On the geometry around an operator between Banach spaces

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Abstract

The aim of this thesis is to study and analyse different notions related to the geometry of the space of all bounded linear operators between Banach spaces around a fixed operator.

We begin with a thorough study of the numerical index with respect to an operator between Banach spaces. Given two Banach spaces X and Y, and a norm-one operator $G \in \mathcal{L}(X,Y)$ (the space of all bounded linear operators from X to Y), the numerical index with respect to G, $n_G(X,Y)$, is the greatest constant $k \geqslant 0$ such that

$$k\|T\| \leqslant \inf_{\delta>0} \sup\big\{|y^*(Tx)| \colon y^* \in Y^*,\, x \in X,\, \|y^*\| = \|x\| = 1,\, \operatorname{Re} y^*(Gx) > 1-\delta\big\}$$

for every $T \in \mathcal{L}(X,Y)$. Firstly, we provide some tools to study this concept and present some results dealing with the numerical index with respect to adjoint operators and rank-one operators. Then, we study the set $\mathcal{N}(\mathcal{L}(X,Y))$ of values of the numerical indices with respect to all norm-one operators between X and Y. We give several examples of spaces having trivial set of values of the numerical indices with respect to operators. For instance, $\mathcal{N}(\mathcal{L}(X,Y)) = \{0\}$ when X or Y is a real Hilbert space of dimension at least two and also when X or Y is the space of bounded or compact operators on an infinite-dimensional real Hilbert space. We also prove that, in the real case,

$$\mathcal{N}(\mathcal{L}(X,\ell_p)) \subseteq [0, M_p] \quad \text{and} \quad \mathcal{N}(\mathcal{L}(\ell_p, Y)) \subseteq [0, M_p]$$

for $1 < p < \infty$ and for all real Banach spaces X and Y, where $M_p = \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$. For complex Hilbert spaces H_1, H_2 with dimension at least 2, we show that $\mathcal{N}(\mathcal{L}(H_1,H_2)) = \{0, 1/2\}$ if H_1 and H_2 are isometrically isomorphic and $\mathcal{N}(\mathcal{L}(H_1,H_2)) = \{0\}$ otherwise. Moreover, for a complex Hilbert space H with dimension greater than 1, $\mathcal{N}(\mathcal{L}(X,H)) \subseteq [0,1/2]$ and $\mathcal{N}(\mathcal{L}(H,Y)) \subseteq [0,1/2]$ for all complex Banach spaces X and Y. We also prove that $\mathcal{N}(\mathcal{L}(C(K_1), C(K_2))) = \{0,1\}$ for many families of compact Hausdorff topological spaces K_1 and K_2, both in the real and complex case. As a consequence, $\mathcal{N}(\mathcal{L}(L_\infty(\mu_1), L_\infty(\mu_2))) \subseteq \{0,1\}$ and $\mathcal{N}(\mathcal{L}(L_1(\mu_1), L_1(\mu_2))) \subseteq \{0,1\}$ for all σ-finite measures μ_1 and μ_2. Additionally, we show that the concept of Lipschitz numerical range for Lipschitz self-maps of a Banach space is a particular case of numerical range with respect to a convenient linear operator between two different Banach spaces. To finish the study of the numerical index with respect to an operator, we provide some results showing the behaviour of this concept when we apply some Banach space operations, such as constructing diagonal operators between c_0-, ℓ_1- or ℓ_∞-sums of Banach spaces, considering composition operators between vector-valued function spaces, taking the adjoint of an operator, and composing two operators.

Next, we deal with the numerical index of Banach spaces, that is, the numerical index with respect to the identity operator on a Banach space X, denoted by $n(X)$. We analyse the behaviour of this concept on operators ideals, showing that the numerical index of any operator ideal endowed with the operator norm is less than or equal to the minimum of the numerical indices of the domain and of the codomain. We present stronger inequalities for the numerical indices of the spaces of compact and weakly compact operators, which allow to give interesting examples. For tensor products of Banach spaces X and Y, we prove that the numerical indices of $X\hat{\otimes}_\pi Y$, the projective tensor product, and $X\hat{\otimes}_\varepsilon Y$, the injective tensor product, are less than or equal to the minimum of $n(X)$ and $n(Y)$. As a consequence, we obtain some inequalities for the spaces of approximable and nuclear operators. Furthermore, we discuss when the Daugavet property of a tensor product passes to the factors. Specifically, we show that if $X\hat{\otimes}_\pi Y$ has the Daugavet property and the unit ball of Y is slicely countably determined or the dual space of Y has a point of Fréchet differentiability of the norm, then X inherits the Daugavet property. For injective tensor products, if $X\hat{\otimes}_\varepsilon Y$ has the Daugavet property and Y has a point of Fréchet differentiability of the norm, then X also has the Daugavet property.

In addition, we address the problem of calculating the numerical index of the real ℓ_p^2, i.e., the real L_p space of dimension two. To do so, we follow two different approaches. The first one is to deal with two-dimensional real spaces endowed with an absolute and symmetric norm and give a lower bound for the numerical index of such spaces. Moreover, we show that in many instances the numerical index coincides with the given bound and, as a consequence, we prove that $n(\ell_p^2) = M_p = \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$ for $3/2 \leqslant p \leqslant 3$ in the real case. In our second approach, we directly work in ℓ_p^2 and show that $n(\ell_p^2) = M_p$ for $6/5 \leqslant p \leqslant 3/2$ and $2 \leqslant p \leqslant 6$ in the real case.

Then, we introduce and study the concept of generating operator: A norm-one operator $G \in \mathcal{L}(X,Y)$ between two Banach spaces X and Y is a generating operator if

$$\|T\| = \inf_{\delta>0} \sup \{\|Tx\| \colon x \in X,\ \|x\| = 1,\ \|Gx\| > 1-\delta\} \quad \text{for every } T \in \mathcal{L}(X,Y),$$

or, equivalently, if $B_X = \overline{\operatorname{conv}}\,(\{x \in X \colon \|x\| = 1,\ \|Gx\| > 1-\delta\})$ for every $\delta > 0$. We show that G is generating if and only if

$$\max_{\theta\in\mathbb{T}} \sup_{y^*\in B_{Y^*}} \|G^*(y^*) + \theta x^*\| = 1 + \|x^*\| \quad \text{for every } x^* \in X^*.$$

Additionally, we study the relationship between generating operators and norm-attainment. While generating operators having rank one and those whose domain has the Radon-Nikodým property attain their norm, there are generating operators which do not attain their norm, even of rank two. We also discuss the possibility for a Banach space X to be the domain of a generating operator which does not attain its norm in terms of the behaviour of some spear sets of X^*. Furthermore, we study the properties of the set $\operatorname{Gen}(X,Y)$ of all generating operators between two Banach spaces X and Y. In this line, we show that the set $\operatorname{Gen}(X,Y)$ generates the unit ball of $\mathcal{L}(X,Y)$ by closed convex hull when X is $\ell_1(\Gamma)$ and that this is the only possibility for real finite-dimensional spaces.

Finally, using its connection with abstract numerical range, we present a widely applicable approach to address Birkhoff-James orthogonality. More precisely, we characterize Birkhoff-James orthogonality and smooth points in a Banach space Z in terms of the actions of functionals in a subset of Z^* which is one-norming for Z. This general approach can be applied in several cases to obtain known results, such as the characterization of Birkhoff-James orthogonality in the space of operators between Banach spaces endowed with the operator norm or with the numerical radius, as well as new results on Birkhoff-James orthogonality for spaces of vector-valued bounded functions and its applications for spaces of vector-valued continuous functions, uniform algebras, polynomials, Lipschitz maps, and injective tensor products. Next, we study possible extensions of the Bhatia-Šemrl Theorem on Birkhoff-James orthogonality of matrices, showing results in for vector-valued continuous functions, compact linear operators on reflexive spaces, and finite Blaschke products. Furthermore, we provide applications to the study of spear vectors, spear operators, and Banach spaces with numerical index one. Specifically, we prove that no smooth point of a Banach space Z can be Birkhoff-James orthogonal to a spear vector of Z. In the case when $Z = \mathcal{L}(X, Y)$, if X is a Banach space with strongly exposed points and Y is a smooth Banach space with dimension at least two, then there are no spear operators in $\mathcal{L}(X, Y)$. Particularizing this result to the identity operator, we obtain an obstructive condition for a Banach space to have numerical index one: the existence of a smooth point which is Birkhoff-James orthogonal to a strongly exposed point. In particular, smooth Banach spaces with dimension at least two containing strongly exposed points do not have numerical index one.

Resumen

El objetivo de esta tesis es estudiar y analizar diferentes conceptos relacionados con la geometría del espacio de los operadores lineales y continuos entre espacios de Banach alrededor de un operador fijado.

Comenzamos con un profundo estudio del índice numérico respecto a un operador entre espacios de Banach. Dados dos espacios de Banach X e Y y un operador de norma uno $G \in \mathcal{L}(X,Y)$ (el espacio de todos los operadores lineales y continuos de X a Y), el índice numérico respecto a G, $n_G(X,Y)$, es la mayor constante $k \geqslant 0$ tal que

$$k\|T\| \leqslant \inf_{\delta>0} \sup\big\{|y^*(Tx)| \colon y^* \in Y^*,\, x \in X,\, \|y^*\| = \|x\| = 1,\, \operatorname{Re} y^*(Gx) > 1-\delta\big\}$$

para todo $T \in \mathcal{L}(X,Y)$. En primer lugar, damos algunas herramientas que serán de ayuda a la hora de estudiar este concepto y presentamos algunos resultados sobre índice numérico respecto a operadores adjuntos y de rango uno. Después, estudiamos el conjunto $\mathcal{N}(\mathcal{L}(X,Y))$ de los posibles valores que puede tomar el índice numérico respecto a todos los operadores de norma uno entre X e Y. Presentamos varios ejemplos de espacios para los que este conjunto es trivial. Por ejemplo, $\mathcal{N}(\mathcal{L}(X,Y)) = \{0\}$ cuando X o Y es un espacio de Hilbert real de dimensión mayor que 1 y cuando X o Y es el espacio de operadores continuos o compactos definidos en un espacio de Hilbert real infinito-dimensional. También probamos que, en caso real,

$$\mathcal{N}(\mathcal{L}(X,\ell_p)) \subseteq [0, M_p] \quad \text{y} \quad \mathcal{N}(\mathcal{L}(\ell_p,Y)) \subseteq [0, M_p]$$

para $1 < p < \infty$ y para cualesquiera espacios de Banach reales X e Y, donde $M_p = \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$. Para espacios de Hilbert H_1, H_2 complejos de dimensión mayor que uno, se tiene que $\mathcal{N}(\mathcal{L}(H_1,H_2)) = \{0, 1/2\}$ si H_1 y H_2 son isométricamente isomorfos y $\mathcal{N}(\mathcal{L}(H_1,H_2)) = \{0\}$ en otro caso. Además, si H es un espacio de Hilbert complejo de dimensión mayor que uno, $\mathcal{N}(\mathcal{L}(X,H)) \subseteq [0,1/2]$ y $\mathcal{N}(\mathcal{L}(H,Y)) \subseteq [0,1/2]$ para cualesquiera espacios de Banach complejos X e Y. Probamos también que $\mathcal{N}(\mathcal{L}(C(K_1),C(K_2))) = \{0,1\}$ para muchas familias de espacios de Hausdorff compactos K_1 y K_2, tanto en caso real como complejo. Como consecuencia, $\mathcal{N}(\mathcal{L}(L_\infty(\mu_1),L_\infty(\mu_2))) \subseteq \{0,1\}$ y $\mathcal{N}(\mathcal{L}(L_1(\mu_1),L_1(\mu_2))) \subseteq \{0,1\}$ para cualesquiera medidas σ-finitas μ_1 y μ_2. Por otro lado, mostramos que el concepto de rango numérico Lipschitz para aplicaciones Lipschitz de un espacio de Banach en

sí mismo se puede ver como un caso particular de rango numérico respecto a un operador lineal entre espacios de Banach convenientemente elegidos. Para finalizar el estudio del índice numérico respecto a un operador, presentamos algunos resultados sobre la estabilidad de este concepto cuando construimos operadores diagonales entre c_0-, ℓ_1- o ℓ_∞-sumas de espacios de Banach, cuando consideramos operadores de composición en algunos espacios de funciones con valores vectoriales, cuando tomamos el adjunto de un operador y, finalmente, cuando componemos dos operadores.

Seguidamente, centramos nuestra atención en el índice numérico de un espacio de Banach X, es decir, el índice numérico respecto al operador identidad, denotado por $n(X)$. Analizamos el comportamiento de este concepto en ideales de operadores, demostrando que el índice numérico de un ideal de operadores con la norma usual de operadores es menor o igual que el mínimo de los índices numéricos del dominio y del codominio. Para el espacio de operadores compactos y el de operadores débilmente compactos obtenemos desigualdades más fuertes que nos permiten dar ejemplos interesantes. Para productos tensoriales de espacios de Banach X e Y, probamos que tanto el índice numérico del producto tensorial proyectivo $X\hat{\otimes}_\pi Y$ como el del producto tensorial inyectivo $X\hat{\otimes}_\varepsilon Y$ es menor o igual que el mínimo de $n(X)$ y $n(Y)$. Como consecuencia, obtenemos desigualdades para el espacio de los operadores aproximables y el de los operadores nucleares. Además, analizamos cuándo la propiedad de Daugavet de un producto tensorial pasa a alguno de sus factores. Concretamente, si $X\hat{\otimes}_\pi Y$ tiene la propiedad de Daugavet y la bola unidad de Y es SCD o el espacio dual Y^* tiene un punto de diferenciabilidad Fréchet para la norma, entonces X hereda la propiedad de Daugavet. En el caso de productos tensoriales inyectivos se tiene que si $X\hat{\otimes}_\varepsilon Y$ tiene la propiedad de Daugavet e Y tiene un punto de diferenciabilidad Fréchet de la norma, entonces X también tiene la propiedad de Daugavet.

Adicionalmente, abordamos el problema del cálculo del índice numérico de ℓ_p^2 en caso real, es decir, del espacio L_p real de dimensión dos. Para ello, seguimos dos enfoques diferentes. En el primero trabajamos en cualquier espacio real de dimensión dos dotado de una norma absoluta y simétrica y damos una cota inferior para el índice numérico de dichos espacios. Además, probamos que en muchos casos el índice numérico coincide con la cota dada y, como consecuencia, demostramos que $n(\ell_p^2) = M_p = \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$ para $3/2 \leqslant p \leqslant 3$ en caso real. En nuestro segundo acercamiento trabajamos directamente en el espacio ℓ_p^2 y probamos que $n(\ell_p^2) = M_p$ para $6/5 \leqslant p \leqslant 3/2$ y $2 \leqslant p \leqslant 6$ en caso real.

Posteriormente, introducimos y estudiamos el concepto de operador generador: Un operador de norma uno $G \in \mathcal{L}(X,Y)$ entre dos espacios de Banach X e Y es generador si

$$\|T\| = \inf_{\delta>0} \sup \{\|Tx\| \colon x \in X,\ \|x\| = 1,\ \|Gx\| > 1-\delta\} \quad \text{para todo } T \in \mathcal{L}(X,Y),$$

o, equivalentemente, si $B_X = \overline{\operatorname{conv}}\,(\{x \in X \colon \|x\| = 1,\ \|Gx\| > 1-\delta\})$ para todo $\delta > 0$. Probamos que G es generador si y solo si

$$\max_{\theta\in\mathbb{T}} \sup_{y^*\in B_{Y^*}} \|G^*(y^*) + \theta x^*\| = 1 + \|x^*\| \quad \text{para todo } x^* \in X^*.$$

Analizamos también la relación de los operadores generadores con la propiedad de alcanzar la norma. Mientras los operadores generadores de rango uno o aquellos cuyo dominio tiene la propiedad de Radon-Nikodým alcanzan su norma, hay operadores generadores que no alcanzan su norma, incluso de rango dos. Además, caracterizamos la posibilidad de que un espacio de Banach X sea el dominio de un operador generador que no alcance su norma en términos del comportamiento de ciertos conjuntos de X^*. Por otro lado, estudiamos el conjunto $\mathrm{Gen}(X,Y)$ de todos los operadores generadores entre dos espacios de Banach X e Y. En esta línea, probamos que el conjunto $\mathrm{Gen}(X,Y)$ genera la bola unidad de $\mathcal{L}(X,Y)$ por envolvente convexa y cerrada cuando X es $\ell_1(\Gamma)$ y, de hecho, esta es la única posibilidad en caso real para espacios de dimensión finita.

