_D^T(\psi) := \int_{C_1^T} \left(|\nabla\psi|^2 - f'(\phi_1)\psi^2\right).$$

We will also need to study the quadratic form $Q^T : H^1_r(C_1^T) \to \mathbb{R}$,

$$Q^T(\psi) := \int_{C_1^T} \left(|\nabla \psi|^2 - f'(\phi_1)\psi^2 \right) + \bar{c} \int_{\partial C_1^T} \psi^2.$$

The main result of this section is next proposition, where we study the behavior of these quadratic forms:

Proposition 3.3. *Define:*

$$\alpha = \inf \left\{ Q_D^T(\psi) : \ \psi \in H^1_{0,r}(C_1^T), \ \|\psi\|_{L^2} = 1, \ \int_{C_1^T} \psi\, z_j = 0, \ j = 1, \dots l. \right\},$$

and

$$\beta = \inf \left\{ Q^T(\psi) : \ \psi \in H^1_r(C_1^T), \ \|\psi\|_{L^2} = 1, \ \int_{\partial C_1^T} \psi = 0, \ \int_{C_1^T} \psi\, z_j = 0, \ j = 1, \dots l. \right\}.$$

Then

$$\alpha = \min \left\{ \gamma_{D_{l+1}}, \gamma_{D_1} + \frac{4\pi^2}{T^2} \right\},$$

and

$$\beta = \min \left\{ \gamma_{D_{l+1}}, \gamma_1 + \frac{4\pi^2}{T^2} \right\}.$$

Moreover, those infima are achieved. If $\gamma_1 + \frac{4\pi^2}{T^2} < \gamma_{D_{l+1}}$, the minimizer is equal to

$$\psi_1(x) \cos\left(\frac{2\pi}{T}(t + \delta) \right),$$

where ψ_1 is the minimizer for (3.6) and $\delta \in [0,1]$.

Proof. We prove the result for β; the result for α is analogous. First, it is rather standard to show that β is achieved by a function ψ. By the Lagrange multiplier rule, there exist θ_1, θ_2 and $\zeta_1, \dots \zeta_l$ real numbers so that for any $\rho \in H^1_r(C_1^T)$,

$$\int_{C_1^T} \left(\nabla\psi \nabla\rho - f'(\phi_1)\psi\rho + \rho \sum_{i=1}^{l} \zeta_i z_i + \theta_1 \psi\rho \right) = \int_{\partial C_1^T} \rho(\theta_2 + \bar{c}\psi).$$

By choosing $\rho = z_j$ we conclude that $\zeta_j = 0$. If we now take $\rho = \psi$, we obtain that $\theta_1 = \beta$. Hence ψ is a solution of the equation

$$\Delta\psi + f'(\phi_1)\psi + \beta\psi = 0 \ \text{ in } C_1^T.$$

Define now:

$$\bar{\psi}(x) = \int_0^T \psi(x,t)\,dt\,.$$

It is immediate that

$$\int_B \bar{\psi}\, z_j = \int_{C_1^T} \psi\, z_j = 0,\ j = 1, \dots, l\,. \tag{3.12}$$

By direct computation

$$\begin{aligned}
\Delta_x \bar{\psi} &= \int_0^T \Delta_x \psi(x,t)dt \\
&= \int_0^T \Delta\psi(x,t)dt - \int_0^T \psi_{tt}(x,t)dt \\
&= \int_0^T \Delta\psi(x,t)dt - (\psi_t(x,T) - \psi_t(x,0)) \\
&= \int_0^T \Delta\psi(x,t)dt \\
&= -\int_0^T (f'(\phi_1) + \beta)\psi(x,t)dt \\
&= -(f'(\phi_1) + \beta)\bar{\psi}.
\end{aligned}$$

As a consequence, we have that $\bar{\psi}$ solves the problem

$$\begin{cases} \Delta\bar{\psi} + f'(\phi_1)\bar{\psi} + \beta\bar{\psi} = 0 & \text{in } B, \\ \bar{\psi} = 0 & \text{on } \partial B. \end{cases}$$

Taking into account (3.12), there are two cases: either $\beta = \gamma_{D_k}$, $k \geq l+1$, or $\bar{\psi} = 0$. In the first case, by plugging z_{l+1} in the definition of β, we conclude that $k = l+1$. In the second case we have,

$$\int_0^T \psi(x,t)dt = 0 \ \ \forall x \in B.$$

Hence we can use the Poincaré-Wirtinger inequality for periodic functions to estimate:

$$\frac{4\pi^2}{T^2} \int_0^T \psi^2 dt \leq \int_0^T \psi_t^2 dt\,.$$

Then, recalling (3.6),

$$\begin{aligned}
\beta = Q^T(\psi) &= \int_0^T \left(\int_B \left(|\nabla_x \psi|^2 - f'(\phi_1)\psi^2 \right) + \bar{c}\omega_d \psi(1,t)^2 \right) dt + \int_B \int_0^T |\psi_t|^2 dt \\
&\geq \gamma_1 \int_0^T dt \int_B \psi^2 + \frac{4\pi^2}{T^2} \int_B \int_0^T \psi^2 dt \\
&= \left(\gamma_1 + \frac{4\pi^2}{T^2} \right) \int_{C_1^T} \psi^2 = \gamma_1 + \frac{4\pi^2}{T^2}\,.
\end{aligned}$$

Moreover, the above inequalities are equalities only if $\psi(x,t)$ is proportional to $\psi_1(x) \cos\left(\frac{2\pi}{T}(t+\delta)\right)$. $\square$

As a consequence, we can state the following:

Corollary 3.4. *Define $\overline{T}$ as:*

$$\overline{T} = \begin{cases} \frac{2\pi}{\sqrt{-\gamma_{D_1}}} & \text{if } \gamma_{D_1} < 0\,, \\ +\infty & \text{if } \gamma_{D_1} > 0\,. \end{cases} \tag{3.13}$$

Then, for any $T \in (0, \overline{T})$, we have that $Q_D^T(\psi) > 0$ for any $\psi \in H^1_{0,r}(C_1^T)$ satisfying the orthogonality conditions:

$$\int_{C_1^T} \psi z_j = 0\,,\; j = 1, 2, \cdots, l\,.$$

As a consequence, L_D^T is nondegenerate for any $T \in (0, \overline{T})$.

3.3 Perturbations of the cylinder and formulation of the problem

The main purpose of this section is to build a nonlinear Dirichlet-to-Neumann operator G associated to (1.1) for any $T \in (0, \overline{T})$.

Given a positive number T and a $C^{2,\alpha}$ function $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ (i.e. periodic of period 1) with small $C^{2,\alpha}$-norm, we define:

$$C_{1+v}^T = \left\{ (x,t) \in \mathbb{R}^d \times \mathbb{R}/\mathbb{Z} \;:\; 0 \le |x| < 1 + v\left(\frac{t}{T}\right) \right\}.$$

Such a domain is in fact a small perturbation of the straight cylinder of radius 1, periodic in the vertical direction with period T. We look at the problem:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } C_{1+v}^T, \\ u > 0 & \text{in } C_{1+v}^T, \\ u = 0 & \text{on } \partial C_{1+v}^T, \\ \partial_\nu u = c & \text{on } \partial C_{1+v}^T. \end{cases} \tag{3.14}$$

Our aim will be to find a curve $(v, T) = (v(T), T)$, with $v \not\equiv 0$, such that problem (3.14) has a solution. We shall write it in the equivalent form:

$$\begin{cases} \Delta_\lambda \phi + f(\phi) = 0 & \text{in } C_{1+v}^1\,, \\ \phi > 0 & \text{in } C_{1+v}^1\,, \\ \phi = 0 & \text{on } \partial C_{1+v}^1, \\ |\nabla^\lambda \phi| = c & \text{on } \partial C_{1+v}^1\,, \end{cases}$$

where $\Delta_\lambda \phi = \Delta_x \phi + \lambda \phi_{tt}$, and $\nabla^\lambda \phi = (\nabla_x \phi, \sqrt{\lambda}\phi_t)$. Indeed, if we set $T = \frac{1}{\sqrt{\lambda}}$, we have that

$$u(x,t) = \phi\left(x, \frac{t}{T}\right) \tag{3.15}$$

is a solution of (3.14).

Since it is clear that (1.1) is invariant under translations, it is natural to require that the function v is even. Moreover, sometimes it will be useful to assume the function v has 0 mean. So, we introduce the Hölder spaces:

$$C_e^{k,\alpha}(\mathbb{R}/\mathbb{Z}) = \left\{ v \in C^{k,\alpha}(\mathbb{R}/\mathbb{Z}) : v(-t) = v(t) \right\},$$

$$C_{e,m}^{k,\alpha}(\mathbb{R}/\mathbb{Z}) = \left\{ v \in C^{k,\alpha}(\mathbb{R}/\mathbb{Z}) : v(-t) = v(t), \ \int_0^1 v \, dt = 0 \right\}$$

for $k \in \mathbb{N}$.

We start with the following result:

Proposition 3.5. *Assume that $\lambda > \frac{1}{\overline{T}^2}$, where $\overline{T}$ is given by (3.13). Then, for all $v \in C_e^{2,\alpha}(\mathbb{R}/\mathbb{Z})$ whose norm is sufficiently small, the problem*

$$\begin{cases} \Delta_\lambda \phi + f(\phi) = 0 & \text{in } C_{1+v}^1, \\ \phi = 0 & \text{on } \partial C_{1+v}^1, \end{cases} \tag{3.16}$$

has a unique positive solution $\phi = \phi_{v,\lambda} \in C^{2,\alpha}(C_{1+v}^1)$. Moreover, ϕ depends smoothly on the function v, and $\phi = \phi_1$ when $v \equiv 0$.

Proof. Let $v \in C_e^{2,\alpha}(\mathbb{R}/\mathbb{Z})$. It will be more convenient to consider the fixed domain C_1^1 endowed with a new metric depending on v. This will be possible by considering the parameterization of C_{1+v}^1 defined by

$$Y(y,t) := \Big(\big(1 + v(t)\big)y, t\Big). \tag{3.17}$$

Therefore, we consider the coordinates $(y,t) \in C_1^1$ from now on, and we can write the new metric in these coordinates as

$$g = \sum_i [1 + v(t)]^2 dy_i^2 + \sum_i [1 + v(t)] v'(t) y_i dy_i dt + [v'(t)^2 y^2 + 1] dt^2.$$

Up to some multiplicative constant, we can now write the problem (3.16) as

$$\begin{cases} \Delta_{\lambda,g} \hat{\phi} + f(\hat{\phi}) = 0 & \text{in } C_1^1, \\ \hat{\phi} = 0 & \text{on } \partial C_1^1, \end{cases} \tag{3.18}$$

where $\Delta_{\lambda,g}$ is the operator Δ_λ rewritten in the metric g. As $v \equiv 0$, the metric g is just the Euclidean metric, and $\hat{\phi} = \phi_1$ is therefore a solution of (3.18). In the general case, the expression between the function ϕ and the function $\hat{\phi}$ can be represented by

$$\hat{\phi} = Y^* \phi.$$

For all $\psi \in C^{2,\alpha}_{0,r}(C^1_1)$, we define:

$$N(v,\psi) := \Delta_{\lambda,g}(\phi_1 + \psi) + f(\phi_1 + \psi). \tag{3.19}$$

We have

$$N(0,0) = 0.$$

The mapping N is C^1 from a neighborhood of $(0,0)$ in $C^{2,\alpha}_e(\mathbb{R}/\mathbb{Z}) \times C^{2,\alpha}_{0,r}(C^1_1)$ into $C^\alpha_r(C^1_1)$. We point out that N could fail to be C^2 with respect to v, since the nonlinearity f is assumed only to be $C^{1,\alpha}$, but in any case it admits the double cross derivative $D_\lambda D_v$. The partial differential of N with respect to ψ at $(0,0)$ is

$$D_\psi N|_{(0,0)}(\psi) = \Delta_\lambda \psi + f'(\phi_1)\psi.$$

Via the change of variables $w(x,t) = \psi\left(x, \frac{t}{T}\right)$, we can use Corollary 3.4 to show that $D_\psi N|_{(0,0)}$ is invertible from $C^{2,\alpha}_{0,r}(C^1_1)$ into $C^\alpha_r(C^1_1)$. The Implicit Function Theorem therefore yields that there exists $\psi(v,\lambda) \in C^{2,\alpha}_{0,r}(C^1_1)$ such that $N(v, \psi(v,\lambda)) = 0$ for v in a neighborhood of 0 in $C^{2,\alpha}_e(\mathbb{R}/\mathbb{Z})$. The function $\hat{\phi} := \phi_1 + \psi$ solves (3.18), and moreover the dependence on λ is C^1. □

For any $\lambda > \frac{1}{\overline{T}^2}$ we define the nonlinear operator G as follows. After the canonical identification of ∂C^1_{1+v} with $\mathbb{S}^{d-1} \times \mathbb{R}/\mathbb{Z}$, we define the following operator G : $\mathcal{U} \times (\frac{1}{\overline{T}^2}, +\infty) \to C^{1,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$, where $\mathcal{U}$ is a neighborhood of 0 in $C^{1,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$, as:

$$G(v,\lambda)(t) = -|\nabla^\lambda \phi_{v,\lambda}|_{\partial C^1_{1+v}} + \frac{1}{\mathrm{Vol}(\partial C^1_{1+v})} \int_{\partial C^1_{1+v}} |\nabla^\lambda \phi_{v,\lambda}|,$$

where $\phi_{v,\lambda}$ is the solution of (3.16) verified by Proposition 3.5. Clearly G is a C^1 operator, and admits also the crossed derivative $D_\lambda D_v G$ since the operator N defined in (3.19) does.

Clearly, G admits the equivalent expression as the Dirichlet-to-Neumann operator:

$$G(v,T)(t) = \partial_\nu(u_{v,T})|_{\partial C^T_{1+v}}(T\,t) - \frac{1}{\mathrm{Vol}(\partial C^T_{1+v})} \int_{\partial C^T_{1+v}} \partial_\nu(u_{v,T}),$$

where $u_{v,T}$ is related to $\phi_{v,\lambda}$ via the formula (3.15). Notice that $G(v,T) = 0$ if and only if $\partial_\nu u$ is constant on the boundary ∂C^T_{1+v}. Obviously, $G(0,T) = 0$ for all $T < \overline{T}$. Our goal is to find a branch of nontrivial solutions (v,T) to the equation $G(v,T) = 0$ bifurcating from some point $(0,T_*)$, $T_* \in (0,\overline{T})$. To this aim, we will use a local bifurcation argument. This leads to the study of the linearization of G around a point $(0,T)$; this study is the purpose of the next section.

3.4 The linearization of the operator G

We will next compute the Fréchet derivative of the operator G. For that aim, we will need the following two lemmas.

Lemma 3.6. *Assume that $T < \overline{T}$, where $\overline{T}$ is given by (3.13). Then for all $v \in C^{2,\alpha}_e(\mathbb{R}/\mathbb{Z})$, there exists a unique solution $\psi_{v,T}$ to the problem*

$$\begin{cases} \Delta\psi_{v,T} + f'(\phi_1)\psi_{v,T} = 0 & \text{in } C_1^T, \\ \psi_{v,T} = \tilde{v} & \text{on } \partial C_1^T, \end{cases} \tag{3.20}$$

where $\tilde{v}(t) := v\left(\frac{t}{T}\right)$.

Proof. Let $\psi_0(x,t) \in C^{2,\alpha}(C_1^T)$ such that $(\psi_0)|_{\partial C_1^T} = \tilde{v}$. If we set $\omega = \psi_{v,T} - \psi_0$, the problem (3.20) is equivalent to the problem

$$\begin{cases} \Delta\omega + f'(\phi_1)\omega = -\Big(\Delta\psi_0 + f'(\phi_1)\psi_0\Big) & \text{in } C_1^T, \\ \omega = 0 & \text{on } \partial C_1^T. \end{cases}$$

Observe that the right hand side of the above equation is in $C^\alpha_r(C_1^T)$. Recall that by Corollary 3.4, L_D^T is nondegenerate. Hence it is a bijection and the result follows. $\square$

For the sake of clarity sometimes we will write ψ_v instead of $\psi_{v,T}$, when the dependence on T is not relevant.

In next lemma we give some orthogonality results on ψ_v defined in Lemma 3.6.

Lemma 3.7. *Let $v \in C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$ and $\psi_v \in C^{2,\alpha}_r(C_1^T)$ be the solution of (3.20). Then*

$$\int_{C_1^T} \psi_v z_j = 0, \qquad \int_{\partial C_1^T} \partial_\nu \psi_v = 0\,, \qquad j = 1, 2, \cdots, l\,.$$

Proof. We multiply the equation in (3.11) by ψ_v, the equation in (3.20) by z_j, and integrate by parts to gain

$$\int_{\partial C_1^T} \Big(\partial_\nu\psi_v\, z_j - \partial_\nu z_j\, \psi_v\Big) = \int_{C_1^T} \tau_j z_j \psi_v\,.$$

Then we can at once gain the first identity by the facts that $z_j = 0$, $\partial_\nu z_j$ is constant and $\psi_v = v(\cdot/T)$ has 0 mean on ∂C_1^T.

We now define $\kappa \in C^{2,\alpha}_r(C_1^T)$ as the unique solution of the problem

$$\begin{cases} \Delta\kappa + f'(\phi_1)\kappa = 0 & \text{in } C_1^T, \\ \kappa = 1 & \text{on } \partial C_1^T. \end{cases} \tag{3.21}$$

whose existence has been verified in Lemma 3.6 for $T < \overline{T}$. Then we multiply the equation in (3.21) by ψ_v, the equation in (3.20) by κ, and integrate by parts to obtain

$$\int_{\partial C_1^T} \Big(\partial_\nu \kappa \, \psi_v - \partial_\nu \psi_v \, \kappa \Big) = 0.$$

Then we can at once gain the second identity by the facts that $\kappa = 1, \partial_\nu \kappa$ is constant and $\psi_v(x,t) = v\left(\frac{t}{T}\right)$ on ∂C_1^T. □

For $T < \overline{T}$ we can define the linear continuous operator $H_T : C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z}) \to C^{1,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$ by

$$H_T(v)(t) = \partial_\nu(\psi_v)|_{\partial C_1^T}(Tt) + \bar{c}\, v,$$

where ψ_v is given in Lemma 3.6 and $\bar{c}$ is given in (3.4). We present some properties of H_T.

Lemma 3.8. *For any $T < \overline{T}$, the operator*

$$H_T : C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z}) \to C^{1,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$$

is a linear essentially self-adjoint operator and has closed range. Moreover, it is also a Fredholm operator of index zero.

Proof. Given $v_i \in C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$, we define $\tilde{v}_i(t) = v_i\left(\frac{t}{T}\right), i = 1, 2$. Let us compute:

$$\begin{aligned}
T\left(\int_0^1 H_T(v_1)v_2 - \int_0^1 H_T(v_2)v_1\right) &= \int_0^T (\partial_\nu \psi_{v_1}\tilde{v}_2 + \bar{c}\tilde{v}_1\tilde{v}_2) - \int_0^T (\partial_\nu \psi_{v_2}\tilde{v}_1 + \bar{c}\tilde{v}_2\tilde{v}_1) \\
&= \int_0^T (\partial_\nu \psi_{v_1}\tilde{v}_2 - \partial_\nu \psi_{v_2}\tilde{v}_1) \\
&= \int_0^T (\psi_{v_2}\partial_\nu \psi_{v_1} - \psi_{v_1}\partial_\nu \psi_{v_2}) \\
&= \frac{1}{\omega_d}\int_{C_1^T} (\psi_{v_2}\Delta\psi_{v_1} - \psi_{v_1}\Delta\psi_{v_2}) \\
&= \frac{1}{\omega_d}\int_{C_1^T} (f'(\phi_1)\psi_{v_2}\psi_{v_1} - f'(\phi_1)\psi_{v_1}\psi_{v_2}) \\
&= 0.
\end{aligned}$$

Therefore, we know that the operator H_T is self-adjoint. In addition, the first part of the operator H_T, the Dirichlet-to-Nenmann operator for $\Delta + f'(\phi_1)$, is lower bounded since 0 is not in the spectrum of $\Delta + f'(\phi_1)$ (see [4]). This yields that there is a constant $C > 0$ such that

$$\|v\|_{C^{2,\alpha}(\mathbb{R}/\mathbb{Z})} \le C\|H_T(v)\|_{C^{1,\alpha}(\mathbb{R}/\mathbb{Z})},$$

for all v that are $L^2(\mathbb{R}/\mathbb{Z})$-orthogonal to Ker (H_T). It follows that the range of H_T is closed. Therefore, H_T is a Fredholm operator of index zero (refer to [50]). □

We show now that the linearization of the operator G with respect to v at $v = 0$ is given by H_T, up to a constant.

Proposition 3.9. *For any $T \in (0, \overline{T})$,*

$$D_v(G)|_{v=0} = -\phi_1'(1)\, H_T\,.$$

Proof. By the C^1 regularity of G, it is enough to compute the linear operator obtained by the directional derivative of G with respect to v, computed at (v, T). Such derivative is given by

$$G'(w) = \lim_{s\to 0} \frac{G(sw,T) - G(0,T)}{s} = \lim_{s\to 0} \frac{G(sw,T)}{s}.$$

Let $v = sw$, for $y \in \mathbb{R}^d$ and $t \in \mathbb{R}$, we consider the parameterization of C^T_{1+v} given in (3.17). Let g be the induced metric such that $\hat{\phi} = Y^*\phi$ (smoothly depending on the real parameter s) solves the problem

$$\begin{cases} \Delta_g \hat{\phi} + f(\hat{\phi}) = 0 & \text{in } C^T_1, \\ \hat{\phi} = 0 & \text{on } \partial C^T_1. \end{cases}$$

Let $\tilde{w}(t) = w\left(\frac{t}{T}\right)$. We remark that $\hat{\phi}_1 = Y^*\phi_1$ is the solution of

$$\Delta_g \hat{\phi}_1 + f(\hat{\phi}_1) = 0$$

in C^T_1, and

$$\hat{\phi}_1(y,t) = \phi_1\big((1 + s\tilde{w})y, t\big)$$

on ∂C^T_1. Let $\hat{\phi} = \hat{\phi}_1 + \hat{\psi}$, we can get that

$$\begin{cases} \Delta_g \hat{\psi} + f(\hat{\phi}_1 + \hat{\psi}) - f(\hat{\phi}_1) = 0 & \text{in } C^T_1\,, \\ \hat{\psi} = -\hat{\phi}_1 & \text{on } \partial C^T_1. \end{cases} \tag{3.22}$$

Obviously, $\hat{\psi}$ is differentiable with respect to s. When $s = 0$, we have $\phi = \phi_1$. Then, $\hat{\psi} = 0$ and $\hat{\phi}_1 = \phi_1$ as $s = 0$. We set

$$\dot{\psi} = \partial_s \hat{\psi}|_{s=0}.$$

Differentiating (3.22) with respect of s and evaluating the result at $s = 0$, we have

$$\begin{cases} \Delta \dot{\psi} + f'(\phi_1)\dot{\psi} = 0 & \text{in } C^T_1, \\ \dot{\psi} = -\phi_1'(1)\tilde{w} & \text{on } \partial C^T_1. \end{cases}$$

Then $\dot{\psi} = -\phi_1'(1)\,\psi_w$ where ψ_w is as given by Lemma 3.6 (with $v = w$). Then, we can write

$$\hat{\phi}(x,t) = \hat{\phi}_1(x,t) - s\dot{\psi}(x,t) + \mathcal{O}(s^2).$$

In particular, in a neighborhood of ∂C_1^T we have

$$\begin{aligned}\hat{\phi}(y,t) &= \phi_1\big((1+s\tilde{w})y,t\big) - s\dot{\psi}(y,t) + \mathcal{O}(s^2)\\ &= \phi_1(y,t) + s\big(\tilde{w}r\partial_r\phi_1 - \dot{\psi}(y,t)\big) + \mathcal{O}(s^2).\end{aligned}$$

In order to complete the proof of the result, it is enough to calculate the normal derivation of the function $\hat{\phi}$ when the normal is calculated with respect to the metric g. By using cylindrical coordinates $(y,t) = (rz,t)$ where $r := |y| > 0$ and $z \in \mathbb{S}^{d-1}$, then the metric g can be expanded in C_1^T as

$$g = (1+s\tilde{w})^2 dr^2 + 2sr\tilde{w}'(1+s\tilde{w})drdt + \big(1+s^2r^2(\tilde{w}')^2\big)dt^2 + r^2(1+s\tilde{w})^2\mathring{h},$$

where $\mathring{h}$ is the metric on $\mathbb{S}^{d-1}$ induced by the Euclidean metric. It follows from this expression that the unit normal vector fields to ∂C_1^T for the metric g is given by

$$\hat{\nu} = \big((1+s\tilde{w})^{-1} + \mathcal{O}(s^2)\big)\partial_r + \mathcal{O}(s)\partial_t.$$

By this, we conclude that

$$g(\nabla\hat{\phi},\hat{\nu}) = \partial_r\phi_1 + s\big(\tilde{w}\partial_r^2\phi_1 - \partial_r\dot{\psi}\big) + \mathcal{O}(s^2)$$

on ∂C_1^T. From the fact that $\partial_r\phi_1$ is constant and the fact that the term $\tilde{w}\partial_r^2\phi_1 - \partial_r\dot{\psi}$ has mean 0 on ∂C_1^T we obtain

$$G'(w) = -\partial_r\dot{\psi}(Tt) + \phi_1''(1)\,w = \phi_1'(1)\,\partial_r\psi_w(Tt) + \phi_1''(1)\,w = -\phi_1'(1)\,H_T(w).$$

This concludes the proof of the result. □

3.5 Study of the linearized operator H_T

In view of Proposition 3.9, a bifurcation of the branch $(0,T)$ of solutions of the equation $G(v,T) = 0$ might appear only at points $(0,T_*)$ such that H_{T_*} becomes degenerate. This will be verified to be true for a precise value $T_* < \overline{T}$. Let us now define the quadratic form associated to H_T, namely:

$$J_T : C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z}) \to \mathbb{R},\ \ J_T(v) = \int_0^1 H_T(v)v.$$

We now study the first eigenvalue of the operator H_T as

$$\sigma(T) = \inf\left\{ J_T(v) : v \in C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})\ ,\quad \int_0^1 v^2 = 1\right\}.$$

Lemma 3.10. *For any $v \in C^{2,\alpha}_{e,m}(\mathbb{R}/\mathbb{Z})$,*

$$Q^T(\psi_v) = T\omega_d J_T(v)\,.$$

Proof. By the divergence formula, we have

$$\begin{aligned}
T\omega_d J_T(v) &= \int_{\partial C_1^T} \psi_v \partial_\nu \psi_v + \bar{c} \int_{\partial C_1^T} (\psi_v)^2 \\
&= \int_{C_1^T} \left(\nabla_x \psi_v \nabla_x \psi_v + \psi_v \Delta_x \psi_v\right) + \bar{c} \int_{\partial C_1^T} (\psi_v)^2 \\
&= \int_{C_1^T} \left(\nabla_x \psi_v \nabla_x \psi_v - \psi_v (\psi_v)_{tt} - f'(\phi_1)\psi_v\psi_v\right) + \bar{c} \int_{\partial C_1^T} (\psi_v)^2 \\
&= \int_{C_1^T} \left(\nabla \psi_v \nabla \psi_v - f'(\phi_1)\psi_v\psi_v\right) + \bar{c} \int_{\partial C_1^T} (\psi_v)^2 \\
&= Q^T(\psi_v).
\end{aligned}$$

□

Next lemma characterizes the eigenvalue $\sigma(T)$ in terms of the quadratic form Q^T.

Lemma 3.11. *For any $T < \overline{T}$, we have*

$$\sigma(T) = \min \left\{ Q^T(\psi) : \psi \in E, \ \int_{\partial C_1^T} \psi^2 = 1 \right\},$$

where

$$E = \left\{ \psi \in H_r^1(C_1^T) : \ \int_{\partial C_1^T} \psi = 0, \int_{C_1^T} \psi z_j = 0, \ j = 1, \dots, l \right\}.$$

Moreover, the infimum is attained.

Proof. Let us define

$$\mu := \inf \left\{ Q^T(\psi) : \psi \in E, \int_{\partial C_1^T} \psi^2 = 1 \right\} \in [-\infty, +\infty).$$

We first show that μ is achieved. On that purpose, take $\psi_n \in E$ such that $Q^T(\psi_n) \to \mu$.

We claim that ψ_n is bounded. Reasoning by contradiction, if $\|\psi_n\|_{H^1} \to +\infty$, we define $\xi_n = \|\psi_n\|_{H^1}^{-1} \psi_n$; we can suppose that up to a subsequence $\xi_n \rightharpoonup \xi_0$. Notice that $\int_{\partial C_1^T} \xi_n^2 \to 0$, which yields that $\xi_0 \in H_{0,r}^1(C_1^T)$. We also point out that

$$\int_{C_1^T} f'(\phi_1)\xi_n^2 \to \int_{C_1^T} f'(\phi_1)\xi_0^2 \,, \ \int_{C_1^T} \xi_0 z_j = 0, \ j = 1, \dots, l \,.$$

Let us consider the following two cases:

Case 1: $\xi_0 = 0$. In this case

$$Q^T(\psi_n) = \|\psi_n\|^2 \int_{C_1^T} \left(|\nabla \xi_n|^2 - f'(\phi_1)\xi_n^2\right) + \bar{c} \to +\infty \,,$$

which is impossible.
Case 2: $\xi_0 \neq 0$. In this case

$$\begin{aligned}\liminf_{n\to\infty} Q^T(\psi_n) &= \liminf_{n\to\infty} \|\psi_n\|^2 \int_{C_1^T} \left(|\nabla \xi_n|^2 - f'(\phi_1)\xi_n^2\right) + \bar{c} \\ &\geq \liminf_{n\to\infty} \|\psi_n\|^2 Q_D^T(\xi_0) + \bar{c}\,,\end{aligned}$$

but $Q_D^T(\xi_0) > 0$ by Proposition 3.3. This is again a contradiction.

Thus, ψ_n is bounded, so up to a subsequence we can pass to the weak limit $\psi_n \rightharpoonup \psi$. Then, ψ is a minimizer for μ and in particular $\mu > -\infty$.

By the Lagrange multiplier rule, there exist θ_1, θ_2 and $\zeta_1, \ldots, \zeta_l$ real numbers so that for any $\rho \in H_r^1(C_1^T)$,

$$\int_{C_1^T} \left(\nabla\psi\nabla\rho - f'(\phi_1)\psi\rho + \rho \sum_{i=1}^{l} \zeta_i z_i \right) = \int_{\partial C_1^T} \rho((\theta_1 + \bar{c})\psi + \theta_2).$$

Taking $\rho = z_j$ above we conclude that $\zeta_j = 0$. Moreover, if we take $\rho = \psi$ and $\rho = \kappa$ (given by (3.21)), we conclude that $\theta_1 + \bar{c} = \mu$ and $\theta_2 = 0$, respectively. In other words, ψ is a (weak) solution of

$$\begin{cases} \Delta\psi + f'(\phi_1)\psi = 0 & \text{in } C_1^T\,, \\ \partial_\nu \psi = \mu\psi & \text{on } \partial C_1^T\,. \end{cases}$$

By the regularity theory, $\psi \in C_r^{2,\alpha}(C_1^T)$. Define $v(t) = \psi|_{\partial C_1^T}(T\,t)$. Observe that:

$$\int_0^1 v^2 = \frac{1}{T\omega_d}, \;\; J_T(v) = \frac{1}{T\omega_d} Q^T(\psi) = \frac{1}{T\omega_d}\mu\,.$$

In the second equality Lemma 3.10 has been used. After a suitable renormalization we obtain that $\sigma(T) \leq \mu$. But, again by Lemma 3.10, the reversed inequality is trivially satisfied, and the proof is concluded. $\square$

We are now in conditions to prove the main result of this section:

Proposition 3.12. *Define $T_* = \frac{2\pi}{\sqrt{-\gamma_1}}$, where γ_1 is given in (3.6). Observe that by Lemma 3.2, T^* is well defined and $T_* \in (0, \overline{T})$. Then:*

(i) if $T \in (0, T_*)$, then $\sigma(T) > 0$;

(ii) if $T = T_*$, then $\sigma(T) = 0$;

(iii) if $T \in (T_*, \overline{T})$ then $\sigma(T) < 0$.