Finalmente, usando su conexión con el rango numérico abstracto, presentamos un método general y extensamente aplicable para abordar la ortogonalidad de Birkhoff-James. De manera más precisa, caracterizamos la ortogonalidad de Birkhoff-James en un espacio de Banach Z en términos de las acciones de los funcionales que pertenecen a un subconjunto de Z^* que es uno-normante para Z. Este método general se puede aplicar en numerosos casos para obtener tanto resultados ya conocidos, como pueden ser las caracterizaciones de ortogonalidad de Birkhoff-James en el espacio de operadores dotado con la norma usual de operadores o con el radio numérico, como nuevos resultados sobre la ortogonalidad de Birkhoff-James para espacios de funciones acotadas con valores vectoriales y sus aplicaciones para funciones continuas con valores vectoriales, algebras uniformes, polinomios, aplicaciones Lipschistz y productos tensoriales inyectivos. Después, estudiamos posibles extensiones del Teorema de Bathia-Šemrl sobre ortogonalidad de Birkhoff-James para matrices, obteniendo resultados de este tipo para funciones continuas con valores vectoriales, operadores compactos en espacios reflexivos y productos de Blaschke finitos. Además, damos aplicaciones para vectores y operadores lanza, así como para espacios de Banach con índice numérico uno. Concretamente, probamos que ningún punto suave de un espacio de Banach Z puede ser Birkhoff-James ortogonal a un vector lanza de Z. Para el caso $Z=\mathcal{L}(X,Y)$, obtenemos que si X es un espacio de Banach con puntos fuertemente expuestos e Y es un espacio de Banach suave de dimensión mayor que uno, entonces no hay operadores lanza en $\mathcal{L}(X,Y)$. Este resultado particularizado al operador identidad nos lleva a una condición obstructiva para que un espacio de Banach tenga índice numérico uno: la existencia de un punto suave que sea Birkhoff-James ortogonal a un punto fuertemente expuesto. En particular, los espacios de Banach suaves de dimensión mayor que uno conteniendo puntos fuertemente expuestos no tienen índice numérico uno.

Contents

Introduction

In this thesis we are interested in studying the geometry of the space of operators between Banach spaces around a fixed operator. To do so, one of the tools that we use is the numerical range with respect to a given operator. The concept of numerical range of operators has its origin in the 1918 paper by Toeplitz [39] dealing with matrices and it was firstly developed within the context of Hilbert spaces (see e.g. the book [16]). Over the years, this notion has been extended to more general settings. A first extension of the concept of numerical range to elements of a unital algebra was given in 1955 by Bohnenblust and Karlin [6] in order to relate the geometric and the alebraic properties of the unit. Later on, in the 1960s, Bauer [3] and Lumer [25] provided independent but essentially equivalent extensions of Toeplitz's numerical range to bounded linear operators on Banach spaces. In 1985 [34], an abstract notion of numerical range was introduced, which implicitly appeared in the aforementioned Bohnenblust–Karlin paper and generalizes all the previous versions. Let us introduce here such abstract version from which we will recover all other numerical ranges. We refer to the classical monographs by Bonsall and Duncan [7, 8] and to the book by Cabrera and Rodríguez-Palacios [10, Sections 2.1 and 2.9] for further information and background.

The basic notation and terminology needed to understand both this introduction and the rest of the dissertation is included in the "Notation and terminology" section on page 33. The bibliography listed from page 199 onwards is specific to this introduction and to the Conclusions chapter, as Chapters I to VI each contain their own separate references sections.

Given a Banach space Z and an element $u \in S_Z$, the *(abstract) numerical range* of $z \in Z$ with respect to (Z, u) is the set of scalars given by

$$V(Z, u, z) := \{\phi(z) \colon \phi \in S_{Z^*},\, \phi(u) = 1\}.$$

It is well-known that the geometry of the space Z around u is related to the numerical range with respect to (Z, u) thanks to the formula

$$\max \operatorname{Re} V(Z, u, z) = \lim_{\alpha \to 0^+} \frac{\|u + \alpha z\| - 1}{\alpha} \tag{1}$$

(see for instance [10, Proposition 2.1.5]). The *numerical radius* of $z \in Z$ with respect to (Z, u) is defined as

$$v(Z, u, z) := \sup\{|\lambda| \colon \lambda \in V(Z, u, z)\}.$$

It is clear that the numerical radius is a seminorm on Z satisfying $v(Z,u,z) \leqslant \|z\|$ for every $z \in Z$. Frequently, the numerical radius is actually an equivalent norm on Z, and this can be measured quantitatively with the (*abstract*) *numerical index* of (Z,u):

$$n(Z,u) := \inf\{v(Z,u,z) \colon z \in S_Z\} = \max\{k \geqslant 0 \colon k\|z\| \leqslant v(Z,u,z)\ \forall z \in Z\}.$$

Clearly, $0 \leqslant n(Z,u) \leqslant 1$ and $n(Z,u) > 0$ if and only if $v(Z,u,\cdot)$ is an equivalent norm on Z.

It is then natural to define the numerical range of an operator as follows: given a Banach space X and an operator $T \in \mathcal{L}(X)$, the *intrinsic numerical range* of T is the set

$$V(\mathcal{L}(X),\mathrm{Id},T) = \{\phi(T) \colon \phi \in \mathcal{L}(X)^*,\ \|\phi\| = \phi(\mathrm{Id}) = 1\}.$$

Notice that this definition deals with the (wild) dual of $\mathcal{L}(X)$, which can be quite complicated. Therefore, it is convenient to have a "spatial" version of the numerical range that allows to work in the ground space and its dual rather than in the dual of the space of operators. We present here Bauer's definition, which is currently the standard one. The *spatial numerical range* of T is the set of scalars given by

$$W(T) := \{x^*(Tx) \colon x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}.$$

These two concepts of numerical range are related by the equality

$$\overline{\mathrm{conv}}(W(T)) = V(\mathcal{L}(X),\mathrm{Id},T) \tag{2}$$

for every $T \in \mathcal{L}(X)$ (see e.g. [10, Proposition 2.1.31]), thus they produce the same numerical radius of operators and so the same numerical index. Namely, the *numerical radius* of an operator $T \in \mathcal{L}(X)$ is

$$v(T) := \sup\{|\lambda| \colon \lambda \in W(T)\} = \sup\{|\lambda| \colon \lambda \in V(\mathcal{L}(X),\mathrm{Id},T)\} = v(\mathcal{L}(X),\mathrm{Id},T)$$

and the *numerical index* of the space X is

$$n(X) := \inf\{v(T) \colon T \in \mathcal{L}(X),\ \|T\| = 1\} = n(\mathcal{L}(X),\mathrm{Id}).$$

It is of course possible to consider the numerical range of operators between (possibly different) Banach spaces with respect to a fixed operator of norm one. The intrinsic version of this concept is clear just particularizing the abstract numerical range to this context. Given two Banach spaces X and Y and a norm-one operator $G \in \mathcal{L}(X,Y)$, the *intrinsic numerical range of $T \in \mathcal{L}(X,Y)$ with respect to G* is the set

$$V(\mathcal{L}(X,Y),G,T) = \{\phi(T) \colon \phi \in \mathcal{L}(X,Y)^*,\ \|\phi\| = \phi(G) = 1\},$$

and we may consider the corresponding numerical radius $v(\mathcal{L}(X,Y),G,T)$ and numerical index $n(\mathcal{L}(X,Y),G)$. Again, this definition forces us to deal with the (even wilder) dual of $\mathcal{L}(X,Y)$, so a more manageable version of this concept would be desirable. A direct extension of the spatial numerical range to this setting using the set

$$\{y^*(Tx) \colon y^* \in S_{Y^*},\ x \in S_X,\ y^*(Gx) = 1\}$$

is not suitable as, for instance, it is empty if G does not attain its norm. Moreover, even in the case when G is an isometric embedding, it is not always representative (see [30, Theorem 2.1]). In 2014, Ardalani [1] provided an approximated version of this set. Given two Banach spaces X and Y and a norm-one operator $G \in \mathcal{L}(X,Y)$, the *approximated spatial numerical range of* $T \in \mathcal{L}(X,Y)$ *with respect to* G is defined as

$$V_G(T) := \bigcap_{\delta>0} \overline{\{y^*(Tx) \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(Gx) > 1-\delta\}}.$$

Ardalani showed in [1, Lemma 4.2] that, in the case when $G = \mathrm{Id}$, this set is just the closure of $W(T)$ and, therefore, recovers $V(\mathcal{L}(X), \mathrm{Id}, T)$ by convex hull using (2). Actually, it is proved in [27, Theorem 2.1] that the same happens for all norm-one operators G, that is,

$$\operatorname{conv}(V_G(T)) = V(\mathcal{L}(X,Y), G, T) \tag{3}$$

for every $T \in \mathcal{L}(X,Y)$. As a consequence, the approximated spatial numerical range and the intrinsic numerical range produce the same numerical radius and numerical index. The *numerical radius of* T *with respect to* G is

$$v_G(T) := \inf_{\delta>0} \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(Gx) > 1-\delta\} = v(\mathcal{L}(X,Y), G, T)$$

and the *numerical index of the pair* (X,Y) *with respect to* G is

$$n_G(X,Y) := \inf\{v_G(T) \colon T \in \mathcal{L}(X,Y),\, \|T\| = 1\} = n(\mathcal{L}(X,Y), G).$$

Equivalently, $n_G(X,Y)$ is the maximum constant $k \geqslant 0$ such that $k\|T\| \leqslant v_G(T)$ for all $T \in \mathcal{L}(X,Y)$.

These notions are natural generalizations of the original concepts by Bauer and Lumer: as already commented, if we restrict to the case $X = Y$ and $G = \mathrm{Id}$, [1, Lemma 4.2] shows that $V_{\mathrm{Id}}(T) = \overline{W(T)}$ by using the Bishop–Phelps–Bollobás Theorem and, in consequence, $v_{\mathrm{Id}}(T) = v(T)$ and $n_{\mathrm{Id}}(X,X) = n(X)$. This opens the way for extending the study of the numerical index to this more general context. Along this process, new questions will arise due to the possibility of moving the operator G and choosing different Banach spaces as domain or codomain.

Thanks to formula (1) in the context of the space of operators and equality (3), it is clear that the numerical range with respect to an operator is related to both the algebraic and the geometric structures of the space $\mathcal{L}(X,Y)$ around a norm-one operator G.

The aim of this thesis is to study and analyse different notions related to the geometry of the space of all bounded linear operators between Banach spaces around a given operator and to make advances in several related problems. More specifically, we conduct a systematic analysis of the concept of numerical index with respect to an operator (Chapter I), we analyse the numerical index and Daugavet property of operators ideals and tensor products (Chapter II), we compute the numerical index of some two-dimensional L_p spaces (Chapters III and IV), we study the property of being a generating operator (Chapter V), and we explore the notion of Birkhoff-James orthogonality (Chapter VI). Finally, the Conclusions chapter (page 191) contains some final remarks, including some open questions on the topics treated in the previous chapters.

The methodology we adopted in our research followed the standard approach used in Functional Analysis. This involved conducting a comprehensive review of existing literature in the field, examining the techniques used by other researchers, and consulting with experts to gain insights into related problems. Our results were documented in research articles and presented through talks and posters at various conferences. Furthermore, our research was significantly enhanced through a three-month research stay supervised by Professor Vladimir Kadets at V. N. Karazin Kharkiv National University in Ukraine.

This dissertation follows a compendium form, where Chapters I to VI consist of independent papers authored by the doctoral candidate and collaborators. Each chapter is self-contained and has its own introduction, development, and bibliography. Therefore, each chapter could be read as an independent entity. Due to the compendium form, different results may be numbered identically across multiple chapters. To avoid confusion when referencing them in this introductory chapter and in the final chapter of conclusions, we will use roman numerals to indicate the chapter in which a particular result appears. For example, Theorem I.2.12 refers to Theorem 2.12 in Chapter I.

Let us provide an overview of the thesis contents.

Chapter I. On the numerical index with respect to an operator

The content of this chapter corresponds to the published paper

[21] V. Kadets, M. Martín, J. Merí, A. Pérez, and A. Quero, On the numerical index with respect to an operator, *Dissertationes Mathematicae* **547** (2020), 1–58.
DOI: 10.4064/dm805-9-2019.

The aim of this chapter is to thoroughly study the numerical index with respect to an operator between Banach spaces.

Section I.2 is devoted to presenting some known and new results on abstract numerical index. When Z is a finite-dimensional real space, we show that the set $\{u \in S_Z \colon n(Z,u) > 0\}$ is countable (i.e. finite or infinite and countable) and give estimations on the sum of $n(Z,u)$ over all elements $u \in S_Z$.

Theorem (Theorem I.2.12)**.** *Let Z be a real space with $\dim(Z) = m \geqslant 2$. Then,*

$$\sum_{u\in S_Z} n(Z,u)^{m-1} < \infty.$$

Additionally, for every subset $A \subseteq [0,1]$ containing 0, we show the existence of a (real or complex) Banach space Z such that $\{n(Z,u)\colon u \in S_Z\} = A$. To finish this section, we give a new expression of $V(Z,u,z)$ in terms of the elements of a subset of B_{Z^*} such that its weak-star closed convex hull is the whole B_{Z^*}, which will be useful to compute numerical radii with respect to operators.

We provide some tools for studying the numerical index with respect to an operator in Section I.3. We begin particularizing some results in the previous section to the setting of the space of operators. Then, we show that the numerical index with respect to an operator is greater than or equal to the numerical index with respect to its adjoint, we give a formula for the numerical index with respect to a rank-one operator, and provide some results which allow to control the numerical index with respect to an operator in terms of the numerical radii of the operators on the domain or on the codomain.

Next, we dedicate Section I.4 to studying the set of values of the numerical indices with respect to all norm-one operators between two given Banach spaces X and Y, that is, the set

$$\mathcal{N}(\mathcal{L}(X,Y)) := \big\{n_G(X,Y)\colon G\in\mathcal{L}(X,Y),\ \|G\|=1\big\}.$$

As a consequence of the results in the previous sections, we obtain that $0\in\mathcal{N}(\mathcal{L}(X,Y))$ unless both X and Y are one-dimensional and that the set $\mathcal{N}(\mathcal{L}(X,Y))$ is countable when X and Y are finite-dimensional real spaces. Additionally, we provide several results for classic Banach spaces. The first result in this line is the following concerning real Hilbert spaces.

Theorem (Theorem I.4.5)**.** *Let H be a real Hilbert space of dimension at least two. Then,*

$$\mathcal{N}(\mathcal{L}(X,H)) = \mathcal{N}(\mathcal{L}(H,Y)) = \{0\}$$

for all real Banach spaces X and Y. In particular, $\mathcal{N}(\mathcal{L}(H))=\{0\}$.

There are other spaces having trivial set of values of the numerical indices with respect to operators, for instance, when the domain or the codomain is $\mathcal{L}(H)$ where H is an infinite-dimensional real Hilbert space.

Theorem (Theorem I.4.7)**.** *Let H be a real Hilbert space of dimension at least two. Then,*

$$\mathcal{N}\big(\mathcal{L}(X,\mathcal{L}(H))\big) = \mathcal{N}\big(\mathcal{L}(X,\mathcal{K}(H))\big) = \{0\}$$

for every Banach space X. In particular,

$$\mathcal{N}\big(\mathcal{L}(\mathcal{L}(H))\big) = \mathcal{N}\big(\mathcal{L}(\mathcal{K}(H))\big) = \{0\}.$$

Moreover, if H is infinite-dimensional or has even dimension, then

$$\mathcal{N}\big(\mathcal{L}(\mathcal{L}(H),Y)\big) = \mathcal{N}\big(\mathcal{L}(\mathcal{K}(H),Y)\big) = \{0\}$$

for every Banach space Y.

Then, we give some inclusions for the set of numerical indices with respect to operators whose domain or codomain is a real ℓ_p space.

Proposition (Proposition I.4.11)**.** *Let $1<p<\infty$ and let $M_p=\sup\limits_{t\in[0,1]}\frac{|t^{p-1}-t|}{1+t^p}$. Then,*

$$\mathcal{N}\big(\mathcal{L}(X,\ell_p(\Gamma))\big)\subseteq[0,M_p] \quad \textit{and} \quad \mathcal{N}\big(\mathcal{L}(\ell_p(\Gamma),Y)\big)\subseteq[0,M_p]$$

hold in the real case for all Banach spaces X and Y.

We also study the set of values of the numerical indices with respect to operators for complex Hilbert spaces.

Proposition (Proposition I.4.13)**.** *Let H be a complex Hilbert space with* $\dim(H)\geqslant 2$*. Then,*

$$\mathcal{N}\big(\mathcal{L}(X,H)\big)\subseteq[0,1/2] \quad \textit{and} \quad \mathcal{N}\big(\mathcal{L}(H,Y)\big)\subseteq[0,1/2]$$

for all complex Banach spaces X and Y.

Moreover, when both domain H_1 and codomain H_2 are complex Hilbert spaces we obtain that $\mathcal{N}\big(\mathcal{L}(H_1,H_2)\big) = \{0,1/2\}$ if H_1 and H_2 are isometrically isomorphic and $\mathcal{N}\big(\mathcal{L}(H_1,H_2)\big) = \{0\}$ otherwise.

In addition, we show that

$$\mathcal{N}(\mathcal{L}(C(K_1),C(K_2))) = \{0,1\}$$

for many families of compact Hausdorff topological spaces K_1 and K_2, both in the real and complex case. As a consequence, we obtain that $\mathcal{N}(\mathcal{L}(L_\infty(\mu_1),L_\infty(\mu_2))) = \{0,1\}$ for all σ-finite measures μ_1 and μ_2 such that at least one of the spaces $L_\infty(\mu_i)$, $i=1,2$, has dimension greater than 1, and $\mathcal{N}(\mathcal{L}(L_1(\mu_1),L_1(\mu_2))) \subseteq \{0,1\}$ for all σ-finite measures μ_1 and μ_2.

In Section I.5, using the tools presented in Section I.3, we prove that the concept of Lipschitz numerical range introduced in [40, 41] for Lipschitz maps from a Banach space to itself can be viewed as a particular case of numerical range with respect to a linear operator between two different Banach spaces.

The last section of this chapter contains some results showing the behaviour of the value of the numerical index when applying some Banach space operations. For instance, the numerical index of a c_0-, ℓ_1- or ℓ_∞-sum of Banach spaces with respect to a direct sum of norm-one operators in the corresponding spaces coincides with the infimum of the numerical indices of the corresponding summands. As an important consequence, we obtain the following example.

Theorem (Theorem I.6.4)**.** *In both the real and the complex case, there exist Banach spaces X such that*

$$\mathcal{N}(\mathcal{L}(X)) = [0,1].$$

We also show that composition operators between vector-valued function spaces $C(K,X)$, $L_1(\mu,X)$ and $L_\infty(\mu,X)$ produce the same numerical index as the original operator. Next, we provide two conditions, each of which ensures that the numerical index with respect to an operator equals the numerical index with respect to its adjoint, namely when the codomain is L-embedded or when the operator has rank-one. Finally, we discuss some results about the numerical index with respect to the composition of two operators, and show how to extend the domain and the codomain of an operator maintaining the value of the numerical index. In particular, these results allow to solve a part of Problem 9.14 posed in [20].

Chapter II. Numerical index and Daugavet property of operator ideals and tensor products

The content of this chapter corresponds to the published paper

[31] M. Martín, J. Merí, and A. Quero, Numerical index and Daugavet property of operator ideals and tensor products, *Mediterranean Journal of Mathematics* **18** (2021), no. 2, 15 pp. DOI: 10.1007/s00009-021-01721-9.

This chapter is dedicated to studying the numerical index of operator ideals and tensor products, and to analysing the behaviour of the Daugavet property in tensor products.

We study in Section II.2 the relationship between the numerical index of subspaces of $\mathcal{L}(X,Y)$ which are ideals and the numerical indices of the spaces X and Y. The first result of the section shows that for every operator ideal $\mathcal{Z}(X,Y) \leqslant \mathcal{L}(X,Y)$ endowed with the operator norm we have that $n(\mathcal{Z}(X,Y)) \leqslant \min\{n(X), n(Y)\}$. In particular, $n(\mathcal{L}(X,Y)) \leqslant \min\{n(X), n(Y)\}$. With the help of suitable representations, we proved stronger inequalities for the numerical indices of the spaces of compact and weakly compact operators.

Theorem (Theorem II.2.3)**.** *Let X, Y be Banach spaces, then the following hold:*

$$n(\mathcal{K}(X,Y)) \leqslant \min\{n(X^*), n(Y)\} \quad \text{and} \quad n(\mathcal{W}(X,Y)) \leqslant \min\{n(X^*), n(Y)\}.$$

As a consequence of this result, we present some interesting examples such as the existence of a real Banach space X with $n(X) = 1$ while $n(\mathcal{K}(X,Y)) = n(\mathcal{W}(X,Y)) = 0$ for every Banach space Y. In particular, $n(X) = 1$ while $n(\mathcal{K}(X,X)) = n(\mathcal{W}(X,X)) = 0$. We also provide an example to show that the previous inequalities can be strict and discuss some cases in which the equality holds.

In Section II.3 we also obtain inequalities for the numerical index of tensor products of Banach spaces, which is the best one can expect as it is known that $n(X\hat{\otimes}_\pi Y)$ and $n(X\hat{\otimes}_\varepsilon Y)$ cannot be computed as a function of $n(X)$ and $n(Y)$ (see [33, Example 10]).

Theorem (Theorem II.3.2)**.** *Let X, Y be Banach spaces. Then, the following hold:*

(a) $n(X\hat{\otimes}_\pi Y) \leqslant \min\{n(X), n(Y)\}$,

(b) $n(X\hat{\otimes}_\varepsilon Y) \leqslant \min\{n(X), n(Y)\}$.

We present some consequences for the space of approximable operators and for the space of nuclear operators using their representations as suitable tensor products. More specifically, we have that

$$n(\mathcal{A}(X,Y)) \leqslant \min\{n(X^*), n(Y)\}$$

and, if X^* or Y has the approximation property,

$$n(\mathcal{N}(X,Y)) \leqslant \min\{n(X^*), n(Y)\}.$$

We finish this chapter with a section devoted to studying when the Daugavet property passes from the tensor product to the factors. Recall that a Banach space X has the *Daugavet property* [24] if the norm equality

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{DE}$$

holds for all rank-one operators $T \in \mathcal{L}(X)$. This property is related to the numerical range of operators as follows: an operator $T \in \mathcal{L}(X)$ satisfies (DE) if and only if $\sup \operatorname{Re} W(T) = \|T\|$ (see [12, Remark, page 483]). A notion related to the Daugavet property is that of slicely countably determined set. A bounded subset A of a Banach space X is said to be *slicely countably determined* if there exists a coutable family of slices $\{S_n \colon n \in \mathbb{N}\}$ of A such that $A \subseteq \overline{\operatorname{conv}}(B)$ for every subset $B \subseteq A$ intersecting all the slices S_n. This concept was introduced in [2], where the authors proved that every operator T defined on a Banach space X with the Daugavet property such that $T(B_X)$ is slicely countably determined satisfies (DE). Thanks to this relation, we are able to prove that the Daugavet property of a projective tensor product passes to one of the factors if the unit ball of the other one is a slicely countably determined set.

Theorem (Theorem II.4.1)**.** *Let X, Y be Banach spaces. Suppose that B_Y is a slicely countably determined set and $X\hat{\otimes}_\pi Y$ has the Daugavet property. Then, X has the Daugavet property.*

For injective tensor products, we obtain the following positive result.

Proposition (Proposition II.4.3)**.** *Let X, Y be Banach spaces such that $X\hat{\otimes}_\varepsilon Y$ has the Daugavet property. Suppose that the norm of Y is Fréchet differentiable at a point $y_0 \in S_Y$. Then, X has the Daugavet property.*

An analogous result is provided for projective tensor products in the case where the space Y^* has a point of Fréchet differentiability of the norm.

Chapter III. On the numerical index of absolute symmetric norms on the plane

The content of this chapter corresponds to the published paper

[35] J. Merí and A. Quero, On the numerical index of absolute symmetric norms on the plane, *Linear and Multilinear Algebra* **69** (2021), no. 5, 971–979.
DOI: 10.1080/03081087.2020.1762532.

The exact computation of the numerical index of concrete Banach spaces is usually a difficult task, even in finite dimension. For instance, the computation of the numerical index of L_p spaces when $p \neq 1, 2, \infty$ remains as an important open problem since the beginning of the theory although it has been addressed by several authors (see [13, 14, 15, 28, 29]). With the aim of advancing in the problem of calculating the numerical index of the real ℓ_p^2 space, we deal in this chapter with the numerical index of two-dimensional real spaces X equipped with an absolute and symmetric norm. Recall that a norm $\|\cdot\|\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is *absolute* if $\|(1,0)\| = \|(0,1)\| = 1$ and

$$\|(a,b)\| = \|(|a|,|b|)\|$$

for every $a, b \in \mathbb{R}$, and that the norm is *symmetric* if $\|(b,a)\| = \|(a,b)\|$ for every $a, b \in \mathbb{R}$. In our approach, we make use of a basis of the space of operators $\mathcal{L}(X)$ formed by onto isometries:

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad I_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and we estimate the norm of the operators using that $\|T\| \leqslant \max\{\|T\|_1, \|T\|_\infty\}$ for every operator $T \in \mathcal{L}(X)$. With the help of these tools, we are able to transform our problem into a linear optimization one, which allows us to give a lower bound for the numerical index of such spaces and to show that, in many instances, the numerical index is attained at the operator I_4.

Theorem (Theorem III.2.2)**.** *Let X be $\mathbb{R}^2$ endowed with an absolute and symmetric norm. Let $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ be such that $x_0^*(x_0) = 1$ and $|x_0^*(I_4x_0)| = v(I_4)$, and write $c_j = |x_0^*(I_jx_0)|$ for every $j = 1, \ldots, 4$. If $c_4 = 0$, then $n(X) = 0$. If otherwise $c_4 > 0$, then*

$$n(X) \geqslant \min\left\{c_4, \frac{2}{1 + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}}\right\}.$$

Moreover, if the inequality $c_4\left(1+\frac{1}{c_2}+\frac{1}{c_3}\right)\leqslant 1$ *holds, then*

$$n(X)=v(I_4).$$

As a major consequence, we compute the numerical index of the real two-dimensional L_p space for $3/2\leqslant p\leqslant 3$.

Theorem (Theorem III.2.3)**.** *Let* $p\in\left[\frac{3}{2},3\right]$*. Then,*

$$n(\ell_p^2)=M_p=\max_{t\in[0,1]}\frac{|t^{p-1}-t|}{1+t^p}.$$

Let us remark that, in order to use Theorem III.2.2 to obtain this result, we show that condition $c_4\left(1+\frac{1}{c_2}+\frac{1}{c_3}\right)\leqslant 1$ holds not only for a particular choice of $x\in S_{\ell_p^2}$ and $x^*\in S_{\ell_q^2}$ satisfying $x^*(x)=1$ but for all of them when $3/2\leqslant p\leqslant 3$. In [37] the authors prove that this condition actually holds for carefully selected pairs for a wider range of values of p, which leads to a slight improvement: the equality $n(\ell_p^2)=M_p$ is proved for $1+\alpha_0\leqslant p\leqslant\alpha_1$, where α_0 is the root of $f(x)=1+x^{-2}-(x^{-\frac{1}{x}}+x^{\frac{1}{x}})$ and $\frac{1}{1+\alpha_0}+\frac{1}{\alpha_1}=1$ ($\alpha_0\approx 0.4547$).

Chapter IV. On the numerical index of the real two-dimensional L_p space

The content of this chapter corresponds to the following paper accepted for publication:

[36] J. Merí and A. Quero, On the numerical index of the real two-dimensional L_p space, *Linear and Multilinear Algebra* (2023), published online, 1–16.
DOI: 10.1080/03081087.2023.2181938.

The aim of this chapter is to calculate the numerical index of ℓ_p^2 for $\frac{6}{5}\leqslant p\leqslant\frac{3}{2}$ and for $3\leqslant p\leqslant 6$. This, together with the previous results, gives the numerical index of ℓ_p^2 for $\frac{6}{5}\leqslant p\leqslant 6$. Since the abstract approach in the previous chapter does not provide a complete solution for the problem of computing the numerical index of the real ℓ_p^2 spaces, we need to explore an alternative method. The main difference here is the use of Riesz–Thorin interpolation theorem to estimate the norm of operators on ℓ_p^2: the inequality $\|T\|\leqslant\|T\|_1^{1/p}\|T\|_\infty^{1/q}$ holds for every operator $T\in\mathcal{L}(\ell_p^2)$.

Theorem (Theorem IV.2.2)**.** *Let* $p\in\left[\frac{6}{5},6\right]$*. Then,*

$$n(\ell_p^2)=M_p=\max_{t\in[0,1]}\frac{|t^{p-1}-t|}{1+t^p}.$$

The proof of this result is rather technical and we want to highlight that, although we only obtain the numerical index of ℓ_p^2 for $6/5\leqslant p\leqslant 3/2$ (and consequently for $3\leqslant p\leqslant 6$ by duality), we actually prove that

$$\frac{v(T)}{\|T\|}\geqslant M_p$$

for a wide class of operators $T\in\mathcal{L}(\ell_p^2)$ for $1<p\leqslant 3/2$. Let us also comment that our techniques can give the equality $n(\ell_p^2)=M_p$ for a slightly wider range of values of p, however our approach does not work for p close to 1 (see Remark IV.2.3).

Chapter V. Generating operators between Banach spaces

The content of this chapter corresponds to the preprint

[22] V. Kadets, M. Martín, J. Merí, and A. Quero, Generating operators between Banach spaces, *Preprint* (2023). Available at arXiv:2306.02645.

In this chapter, we introduce a new seminorm on the space of bounded linear operators which is between the numerical radius with respect to an operator and the operator norm. Let X, Y and Z be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator, and let $T \in \mathcal{L}(X,Z)$. We define the *norm of T relative to G* by

$$\|T\|_G := \inf_{\delta>0} \sup \{\|Tx\| \colon x \in \operatorname{att}(G,\delta)\}.$$

where, for each $\delta > 0$, $\operatorname{att}(G,\delta) := \{x \in S_X \colon \|Gx\| > 1-\delta\}$ is the *δ-attainment set* of G.

Clearly, if $Y = Z$, we have that

$$v_G(T) \leqslant \|T\|_G \leqslant \|T\|$$

for all operators $T \in \mathcal{L}(X,Y)$. This chapter aims to study the relationship between $\|\cdot\|_G$ and the usual operator norm, especially focusing on the case when they coincide. We say that a norm-one operator $G \in \mathcal{L}(X,Y)$ between two Banach spaces X and Y is *generating* if $\|T\|_G = \|T\|$ for every $T \in \mathcal{L}(X,Y)$.

In many instances, the relative norm and the usual norm of operators are equivalent. The first result in Section V.2 characterizes this fact in several ways.

Proposition (Proposition V.2.1). *Let X, Y be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator, and let $r \in (0,1]$. Then, the following are equivalent:*

(*i*) $\|T\|_G \geqslant r\|T\|$ *for every Banach space Z and every $T \in \mathcal{L}(X,Z)$.*

(*ii*) *There is a (non null) Banach space Z such that $\|T\|_G \geqslant r\|T\|$ for every $T \in \mathcal{L}(X,Z)$.*

(*iii*) *There is a (non null) Banach space Z such that $\|T\|_G \geqslant r\|T\|$ for every rank-one operator $T \in \mathcal{L}(X,Z)$.*

(*iv*) $\|x^*\|_G \geqslant r\|x^*\|$ *for every $x^* \in X^*$.*

(*v*) $\overline{\operatorname{conv}}(\operatorname{att}(G,\delta)) \supseteq rB_X$ *for every $\delta > 0$.*

As a consequence, we obtain the following characterization of generating operators.

Corollary (Corollary V.2.3). *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then, the following are equivalent:*

(*i*) *G is generating.*

(*ii*) $\|T\|_G = \|T\|$ *for every $T \in \mathcal{L}(X,Z)$ and every Banach space Z.*

(*iii*) *There is a (non null) Banach space* Z *such that* $\|T\|_G = \|T\|$ *for every rank-one operator* $T \in \mathcal{L}(X, Z)$*.*

(*iv*) $B_X = \overline{\text{conv}}(\text{att}(G, \delta))$ *for every* $\delta > 0$*.*

Thanks to this result, it is clear that the property of being generating does not depend on the codomain. When X is reflexive and G is compact, the affirmations in the previous result are also equivalent to the fact that $B_X = \overline{\text{conv}}(\text{att}(G))$. We also study the relationship between the concept of generating operator and denting point, showing that any generating operator attains its norm at every denting point of the unit ball of the domain. Actually, this necessary condition is also sufficient when B_X is the closed convex hull of its denting points and, in particular, when X has the Radon-Nikodým property. We provide another useful characterization in terms of some spear sets of the dual of the domain. Recall that $F \subset B_X$ is a *spear set* if $\max_{\theta \in \mathbb{T}} \sup_{z \in F} \|z + \theta x\| = 1 + x$ for every $x \in X$. If $z \in S_Z$ satisfies that $F = \{z\}$ is a spear set, we just say that z is a *spear vector*.