Moreover, $\mathrm{Ker}(H_{T_*}) = \mathbb{R}\cos(2\pi t)$. *In particular,* $\dim \mathrm{Ker}(H_{T_*}) = 1$.

Proof. It follows from Lemma 3.11 and Proposition 3.3, taking into account that $C_{e,m}^{2,\alpha}(\mathbb{R}/\mathbb{Z})$ contains only even functions. $\square$

3.6 The bifurcation argument

In this section, we are in position to prove our main Theorem 1.1 by the bifurcation argument. Theorem 1.1 follows immediately from the following proposition and the Crandall-Rabinowitz Theorem 1.7.

Proposition 3.13. *The linearized operator $D_vG(0,T_*)$ has a 1-dimensional kernel spanned by the function $w=\cos(2\pi t)$, that is,*

$$\text{Ker } D_vG(0,T_*)=\mathbb{R}w.$$

The cokernel of $D_vG(0,T_*)$ is also 1-dimensional, and

$$D_TD_vG(0,T_*)(w)\notin \text{Im } D_vG(0,T_*).$$

Proof. Recall from the Proposition 3.9, we know that $D_vG(0,T_*)=-\phi_1'(1)H_{T_*}$. Then we have

$$\text{Im } D_vG(0,T_*)=\text{Im } H_{T_*}.$$

By the Proposition 3.12, we have that the kernel of the linearized operator $D_vG(0,T_*)$ has dimension 1 and can be spanned by the function $w(t)=\cos(2\pi t)$:

$$\text{Ker } D_vG(0,T_*)=\mathbb{R}\,w.$$

Then, codim Im $(H_{T_*})=1$ follows from the fact that H_T is a Fredholm operator of index zero by Lemma 3.8.

Here, we are ready to prove $D_TD_vG(0,T_*)(w)\notin$ Im $D_vG(0,T_*)$. Taking $\xi\in$ Im $D_vG(0,T_*)=$ Im $(H_{T_*}),\xi=H_{T_*}(v)$, then we have

$$\int_0^1 \xi w=\int_0^1 H_{T_*}(v)w=\int_0^1 H_{T_*}(w)v=0,$$

because of the fact $H_{T_*}(w)=0$. By Lemma 3.8 we have

$$\text{Im } (H_{T_*})=\left\{\xi:\int_0^1 \xi w=0\right\}.$$

Recall that $D_TD_vG(0,T_*)(w)=-\phi_1'(1)D_T|_{T=T_*}H_T(w)$, then, in order to prove $D_TD_vG(0,T_*)(w)\notin$ Im $D_vG(0,T_*)$, we just need to prove that

$$\int_0^1 \big(D_T|_{T=T_*}H_T(w)\big)w\neq 0.$$

Actually, by Lemma 3.10,

$$\begin{aligned}
\int_0^1 \big(D_T|_{T=T_*} H_T(w)w\big) &= \frac{d}{dT}\Big|_{T=T_*} \int_0^1 H_T(w)w \\
&= \frac{1}{\omega_d}\frac{d}{dT}\Big|_{T=T_*}\left(\frac{1}{T}Q^T(\psi_w)\right) \\
&= \frac{1}{\omega_d}\frac{d}{dT}\Big|_{T=T_*}\left(\frac{1}{2}Q(\psi_1) + \frac{2\pi^2}{T^2}\int_B \psi_1^2\right) \\
&= -\frac{4\pi^2}{\omega_d T_*^3}\int_B \psi_1^2 \neq 0,
\end{aligned}$$

where the passage from the second to third equality is given by the following computation of $Q^T(\psi)$ with the function $\psi_w(x,t) = \psi_1(x)\cos\left(\frac{2\pi t}{T}\right)$, with ψ_1 as in (3.7):

$$\begin{aligned}
Q^T(\psi) &= Q_D^T(\psi) + \bar{c}\int_{\partial C_1^T} \psi^2 \\
&= \int_{C_1^T} \big(|\nabla\psi|^2 - f'(\phi_1)\psi^2\big) + \bar{c}\int_{\partial C_1^T}\psi^2 \\
&= \int_{C_1^T}\left[|\nabla\psi_1|^2\cos^2\left(\frac{2\pi t}{T}\right) + \psi_1^2\left(\frac{2\pi}{T}\right)^2\sin^2\left(\frac{2\pi t}{T}\right) - f'(\phi_1)\psi_1^2\cos^2\left(\frac{2\pi t}{T}\right)\right] \\
&\quad + \bar{c}\int_{\partial C_1^T}\psi_1^2\cos^2\left(\frac{2\pi t}{T}\right) \\
&= \left[\int_B \big(|\nabla\psi_1|^2 - f'(\phi_1)\psi_1^2\big) + \bar{c}\,\omega_d\psi_1^2(1)\right]\int_0^T \cos^2\left(\frac{2\pi t}{T}\right) \\
&\quad + \left(\frac{2\pi}{T}\right)^2\int_B \psi_1^2\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) \\
&= Q(\psi_1)\int_0^T \cos^2\left(\frac{2\pi t}{T}\right) + \left(\frac{2\pi}{T}\right)^2\int_B\psi_1^2\int_0^T\sin^2\left(\frac{2\pi t}{T}\right) \\
&= \frac{T}{2}Q(\psi_1) + \frac{2\pi^2}{T}\int_B\psi_1^2.
\end{aligned}$$

□

Chapter 4

Exceptional domains in $\mathbb{R}^d$

In this chapter, we prove the existence of exceptional domains, in which the overdetermined problem (1.9) admits a solution. The pull-back problem corresponding to the problem (1.9) is shown first such that we just need to study the equivalent problem on a fixed domain. In order to do this, we need to analyze the pull-back operator in weighted Hölder spaces because our analysis strongly relies on the decay assumptions. Next, we reformulate our problem as a nonlinear operator equation and prove the existence of a unique solution to this problem under such a functional analytic setting. After this, the computation of the linearised operator and the study of its spectral properties are presented. Based on all these results, the proof of Theorem 1.2 is then completed by making use of the Crandall-Rabinowitz bifurcation theorem. We end this chapter with an appendix where we collect some useful scale-invariant Hölder estimates for solutions of the Poisson equation and properties of modified Bessel functions.

4.1 The pull-back problem

We begin by fixing some notations. For $k \in \mathbb{N} \cup \{0\}$, we let

$$C^{k,\alpha}_{p,e}(\mathbb{R}) := \left\{ u \in C^{k,\alpha}(\mathbb{R}) : \quad u \text{ is } 2\pi\text{-periodic and even} \right\}.$$

Moreover, we consider the open set

$$\mathcal{U} := \{\varphi \in C^{2,\alpha}_{p,e}(\mathbb{R}) : \|\varphi\|_\infty < 1\}.$$

Recalling our problem, we are looking for a number $T > 0$ and a 2π-periodic positive function $\varphi : \mathbb{R} \to (0, \infty)$ of class $C^{2,\alpha}$ such that the overdetermined problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega_{T,\varphi}, \\ u = 1 & \text{on} \quad \partial\Omega_{T,\varphi}, \\ \lim_{|z|\to\infty} u(z,t) = 0 & \text{uniformly in } t \in \mathbb{R}, \\ \dfrac{\partial u}{\partial \nu} = d-3 & \text{on} \quad \partial\Omega_{T,\varphi}, \end{cases} \tag{4.1}$$

is solvable in the perturbed domain $\Omega_{T,\varphi}$ defined in (1.10).

In order to find a suitable variational framework for this problem, it is convenient to pull-back (4.1) on the fixed domain $\Omega_1 := B_1^c \times \mathbb{R}$ via a suitable diffeomorphism of the form

$$\Omega_1 \to \Omega_{T,\varphi}, \qquad (y,\tau) \mapsto \Big((1+\varphi(\tau)|y|^s)y, \frac{T}{2\pi}\tau\Big). \tag{4.2}$$

Note that, since the function $r \mapsto (1+cr^s)r$ is strictly increasing on $(1,\infty)$ for $|c| \le 1$ and $0 \ge s \ge -2$, it is easy to see that (4.2) defines a diffeomorphism if $T > 0$, $0 \ge s \ge -2$ and $\|\varphi\|_\infty \le 1$. It will turn out to be important in our functional analytic framework to minimize the effect of φ for large values of $|y|$, which leads us to choose $s = -2$ in the following. Hence we consider, for $T > 0$ and a 2π-periodic positive function $\varphi \in C^{2,\alpha}(\mathbb{R})$ with $\|\varphi\|_\infty < 1$, the diffeomorphism

$$\Psi_{T,\varphi} : \Omega_1 \to \Omega_{T,\varphi}, \quad (y,\tau) \mapsto \left(\kappa(|y|^2, \varphi(\tau))y, \frac{T}{2\pi}\tau\right) \tag{4.3}$$

with

$$\kappa(a,b) = 1 + \frac{b}{a}, \qquad \text{for } a \ge 1 \text{ and } |b| \le 1\ .$$

Furthermore,

$$z = \kappa(|y|^2, b)y \Longleftrightarrow y = \zeta(|z|^2, b)z, \tag{4.4}$$

where ζ is the unique function given by

$$\zeta(a,b) = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{b}{a}}, \qquad \text{for } |b| \le 1 \text{ and } a \ge 4b. \tag{4.5}$$

In order to pull back the problem (4.1) in Ω_1, we consider the ansatz

$$u(z,t) = w\left(\zeta\left(|z|^2, \varphi\left(\frac{2\pi}{T}t\right)\right)z, \frac{2\pi}{T}t\right) = w(y,\tau) \tag{4.6}$$

for some functions $w : \Omega_1 \to \mathbb{R}$, and we look for the operator $L_{T,\varphi}$ such that

$$L_{T,\varphi}w(y,\tau) = \Delta u(z,t) \qquad \text{for } (z,t) \in \Omega_{T,\varphi}, \tag{4.7}$$

where $\tau = \frac{2\pi}{T}t$. To write down the operator $L_{T,\varphi}$ explicitly, we need the partial derivatives $\zeta_i = \partial_i \zeta$, $\zeta_{ii} = \partial_{ii}\zeta$, $i = 1, 2$ of the function ζ in (4.5), which are given as follows for $|b| < 1$ and $a > 4b$:

$$\begin{cases} \zeta_1(a,b) = \dfrac{b}{2a^2(\zeta(a,b) - \frac{1}{2})}, \\ \zeta_{11}(a,b) = -\dfrac{b}{a^3(\zeta(a,b) - \frac{1}{2})}\left(1 + \dfrac{b}{4a(\zeta(a,b) - \frac{1}{2})^2}\right), \\ \zeta_2(a,b) = -\dfrac{1}{2a(\zeta(a,b) - \frac{1}{2})}, \\ \zeta_{22}(a,b) = -\dfrac{1}{4a^2(\zeta(a,b) - \frac{1}{2})^3}, \end{cases}$$

Lemma 4.1. *For every $T > 0$ and $\varphi \in C^{2,\alpha}(\mathbb{R})$, the operator $L_{T,\varphi}$ in (4.7) is given by*

$$\begin{aligned} L_{T,\varphi} w = \zeta^2 \Delta_y w &+ \left(\frac{2\pi}{T}\right)^2 \frac{\partial^2 w}{\partial \tau^2} + \frac{4}{\zeta^2}\left(\zeta_1\left(\zeta + |y|^2 \frac{\zeta_1}{\zeta^2}\right) + \left(\frac{2\pi}{T}\right)^2 \zeta_2^2 \varphi'^2\right) \sum_{k,\ell=1}^{d-1} y_\ell y_k \frac{\partial^2 w}{\partial y_\ell \partial y_k} \\ &+ \left(2(N+1)\frac{\zeta_1}{\zeta} + 4\frac{\zeta_{11}}{\zeta^3}|y|^2 + \left(\frac{2\pi}{T}\right)^2 \left(\frac{\varphi''\zeta_2 + \varphi'^2 \zeta_{22}}{\zeta}\right)\right) y \cdot \nabla_y w \\ &+ \left(\frac{2\pi}{T}\right)^2 \frac{\zeta_2 \varphi'}{\zeta} \sum_{\ell=1}^{d-1} y_\ell \frac{\partial^2 w}{\partial y_\ell \partial \tau}, \end{aligned}$$

where, for abbreviation, we merely write ζ in place of the function

$$(y,\tau) \mapsto \zeta(|\kappa(|y|^2, \varphi(\tau))y|^2, \varphi(\tau)) \tag{4.8}$$

and similarly for the partial derivatives ζ_i, ζ_{ii}, $i = 1, 2$.

Proof. We set

$$y_\ell(z,t) := \zeta\left(|z|^2, \varphi\left(\frac{2\pi t}{T}\right)\right) z_\ell, \quad \ell = 1, \cdots, d-1.$$

Then, on $\Omega_{T,\varphi}$, we find after computation,

$$\begin{aligned} \frac{\partial y_\ell}{\partial z_i} &= 2z_i \zeta_1 z_\ell + \zeta \delta_{i\ell}, \\ \frac{\partial^2 y_\ell}{\partial z_i^2} &= 2\frac{\zeta_1}{\zeta} y_\ell + 4y_i^2 \frac{\zeta_{11}}{\zeta^3} y_\ell + 4\frac{\zeta_1}{\zeta}\delta_{i\ell} y_i, \end{aligned} \tag{4.9}$$

where, here and in the following, we simply write ζ in place of the function

$$(z,t) \mapsto \zeta\left(|z|^2, \varphi\left(\frac{2\pi t}{T}\right)\right) \tag{4.10}$$

and similarly for ζ_i, ζ_{ii}, $i = 1, 2$. We also have

$$\frac{\partial}{\partial t}\Big(\frac{\zeta_2(\tau)}{\zeta(\tau)}\Big) = \Big(\frac{2\pi}{T}\Big)\frac{\partial}{\partial \tau}\Big(\frac{\zeta_2(\tau)}{\zeta(\tau)}\Big) = \Big(\frac{2\pi}{T}\Big)\frac{\varphi'}{\zeta}\Big(\zeta_{22} - \frac{\zeta_2^2}{\zeta}\Big),$$

and hence

$$\begin{aligned}
\frac{\partial y_\ell}{\partial t} &= \frac{2\pi}{T}\varphi'\zeta_2 z_\ell = \frac{2\pi}{T}\varphi'\frac{\zeta_2}{\zeta}y_\ell, \\
\frac{\partial^2 y_\ell}{\partial t^2} &= \Big(\frac{2\pi}{T}\Big)^2\Big(\varphi''\frac{\zeta_2}{\zeta} + \varphi'\frac{\partial}{\partial\tau}\Big(\frac{\zeta_2(\tau)}{\zeta(\tau)}\Big) + \varphi'^2\frac{\zeta_2^2}{\zeta^2}\Big)y_\ell \\
&= \Big(\frac{2\pi}{T}\Big)^2\Big(\frac{\varphi''\zeta_2 + \varphi'^2\zeta_{22}}{\zeta}\Big)y_\ell. \qquad (4.11)
\end{aligned}$$

Next we compute

$$\begin{aligned}
\frac{\partial u}{\partial z_i} &= \sum_{\ell=1}^{d-1}\frac{\partial y_\ell}{\partial z_i}\left(\frac{\partial w}{\partial y_\ell}\circ\Psi_{T,\varphi}^{-1}\right), \\
\frac{\partial^2 u}{\partial z_i^2} &= \sum_{\ell=1}^{d-1}\frac{\partial^2 y_\ell}{\partial z_i^2}\left(\frac{\partial w}{\partial y_\ell}\circ\Psi_{T,\varphi}^{-1}\right) + \sum_{k,\ell=1}^{d-1}\frac{\partial y_\ell}{\partial z_i}\frac{\partial y_k}{\partial z_i}\left(\frac{\partial^2 w}{\partial y_\ell\partial y_k}\circ\Psi_{T,\varphi}^{-1}\right) =: (A) + (B).
\end{aligned}$$

Using (4.9), we find

$$\begin{aligned}
(A) :&= \Big(2\frac{\zeta_1}{\zeta} + 4\frac{\zeta_{11}}{\zeta^3}y_i^2\Big)y\cdot\Big(\nabla_y w\circ\Psi_{T,\varphi}^{-1}\Big) + 4\frac{\zeta_1}{\zeta}y_i\Big(\frac{\partial w}{\partial y_i}\circ\Psi_{T,\varphi}^{-1}\Big), \\
(B) :&= \zeta^2\Big(\frac{\partial^2 w}{\partial y_i^2}\circ\Psi_{T,\varphi}^{-1}\Big) + 4\frac{\zeta_1}{\zeta}\sum_{k=1}^{d-1}y_i y_k\Big(\frac{\partial^2 w}{\partial y_i\partial y_k}\circ\Psi_{T,\varphi}^{-1}\Big) \\
&\quad + 4y_i^2\frac{\zeta_1^2}{\zeta^4}\sum_{k,\ell=1}^{d-1}y_\ell y_k\Big(\frac{\partial^2 w}{\partial y_\ell\partial y_k}\circ\Psi_{T,\varphi}^{-1}\Big).
\end{aligned}$$

In addition,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{\ell=1}^{d-1}\frac{\partial y_\ell}{\partial t}\Big(\frac{\partial w}{\partial y_\ell}\circ\Psi_{T,\varphi}^{-1}\Big) + \frac{2\pi}{T}\Big(\frac{\partial w}{\partial\tau}\circ\Psi_{T,\varphi}^{-1}\Big), \\
\frac{\partial^2 u}{\partial t^2} &= \sum_{\ell=1}^{d-1}\frac{\partial^2 y_\ell}{\partial t^2}\Big(\frac{\partial w}{\partial y_\ell}\circ\Psi_{T,\varphi}^{-1}\Big) + \sum_{k,\ell=1}^{d-1}\frac{\partial y_\ell}{\partial t}\frac{\partial y_k}{\partial t}\Big(\frac{\partial^2 w}{\partial y_\ell\partial y_k}\circ\Psi_{T,\varphi}^{-1}\Big) \\
&\quad + \frac{2\pi}{T}\sum_{\ell=1}^{d-1}\frac{\partial y_\ell}{\partial t}\Big(\frac{\partial^2 w}{\partial\tau\partial y_\ell}\circ\Psi_{T,\varphi}^{-1}\Big) + \left(\frac{2\pi}{T}\right)^2\Big(\frac{\partial^2 w}{\partial\tau^2}\circ\Psi_{T,\varphi}^{-1}\Big) \\
&=: (I) + (J) + (K).
\end{aligned}$$

We now use (4.11) and find

$$
\begin{aligned}
(I) &= \Big(\frac{2\pi}{T}\Big)^2 \Big(\frac{\varphi''\zeta_2 + \varphi'^2\zeta_{22}}{\zeta}\Big) y \cdot \Big(\nabla_y w \circ \Psi_{T,\varphi}^{-1}\Big),\\
(J) &= \Big(\frac{2\pi}{T}\Big)^2 \frac{\zeta_2^2 \varphi'^2}{\zeta^2} \sum_{k,\ell=1}^{d-1} y_k y_\ell \Big(\frac{\partial^2 w}{\partial y_k y_\ell} \circ \Psi_{T,\varphi}^{-1}\Big),\\
(K) &= \frac{2\pi}{T} \sum_{\ell=1}^{d-1} \frac{\partial y_\ell}{\partial t} \Big(\frac{\partial^2 w}{\partial \tau \partial y_\ell} \circ \Psi_{T,\varphi}^{-1}\Big) + \left(\frac{2\pi}{T}\right)^2 \left(\frac{\partial^2 w}{\partial \tau^2} \circ \Psi_{T,\varphi}^{-1}\right)\\
&= \left(\frac{2\pi}{T}\right)^2 \left[\frac{\zeta_2 \varphi'}{\zeta} \sum_{\ell=1}^{d-1} y_\ell \Big(\frac{\partial^2 w}{\partial y_\ell \partial \tau} \circ \Psi_{T,\varphi}^{-1}\Big) + \left(\frac{\partial^2 w}{\partial \tau^2} \circ \Psi_{T,\varphi}^{-1}\right)\right].
\end{aligned}
$$

Collecting these identities, which we have derived on the domain $\Omega_{T,\varphi}$ in the variables (z,t), and passing to the variables $(y,\tau) \in \Omega_1$, we obtain the claim. Note here that we have to write $(z,t) = \big(\kappa(|y|^2, \varphi(\tau))y, \frac{T}{2\pi}\tau\big)$ to pass from (4.10) to (4.8) and similarly for the partial derivatives ζ_i, ζ_{ii}, $i = 1,2$. $\square$

We now use Lemma 4.1 and rephrase the original problem (4.1) and the fixed domain $B_1^c \times \mathbb{R}$.

We recall that the boundary $\partial\Omega_{T,\varphi}$ of $\Omega_{T,\varphi}$ is given by

$$
\partial\Omega_{T,\varphi} = \left\{ \left((1+\varphi(\tau))\sigma, \frac{T}{2\pi}\tau \right) : \sigma \in \mathbb{S}^{d-2}, t \in \mathbb{R} \right\} \tag{4.12}
$$

and its outer normal vector field is given by

$$
\Upsilon\left((1+\varphi(\tau))\sigma, \frac{T}{2\pi}\tau \right) = \frac{\Big(-\sigma, 2\pi\varphi'(\tau)/T\Big)}{\sqrt{1 + \left(\frac{2\pi}{T}\right)^2 \varphi'^2(\tau)}}. \tag{4.13}
$$

Let the metric $g_{T,\varphi}$ be defined as the pull back of the euclidean metric g_{eucl} under the map $\Psi_{T,\varphi}$, so that $\Psi_{T,\varphi} : (\overline{\Omega}_1, g_{T,\varphi}) \to (\overline{\Omega_{T,\varphi}}, g_{eucl})$ is an isometry. Denote by

$$
\nu_{T,\varphi} : \partial\Omega_1 \to \mathbb{R}^d
$$

the unit outer normal vector field on $\partial\Omega_1$ with respect to $g_{T,\varphi}$. Since $\Psi_{T,\varphi} : (\overline{\Omega}_1, g_{T,\varphi}) \to (\overline{\Omega_{T,\varphi}}, g_{eucl})$ is an isometry, we have

$$
\nu_{T,\varphi} = [d\Psi_{T,\varphi}]^{-1} \Upsilon \circ \Psi_{T,\varphi} \qquad \text{on } \partial\Omega_1. \tag{4.14}
$$

Moreover, by (4.14) we have $\Upsilon(\Psi_\varphi(y,\tau)) = d\Psi_\varphi(y,\tau)\nu_\varphi(y,\tau)$ and therefore

$$
\begin{aligned}
\partial_{\nu_{T,\varphi}} w(y,\tau) &= dw(y,\tau)\nu_\varphi(y,\tau) = du(\Psi_\varphi(y,\tau)) d\Psi_\varphi(y,\tau)\nu_{T,\varphi}(y,\tau)\\
&= du(\Psi_\varphi(y,\tau))\Upsilon(\Psi_\varphi(y,\tau)) = \langle \Upsilon(\Psi_\varphi(y,\tau)), (\nabla_{(z,t)} u)(\Psi_\varphi(y,\tau))\rangle_{g_{eucl}}.
\end{aligned}
$$

That is

$$\partial_{\nu_{T,\varphi}} w(y,\tau) = \langle \Upsilon(\Psi_\varphi(y,\tau)), (\nabla_{(z,t)} u)(\Psi_\varphi(y,\tau))\rangle_{g_{eucl}}, \tag{4.15}$$

where u and w are related by (4.6).

Taking into account (4.12), our aim is to show that for some values of the parameter $T > 0$ and $\varphi > -1$, we can find a solution w to the overdetermined boundary value problem

$$\begin{cases} L_{T,\varphi} w = 0 & \text{in} \quad B_1^c \times \mathbb{R}, \\ w = 1 & \text{on} \quad \partial(B_1^c \times \mathbb{R}), \\ \lim_{|y|\to\infty} w(y,\tau) = 0 & \text{uniformly in } \tau \in \mathbb{R}. \end{cases} \tag{4.16}$$

and

$$\frac{\partial w}{\partial \nu_{T,\varphi}} = d - 3 \quad \text{on} \quad \partial(B_1^c \times \mathbb{R}). \tag{4.17}$$

We start proving the existence of a solution to the problem (4.16) by analyzing the operator $L_{T,\varphi}$ in weighted Hölder spaces.

4.2 Analysis of the operator $L_{T,\varphi}$ on weighted Hölder spaces

Our setting in analysing the operator $L_{T,\varphi}$ is that of the weighted Hölder spaces introduced by Pacard and Rivière [62]. We emphasise that Pacard and Rivière performed their analysis on $B_1\backslash\{0\}$ whereas in contrast we wish to carry our study on the open set $\Omega_1 = B_1^c \times \mathbb{R}$. More generally, we consider, for $r > 0$, the sets

$$\Omega_r := \Big\{(y,\tau) \in \mathbb{R}^{d-1} \times \mathbb{R} : |y| > r\Big\} \subset \mathbb{R}^d,$$

and we set $\Omega_0 = \mathbb{R}^d$. Let $\alpha \in (0,1)$ be fixed in the following.

Definition 4.2. *Let $\mu < 0$, $k \in \mathbb{N}$.*

(i) For a set $K \subset \mathbb{R}^d$ and a function $v \in C^{0,\alpha}(K)$ we put

$$[v]_{C^{0,\alpha}(K)} := \sup_{z,z'\in K} \frac{|v(z) - v(z')|}{|z - z'|^\alpha}$$

and

$$\|v\|_{C^{0,\alpha}(K)} := \|v\|_{L^\infty(K)} + [v]_{C^{0,\alpha}(K)}.$$

(ii) *We say* $u \in C^{k,\alpha}_{\mu}(\overline{\Omega}_r)$ *if* $u \in C^{k,\alpha}_{loc}(\overline{\Omega}_r)$ *and*

$$\|u\|_{k,\alpha,\mu} = \sup_{s>r} \left(s^{-\mu}[u]_{k,\alpha,s} \right) < \infty,$$

where $A_s = \{(x,t) \in \mathbb{R}^{d-1} \times \mathbb{R} : \quad s \leqslant |x| \leqslant 2s\}$ *for* $s > 0$ *and*

$$[u]_{k,\alpha,s} := \sum_{i=0}^{k} s^i \|\nabla^i u\|_{L^\infty(A_s)} + s^{k+\alpha}[\nabla^k u]_{C^\alpha(A_s)}.$$

(iii) *We also define the following function spaces:*

$$\begin{aligned} C^{k,\alpha}_{\mu,\mathcal{D}}(\Omega_r) &:= \left\{ u \in C^{k,\alpha}_{\mu}(\Omega_r) : \ u|_{\partial\Omega_r} = 0 \right\}, \\ C^{k,\alpha}_{\mu,p,e}(\Omega_r) &:= \left\{ u \in C^{k,\alpha}_{\mu}(\Omega_r) : \ u \text{ is } 2\pi\text{-periodic and even in the coordinate } t \right\}, \\ C^{k,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_r) &:= \left\{ u \in C^{k,\alpha}_{\mu,\mathcal{D}}(\Omega_r) : \ u \text{ is } 2\pi\text{-periodic and even in the coordinate } t \right\}. \end{aligned}$$

Remark 4.3. *Let* $r \geq 0$ *and* $u \in C^{k,\alpha}_{\mu}(\Omega_r)$. *By definition, we then have*

$$\sup_{A_s} |u| \leqslant [u]_{k,\alpha,s} \leqslant s^{\mu}\|u\|_{k,\alpha,\mu} \qquad \text{for all } s > r$$

and therefore, in particular,

$$|u(y,\tau)| \leqslant |y|^{\mu}\|u\|_{k,\alpha,\mu} \qquad \text{for all } (y,\tau) \in \Omega_r. \tag{4.18}$$

Consequently,

$$|u(y,\tau)| \to 0 \qquad \text{as } |y| \to \infty \text{ uniformly in } \tau \in \mathbb{R} \text{ if } \mu < 0. \tag{4.19}$$

It can be deduced from the specific form of the operator $L_{T,\varphi}$ given in Lemma 4.1 that $L_{T,\varphi}$ maps $C^{2,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1)$ to $C^{0,\alpha}_{\mu-2,p,e}(\Omega_1)$. The following is the main result of this section.

Theorem 4.4. *Let* $3 - d < \mu < 0$. *Then we have a smooth map*

$$(0,+\infty) \times \mathcal{U} \to \mathcal{L}(C^{2,\alpha}_{\mu,p,e}(\Omega_1), C^{0,\alpha}_{\mu-2,p,e}(\Omega_1)), \qquad (T,\varphi) \to L_{T,\varphi}. \tag{4.20}$$

Moreover, there exists an open neighborhood $\mathcal{O} \subset (0,+\infty) \times \mathcal{U}$ *of* $(0,\infty) \times \{0\}$ *with the property that*

$$L^D_{T,\varphi} := L_{T,\varphi}\Big|_{C^{k,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1)} \in \mathcal{I}\left(C^{2,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1), C^{0,\alpha}_{\mu-2,p,e}(\Omega_1)\right) \quad \text{for } (T,\varphi) \in \mathcal{O}. \tag{4.21}$$

Here and in the following, for two Banach spaces X and Y, we let $\mathcal{L}(X,Y)$ denote the space of bounded and linear operators from X to Y, and $\mathcal{I}(X,Y)$ the subset of topological isomorphisms $X \to Y$.

The main ingredient in the proof of Theorem 4.4 is the following proposition.

Proposition 4.5. *Let $3-d<\mu<0$ and $T>0$. Then the operator*

$$L^D_{T,0}=\Delta_y+\left(\frac{2\pi}{T}\right)^2\frac{\partial^2}{\partial\tau^2}:\ C^{2,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1)\to C^{0,\alpha}_{\mu-2,p,e}(\Omega_1)$$

is a topological isomorphism.

Let us postpone the proof of Proposition 4.5 for a moment and first quickly finish the proof of Theorem 4.4. Since

$$\zeta_1(a,b)=O(a^{-2}),\ \zeta_{11}(a,b)=O(a^{-3}),\ \zeta_2(a,b)=O(a^{-1})\text{ and }\zeta_{11}(a,b)=O(a^{-2})$$

as $a\to+\infty$, it follows by a straightforward computation from Lemma 4.1 that (4.20) defines a smooth map. Moreover, since, for any Banach spaces X,Y, the set $\mathcal{I}(X,Y)$ is open in $\mathcal{L}(X,Y)$, it follows directly from Proposition 4.5 and the continuity of the map $(T,\varphi)\to L_{T,\varphi}$ that there exists an open neighborhood $\mathcal{O}\subset(0,+\infty)\times\mathcal{U}$ of $(0,\infty)\times\{0\}$ with the property that (4.21) holds.