Corollary (Corollary V.2.17)**.** *Let* X*,* Y *be Banach spaces and let* $G \in \mathcal{L}(X, Y)$ *with* $\|G\| = 1$*. Then,* G *is generating if and only if* $G^*(B_{Y^*})$ *is a spear set of* X^**.*

Additionally, we analyse how the property of being generating behaves with respect to the operation of taking the adjoint G^* of an operator G and show that this property does not pass in general from an operator to its adjoint, nor the other way around. Nevertheless, if the second adjoint is generating, then so is G. To finish this section, we study the stability of generating operators by taking c_0-, ℓ_1-, and ℓ_∞-sums, and provide some examples in classical Banach spaces.

We study in Section V.3 the relationship between generating operators and norm-attainment. While generating operators having rank one and those whose domain has the Radon-Nikodým property attain their norm, there are generating operators, even of rank two, which do not attain their norm. Then, we characterize when it is possible to construct an operator from a given Banach space which is generating but does not attain its norm.

Theorem (Theorem V.3.5)**.** *Let* X *be a Banach space. Then, the following statements are equivalent:*

(*i*) *There exists a Banach space* Y *and a norm-one operator* $G \in \mathcal{L}(X, Y)$ *such that* G *is generating but it does not attain its norm.*

(*ii*) *There exists a spear set* $\mathcal{B} \subseteq B_{X^*}$ *such that* $\sup\limits_{x^* \in \mathcal{B}} |x^*(x)| < 1$ *for every* $x \in S_X$*.*

In Section V.4, we consider the set $\text{Gen}(X, Y)$ of all generating operators between two Banach spaces X and Y. We show that this set is closed and that for every Banach space Y, there exists a Banach space X such that $\text{Gen}(X, Y) = \emptyset$. However, if we restrict the space X to be separable, this result is not longer true. Then, we study some properties of $\text{Gen}(X, Y)$ when X is fixed. We show that $\text{Gen}(X, Y) \neq \emptyset$ for every Y if and only if X^* contains spear vectors and that the only case in which there is Y such that $\text{Gen}(X, Y) = S_{\mathcal{L}(X,Y)}$ is when X has dimension one. We next study when the set $\text{Gen}(X, Y)$ generates the unit ball of $\mathcal{L}(X, Y)$ by closed convex hull. The next result shows that this is the case when $X = L_1(\mu)$ for a finite measure μ and Y has the Radon-Nykodým property.

Theorem (Theorem V.4.10)**.** *Let* (Ω, Σ, μ) *be a finite measure space and let* Y *be a Banach space. Then,*

$$\{T \in \mathcal{L}(L_1(\mu), Y) \colon \|T\| \leqslant 1,\ T \text{ is representable}\} \subseteq \overline{\text{conv}}\left(\text{Gen}(L_1(\mu), Y)\right).$$

As a consequence, if Y has the Radon-Nikodým property, then

$$B_{\mathcal{L}(L_1(\mu),Y)} = \overline{\operatorname{conv}}\left(\operatorname{Gen}(L_1(\mu), Y)\right).$$

Moreover, $B_{\mathcal{L}(\ell_1(\Gamma),Y)} = \overline{\operatorname{conv}}\left(\operatorname{Gen}(\ell_1(\Gamma), Y)\right)$ for every Banach space Y and the only *real* finite-dimensional spaces with this property are ℓ_1^n for $n \in \mathbb{N}$.

Chapter VI. A numerical range approach to Birkhoff-James orthogonality with applications

The content of this chapter corresponds to the preprint

[32] M. Martín, J. Merí, A. Quero, S. Roy, and D. Sain, A numerical range approach to Birkhoff-James orthogonality with applications, *Preprint* (2023). Available at arXiv:2306.02638.

Let Z be a Banach space. Given $x, y \in Z$, we say that x is *Birkhoff-James orthogonal* to y, denoted by $x \perp_B y$, if

$$\|x + \lambda y\| \geqslant \|x\| \quad \forall \lambda \in \mathbb{K}.$$

This concept was introduced by Birkhoff [5] and thoroughly studied by James [17, 18], and it extends the standard definition of orthogonality in Hilbert spaces. A general approach to study Birkhoff-James orthogonality in any Banach space Z was given by James in [18, Corollary 2.2]:

$$x \perp_B y \iff \text{there exists } \phi \in Z^* \text{ with } \|\phi\| = 1 \text{ such that } \phi(x) = \|x\| \text{ and } \phi(y) = 0.$$

Observe that this characterization can be easily written using the abstract numerical range as follows:

$$x \perp_B y \iff 0 \in V(Z, u, z).$$

The notion of Birkhoff-James orthogonality has been extensively studied in specific Banach spaces by several authors, specially the case when Z is a space of bounded linear operators (see for instance [4, 26, 38]). The main disadvantage of James' approach is that involves working in the dual space, which can be difficult in certain cases. Consequently, in order to give characterizations of Birkhoff-James orthogonality, the authors usually needed to use specific techniques for each of the particular cases. With the help of the numerical range, the main aim of this chapter is to provide a widely applicable approach to address Birkhoff-James orthogonality that unifies all these techniques.

We begin Section VI.2 showing that it is also possible to express the numerical range in terms of Birkhoff-James orthogonality. Then, we present in Theorem VI.2.4 different expressions of the numerical range in terms of the elements on an arbitrary subset of the dual which is one-norming for the space, which extend a previous result from Chapter I, specifically Proposition I.2.14. Recall that a subset $\Lambda \subset S_{Z^*}$ is said to be *one-norming* for Z if $\|z\| = \sup\{|\phi(z)| : \phi \in \Lambda\}$ for all $z \in Z$ (equivalently, if B_{Z^*} equals the absolutely weak-star closed convex hull of Λ).

Theorem (Theorem VI.2.4)**.** *Let Z be a Banach space, let $u \in S_Z$, and let $\Lambda \subset B_{Z^*}$ be one-norming for Z. Then, for every $z \in Z$,*

$$\begin{aligned} V(Z,u,z) &= \operatorname{conv}\left(\left\{\theta_0 \lim \psi_n(z)\colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \theta_0 \in \mathbb{T},\ \lim \psi_n(u) = \overline{\theta_0}\right\}\right) \\ &= \operatorname{conv}\left(\left\{\lim \psi_n(z)\overline{\psi_n(u)}\colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \lim |\psi_n(u)| = 1\right\}\right) \\ &= \operatorname{conv} \bigcap\nolimits_{\delta>0} \overline{\left\{\psi(z)\overline{\psi(u)}\colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}} \\ &= \bigcap\nolimits_{\delta>0} \overline{\operatorname{conv}\left\{\psi(z)\overline{\psi(u)}\colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}}. \end{aligned}$$

This is the key to obtain the main result of this section, which provides a characterization of Birkhoff-James orthogonality in a Banach space in terms of the actions of functionals on an arbitrary one-norming subset. One of the equivalences presented in this result is the following.

Corollary (Corollary VI.2.6)**.** *Let Z be a Banach space, let $u \in S_Z$, and let $\Lambda \subset B_{Z^*}$ be one-norming for Z. Then, for $z \in Z$,*

$$u \perp_B z \iff 0 \in \operatorname{conv}\left(\left\{\lim \psi_n(z)\overline{\psi_n(u)}\colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \lim |\psi_n(u)| = 1\right\}\right).$$

Additionally, we characterize smooth points following the same spirit.

Corollary (Corollary VI.2.11)**.** *Let Z be a Banach space, let $u \in S_Z$, and let $\Lambda \subset B_{Z^*}$ be one-norming for Z. Then, u is a smooth point if and only if $\left\{\lim \psi_n(z)\overline{\psi_n(u)}\colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \lim |\psi_n(u)| = 1\right\}$ is a singleton set for every $z \in Z$.*

Section VI.3 contains a collection of particular cases in which the results of Section VI.2 apply. Even though some of the results in this section were already known, the techniques previously used depended on the particular case, while our current approach is applicable to all of them. The new results include general characterizations of Birkhoff-James orthogonality and smoothness in the space of vector-valued bounded functions.

Theorem (Theorem VI.3.2)**.** *Let Γ be a non-empty set, let Y be a Banach space, let $C \subset S_{Y^*}$ such that $B_{Y^*} = \overline{\operatorname{conv}}^{w^*}(C)$, and let $f, g \in \ell_\infty(\Gamma, Y)$. Then,*

$$f \perp_B g \iff 0 \in \operatorname{conv}\left\{\lim y_n^*(g(\gamma_n))\colon \gamma_n \in \Gamma,\ y_n^* \in C\ \forall n \in \mathbb{N},\ \lim y_n^*(f(\gamma_n)) = \|f\|\right\}.$$

This result also applies to Banach spaces which can be viewed as closed subspaces of $\ell_\infty(\Gamma, Y)$, and this allows to present new applications for spaces of vector-valued continuous functions, uniform algebras, polynomials, Lipschitz maps, and injective tensor products. For bounded linear operators, we present several results with respect to the operator norm as well as with respect to the numerical radius. Most of them were previously known but there are some improvements for compact operators.

In Section VI.4, we present some cases in which it is possible to remove the convex hull and the limits when characterizing Birkhoff-James orthogonality. The main result in this section deals with vector-valued continuous functions on a compact Hausdorff topological space and uses the notion of directional orthogonality. Given a Banach space Z and $x, y \in Z$, we say that x is *orthogonal* to y *in the direction of* $\gamma \in \mathbb{T}$, denoted by $x \perp_\gamma y$, if $\|x + t\gamma y\| \geqslant \|x\|$ for every $t \in \mathbb{R}$.

Theorem (Theorem VI.4.3)**.** *Let K be a compact Hausdorff topological space and let Y be a Banach space. Let $f, g \in C(K, Y)$ be such that the set $\{t \in K : \|f(t)\| = \|f\|\}$ is connected. Then,*

$$f \perp_B g \iff \forall \mu \in \mathbb{T}\ \exists t \in K \text{ such that } \|f(t)\| = \|f\| \text{ and } f(t) \perp_\mu g(t).$$

In the real case, we actually have

$$f \perp_B g \iff \exists t \in K \text{ such that } \|f(t)\| = \|f\| \text{ and } f(t) \perp_B g(t).$$

As a consequence, we obtain analogous results for compact operators on reflexive Banach spaces, which is new for complex infinite-dimensional spaces. We finish this section with a nice characterization of Birkhoff-James orthogonality for finite Blaschke products.

Finally, the last section contains applications of the results in the chapter to the study of spear vectors, spear operators, and Banach spaces with numerical index one. We say that a norm-one operator $G \in \mathcal{L}(X, Y)$ between two Banach spaces X and Y is a *spear operator* if $\max_{\theta \in \mathbb{T}} \|G + \theta T\| = 1 + \|T\|$ for every $T \in \mathcal{L}(X, Y)$. All the applications followed from Theorem VI.5.1, which connects the concepts of smoothness and Birkhoff-James orthogonality with respect to the abstract numerical radius.

Theorem (Theorem VI.5.1)**.** *Let Z be a Banach space and let $u \in S_Z$ be such that $v(Z, u, \cdot)$ is a norm on Z. Then, no smooth point is Birkhoff-James orthogonal to u in $(Z, v(Z, u, \cdot))$.*

As a consequence, we obtain the next result for spear vectors.

Corollary (Corollary VI.5.3)**.** *Let Z be a Banach space and $u \in S_Z$. If there exists a smooth point z_0 in Z such that $z_0 \perp_B u$, then u is not a spear vector.*

In the case when $Z = \mathcal{L}(X, Y)$ for Banach spaces X and Y, this leads to obstructive results for the existence of spear operators.

Corollary (Corollary VI.5.5)**.** *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$. Suppose that there is a strongly exposed point $x_0 \in B_X$ and a smooth point u_0 in Y satisfying that $u_0 \perp_B Gx_0$. Then, G is not a spear operator. As a consequence, if X is a Banach space with strongly exposed points and Y is a smooth Banach space with dimension at least two, then there are no spear operators in $\mathcal{L}(X, Y)$.*

This result somehow extends [20, Proposition 6.5.a] and provides a partial answer to [20, Problem 9.12]. When $X = Y$ and $G = \mathrm{Id}$, we get an obstructive condition for a Banach space to have numerical index one: the existence of a smooth point which is Birkhoff-James orthogonal to a strongly exposed point. In particular, smooth Banach spaces with dimension at least two containing strongly exposed points do not have numerical index one. This partially answers the question of whether a smooth Banach space of dimension at least two may have numerical index one [19, page 166].

Notation and terminology

Most of our notation and terminology is standard. We gather here some basics, while the rest is explained in the chapters when it is required.

We use $\mathbb{K}$ to denote the scalar field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers, and we use the standard notation $\mathbb{T} := \{\theta \in \mathbb{K} \colon |\theta| = 1\}$ for its unit sphere. We denote by $\mathrm{Re}(\cdot)$ the real part function, which is nothing more than the identity if we are dealing with real numbers.

We use the letters X, Y, Z for Banach spaces over $\mathbb{K}$. In some cases, we have to distinguish between the real and the complex case. For $x \in X$ and $\delta > 0$, we denote by $B(x, \delta)$ the closed ball centered at x of radius $\delta > 0$. For simplicity, we write B_X and S_X to denote the closed unit ball and the unit sphere of X respectively.

Given a non-empty subset $A \subset X$ and $x \in X$, we write

$$\mathbb{T}A := \{\theta a \colon \theta \in \mathbb{T},\, a \in A\} \quad \text{and} \quad \mathbb{T}x := \{\theta x \colon \theta \in \mathbb{T}\}.$$

We write $\mathrm{conv}(A)$, $\mathrm{aconv}(A)$, and $\mathrm{span}(A)$ to denote the convex hull, absolutely convex hull, and (linear) span of A respectively, while $\overline{\mathrm{conv}}(A)$, $\overline{\mathrm{aconv}}(A)$, and $\overline{\mathrm{span}}(A)$ denote their respective closures.

The *diameter* of a (bounded) set $A \subset X$ is

$$\mathrm{diam}(A) := \sup\{\|x - y\| \colon x, y \in A\},$$

and the *distance* between two subsets $A, B \subset X$ is

$$\mathrm{dist}(A, B) := \inf\{\|a - b\| \colon a \in A,\, b \in B\}.$$

We denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators from X to Y endowed with the operator norm

$$\|T\| := \sup\{\|Tx\| \colon x \in S_X\}.$$

We say that an operator $T \in \mathcal{L}(X, Y)$ *attains its norm*, or that it is *norm-attaining*, if there exists $x_0 \in S_X$ such that $\|Tx_0\| = 1$. We just write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$ and the identity operator is denoted by Id, or Id_X if it is necessary to precise the space. The topological dual of X is $X^* := \mathcal{L}(X, \mathbb{K})$, and $J_X \colon X \longrightarrow X^{**}$ denotes the natural isometric inclusion of X into its bidual X^{**}. If $T \in \mathcal{L}(X, Y)$, the operator $T^* \in \mathcal{L}(Y^*, X^*)$ defined by $(T^*y^*)(x) := y^*(Tx)$ for every $y^* \in Y^*$ and $x \in X$ is called the *adjoint operator* of T. For $x_0^* \in X^*$ and $y_0 \in Y$, we write $x_0^* \otimes y_0$ to denote the rank-one operator defined by $(x_0^* \otimes y_0)(x) := x_0^*(x)y_0$ for every $x \in X$.

A subset $\Lambda \subseteq B_{X^*}$ is said to be *r-norming for X* $(0 < r \leqslant 1)$ if $r\|x\| \leqslant \sup\{|x^*(x)|\colon x^* \in \Lambda\}$ for every $x \in X$; equivalently, if $rB_{X^*} \subseteq \overline{\mathrm{aconv}}^{w^*}(\Lambda)$. In the case when $r = 1$, we say that Λ is *one-norming for X* if $\|x\| = \sup\{|x^*(x)|\colon x^* \in \Lambda\}$ for every $x \in X$; equivalently, if $B_{X^*} = \overline{\mathrm{aconv}}^{w^*}(\Lambda)$.

An *operator ideal* $\mathcal{Z}$ is a "rule" (formally a subclass of the class of all bounded linear operators between Banach spaces) assigning to every pair of Banach spaces X and Y a linear subspace $\mathcal{Z}(X,Y)$ of $\mathcal{L}(X,Y)$ (called a *component* of $\mathcal{Z}$) which contains the finite rank operators and satisfies that

$$\mathcal{L}(F,Y) \circ \mathcal{Z}(E,F) \circ \mathcal{L}(X,E) \subseteq \mathcal{Z}(X,Y)$$

for all Banach spaces E, F, X, Y. We only consider ideals whose components are closed subspaces. We write $\mathcal{K}(X,Y)$, $\mathcal{W}(X,Y)$, and $\mathcal{A}(X,Y)$ to denote, respectively, the space of all *compact* operators, *weakly compact* operators, and *approximable* operators (i.e. norm limits of finite-rank operators), from X to Y, all of them endowed with the operator norm. We also consider the space of all nuclear operators. We say that an operator $T \in \mathcal{L}(X,Y)$ is *nuclear* if there exist $x_n^* \in X$ and $y_n \in Y$ for every $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} \|x_n^*\|\,\|y_n\| < \infty$ and

$$Tx = \sum_{n=1}^{\infty} x_n^*(x) y_n \qquad (x \in X).$$

The space of all nuclear operators, denoted by $\mathcal{N}(X,Y)$, is a Banach space endowed with the norm

$$N(T) := \inf\left\{\sum_{n=1}^{\infty} \|x_n^*\|\,\|y_n\|\colon Tx = \sum_{n=1}^{\infty} x_n^*(x) y_n\right\},$$

where the infimum is taken over all the representations of T as above.

Given $x \in X$ and $y \in Y$, $x \otimes y$ denotes the evaluation mapping acting on elements $T \in \mathcal{L}(X, Y^*)$ given by $(x \otimes y)(T) = (Tx)(y)$. The *(algebraic) tensor product* of X and Y, denoted by $X \otimes Y$, is the vector space spanned by $\{x \otimes y\colon x \in X,\ y \in Y\}$. Observe that every element u of $X \otimes Y$ is of the form

$$u = \sum_{i=1}^{n} x_i \otimes y_i,$$

where $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $y_1, \dots, y_n \in Y$, and the representation above is not unique in general. We introduce two different norms on $X \otimes Y$. The *projective norm* is defined for every $u \in X \otimes Y$ by

$$\|u\|_\pi := \inf\left\{\sum_{i=1}^{n} \|x_i\|\,\|y_i\|\colon u = \sum_{i=1}^{n} x_i \otimes y_i\right\},$$

where the infimum is taken over all the representations of $u = \sum_{i=1}^{n} x_i \otimes y_i$. We define the *projective tensor product* of X and Y, denoted by $X \hat{\otimes}_\varepsilon Y$, as the completion of $X \otimes Y$ under the projective norm. The *injective norm* is defined for each $u \in X \otimes Y$ as

$$\|u\|_\varepsilon := \sup\left\{\left|\sum_{i=1}^{n} x^*(x_i) y^*(y_i)\right| : x^* \in B_{X^*},\ y^* \in B_{Y^*}\right\},$$

where $\sum_{i=1}^{n} x_i \otimes y_i$ is any representation of u. The *injective tensor product* of X and Y, denoted by $X\hat{\otimes}_\varepsilon Y$, is the completion of $X \otimes Y$ under the injective norm.

Given $u \in S_X$, the *face* of B_{X^*} generated by u is the (non-empty) set

$$\mathrm{F}(B_{X^*}, u) := \{x^* \in S_{X^*} \colon x^*(u) = 1\}.$$

Let $A \subset X$ be a non-empty subset. A *slice* of A is a non-empty intersection of A with an open half space, and for $x^* \in X^*$ and $\delta > 0$, we write

$$\mathrm{Slice}(A, x^*, \delta) := \{x \in A \colon \operatorname{Re} x^*(x) > \sup_A \operatorname{Re} x^* - \delta\}.$$

A point $x \in A$ is said to be *extreme* if whenever $x = \lambda y + (1-\lambda)z$ with $y, z \in A$ and $0 < \lambda < 1$, then $y = z = x$. We denote by $\mathrm{ext}(A)$ the set of extreme points of A. A point $x \in A$ is *denting* if $x_0 \notin \overline{\mathrm{conv}}(A \setminus B(x, \delta))$ for every $\delta > 0$ (equivalently, if it belongs to slices of A of arbitrarily small diameter). We write $\mathrm{dent}(A)$ to denote the set of denting points of A.

We say that a closed convex subset A of X has the *Radon-Nikodým property* (*RNP* in short), if all its closed convex bounded subsets contain slices of arbitrarily small diameter or, equivalently, if all its closed convex bounded subsets are the closed convex hull of their denting points (see e.g. [9, 11]). In particular, the whole space X may also have this property.

A point $x \in X \setminus \{0\}$ is said to be *smooth* if the mapping $x \mapsto \|x\|$ is Gâteaux differentiable at x (equivalently, if there is a unique $x^* \in S_{X^*}$ with $x^*(x) = \|x\|$). We say that the Banach space X is *smooth* if every point $x \in X \setminus \{0\}$ is smooth. An element $x \in B_X$ is said to be a *strongly exposed point* of B_X if there is $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and whenever a sequence $\{x_n\}$ in B_X satisfies that $\operatorname{Re} x^*(x_n) \to 1$, we have that $\|x_n - x\| \to 0$ (equivalently, if $x^*(x) = 1$ and the diameter of $\mathrm{Slice}(B_X, x^*, \delta)$ tends to zero as $\delta \to 0^+$). We denote by $\mathrm{StrExp}(B_X)$ the set of strongly exposed points of B_X. Observe that strongly exposed points are denting points and denting points are extreme points, but none of the implications reverses in general.

Let Γ be a non-empty index set, and $\{X_\gamma \colon \gamma \in \Gamma\}$ be a collection of Banach spaces. We write

$$\Big[\bigoplus\nolimits_{\lambda\in\Lambda} X_\lambda\Big]_{c_0}, \qquad \Big[\bigoplus\nolimits_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_1}, \qquad \Big[\bigoplus\nolimits_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_\infty},$$

to denote, respectively, the c_0-, ℓ_1-, and ℓ_∞-sum of the family. If E is $\mathbb{R}^n$ endowed with an absolute norm $|\cdot|_E$ and $X_1, \ldots, X_n$ are Banach spaces, we write $X = [X_1 \oplus \cdots \oplus X_n]_E$ to denote the product space $X_1 \times \cdots \times X_n$ endowed with the norm

$$\|(x_1, \ldots, x_n)\| = \big|(\|x_1\|, \ldots, \|x_n\|)\big|_E$$

for all $x_i \in X_i$, $i = 1, \ldots, n$.

Given a Hausdorff topological space Ω and a Banach space X, we write $C_b(\Omega, X)$ to denote the Banach space of all bounded continuous functions from Ω to X, endowed with the supremum norm. If K is a compact Hausdorff topological space and X is a Banach space, we write $C(K, X) = C_b(K, X)$. Let (Ω, Σ, μ) be a positive measure space. We write $L_\infty(\mu, X)$ to denote the Banach space of all (classes of) strongly measurable functions from Ω to X which are essentially bounded, endowed with the essential supremum norm

$$\|f\|_\infty := \inf\{c > 0 \colon \|f(t)\| \leqslant c \text{ for a.e. } t \in \Omega\}.$$

For each $1 \leqslant p < \infty$, $L_p(\mu, X)$ is the Banach space of all (classes of) p-Bochner integrable functions from Ω to X, endowed with the integral norm

$$\|f\|_p := \left(\int_\Omega \|f(t)\|^p d\mu \right)^{\frac{1}{p}}.$$

To simplify, we just write $C_b(\Omega) = C_b(\Omega, \mathbb{K})$, $C(K) = C(K, \mathbb{K})$ and $L_p(\mu) = L_p(\mu, \mathbb{K})$. Given $1 \leqslant p \leqslant \infty$ and a non-empty set Γ, we write $\ell_p(\Gamma)$ to denote the L_p space associated to the counting measure on Γ. For $n \in \mathbb{N}$, we just write ℓ_p^n to denote $\ell_p(\{1, \ldots, n\})$.

Chapter I

On the numerical index with respect to an operator

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On the numerical index with respect to an operator

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Contents

Abstract

The aim of this paper is to study the numerical index with respect to an operator between Banach spaces. Given Banach spaces X and Y, and a norm-one operator $G \in \mathcal{L}(X,Y)$ (the space of all bounded linear operators from X to Y), the numerical index with respect to G, $n_G(X,Y)$, is the greatest constant $k \geq 0$ such that

$$k\|T\| \leq \inf_{\delta>0} \sup\{|y^*(Tx)|\colon y^* \in Y^*,\, x \in X,\, \|y^*\| = \|x\| = 1,\, \operatorname{Re} y^*(Gx) > 1-\delta\}$$

for every $T \in \mathcal{L}(X,Y)$. Equivalently, $n_G(X,Y)$ is the greatest constant $k \geq 0$ such that

$$\max_{|w|=1} \|G + wT\| \geq 1 + k\|T\|$$

for all $T \in \mathcal{L}(X,Y)$. Here, we first provide some tools to study the numerical index with respect to G. Next, we present some results on the set $\mathcal{N}(\mathcal{L}(X,Y))$ of the values of the numerical indices with respect to all norm-one operators in $\mathcal{L}(X,Y)$. For instance, $\mathcal{N}(\mathcal{L}(X,Y)) = \{0\}$ when X or Y is a real Hilbert space of dimension greater than 1 and also when X or Y is the space of bounded or compact operators on an infinite-dimensional real Hilbert space. In the real case

$$\mathcal{N}(\mathcal{L}(X,\ell_p)) \subseteq [0, M_p] \quad \text{and} \quad \mathcal{N}(\mathcal{L}(\ell_p,Y)) \subseteq [0, M_p]$$

for $1 < p < \infty$ and for all real Banach spaces X and Y, where $M_p = \sup_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$. For complex Hilbert spaces H_1, H_2 of dimension greater than 1, $\mathcal{N}(\mathcal{L}(H_1,H_2)) \subseteq \{0, 1/2\}$ and the value $1/2$ is taken if and only if H_1 and H_2 are isometrically isomorphic. Moreover, $\mathcal{N}(\mathcal{L}(X,H)) \subseteq [0,1/2]$ and $\mathcal{N}(\mathcal{L}(H,Y)) \subseteq [0,1/2]$ when H is a complex infinite-dimensional Hilbert space and X and Y are arbitrary complex Banach spaces. Also, $\mathcal{N}(\mathcal{L}(L_1(\mu_1), L_1(\mu_2))) \subseteq \{0,1\}$ and $\mathcal{N}(\mathcal{L}(L_\infty(\mu_1), L_\infty(\mu_2))) \subseteq \{0,1\}$ for arbitrary σ-finite measures μ_1 and μ_2, in both the real and the complex cases. Also, we show that the Lipschitz numerical range of Lipschitz maps from a Banach space to itself can be viewed as the numerical range of convenient bounded linear operators with respect to a bounded linear operator. Further, we provide some results which show the behaviour of the value of the numerical index when we apply some Banach space operations, such as constructing diagonal operators between c_0-, ℓ_1-, or ℓ_∞-sums of Banach spaces, composition operators on some vector-valued function spaces, taking the adjoint to an operator, and composition of operators.

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1. Introduction

The study of isometric properties of the space $\mathcal{L}(X,Y)$ of all bounded linear operators between two Banach spaces X and Y and their impact on the domain and range spaces is a traditional subject of Banach space theory, and it remains to be an active area of research. For instance, in the second part of the twentieth century there were a number of results [5, 22, 40, 47, 48, 49, 50] on the structure of extreme points of the unit ball of $\mathcal{L}(X,Y)$ (sometimes known as extreme operators or extreme contractions), but the subject attracts researchers until now, see for instance [10, 11, 32, 41, 46] and references therein. When $X = Y$, the space $\mathcal{L}(X) := \mathcal{L}(X,X)$ is a Banach algebra with unit Id (or Id_X if it is necessary to mention), and there are many deep results in this case (see, for instance, the classical references [42, 45]). The starting point of all these results is a celebrated result of 1955 by Bohnenblust and Karlin [6] which related the geometric and the algebraic properties of the unit. To state their result, they introduce and study a numerical range of elements of a unital algebra which generalized the classical Toeplitz numerical range of operators on Hilbert spaces from 1918. Let us state here an extension of this numerical range, which implicitly appeared in Bohnenblust–Karlin paper, and which was introduced in the 1985 paper [39]. We refer the reader to the classical books [7, 8] by Bonsall and Duncan, and to the recent book [9, Sections 2.1 and 2.9] for more information and background. Given a Banach space Z, we write B_Z and S_Z to denote the closed unit ball and the unit sphere of Z, respectively. If $u \in Z$ is a norm-one element, the (*abstract*) *numerical range* of $z \in Z$ with respect to (Z,u) is given by

$$V(Z,u,z) := \{\phi(z) \colon \phi \in \mathrm{F}(B_{Z^*},u)\},$$

where Z^* denotes the topological dual of Z and

$$\mathrm{F}(B_{Z^*},u) := \{\phi \in S_{Z^*} \colon \phi(u) = 1\}$$

is the *face* of B_{Z^*} generated by $u \in S_Z$ (also known as the set of *states* of Z relative to u). Let us mention that when $Z = A$ is a unital Banach algebra and u is the unit of A, then $V(A,u,a)$ is the *algebra numerical range* of the element $a \in A$. The well-known formula

$$\sup \operatorname{Re} V(Z,u,z) = \lim_{\alpha \to 0^+} \frac{\|u + \alpha z\| - 1}{\alpha}$$

(see Lemma 2.2) connects the geometry of the space Z around u with the numerical range with respect to (Z,u). The *numerical radius* of $z \in Z$ with respect to (Z,u) is

$$v(Z,u,z) := \sup\{|\lambda| \colon \lambda \in V(Z,u,z)\},$$

which is obviously a seminorm on Z satisfying $v(Z,u,z) \leq \|z\|$ for every $z \in Z$. Sometimes the numerical radius is an equivalent norm on Z. The constant

$$n(Z,u) := \inf\{v(Z,u,z) \colon z \in S_Z\} = \max\{k \geq 0 \colon k\|z\| \leq v(Z,u,z)\ \forall z \in Z\}$$

clearly measures this fact quantitatively. This constant is called the (*abstract*) *numerical index* of (Z,u) or the *numerical index of Z with respect to u*. Clearly, $0 \leq n(Z,u) \leq 1$ and $n(Z,u) > 0$ if and only if $v(Z,u,\cdot)$ is an equivalent norm on Z (and this is equivalent to the fact that u is a geometrically unitary element of B_Z, see the beginning of Chapter 2). When $n(Z,u) = 0$, it is possible that $v(Z,u,\cdot)$ is not a norm, or that $v(Z,u,\cdot)$ is a non-equivalent norm on Z (and in this case, u is a vertex of B_Z which is not a geometrically unitary element, see also the beginning of Chapter 2). The value $n(Z,u) = 1$ means that the numerical radius with respect to (Z,u) coincides with the given norm of Z (and in this case, we say that u is a *spear* element of Z, see Proposition 2.5 and the paragraph after it for some equivalent formulations). With this language in mind, the announced result of Bohnenblust and Karlin states that unitary elements of a unital complex algebra A (a purely algebraic concept) are geometrically unitary elements of A (a purely geometric concept), actually $n(A,u) \geq 1/\mathrm{e}$ if u is a unitary element of the complex Banach algebra A, see [9, Corollary 2.1.21]. This is no longer true in the real case as, for instance, the identity is not even a vertex of $\mathcal{L}(H)$ when H is any real Hilbert space of dimension greater than 1. Nevertheless, by numerical range arguments, the unit of a unital real Banach algebra is a strongly extreme point (see [9, Corollary 2.1.42] and [25] for a quantitative version). For (complex) C^*-algebras, the concepts of unitary element and geometrically unitary element coincide (see [9, Theorem 2.1.27] for the details). Let us also comment that the study of the algebra numerical range was crucial to state very important results in the theory of Banach algebras such as Vidav's characterization of C^*-algebras (see [7] or [9]). More recently, geometric characterizations of algebraic properties of elements of C^*-algebras have been given by Akeman and Weaver [2], some of which can be expressed in terms of the numerical ranges (see [43]). Let us observe that geometrically unitary elements (and even vertices) of the unit ball of a Banach space are extreme points of the unit ball (see Lemma 2.3, for instance) so, when non-zero, the abstract numerical index measures "how extreme" a point of the unit ball of a Banach space is. Finally, let us recall that the concept of numerical range (and so the ones of numerical radius and numerical index) depends on the base field, as for a complex Banach space Z and a norm-one element $u \in Z$, $V(Z_{\mathbb{R}},u,z) = \operatorname{Re} V(Z,u,z)$, where $Z_{\mathbb{R}}$ is the real space underlying the space Z and Re represents the real part function.