So the proof of Theorem 4.4 will be completed by proving Proposition 4.5, and this will be done in the remainder of this section. Without loss of generality, we may restrict our attention to the special case $T=2\pi$, in which we have

$$L_{2\pi,0}=\Delta$$

is merely the Laplace operator in the variables $(y,\tau)\in\mathbb{R}^d$. The general case will then follow by rescaling the τ-variable. This will change the period length in the spaces $C^{2,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1)$ and $C^{0,\alpha}_{\mu-2,p,e}(\Omega_1)$ but does not require further changes as the arguments below do not depend on the period length.

We first note the following.

Lemma 4.6. *Let $\mu<0$. Then the operator*

$$\Delta:\ C^{2,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1)\to C^{0,\alpha}_{\mu-2,p,e}(\Omega_1)$$

is injective.

Proof. Let $w\in C^{2,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1)$. Then $w=0$ on $\partial\Omega_1$ and $w(y,\tau)\to 0$ as $|y|\to\infty$ uniformly in τ by (4.19). Since w is continuous and also periodic in the τ-variable, w attains its maximum and minimum on $\overline{\Omega}_1$. Moreover, if $\Delta w=0$, then neither the maximum nor the mimimum can be attained in Ω_1 unless w is constant. In any case, we therefore conclude that $\Delta w=0$ implies $w=0$, and thus the the lemma is proved. □

As a consequence of the open mapping theorem, the proof of Proposition 4.5 is completed once we have shown that

$$\Delta:\ C^{2,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1)\to C^{0,\alpha}_{\mu-2,p,e}(\Omega_1)\quad\text{is surjective for }3-d<\mu<0.$$

To prove this, we let $3-d<\mu<0$ and $f \in C^{0,\alpha}_{\mu-2,p,e}(\Omega_1)$ be fixed in the following. We are looking for a function $w \in C^{2,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1)$ such that $\Delta w = f$. We shall find this function in the form $w = w_1 - w_2$, where

$$w_1 := \varphi * \tilde{f} : \overline{\Omega}_1 \to \mathbb{R},$$

where $\tilde{f} \in C^{0,\alpha}_{\mu-2,p,e}(\mathbb{R}^d)$ is an arbitrary τ-periodic, even and Hölder continuous extension of f to $\mathbb{R}^d$, and w_2 is a τ-periodic and even solution of

$$\Delta w_2 = 0 \quad \text{in } \Omega_1, \qquad w_2 = w_1 \quad \text{on } \partial\Omega_1.$$

Here $x \mapsto \varphi(x) = c_d|x|^{2-d}$ is the fundamental solution associated with $-\Delta$ in $\mathbb{R}^d$, where $c_d = \frac{1}{d-2}|\mathbb{S}^{d-1}|$. The surjectivity is therefore a consequence of the following two lemmas.

Lemma 4.7. *Let $3-d<\mu<0$ and $\tilde{f} \in C^{0,\alpha}_{\mu-2,p,e}(\mathbb{R}^d)$ be as above. Then*

$$\frac{1}{|\cdot|^{d-2}} * \tilde{f} \in C^{2,\alpha}_{\mu,p,e}(\mathbb{R}^d).$$

and therefore

$$\Big(\frac{1}{|\cdot|^{d-2}} * \tilde{f}\Big)\Big|_{\overline{\Omega}_1} \in C^{2,\alpha}_{\mu,p,e}(\Omega_1).$$

Proof. By (4.18) we have

$$|\tilde{f}(y,\sigma)| \le |y|^{\mu-2}\|\tilde{f}\|_{0,\alpha,\mu-2} \qquad \text{for } y \in \mathbb{R}^{d-1},\ \sigma \in \mathbb{R}.$$

For $x \in \mathbb{R}^{d-1}\setminus\{0\}$ and $t \in \mathbb{R}$ we then find, by a change of variable, that

$$\begin{aligned}
\Big|\Big(\frac{1}{|\cdot|^{d-2}} * \tilde{f}\Big)(x,t)\Big| &\le \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\Big(|x-y|^2+(t-\sigma)^2\Big)^{\frac{2-d}{2}}|\tilde{f}(y,\sigma)|\,d\sigma dy \\
&\le \|\tilde{f}\|_{0,\alpha,\mu-2}\int_{\mathbb{R}^{d-1}}|y|^{\mu-2}|x-y|^{2-d}\int_{\mathbb{R}}\Big(1+\Big(\frac{t-\sigma}{|x-y|}\Big)^2\Big)^{\frac{2-d}{2}}\,d\sigma dy \\
&= C_d\|\tilde{f}\|_{0,\alpha,\mu-2}\int_{\mathbb{R}^{d-1}}|y|^{\mu-2}|x-y|^{3-d}dy,
\end{aligned}$$

where $C_d = \int_{\mathbb{R}}(1+\tau^2)^{\frac{2-d}{2}}\,d\tau < \infty$. By rotational invariance, we thus find that

$$\begin{aligned}
\Big|\Big(\frac{1}{|\cdot|^{d-2}} * \tilde{f}\Big)(x,t)\Big| &\le C_d\|\tilde{f}\|_{0,\alpha,\mu-2}\int_{\mathbb{R}^{d-1}}|y|^{\mu-2}\big||x|e_1-y\big|^{3-d}dy \\
&= C_d\|\tilde{f}\|_{0,\alpha,\mu-2}|x|^{\mu+1-d}\int_{\mathbb{R}^{d-1}}\Big|\frac{y}{|x|}\Big|^{\mu-2}\Big|e_1-\frac{y}{|x|}\Big|^{3-d}dy \\
&= C_d D_{d,\mu}\|\tilde{f}\|_{0,\alpha,\mu-2}|x|^{\mu},
\end{aligned}$$

where $D_{d,\mu} := \int_{\mathbb{R}^{d-1}} |z|^{\mu-2}|e_1 - z|^{3-d} dz$ is finite since $3 - d < \mu < 0$. Therefore, the function $u := \frac{1}{|\cdot|^{d-2}} * \tilde{f}$ satisfies the estimate

$$\|u\|_{L^\infty(A_s)} \leq C_{d,\mu} \|\tilde{f}\|_{0,\alpha,\mu-2} s^\mu \tag{4.22}$$

for some constant $C_{d,\mu} > 0$.

Moreover, by (4.65) in the appendix, using the fact that u solves $-\Delta u = \tilde{f}$ in $A_{2s} \cup A_s \cup A_{\frac{s}{2}}$, we find that for every $s > 0$,

$$\begin{aligned}
&\sum_{i=0}^{2} s^i \|\nabla^i u\|_{A_s} + s^{2+\alpha}[\nabla^2 u]_{C^{0,\alpha}(A_s)} \\
&\leq C\Big(s^2 \|\tilde{f}\|_{L^\infty(A_{2s}\cup A_s \cup A_{\frac{s}{2}})} + s^{2+\alpha}[\tilde{f}]_{C^{0,\alpha}(A_{2s}\cup A_s\cup A_{\frac{s}{2}})} + \|u\|_{L^\infty(A_{2s}\cup A_s\cup A_{\frac{s}{2}})}\Big) \\
&\leq C s^\mu \Big(\|\tilde{f}\|_{0,\alpha,\mu-2} + \|u\|_{0,\alpha,\mu}\Big) \leq C s^\mu \|\tilde{f}\|_{0,\alpha,\mu-2},
\end{aligned}$$

where we used (4.22) in the last step. Hence $u \in C^{2,\alpha}_\mu(\mathbb{R}^d)$. Moreover, u is even and periodic in the t-variable, and therefore $u \in C^{2,\alpha}_{\mu,p,e}(\mathbb{R}^d)$, as claimed. □

Lemma 4.8. *Let $3 - d < \mu < 0$ and let $\varphi \in C^{2,\alpha}_{p,e}(\overline{\Omega}_1)$. Then there exists $w \in C^{2,\alpha}_{\mu,p,e}(\overline{\Omega_1})$ satisfying*

$$\Delta w = 0 \quad \text{in } \Omega_1, \qquad w = \varphi \quad \text{on } \partial\Omega_1. \tag{4.23}$$

Proof. With the help of Perron's method, it is easy to see that (4.23) admits an even and periodic solution $w \in C^{2,\alpha}_{loc}(\overline{\Omega_1})$ with respect to τ satisfying

$$w(y,\tau) \to 0 \qquad \text{as } |y| \to \infty \text{ uniformly in } \tau \in \mathbb{R}.$$

To see that $w \in C^{2,\alpha}_{\mu,p,e}(\overline{\Omega_1})$, we first note that

$$|w(y,\tau)| \leq \|\varphi\|_{C^{2,\alpha}_{p,e}(\overline{\Omega}_1)} |y|^{3-d} \qquad \text{for } (y,\tau) \in \Omega_1,$$

by comparison (see [44, Theorem 3.3]) with the functions

$$w_\pm : \overline{\Omega_1} \to \mathbb{R}, \qquad w_\pm(y,\tau) = \pm\|\varphi\|_{C^{2,\alpha}_{p,e}(\overline{\Omega}_1)} |y|^{3-d},$$

which are harmonic in Ω_1 and satisfy $w_- \leq w \leq w_+$ on $\partial\Omega_1$.

Moreover, for $s > 2$, there exists $R_0 > 1$ such that $B_{R_0 s}(x) \subset A_{2s} \cup A_s \cup A_{\frac{s}{2}}$ for any $x \in A_s$. Using this and the fact that w solves $-\Delta w = 0$ in $A_{2s} \cup A_s \cup A_{\frac{s}{2}}$, we may apply (4.66) in the appendix with $f \equiv 0$ to see that

$$\begin{aligned}
&\sum_{i=0}^{2} s^i \|\nabla^i w\|_{A_s} + s^{2+\alpha}[\nabla^2 w]_{C^{0,\alpha}(A_s)} \leq C\|w\|_{L^\infty(A_{2s}\cup A_s\cup A_{\frac{s}{2}})} \\
&\leq C\|\varphi\|_{L^\infty(\Omega_1)} s^{3-d} \leq C\|\varphi\|_{L^\infty(\Omega_1)} s^\mu.
\end{aligned} \tag{4.24}$$

Here we used that $\mu > 3 - d$. Using also the fact that, by standard elliptic estimates we have $w \in C^{2,\alpha}(\overline{\Omega_1 \setminus \Omega_2})$ with $\|w\|_{C^{2,\alpha}(\overline{\Omega_1 \setminus \Omega_2})} \le C\|\varphi\|_{C^{2,\alpha}(\overline{\Omega}_1)}$, we see that for $1 < s \le 2$,

$$\sum_{i=0}^{2} s^i \|\nabla^i w\|_{A_s} + s^{2+\alpha}[\nabla^2 w]_{C^{0,\alpha}(A_s)}$$
$$\le C\|w\|_{C^{2,\alpha}(\overline{\Omega_1 \setminus \Omega_2})} \le C\|\varphi\|_{C^{2,\alpha}(\overline{\Omega}_1)} \le \frac{C}{2^\mu} s^\mu \|\varphi\|_{C^{2,\alpha}(\overline{\Omega}_1)}, \tag{4.25}$$

where C is a constant only depending on α.

Combining (4.24) and (4.25), we obtain

$$\|w\|_{2,\alpha,\mu} \le C\|\varphi\|_{C^{2,\alpha}(\overline{\Omega}_1)},$$

as required. $\square$

4.3 Solution to the Dirichlet problem

In this section, we wish to use Theorem 4.4 to formulate problem (4.16), (4.17) as a nonlinear operator equation. For this, we need the following Lemma. Throughout the remainder of this chapter, we let $\mathcal{O}$ be given as in Theorem 4.4.

Lemma 4.9. *Let $3 - d < \mu < 0$. Then there exists a smooth map*

$$\mathcal{O} \to C^{2,\alpha}_{\mu,p,e}(\Omega_1), \quad (T, \varphi) \mapsto w_{T,\varphi} \tag{4.26}$$

with the property that, for every $(T, \varphi) \in \mathcal{O}$, the function $w_{T,\varphi}$ is the unique solution of the problem

$$\begin{cases} L_{T,\varphi} w_{T,\varphi} = 0 & \text{in} \quad B_1^c \times \mathbb{R}, \\ w_{T,\varphi} = 1 & \text{on} \quad \partial(B_1^c \times \mathbb{R}), \\ \lim\limits_{|y|\to\infty} w_{T,\varphi}(y,\tau) = 0 & \text{uniformly in } \tau \in \mathbb{R}. \end{cases} \tag{4.27}$$

Moreover, the functions $w_{T,\varphi} : \overline{\Omega}_1 \to \mathbb{R}$ and $\partial_{\nu_\varphi} w_{T,\varphi} : \partial\Omega_1 \to \mathbb{R}$ are radially symmetric in the y-variable, and

$$w_{T,0}(y,\tau) = u_1(y) = |y|^{3-d} = u_1(y,\tau) \qquad \text{for every } T > 0. \tag{4.28}$$

Proof. We first note that $u_1 \in C^{2,\alpha}_\mu(\Omega_1)$ since $3 - d < \mu < 0$, as $|\partial^i u_1(y,\tau)| \le (d-3)|y|^{2-d}$ and $|\partial^{ij} u_1(y,\tau)| \le d(d-3)|y|^{1-d}$ for $i, j = 1, \dots, d$. We consider the map

$$\mathcal{O} \to C^{2,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1), \qquad (T,\varphi) \mapsto m_{T,\varphi} := \left(L^D_{T,\varphi}\right)^{-1}\left(L_{T,\varphi} u_1\right), \tag{4.29}$$

which is well-defined by Theorem 4.4. For $(T,\varphi) \in \mathcal{O}$, the function $m_{T,\varphi} \in C^{2,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1)$ is the unique solution of the problem

$$\begin{cases} L_{T,\varphi} m_{T,\varphi} = L_{T,\varphi} u_1 & \text{in} \quad \Omega_1, \\ m_{T,\varphi} = 0 & \text{on} \quad \partial\Omega_1, \\ \lim_{|y|\to\infty} m_{T,\varphi}(y,\tau) = 0 & \text{uniformly in } \tau \in \mathbb{R}. \end{cases}$$

Hence the function

$$w_{T,\varphi} := -m_{T,\varphi} + u_1 \ \in \ C^{2,\alpha}_{\mu,p,e}(\Omega_1)$$

is the unique solution of (4.27). Moreover, both $w_{T,\varphi} : \overline{\Omega}_1 \to \mathbb{R}$ and $\partial_{\nu_\varphi} w_{T,\varphi} : \partial\Omega_1 \to \mathbb{R}$ are radially symmetric in the x-variable by uniqueness and the fact that both Ω_1 and the operator $L_{T,\varphi}$ are radial in the x-variable. In addition, for $T > 0$ we have $m_{T,0} \equiv 0$ in Ω_1 since $L_{T,0} u_1 = \Delta_y u_1 \equiv 0$ and therefore $w_{T,0} = u_1$.

It thus remains to show that the map $(T,\varphi) \mapsto m_{T,\varphi}$ in (4.29) is smooth. For this we first note that, for every pair of Banach spaces X, Y, the inversion map

$$\mathcal{I}(X,Y) \to \mathcal{I}(Y,X), \qquad T \mapsto T^{-1}$$

is smooth in the open set $\mathcal{I}(X,Y) \subset \mathcal{L}(X,Y)$ of topological isomorphisms. Hence the smoothness of the maps $(T,\varphi) \mapsto m_{T,\varphi}$ follows by the smoothness of the maps

$$\mathcal{O} \to \mathcal{L}(C^{2,\alpha}_{\mu,p,e}(\Omega_1), C^{0,\alpha}_{\mu-2,p,e}(\Omega_1)), \qquad (T,\varphi) \to L_{T,\varphi},$$
$$\mathcal{O} \to \mathcal{L}(C^{2,\alpha}_{\mu,\mathcal{D},p,e}(\Omega_1), C^{0,\alpha}_{\mu-2,p,e}(\Omega_1)), \qquad (T,\varphi) \to L^{D}_{T,\varphi}$$

asserted in Theorem 4.4. The proof is thus finished. □

The aim now is to prove that for some parameter values $(T,\varphi) \in \mathcal{O}$ with $\varphi \not\equiv 0$, the function $w_{T,\varphi}$ satisfies the overdetermined condition

$$\frac{\partial w_{T,\varphi}}{\partial \nu_{T,\varphi}} = d - 3 \quad \text{on} \quad \partial(B_1^c \times \mathbb{R}). \tag{4.30}$$

We thus define the map

$$F : \mathcal{O} \subset \mathbb{R} \times C^{2,\alpha}_{p,e}(\mathbb{R}) \to C^{1,\alpha}_{p,e}(\mathbb{R}), \quad F(T,\varphi)(\tau) := \frac{\partial w_{T,\varphi}}{\partial \nu_{T,\varphi}}(e_1,\tau) - (d-3). \tag{4.31}$$

By radial symmetry of $\dfrac{\partial w_{T,\varphi}}{\partial \nu_{T,\varphi}}$, the condition (4.30) is therefore equivalent to $F(T,\varphi) = 0$. Our aim is to apply the Crandall-Rabinowitz bifurcation theorem 1.7 to solve the equation

$$F(T,\varphi) \equiv 0 \quad \text{in} \quad C^{1,\alpha}_{p,e}(\mathbb{R}).$$

Observe that from (4.13) and (4.14) that the map $\mathcal{O} \to C^{1,\alpha}(\partial\Omega_1, \mathbb{R}^d)$, $(T,\varphi) \mapsto \nu_{T,\varphi}$ is smooth. This together with the smoothness of the map in (4.26) guarantees that $(T,\varphi) \mapsto F(T,\varphi)$ is smooth. Indeed, we have the following observation.

Lemma 4.10. *(see [32, Lemma 2.3])*
Let

$$h : \mathcal{O} \to C^{2,\alpha}(\overline{\Omega}_1), \qquad (T, \varphi) \mapsto h_{T,\varphi}$$

be a smooth map. Then the map

$$\mathcal{G}(T, \varphi) : \mathcal{O} \to C^{1,\alpha}(\partial\Omega_1), \qquad \mathcal{G}(\varphi) = \frac{\partial h_{T,\varphi}}{\partial \nu_{T,\varphi}}$$

is smooth as well and satisfies

$$D_\varphi \mathcal{G}(T, \varphi)v = \frac{\partial h_{T,\varphi}}{\partial \tilde{\nu}_{T,\varphi}(v)} + \frac{\partial\Big([D_\varphi h_{T,\varphi}]v\Big)}{\partial \nu_{T,\varphi}} \qquad \text{for } v \in C^{2,\alpha}_{p,e}(\mathbb{R}), \tag{4.32}$$

where

$$\tilde{\nu}_{T,\varphi}(v) := [D_\varphi \nu_{T,\varphi}]v \in C^{1,\alpha}_{p,e}(\partial\Omega_1, \mathbb{R}^d) \qquad \text{for } (T, \varphi) \in \mathcal{O},\ v \in C^{2,\alpha}_{p,e}(\mathbb{R}).$$

In addition, by (4.15), (4.31) and (4.6) it reads

$$F(T, \varphi)(t) = \nabla u_{T,\varphi}\left((1 + \varphi(\tau))e_1, \frac{T}{2\pi}\tau\right) \cdot \Upsilon\left((1 + \varphi(\tau))e_1, \frac{T}{2\pi}\tau\right) - (d-3). \tag{4.33}$$

Hence applying (4.33) with $\varphi = 0$ and using (4.28), we see that

$$F(T, 0) = 0 \qquad \text{for every } T > 0\ .$$

4.4 Study of the linearized operator

The aim of this section is to study the spectral properties of the linearized operator

$$H_T \in \mathcal{L}(C^{2,\alpha}_{p,e}(\mathbb{R}), C^{1,\alpha}_{p,e}(\mathbb{R})), \qquad H_T(v) = D_\varphi\big|_{\varphi=0} F(T, \varphi)v. \tag{4.34}$$

Proposition 4.11. *For every $T > 0$, the linearised operator H_T defined by (4.34) is given by*

$$H_T(v)(\tau) = (d-3)\Big(\nabla \dot{u}(e_1, \tau) \cdot (-e_1, 0) - (d-2)v(\tau)\Big), \tag{4.35}$$

where $\dot{u}$ is solution to

$$\begin{cases} \Big(\Delta_y + \left(\frac{2\pi}{T}\right)^2 \frac{\partial^2}{\partial \tau^2}\Big)\dot{u} = 0 & \text{in} \quad B_1^c \times \mathbb{R}, \\ \dot{u}(y, \tau) = v(\tau) & (y, \tau) \in \partial(B_1^c \times \mathbb{R}), \\ \dot{u} \to 0 & \text{as } |z| \to \infty \text{ uniformly in } \tau \in \mathbb{R}. \end{cases} \tag{4.36}$$

Furthermore, the eigenfunctions of H_T are given by $v_k(\tau) := \cos(k\tau)$, $k \in \mathbb{N} \cup \{0\}$, and we have

$$H_T v_k = \lambda_k(T) v_k, \tag{4.37}$$

where

$$\lambda_k(T) = -(d-3)\Lambda\left(\frac{2k\pi}{T}\right) \tag{4.38}$$

with $\Lambda(0) = 1$ and

$$\Lambda(\rho) = \left(d - 2 - \frac{\rho K_{\eta+1}(\rho)}{K_\eta(\rho)}\right) \quad \textit{for } \rho > 0 \textit{ with } \eta = \frac{d-3}{2}. \tag{4.39}$$

Proof. To prove (4.35) and (4.36), we consider the functions

$$\begin{aligned} W_{T,\varphi}(y,\tau) &:= u_1\left(\kappa(|y|^2, \varphi(\tau))y, \frac{T}{2\pi}\tau\right) = |y|^{3-d}\Big(1 + \frac{\varphi(\tau)}{|y|^2}\Big)^{3-d}, \\ V_{T,\varphi}(y,\tau) &:= W_{T,\varphi}(y,\tau) - w_{T,\varphi}(y,\tau). \end{aligned} \tag{4.40}$$

Since $u_1(y,\tau) = |y|^{3-d}$ is harmonic in $\mathbb{R}^d \setminus (\{0\} \times \mathbb{R})$, we have, by (4.6) and (4.4),

$$L_{T,\varphi} W_{T,\varphi} = 0 \quad \text{in} \quad \Omega_1.$$

Moreover,

$$\begin{cases} L_{T,\varphi} V_{T,\varphi} = 0 & \text{in} \quad \Omega_1, \\ V_{T,\varphi}(y,\tau) = (1+\varphi(\tau))^{3-d} - 1 & \text{for } (y,\tau) \in \partial\Omega_1, \end{cases} \tag{4.41}$$

and

$$V_{T,0} \equiv 0 \quad \text{in} \quad \overline{\Omega}_1.$$

Differentiating (4.41) with respect to φ at $\varphi \equiv 0$ and setting

$$\psi_{T,v} := \frac{1}{3-d}[D_\varphi\big|_{\varphi=0} V_{T,\varphi}]v \qquad \text{for } v \in C^{2,\alpha}_{p,e}(\mathbb{R}),$$

we find that

$$\begin{cases} \left(\Delta_y + \left(\frac{2\pi}{T}\right)^2 \frac{\partial^2}{\partial\tau^2}\right)\psi_{T,v} = 0 & \text{in } \Omega_1, \\ \psi_{T,v} = v & \text{on } \partial\Omega_1, \\ \psi_{T,v} \to 0 \quad \textit{as } |y| \to \infty \textit{ uniformly in } \tau \in \mathbb{R}. \end{cases}$$

We now put $G(T,\varphi)(\tau) := \dfrac{\partial W_{T,\varphi}}{\partial \nu_{T,\varphi}}(e_1,\tau)$ for $\tau \in \mathbb{R}$. By (4.13), (4.15) and (4.40), we have

$$\begin{aligned}
G(T,\varphi)(\tau) &= (\nabla u_1)((1+\varphi(\tau))e_1,\tau)) \cdot \Upsilon((1+\varphi(\tau))e_1,\tau) \\
&= \Big(-(d-3)(1+\varphi(\tau))^{2-d}e_1\Big) \cdot \Big(-\Big(1+(\frac{2\pi}{T})^2\varphi'^2(\tau)\Big)^{-\frac{1}{2}}e_1\Big) \\
&= (d-3)(1+\varphi(\tau))^{2-d}\Big(1+(\frac{2\pi}{T})^2\varphi'^2(\tau)\Big)^{-\frac{1}{2}}.
\end{aligned}$$

Consequently,

$$\Big[D_\varphi\big|_{\varphi=0} G(T,\varphi)v\Big](\tau) = -(d-2)(d-3)v(\tau). \tag{4.42}$$

Moreover, by (4.32) and since $V_{T,0} \equiv 0$ in $\overline{\Omega}_1$, we have

$$\Big[D_\varphi\big|_{\varphi=0} \frac{\partial V_{T,\varphi}}{\partial \nu_{T,\varphi}}\Big]\omega = \partial_{\tilde{\nu}_\varphi(\omega)}V_{T,0} + \partial_\nu \psi_{T,v} = -(d-3)\partial_\nu \psi_{T,v} \qquad \text{on } \partial\Omega_1. \tag{4.43}$$

Using (4.40), (4.42) and (4.43), we get

$$\Big[D_\varphi\big|_{\varphi=0} \frac{\partial w_{T,\varphi}}{\partial \nu_{T,\varphi}}\Big]v = \Big[D_\varphi\big|_{\varphi=0} \frac{\partial}{\partial \nu_{T,\varphi}}(W_{T,\varphi}-V_{T,\varphi})\Big]v = (d-3)\Big(\partial_\nu \psi_{T,v} - (d-2)v\Big),$$

which combined with (4.31) and (4.34) yields

$$H_T(v)(\tau) = \Big[D_\varphi\big|_{\varphi=0} F(T,\varphi)v\Big](\tau) = (d-3)\Big(\nabla \dot{u}(e_1,\tau)\cdot(-e_1,0) - (d-2)v(\tau)\Big),$$

as claimed in (4.35). This proves the first part of Proposition 4.11.

Next we claim that the functions $\tau \mapsto v_k(\tau) := \cos(k\tau)$, $k \in \mathbb{N}\cup\{0\}$ are the eigenfunctions of the operator $v \mapsto H_T(v)$. This is clear for $k=0$, as the unique solution of (4.36) with $v \equiv 1$ is merely given by $(y,\tau) \to u_1(y,\tau) = |y|^{3-d}$, and therefore $H_T(v) \equiv -(d-3)$ by (4.35).

Assuming $k \neq 0$ from now on, we see that the unique solution $\dot{u}_k$ to (4.36) with $v = v_k$ can be expressed, by separation of variables, as $\dot{u}_k(y,\tau) = u_k(|y|)\cos(k\tau)$ where u_k solves the ODE boundary value problem

$$\begin{cases}
u_k'' + \dfrac{d-2}{r}u_k' - (\frac{2\pi k}{T})^2 u_k = 0 & \textit{in } (1,\infty), \\
u_k(1) = 1, & \\
u_k(r) \to 0 & \textit{as } r \to \infty.
\end{cases}$$

We set $\rho_k := \frac{2\pi k}{T}$ and consider the function $w : [\rho_k,\infty) \to \mathbb{R}$ defined by

$$w(\rho) = u_k\left(\frac{\rho}{\rho_k}\right). \tag{4.44}$$

We have

$$\begin{cases} w''(\rho) + \frac{d-2}{\rho} w'(\rho) - w(\rho) = 0 & \textit{in } (\rho_k, \infty), \\ w(\rho_k) = 1, \\ w(\rho) \to 0 & \textit{as } \rho \to \infty, \end{cases}$$

and setting $g(\rho) := \rho^\eta w(\rho)$, where $\eta = \frac{d-3}{2}$, we see that function g satisfies

$$\begin{cases} g(\rho) + \frac{1}{\rho} g'(\rho) - \left(1 + \frac{\eta 2}{\rho^2}\right) g(\rho) = 0 & \textit{in } (\rho_k, \infty), \\ g(\rho) \to 0 & \textit{as } \rho \to \infty, \\ g(\rho_k) = \rho_k^\eta. \end{cases} \tag{4.45}$$

Up to a multiplicative constant, the modified Bessel function of second kind K_η is the unique solution to (4.45). Since the function K_η is positive on $(0, +\infty)$ it follows that

$$w(\rho) = C\rho^{-\eta} K_\eta(\rho), \quad \rho \in (\rho_k, +\infty), \quad \text{for some constant } C > 0. \tag{4.46}$$

Furthermore, combining (4.44) with (4.46),

$$u_k(r) = w(K_0 r) = C\rho_k^{-\eta} r^{-\eta} K_\eta(\rho_k r)$$

and it follows from $u_k(1) = 1$ that $C = \rho_k^\eta / K_\eta(\rho_k)$ and

$$u_k'(1) = C\rho_k^{-\eta} K_\eta(\rho_k) \left(-\eta + \rho_k \frac{K_\eta'(\rho_k)}{K_\eta(\rho_k)}\right) = -\eta + \rho_k \frac{K_\eta'(\rho_k)}{K_\eta(\rho_k)}. \tag{4.47}$$

Recalling (4.35) and (4.47), we find that

$$\begin{aligned} H_T(v_k)(\tau) &= (d-3)\Big(\nabla u_k(e_1, \tau) \cdot (-e_1, 0) - (d-2) v_k(\tau)\Big) \\ &= -(d-3)\left(\frac{d-1}{2} + \rho_k \frac{K_\eta'(\rho_k)}{K_\eta(\rho_k)}\right) v_k(\tau) \\ &= -(d-3)\left(d - 2 - \rho_k \frac{K_{\eta+1}(\rho_k)}{K_\eta(\rho_k)}\right) v_k(\tau), \end{aligned}$$

where we have used the relation

$$\rho K_\eta'(\rho) = \eta K_\eta(\rho) - \rho K_{\eta+1}(\rho).$$

Consequently,

$$H_T(v_k) = -(d-3)\Lambda(\rho_k) v_k = -(d-3)\Lambda(\frac{2k\pi}{T}) v_k$$

with Λ given in (4.39). Hence (4.37) follows with $\lambda_k(T)$ given in (4.38), and the proof is finished. $\square$

In the following result, we study the behaviour of the eigenvalue in $\lambda_1(T)$ in (4.38).

Lemma 4.12. *For $k > 0$ we have*

$$\mu_k(T) \to \begin{cases} -(d-3), & T \to +\infty, \\ +\infty, & T \to 0^+. \end{cases} \tag{4.48}$$

and for every $T > 0$ we have

$$\lim_{k\to\infty} \frac{\lambda_k(T)}{k} = \frac{2\pi(d-3)}{T}. \tag{4.49}$$

Moreover, there exists a unique $T_* > \dfrac{2\pi}{\sqrt{d-2}}$ satisfying

$$\lambda_1(T_*) = 0,\ \lambda_1'(T_*) < 0 \text{ and } \lambda_k(T_*) \neq 0 \text{ for } k \neq 1. \tag{4.50}$$

Proof. The proof of the above result is achieved studying the asymptotics for the function Λ defined in (4.39) with $\eta = \frac{d-3}{2}$. By (4.70) and (4.71) in the appendix, we have

$$\rho \frac{K_{\eta+1}(\rho)}{K_\eta(\rho)} \to \begin{cases} 2\eta = d-3, & \rho \to 0^+, \\ +\infty, & \rho \to +\infty. \end{cases}$$

and therefore (4.48) follows. Furthermore, since

$$\frac{K_{\eta+1}(\rho)}{K_\eta(\rho)} \to 1 \qquad \text{as} \quad \rho \to +\infty$$

by (4.70), we have

$$\lim_{\rho\to\infty} \frac{\Lambda(\rho)}{\rho} = \frac{d-2}{\rho} - \frac{K_{\eta+1}(\rho)}{K_\eta(\rho)} = -1$$

and hence

$$\lim_{k\to\infty} \frac{\lambda_k(T)}{k} = -\frac{2\pi(d-3)}{T} \frac{T}{2\pi k} \Lambda(\frac{2\pi k}{T}) = \frac{2\pi(d-3)}{T},$$

proving (4.49).