Let us now return to our aim of studying the geometry of $\mathcal{L}(X,Y)$ around a norm-one operator G. For this to be done, we introduce the numerical range with respect to G. If X and Y are Banach spaces and $G \in \mathcal{L}(X,Y)$ is a norm-one operator, we consider the *numerical range* of $T \in \mathcal{L}(X,Y)$ *with respect to* G, which is the set

$$V(\mathcal{L}(X,Y),G,T) = \{\phi(T) \colon \phi \in \mathcal{L}(X,Y)^*,\ \|\phi\| = \phi(G) = 1\}.$$

Analogously, we may consider the corresponding *numerical radius with respect to* G:

$$v(\mathcal{L}(X,Y),G,T) = \sup\{|\lambda| \colon \lambda \in V(\mathcal{L}(X,Y),G,T)\} \qquad (T \in \mathcal{L}(X,Y)),$$

and the *numerical index* of $(\mathcal{L}(X,Y),G)$ (or the numerical index of $\mathcal{L}(X,Y)$ with respect to G):

$$n_G(X,Y) := n(\mathcal{L}(X,Y),G) = \inf\{v(\mathcal{L}(X,Y),G,T)\colon T\in\mathcal{L}(X,Y),\ \|T\|=1\}.$$

This will be the central concept of study in this paper. Note that $n_G(X,Y)$ is the greatest constant $k\geq 0$ such that

$$\max_{|w|=1}\|G+wT\|\geq 1+k\|T\|$$

for every $T\in\mathcal{L}(X,Y)$ (see Proposition 3.3). The case $k=1$ in the inequality above gives the concept of spear operator, introduced in [3] and deeply studied in [26].

Usually, when one deals with the geometry of spaces of operators, it is convenient to have tools which allow to describe this geometry in terms of the geometry of the domain and range spaces, allowing us to work on these spaces and not on the whole space of operators and, even more, on its wild dual space. In the case of the numerical range of operators on a Banach space (with respect to the identity operator), this tool is the "spatial" version of the numerical range. For a Banach space X and $T\in\mathcal{L}(X)$, the *spatial numerical range* of T was introduced by Bauer (and in a somehow equivalent reformulation by Lumer) in the 1960s (see [7] for instance) as the set

$$W(T) := \{x^*(Tx)\colon x^*\in S_{X^*},\ x\in S_X,\ x^*(x)=1\}, \tag{1.1}$$

which is the direct extension of Toeplitz's numerical range of operators on Hilbert spaces. There is a straightforward inclusion $W(T)\subseteq V(\mathcal{L}(X),\mathrm{Id},T)$ and, actually, one has

$$\overline{\mathrm{conv}}(W(T)) = V(\mathcal{L}(X),\mathrm{Id},T)$$

for every $T\in\mathcal{L}(X)$ (see [9, Proposition 2.1.31], for instance). Hence, the *spatial numerical radius* $v(T)$ of an operator $T\in\mathcal{L}(X)$ coincides with the numerical radius with respect to Id, that is,

$$v(T) := \sup\{|\lambda|\colon \lambda\in W(T)\} = v(\mathcal{L}(X),\mathrm{Id},T).$$

Therefore, the same happens with the corresponding numerical index:

$$n(X) := \inf\{v(T)\colon T\in\mathcal{L}(X),\ \|T\|=1\} = n(\mathcal{L}(X),\mathrm{Id}).$$

With this tool it has been possible to construct an example of a Banach space X such that the identity operator is a vertex but not a geometrically unitary element (see [9, Proposition 2.1.39] for instance). For a detailed study of the Banach space numerical index, we refer the reader to the expository paper [27] and to Subsection 1.1 of the very recent paper [28].

When dealing with a general operator $G\in\mathcal{L}(X,Y)$, it is not possible to get a spatial numerical range with respect to G with a formula analogous to (1.1). Indeed, for the set

$$\{(x,y^*)\colon x\in S_X,\ y^*\in S_{Y^*},\ y^*(Gx)=1\}$$

to be non-empty, we need the operator G to attain its norm; but even in the case of G being an inclusion operator, the above set is not always representative (see [34]). Nevertheless, there is an "approximate spatial" numerical range with respect to an operator recently introduced by Ardalani [3] which does the job. Given two Banach spaces X

and Y and a norm-one operator $G \in \mathcal{L}(X,Y)$, the *approximate spatial numerical range* of $T \in \mathcal{L}(X,Y)$ *with respect to* G is the set

$$V_G(T) := \bigcap_{\delta>0} \overline{\{y^*(Tx) \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(Gx) > 1-\delta\}}.$$

It was shown in [3], using the Bishop–Phelps–Bollobás theorem, that $V_{\mathrm{Id}}(T) = \overline{W(T)}$ for every $T \in \mathcal{L}(X)$ and every Banach space X, so both numerical ranges produce the same associated numerical radii. Moreover, the equality

$$\operatorname{conv}(V_G(T)) = V(\mathcal{L}(X,Y), G, T) \tag{1.2}$$

holds [33, Theorem 2.1] for all Banach spaces X, Y and all operators $G, T \in \mathcal{L}(X,Y)$. Consequently,

$$v_G(T) := \inf_{\delta>0} \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(Gx) > 1-\delta\} = v(\mathcal{L}(X,Y), G, T),$$

and

$$n_G(X,Y) = \inf\{v_G(T) \colon T \in \mathcal{L}(X,Y),\, \|T\| = 1\} = n(\mathcal{L}(X,Y), G).$$

This provides a "spatial" way to deal with the numerical radius and the numerical index with respect to an arbitrary operator, which is especially interesting when we work in concrete Banach spaces and when we study the behaviour of these concepts with respect to Banach space operations on the domain and range spaces.

The aim of this paper is to present a number of results on the numerical indices with respect to operators. Let us detail the content of the paper. First, we finish this introduction with a short section containing the needed terminology and notation. Next, we provide in Chapter 2 some basic results on abstract numerical index. Some of the results were previously known, but some others are new. Among the new ones, we may stress the fact that the set $\{u \in S_Z \colon n(Z,u) > 0\}$ is countable (i.e. finite or infinite and countable) when Z is a finite-dimensional real space, and we provide some estimations on the sum of the values $n(Z,u)$ with varying $u \in S_Z$. On the other hand, for every subset $A \subseteq [0,1]$ containing 0, we show that there is a (real or complex) Banach space Z such that $\{n(Z,u) \colon u \in S_Z\} = A$. Moreover, an extension of the formula (1.2) is given, which provides some useful ways to calculate numerical radii with respect to operators. Next, we particularize these results to numerical indices with respect to operators and also give some more important tools in Chapter 3. Namely, we show that the numerical index with respect to an operator always dominates the numerical index with respect to its adjoint, we calculate the value of the numerical index with respect to a rank-one operator and we show some estimations of the numerical index with respect to an operator in terms of the numerical radii of operators on the domain space or on the range space. In Chapter 4 we provide results on the set of values of the numerical indices with respect to all norm-one operators between two fixed Banach spaces, that is, on the set

$$\mathcal{N}(\mathcal{L}(X,Y)) := \{n_G(X,Y) \colon G \in \mathcal{L}(X,Y),\, \|G\| = 1\}$$

for given Banach spaces X and Y (this notation is coherent with the one that we will introduce at the beginning of Section 2.2 for the abstract numerical index). For example, $0 \in \mathcal{N}(\mathcal{L}(X,Y))$ unless both X and Y are one-dimensional, and the set $\mathcal{N}(\mathcal{L}(X,Y))$ is countable if X and Y are finite-dimensional real spaces. In addition, for a real Hilbert space H with $\dim(H) \geq 2$ one has

$$\mathcal{N}(\mathcal{L}(X,H)) = \mathcal{N}(\mathcal{L}(H,Y)) = \{0\}$$

for all Banach spaces X and Y. The role of the space H can also be played by some non-Hilbertian real Banach spaces like $\mathcal{L}(H)$ where H is an infinite-dimensional real Hilbert space. Estimations of the numerical indices with respect to operators whose domain or range is a real space ℓ_p are also given: for $1 < p < \infty$,

$$\mathcal{N}(\mathcal{L}(X,\ell_p)) \subseteq [0, M_p] \quad \text{and} \quad \mathcal{N}(\mathcal{L}(\ell_p, Y)) \subseteq [0, M_p]$$

for all real Banach spaces X and Y, where $M_p = \sup_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$. For complex Hilbert spaces H_1, H_2 of dimension greater than 1, $\mathcal{N}(\mathcal{L}(H_1,H_2)) \subseteq \{0, 1/2\}$ and the value $1/2$ is taken if and only if H_1 and H_2 are isometrically isomorphic. Moreover, $\mathcal{N}(\mathcal{L}(X,H)) \subseteq [0,1/2]$ and $\mathcal{N}(\mathcal{L}(H,Y)) \subseteq [0,1/2]$ when H is a complex infinite-dimensional Hilbert space and X and Y are arbitrary complex Banach spaces. Also

$$\mathcal{N}(\mathcal{L}(C(K_1), C(K_2))) = \{0, 1\}$$

for many families of Hausdorff topological compact spaces K_1 and K_2, both in the real and the complex cases. As a consequence, we demonstrate the inclusions

$$\mathcal{N}(\mathcal{L}(L_\infty(\mu_1), L_\infty(\mu_2))) \subseteq \{0,1\} \quad \text{and} \quad \mathcal{N}(\mathcal{L}(L_1(\mu_1), L_1(\mu_2))) \subseteq \{0,1\}$$

for all σ-finite positive measures μ_1 and μ_2.

In Chapter 5 we use the tools presented in Chapter 3 to prove that the concept of Lipschitz numerical range introduced in [51, 52] for Lipschitz self-maps of a Banach space can be viewed as a particular case of numerical range with respect to a linear operator between two different Banach spaces.

Finally, we collect in Chapter 6 some results which show the behaviour of the value of the numerical index when we apply some Banach space operations. For instance, the numerical index of a c_0-, ℓ_1- or ℓ_∞-sum of Banach spaces with respect to a direct sum of norm-one operators in the corresponding spaces coincides with the infimum of the numerical indices of corresponding summands. As a consequence, we show the existence of real and complex Banach spaces X for which $\mathcal{N}(\mathcal{L}(X)) = [0,1]$. We also show that a composition operator between spaces of vector-valued continuous, integrable, or essentially bounded functions produces the same numerical index as the original operator. Next, we provide two independent conditions ensuring that the numerical index with respect to an operator and the numerical index with respect to its adjoint coincide: that the range space is L-embedded or that the operator is rank-one. Finally, we discuss some results on the value of the numerical index with respect to a composition of two operators, and then we show how to extend the domain of an operator retaining the value of the numerical index, and an analogous result for the codomain. In particular, the results of the chapter allow us to solve Problem 9.14 of [26].

1.1. Notation and terminology. By $\mathbb{K}$ we denote the scalar field ($\mathbb{R}$ or $\mathbb{C}$), and we use the standard notation $\mathbb{T} := \{\lambda \in \mathbb{K} \colon |\lambda| = 1\}$ for its unit sphere. We use the letters X, Y, Z for Banach spaces over $\mathbb{K}$ and by a subspace we always mean a closed subspace.

In some cases, we have to distinguish between the real and the complex case, but for most results this difference is insignificant. We write $J_X\colon X \to X^{**}$ to denote the natural isometric inclusion of X into its bidual X^{**}. Given a subset C of X we denote by $\operatorname{conv}(C)$ and $\operatorname{aconv}(C)$ the convex hull and the absolutely convex hull of C, respectively.

Let Γ be a non-empty index set, and $\{X_\gamma\colon \gamma\in\Gamma\}$ be a collection of Banach spaces. We write

$$\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{c_0}, \qquad \Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_1}, \qquad \Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_\infty},$$

to denote, respectively, the c_0-, ℓ_1-, and ℓ_∞-sum of the family. If E is $\mathbb{R}^n$ endowed with an absolute norm $|\cdot|_E$ and $X_1,\ldots,X_n$ are Banach spaces, we write $X=[X_1\oplus\cdots\oplus X_n]_E$ to denote the product space $X_1\times\cdots\times X_n$ endowed with the norm

$$\|(x_1,\ldots,x_n)\| = \big|(\|x_1\|,\ldots,\|x_n\|)\big|_E$$

for all $x_i\in X_i$, $i=1,\ldots,n$.

Given $1\leq p\leq\infty$ and a non-empty set Γ, we write $\ell_p(\Gamma)$ to denote the L_p-space associated to the counting measure on Γ. For $n\in\mathbb{N}$, we just write ℓ_p^n to denote $\ell_p(\{1,\ldots,n\})$. Given a Banach space X, a compact Hausdorff topological space K, and a σ-finite measure space (Ω,Σ,μ), we write $C(K,X)$, $L_1(\mu,X)$, and $L_\infty(\mu,X)$ to denote, respectively, the spaces of continuous functions from K to X, (classes of) Bochner-integrable functions from Ω to X, and (classes of) strongly measurable and essentially bounded functions from Ω to X.

2. Some old and new results on abstract numerical index

Our aim here is to collect a few basic facts about the abstract numerical range, some of which seem to be new. We start by recalling some related definitions which were already mentioned in the introduction.

DEFINITION 2.1. Let Z be a Banach space and let $u \in S_Z$.

(a) We say that u is a *vertex* of B_Z if $\mathrm{F}(B_{Z^*}, u)$ separates the points of Z (i.e. for every $z \in Z \setminus \{0\}$, there is $\phi \in \mathrm{F}(B_{Z^*}, u)$ such that $\phi(z) \neq 0$). This is clearly equivalent to the fact that $v(Z, u, z) = 0$ for $z \in Z$ implies $z = 0$.

(b) We say that u is a *geometrically unitary element* of B_Z if the linear span of $\mathrm{F}(B_{Z^*}, u)$ is equal to the whole Z^*. It is known (see [9, Theorem 2.1.17]) that u is a geometrically unitary element if and only if $n(Z, u) > 0$.

We refer the reader to the already cited book [9], and to the papers [4, 19, 21, 43] for more information and background on these concepts.

2.1. A few known elementary results. First, we present some known results on abstract numerical index which we will use throughout the paper. They are elementary and come from many sources, but we use the recent monograph [9] as reference for them for the convenience of the reader.

The first result allows us to relate the numerical range to a directional derivative.

LEMMA 2.2. *Let Z be a Banach space and let $u \in S_Z$. Then*

$$\max \operatorname{Re} V(Z, u, z) = \lim_{t \to 0^+} \frac{\|u + tz\| - 1}{t}$$

for every $z \in Z$. Therefore,

$$v(Z, u, z) = \max_{\theta \in \mathbb{T}} \lim_{t \to 0^+} \frac{\|u + t\theta z\| - 1}{t} = \lim_{t \to 0^+} \max_{\theta \in \mathbb{T}} \frac{\|u + t\,\theta z\| - 1}{t}.$$

The first part of the above lemma is folklore and can be found in [9, Proposition 2.1.5]. The first equality for the numerical radius is an immediate consequence, and the second equality follows routinely from the compactness of $\mathbb{T}$. Indeed, let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of positive scalars converging to 0 and for each $n \in \mathbb{N}$, take $\theta_n \in \mathbb{T}$ such that

$$\max_{\theta \in \mathbb{T}} \frac{\|u + t_n\,\theta z\| - 1}{t_n} = \frac{\|u + t_n\,\theta_n\,z\| - 1}{t_n}.$$

Extract a subsequence $\{\theta_{\sigma(n)}\}_{n \in \mathbb{N}}$ which is convergent to, say, $\theta_0 \in \mathbb{T}$. Then

$$\frac{\|u + t_{\sigma(n)}\theta_0 z\| - 1}{t_{\sigma(n)}} \geq \frac{\|u + t_{\sigma(n)}\theta_{\sigma(n)} z\| - 1}{t_{\sigma(n)}} - |\theta_{\sigma(n)} - \theta_0|\|z\|.$$

Finally,

$$v(Z,u,z) \geq \lim_{n\to\infty} \frac{\|u + t_{\sigma(n)}\theta_0 z\| - 1}{t_{\sigma(n)}} \geq \lim_{n\to\infty} \max_{\theta\in\mathbb{T}} \frac{\|u + t_{\sigma(n)}\,\theta z\| - 1}{t_{\sigma(n)}}.$$

The next result relates the numerical index with respect to a point to the geometry at the point. Recall that a norm-one element u of a Banach space Z is said to be a *strongly extreme point* of B_Z if whenever $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences in B_Z such that $\lim(x_n + y_n) = 2u$, then $\lim(x_n - y_n) = 0$. Strongly extreme points are extreme points, but the converse result is not true (see [29] for instance).

LEMMA 2.3. *Let Z be a Banach space and $u \in S_Z$.*

(a) *If u is a vertex of B_Z, then u is an extreme point, and if moreover* $\dim(Z) \geq 2$, *then the norm of Z is not smooth at u.*
(b) *If u is a geometrically unitary element of B_Z (i.e. $n(Z,u) > 0$), then u is a strongly extreme point of B_Z.*

The extreme point condition appears in [9, Lemma 2.1.25]; if the norm of Z is smooth at u, then $\mathrm{F}(B_{Z^*}, u)$ is a singleton, so either $\dim(Z) = 1$ or u cannot be a vertex. The result in (b) appears in [9, Proposition 2.1.41]. There are vertices which are not strongly extreme points [9, Example 2.1.43].

The next result, which can be found in [9, Corollary 2.1.2], is elementary and very useful.

LEMMA 2.4. *Let $\psi\colon Z_1 \to Z_2$ be a linear operator between Banach spaces Z_1 and Z_2, let $u \in S_{Z_1}$ be such that $\|\psi(u)\| = 1$.*

(a) *If $\|\psi\| = 1$, then $v(Z_2, \psi(u), \psi(z)) \leq v(Z_1, u, z)$ for every $z \in Z_1$.*
(b) *If ψ is an isometric embedding, then $v(Z_2, \psi(u), \psi(z)) = v(Z_1, u, z)$ for every $z \in Z_1$; therefore, $n(Z_2, \psi(u)) \leq n(Z_1, u)$ in this case.*

We next would like to present a pair of characterizations of the abstract numerical index.

PROPOSITION 2.5. *Let Z be a Banach space, $u \in S_Z$, and $0 < \lambda \leq 1$. Then the following statements are equivalent:*

(i) $n(Z,u) \geq \lambda$.
($\mathrm{ii}_{\mathbb{R}}$) *In the real case,* $\lambda B_{Z^*} \subseteq \operatorname{conv}(\mathrm{F}(B_{Z^*}, u) \cup -\mathrm{F}(B_{Z^*}, u))$.
($\mathrm{ii}_{\mathbb{C}}$) *In the complex case, given $\varepsilon > 0$, $\theta_1, \ldots, \theta_k \in B_{\mathbb{C}}$ satisfying*

$$B_{\mathbb{C}} \subseteq (1+\varepsilon)\operatorname{conv}\{\theta_1, \ldots, \theta_k\},$$

we have

$$\lambda B_{Z^*} \subseteq (1+\varepsilon)\operatorname{conv}\Big(\bigcup_{j=1}^{k} \theta_j\,\mathrm{F}(B_{Z^*}, u)\Big).$$

(iii) $\max_{\theta\in\mathbb{T}} \|u + \theta z\| \geq 1 + \lambda\|z\|$ *for every $z \in Z$.*

The equivalence between (i) and (ii) is well known and can be found, for instance, in [9, Theorem 2.1.17]. The implication (i)$\Rightarrow$(iii) is immediate from the Hahn–Banach theorem. The converse result follows straightforwardly from the last equality in Lemma 2.2.

The strongest possibility in Proposition 2.5, that is, $\lambda = 1$, gives rise to the concept of spear vector introduced in [26]. A norm-one element u of a Banach space Z is a *spear vector* if

$$\max_{\theta\in\mathbb{T}} \|u+\theta z\| = 1+\|z\| \quad \text{for every } z\in Z.$$

The previous proposition shows that this is equivalent to $n(Z,u)=1$. We refer the reader to [26, Chapter 2] for more information and background.

Finally, we present a result relating the numerical index of a Banach space with respect to a point to the numerical index of its bidual with respect to the same point which can be found in [9, Theorem 2.1.17.v].

LEMMA 2.6. *Let Z be a Banach space and let $u\in S_Z$. Then $n(Z^{**},J_Z(u))=n(Z,u)$.*

2.2. On the set of values of the abstract numerical indices with respect to all unit vectors of a given space. For a given Banach space Z, denote

$$\mathcal{N}(Z) := \{n(Z,u)\colon u\in S_Z\}.$$

In this section we concentrate on the properties of $\mathcal{N}(Z)$ for various classes of Banach spaces Z.

Let us start with a general important observation.

PROPOSITION 2.7. *Let Z be a Banach space with $\dim(Z)\geq 2$. Then $0\in\mathcal{N}(Z)$.*

Proof. Let Y be a two-dimensional subspace of Z. Then there is a smooth point $u\in S_Y$ and we have $n(Y,u)=0$ by Lemma 2.3(a). Now, Lemma 2.4(b) gives $n(Z,u)=0$. ∎

For many Banach spaces Z, zero is the only element of $\mathcal{N}(Z)$. Say, this happens for smooth spaces of dimension greater than 1, a fact which follows immediately from the above proof. In Chapter 4 the reader will find many examples of operator spaces $Z=\mathcal{L}(X_1,X_2)$ with the property that $\mathcal{N}(Z)=\{0\}$. On the other hand, for "big bad" spaces Z, the corresponding set $\mathcal{N}(Z)$ can be big. Moreover, it is possible to show that this set can be any subset of $[0,1]$ containing 0.

PROPOSITION 2.8. *For every subset A of $[0,1]$ with $0\in A$, one can find a (real or complex) Banach space Z with $\mathcal{N}(Z)=A$.*

In order to demonstrate this result, we need some preparatory work.

EXAMPLE 2.9. *For every $a\in[0,1]$ there is a two-dimensional (real or complex) space Z_a with $\mathcal{N}(Z_a)=\{0,a\}$.*

Indeed, for $r\in[0,1]$ denote by Z_r^* the two-dimensional space $\mathbb{K}^2$ equipped with the norm

$$\|(x_1,x_2)\| = \max\{|x_1|, \sqrt{r|x_1|^2+|x_2|^2}\}.$$

Then the intersections of $B_{Z_r^*}$ with the lines $\{x_1=\theta\}$ for $\theta\in\mathbb{T}$ are the only non-trivial faces of $B_{Z_r^*}$ (see Figure 1). Therefore, in the predual space Z_r the only elements u of S_{Z_r} with $n(Z_r,u)\neq 0$ are $u=(\theta,0)$ with $\theta\in\mathbb{T}$. As Z_r has the same abstract numerical index with respect to all these elements, $\mathcal{N}(Z_r)$ consists of two points: 0 and some $h(r)\geq 0$. The value $h(r)$ varies continuously from 1 to 0 as r varies from 0 to 1 (because $Z_0=\ell_\infty^2$ and $Z_1=\ell_2^2$).

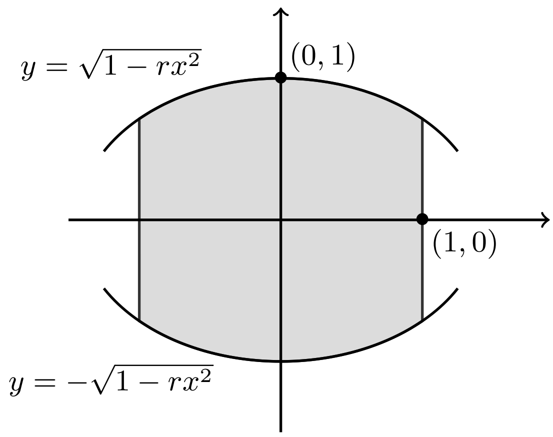

Fig. 1. The unit ball of Z_r^*

The next result may be known, but we include the easy proof as we have not found it in the literature.

LEMMA 2.10. *Let $\{Z_\gamma\colon \gamma\in\Gamma\}$ be a family of Banach spaces. Then*

$$\mathcal{N}\Big(\Big[\bigoplus_{\gamma\in\Gamma} Z_\gamma\Big]_{\ell_1}\Big)=\bigcup_{\gamma\in\Gamma}\mathcal{N}(Z_\gamma).$$

Proof. If a norm-one element $u=(u_\gamma)_{\gamma\in\Gamma}\in[\bigoplus_{\gamma\in\Gamma} Z_\gamma]_{\ell_1}$ has more than one non-zero coordinate, then $n([\bigoplus_{\gamma\in\Gamma} Z_\gamma]_{\ell_1},u)=0$ as u is then not an extreme point. In the case of u having just one non-zero coordinate u_τ, one has $n([\bigoplus_{\gamma\in\Gamma} Z_\gamma]_{\ell_1},u)=n(Z_\tau,u_\tau)$ routinely. ∎

We are now ready to provide the pending proof.

Proof of Proposition 2.8. For every $a\in A$, select a two-dimensional Z_a such that $\mathcal{N}(Z_a)=\{0,a\}$ provided by Example 2.9 and then the desired example is $Z=[\bigoplus_{a\in A} Z_a]_{\ell_1}$ by Lemma 2.10. ∎

Our next goal is to find the restrictions on $\mathcal{N}(Z)$ which appear in the finite-dimensional case. We start by showing that, in this case, the corresponding $\mathcal{N}(Z)$ is at most countable.

PROPOSITION 2.11. *Let Z be a finite-dimensional real Banach space. Then the set of points $u\in S_Z$ satisfying $n(Z,u)>0$ is countable. As a consequence, $\mathcal{N}(Z)$ is countable.*

Proof. Let $u\in S_Z$ be such that $n(Z,u)>0$. By Proposition 2.5, the set

$$\operatorname{conv}(\mathrm{F}(B_{Z^*},u)\cup-\mathrm{F}(B_{Z^*},u))$$

has non-empty interior so, being Z^* finite-dimensional, $\mathrm{F}(B_{Z^*},u)$ has non-empty interior relative to S_{Z^*}. Indeed, otherwise $\mathrm{F}(B_{Z^*},u)$ has affine dimension at most $\dim(Z^*)-2$, so its linear span has dimension at most $\dim(Z^*)-1$, and so $\operatorname{conv}(\mathrm{F}(B_{Z^*},u)\cup-\mathrm{F}(B_{Z^*},u))$ has empty interior, a contradiction. Furthermore, for $u_1,u_2\in S_Z$, as

$$\mathrm{F}(B_{Z^*},u_1)\cap\mathrm{F}(B_{Z^*},u_2)\subseteq\ker(u_1-u_2),\tag{2.1}$$

the relative interiors of $\mathrm{F}(B_{Z^*},u_1)$ and $\mathrm{F}(B_{Z^*},u_2)$ are disjoint if $u_1\neq u_2$. Hence, by separability, the set of those $u\in S_Z$ satisfying $n(Z,u)>0$ has to be countable and, a fortiori, so is $\mathcal{N}(Z)$. ∎

We do not know if the above corollary remains valid for "small" infinite-dimensional spaces, such as Banach spaces with separable dual. We also do not know whether $\mathcal{N}(Z)$ is countable for every finite-dimensional complex Banach space Z.

Our next aim is to give a strengthening of Proposition 2.11 for real finite-dimensional spaces, where some techniques from combinatorial geometry are applicable. Note that neither Theorem 2.12 nor Proposition 2.13 below are needed in the rest of the paper. We introduce some notation. For a convex body $K \subseteq \mathbb{R}^n$ let us denote its *inradius* by

$$r(K) := \sup\{r > 0\colon \exists x \in K \text{ such that } x + rB_{\ell_2^n} \subseteq K\}.$$

Note that in the case of $K = -K$, the above formula simplifies to

$$r(K) = \sup\{r > 0\colon rB_{\ell_2^n} \subseteq K\}.$$

We denote by $\mathrm{vol}_n[K]$ and $S(K)$ the volume and the surface area of K, respectively.

THEOREM 2.12. *Let Z be a real space with* $\dim(Z) = m \geq 2$. *Then*

$$\sum_{u \in S_Z} n(Z, u)^{m-1} < \infty.$$

Proof. Let us identify, as usual, Z with $(\mathbb{R}^m, \|\cdot\|)$, Z^* with $(\mathbb{R}^m, \|\cdot\|^*)$ and B_{Z^*} with the polar body of B_Z. Given a finite set F of points in S_Z, we evidently have

$$\sum_{u \in F} \mathrm{vol}_{m-1}[\mathrm{F}(B_{Z^*}, u)] \leq S(B_{Z^*}) \tag{2.2}$$

by (2.1). Using Proposition 2.5, for every $u \in F$, we have

$$n(Z, u) r(B_{Z^*}) B_{\ell_2^m} \subseteq n(Z, u) B_{Z^*} \subseteq \mathrm{conv}(\mathrm{F}(B_{Z^*}, u) \cup -\mathrm{F}(B_{Z^*}, u))$$

and so,

$$n(Z, u) r(B_{Z^*}) B_{\ell_2^m} \cap \ker(u) \subseteq [\mathrm{conv}(\mathrm{F}(B_{Z^*}, u) \cup -\mathrm{F}(B_{Z^*}, u))] \cap \ker(u).$$

For an arbitrary $z^* \in \mathrm{F}(B_{Z^*}, u)$, the latter set can be rewritten as

$$\tfrac{1}{2}[\mathrm{F}(B_{Z^*}, u) - \mathrm{F}(B_{Z^*}, u)] = \tfrac{1}{2}[(\mathrm{F}(B_{Z^*}, u) - z^*) - (\mathrm{F}(B_{Z^*}, u) - z^*)].$$

According to the Rogers–Shephard theorem [44, Theorem 1],

$$\mathrm{vol}_n[K - K] \leq \binom{2n}{n} \mathrm{vol}_n[K]$$

for every convex body K in an n-dimensional space. Applying this to the convex body $(\mathrm{F}(B_{Z^*}, u) - z^*)$ of the $(m-1)$-dimensional space $\ker(u)$, we obtain the inequality

$$\mathrm{vol}_{m-1}[n(Z, u) r(B_{Z^*}) B_{\ell_2^m} \cap \ker(u)] \leq \frac{1}{2^{m-1}} \binom{2(m-1)}{m-1} \mathrm{vol}_{m-1}[\mathrm{F}(B_{Z^*}, u)].$$

Therefore, we can write

$$\begin{aligned} n(Z, u)^{m-1} r(B_{Z^*})^{m-1} \mathrm{vol}_{m-1}[B_{\ell_2^{m-1}}] &= \mathrm{vol}_{m-1}[n(Z, u) r(B_{Z^*}) B_{\ell_2^m} \cap \ker(u)] \\ &\leq \frac{1}{2^{m-1}} \binom{2(m-1)}{m-1} \mathrm{vol}_{m-1}[\mathrm{F}(B_{Z^*}, u)] \end{aligned}$$

which, combined with (2.2), gives

$$\sum_{u\in F} n(Z,u)^{m-1} \leq \frac{1}{2^{m-1}}\binom{2(m-1)}{m-1}\frac{S(B_{Z^*})}{\mathrm{vol}_{m-1}[B_{\ell_2^{m-1}}]\cdot r(B_{Z^*})^{m-1}}. \tag{2.3}$$

As F was arbitrary, we get the desired result. ∎

For a finite-dimensional polyhedral space (i.e. finite-dimensional real space whose unit ball has finitely many faces), we can give a lower bound for the sum of numerical indices of the elements of the unit sphere.