To prove (4.50), we note that the function $T \mapsto \mu_1(T)$ has a positive zero by (4.49) and (4.48). We then use (4.68) and (4.69) to compute

$$\begin{aligned} -\Lambda'(\rho) &= \frac{K_{\eta+1}(\rho)}{K_v(\rho)} + \rho \frac{K'_{\eta+1}(\rho)}{K_\eta(\rho)} - \rho \frac{K'_\eta(\rho)}{K_\eta(\rho)} \frac{K_{\eta+1}(\rho)}{K_\eta(\rho)}, \\ &= \frac{K_{\eta+1}(\rho)}{K_\eta(\rho)} - \rho - (\eta+1) \frac{K_{\eta+1}(\rho)}{K_\eta(\rho)} + \Big(-\eta + \rho \frac{K_{\eta+1}(x)}{K_\eta(\rho)}\Big) \frac{K_{\eta+1}(\rho)}{K_\eta(\rho)}, \\ &= \frac{K_{\eta+1}(\rho)}{K_\eta(\rho)} \Big(-2\eta + \rho \frac{K_{\eta+1}(x)}{K_\eta(\rho)}\Big) - \rho \qquad \text{for } \rho > 0. \end{aligned} \tag{4.51}$$

Next, we consider a point $\rho > 0$ with $\Lambda(\rho) = 0$. Then

$$\frac{K_{\eta+1}(\rho)}{K_\eta(\rho)} = \frac{d-2}{\rho} \tag{4.52}$$

and, by (4.72),

$$\eta + \sqrt{\rho^2+\eta^2} < \rho\frac{K_{\eta+1}(\rho)}{K_\eta(\rho)} = d-2 \Longleftrightarrow \sqrt{\rho^2+\eta^2} < -\eta + d - 2 = \frac{d-1}{2}$$

since $\eta = \frac{d-3}{2}$, which gives

$$\rho < \sqrt{d-2}. \tag{4.53}$$

Plugging (4.52) in (4.51) and using (4.53) yields

$$-\Lambda'(\rho) = \frac{d-2}{\rho} - \rho > 0.$$

It thus follows that the function Λ has a unique zero ρ_* on $(0,\infty)$ satisfying $\rho* < \sqrt{d-2}$. Hence (4.38) gives (4.50) with $T_* > \dfrac{2\pi}{\sqrt{d-2}}$. □

4.5 Proof of the main result

In this section we consider the fractional Sobolev spaces

$$H^\sigma_{p,e} := \Big\{ v \in H^\sigma_{loc}(\mathbb{R}) \ : \ \ v \text{ even and } 2\pi\text{-periodic} \Big\}$$

for $\sigma \geq 0$, and we put $L^2_{p,e} := H^0_{p,e}$. Note that $L^2_{p,e}$ is a Hilbert space with scalar product

$$(u,v) \mapsto \langle u,v\rangle_{L^2_{p,e}} := \int_{-\pi}^{\pi} u(t)v(t)\,dt \qquad \text{for } u,v \in L^2_{p,e},$$

and induced norm denoted by $\|\cdot\|_{L^2_{p,e}}$. We define for all $k\in\mathbb{N}$, $v_k(t) := \cos(kt)$. $\|v_k\|_{L^2_{p,e}} = \sqrt{\pi}$, the set $\{\frac{v_k}{\sqrt{\pi}},\ k\in\mathbb{N}\}$ forms a complete orthonormal basis of $L^2_{p,e}$. Moreover, $H^\sigma_{p,e} \subset L^2_{p,e}$ is characterized as the subspace of all functions $v \in L^2_{p,e}$ such that

$$\sum_{k\in\mathbb{N}} (1+k^2)^\sigma \langle v, v_k\rangle^2_{L^2_{p,e}} < \infty. \tag{4.54}$$

Therefore, $H^\sigma_{p,e}$ is also a Hilbert space with scalar product

$$(u,v) \mapsto \sum_{k\in\mathbb{N}} (1+k^2)^\sigma \langle u, v_k\rangle_{L^2_{p,e}} \langle v, v_k\rangle_{L^2_{p,e}} \qquad \text{for } u,v \in H^\sigma_{p,e}.$$

Set
$$W_k := \operatorname{span}\{v_k\} \subset \bigcap_{j\in\mathbb{N}} H^j_{p,e}$$
for $k \in \mathbb{N}$. Then from Proposition 4.11, the spaces W_k are the eigenspaces of the operator H_T in (4.34) corresponding to the eigenvalues $\lambda_k(T)$, i.e., we have
$$H_T v = \lambda_k(T) v \qquad \text{for every } v \in W_k. \tag{4.55}$$
We also consider their orthogonal complements in $L^2_{p,e}$, given by
$$W_k^\perp := \left\{ w \in L^2_{p,e} \,:\, \int_{-\pi}^{\pi} \cos(ks) w(s)\, ds = 0 \right\},$$
as well as the the spaces
$$X := \left\{ \varphi : \mathbb{R} \to \mathbb{R}, \quad \varphi \in C^{2,\alpha}(\mathbb{R}) \text{ is even and } 2\pi\text{-periodic} \right\},$$
and
$$Y := \left\{ \varphi : \mathbb{R} \to \mathbb{R}, \quad \varphi \in C^{1,\alpha}(\mathbb{R}) \text{ is even and } 2\pi\text{-periodic} \right\}.$$

Proposition 4.13. *There exists a unique $T_* > 0$ such that the linear operator $H_{T_*} : X \to Y$ has the following properties:*

(i) The kernel $N(H_{T_*})$ of H_{T_*} is spanned by the function $\cos(\cdot)$,

(ii) $H_{T_*} : X \cap W_1^\perp \to Y \cap W_1^\perp$ is an isomorphism.

Moreover
$$\partial_T \Big|_{T=T_*} H_T v_1 = \lambda_1'(T_*) v_1 \notin Y \cap W_1^\perp. \tag{4.56}$$

Proof. By Lemma 4.12, we have the existence of a unique $T_* > 0$ such that $\lambda_1(T_*) = 0$, $\mu_1'(T_*) < 0$ and $\lambda_k(T_*) \neq 0$ for all $k \neq 1$. This with (4.55) imply that $N(H_*) = \operatorname{span}\{v_1\} = W_1$ and we obtain (i) and (4.56).

To prove (ii), we pick $g \in Y \cap W_1^\perp$ and consider the equation
$$H_* w = g. \tag{4.57}$$
Using (4.37), the equation (4.57) is uniquely solved by the function
$$w(s) = \sum_{\ell \in \mathbb{N}\setminus\{1\}} w_\ell v_\ell(s),$$
where
$$w_\ell = \frac{1}{\pi \lambda_\ell(T_*)} \langle g, v_\ell \rangle_{L^2_{p,e}}, \quad \ell \neq 1. \tag{4.58}$$

In addition,

$$\sum_{\ell \in \mathbb{N}\setminus\{1\}} (1+\ell^2)^2 \langle w, v_\ell\rangle^2_{L^2_{p,e}} = \frac{1}{\pi} \sum_{\ell \in \mathbb{N}\setminus\{1\}} \frac{(1+\ell^2)}{\lambda^2_\ell(T_*)} (1+\ell^2)\langle g, v_\ell\rangle^2_{L^2_{p,e}}. \tag{4.59}$$

Since $g \in C^{1,\alpha}_{p,e}(\mathbb{R}) \subset H^1_{loc}(\mathbb{R})$, we have $g \in H^1_{p,e}$. This combined with (4.54) and the first asymptotic in Lemma 4.12 allow to see that the right hand side in (4.59) is bounded, which implies $w \in H^2_{p,e}$. We now show that $w \in C^{2,\alpha}_{p,e}(\partial\Omega_1)$.

Recalling Proposition 4.11 (4.35), (4.58) reads

$$(d-3)\nabla \dot{u}(e_1,\tau)\cdot(-e_1,0) = (d-3)(d-2)w(\tau) + g(\tau), \tag{4.60}$$

where $\dot{u}$ is the unique even and 2π periodic solution in τ of

$$\begin{cases} \Delta \dot{u} = 0 & \text{in} \quad \Omega_1, \\ \dot{u}(y,\tau) = w(\tau) & \text{in} \quad \partial\Omega_1, \\ \dot{u} \to 0 & \textit{as } |y| \to \infty \textit{ uniformly in } \tau \in \mathbb{R}. \end{cases} \tag{4.61}$$

Furthermore since $w \in H^2_{p,e}$, we have by standard elliptic regularity that $\dot{u} \in W^{2,2}_{loc}(\Omega_1)$.

We now show that

$$\dot{u} \in C^{2,\alpha}_{p,e}(\overline{\Omega}_1). \tag{4.62}$$

The fact that (4.62) holds follows from a similar argument as in the proof of [32, Proposition 4.1] (see also the proof of [33, Proposition 5.1]). We give the details here for the reader's convenience. The regularity property in (4.62) is obtained from [45, Theorem 6.3.2.1] once we show that

$$\dot{u} \in W^{2,p}_{loc}(\Omega_1) \quad \text{for any } p \in (1,\infty). \tag{4.63}$$

Indeed if (4.63) holds then by Sobolev embedding, we get $\dot{u} \in C^{1,\alpha}_{p,e}(\overline{\Omega}_1)$ and hence $w \in C^{1,\alpha}_{p,e}(\partial\Omega_1)$ by (4.61). Then applying [45, Theorem 6.3.2.1] with the order $d = 1$ to the boundary operator (see [45, Section 2.1]) yields (4.62).

To see (4.63), we prove by induction that

$$\dot{u} \in W^{2,p_\ell}_{loc}(\Omega_1)$$

for a sequence of numbers $p_\ell \in [2,\infty)$ with $p_0 = 2$ and $p_{\ell+1} \geq \frac{d-1}{d-p_\ell} p_\ell$ for $\ell \geq 0$. Clearly, it holds for $p_0 = 2$. So let us assume that $\dot{u} \in W^{2,p_\ell}_{loc}(\Omega_1)$ also holds for some p_ℓ. We consider the following two cases:
Case 1: $p_\ell < d$. By the trace Theorem, [1, Theorem 5.4], we can get that

$$\dot{u}|_{\partial\Omega_1} \in W^{1,p_{\ell+1}}_{loc}(\partial\Omega_1) \quad \text{with } p_{\ell+1} := \frac{d-1}{d-p_\ell} p_\ell \geq \frac{d-1}{d-2} p_\ell,$$

therefore $w \in W^{1,p_{\ell+1}}_{loc}(\partial\Omega_1)$. Since $g \in C^{1,\alpha}(\mathbb{R}) \subset W^{1,p_{\ell+1}}_{loc}(\partial\Omega_1)$ and by (4.60),

$$(d-3)\partial_\nu \dot{u}(\pm e_1, \tau) + \dot{u}(\pm e_1, \tau) = f(\tau),$$

where

$$f(\tau) := (1 + (d-3)(d-2))w(\tau) + g(\tau) \in W^{1,p_{\ell+1}}_{loc}(\partial\Omega_1).$$

Therefore, $\dot{u} \in W^{2,p_{\ell+1}}_{loc}(\Omega_1)$ by [45, Theorem 2.4.2.6].
Case 2: $p_\ell \geq d$. The trace theorem implies that $w \in W^{1,p}_{loc}(\partial\Omega_1)$ for any $p > 2$, and then we repeat the above argument to deduce that $\dot{u} \in W^{2,p_{\ell+1}}_{loc}(\Omega_1)$ for arbitrarily chosen $p_{\ell+1} \geq \frac{d-1}{d-2} p_\ell$.

Finally, we conclude that (4.63) holds and (4.62) follows. By passing to the trace, we see with (4.62) that $w \in C^{2,\alpha}_{p,e}(\partial\Omega_1)$ and the proof is complete. □

We are now in position to apply the Crandall-Rabinowitz Theorem 1.7, which will give rise to the following bifurcation property.

4.5.1 Proof of Theorem 1.2

We define

$$\mathcal{X}^\perp := \left\{ v \in X : \int_{-\pi}^{\pi} v(\tau)\cos(\tau)\, d\tau = 0 \right\}.$$

By Proposition 4.13 and the Crandall-Rabinowitz Theorem 1.7, we then find $\varepsilon > 0$ and a smooth curve

$$(-\varepsilon, \varepsilon) \to (0, +\infty) \times \mathcal{U} \subset \mathbb{R}_+ \times X, \qquad s \mapsto (T(s), \varphi_s)$$

such that

(i) $F(T(s), \varphi_s) = 0$ for $s \in (-\varepsilon, \varepsilon)$,

(ii) $T(0) = T_*$,

(iii) $\varphi_s = s\cos(\cdot) + s v_s$ for $s \in (-\varepsilon, \varepsilon)$ with a smooth curve

$$(-\varepsilon, \varepsilon) \to \mathcal{X}^\perp, \qquad s \mapsto v_s$$

satisfying $v_0 = 0$ and

$$\int_{-\pi}^{\pi} v_s(\tau)\cos(\tau)\, d\tau = 0.$$

Finally, since $F(T(s), \varphi_s) = 0$ for $s \in (-\varepsilon, \varepsilon)$, we see from (4.31) and Lemma 4.9 that the function $w_s := w_{T_s,\varphi_s}$ solves (4.17). Furthermore recalling (4.6), the function

$$u_s(z, t) = w_s\Big(\zeta(|z|^2, \varphi_s(\frac{2\pi t}{T_s}))z, \frac{2\pi t}{T_s}\Big)$$

solves (4.1) on Ω_{T_s,φ_s}. The proof is complete. □

4.6 Appendix

4.6.1 Scale invariant Hölder estimates for solutions of the Poisson equation

In this section, we provide $C^{2,\alpha}_{\mu}$ estimate for solutions of the Poisson problem $\Delta u = f$. We first recall the following classical regularity results (see [44, Theorem 4.6] and [44, Theorem 6.6]).

Lemma 4.14. *Let $f \in C^{0,\alpha}(B_1)$ and $u \in C^{2,\alpha}(B_1)$ solve the equation $-\Delta u = f$ in B_1. Then there exists a constant $C = C(N, \alpha) > 0$ such that*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C(\|f\|_{L^\infty(B_1)} + [f]_{C^{0,\alpha}(B_1)} + \|u_R\|_{L^\infty(B_1)}). \tag{4.64}$$

Lemma 4.15. *Let $f \in C^{0,\alpha}(\overline{B}_1)$ and $u \in C^{2,\alpha}(\overline{B}_1)$ solve the equation $-\Delta u = f$ in B_1.*

Assume there exists $\varphi \in C^{2,\alpha}(\overline{B}_1)$ such that $u = \varphi$ on ∂B_1. Then there exists a constant $C = C(N, \alpha) > 0$ such that

$$\|u\|_{C^{2,\alpha}(B_1)} \le C(\|\varphi\|_{C^{2,\alpha}(B_1)} + \|f\|_{L^\infty(B_1)} + [f]_{C^{0,\alpha}(B_1)} + \|u_R\|_{L^\infty(B_1)}).$$

Lemma 4.16. *Let $z \in \mathbb{R}^d$, $f \in C^{0,\alpha}(B_R(z))$, $u \in C^{2,\alpha}_{loc}(B_R(z)) \cap L^\infty(B_R(z))$ solves $-\Delta u = f$ in $B_R(z), R > 0$. Then there exists a constant $C > 0$, independent of R, with the property that*

$$\begin{aligned}&\sum_{i=0}^{2} R^i \|\nabla^i u\|_{L^\infty(B_{\frac{R}{2}}(0))} + R^{2+\alpha}[\nabla^2 u]_{C^{0,\alpha}(B_{\frac{R}{2}}(0))}\\ &\le C(R^2\|f\|_{L^\infty(B_R(0))} + R^{2+\alpha}[f]_{C^{0,\alpha}(B_R(0))} + \|u\|_{L^\infty(B_R(0))}).\end{aligned} \tag{4.65}$$

Furthermore if $f \in C^{0,\alpha}(\overline{B}_R(z))$ and $u \in C^{2,\alpha}_{loc}(\overline{B}_R(z)) \cap L^\infty(B_R(z))$ solves $-\Delta u = f$ in $B_R(z)$, with $u = \varphi$ on $\partial B_R(z)$, then

$$\begin{aligned}&\sum_{i=0}^{2} R^i \|\nabla^i u\|_{L^\infty(B_R(0))} + R^{2+\alpha}[\nabla^2 u]_{C^{0,\alpha}(B_R(0))}\\ &\le C\left(\sum_{i=0}^{2} R^i \|\nabla^i \varphi\|_{L^\infty(B_R(0))} + R^{2+\alpha}[\nabla^2 \varphi]_{C^{0,\alpha}(B_R(0))} + \|u\|_{L^\infty(B_R(0))}\right)\\ &+ C(R^2\|f\|_{L^\infty(B_R(0))} + R^{2+\alpha}[f]_{C^{0,\alpha}(B_R(0))}).\end{aligned} \tag{4.66}$$

Here, $[\cdot]$ is the Hölder semi-norm defined in Definition 4.2.

Proof. Without loss of generality, we may take $z = 0$. Hence we assume that

$$-\Delta u = f \qquad \text{in } B_R(0), \tag{4.67}$$

and we let $u_R, f_R : B_1(0) \to \mathbb{R}$ be defined by $u_R(x) = u(Rx)$, $f_R(x) = f(Rx)$. Then we have

$$\begin{aligned} \nabla u_R &= R(\nabla u)(R\,\cdot\,) && \text{in } B_1(0), \\ \nabla^2 u_R &= R^2(\nabla^2 u)(R\,\cdot\,) && \text{in } B_1(0). \end{aligned}$$

Since by (4.67) we have

$$-\Delta u_R = R^2 f_R \qquad \text{in } B_1(0),$$

we can apply (4.64) to get

$$\|u_R\|_{C^{2,\alpha}(B_{1/2})} \leq C(\|R^2 f_R\|_{L^\infty(B_1)} + [R^2 f_R]_{C^{0,\alpha}(B_1)} + \|u_R\|_{L^\infty(B_1)}),$$

where C is a constant independent of R and f. Combining this estimate with the scaling identities listed above, we obtain

$$\begin{aligned} &\sum_{i=0}^{2} R^i \|\nabla^i u\|_{L^\infty(B_{\frac{R}{2}}(0))} + R^{2+\alpha}[\nabla^2 u]_{C^{0,\alpha}(B_{\frac{R}{2}}(0))} \\ &\leq C(R^2\|f\|_{L^\infty(B_R(0))} + R^{2+\alpha}[f]_{C^{0,\alpha}(B_R(0))} + \|u\|_{L^\infty(B_R(0))}), \end{aligned}$$

which gives (4.65) in the case $z = 0$. Similarly, we obtain (4.66) using Lemma 4.15. □

4.6.2 Identities and inequalities involving modified Bessel functions

Here we collect some properties on the modified Bessel functions K_η.

General properties

For $\eta \geq 0$, the modified Bessel function K_η is defined on $(0, \infty)$ by the integral representation

$$K_\eta(x) = \int_0^\infty e^{-x\cosh(t)} \cosh(\eta t)dt \qquad \text{for } x > 0.$$

Derivatives

For all $x \in (0, +\infty)$, we have

$$x\frac{K'_{\eta+1}(x)}{K_\eta(x)} = -x - (\eta+1)\frac{K_{\eta+1}(x)}{K_\eta(x)}, \tag{4.68}$$

$$x\frac{K'_\eta(x)}{K_\eta(x)} = \eta - x\frac{K_{\eta+1}(x)}{K_\eta(x)}, \tag{4.69}$$

see e.g. [91, Page 6]) or and [7] and [27].

Asymptotic behaviour

Asymptotics of K_η are given e.g. in [91, Page 4]). In particular, we have for all $\eta > 0$,

$$K_\eta(x) \sim \frac{\sqrt{\pi}}{\sqrt{2}} x^{-\frac{1}{2}} e^{-x} \quad \text{as} \quad x \longrightarrow +\infty, \tag{4.70}$$

$$K_\eta(x) \sim \frac{1}{2}\Gamma(\eta)\Big(\frac{x}{2}\Big)^{-\eta} \quad \text{as} \quad x \longrightarrow 0. \tag{4.71}$$

Inequalities

The following inequality identity can be found in [91]: For every $\rho > 0$ and $\eta \geq 0$,

$$\frac{K_{\eta+1}(\rho)}{K_\eta(\rho)} > \frac{\eta + \sqrt{\rho^2 + \eta^2}}{\rho}. \tag{4.72}$$

Chapter 5

Nontrivial contractible domains in $\mathbb{S}^d$

This chapter is concerned with the problem (1.7) on the sphere. Firstly, we give a more precise statement of Theorem 1.3 and some notations which will be used later. Then we show that there exists a radial family of solutions to the related Dirichlet problem. Then, we construct the nonlinear Dirichlet-to-Neumann operator and compute its linearization under certain nondegeneracy assumptions, which will be verified. Furthermore, we will study the spectral properties of the linearized operator computed before. With all those ingredients we can apply a local bifurcation argument to prove our main result. Finally, we complete the proof of the quantitative version of the Implicit Function Theorem and two important inequalities in the appendix.

5.1 Notations and statement of the main result

If $k > 0$, let $\mathbb{S}^d(k)$ be the d-dimensional sphere of radius $\frac{1}{k}$ naturally embedded in $\mathbb{R}^{d+1}$ ($d \geq 2$). We consider $\mathbb{S}^d(k)$ as a Riemannian manifold with the metric g_k endowed by its embedding in $\mathbb{R}^{d+1}$. The sectional curvature of such manifold is equal to k. When $k = 1$ we write directly $\mathbb{S}^d$ as the usual notation. We fix two opposite points $S, N \in \mathbb{S}^d(k)$ (let's say respectively the south and the north pole) and we use the exponential map of $\mathbb{S}^d(k)$ centered at S,

$$\exp_S : B\left(0, \frac{\pi}{k}\right) \to \mathbb{S}^d(k) \setminus \{N\},$$

where $B\left(0, \frac{\pi}{k}\right) \subset \mathbb{R}^d$ is the Euclidean ball of radius $\frac{\pi}{k}$ centered at the origin. Given any continuous function $v : \mathbb{S}^{d-1} \to \left(0, \frac{\pi}{k}\right)$, we define the domain

$$B_v = \exp_S\left(\left\{x \in \mathbb{R}^d : 0 \leq |x| < v\left(\frac{x}{|x|}\right)\right\}\right) \subset \mathbb{S}^d(k)\,.$$

Through this chapter we take $\alpha \in (0, \min\{p-1, 1\})$ fixed. The precise statement of our result is the following:

Theorem 5.1. *Let $d \in \mathbb{N}$, $d \geq 2$, let $1 < p < \frac{d+2}{d-2}$ ($p > 1$ if $d = 2$). Then, there exists a real number $k_0 > 0$, such that for any $0 < k < k_0$ the following holds true: there exist a sequence of real parameters $\lambda_m = \lambda_m(k)$ converging to some $\lambda_*(k) > 0$, a sequence of nonconstant functions $v_m = v_m(k) \in C^{2,\alpha}(\mathbb{S}^{d-1})$ converging to 0 in $C^{2,\alpha}$ sense, and a sequence of positive functions $u_m \in C^{2,\alpha}(\mathbb{S}^d(k) \setminus B_{1+v_m})$, such that the problem*

$$\begin{cases} -\lambda_m \Delta u_m + u_m - u_m^p = 0 & \text{in } \mathbb{S}^d(k) \setminus B_{1+v_m} \ , \\ u_m = 0 & \text{on } \partial B_{1+v_m} \ , \\ \partial_\nu u_m = c & \text{on } \partial B_{1+v_m} \ , \end{cases}$$

is satisfied.

Observe that Theorem 1.3 follows at once from the previous result, by a scale change transforming $\mathbb{S}^d(k)$ into $\mathbb{S}^d$.

Throughout this chapter we shall use the coordinates in $\mathbb{S}^d(k)$ given by the exponential map centered at the south pole composed with polar coordinates in $\mathbb{R}^d$. In other words, we write:

$$\begin{aligned} X : \big[0, \tfrac{\pi}{k}\big) \times \mathbb{S}^{d-1} \to \mathbb{S}^d(k) \setminus \{N\} \\ X(r, \theta) = \exp_S(r\,\theta). \end{aligned} \tag{5.1}$$

Observe that X is well defined but singular at the south pole S.

With this notation, the set B_v can be written as:

$$B_v = \left\{ (r, \theta) \in \left[0, \frac{\pi}{k}\right) \times \mathbb{S}^{d-1} : \ r < v(\theta) \right\}.$$

Moreover, the standard metric g_k on $\mathbb{S}^d(k)$ and its corresponding Laplace-Beltrami operator can be written in (r, θ) coordinates as:

$$g_k = dr^2 + S_k^2(r) d\theta^2,$$

$$\Delta := \Delta_{g_k} = \partial^2 r + (d-1)\frac{C_k(r)}{S_k(r)}\partial_r + \frac{1}{S_k^2(r)}\Delta_{\mathbb{S}^{d-1}},$$

where

$$S_k(r) = \frac{\sin(kr)}{k}, \ C_k(r) = \cos(kr). \tag{5.2}$$

See for instance [14].

Sometimes it will be useful to consider $S_k(r)$, $C_k(r)$ defined for all $r > 0$; in such case, $S_k(r) = C_k(r) = 0$ for all $r \geq \pi/k$.

In order to prove Theorem 5.1 we will make use of symmetry groups. Given a group of isometries G acting on $\mathbb{S}^{d-1}$, we say that $\Omega \subset \mathbb{S}^d(k)$ is G-symmetric if, working in coordinates:

$$(r,\theta) \in \Omega \Rightarrow (r, g(\theta)) \in \Omega$$

for any $g \in G$. Let us point out that in the case in which $G = \mathcal{G}$ the group of all possible symmetries of $\mathbb{S}^{d-1}$, then G-symmetry is just axial symmetry in $\mathbb{S}^d(k)$ (with respect to the axis of $\mathbb{R}^{d+1}$ passing through the south and the north poles).

We define:

$$C^{k,\alpha}_G(\mathbb{S}^{d-1}) = \{u \in C^{k,\alpha}(\mathbb{S}^{d-1}) : u = u \circ g \ \forall g \in G\}.$$

For later purposes we also define the set of functions in $C^{k,\alpha}_G(\mathbb{S}^{d-1})$ whose mean is 0:

$$C^{k,\alpha}_{G,0}(\mathbb{S}^{d-1}) = \left\{ u \in C^{k,\alpha}_G(\mathbb{S}^{d-1}) : \int_{\mathbb{S}^{d-1}} u = 0 \right\}.$$

If $\Omega \subset \mathbb{S}^d(k)$ is G-symmetric, we define the following Hölder spaces of G-symmetric functions:

$$\begin{aligned} C^{k,\alpha}_G(\Omega) &= \{u \in C^{k,\alpha}(\Omega) : u(r,\theta) = u(r,g(\theta)) \ \forall g \in G\}, \\ C^{k,\alpha}_{G,0}(\Omega) &= \{u \in C^{k,\alpha}_G(\Omega) : \ u = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

In addition, we denote the Sobolev spaces of G-symmetric functions as follows:

$$\begin{aligned} H^1_G(\Omega) &= \{u \in H^1(\Omega) : \ u(r,\theta) = u(r,g(\theta)) \ \forall g \in G\}, \\ H^1_{0,G}(\Omega) &= \{u \in H^1_0(\Omega) : \ u(r,\theta) = u(r,g(\theta)) \ \forall g \in G\}. \end{aligned}$$

Let us recall that the norm of the Sobolev space $H^1(\Omega)$ is given by:

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega \left(|\nabla u|^2 + u^2\right) \mathrm{dvol}_{g_k} \right)^{1/2}$$

where the gradient depends on the metric g_k. If $\Omega = \mathbb{S}^d(k) \setminus B_v$, we can write this expression in coordinates (r,θ) as:

$$\int_\Omega \left(|\nabla u|^2 + u^2\right) \mathrm{dvol}_{g_k} = \int_{v(\theta)}^{\pi/k} \int_{\mathbb{S}^{d-1}} S_k^{d-1}(r) \left[(\partial_r u)^2 + \frac{|\nabla_\theta u|^2}{S_k^2(r)} + u^2 \right] d\theta \, dr.$$

In the axisymmetric case of $\mathbb{S}^d(k)$ with respect to the two poles (that is, $G = \mathcal{G}$ the group of all possible isometries in $\mathbb{S}^{d-1}$) we write a subscript r instead of G, to highlight that the spaces depend only on the r variable. Hence, we shall write:

$$H^1_r(\mathbb{S}^d(k) \setminus B_1) = \left\{ u : \left[1, \frac{\pi}{k}\right) \to \mathbb{R}, \ \|u\|_k < \infty \right\},$$

where

$$\|u\|_k = \left(\int_1^{\pi/k} S_k^{d-1}(r) \left[(\partial_r u)^2 + u^2 \right] dr \right)^{1/2}. \tag{5.3}$$

Clearly,

$$\|u\|_{H_r^1(\mathbb{S}^d(k)\setminus B_1)} = \sqrt{\omega_{d-1}}\, \|u\|_k,$$

where ω_{d-1} is the $(d-1)$-dimensional measure of the unit sphere $\mathbb{S}^{d-1}$.

Moreover we shall make use of the notation:

$$H_{0,r}^1(\mathbb{S}^d(k) \setminus B_1) = \left\{ u \in H_r^1(\mathbb{S}^d(k) \setminus B_1),\ u(1) = 0 \right\}.$$

Observe that in the above definitions the functions $u(r)$ are absolutely continuous in $\left[1, \frac{\pi}{k}\right)$ (possibly singular at $\frac{\pi}{k}$).

We denote by $\mu_i = i(i+d-2), i \in \mathbb{N}$ the eigenvalues of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^{d-1}}$ in $\mathbb{S}^{d-1}$. From now on, we shall fix a symmetry group G in $\mathbb{S}^{d-1}$ satisfying the following property:

(G) Defining by $\{\mu_{i_l}\}_{l\in\mathbb{N}}$ the eigenvalues of $\Delta_{\mathbb{S}^{d-1}}$ restricted to G-symmetric functions and by m_l their multiplicities, we require $i_1 \geq 2$ and m_1 odd.

A group satisfying those properties is the dihedric group $\mathbb{D}_n$, $n \geq 2$, if $d = 2$. For $d > 2$ one can take for instance $G = O(2) \times O(d-2)$. Other examples are possible, see [69, Remark 2.2].

5.2 Existence of the axisymmetric solution to the Dirichlet problem

As mentioned before, we will use a local bifurcation argument. On that purpose, we first need to build an axially symmetric solution to the problem:

$$\begin{cases} -\lambda \Delta u + u - u^p = 0 & \text{in } \mathbb{S}^d(k) \setminus B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{5.4}$$

for suitable values of λ. This is the goal of this section. We will do it for k sufficiently small (i.e. for spheres $\mathbb{S}^d(k)$ with large radius, and then small curvature).

By using the coordinates (r, θ) as in (5.1), we have to find a solution $u(r)$ of the ODE problem:

$$\begin{cases} -\lambda \left[\partial_r^2 + (d-1)\frac{C_k(r)}{S_k(r)} \partial_r \right] u + u - u^p = 0 & r \in \left(1, \frac{\pi}{k}\right), \\ u(1) = 0, \\ u'(\frac{\pi}{k}) = 0, \end{cases} \tag{5.5}$$

where $C_k(r)$ and $S_k(r)$ are defined in (5.2). Observe that, for any fixed $r>0$,

$$\frac{C_k(r)}{S_k(r)} \to \frac{1}{r} \quad \text{as} \quad k \to 0.$$

Hence, at least formally, a limit problem for (5.5) as $k \to 0$ is:

$$\begin{cases} -\lambda \left(\partial_r^2 + \frac{d-1}{r}\partial_r\right) u + u - u^p = 0 & r > 1, \\ u(1) = 0. \end{cases} \tag{5.6}$$

Those are just radially symmetric functions of the Dirichlet problem:

$$\begin{cases} -\lambda\Delta u + u - u^p = 0 & \text{in } \mathbb{R}^d \setminus B_1, \\ u = 0 & \text{on } \partial B_1 \ . \end{cases} \tag{5.7}$$

In the proposition below we list some known properties of this problem.

Proposition 5.2. *We have:*

a) *For any $\lambda > 0$, there exists a positive radially symmetric C^∞ solution of (5.7). This solution increases in the radius up to a certain maximum, and then it decreases and converges to 0 at infinity exponentially.*

b) *Such positive and radial solution to (5.7) is unique: we denote it by $\tilde{u}_\lambda$. Moreover it has an exponential decay (see [84], for instance):*

$$\tilde{u}_\lambda(x) \sim |x|^{\frac{d-1}{2}} e^{-\frac{1}{\sqrt{\lambda}}|x|} \ \textit{as}\ |x| \to +\infty.$$

c) *Set $B_1^c = \mathbb{R}^d \setminus B_1$, let $H^1_{0,r}(B_1^c)$ be the classical Sobolev space $H^1_0(B_1^c)$ restricted to radial functions, and let $H^{-1}_r(B_1^c)$ be its dual. Let us define the linearized operator $L_\lambda : H^1_{0,r}(B_1^c) \to H^{-1}_r(B_1^c)$,*

$$L_\lambda(\phi) = -\lambda\Delta\phi + \phi - p\tilde{u}_\lambda^{p-1}\phi\,,$$

and consider the eigenvalue problem:

$$L_\lambda(\phi) = \tau\,\phi.$$

This problem has a unique negative eigenvalue and no zero eigenvalues. In other words, $\tilde{u}_\lambda$ is nondegenerate in $H^1_{0,r}(B_1^c)$ and has Morse index 1. We denote by $\tilde{z}_\lambda \in H^1_{0,r}(B_1^c)$ (normalized by $\|\tilde{z}_\lambda\| = 1$) the positive eigenfunction with negative eigenvalue, i.e.

$$\begin{cases} -\lambda\,\Delta\tilde{z}_\lambda + \tilde{z}_\lambda - p\tilde{u}_\lambda^{p-1}\tilde{z}_\lambda = \tilde{\tau}_\lambda\tilde{z}_\lambda & \textit{in}\ \ B_1^c, \\ \tilde{z}_\lambda = 0 & \textit{on}\ \ \partial B_1, \end{cases} \tag{5.8}$$

where $\tilde{\tau}_\lambda < 0$. Moreover $\tilde{z}_\lambda$ is a C^∞ function.

Statement a) is quite well known and has been proved in [29], for instance. The results b) and c) are more recent and have been obtained in [40, 87].

We now state the main result of this section.

Proposition 5.3. *Given $\varepsilon \in (0,1)$, there exists $k_0 > 0$ such that for any $k \in (0, k_0)$ and any $\lambda \in [\varepsilon, 1/\varepsilon]$ there exists a positive solution $u_{k,\lambda} \in C^{2,\alpha}(\mathbb{S}^d(k) \setminus B_1)$ to the problem (5.5). Moreover, for any $\lambda \in [\varepsilon, 1/\varepsilon]$,*

$$\lim_{k \to 0} \|u_{k,\lambda} - \tilde{u}_\lambda\|_k = 0, \tag{5.9}$$

where $\tilde{u}_\lambda$ is the unique positive radial solution of (5.7) and $\|\cdot\|_k$ is given by (5.3).

In order to prove the Proposition 5.3, we use the following version of the Inverse Function Theorem. We include its proof in the Appendix for the sake of completeness.

Proposition 5.4. *Let Y be a Hilbert space, $v \in Y$ and $F \in C^1(Y, Y)$. Suppose that:*

(A1) $\|F(v)\| < \delta$ for some fixed $\delta > 0$;

(A2) The derivative operator $F' : Y \to Y$ is invertible and $\|F'(v)^{-1}\| \leq c_0$, for some $c_0 > 0$;

(A3) Define $U = \{z \in Y : \|z\| \leq 2c_0\, \delta\}$ and assume that:

$$\|F'(v+z) - F'(v)\| < \frac{1}{2c_0}, \forall z \in U.$$

Then there exists a unique $z \in U$ such that $F(v+z) = 0$. Moreover, $\|F'(v+z)^{-1}\| \leq 2c_0$.

In order to apply the previous proposition to our purpose we will need the following lemma, whose proof again is postponed to the Appendix.

Lemma 5.5. *For any $u \in H_0^1(\mathbb{S}^d(k) \setminus B_1), 0 < k < \frac{\pi^2}{2}$, we have*

$$\|u\|_{L^s} \leq C\|u\|_{H^1},$$

where $2 < s \leq 2^* = \frac{2d}{d-2}$ if $d \geq 3$, $s > 2$ if $d = 2$, and $C = C(d,s) > 0$ is a constant independent of k.

With these two results we are now able to prove Proposition 5.3.

Proof of Proposition 5.3. We will apply the Proposition 5.4 in the Sobolev space

$$Y = H^1_{0,r}(\mathbb{S}^d(k) \setminus B_1) \tag{5.10}$$

via the coordinates given in (5.1). We define the operator

$$F(u) = \Phi(\tilde{F}(u)) : Y \to Y,$$

where $\Phi : Y^{-1} \to Y$ is the isomorphism given by the Riesz Representation Theorem and the functional $\tilde{F} : Y \to Y^{-1}$ is defined by

$$\tilde{F}(u)w = \omega_{d-1} \int_1^{\frac{\pi}{k}} S_k^{d-1}(r)(\lambda u'w' + uw - (u^+)^p w)\, dr,$$

where Y^{-1} is the dual space of Y.

Observe that if $F(u) = 0$, then u is a solution of problem (5.4) by the maximum principle. In what follows we fix $\lambda \in [\varepsilon, 1/\varepsilon]$ for some $\varepsilon \in (0,1)$ and define $v := v_{k,\lambda} = \tilde{u}_\lambda \chi_k$, where $\tilde{u}_\lambda$ is the positive solution of (5.6) and $0 \le \chi_k \le 1$ is a cut-off function such that:

$$\begin{cases} \chi_k(r) = 1 & r \in \left(1, \frac{\pi}{\sqrt{k}}\right), \\ \chi_k(r) = 0 & r \in \left(\frac{2\pi}{\sqrt{k}}, \frac{\pi}{k}\right), \\ |\chi_k'(r)| \le k^{-1/4} & r \in \left(1, \frac{\pi}{k}\right). \end{cases}$$

This function will play the role of v in Proposition 5.4. Now, let us verify that the assumptions of Proposition 5.4 are satisfied. For that, the exponential decay of $\tilde{u}_\lambda$ will be essential. First, it is clear that $\tilde{F}$ is a C^1 map.

(A1) Let's prove that $\|\tilde{F}(v)\| = o_k(1)$. We compute

$$
\begin{aligned}
\left|\frac{\tilde{F}(v)w}{\omega_{d-1}}\right| &= \left|\int_1^{\frac{\pi}{k}} S_k^{d-1}(r)(\lambda v'w' + vw - v^p w)dr\right| \\
&= \left|\int_1^{\frac{\pi}{k}} S_k^{d-1}(r)\left[\lambda(\tilde{u}_\lambda\chi_k)'w' + (\tilde{u}_\lambda\chi_k w - \tilde{u}_\lambda^p\chi_k^p w + \tilde{u}_\lambda^p\chi_k w - \tilde{u}_\lambda^p\chi_k w)\right]dr\right| \\
&= \left|\int_1^{\frac{\pi}{k}} S_k^{d-1}(r)\left[\lambda(\tilde{u}_\lambda\chi_k' w' - \tilde{u}_\lambda'\chi_k' w) + \tilde{u}_\lambda^p(\chi_k - \chi_k^p)w\right.\right. \\
&\qquad \left.\left. + (d-1)\lambda\tilde{u}_\lambda'\chi_k w\left(\frac{1}{r} - \frac{C_k(r)}{S_k(r)}\right)\right]dr\right| \\
&\le \left|\int_1^{\frac{2\pi}{\sqrt{k}}} (d-1)\lambda\tilde{u}_\lambda' w S_k^{d-1}(r)\left(\frac{1}{r} - \frac{C_k(r)}{S_k(r)}\right)dr\right| \\
&\qquad + \left|\int_{\frac{\pi}{\sqrt{k}}}^{\frac{\pi}{k}} S_k^{d-1}(r)\left[\lambda(\tilde{u}_\lambda\chi_k' w' - \tilde{u}_\lambda'\chi_k' w) + \tilde{u}_\lambda^p(\chi_k - \chi_k^p)w\right]dr\right| \\
&\le c_1 B\|w\|_k + c_2\frac{1}{\varepsilon}\left(\frac{e^{-\frac{\pi}{\sqrt{k\lambda}}}}{k^{\frac{d-1}{4}}} + \frac{e^{-\frac{p\pi}{\sqrt{k\lambda}}}}{k^{\frac{d-1}{2}}} + \frac{e^{-\frac{\pi}{\sqrt{k\lambda}}}}{k^{\frac{d-3}{2}}}\right)\|w\|_k \\
&\le c_1 B\|w\|_k + c_2\frac{1}{\varepsilon}\left(\frac{e^{-\frac{\sqrt{\varepsilon}\pi}{\sqrt{k}}}}{k^{\frac{d-1}{4}}} + \frac{e^{-\frac{p\sqrt{\varepsilon}\pi}{\sqrt{k}}}}{k^{\frac{d-1}{2}}} + \frac{e^{-\frac{\sqrt{\varepsilon}\pi}{\sqrt{k}}}}{k^{\frac{d-3}{2}}}\right)\|w\|_k,
\end{aligned}
$$

where $B = \left|1 - \frac{2\sqrt{k}\pi}{\tan(2\sqrt{k}\pi)}\right|$.

(A2) Let's prove that $\tilde{F}'$ is invertible and $\|\tilde{F}'(v)^{-1}\| \le c_0, c_0 > 0$. By contradiction, we suppose that either $\tilde{F}'(v)$ is not invertible or $\|\tilde{F}'(v)^{-1}\|$ is not bounded. We first assume that $\tilde{F}'(v)$ is not invertible, then we can have that $\tilde{F}'(v)$ is not injective (see [4, Lemma 4.1]). Suppose that $\tilde{F}'(v)$ is injective, let us write $\tilde{F}'(v)(\psi_1,\psi_2) = (\mathcal{A}\psi_1,\psi_2)$ for all $\psi_1,\psi_2 \in Y$ that is given in (5.10), where the operator

$$
\begin{aligned}
\mathcal{A} &: Y \to Y^{-1} \\
&\psi \mapsto -\lambda\Delta\psi + \psi - pv^{p-1}\psi.
\end{aligned}
$$

Notice that the operator $\mathcal{A}_0 : \psi \to -\lambda\Delta\psi + \psi$ is an isomorphism from Y to Y^{-1}. And the operator $K : \psi \to pv^{p-1}\psi$ is compact from Y to Y^{-1} since $v := v_{k,\lambda} = \tilde{u}_\lambda\chi_k$ and $\tilde{u}_\lambda$ converges to 0 at infinity exponentially. Since $\mathcal{A} = \mathcal{A}_0(I - \mathcal{A}_0^{-1}K)$ is injective and $\mathcal{A}_0^{-1}K$ is compact, the operator $\mathcal{A}$ is invertible by the Fredholm alternative for $I - \mathcal{A}_0^{-1}K$. This is a contradiction. Then there exists a non-zero function $\varphi_k \in Y$ such that

$$\tilde{F}'(v)\varphi_k = 0.$$

Now, in the case $\|\tilde{F}'(v)^{-1}\|$ is unbounded, we assume that $\|\tilde{F}'(v)^{-1}\| \to \infty$ as $k \to 0$. Taking $\phi_k \in Y^{-1}$ so that $\tilde{F}'(v)^{-1}\phi_k = \xi_k$ is a divergence sequence of Y.

Let $\varphi_k = \frac{\xi_k}{\|\xi_k\|_k}$, then $\|\varphi_k\|_k = 1$ and define $\tilde{F}'(v)^{-1}\frac{\phi_k}{\|\xi_k\|_k} = \varphi_k$, one has that

$$\|\tilde{F}'(v)\varphi_k\| = \frac{\|\phi_k\|}{\|\xi_k\|_k} \to 0.$$

Therefore, it suffices to consider

$$\tilde{F}'(v)\varphi_k \to 0,$$

with $\|\varphi_k\|_k = 1$. First, we verify that $\varphi_k \rightharpoonup \varphi_0$ in $H^1_{r,loc}(\mathbb{R}^d \setminus B_1)$ and $\varphi_0 \in H^1_{0,r}(\mathbb{R}^d \setminus B_1)$.

In fact, for $k = k(i,j)$ sufficiently small and a some fixed constant $M > 1$, we use the Cantor's diagonal argument

$$\begin{array}{cccccc}
\varphi_{k(1,1)} & \varphi_{k(1,2)} & \cdots & \varphi_{k(1,n)} \rightharpoonup \varphi_0^1 & \text{in } H^1_r(B_M \setminus B_1) \\
\varphi_{k(2,1)} & \varphi_{k(2,2)} & \cdots & \varphi_{k(2,n)} \rightharpoonup \varphi_0^2 & \text{in } H^1_r(B_{2M} \setminus B_1) \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\varphi_{k(n,1)} & \varphi_{k(n,2)} & \cdots & \varphi_{k(n,n)} \rightharpoonup \varphi_0^n & \text{in } H^1_r(B_{nM} \setminus B_1) \\
\vdots & \vdots & \cdots & \vdots & \vdots
\end{array}$$

We consider the diagonal subsequence $\varphi_k = \varphi_{k(n,n)}$. Fix $m \in \mathbb{N}$, we can check that $(\varphi_{k(n,n)})_{n\geq m} \subset (\varphi_{k(m,n)})_{n\geq 1}$ with $\varphi_{k(m,n)} \rightharpoonup \varphi_0^m$ in $H^1_r(B_{mM} \setminus B_1)$ since each row is a subsequence of all previous rows. Then, by the uniqueness of limits, the function $\varphi_0 = \varphi_0^m$ is well-defined. Thus we have $\varphi_k \rightharpoonup \varphi_0$ in $H^1_{r,loc}(\mathbb{R}^d \setminus B_1)$. And we know that

$$\begin{aligned}
\int_1^\infty r^{d-1}(\varphi_0'^2 + \varphi_0^2)dr &= \lim_{M\to\infty} \int_1^M r^{d-1}(\varphi_0'^2 + \varphi_0^2)dr \\
&\leq (1+\epsilon)^{d-1} \lim_{M\to\infty} \liminf_{k\to 0} \int_1^M \left(\frac{r}{1+\epsilon}\right)^{d-1} (\varphi_k'^2 + \varphi_k^2)dr \\
&\leq (1+\epsilon)^{d-1} \lim_{M\to\infty} \liminf_{k\to 0} \int_1^M S_k^{d-1}(r)(\varphi_k'^2 + \varphi_k^2)dr \\
&\leq (1+\epsilon)^{d-1} \lim_{M\to\infty} \liminf_{k\to 0} \int_1^{\frac{\pi}{k}} S_k^{d-1}(r)(\varphi_k'^2 + \varphi_k^2)dr \\
&= (1+\epsilon)^{d-1},
\end{aligned}$$

with small ϵ. This shows that $\varphi_0 \in H^1_{0,r}(\mathbb{R}^d \setminus B_1)$.

Now we prove that φ_0 differs from 0 by contradiction. If $\varphi_0 = 0$,

$$\begin{aligned}
\frac{\tilde{F}'(v)(\varphi_k, \varphi_k)}{\omega_{d-1}} &= \int_1^{\frac{\pi}{k}} S_k^{d-1}(r)(\lambda\varphi_k'^2 + \varphi_k^2 - pv^{p-1}\varphi_k^2)dr \\
&\geq \min\{\lambda, 1\}\|\varphi_k\|_k^2 - p\int_1^{\frac{\pi}{k}} S_k^{d-1}(r)v^{p-1}\varphi_k^2 dr \\
&= \min\{\lambda, 1\} - p\int_1^{\frac{\pi}{k}} S_k^{d-1}(r)v^{p-1}\varphi_k^2 dr.
\end{aligned}$$

Then we can get a contradiction that the left-hand side converges to 0 but the right-hand side is greater and equal to $\min\{\lambda, 1\} > 0$. The latter result comes from the fact that

$$
\begin{aligned}
&\int_1^{\frac{\pi}{k}} S_k^{d-1}(r) v^{p-1} \varphi_k^2 \\
&= \int_1^M S_k^{d-1}(r) \tilde{u}_\lambda^{p-1} \varphi_k^2 dr + \int_M^{\frac{\pi}{k}} S_k^{d-1}(r) \tilde{u}_\lambda^{p-1} \chi_k^{p-1} \varphi_k^2 dr \\
&= \int_1^M r^{d-1} \tilde{u}_\lambda^{p-1} \varphi_0^2 dr + o_k(1) + \int_M^{\frac{\pi}{k}} S_k^{d-1}(r) \tilde{u}_\lambda^{p-1} \chi_k^{p-1} \varphi_k^2 dr \\
&\le \int_1^M r^{d-1} \tilde{u}_\lambda^{p-1} \varphi_0^2 dr + C e^{\frac{-(p-1)M}{\sqrt{\lambda}}} \|\varphi_k\|_k^2 + o_k(1),
\end{aligned}
$$

which can be taken arbitrarily small by choosing M and k appropriately.

Finally, for all $w \in C_0^\infty(B_M \setminus B_1)$,

$$
\begin{aligned}
\frac{\tilde{F}'(v)(\varphi_k, w)}{\omega_{d-1}} &= \int_1^{\frac{\pi}{k}} S_k^{d-1}(r)(\lambda \varphi_k' w' + \varphi_k w - p v^{p-1} \varphi_k w) dr \\
&= \int_1^M S_k^{d-1}(r)(\lambda \varphi_k' w' + \varphi_k w - p v^{p-1} \varphi_k w) dr \\
&\to \int_1^M r^{d-1}(\lambda \varphi_0' w' + \varphi_0 w - p \tilde{u}_\lambda^{p-1} \varphi_0 w) dr \\
&= \int_1^\infty r^{d-1}(\lambda \varphi_0' w' + \varphi_0 w - p \tilde{u}_\lambda^{p-1} \varphi_0 w) dr = 0.
\end{aligned}
$$

As a consequence, $\varphi_0 \neq 0$ solves the linearized problem

$$
\begin{cases}
-\lambda \varphi_0'' - (d-1)\frac{\lambda}{r}\varphi_0' + \varphi_0 - p\tilde{u}_\lambda^{p-1}\varphi_0 = 0 & \text{in } (1, \infty), \\
\varphi_0(1) = 0.
\end{cases}
$$

But this is a contradiction with Proposition 5.2, c).

(A3) Let's show that $\|\tilde{F}'(v) - F'(s)\| < \frac{1}{2c_0}$ for any $v, s \in Y$ with $\|v - s\| \le 2c_0\,\delta$. For any $\phi \in Y$, we have that

$$
\begin{aligned}
\left| \frac{\tilde{F}'(v)(\phi)w - \tilde{F}'(s)(\phi)w}{\omega_{d-1}} \right| &= p \left| \int_1^{\frac{\pi}{k}} S_k^{d-1}(r)(v^{p-1} - (s^+)^{p-1}) \phi w dr \right| \\
&\le p \int_1^{\frac{\pi}{k}} S_k^{d-1}(r) |\phi w| (\epsilon |s^+|^{p-1} + C_\epsilon |v - s^+|^{p-1}) \\
&\le p \int_1^{\frac{\pi}{k}} S_k^{d-1}(r) |\phi w| (\epsilon |s|^{p-1} + C_\epsilon |v - s|^{p-1}) \\
&\le C \|\phi\|_k \|w\|_k \left(\epsilon \|s\|_k^{p-1} + \|v - s\|_k^{p-1} \right) \\
&\le C(\epsilon + 2c_0\delta)^{p-1} \|\phi\|_k \|w\|_k \\
&< \frac{1}{2c_0} \|\phi\|_k \|w\|_k,
\end{aligned}
$$

where ϵ is arbitrary and C_ϵ is a constant.

The $C^{2,\alpha}$ regularity of $u_{k,\lambda}$ follows from usual Schauder regularity estimates. This concludes the proof of the proposition. $\square$

In next result we gather some properties of the boundedness and decay of the solution $u_{k,\lambda}$ that will be useful later.

Lemma 5.6. *Given $\varepsilon > 0$, consider $k_0 > 0$, k and λ as in Proposition 5.3. Then the following properties hold:*

1) *There exists $M > 0$ independent of k, λ such that $\|u_{k,\lambda}\|_{L^\infty} \leq M$.*

2) *For any $\delta > 0$ there exists $R > 0$ independent of k, λ such that $u_{k,\lambda}(r) < \delta$ for any $k < \pi/R$, $r \in (R, \pi/k)$.*

Proof. The proof of 1) follows immediately from the classical blow-up analysis of Gidas and Spruck, together with their Liouville theorem, see [42,43]. Observe that here we use in an essential way that $1 < p < \frac{d+2}{d-2}$ (if $d \geq 3$).

We now turn our attention to 2). Observe that by Proposition 5.3, $\|u_{k,\lambda}\|_k \leq C$. In particular, taking into account that $S_k(r) \geq 2r/\pi$ for all $r \in (1, \frac{\pi}{2k}]$,

$$\int_1^{\frac{\pi}{2k}} r^{d-1}(u_{k,\lambda}'(r)^2 + u_{k,\lambda}(r)^2)\, dr \leq C.$$

Then, the Radial Lemma of Strauss [84] gives us the desired decay for any $r \in (1, \frac{\pi}{2k}]$.

We now consider the estimate in the north hemisphere $\mathbb{S}^d_+(k)$. If it does not hold, then $u_{k,\lambda}$ attains a local maximum in such hemisphere for a sequence $k = k_n \to 0$. By the maximum principle, this local maximum is strictly bigger than 1.

We now multiply (5.4) by $v = (u_{k,\lambda} - 1)^+$ and integrate in $\mathbb{S}^d_+(k)$, to obtain that:

$$\lambda \int_{\mathbb{S}^d_+(k)} |\nabla v|^2 = \int_{\mathbb{S}^d_+(k)} \frac{u_{k,\lambda}^p - u_{k,\lambda}}{u_{k,\lambda} - 1} v^2 \leq C \int_{\mathbb{S}^d_+(k)} v^2,$$

by 1). We now define $\Sigma = \{x \in \mathbb{S}^d_+(k) : \; u_{k,\lambda}(x) > 1\}$. Therefore, we can use Hölder inequality for any $2 < q \leq \frac{2d}{d-2}$ (any $q > 2$ if $d = 2$), and Lemma 5.5, to obtain:

$$\lambda \int_{\mathbb{S}^d_+(k)} (|\nabla v|^2 + v^2) \leq (C+\lambda) \int_{\mathbb{S}^d_+(k)} v^2$$
$$\leq (C+\lambda)\Big(\int_{\mathbb{S}^d_+(k)} v^q\Big)^{2/q} |\Sigma|^{\frac{q-2}{q}} \leq C\Big(\int_{\mathbb{S}^d_+(k)} (|\nabla v|^2 + v^2)\Big)|\Sigma|^{\frac{q-2}{q}}.$$

This implies that the measure of Σ is uniformly bounded from below. Observe moreover that:

$$\int_{\mathbb{S}^d_+(k)} u_{k,\lambda}(x)^2\, dx \geq |\Sigma|,$$

which is bounded from below. But this is in contradiction with (5.9) and the exponential decay of $\tilde{u}_\lambda$. □

5.3 The Dirichlet-to-Neumann operator and its linearization

In this section we build the Dirichlet-to-Neumann operator to which we intend to apply a local bifurcation argument. In order to do this, some definitions are in order. First, let us fix a symmetry group G satisfying assumption (G).

For any solution given in Proposition 5.3, we define the linearized operator of the Dirichlet problem: $L^D = L^D_{k,\lambda} : H^1_{0,G}(\mathbb{S}^d(k) \setminus B_1) \to H^{-1}_G(\mathbb{S}^d(k) \setminus B_1)$ associated to (5.4) by

$$L^D(\phi) = -\lambda\Delta\phi + \phi - p u^{p-1}_{k,\lambda}\phi. \tag{5.11}$$

Proposition 5.7. *The operator L^D has a negative eigenvalue $\tau = \tau_{k,\lambda}$ with a radially symmetric eigenfunction $z = z_{k,\lambda}$.*

The proof is rather easy and will be presented in Section 5, when we introduce quadratic forms associated to L^D.

We also need the following result, that establishes the nondegeneracy of the Dirichlet operator:

Proposition 5.8. *There exist $0 < \lambda_0 < \lambda_1$ such that, by taking smaller $k_0 > 0$, if necessary, we have that for any $k \in (0, k_0)$, $\lambda \in [\lambda_0, \lambda_1]$, the eigenvalues of L^D different from τ are all strictly bigger than a positive constant ε independent of λ, k. In particular, the operator L^D is an isomorphism.*

The proof of this proposition is an immediate consequence of Proposition 5.14, iii). We emphasize that the values λ_0, λ_1, which will be fixed in the rest of the section, are given in Proposition 5.13 and depend only on d, p and G.

The main result of this section is the following:

Proposition 5.9. *Assume that $\lambda \in [\lambda_0, \lambda_1]$, and $k \in (0, k_0)$, where k_0 is given in Proposition 5.8. Then, there exists a neighborhood $\mathcal{U}$ of 0 in $C^{2,\alpha}_{G,0}(\mathbb{S}^{d-1})$, independent of λ, k, such that for any $v \in \mathcal{U}$, the problem*

$$\begin{cases} -\lambda\Delta u + u - u^p = 0 & \text{in } \mathbb{S}^d(k) \setminus B_{1+v}, \\ u > 0 & \text{in } \mathbb{S}^d(k) \setminus B_{1+v}, \\ u = 0 & \text{on } \partial B_{1+v}, \end{cases} \tag{5.12}$$

has a unique positive solution $u = u^v_{k,\lambda} \in C^{2,\alpha}(\mathbb{S}^d(k) \setminus B_{1+v})$ in a neighborhood of $u_{k,\lambda}$. Moreover the dependence of u on the function v is C^1 and $u^0_{k,\lambda} = u_{k,\lambda}$.

Proof. Let $v \in C^{2,\alpha}_G(\mathbb{S}^{d-1})$. It will be more convenient to consider the fixed domain $\mathbb{S}^d(k) \setminus B_1$ endowed with a new metric depending on v. This will be possible by considering the parameterization of $\mathbb{S}^d(k) \setminus B_{1+v}$ defined by $\Xi : \mathbb{S}^d(k) \setminus B_1 \to \mathbb{S}^d(k) \setminus B_{1+v}$,

$$\Xi(r, \theta) := \left((1 + \chi(r)v(\theta))\, r, \theta \right), \tag{5.13}$$

where χ is a cut-off function

$$\chi(r) = \begin{cases} 0, & r \geq \frac{3}{2}, \\ 1, & r \leq \frac{5}{4}. \end{cases}$$

Therefore, we consider the coordinates $(r, \theta) \in \left(1, \frac{\pi}{k}\right) \times \mathbb{S}^{d-1}$ from now on, and we can write the new metric in these coordinates as

$$g_v = a^2 dr^2 + 2ab\,dr\,dv + b^2 dv^2 + S_k^2\left((1 + \chi(r)v(\theta))\, r \right) \mathring{h},$$

where $a = 1 + \chi(r)v(\theta) + \chi'(r)v(\theta)r$, $b = \chi(r)r$, and $\mathring{h}$ is the standard metric on $\mathbb{S}^{d-1}$ induced by the Euclidean one. It is clear that $a = 1$ if $v = 0$. Up to some multiplicative constant, we can now write the problem (5.12) as

$$\begin{cases} -\lambda\Delta_{g_v}\hat{u} + \hat{u} - \hat{u}^p = 0 & \text{in } \mathbb{S}^d(k) \setminus B_1\ , \\ \hat{u} = 0 & \text{on } \partial B_1. \end{cases} \tag{5.14}$$

As $v \equiv 0$, the metric g_v is just the round metric g_k, and $\hat{u} = u_{k,\lambda}$ is therefore a solution of (5.14). In the general case, the expression between the function u and the function $\hat{u}$ can be represented by

$$\hat{u} = \Xi^* u.$$

For all $\psi \in C^{2,\alpha}_{G,0}(\mathbb{S}^d(k) \setminus B_1)$, we define:

$$N(v, \psi, \lambda) := -\lambda\Delta_{g_v}\psi + \psi - [\psi^+]^p,$$

where ψ^+ is the positive part of ψ. By Proposition 5.3 we have that

$$N(0, u_{k,\lambda}, \lambda) = 0.$$

The mapping N is C^1 from a neighborhood of $(0, u_{k,\lambda}, \lambda)$ in $C^{2,\alpha}_G(\mathbb{S}^{d-1})\times C^{2,\alpha}_{G,0}(\mathbb{S}^d(k)\setminus B_1) \times [\lambda_0, \lambda_1]$ into $C^{0,\alpha}_G(\mathbb{S}^d(k) \setminus B_1)$. The partial differential of N with respect to ψ at $(0, u_{k,\lambda}, \lambda)$ is

$$D_\psi N|_{(0,u_{k,\lambda},\lambda)}(\psi) = -\lambda\Delta\psi + \psi - pu^{p-1}_{k,\lambda}\psi.$$

Observe that $D_\psi N|_{(0,0,\lambda)}(\psi)$ is precisely invertible by the fact that the operator $L^D_{k,\lambda}$ is nondegenerate, see Proposition 5.8. This is true in the framework of Sobolev spaces, and also in the framework of Holder spaces by Schauder regularity.

The Implicit Function Theorem yields that there exists $\psi(v, \lambda) \in C^{2,\alpha}_{G,0}(\mathbb{S}^d(k) \setminus B_1)$ such that $N(v, \psi(v, \lambda), \lambda) = 0$ for v in a neighborhood of 0 in $C^{2,\alpha}_G(\mathbb{S}^{d-1})$. Observe that the neighborhood $\mathcal{U}$ can be taken uniformly in k by the quantitative version of the Implicit Function Theorem, see Proposition 5.4. By the maximum principle, $\psi(v, \lambda)$ is positive and solves (5.14). □

Let $k \in (0, k_0)$. After the canonical identification of ∂B_{1+v} with $\mathbb{S}^{d-1}$, we define $F_k : \mathcal{U} \times [\lambda_0, \lambda_1] \to C^{1,\alpha}_{G,0}(\mathbb{S}^{d-1})$,

$$F_k(v, \lambda) = \partial_\nu u\Big|_{\partial B_{1+v}} - \frac{1}{\mathrm{Vol}(\partial B_{1+v})}\int_{\partial B_{1+v}} \partial_\nu u.$$

Here $\mathcal{U}$ and $u = u^v_{k,\lambda}$ are as given by Proposition 5.9. Notice that $F_k(v, \lambda) = 0$ if and only if $\partial_\nu u$ is constant on the boundary ∂B_{1+v}. Obviously, $F_k(0, \lambda) = 0$ for all $\lambda \in [\lambda_0, \lambda_1]$. Our goal is to find a branch of nontrivial solutions (v, λ) to the equation $F_k(v, \lambda) = 0$ bifurcating from some point $(0, \lambda_*(k))$, $\lambda_*(k) \in [\lambda_0, \lambda_1]$. For this aim, we will use a local bifurcation argument. This leads to the study of the linearization of F_k around a point $(0, \lambda)$. To start that, we first show the following useful lemmas.

Lemma 5.10. *Assume that $\lambda \in [\lambda_0, \lambda_1]$, and $k \in (0, k_0)$, where k_0 is given in Proposition 5.8. Then for all $v \in C^{2,\alpha}_G(\mathbb{S}^{d-1})$, there exists a unique solution $\psi = \psi^v_{k,\lambda} \in C^{2,\alpha}_G(\mathbb{S}^d(k) \setminus B_1)$ to the problem*

$$\begin{cases} -\lambda\Delta\psi + \psi - p\, u^{p-1}_{k,\lambda}\psi = 0 & \text{in } \mathbb{S}^d(k) \setminus B_1, \\ \psi = v & \text{on } \partial B_1. \end{cases} \tag{5.15}$$

Proof. Let $\psi_0(x) \in C^{2,\alpha}_G(\mathbb{S}^d(k) \setminus B_1)$ such that $\psi_0|_{\partial B_1} = v$. If we set $w = \psi - \psi_0$, the problem (5.15) is equivalent to the problem

$$\begin{cases} -\lambda\Delta w + w - pu^{p-1}_{k,\lambda}w = -\Big(-\lambda\Delta\psi_0 + \psi_0 - pu^{p-1}_{k,\lambda}\psi_0\Big) & \text{in } \mathbb{S}^d(k) \setminus B_1, \\ w = 0 & \text{on } \partial B_1. \end{cases}$$

Observe that the right hand side of the above equation is in $H_G^{-1}(\mathbb{S}^d(k)\setminus B_1)$. Since by Proposition 5.8 the operator $L^D_{k,\lambda}$ is a bijection, there exists a solution w. By Schauder estimates, w has the required regularity and the result follows. □

Lemma 5.11. *Assume that $\lambda \in [\lambda_0, \lambda_1]$, and $k \in (0, k_0)$, where k_0 is given in Proposition 5.8. Let $v \in C^{2,\alpha}_{G,0}(\mathbb{S}^{d-1})$ and $\psi \in C^{2,\alpha}_G(\mathbb{S}^d(k) \setminus B_1)$ be the solution of (5.15). Then*

$$\int_{\mathbb{S}^d(k)\setminus B_1} \psi z = 0, \qquad \int_{\partial B_1} \partial_\nu \psi = 0.$$

Here z stands for the eigenfunction associated to the negative eigenvalue of L^D, as given in Proposition 5.7.

Proof. Let

$$L^D(z) = \tau z \ \text{ in } \mathbb{S}^d(k) \setminus B_1, \ z = 0 \text{ on } \partial B_1,$$

where L^D is given in (5.11). We now multiply the above equation by $\psi = \psi^v_{k,\lambda}$, the equation in (5.15) by z, and integrate by parts to gain

$$\int_{\partial B_1} \Big(\partial_\nu \psi \, z - \partial_\nu z \, \psi \Big) = \int_{\mathbb{S}^d(k)\setminus B_1} \tau z \psi \, .$$

Then we can at once gain the first identity by the facts that $z = 0$, $\partial_\nu z$ is constant and $\psi = v$ has 0 mean on ∂B_1.

We now define $\vartheta \in H^1_G(\mathbb{S}^d(k) \setminus B_1)$ as the unique solution of the problem

$$\begin{cases} -\lambda \Delta \vartheta + \vartheta - p u^{p-1}_{k,\lambda} \vartheta = 0 & \text{in } \mathbb{S}^d(k) \setminus B_1, \\ \vartheta = 1 & \text{on } \partial B_1, \end{cases} \tag{5.16}$$

whose existence has been proved in Lemma 5.10. Observe that, by uniqueness, ϑ is radially symmetric. Then we multiply the equation in (5.16) by ψ, the equation in (5.15) by ϑ, and integrate by parts to obtain

$$\int_{\partial B_1} \Big(\partial_\nu \psi \, \vartheta - \partial_\nu \vartheta \, \psi \Big) = 0.$$

Then we can immediately gain the second identity by the facts that $\vartheta = 1, \partial_\nu \vartheta$ is constant and $\psi = v$ on ∂B_1. □

For $\lambda \in [\lambda_0, \lambda_1]$, $k \in (0, k_0)$, we can define the linear continuous operator $H_{k,\lambda} : C^{2,\alpha}_{G,0}(\mathbb{S}^{d-1}) \to C^{1,\alpha}_{G,0}(\mathbb{S}^{d-1})$ by

$$H_{k,\lambda}(v) = \partial_\nu(\psi_v) - \frac{(d-1)k}{\tan(k)} \, v,$$

where ψ_v is given by Lemma 5.10. It is worth pointing out that the constant

$$\frac{(d-1)k}{\tan(k)}$$

is nothing but the mean curvature of $\partial B_1 \subset \mathbb{S}^d(k)$.

We show now that the linearization of the operator F_k with respect to v at $v = 0$ is given by $H_{k,\lambda}$, up to a constant.

Proposition 5.12. *For any $\lambda \in [\lambda_0, \lambda_1]$ and $k \in (0, k_0)$ we have*

$$D_v(F_k)(\lambda, 0) = \partial_r u(1)\, H_{k,\lambda}\,,$$

where $u = u_{k,\lambda}$.

Proof. By the C^1 regularity of F_k, it is enough to compute the linear operator obtained by the directional derivative of F_k with respect to v, computed at $(0, \lambda)$. Such derivative is given by

$$F_k'(w) = \lim_{s\to 0} \frac{F_k(sw,\lambda) - F_k(0,\lambda)}{s} = \lim_{s\to 0} \frac{F_k(sw,\lambda)}{s}.$$

Let $v = sw$, we consider the parameterization of $\mathbb{S}^d(k) \setminus B_{1+v}$ given in (5.13) for $(r, \theta) \in \left(1, \frac{\pi}{k}\right) \times \mathbb{S}^{d-1}$. Let g_v be the induced metric such that $\hat{u} = \hat{u}^v_{k,\lambda} = \Xi^* u^v_{k,\lambda}$ (smoothly depending on the real parameter s) solves the problem

$$\begin{cases} -\lambda\Delta_{g_v}\hat{u} + \hat{u} - \hat{u}^p = 0 & \text{in } \mathbb{S}^d(k)\setminus B_1,\\ \hat{u} = 0 & \text{on } \partial B_1. \end{cases}$$

We define that $\hat{u}_{k,\lambda} = \Xi^* u_{k,\lambda}$, which is a solution of

$$-\lambda\Delta_{g_v}\hat{u}_{k,\lambda} + \hat{u}_{k,\lambda} - \hat{u}^p_{k,\lambda} = 0$$

in $\mathbb{S}^d(k) \setminus B_1$ (notice that $u_{k,\lambda}$ is radial and then can be extended as a solution of (5.5) in a small inner neighborhood of ∂B_1), and

$$\hat{u}_{k,\lambda}(r,\theta) = u_{k,\lambda}\big((1+sw)r, \theta\big)$$

on ∂B_1. Let $\hat{u} = \hat{u}_{k,\lambda} + \hat{\psi}$, we can get that

$$\begin{cases} -\lambda\Delta_{g_v}\hat{\psi} + (\hat{u}_{k,\lambda} + \hat{\psi}) - (\hat{u}_{k,\lambda} + \hat{\psi})^p - \hat{u}_{k,\lambda} + \hat{u}^p_{k,\lambda} = 0 & \text{in } \mathbb{S}^d(k)\setminus B_1\ ,\\ \hat{\psi} = -\hat{u}_{k,\lambda} & \text{on } \partial B_1. \end{cases} \tag{5.17}$$

Obviously, $\hat{\psi}$ is differentiable with respect to s. When $s = 0$, we have $\hat{u} = u_{k,\lambda}$. Then, $\hat{\psi} = 0$ as $s = 0$. We set

$$\dot{\psi} = \partial_s \hat{\psi}|_{s=0}.$$

Differentiating (5.17) with respect to s and evaluating the result at $s = 0$, we get that

$$\begin{cases} -\lambda\Delta\dot{\psi} + \dot{\psi} - pu_{k,\lambda}^{p-1}\dot{\psi} = 0 & \text{in } \mathbb{S}^d(k) \setminus B_1, \\ \dot{\psi} = -\partial_r u_{k,\lambda}(1) w & \text{on } \partial B_1. \end{cases}$$

Then $\dot{\psi} = -\partial_r u_{k,\lambda}(1)\, \psi$ where $\psi = \psi_{k,\lambda}^v$ is as given in Lemma 5.10 (with $v = w$). Then, we can write

$$\hat{u}(r,\theta) = \hat{u}_{k,\lambda}(r,\theta) - s\partial_r u_{k,\lambda}(1)\, \psi(r,\theta) + o(s).$$

In particular, in $B_{5/4} \setminus B_1$ we have

$$\begin{aligned} \hat{u}(r,\theta) &= u_{k,\lambda}((1+sw(\theta))r,\theta) - s\partial_r u_{k,\lambda}(1)\, \psi(r,\theta) + o(s) \\ &= u_{k,\lambda}(r,\theta) + s\big(rw(\theta)\partial_r u_{k,\lambda}(r,\theta) - \partial_r u_{k,\lambda}(1)\, \psi(r,\theta)\big) + o(s). \end{aligned}$$

In order to complete the proof of the result, it is enough to calculate the normal derivation of the function $\hat{u}$ when the normal is calculated with respect to the metric g_v. Since the coordinates $(r,\theta) \in \big(1, \frac{\pi}{k}\big) \times \mathbb{S}^{d-1}$, the metric g_v can be expanded in $B_{5/4} \setminus B_1$ as

$$g_v = (1+sw)^2 dr^2 + 2sr(1+sw)drdw + s^2r^2dw^2 + S_k^2((1+sw)r)\, \mathring{h}\,,$$

where again $\mathring{h}$ is the standard metric on $\mathbb{S}^{d-1}$ induced by the Euclidean one. It follows from this expression that the unit normal vector field to ∂B_1 for the metric g_v is given by

$$\hat{\nu} = \big((1+sw)^{-1} + o(s)\big)\partial_r + o(s)\partial_{\theta_i},$$

where θ_i are the vector fields induced by a parameterization of $\mathbb{S}^{d-1}$. As a result,

$$g_v(\nabla\hat{u}, \hat{\nu}) = \partial_r u_{k,\lambda} + s\big(w\partial_r^2 u_{k,\lambda} - \partial_r u_{k,\lambda}\partial_r\psi\big) + o(s)$$

on ∂B_1. From the fact that $\partial_r u_{k,\lambda}$ and $\partial_r^2 u_{k,\lambda}$ are constant, while the term w and ψ have mean 0 on ∂B_1, and

$$-\lambda\left(\partial_r^2 u_{k,\lambda} + \frac{(d-1)k}{\tan(k)}\partial_r u_{k,\lambda}\right) = 0$$

on ∂B_1, we can conclude the proof of the result by using Lemma 5.11. □

5.4 Study of the linearized operator $H_{k,\lambda}$

In view of Proposition 5.12, a bifurcation of the branch $(0,\lambda)$ of solutions to the equation $F_k(v,\lambda) = 0$ might appear only at point $(0,\lambda_*)$ such that H_{k,λ_*} becomes degenerate. We will see that this is true for some λ_*.

As we shall see in Lemma 5.16, the behavior of $H_{k,\lambda}$ is related to the following quadratic form:

$$Q_{k,\lambda} : H^1_G(\mathbb{S}^d(k) \setminus B_1) \to \mathbb{R},$$

$$Q_{k,\lambda}(\psi) = \int_{\mathbb{S}^d(k)\setminus B_1} \left(\lambda|\nabla\psi|^2 + \psi^2 - pu_{k,\lambda}^{p-1}\psi^2\right) - \lambda\frac{(d-1)k}{\tan(k)}\int_{\partial B_1} \psi^2.$$

When restricted to functions that vanish at the boundary, we obtain the quadratic form associated to the Dirichlet problem:

$$Q^D_{k,\lambda} := Q_{k,\lambda}|_{H^1_{0,G}(\mathbb{S}^d(k)\setminus B_1)}.$$

Sometimes we will simply write Q^D instead of $Q^D_{k,\lambda}$.

We first show that Q^D attains negative values, which shows the validity of Proposition 5.7.

Proof of Proposition 5.7. In this proof we drop the subindices k, λ. Observe that by multiplying equation (5.4) by u and integrating by parts we obtain that:

$$\int_{\mathbb{S}^d(k)\setminus B_1} \lambda|\nabla u|^2 + |u|^2 - u^{p+1} = 0.$$

As a consequence,

$$Q^D(u) = -(p-1)\int_{\mathbb{S}^d(k)\setminus B_1} u^{p+1} < 0.$$

Then, the first eigenvalue of L^D is strictly negative. Since the first eigenfunction is simple, it is radially symmetric. □

In what follows it will be necessary to restrict those quadratic forms to the following spaces:

$$E_{k,\lambda} = \left\{\psi \in H^1_G(\mathbb{S}^d(k) \setminus B_1) : \int_{\partial B_1} \psi = 0, \int_{\mathbb{S}^d(k)\setminus B_1} \psi z_{k,\lambda} = 0\right\},$$

$$E^D_{k,\lambda} = \left\{\psi \in H^1_{0,G}(\mathbb{S}^d(k) \setminus B_1) : \int_{\mathbb{S}^d(k)\setminus B_1} \psi z_{k,\lambda} = 0\right\}.$$

As k tends to 0 we have formally limit quadratic forms related to the problem in

$\mathbb{R}^d \setminus B_1$. On that purpose, let us define:

$$\tilde{Q}_\lambda : H^1_G(\mathbb{R}^d \setminus B_1) \to \mathbb{R}$$

$$\tilde{Q}_\lambda(\psi) := \int_{\mathbb{R}^d \setminus B_1} \lambda |\nabla \psi|^2 + \psi^2 - p\tilde{u}_\lambda^{p-1}\psi^2 - \lambda(d-1)\int_{\partial B_1} \psi^2,$$

$$\tilde{Q}^D_\lambda = \tilde{Q}_\lambda|_{H^1_{0,G}(\mathbb{R}^d \setminus B_1)}.$$

Here $\tilde{u}_\lambda$ is the solution given in Proposition 5.2. We also define the analogous functional spaces:

$$\tilde{E}_\lambda = \left\{ \psi \in H^1_G(\mathbb{R}^d \setminus B_1) : \int_{\partial B_1} \psi = 0, \int_{\mathbb{R}^d \setminus B_1} \psi \tilde{z}_\lambda = 0 \right\},$$

$$\tilde{E}^D_\lambda = \left\{ \psi \in H^1_{0,G}(\mathbb{R}^d \setminus B_1) : \int_{\mathbb{R}^d \setminus B_1} \psi \tilde{z}_\lambda = 0 \right\}.$$

In order to facilitate the reading, sometimes we will drop the subscripts k, λ of the above definitions.

The behavior of the quadratic forms $\tilde{Q}_\lambda$ and $\tilde{Q}^D_\lambda$ has been studied in [69], and the following result holds true.

Proposition 5.13. . *There exist $0 < \lambda_0 < \lambda_1$ and $\varepsilon > 0$ such that:*

(i) $\tilde{Q}_{\lambda_0}(\psi) < -\varepsilon$ *for some* $\psi \in \tilde{E}_{\lambda_0}$, $\|\psi\|_{L^2} = 1$;

(ii) $\tilde{Q}_{\lambda_1}(\psi) > \varepsilon$ *for any* $\psi \in \tilde{E}_{\lambda_1}$, $\|\psi\|_{L^2} = 1$;

(iii) $\tilde{Q}^D_\lambda(\psi) > \varepsilon$ *for any* $\psi \in \tilde{E}^D_\lambda$, $\|\psi\|_{L^2} = 1$, *and any* $\lambda > \lambda_0$.

Proof. The proof is basically contained in [69]. We first recall the definitions given in [69, (3.6) and (5.7)] under our notations:

$$\Lambda_0 = \sup \left\{ \lambda > 0 : \tilde{Q}^D_\lambda(\psi) \le 0 \text{ for some } \psi \in \tilde{E}^D_\lambda,\ \psi \neq 0 \right\}.$$

$$\Lambda^* = \sup\{\lambda > 0 : \tilde{Q}_\lambda(\psi) < 0 \ \text{ for some } \psi \in \tilde{E}_\lambda\}.$$

It is proved in [69] that the above suprema exist and that $0 < \Lambda_0 < \Lambda^*$. Then, we can take $\lambda_0 \in (\Lambda_0, \Lambda^*)$ such that i) and iii) hold.

Moreover, [69, Proposition 5.3] implies that $\tilde{Q}(\psi) > 0$ for any $\psi \in \tilde{E}_\lambda$, $\psi \neq 0$, provided that λ is sufficiently large. Then we can take $\lambda_1 > \Lambda^*$ such that ii) is satisfied. $\square$

In next proposition we use a perturbation argument to prove an analogous result for small $k > 0$.

Proposition 5.14. *Fix λ_0 and λ_1 as given in Proposition 5.13. By taking $k_0 > 0$ smaller if necessary, and for any $k \in (0, k_0)$ there exists $\varepsilon > 0$ independent of k, λ such that:*

(i) $Q_{k,\lambda_0}(\psi) < -\varepsilon$ *for some* $\psi \in E_{k,\lambda_0}$, $\|\psi\|_{L^2} = 1$*:*

(ii) $Q_{k,\lambda_1}(\psi) > \varepsilon$ *for any* $\psi \in E_{k,\lambda_1}$, $\|\psi\|_{L^2} = 1$*:*

(iii) $Q^D_{k,\lambda}(\psi) > \varepsilon$ *for any* $\psi \in E^D_{k,\lambda}$, $\|\psi\|_{L^2} = 1$ *and any* $\lambda \in [\lambda_0, \lambda_1]$.

In particular, (iii) implies Proposition 5.8.

In order to prove the above Proposition, we need the following lemma, whose proof will be given in the Appendix.

Lemma 5.15. *For any function $\psi \in H^1_G(\mathbb{S}^d(k) \setminus B_1)$, we can verify that*

$$\|\psi\|^2_{L^2(\partial B_1)} \leq C\|\nabla\psi\|_{L^2(\mathbb{S}^d(k)\setminus B_1)}\|\psi\|_{L^2(\mathbb{S}^d(k)\setminus B_1)} + C\|\psi\|^2_{L^2(\mathbb{S}^d(k)\setminus B_1)}$$

for some constant $C > 0$, which does not depend on k.

Proof of Proposition 5.14. First, we prove that (i) holds. Let us give the following min-max characterization of the second eigenvalue related to the quadratic forms $Q_{k,\lambda}$ and $\tilde{Q}_\lambda$ as follows. Define $\mathcal{A}_k$ and $\tilde{\mathcal{A}}$ the class of 2-dimensional vector spaces in

$$\left\{\psi \in H^1_G(\mathbb{S}^d(k) \setminus B_1) : \int_{\partial B_1} \psi = 0\right\},$$

and

$$\left\{\psi \in H^1_G(\mathbb{R}^d \setminus B_1) : \int_{\partial B_1} \psi = 0\right\},$$

respectively. Then we are concerned with the infimum:

$$\inf_U\{\max\{Q_{k,\lambda}(\psi) : \psi \in U, \|\psi\|_{L^2(\mathbb{S}^d(k)\setminus B_1)} = 1\} : U \in \mathcal{A}_k\},$$

and

$$\inf_{\tilde{U}}\{\max\{\tilde{Q}_\lambda(\psi) : \psi \in \tilde{U}, \|\psi\|_{L^2(\mathbb{R}^d\setminus B_1)} = 1\} : U \in \tilde{\mathcal{A}}\}.$$

Observe that the last infimum is strictly negative for $\lambda = \lambda_0$ by Proposition 5.13, i). By density, there exists a 2-dimensional vector space of U of functions in $C^\infty_G(\mathbb{R}^d \setminus B_1)$ with support contained in a fixed compact set K such that

$$\max\{\tilde{Q}_{\lambda_0}(\psi) : \psi \in \tilde{U}, \|\psi\|_{L^2(\mathbb{R}^d\setminus B_1)} = 1\} < 0.$$

We also have

$$\max\{\tilde{Q}_{\lambda_0}(\psi) : \psi \in \tilde{U}, \|\psi\|_{L^2(\mathbb{S}^d(k)\setminus B_1)} = 1\} < 0$$

for k small, since both norms are equivalent in the 2-dimensional vector space $\tilde{U}$.

Therefore, for any $\psi \in \tilde{U}$ with $\|\psi\|_{L^2(\mathbb{S}^d(k)\setminus B_1)} = 1$, one has that

$$Q_{k,\lambda_0}(\psi) = \tilde{Q}_{\lambda_0}(\psi) + o_k(1) < 0.$$

Clearly, we can choose such ψ orthogonal to $z_{k,\lambda}$, so that

$$Q_{k,\lambda_0}(\psi) < 0,$$

for some $\psi \in E_{k,\lambda_0}$ as k small.

We now prove (ii). Since in what follows $\lambda = \lambda_1$ is fixed, we drop the subscripts of its dependence for the sake of clarity.

To reach a contradiction, we assume that there exists a sequence k_n, which converges to 0 as $n \to \infty$, such that $\tau_{2,n} := \tau_2(k_n) \leq o_n(1)$, where τ_2 is defined as:

$$\tau_2(k) = \inf\{Q_{k,\lambda}(\psi) : \psi \in E_{k,\lambda}, \|\psi\|_{L^2(\mathbb{S}^d(k)\setminus B_1)} = 1\}.$$

By a standard minimization procedure, there exists a sequence of functions $\psi_n \in H^1_G(\mathbb{S}^d(k_n) \setminus B_1)$ with $\|\psi_n\|_{H^1_G(\mathbb{S}^d(k_n)\setminus B_1)} = 1$ satisfying $\int_{\mathbb{S}^d(k_n)\setminus B_1} \psi_n z_{k_n} = 0$ solving the eigenvalue problem:

$$\begin{cases} -\lambda_1 \Delta\psi_n + \psi_n - p u_{k_n}^{p-1}\psi_n = \tau_{2,n}\psi_n & \text{in } \mathbb{S}^d(k_n) \setminus B_1 \ , \\ \partial_\nu \psi_n = \frac{(d-1)k_n}{\tan(k_n)}\psi_n + \mu_n & \text{on } \partial B_1, \end{cases} \tag{5.18}$$

for some $\mu_n \in \mathbb{R}$.

In what follows we just want to pass to the limit in ψ_n, $\tau_{2,n}$ and z_{k_n}. This will give a contradiction with Proposition 5.13, ii). The limit argument is technically intricate and will be divided in several steps.

Step 1: Up to a subsequence, ψ_n converges weakly (in a sense to be specified) to some ψ_0 in $H^1_G(\mathbb{R}^d \setminus B_1)$.

First, let us consider ψ_n in coordinates $(r, \theta), r \in (1, \frac{\pi}{k_n}), \theta \in \mathbb{S}^{d-1}$.

For any compact set $K \subset \mathbb{R}^d \setminus B_1$, we have that $\|\psi_n\|_{H^1(K)}$ is bounded. Via a diagonal argument, we can take a subsequence ψ_n such that $\psi_n \rightharpoonup \psi_0$ in $H^1_{loc}(\mathbb{R}^d \setminus B_1)$. By using compactness and taking a convenient subsequence, if necessary, we can assume also that $\psi_n \to \psi_0$ in $L^2_{loc}(\mathbb{R}^d \setminus B_1)$ and also pointwise. We now claim that $\psi_0 \in H^1(\mathbb{R}^d \setminus B_1)$. We prove this by duality, showing that:

$$\sup\left\{\int_{\mathbb{R}^d\setminus B_1} \nabla\psi_0\cdot\nabla\xi+\psi_0\,\xi :\ \xi\in H^1(\mathbb{R}^d\setminus B_1) \text{ with compact support, } \|\xi\|_{H^1}\leq 1\right\}\leq 1.$$

Observe that:

$$\frac{S_{k_n}(r)^{d-1}}{r^{d-1}}\xi(r,\theta)\to\xi(r,\theta),\quad \frac{S_{k_n}(r)^{d-1}}{r^{d-1}}\partial_r\xi(r,\theta)\to\partial_r\xi(r,\theta),$$

$$\frac{S_{k_n}(r)^{d-3}}{r^{d-3}}\nabla_\theta\xi(r,\theta)\to\nabla_\theta\xi(r,\theta),$$

in $L^2(\mathbb{R}^d\setminus B_1)$. Recall also that $\psi_n\rightharpoonup\psi_0$ in $H^1_{loc}(\mathbb{R}^d\setminus B_1)$. Then,

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B_1}\nabla\psi_0\cdot\nabla\xi+\psi_0\,\xi &= \lim_{n\to\infty}\int_{\mathbb{S}^d_{k_n}\setminus B_1}\nabla\psi_n\cdot\nabla\xi+\psi_n\,\xi\\ &\leq \lim_{n\to\infty}\|\xi\|_{H^1(\mathbb{S}^d(k_n)\setminus B_1)}=\|\xi\|_{H^1(\mathbb{R}^d\setminus B_1)}\leq 1.\end{aligned}$$

For later use we point out that the restriction of ψ_0 to $r\in[1,\frac{\pi}{k_n}]$ belongs to $L^2(\mathbb{S}^2(k_n)\setminus B_1)$, since

$$\int_1^{\frac{\pi}{k_n}}\int_{\mathbb{S}^{d-1}}S_{k_n}^{d-1}(r)(\psi_0)^2\leq\int_1^{+\infty}\int_{\mathbb{S}^{d-1}}r^{d-1}(\psi_0)^2\leq 1.$$

Moreover, by local weak convergence,

$$\psi_n|_{\partial B_1}\to\psi_0|_{\partial B_1}\text{ strongly in }L^2(\partial B_1). \tag{5.19}$$

Step 2: The sequences $\tau_{2,n}$, μ_n are bounded. In particular, $\tau_{2,n}\to\tau_2\leq 0$ and $\mu_n\to\mu_0$ up to a subsequence.

We argue by a contradiction and assume that $\tau_{2,n}\to-\infty$. Multiply the equation (5.18) with ψ_n,

$$\begin{aligned}&\int_{\mathbb{S}^d(k_n)\setminus B_1}\lambda_1|\nabla\psi_n|^2+\psi_n^2-\int_{\mathbb{S}^d(k_n)\setminus B_1}pu_{k_n}^{p-1}\psi_n^2-\lambda_1\frac{(d-1)k_n}{\tan(k_n)}\int_{\partial B_1}\psi_n^2\\ &=\tau_{2,n}\int_{\mathbb{S}^d(k_n)\setminus B_1}\psi_n^2.\end{aligned}\tag{5.20}$$

It is clear that the left-hand side of (5.20) is bounded, so $\|\psi_n\|$ is supposed to converge to 0. Therefore, by Lemma 5.6,

$$\int_{\mathbb{S}^d(k_n)\setminus B_1}pu_{k_n}^{p-1}\psi_n^2\leq C\|\psi_n\|^2_{L^2}\to 0,$$

Moreover, by the Lemma 5.15,

$$\int_{\partial B_1} \psi_n^2 \to 0.$$

Then we can see that left-hand side of (5.20) converges to $\lambda_1 > 0$ but the right-hand side is non-positive, which gives a contradiction.

For the estimate on μ_n, take a test function $\phi(r) = (2-r)^+$. Clearly ϕ has compact support, is bounded in $H^1(\mathbb{S}^d(k_n) \setminus B_1)$ and $\phi = 1$ on ∂B_1. Multiplying equation (5.18) by ϕ and integrating, we obtain

$$\int_{\mathbb{S}^d(k_n)\setminus B_1} \lambda_1 \nabla\psi_n \cdot \nabla\phi + (1-\tau_{2,n})\psi_n\phi - pu_{k_n}^{p-1}\psi\phi = \mu_n|\partial B_1|.$$

Taking into account the L^∞ bound of u_{k_n} (see Lemma 5.6), we conclude.

Step 3: ψ_0 is a nonzero weak solution of the problem:

$$\begin{cases} -\lambda_1\Delta\psi_0 + \psi - p\tilde{u}^{p-1}\psi_0 = \tau_2\psi_0 & \text{in } \mathbb{R}^d \setminus B_1 \ , \\ \partial_\nu\psi_0 = (d-1)\psi_0 + \mu_0 & \text{on } \partial B_1. \end{cases} \tag{5.21}$$

Multiplying (5.18) by a test function with compact support and passing to the limit, we obtain that ψ_0 is a solution of (5.21). We now show that $\psi_0 \neq 0$.

By multiplying the equation (5.18) with ψ_n, we obtain

$$\begin{aligned} &\int_{\mathbb{S}^d(k_n)\setminus B_1} \lambda_1|\nabla\psi_n|^2 + (1-\tau_{2,n})\psi_n^2 \\ &\quad - \int_{\mathbb{S}^d(k_n)\setminus B_1} pu_{k_n}^{p-1}\psi_n^2 - \lambda_1\frac{(d-1)k_n}{\tan(k_n)}\int_{\partial B_1}\psi_n^2 = 0, \end{aligned} \tag{5.22}$$

By (5.19), we have that:

$$\int_{\partial B_1} \psi_n^2 \to \int_{\partial B_1} \psi_0^2.$$

We now claim that

$$\int_{\mathbb{S}^d(k_n)\setminus B_1} pu_{k_n}^{p-1}(\psi_n^2 - \psi_0^2) \to 0. \tag{5.23}$$

Let us fix an arbitrary $\delta > 0$, and $M > 0$ sufficiently large as in Lemma 5.6. We can compute:

$$\begin{aligned}&\left|\int_{\mathbb{S}^d(k_n)\setminus B_1} u_{k_n}^{p-1}(\psi_n-\psi_0)^2\right| = \left|\int_1^{\frac{\pi}{k_n}}\int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)u_{k_n}^{p-1}(\psi_n-\psi_0)^2\right|\\ &\leq \left|\int_1^{M}\int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)u_{k_n}^{p-1}(\psi_n-\psi_0)^2\right| + \left|\int_M^{\frac{\pi}{k_n}}\int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)u_{k_n}^{p-1}(\psi_n-\psi_0)^2\right|\\ &=: (I)+(II).\end{aligned}$$

We are first concerned with the estimate of (I).

$$\left|\int_1^{M}\int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)u_{k_n}^{p-1}(\psi_n-\psi_0)^2\right| \leq C\left|\int_1^{M}\int_{\mathbb{S}^{d-1}} r^{d-1}(\psi_n-\psi_0)^2\right| = o_n(1),$$

by the L^2_{loc} convergence of ψ_n and the uniform L^∞ bound on u_{k_n}, see Lemma 5.6. We now estimate (II) by making use of Lemma 5.6, 2), to obtain:

$$\begin{aligned}&\int_M^{\frac{\pi}{k_n}}\int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)u_{k_n}^{p-1}(\psi_n-\psi_0)^2 \leq \delta^{p-1}\int_M^{\frac{\pi}{k_n}}\int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)(\psi_n^2+\psi_0^2)\\ &\leq C\delta^{p-1}\Big(\int_M^{\frac{\pi}{k_n}}\int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)\psi_n^2 + \int_M^{+\infty}\int_{\mathbb{S}^{d-1}} r^{d-1}\psi_0^2\Big) \leq 2C\delta^{p-1}.\end{aligned}$$

Since $\delta > 0$ is arbitrary, we conclude the proof of (5.23).

Then, if $\psi_0 = 0$, (5.22) implies that:

$$\int_{\mathbb{S}^d(k_n)\setminus B_1} \lambda_1|\nabla\psi_n|^2 + (1-\tau_{2,n})\psi_n^2 \to 0,$$

which is in contradiction with $\|\psi_n\|_{H^1(\mathbb{S}^d(k_n)\setminus B_1)} = 1$.

Step 4: Defining $z_n = z_{k_n}$, we show that $\|z_n - \tilde{z}\|_{H^1_{0,r}(\mathbb{S}^d(k_n)\setminus B_1)} \to 0$.

By definition,

$$\begin{cases} -\lambda_1\Delta z_n + z_n - pu_{k_n}^{p-1}z_n = \tau_n z_n & \text{in } \mathbb{S}^d(k_n)\setminus B_1\ ,\\ z_n = 0 & \text{on } \partial B_1, \end{cases} \tag{5.24}$$

where $\tau_n < 0$ by Proposition 5.7. Recall moreover that $\|z_n\|_{H^1(\mathbb{S}^d(k_n)\setminus B_1)} = 1$. Hence we can use the same ideas in Steps 1, 2 and 3 to prove that $\tau_n \to \tilde{\tau}$ and also z_{k_n} converges to $\tilde{z}$ weakly in $H^1_{0,loc}(\mathbb{R}^d\setminus B_1)$, strongly in $L^2_{loc}(\mathbb{R}^d\setminus B_1)$. Moreover $\tilde{z} \neq 0$ belongs to $H^1_0(\mathbb{R}^d\setminus B_1)$ with norm smaller or equal than 1, and $\tilde{z}$ is a solution of (5.8).

Observe that the restriction of $\tilde{z}_n$ to $r \in [1, \frac{\pi}{k_n}]$ can be seen as an axially symmetric function in $H^1(\mathbb{S}^d(k_n) \setminus B_1)$, since:

$$\|\tilde{z}\|^2_{H^1(\mathbb{S}^d(k_n)\setminus B_1)} = \omega_d \int_1^{\frac{\pi}{k_n}} (\tilde{z}'(r)^2 + \tilde{z}(r)^2) S_{k_n}(r)^{d-1} \leq \omega_d \int_1^{+\infty} (\tilde{z}'(r)^2 + \tilde{z}(r)^2) r^{d-1} \leq 1.$$

Claim 1: $\displaystyle\int_{\mathbb{S}^d(k_n)\setminus B_1} \lambda_1 \nabla z_n \cdot \nabla(z_n - \tilde{z}) + (1-\tau_n) z_n (z_n - \tilde{z}) = o_n(1).$

By multiplying the equation (5.24) by $z_n - \tilde{z}$ and integrating, we obtain

$$\int_{\mathbb{S}^d(k_n)\setminus B_1} \lambda_1 \nabla z_n \cdot \nabla(z_n - \tilde{z}) + (1-\tau_n) z_n (z_n - \tilde{z}) - \int_{\mathbb{S}^d(k_n)\setminus B_1} p u_{k_n}^{p-1} z_n (\tilde{z} - z_n) = 0.$$

Hence, it suffices to show that:

$$\int_{\mathbb{S}^d(k_n)\setminus B_1} p u_{k_n}^{p-1} z_n (z_n - \tilde{z}) \to 0.$$

Indeed, by Hölder inequality and Lemma 5.6,

$$\int_{\mathbb{S}^d(k_n)\setminus B_1} p u_{k_n}^{p-1} z_n (z_n - \tilde{z}) \leq \Big(\int_{\mathbb{S}^d(k_n)\setminus B_1} p u_{k_n}^{p-1} (z_n - \tilde{z})^2 \Big)^{1/2} \Big(\int_{\mathbb{S}^d(k_n)\setminus B_1} p u_{k_n}^{p-1} z_n^2 \Big)^{1/2}$$
$$\leq \Big(\int_{\mathbb{S}^d(k_n)\setminus B_1} p u_{k_n}^{p-1} (z_n - \tilde{z})^2 \Big)^{1/2} C \|z_n\|_{L^2}.$$

Moreover, the same argument of the proof of (5.23) implies that

$$\int_{\mathbb{S}^d(k_n)\setminus B_1} p u_{k_n}^{p-1} (z_n - \tilde{z})^2 \to 0.$$

Claim 2: $\displaystyle\int_{\mathbb{S}^d(k_n)\setminus B_1} \nabla \tilde{z} \cdot \nabla(z_n - \tilde{z}) \to 0, \ \int_{\mathbb{S}^d(k_n)\setminus B_1} \tilde{z}(z_n - \tilde{z}) \to 0.$

In order to show the first convergence, write:

$$\int_{\mathbb{S}^d(k_n)\setminus B_1} \nabla \tilde{z} \cdot \nabla(z_n - \tilde{z})$$
$$= \int_1^M \int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r) \partial_r \tilde{z}\, \partial_r (z_n - \tilde{z}) + \int_M^{\frac{\pi}{k_n}} \int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r) \partial_r \tilde{z}\, \partial_r (z_n - \tilde{z}) = (I) + (II).$$

Clearly, $|(I)| \to 0$ because of the weak convergence H^1_{loc} of z_n to $\tilde{z}$. By Hölder inequality,

$$|(II)| \leq C\Big(\int_M^{\frac{\pi}{k_n}} \int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)|\partial_r \tilde{z}|^2\Big)^{1/2}.$$

Moreover,

$$\limsup_{n\to+\infty} \int_M^{\frac{\pi}{k_n}} \int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)|\partial_r \tilde{z}|^2 \leq \int_M^{+\infty} \int_{\mathbb{S}^{d-1}} r^{d-1}|\partial_r \tilde{z}|^2,$$

which can be made arbitrarily small by choosing M appropriately.

The second convergence of claim 2 can be justified in the same way.

We are now in conditions of proving Step 4:

$$\begin{aligned}
&\int_{\mathbb{S}^d(k_n)\setminus B_1} \lambda_1|\nabla(z_n-\tilde{z})|^2 + (1-\tau_n)(z_n-\tilde{z})^2 \\
&= \int_{\mathbb{S}^d(k_n)\setminus B_1} \lambda_1 \nabla(z_n-\tilde{z})\cdot\nabla z_n + (1-\tau_n)(z_n-\tilde{z})z_n \\
&- \int_{\mathbb{S}^d(k_n)\setminus B_1} \lambda_1 \nabla(z_n-\tilde{z})\cdot\nabla \tilde{z} + (1-\tau_n)(z_n-\tilde{z})\tilde{z} = o_n(1),
\end{aligned}$$

by claims 1 and 2.

Step 5: Conclusion. We now show that

$$\int_{\mathbb{R}^d\setminus B_1} \psi_0 \tilde{z} = 0.$$

Indeed,

$$\begin{aligned}
0 &= \int_{\mathbb{S}^d(k_n)\setminus B_1} \psi_n z_n \\
&= \int_{\mathbb{S}^d(k_n)\setminus B_1} \psi_n(z_n-\tilde{z}) + \int_{\mathbb{S}^d(k_n)\setminus B_1} \tilde{z}(\psi_n-\psi_0) + \int_{\mathbb{S}^d(k_n)\setminus B_1} \psi_0\tilde{z} \\
&= (I) + (II) + (III).
\end{aligned}$$

By Hölder inequality,

$$(I) \leq \Big(\int_{\mathbb{S}^d(k_n)\setminus B_1} (z_n-\tilde{z})^2\Big)^{1/2}\Big(\int_{\mathbb{S}^d(k_n)\setminus B_1} (\psi_n)^2\Big)^{1/2} \to 0.$$

Moreover, taking $M > 0$ sufficiently large,

$$(II) = \int_1^M \int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)\tilde{z}(\psi_n - \psi_0) + \int_M^{\frac{\pi}{k_n}} \int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)\tilde{z}(\psi_n - \psi_0)$$

The first term above converges to 0 by the local weak convergence of ψ_n to ψ_0. Moreover, the second term can be estimated via Hölder inequality:

$$\begin{aligned}
&\int_M^{\frac{\pi}{k_n}} \int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)\tilde{z}(\psi_n - \psi_0) \\
\leq&\Big(\int_M^{\frac{\pi}{k_n}} \int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)(\psi_n - \psi_0)^2\Big)^{1/2}\Big(\int_M^{\frac{\pi}{k_n}} \int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)\tilde{z}^2\Big)^{1/2} \\
\leq& C\Big(\int_M^{\frac{\pi}{k_n}} \int_{\mathbb{S}^{d-1}} S_{k_n}^{d-1}(r)\tilde{z}^2\Big)^{1/2},
\end{aligned}$$

which can be made arbitrarily small if M is sufficiently large.

As a consequence,

$$(III) = \int_{\mathbb{S}^d(k_n)\setminus B_1} \psi_0\tilde{z} \to 0.$$

Passing to the limit, we conclude that

$$\int_{\mathbb{R}^d\setminus B_1} \psi_0\tilde{z} = 0.$$

This, together with step 3 and the inequality $\tau_2 \leq 0$, gives a contradiction with Proposition 5.13 (ii).

The proof of iii) follows the same arguments as that of ii). In some places the computations are easier since there are no boundary terms. $\square$

Let us now define the quadratic form associated to $H_{k,\lambda}$, namely:

$$J_{k,\lambda} : C_{G,0}^{k,\alpha}(\mathbb{S}^{d-1}) \to \mathbb{R},\ J_{k,\lambda}(v) = \int_{\mathbb{S}^{d-1}} vH_{k,\lambda}(v).$$

In this case also, sometimes we will drop the subindices k, λ. Let us also denote the first eigenvalue of the operator $H_{k,\lambda}$ as

$$\sigma_1(H_{k,\lambda}) = \inf\left\{ J_{k,\lambda}(v) : v \in C_{G,0}^{k,\alpha}(\mathbb{S}^{d-1})\ ,\quad \int_{\mathbb{S}^{d-1}} v^2 = 1 \right\}.$$

By the divergence formula, one can get

$$J_{k,\lambda}(v) = \frac{1}{\lambda} Q_{k,\lambda}(\psi_v).$$

Next lemma characterizes the eigenvalue $\sigma_1(H_{k,\lambda})$ in terms of the quadratic form $Q_{k,\lambda}$.

Lemma 5.16. *For any $\lambda \in [\lambda_0, \lambda_1]$, we have*

$$\sigma_1(H_{k,\lambda}) = \min\left\{ \frac{1}{\lambda} Q_{k,\lambda}(\psi) : \psi \in E_{k,\lambda}, \ \int_{\partial B_1} \psi^2 = 1 \right\}.$$

Moreover the infimum is attained.

Proof. Let us define

$$\eta := \inf\left\{ Q_{k,\lambda}(\psi) : \psi \in E_{k,\lambda}, \int_{\partial B_1} \psi^2 = 1 \right\} \in [-\infty, +\infty).$$

We first show that η is achieved. On that purpose, let us take $\psi_m \in E_{k,\lambda}$ such that $Q_{k,\lambda}(\psi_m) \to \eta$. We claim that ψ_m is bounded. By contradiction, if $\|\psi_m\|_{H^1_G} \to +\infty$, we define $\xi_m = \|\psi_m\|^{-1}_{H^1_G} \psi_m$; we can suppose that up to a subsequence $\xi_m \rightharpoonup \xi_0$. Notice that $\int_{\partial B_1} \xi_m^2 \to 0$, which yields that $\xi_0 \in H^1_{0,G}(\mathbb{S}^d(k) \setminus B_1)$. We also point out that

$$\int_{\mathbb{S}^d(k)\setminus B_1} u_{k,\lambda}^{p-1} \xi_m^2 \to \int_{\mathbb{S}^d(k)\setminus B_1} u_{k,\lambda}^{p-1} \xi_0^2 \, .$$

Let us distinguish two cases:
Case 1: $\xi_0 = 0$. In this case

$$Q_{k,\lambda}(\psi_m) = \|\psi_m\|^2_{H^1_G} \int_{\mathbb{S}^d(k)\setminus B_1} \left(\lambda |\nabla \xi_m|^2 + \xi_m^2 - p u_{k,\lambda}^{p-1} \xi_m^2\right) - \lambda \frac{(d-1)k}{\tan(k)} \to +\infty \, ,$$

which is impossible.
Case 2: $\xi_0 \neq 0$. In this case

$$\begin{aligned} \liminf_{m\to\infty} Q_{k,\lambda}(\psi_m) &= \liminf_{m\to\infty} \|\psi_m\|^2_{H^1_G} \int_{\mathbb{S}^d(k)\setminus B_1} \left(\lambda |\nabla \xi_m|^2 + \xi_m^2 - p u_{k,\lambda}^{p-1} \xi_m^2\right) - \lambda \frac{(d-1)k}{\tan(k)} \\ &\geq \liminf_{m\to\infty} \|\psi_m\|^2_{H^1_G} Q^D_{k,\lambda}(\xi_0) - \lambda \frac{(d-1)k}{\tan(k)} \, , \end{aligned}$$

but $Q^D_{k,\lambda}(\xi_0) > 0$ for $\lambda \in [\lambda_0, \lambda_1]$ by the Proposition 5.14. This is again a contradiction.

Thus, ψ_m is bounded, so up to a subsequence we can pass to the weak limit $\psi_m \rightharpoonup \psi$. Then, ψ is a minimizer for $Q_{k,\lambda}$ and in particular $\eta > -\infty$.

By the Lagrange multiplier rule, there exist real numbers $\theta_0, \theta_1, \theta_2$ so that for any $\rho \in H^1_G(\mathbb{S}^d(k) \setminus B_1)$,

$$\int_{\mathbb{S}^d(k)\setminus B_1} \left(\lambda \nabla\psi \nabla\rho + \psi\rho - p u_{k,\lambda}^{p-1}\psi\rho - \theta_0 \rho z_{k,\lambda}\right) = \int_{\partial B_1} \rho((\theta_1 + \tilde{c})\psi + \theta_2),$$

where $\tilde{c} = \lambda \frac{(d-1)k}{\tan(k)}$. Taking $\rho = z_{k,\lambda}$ above we conclude that $\theta_0 = 0$. Moreover, if we take $\rho = \psi$ and $\rho = \vartheta$ (given by (5.16)), we conclude that $\theta_1 + \tilde{c} = \eta$ and $\theta_2 = 0$, respectively. In other words, ψ is a (weak) solution of

$$\begin{cases} -\lambda\Delta\psi + \psi - p u_{k,\lambda}^{p-1}\psi = 0 & \text{in } \mathbb{S}^d(k) \setminus B_1\,, \\ \partial_\nu \psi = \eta\psi & \text{on } \partial B_1\,. \end{cases}$$

By the regularity theory, $\psi \in C^{2,\alpha}_G(\mathbb{S}^d(k) \setminus B_1)$. Define $v = \psi|_{\partial B_1}$ and $\psi \in C^{2,\alpha}_G(\mathbb{S}^d(k) \setminus B_1) \cap E_{k,\lambda}$ by the Lemma 5.11. Observe that:

$$\int_{\mathbb{S}^{d-1}} v^2 = 1, \;\; J_{k,\lambda}(v,v) = \frac{1}{\lambda} Q_{k,\lambda}(\psi) = \frac{1}{\lambda}(\eta - \tilde{c})\,.$$

The proof is completed. □

For any $k \in (0, k_0)$, we define

$$\lambda_*(k) := \sup\{\lambda \in [\lambda_0, \lambda_1] : Q_{k,\lambda}(\psi) < 0 \text{ for some } \psi \in E_{k,\lambda}\}\,. \tag{5.25}$$

The above set is non empty since λ_0 belongs to it, so the supremum is well defined. It is clear that $\lambda_0 < \lambda_*(k) < \lambda_1$ by the Proposition 5.14. We now can state the main result of this section as the following:

Proposition 5.17. *We have:*

(i) if $\lambda = \lambda_1$, then $\sigma_1(H_{k,\lambda}) > 0$;

(ii) if $\lambda \geq \lambda_*(k)$, then $\sigma_1(H_{k,\lambda}) \geq 0$;

(iii) if $\lambda = \lambda_*(k)$, then $\sigma_1(H_{k,\lambda}) = 0$;

(iv) for any $\varepsilon > 0$ there exists $\lambda \in (\lambda_* - \varepsilon, \lambda_*)$, with $\sigma_1(H_{k,\lambda}) < 0$.

Proof. This proposition follows at once from the Lemma 5.16, Proposition 5.14 and from the definition of λ_* given by (5.25). □

5.5 Bifurcation argument

In this section, we are ready to prove our main result, Theorem 5.1, by the bifurcation argument. In order to prove our result, we reformulate the problem. For this aim, we need the following lemma:

Lemma 5.18. *There exists $\epsilon > 0$ such that for any $\lambda \in (\lambda_*(k)-\epsilon, \lambda_1)$, the operator*

$$H_{k,\lambda} + Id : C^{2,\alpha}_{G,0}(\mathbb{S}^{d-1}) \to C^{1,\alpha}_{G,0}(\mathbb{S}^{d-1}),\ v \mapsto H_{k,\lambda}(v) + v,$$

is invertible.

Proof. Observe that, by choosing $\epsilon > 0$ sufficiently small, we can assume that $\sigma_1(H_{k,\lambda}) > -1$. We define the quadratic forms $\hat{Q}_{k,\lambda} : E_{k,\lambda} \to \mathbb{R}$ by

$$\hat{Q}_{k,\lambda}(\psi) = Q_{k,\lambda}(\psi) + \lambda \int_{\partial B_1} \psi^2,$$

and $\hat{J}_{k,\lambda} : C^{2,\alpha}_{G,0}(\mathbb{S}^{d-1}) \to \mathbb{R}$ by

$$\hat{J}_{k,\lambda}(v) = \int_{\mathbb{S}^{d-1}} \left(v\partial_\nu \psi_v - \frac{(d-1)k}{\tan k} v^2 + v^2 \right).$$

Actually, these quadratic forms are positive definite since $\sigma_1 > -1$. We state that they are also coercive. Let's first show that

$$\beta := \inf\{\hat{Q}_{k,\lambda}(\psi) :\ \psi \in E_{k,\lambda},\ \|\psi\| = 1\} > 0$$

is achieved. On that purpose, take $\psi_n \in E_{k,\lambda}$ with $\|\psi_n\|_{H^1_G} = 1$ such that $\hat{Q}_{k,\lambda}(\psi_n) \to \beta$, and suppose that $\psi_n \rightharpoonup \psi_0$. If the convergence is strong, then the infimum β is attained, which implies that $\beta > 0$. Otherwise,

$$\begin{aligned} \beta &= \limsup_{n\to\infty} \int_{\mathbb{S}^d(k)\setminus B_1} \left(\lambda|\nabla\psi_n|^2 + \psi_n^2 - pu_{k,\lambda}^{p-1}\psi_n^2\right) - \lambda\frac{(d-1)k}{\tan k}\int_{\partial B_1} \psi_n^2 + \lambda\int_{\partial B_1} \psi_n^2 \\ &> \int_{\mathbb{S}^d(k)\setminus B_1} \left(\lambda|\nabla\psi_0|^2 + \psi_0^2 - pu_{k,\lambda}^{p-1}\psi_0^2\right) - \lambda\frac{(d-1)k}{\tan k}\int_{\partial B_1} \psi_0^2 + \lambda\int_{\partial B_1} \psi_0^2 \ge 0. \end{aligned}$$

Therefore, $\hat{Q}_{k,\lambda}$ is coercive. Thus, we can obtain that

$$\hat{J}_{k,\lambda}(v) = \frac{1}{\lambda}\hat{Q}_{k,\lambda}(\psi_v) \ge C\|\psi_v\|^2_{H^1_G(\mathbb{S}^d(k)\setminus B_1)}.$$

Observe that $\hat{J}_{k,\lambda}$ is naturally defined in the space:

$$X = \left\{ u \in H^{1/2}_G(\mathbb{S}^{d-1}) : \int_{\mathbb{S}^{d-1}} u = 0 \right\},$$

where $H_G^{1/2}(\mathbb{S}^{d-1})$ denotes the Sobolev space of G-symmetric functions, i.e.

$$H_G^{1/2}(\mathbb{S}^{d-1}) = \{v \in H^{1/2}(\mathbb{S}^{d-1}) : \ v(\theta) = v(g(\theta)) \ \forall g \in G\}\,.$$

By the trace estimate $\hat{J}_{k,\lambda}$ is coercive in X. According to the Lax-Milgram theorem, the regularity theory and the fact that the mean property is preserved, the operator

$$v \mapsto \partial_\nu \psi_v\big|_{\partial B_1} - \frac{(d-1)k}{\tan k} v + v$$

is invertible from $C^{2,\alpha}_{G,0}(\mathbb{S}^{d-1})$ to $C^{1,\alpha}_{G,0}(\mathbb{S}^{d-1})$. The regularity of elliptic operators under non-homogenous Robin boundary condition can be found in [53, Chapters 4 and 5]. □

By the Proposition 5.17, we can take $\bar{\lambda} \in (\lambda_0, \lambda_*(k))$ sufficiently close to $\lambda_*(k)$ such that $\sigma_1(H_{\bar{\lambda}}) < 0$. Define the operator $Z : \mathcal{U} \times [\bar{\lambda}, \lambda_1] \to \mathcal{V}$ by

$$Z(v, \lambda) = F_k(v, \lambda) + v,$$

where $\mathcal{U} \subset C^{2,\alpha}_{G,0}(\mathbb{S}^{d-1}), \mathcal{V} \subset C^{1,\alpha}_{G,0}(\mathbb{S}^{d-1})$ are open neighborhoods of 0. By the Lemma 5.18, taking $\bar{\lambda}$ close enough to $\lambda_*(k)$ so that we can assume that $D_v Z(0, \lambda)$ is an isomorphism for all $\lambda \in [\bar{\lambda}, \lambda_1]$. Then, we can further restrict $\mathcal{U}$ and $\mathcal{V}$ so that $Z(\cdot, \lambda)$ is invertible for all $\lambda \in [\bar{\lambda}, \lambda_1]$ according to the Inverse Function theorem.

Now define the operator $W : \mathcal{V} \times [\bar{\lambda}, \lambda_1] \to C^{1,\alpha}_{G,0}(\mathbb{S}^{d-1})$ by $W(v, \lambda) = v - \hat{v}$ with $Z(\hat{v}, \lambda) = v$. By the compactness of the inclusion of $C^{2,\alpha}_{G,0}(\mathbb{S}^{d-1})$ into $C^{1,\alpha}_{G,0}(\mathbb{S}^{d-1})$, we can point out that W is the operator by the sum of an identity and a compact operator. Obviously, $F_k(v, \lambda) = 0 \Leftrightarrow W(v, \lambda) = 0$. Theorem 5.1 follows if we show the local bifurcation of solutions to the equation $W(v, \lambda) = 0$.

We have

$$D_v W\big|_{(0,\lambda)}(v) = v - D_v Z^{-1}\big|_{(0,\lambda)}(v).$$

Thus

$$D_v W\big|_{(0,\lambda)}(v) = \delta v \Leftrightarrow H_{k,\lambda}(v) = \frac{\delta}{(1-\delta)\partial_r u_{k,\lambda}(1)} v.$$

Recall from the proof of the Lemma 5.18, we have that $\delta < 1$ if $\lambda \geq \bar{\lambda}$. Hence, $D_v W(0, \lambda)$ has the same number of negative eigenvalues as $H_{k,\lambda}$.

Under this framework, Theorem 5.1 follows immediately from the following lemma and the Krasnoselskii Theorem 1.8.

Lemma 5.19. *The index of the linearized operator $D_v W(0, \lambda)$ is odd for some $\lambda < \lambda_*(k)$ sufficiently close to $\lambda_*(k)$.*

Proof. In view of Proposition 5.17, it is sufficient to prove that $H_{k,\lambda_*(k)}$ has odd-dimensional kernel. For any $\psi \in E_{k,\lambda}$, there exist functions $\psi_0, \psi_{l,j}$ defined in $[1, \frac{\pi}{k})$ such that

$$\psi(r,\theta) = \psi_0(r) + \sum_{l=1}^{+\infty} \sum_{j=1}^{m_l} \psi_{l,j}(r)\xi_{l,j}(\theta),$$

where $(r,\theta) \in [1, \frac{\pi}{k}) \times \mathbb{S}^{d-1}$, and $\xi_{l,j}$ are the G-symmetric spherical harmonics, normalized to 1 in the L^2-norm, with the eigenvalue μ_{i_l} of multiplicity m_l. Then the quadratic form $\psi \mapsto Q_{k,\lambda}(\psi)$ defined in $E_{k,\lambda}$ can be given by

$$Q_{k,\lambda}(\psi) = Q^0_{k,\lambda}(\psi_0) + \sum_{l=1}^{+\infty} \sum_{j=1}^{m_l} Q^l_{k,\lambda}(\psi_{l,j}), \tag{5.26}$$

where the functional $Q^l_{k,\lambda}$ is defined as

$$\begin{aligned} Q^l_{k,\lambda}(\phi) = &\int_1^{\frac{\pi}{k}} (\lambda\phi'^2 + \phi^2 - pu^{p-1}_{k,\lambda}\phi^2) S_k^{d-1} dr \\ &- \lambda\frac{(d-1)k}{\tan(k)}\phi(1)^2 + \lambda\mu_{i_l} \int_1^{\frac{\pi}{k}} \phi^2 S_k^{d-3} dr \end{aligned}$$

for a function $\phi : \left(1, \frac{\pi}{k}\right) \to \mathbb{R}$. By convention, we choose $\mu_{i_0} = 0$. Also $\psi_0(1) = 0$ and ψ_0 is orthogonal to the function $z_{k,\lambda}$ restricted to the radial variable since $\psi \in E_{k,\lambda}$. We know that $Q^0_{k,\lambda}(\psi_0) > 0$ in the radial case. For $\lambda = \lambda_*(k)$, $Q_{k,\lambda}(\psi) \geq 0$, and then $Q^l_{k,\lambda} \geq 0$ by (5.26). Moreover, it is obvious that

$$Q^{l_1}_{k,\lambda} < Q^{l_2}_{k,\lambda} \text{ if } 1 \leq l_1 < l_2.$$

We also know that there exists a $\psi \in E_{k,\lambda}$ such that $Q_{k,\lambda}(\psi) = 0$. Therefore $Q^1_{k,\lambda} \geq 0$ and $Q^l_{k,\lambda} > 0$ for $l > 1$. This implies that the dimension of the kernel of the operator $H_{\lambda_*(k)}$ is m_1, which is odd by the assumption (G) on the symmetry group. □

5.6 Appendix

In this appendix we prove the quantitative version of the Implicit Function Theorem given in Proposition 5.4, the uniformity of the Sobolev constant given in Lemma 5.5, and also the proof of Lemma 5.15.

Proof of Proposition 5.4. Let us define the map $T : U \to Y$ by setting

$$T(w) = w - [F'(v)]^{-1}(F(v+w)).$$

Clearly, a fixed point z of T will give rise to a solution to the equation $F(v+z) = 0$. We apply now the Banach contraction theorem to the operator T.

For any $\psi \in Y, w \in U$, one has

$$T'(w)[\psi] = \psi - [F'(v)]^{-1}(F'(v+w)[\psi]) = [F'(v)]^{-1}(F'(v)[\psi] - F'(v+w)[\psi]).$$

Thus we find

$$\|T'(w)[\psi]\| \leq \frac{c_0}{2c_0}\|\psi\| = \frac{1}{2}\|\psi\|. \tag{5.27}$$

Therefore we conclude that T is a contraction. We finish the proof if we show that T maps U into itself. With this purpose, let us compute:

$$\|T(0)\| = \|[F'(v)]^{-1}(F(v))\| \leq c_0\delta.$$

On the other hand, for any $w \in U$ we can use (5.27) to deduce

$$\|T(w) - T(0)\| \leq \frac{c_0}{2c_0}\|w\| \leq c_0\delta.$$

By using the triangular inequality of the norm, we get

$$\|T(w)\| \leq 2c_0\delta.$$

By the Banach contraction Theorem, T has a fixed point in U. Moreover, by (A2) and (A3) we conclude that $\|F'(v+z)^{-1}\| \leq 2c_0$. □

Proof of Lemma 5.5. We first give the facts that

$$\|u\|_{L^s} \leq C(q)\left(\|\nabla u\|_{L^q} + \|u\|_{L^q}\right) \;\; \forall \;\; u \in H^{1,q}(\mathbb{S}^2_1),\; \frac{1}{s} = \frac{2-q}{2q},\; 1 < q < 2,$$

and

$$\|u\|_{L^{2^*}} \leq C(d)\left(\|\nabla u\|_{L^2} + \|u\|_{L^2}\right) \;\; \forall \;\; u \in H^1(\mathbb{S}^d_1), d \geq 3,$$

see [5,8]. And we expand that

$$\bar{u} = \begin{cases} u, & x \in \mathbb{S}^d(k) \setminus B_1, \\ 0, & x \in B_1, \end{cases}$$

so that $\bar{u} \in H^1_r(\mathbb{S}^d(k)) = H^1_r(\mathbb{S}^d_R), R = \frac{1}{k}$. Taking $\hat{u}(y) = \bar{u}(Ry), y \in \mathbb{S}^d_1$, we now consider two cases:

Case 1: $d = 2$.

Let $f(t) = \frac{2t}{2-t}$, with $f(1) = 2, f(2_-) = +\infty$, then for any $s \in (2, +\infty)$, there exists a unique $t \in (1, 2)$ such that $f(t) = \frac{2t}{2-t} = s$. Then we have

$$\|u\|_{L^s} \leq C(\|u\|_{L^t} + \|\nabla u\|_{L^t})$$

by the fact that

$$\begin{aligned}\left(\int_{\mathbb{S}^2_R} |\bar{u}|^s\right)^{\frac{t}{s}} &= R^{2-t}\left(\int_{\mathbb{S}^2_1} |\hat{u}|^s\right)^{\frac{t}{s}} \\ &\leq R^{2-t}C\left(\int_{\mathbb{S}^2_1} |\hat{u}|^t + \int_{\mathbb{S}^2_1} |\nabla \hat{u}|^t\right) \\ &= R^{2-t}C\left(R^{-2}\int_{\mathbb{S}^2_R} |\bar{u}|^t + R^{t-2}\int_{\mathbb{S}^2_R} |\nabla \bar{u}|^t\right) \\ &\leq C\left(\int_{\mathbb{S}^2_R} |\bar{u}|^t + \int_{\mathbb{S}^2_R} |\nabla \bar{u}|^t\right),\end{aligned}$$

where $C = C(s)$ is a positive constant. We take u^2 instead of u, then we can get that

$$\begin{aligned}\|u^2\|_{L^s} &\leq C(\|u^2\|_{L^t} + \|\nabla u^2\|_{L^t}) \\ &\leq C(\|u\|_{L^2} + 2\|\nabla u\|_{L^2})\|u\|_{L^s} \\ &\leq C(\|u\|_{L^2} + 2\|\nabla u\|_{L^2})\|u\|_{L^2}^{1-\beta}\|u\|_{L^{2s}}^{\beta},\end{aligned}$$

where $\beta = \frac{2}{s}$. Then one can get that

$$\|u\|_{L^{2s}}^{2-\beta} \leq C(\|u\|_{L^2} + 2\|\nabla u\|_{L^2})\|u\|_{L^2}^{1-\beta}.$$

Therefore, for any $s > 4$, we have

$$\|u\|_{L^s} \leq C\|u\|_{H^1}.$$

As $2 < s \leq 4$, we obtain

$$\|u\|_{L^s} \leq \|u\|_{L^2}^{\gamma}\|u\|_{L^{2s}}^{1-\gamma},$$

where $\gamma = \frac{2}{s}$. Also, we can gain

$$\|u\|_{L^s} \leq C\|u\|_{H^1}.$$

Case 2: $d \geq 3$.
By the interpolation inequality, one has that

$$\|u\|_{L^s} \leq \|u\|_{L^2}^{\alpha}\|u\|_{L^{2^*}}^{1-\alpha},$$

where α satisfies $\frac{\alpha}{2}+(1-\alpha)(\frac{1}{2}-\frac{1}{d})=\frac{1}{s}$. We also have

$$\begin{aligned}\left(\int_{\mathbb{S}^d_R}|\bar{u}|^{2^*}\right)^{\frac{2}{2^*}} &= R^{d-2}\left(\int_{\mathbb{S}^d_1}|\hat{u}|^{2^*}\right)^{\frac{2}{2^*}}\\ &\le R^{d-2}C\left(\int_{\mathbb{S}^d_1}|\hat{u}|^2+\int_{\mathbb{S}^d_1}|\nabla\hat{u}|^2\right)\\ &= R^{d-2}C\left(R^{-d}\int_{\mathbb{S}^d_R}|\bar{u}|^2+R^{2-d}\int_{\mathbb{S}^d_R}|\nabla\bar{u}|^2\right)\\ &\le C\left(\int_{\mathbb{S}^d_R}|\bar{u}|^2+\int_{\mathbb{S}^d_R}|\nabla\bar{u}|^2\right),\end{aligned}$$

where the positive constant C just depends on d. Then one can obtain that $\|u\|_{L^s}\le C\|u\|_{H^1_{0,r}}$. The proof of the lemma is done. □

Proof of Lemma 5.15. Take a smooth vector field $M(x)$ in $\mathbb{S}^d(k)\setminus B_1$ such that $M(x)=\nu(x)$ on the boundary ∂B_1, $|M(x)|\le 1$. Then, we apply the divergence theorem and Hölder inequality to get

$$\begin{aligned}&\int_{\partial B_1}\psi^2=\int_{\partial B_1}\psi^2\cdot M(x)\cdot\nu(x)\\ &=\int_{\mathbb{S}^d(k)\setminus B_1}\big(2\psi\nabla\psi\cdot M(x)+\psi^2\cdot divM(x)\big)\\ &\le 2\|\nabla\psi\|_{L^2}\|\psi\|_{L^2}+\|\psi\|^2_{L^2}\|divM(x)\|_{L^\infty}\\ &\le C\|\nabla\psi\|_{L^2}\|\psi\|_{L^2}+C\|\psi\|^2_{L^2}\end{aligned}$$

since $|divM(x)|$ is uniformly bounded. In order to prove that, let us now consider the vector field $M(x)$ in the coordinates $(r,\theta), r\in(1,\frac{\pi}{k}),\theta\in\mathbb{S}^{d-1}$, and give by

$$M(r,\theta)=-\frac{\partial}{\partial r}X(r,\theta)\cdot\chi(r)=-\mathbf{v}_r\cdot\chi(r)=-\left(M_r\mathbf{v}_r+\sum_{i=1}^{d-1}M_{\theta_i}\mathbf{v}_{\theta_i}\right),$$

where $\mathbf{v}_r:=\mathbf{v}_r(\theta),\mathbf{v}_{\theta_i}$ are orthonormal vectors along X and $\chi(r)$ is a cut-off function

$$\chi(r)=\begin{cases}0, & r\ge\frac{3}{2},\\ 1, & r\le\frac{5}{4}.\end{cases}$$

Then the divergence can be written as

$$\begin{aligned}divM(x)&=\frac{1}{S_k^{d-1}(r)}\frac{\partial}{\partial_r}(S_k^{d-1}(r)M_r(r,\theta))+\frac{1}{S_k(r)}div_\theta M_\theta(r,\theta)\\ &=\partial_r M_r(r,\theta)+(d-1)\frac{C_k(r)}{S_k(r)}M_r(r,\theta)+\frac{1}{S_k(r)}div_\theta M_\theta(r,\theta)\\ &=-\chi'(r)-(d-1)\frac{C_k(r)}{S_k(r)}\chi(r).\end{aligned}$$

Therefore,

$$|divM(x)| \leq C(d)$$

as $k \to 0$, and the proof is finished. $\square$

Chapter 6

Modica type estimate and curvature results

In this chapter, we establish a Modica type estimate and give some curvature results for the overdetermined problem (1.1). In order to verify the main result, we first present some facts about the function P. Then we prove a uniform gradient bound on the solution. This is the key to proving the main theorem. Once this is done, we then can conclude the proof of the Modica type estimate by a scaling argument and passing to a limit. The results about mean curvature and the rigidity result will be proved afterwards.

6.1 Gradient's estimate

Let u be a C^3 solution of (1.1), $f \in C^1$, and P as in (1.12) where F is a primitive of f (no assumption on its sign at the moment). We begin this section by showing that the function P is a subsolution of an elliptic PDE. This is a very well known result (see for instance [83]), which we present here for the sake of completeness.

Lemma 6.1. *The function P satisfies:*

$$L(P) := \Delta P + 2f(u)\frac{\nabla u}{|\nabla u|^2} \cdot \nabla P \geq 0.$$

for any $x \in \Omega$ such that $\nabla u(x) \neq 0$.

Proof. We have that $P \in C^2(\Omega)$ and following Sperb [83], we find that

$$D_i P = 2\sum_{j=1}^{d} D_j u D_{ij} u + 2f(u) \cdot D_i u$$

for every $i = 1, \cdots, d$. Therefore,

$$D_i P - 2f(u) \cdot D_i u = 2 \sum_{j=1}^{d} D_j u D_{ij} u \le 2|\nabla u| \left[\sum_{j=1}^{d} (D_{ij} u)^2 \right]^{1/2}.$$

Using $\Delta u + f(u) = 0$, the Laplacian of P can be given by

$$\begin{aligned} \Delta P &= 2 \sum_{i,j=1}^{d} (D_{ij} u)^2 + 2 \sum_{j=1}^{d} D_j u D_j (\Delta u) + 2f'(u)|\nabla u|^2 - 2f^2(u) \\ &= 2 \sum_{i,j=1}^{d} (D_{ij} u)^2 - 2f^2(u), \end{aligned}$$

so that

$$|\nabla u|^2 \Delta P \ge \frac{1}{2} |\nabla P|^2 - 2f(u) \nabla u \cdot \nabla P,$$

and then

$$L(P) := \Delta P + 2f(u) \frac{\nabla u}{|\nabla u|^2} \cdot \nabla P \ge \frac{\frac{1}{2}|\nabla P|^2}{|\nabla u|^2} \ge 0.$$

This concludes the proof of the lemma. □

We finish this section with a uniform estimate on the gradient of u, under mild conditions on f. This will be essential in the proof of Theorem 1.5.

Lemma 6.2. *Let u be a bounded solution of the problem (1.1), then there exists a constant $M > 0$ such that $|\nabla u| \le M$ in Ω. Moreover, M depends only on d, $\|f(u)\|_{L^\infty}$ and c.*

Proof. Let us observe that the above gradient estimate would be a immediate consequence of interior regularity estimates for elliptic equations in

$$\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta\},$$

for any $\delta > 0$ that will be fixed later. We now prove that this estimates holds also up to the boundary.

Given any $x_0 \in \Omega$, denote by $h := \mathrm{dist}(x_0, \partial\Omega)$ and assume that this distance is attained at $y_0 \in \partial\Omega$. By the previous comment, we can focus on the case $h \le \delta$. We define:

$$\tilde{u}(x) := \frac{1}{h} u(x_0 + hx),$$

then in $B_1(0)$,

$$-\Delta \tilde{u} = h f(h\tilde{u}), \ \ \tilde{u} > 0.$$

Let $z_0 := \frac{y_0 - x_0}{h}$,

$$|\nabla \tilde{u}(z_0)| = |\nabla u(y_0)| = |c|,$$

since the ball $B_h(x_0)$ is tangent to $\partial\Omega$ at y_0.

We now decompose $\tilde{u} = v + w$, where the functions v, w solve the following two problems, respectively:

$$\begin{cases} \Delta v = 0 & \text{in } B_1(0), \\ v = \tilde{u} & \text{on } \partial B_1(0), \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w = hf(h\tilde{u}) & \text{in } B_1(0), \\ w = 0 & \text{on } \partial B_1(0). \end{cases}$$

For the function w, we can use regularity estimates to conclude that

$$\|w\|_{C^{1,\alpha}(\bar{B}_1)} \leq Ch\|f\|_{L^\infty}.$$

Here C is a constant depending only on d. We now turn our attention to v. By applying an explicit version of the Harnack's inequality for harmonic functions v (see, for instance, [44, Chapter 2, exercise 2.6]), one can get that

$$\frac{1-r}{(1+r)^{d-1}} v(0) \leq v(x) \leq \frac{1+r}{(1-r)^{d-1}} v(0) \tag{6.1}$$

where $r = |x| < 1$. Using the first inequality and computing boundary derivatives on z_0, we have:

$$2^{1-d} v(0) \leq \left|\partial_\nu v(z_0)\right| \leq |\nabla\tilde{u}(z_0)| + |\partial_\nu w(z_0)| \leq c + C\,h\|f(u)\|_{L^\infty}.$$

If h is sufficiently small, we have that

$$v(0) \leq M,$$

We now use the second inequality in (6.1) to conclude that

$$v(x) \leq M \frac{1+r}{(1-r)^{d-1}}.$$

Based on these results, we get that

$$\tilde{u}(x) = v(x) + w(x) \leq M \frac{1+r}{(1-r)^{d-1}} + Ch\|f(u)\|_{L^\infty}.$$

Therefore,

$$\sup_{B_{1/2}(0)} \tilde{u}(x) \leq M,$$

taking a bigger M if necessary. By standard interior gradient estimates,

$$|\nabla u(x_0)| = |\nabla\tilde{u}(0)| \leq CM,$$

where, again, this constant C depends only on d and $\|f(u)\|_{L^\infty}$. □

6.2 Proof of theorems

In this section, we shall present the proof of our main results, Theorem 1.4, Theorem 1.5 and Theorem 1.6. Define

$$\alpha := \max\{0, c^2 + 2F(0).\}$$

Proposition 6.3. *Under the assumptions of Theorem 1.5, we have that $P(x) \leq \alpha$ for all $x \in \Omega$.*

Proof. By Lemma 6.2 we know that P is bounded. Reasoning by contradiction, assume that

$$\beta := \sup_{\Omega} P(x) > \alpha.$$

There exists a sequence $(x_k)_{k\in\mathbb{N}} \subset \Omega$ such that

$$P(x_k) \to \beta \ \text{ as } \ k \to \infty.$$

If x_k is bounded, up to a subsequence we can assume that $x_k \to x_0$, with $P(x_0) = \beta$. Since $\beta > 0$, we conclude that x_0 is not a critical point of u. Moreover, since $\beta > c^2 + 2F(0)$, then $x_0 \in \Omega$. As a consequence, P attains a local maximum at x_0, and $\nabla u(x_0) \neq 0$; the maximum principle applied to P implies that P is constant on a neighborhood of x_0. This argument implies that the set:

$$\{x \in \Omega : \ P(x) = \beta\}$$

is a non-empty open subset of Ω. Since it is obviously closed, then $P(x) = \beta$ for all $x \in \Omega$. But $P(x) = c^2 + 2F(0) < \beta$ if $x \in \partial\Omega$, and this is a contradiction with the continuity of P.

We now assume that x_k is unbounded, and discuss the following two cases.

Case 1: $\limsup_{k\to+\infty} \operatorname{dist}(x_k, \partial\Omega) > \delta$.
Let us consider u_k extended by 0 outside Ω, so that $u_k : \mathbb{R}^d \to \mathbb{R}$ is a globally Lipschitz function. We consider $u_k(x) = u(x + x_k)$ and $P_k(x) = P(x + x_k)$, so that

$$P_k(0) = P(x_k) \to \beta.$$

By a Cantor diagonal argument we can take a subsequence, still denoted by u_k, so that on compact sets of $\mathbb{R}^d$ the sequence u_k converges uniformly to a certain Lipschitz function $u_\infty \geq 0$. Moreover $\nabla u_k \stackrel{*}{\rightharpoonup} \nabla u_\infty$, where $\stackrel{*}{\rightharpoonup}$ denotes convergence with the weak star topology in L^∞.

Denoting $P_\infty = |\nabla u_\infty|^2 + 2F(u_\infty) \in L^\infty(\mathbb{R}^d)$, we have that $P_\infty \leq \beta$. Recall that by Lemma 6.2, $f(u_k)$ is a globally Lipschitz function in $\mathbb{R}^d$. By interior regularity estimates we have that u_k is bounded in $C^{2,\alpha}$ norm on any compact subset of

$B_\delta(0)$. Then u_k converges to u_∞ in $C^{2,\alpha}$ sense in compact subsets of $B_\delta(0)$. In particular,

$$0 < \beta \leftarrow P_k(0) \rightarrow P_\infty(0).$$

Since $\beta > 0$, then 0 is not a critical point of u_∞. This, in particular, excludes the possibility $u_\infty(0) = 0$. Define the open set $\Omega_\infty = \{x \in \mathbb{R}^d : \ u_\infty(x) > 0\}$. Clearly Ω_∞ is not empty since $0 \in \Omega_\infty$. We first claim that inside Ω_∞ the convergence is $C^{2,\alpha}_{loc}$, and $\Delta u_\infty = f(u_\infty)$. Indeed, given $p \in \Omega_\infty$, by uniform convergence there exists $r > 0$ such that $u_k(x) > 0$ for any $x \in B_r(p)$. Reasoning as above, interior regularity estimates allow us to conclude the claim.

By applying the maximum principle to P_∞, we conclude that P is constant on a neighborhood of 0. Indeed, we have that the set $\{x \in \Omega : \ P(x) = \beta\}$ is an open subset of Ω_∞. By continuity of P in Ω_∞, it is also closed. As a consequence,

$$P(x) = \beta \text{ for all } x \in \tilde{\Omega}_\infty, \tag{6.2}$$

where $\tilde{\Omega}_\infty$ is the connected component of Ω_∞ containing 0.

Take now $y_k \in \tilde{\Omega}_\infty$ such that $u_\infty(y_k) \rightarrow \xi$, where

$$\xi = \sup\{u_\infty(x) : \ x \in \tilde{\Omega}_\infty\} > 0.$$

By the Lipschitz regularity of u_∞, there exists $r > 0$ such that $B_r(y_k) \subset \tilde{\Omega}_\infty$. Thanks to the Ekeland Variational Principle (see for instance [85, Chapter 5]) there exists $z_k \in \tilde{\Omega}_\infty$ such that:

- $u_\infty(z_k) \rightarrow \xi$;
- $|y_k - z_k| \rightarrow 0$;
- $\nabla u_\infty(z_k) \rightarrow 0$.

In particular, $P(z_k) \rightarrow 2F(\xi) \leq 0$, which is a contradiction with (6.2).

Case 2: $\lim_{k\rightarrow+\infty} \operatorname{dist}(x_k, \partial\Omega) = 0$.
As in case 1, we consider u_k extended by 0 outside Ω, so that $u_k : \mathbb{R}^d \rightarrow \mathbb{R}$ is a globally Lipschitz function. Observe that $u(x_k) \rightarrow 0$ since $|\nabla u|$ is bounded by Lemma 6.2. Then,

$$|\nabla u(x_k)|^2 = P(x_k) - 2F(u(x_k)) \rightarrow \beta - 2F(0) > c^2. \tag{6.3}$$

Denote $h_k := \operatorname{dist}(x_k, \partial\Omega) \rightarrow 0$ and assume that this distance is attained at $y_k \in \partial\Omega$. Let

$$u_k(x) := \frac{1}{h_k} u(y_k + h_k x).$$

Observe that $u_k(0) = 0$ and ∇u_k is uniformly bounded by Lemma 6.2. Therefore, we can take a subsequence such that u_k converges uniformly in compact sets to a limit function $u_\infty \geq 0$, which is Lipschitz continuous (but possibly unbounded). Moreover, $\nabla u_k \overset{*}{\rightharpoonup} \nabla u_\infty$ in L^∞.

Let us denote $\Omega_\infty = \{x \in \mathbb{R}^d : \ u_\infty(x) > 0\}$. First, we show that in compact sets of Ω_∞, u_k converges to u_∞ in $C^{2,\alpha}$ sense and:

$$\Delta u_\infty = 0, \quad x \in \Omega_\infty.$$

Indeed, take $p \in \Omega_\infty$. By uniform convergence, there is $r > 0$ such that $u_k > 0$ in $B_r(p)$. Recall that $f(u_k)$ is uniformly Lipschitz continuous in $\mathbb{R}^d$, by interior regularity estimates, we conclude that $u_k \to u_\infty$ in $C^{2,\alpha}$ sense in compact sets of $B_r(p)$. Passing to the limit, u_∞ is harmonic in Ω_∞.

Now, let $z_k := \frac{x_k - y_k}{h_k}$, then we can assume that z_k converges to z_∞ since $|z_k| = 1$. The same argument as above works in $B_r(z_\infty)$ for any $r \in (0,1)$, and then $u_k \to u_\infty$ in $C^{2,\alpha}$ sense in compact sets of $B_1(z_\infty)$. Hence $u_\infty \geq 0$ is harmonic in $B_1(z_\infty)$. It is clear that, by (6.3),

$$|c| < |\nabla u(x_k)| = |\nabla u_k(z_k)| \to |\nabla u_\infty(z_\infty)| := a. \tag{6.4}$$

By the maximum principle we conclude that $u_\infty > 0$ in $B_1(z_\infty)$, that is, $B_1(z_\infty) \subset \Omega_\infty$.

Observe now that:

$$\begin{aligned}
\beta &= |\nabla u_k(z_k)|^2 + 2F(u_k(z_k)) + o_k(1) \\
&= |\nabla u(x_k)|^2 + 2F(u(x_k)) + o_k(1) \\
&\geq |\nabla u(y_k + xh_k)|^2 + 2F(u(y_k + xh_k)) \\
&= |\nabla u_k(x)|^2 + 2F(u(y_k + xh_k))
\end{aligned}$$

where $F(u(x_k)) \to F(0)$ and $F(u(y_k + xh_k)) \to F(0)$ since $h_k \to 0, u(y_k) = 0$. As a consequence, $|\nabla u_\infty(z_\infty)| \geq |\nabla u_\infty(x)|$ for all $x \in B_1(z_\infty)$. Recall moreover that u_∞ is Lipschitz continuous in $\mathbb{R}^d$ and $u_\infty(0) = 0$.

Therefore, up to a rotation,

$$u_\infty(x) = ax_d^+,$$

where a is given in (6.4).

Fix $\epsilon > 0$ small enough, by the uniform convergence of u_k to u_∞,

$$|u_k - ax_d^+| < \frac{a\epsilon}{2}$$

in $B_1(0)$ for all large enough k. Following [48, Lemma 4.4], let us consider the domain D_t, the perturbation of $B_1(0) \cap \{x_d > \epsilon\}$, given by

$$D_t = \{x \in B_1(0) : x_d > \epsilon - t\eta(x')\},$$

where $x' = (x_1, \cdots, x_{d-1}), 0 \le \eta(x') \le 1$ is a smooth bump function supported in $|x'| \le \frac{1}{2}$ with $\eta(x') = 1$ for $|x'| \le \frac{1}{4}$. It is clear that $u_k > 0$ in D_0.

Now, we denote a function w which solves the following problem

$$\begin{cases} -\Delta w(x) = -\epsilon & \text{in } D_{t_0}, \\ w(x) = ax_d - a\epsilon & \text{on } \partial B_1(0) \cap \{x_d > \epsilon\}, \\ w(x) = 0 & \text{on } \partial B_1(0) \cap \{x_d = \epsilon - t_0\eta(x')\}, \end{cases}$$

for some $0 < t_0 < 2\epsilon$, where $D_{t_0} \subseteq B_2(0) \cap \{u_k > 0\}$ will touch

$$J(u_k) := \partial\Big(B_2(0) \cap \{u_k > 0\}\Big) \cap B_2(0)$$

at some point $p \in J(u_k) \cap \{|x'| < \frac{1}{2}\}$. Thus, $w(x) \le u_k(x)$ in D_{t_0} by the maximum principle, and by the Hopf lemma,

$$|w_\nu(p)| \le |(u_k)_\nu(p)| = |c|,$$

where ν is the outer normal to ∂D_{t_0} at p. On the other hand,

$$|w_\nu(p)| = a + O(\epsilon),$$

by the standard perturbation argument. We therefore conclude $a \le |c|$ since ϵ is arbitrary. This is a contradiction with (6.4). □

In order to conclude the proof of Theorem 1.5 we only need to show the following result.

Proposition 6.4. *If, under the assumptions of Theorem 1.5, $P(x) = \alpha$ at a point $x \in \Omega$, then P is constant, u is 1-dimensional and Ω is a half-space.*

Proof. The proof of his proposition follows the ideas of [13], and is divided in two steps.

Step 1: If $P(x_0) = \alpha$, then $\nabla u(x_0) \neq 0$.

Reasoning by contradiction, if $\nabla u(x_0) = 0$, then $0 \le \alpha = 2F(u(x_0)) \le 0$, so that $\alpha = 0 = F(u_0)$, where $u_0 = u(x_0)$. We consider the set

$$U = \{x \in \Omega : u(x) = u_0\},$$

where $u_0 = u(x_0)$. Clearly, U is closed and $U \neq \emptyset$. Then we take $x_1 \in U$ and consider the function $\varphi(t) = u(x_1 + tw) - u_0$, where $w \in \mathbb{S}^{d-1}$ is arbitrarily fixed. We have

$$\varphi'(t) = \nabla u(x_1 + tw)w.$$

Then

$$\begin{aligned}|\varphi'(t)|^2 &\leq |\nabla u(x_1+tw)|^2 \leq \alpha - 2F(u(x_1+tw)) \\ &= 2F(u_0) - 2F(u(x_1+tw)).\end{aligned}$$

Since $F(u_0) = 0$, then $F'(u_0) = 0$ and $F''(u_0) \leq 0$ by the C^2 regularity of F. Hence $F(u) = O((u-u_0)^2)$ as $|u-u_0| \to 0$. Therefore we can get that

$$|\varphi'(t)| \leq C|\varphi(t)|,$$

as t small enough. It follows that $\varphi \equiv 0$ in $[-\epsilon, \epsilon]$, for some $\epsilon > 0$, since $\varphi(0) = 0$. It shows that the set U given above is open. By continuity it is clear that U is also closed, and then $U = \Omega$, which implies that u is a constant, a contradiction.

Step 2: Conclusion.

Then, we have that $\nabla u(x) \neq 0$ for any $x \in \Omega$ with $P(x) = \alpha$. By the maximum principle applied to P, we conclude that the set

$$\{x \in \Omega : \; P(x) = \alpha\}$$

is a non-empty open set, which is also closed by continuity. Then $P(x) = \alpha$ for all $x \in \Omega$.

We now set $v = G(u)$, where $G \in C^2(\mathbb{R})$ is suitably determined. By the straightforward computation, one has

$$\Delta v = G''(u)|\nabla u(x)|^2 + G'(u)\Delta u = G''(u)(\alpha - 2F(u)) - G'(u)F'(u). \tag{6.5}$$

Then we can attain that

$$\Delta v = 0, \tag{6.6}$$

if we choose

$$G(u) = \int_{u_0}^{u} (\alpha - 2F(s))^{-\frac{1}{2}} ds,$$

for some fixed $u_0 \in u(\Omega)$. With this choice, one has that

$$|\nabla v|^2 = G'(u)^2 |\nabla u|^2 = 1. \tag{6.7}$$

We can infer that $v(x) = \vec{a} \cdot x + b$ for some $\vec{a} \in \mathbb{R}^d$ with $|\vec{a}| = 1$ and $b \in \mathbb{R}$.

Then, we can obtain that $u(x) = G^{-1}(v(x)) = g(\vec{a} \cdot x + b)$ with $g = G^{-1}$, since G is invertible. Then u is 1-dimensional. A priori, Ω could be the inner space between two parallel hyperplanes, but this is impossible since u has no critical points by Step 1. This concludes the proof. □

We are now in conditions to conclude the proofs of Theorem 1.6 and Theorem 1.4.

Proof of Theorem 1.6. Since $c \neq 0$, we have that u has no critical points close to $\partial\Omega$. By the assumption that P attains its maximum at $\partial\Omega$, then

$$P_\nu \geq 0, \quad \text{on } \partial\Omega.$$

Let H be the mean curvature of $\partial\Omega$ at a given point. On the other hand,

$$\begin{aligned}
\frac{\partial P}{\partial \nu} &= \frac{\partial}{\partial \nu}\left(|\nabla u|^2 + 2F(u)\right) \\
&= 2\frac{\partial u}{\partial \nu}\frac{\partial^2 u}{\partial \nu^2} + 2f(u)\frac{\partial u}{\partial \nu} \\
&= 2\frac{\partial u}{\partial \nu}\left(\Delta u - (d-1)H\frac{\partial u}{\partial \nu} + f(u)\right) \\
&= -2(d-1)\left|\frac{\partial u}{\partial \nu}\right|^2 H,
\end{aligned}$$

by using the fact that $\frac{\partial^2 u}{\partial \nu^2} = \Delta u - (d-1)H\frac{\partial u}{\partial \nu}$ on $\partial\Omega$, see [83, Section 5.4]. Therefore,

$$H \leq 0.$$

If $H(p) = 0$ for some $p \in \partial\Omega$, one has that $\frac{\partial P}{\partial \nu}(p) = 0$. By Hopf's lemma, we can get that P is constant, at least in a neighborhood of p. We can now argue as in Step 2 of the proof of Proposition 6.4 (see (6.5), (6.6), (6.7)) to conclude that u is 1-dimensional in such neighborhood. By unique continuation, u is 1-dimensional. As a consequence, Ω is either a half-space or the open set placed between two parallel hyperplanes. □

Proof of Theorem 1.4. By Theorem 1.5, we have that $P(x) \leq c^2 + 2F(0)$ for all $x \in \Omega$. Let us consider first the case $c \neq 0$. In this case we can apply Theorem 1.6 to deduce $H \leq 0$. Moreover, if $H(p) = 0$ at some point of $\partial\Omega$, then P is constant and u is 1-dimensional. Moreover, by Step 1 of Proposition 6.4, u has no critical points, and then Ω is a half-space.

We now address the case $c = 0$. Observe that:

$$0 \leq c^2 + 2F(0) = 2F(0) \leq 0.$$

As a consequence, $F(0) = 0$. Since F is nonnegative, this implies that $f(0) = F'(0) = 0$. Then, 0 is a solution of the equation $-\Delta u = f(u)$ with 0 Dirichlet boundary data. By unique continuation $u = 0$, which is a contradiction. □

Chapter 7

Conclusions and future perspectives

In this thesis, we have proved some results of local bifurcation of solutions to overdetermined elliptic problems, which give nontrivial solutions. Moreover, a gradient estimate in the spirit of Modica's classical result is presented.

In $\mathbb{R}^d$, we first obtained the existence of nontrivial unbounded domains, bifurcating from the straight cylinder such that the overdetermined elliptic problem (1.1) has a positive bounded solution. We proved such a result for a very general class of functions f by making use of the Crandall-Rabinowitz Bifurcation Theorem (see [73]). We also have treated the case of the complement of a cylinder for $f = 0$ in [57], where we proved the existence of nontrivial unbounded exceptional domains in the Euclidean space $\mathbb{R}^d$, $d \geq 4$. Moreover, we have established a kind of Modica type estimate for bounded solutions to the overdetermined elliptic problem. As we have seen, the presence of the boundary changes the usual form of the Modica estimate for entire solutions. The case of equality has also been discussed. From such estimates we will derive information about the curvature of $\partial\Omega$ under a certain condition on c and f. The proof uses the maximum principle together with scaling arguments and a careful passage to the limit in the arguments by contradiction, see [74].

Concerning the problem on the sphere $\mathbb{S}^d$, we have constructed nontrivial contractible domains where the overdetermined elliptic problem could admit a positive solution. These domains are perturbations of $\mathbb{S}^d \setminus D$, where D is a small geodesic ball. This shows in particular that Serrin's theorem for overdetermined problems in the Euclidean space cannot be generalized to the sphere even for contractible domains (see [72]).

This thesis has addressed positive bounded solutions to the overdetermined problems. A natural question is to ask what happens on sign-changing solutions? And how about the cases of unbounded solutions? Taking this into consideration, some interesting research objectives could be as follows.

Regarding Modica type estimate for overdetermined elliptic problems, another point of interest could be to consider unbounded solutions, and whether a Modica type estimate is possible in this situation. Here the situation changes, since the Modica result does not hold as it is (consider e.g. $F = 0$ and $u(x) = x_1$). Hence, a new formulation is needed here. A related important question is to find out whether the inequality $|\nabla u| < 1$ holds or not in the harmonic case. We point that all examples found so far satisfy this inequality. In [88] it is shown that the inequality holds in dimension 2 under some extra (mild) assumptions, but there is no result in higher dimensions.

With respect to sign-changing solutions, we would like to mention some recent work [56, 71] by Minlend and Ruiz, where the authors obtained sign-changing solutions to some overdetermined problems via the local bifurcation theorem. In [71], the choice of trivial sign-changing solutions depend closely on the form of the equation, which is indeed a delicate issue. Moreover, the nontrivial domains constructed are bounded. The unbounded domains constructed in [56] are periodic in the first coordinate and they bifurcate from suitable strips in $\mathbb{R}^2$, the Neumann boundary condition considered in [56] varies from top to bottom.

These results on changing-sign solutions are important in application and will be the object of intensive research.

Finally, we mention another kind of overdetermined problem that can become a subject of study in the future. Let us recall the following conjecture which is still open:

Schiffer conjecture: Let $\Omega \in \mathbb{R}^d$ be a bounded smooth domain, and $u : \Omega \to \mathbb{R}$ a non-constant solution of the overdetermined elliptic problem:

$$\begin{cases} \Delta_g u + \lambda u = 0 & \text{in } \Omega, \\ u = c & \text{on } \partial\Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

for some $\lambda > 0$. Then Ω is a ball.

The study of the Schiffer conjecture is now considered one of the outstanding problems in Analysis since Williams proved in 1976 that the conjecture is equivalent to the famous Pompeiu problem in integral geometry, see [90]. For a survey on this subject we remind to [92]. Very recently, Fall, Minlend and Weth [35] provide the first example of a Schiffer domain in a Riemannian manifold given as the flat cylinder endowed with the flat metric with nonconstant principal curvatures on the boundary. They also construct regular Schiffer domains which are given as open neighborhoods of the equator in 2-sphere bounded by pairs of curves.

What we plan to study in the near future is the existence of solutions to overdetermined elliptic problems (7.1) with nonlinearity $f(u) = |u|^{p-2}u, 2 < p < 2^* = \frac{2d}{d-2}$

($p > 2$ if $d = 1, 2$) instead of λu. Indeed, we plan to construct a domain, which is a bifurcated straight cylinder. This work will be also addressed by taking advantage of a bifurcation argument and the idea of frameworks of Banach spaces set in [35], but here the situation is more complicated due to the nonlinear term. This will be joint work with Ignace Aristide Minlend at the University of Goethe Frankfurt.

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