Proposition 2.13. *Let Z be $\mathbb{R}^m$ endowed with a polyhedral norm such that $B_{Z^*}\subseteq B_{\ell_2^m}$. Then*

$$\sum_{u\in S_Z} n(Z,u) \geq r(B_{Z^*}). \tag{2.4}$$

Proof. Since Z^* is also polyhedral, S_{Z^*} is the union of finitely many sets of the form $\mathrm{F}(B_{Z^*},u)\cup -\mathrm{F}(B_{Z^*},u)$ for some $u\in S_Z$. Let us denote by F the set of corresponding $u\in S_Z$. Then obviously

$$B_{Z^*}\subseteq \bigcup_{u\in F}\mathrm{conv}(\mathrm{F}(B_{Z^*},u)\cup -\mathrm{F}(B_{Z^*},u)).$$

Since

$$\begin{aligned}\mathrm{conv}(\mathrm{F}(B_{Z^*},u)\cup -\mathrm{F}(B_{Z^*},u)) &\supset r(\mathrm{conv}(\mathrm{F}(B_{Z^*},u)\cup -\mathrm{F}(B_{Z^*},u)))B_{\ell_2^m}\\ &\supset r(\mathrm{conv}(\mathrm{F}(B_{Z^*},u)\cup -\mathrm{F}(B_{Z^*},u)))B_{Z^*},\end{aligned}$$

Proposition 2.5 implies $n(Z,u)\geq r(\mathrm{conv}\,(\mathrm{F}(B_{Z^*},u)\cup -\mathrm{F}(B_{Z^*},u)))$. As the convex body B_{Z^*} is covered by a finite number of convex bodies, we can use [24, Theorem 2.1] to get

$$\sum_{u\in F} n(Z,u)\geq \sum_{u\in F} r(\mathrm{conv}(\mathrm{F}(B_{Z^*},u)\cup -\mathrm{F}(B_{Z^*},u)))\geq r(B_{Z^*}). \ ∎$$

Let us remark that the estimates in (2.3) and (2.4) depend on the particular chosen representation of Z as $\mathbb{R}^m$, and they do not pretend to be optimal. It would be interesting to find the sharp estimates in both inequalities.

2.3. A new result on abstract numerical ranges. Our goal here is to present a very general result about numerical range spaces which extends and generalizes the results of [33]. It will be useful to study the behaviour of the numerical ranges with respect to operators when dealing with some Banach space operations on the domain and range spaces (see Chapter 6) and also to study Lipschitz numerical ranges (see Chapter 5).

Proposition 2.14. *Let Z be a Banach space, let $u\in S_Z$, and let $C\subseteq B_{Z^*}$ be such that $B_{Z^*}=\overline{\mathrm{conv}}^{w^*}(C)$. Then*

$$V(Z,u,z)=\mathrm{conv}\bigcap_{\delta>0}\overline{\{z^*(z)\colon z^*\in C,\ \mathrm{Re}\,z^*(u)>1-\delta\}}$$

for every $z\in Z$. Consequently,

$$v(Z,u,z)=\inf_{\delta>0}\sup\{|z^*(z)|\colon z^*\in C,\ \mathrm{Re}\,z^*(u)>1-\delta\}$$

for every $z\in Z$.

Let us first observe that the inclusion "$\supseteq$" is a straightforward application of the Banach–Alaoglu theorem. Indeed, given $\lambda_0 \in \bigcap_{\delta>0} \overline{\{z^*(z)\colon z^* \in C,\ \operatorname{Re} z^*(u) > 1-\delta\}}$, for every $n \in \mathbb{N}$ there is $z_n^* \in C$ such that

$$\operatorname{Re} z_n^*(u) > 1 - 1/n \quad \text{and} \quad |\lambda_0 - z_n^*(z)| < 1/n.$$

If $z_0^* \in B_{Z^*}$ is a limiting point of the sequence $\{z_n^*\}_{n\in\mathbb{N}}$, we have $z_0^*(u) = 1$ and $z_0^*(z) = \lambda_0$, so $\lambda_0 \in V(Z,u,z)$. As the latter set is convex, the inclusion follows.

To prove the more intriguing reverse inequality, we need a couple of preliminary results. The first one is a general version of [33, Lemma 2.5].

Lemma 2.15. *Let Z be a Banach space, let $C \subseteq B_{Z^*}$ be such that $B_{Z^*} = \overline{\operatorname{conv}}^{w^*}(C)$, and let $u \in S_Z$ and $z \in Z$. Then for every $z_0^* \in S_{Z^*}$ with $z_0^*(u) = 1$ and every $\delta > 0$, there is $z^* \in C$ such that*

$$\operatorname{Re} z^*(u) > 1 - \delta \quad \textit{and} \quad \operatorname{Re} z^*(z) > \operatorname{Re} z_0^*(z) - \delta.$$

Proof. As $B_{Z^*} = \overline{\operatorname{conv}}^{w^*}(C)$, for $\delta' > 0$ satisfying $2\|z\|\delta' < \delta$, we may find $n \in \mathbb{N}$, $z_1^*, \ldots, z_n^* \in C$, $\alpha_1, \ldots, \alpha_n \in [0,1]$ with $\sum_{k=1}^n \alpha_k = 1$ such that

$$\sum_{k=1}^n \alpha_k \operatorname{Re} z_k^*(u) > 1 - (\delta')^2 \quad \text{and} \quad \sum_{k=1}^n \alpha_k \operatorname{Re} z_k^*(z) > \operatorname{Re} z_0^*(z) - \delta/2.$$

Now, consider

$$J = \{k \in \{1,\ldots,n\}\colon \operatorname{Re} z_k^*(u) > 1 - \delta'\}$$

and let $L = \{1,\ldots,n\} \setminus J$. We have

$$1 - (\delta')^2 < \sum_{k=1}^n \alpha_k \operatorname{Re} z_k^*(u) \le \sum_{k\in J} \alpha_k + \sum_{k\in L} \alpha_k (1-\delta') = 1 - \delta' \sum_{k\in L} \alpha_k,$$

from which we deduce

$$\sum_{k\in L} \alpha_k < \delta'.$$

Now, we have

$$\begin{aligned} \operatorname{Re} z_0^*(z) - \delta/2 &< \sum_{k=1}^n \alpha_k \operatorname{Re} z_k^*(z) \\ &\le \sum_{k\in J} \alpha_k \operatorname{Re} z_k^*(z) + \|z\| \sum_{k\in L} \alpha_k < \sum_{k\in J} \alpha_k \operatorname{Re} z_k^*(z) + \delta/2. \end{aligned}$$

Therefore,

$$\sum_{k\in J} \alpha_k \operatorname{Re} z_k^*(z) > \operatorname{Re} z_0^*(z) - \delta,$$

and an obvious convexity argument provides the existence of $k \in J$ such that

$$\operatorname{Re} z_k^*(z) > \operatorname{Re} z_0^*(z) - \delta.$$

On the other hand, $\operatorname{Re} z_k^*(u) > 1 - \delta$ as $k \in J$, so the proof is finished. ∎

The next preliminary result follows straightforwardly from [33, Lemma 2.4].

Lemma 2.16. *Let* $\{W_\delta\}_{\delta>0}$ *be a monotone family of compact subsets of* $\mathbb{K}$ *(i.e.* $W_{\delta_1} \subseteq W_{\delta_2}$ *when* $\delta_1 < \delta_2$*). Then*

$$\sup \operatorname{Re} \bigcap_{\delta>0} W_\delta \;=\; \inf_{\delta>0} \sup \operatorname{Re} W_\delta .$$

Proof of the main part of Proposition 2.14. For $z \in Z$, write

$$W_\delta(z) := \overline{\{z^*(z)\colon z^* \in C,\ \operatorname{Re} z^*(u) > 1-\delta\}} \quad \text{and} \quad W(z) := \bigcap_{\delta>0} W_\delta(z).$$

To get the desired inclusion $V(Z,u,z) \subseteq \operatorname{conv} W(z)$ for every $z \in Z$, it is enough to prove that for every $\delta > 0$ and every $z \in Z$,

$$\sup \operatorname{Re} V(Z,u,z) \leq \sup \operatorname{Re} W_\delta(z) + \delta. \tag{2.5}$$

Indeed, it then follows from Lemma 2.16 that $\sup \operatorname{Re} V(Z,u,z) \leq \sup \operatorname{Re} W(z)$ for every $z \in Z$. Now, as for every $\theta \in \mathbb{T}$, we have

$$V(Z,u,\theta z) = \theta V(Z,u,z) \quad \text{and} \quad W(\theta z) = \theta W(z),$$

the desired inclusion follows easily.

So let us prove that inequality (2.5) holds. Fix $z \in Z$ and $\delta > 0$. Given $z_0^* \in \mathrm{F}(B_{Z^*},u)$, we may use Lemma 2.15 to get $z^* \in C$ such that

$$\operatorname{Re} z^*(u) > 1-\delta \quad \text{and} \quad \operatorname{Re} z_0^*(z) < \operatorname{Re} z^*(z) + \delta.$$

So, $\operatorname{Re} z_0^*(z) \leq \sup \operatorname{Re} W_\delta(z) + \delta$. Varying z_0^* in $\mathrm{F}(B_{Z^*},u)$, we get

$$\sup \operatorname{Re} V(Z,u,z) \leq \sup \operatorname{Re} W_\delta(z) + \delta,$$

as desired. ∎

3. Tools to study the numerical index with respect to an operator

Our aim in this chapter is to provide some tools to calculate, or at least estimate, the numerical indices with respect to operators. Some of the results are just direct translation to the operator spaces setting of the abstract results contained in the previous chapter, but other ones rely on specifics of the operator case.

We need some notation. Let X and Y be Banach spaces. For a norm-one operator $G \in \mathcal{L}(X,Y)$ and $\delta > 0$, we write

$$v_{G,\delta}(T) := \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(Gx) > 1-\delta\}$$

for every $T \in \mathcal{L}(X,Y)$. It then follows from [33] (or from Proposition 2.14) that

$$v(\mathcal{L}(X,Y), G, T) = v_G(T) = \inf_{\delta>0} v_{G,\delta}(T)$$

for every $T \in \mathcal{L}(X,Y)$, a result which we will use without any further mention (see Lemma 3.4 for details).

We include first some results which directly follow from those of Chapter 2. The first one is the translation of Lemma 2.3 to the setting of the spaces of operators. For a simpler notation, let us say that a norm-one operator $G \in \mathcal{L}(X,Y)$ is an *extreme operator* (or *extreme contraction*) if G is an extreme point of the unit ball of $\mathcal{L}(X,Y)$.

LEMMA 3.1. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator with $n_G(X,Y) > 0$. Then G is a strongly extreme point of $B_{\mathcal{L}(X,Y)}$; in particular, G is an extreme operator. Moreover, if* $\dim(X) \geq 2$ *or* $\dim(Y) \geq 2$, *then the norm of* $\mathcal{L}(X,Y)$ *is not smooth at G.*

Next, we particularize Lemma 2.2 to our setting.

LEMMA 3.2. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then*

$$v_G(T) = \max_{\theta\in\mathbb{T}} \lim_{\alpha\to 0^+} \frac{\|G+\alpha\theta T\|-1}{\alpha} = \lim_{\alpha\to 0^+} \max_{\theta\in\mathbb{T}} \frac{\|G+\alpha\theta T\|-1}{\alpha}$$

for every $T \in \mathcal{L}(X,Y)$.

We now include a part of Proposition 2.5, particularized to spaces of operators, which allows us to characterize the numerical index in terms of the norm of the space of operators.

PROPOSITION 3.3. *Let X, Y be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator, and $0 < \lambda \leq 1$. Then the following statements are equivalent:*

(i) $n_G(X,Y) \geq \lambda$.
(ii) $\max_{\theta\in\mathbb{T}} \|G+\theta T\| \geq 1+\lambda\|T\|$ *for every* $T \in \mathcal{L}(X,Y)$.

The case $\lambda = 1$ in the previous result gives us the concept of spear operator. A norm-one operator $G \in \mathcal{L}(X,Y)$ is said to be a *spear operator* if

$$\max_{\theta\in\mathbb{T}} \|G + \theta T\| = 1 + \|T\|$$

for every $T \in \mathcal{L}(X,Y)$. This concept was introduced in [3] and deeply studied in [26], where we refer for more information and background. Observe that Proposition 3.3 says, in particular, that G is a spear operator if and only if $n_G(X,Y) = 1$.

The next result is a direct consequence of Proposition 2.14 and will be very useful later on.

LEMMA 3.4. *Let X, Y be Banach spaces. Suppose that $A \subseteq B_X$ and $B \subseteq B_{Y^*}$ satisfy $\overline{\mathrm{conv}}(A) = B_X$ and $\overline{\mathrm{conv}}^{w^*}(B) = B_{Y^*}$. Then given $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, we have*

$$V(\mathcal{L}(X,Y),G,T) = \mathrm{conv} \bigcap_{\delta>0} \overline{\{y^*(Tx) \colon y^* \in B,\, x \in A,\, \mathrm{Re}\, y^*(Gx) > 1-\delta\}}$$

for every $T \in \mathcal{L}(X,Y)$. Accordingly,

$$v_G(T) = \inf_{\delta>0} \sup\{|y^*(Tx)| \colon y^* \in B,\, x \in A,\, \mathrm{Re}\, y^*(Gx) > 1-\delta\}.$$

Proof. The result follows from Proposition 2.14 as the hypotheses on A and B give $B_{\mathcal{L}(X,Y)^*} = \overline{\mathrm{conv}}^{w^*}(A \otimes B)$. Indeed, for every $G \in \mathcal{L}(X,Y)$, we have

$$\begin{aligned}\sup_{x\in A,\, y^*\in B} \mathrm{Re}\, y^*(Gx) &= \sup_{y^*\in B} \sup_{x\in A} \mathrm{Re}\, y^*(Gx) = \sup_{y^*\in B} \sup_{x\in B_X} \mathrm{Re}\, y^*(Gx)\\ &= \sup_{x\in B_X} \sup_{y^*\in B} \mathrm{Re}\, y^*(Gx) = \sup_{x\in B_X} \sup_{y^*\in B_{Y^*}} \mathrm{Re}\, y^*(Gx) = \|G\|,\end{aligned}$$

as desired. ∎

We may also relate the numerical index with respect to an operator to the numerical index with respect to its adjoint.

LEMMA 3.5. *Let X, Y be Banach spaces. Then*

$$n_{G^*}(Y^*, X^*) \le n_G(X,Y)$$

for every norm-one $G \in L(X,Y)$.

Proof. The result follows immediately from Lemma 3.2 and the fact that the norm of an operator and the norm of its adjoint coincide. Alternatively, it also follows from Lemma 2.4 as the operator $\Psi \colon \mathcal{L}(X,Y) \to \mathcal{L}(Y^*,X^*)$ given by $T \mapsto T^*$ is an isometric embedding. ∎

In the case of a rank-one operator, we may provide a formula for the numerical index with respect to it.

PROPOSITION 3.6. *Let X, Y be Banach spaces, $x_0^* \in S_{X^*}$, and $y_0 \in S_Y$. Then the rank-one operator $G = x_0^* \otimes y_0$ satisfies*

$$n_G(X,Y) = n(X^*, x_0^*)\, n(Y, y_0).$$

We need to introduce some notation, just for this proof. Given a Banach space Z, $u \in S_Z$, and $\delta \in (0,1)$, we write

$$v_\delta(Z,u,z) := \sup\{|z^*(z)| \colon z^* \in S_{Z^*},\, \mathrm{Re}\, z^*(u) > 1-\delta\}.$$

Then (use Proposition 2.14, for instance) $v(Z,u,z) = \inf_{\delta>0} v_\delta(Z,u,z)$.

Proof of Proposition 3.6. Given $x^* \in S_{X^*}$ and $y \in S_Y$, we consider the norm-one operator $T = x^* \otimes y$ and show

$$v_{G,\delta}(T) \leq v_\delta(X^*, x_0^*, x^*) v_\delta(Y, y_0, y)$$

for every $\delta > 0$. To do so, we first observe that

$$v_\delta(X^*, x_0^*, x^*) = \sup\{|x^*(x)| \colon x \in S_X, \operatorname{Re} x_0^*(x) > 1 - \delta\}$$

as $J_X(B_X)$ is weak* dense in $B_{X^{**}}$. Therefore, we can write

$$\begin{aligned}
v_{G,\delta}(T) &= \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re}(y^*(y_0)x_0^*(x)) > 1 - \delta\} \\
&\leq \sup\{|y^*(y)||x^*(x)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(y_0) > 1 - \delta,\, \operatorname{Re} x_0^*(x) > 1 - \delta\} \\
&\leq \sup\{|x^*(x)| \colon x \in S_X,\, \operatorname{Re} x_0^*(x) > 1 - \delta\}\ \sup\{|y^*(y)| \colon y^* \in S_{Y^*},\, \operatorname{Re} y^*(y_0) > 1 - \delta\} \\
&= v_\delta(X^*, x_0^*, x^*)\, v_\delta(Y, y_0, y).
\end{aligned}$$

This clearly gives $n_G(X, Y) \leq n(X^*, x_0^*) n(Y, y_0)$. To prove the reverse inequality, fixed $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $\delta > 0$, observe that

$$\begin{aligned}
&\sup\{\|Tx\| \colon x \in S_X,\, \operatorname{Re} x_0^*(x) > 1 - \delta\} \\
&\qquad = \sup\{|z^*(Tx)| \colon z^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} x_0^*(x) > 1 - \delta\} \\
&\qquad = \sup\{|[T^*z^*](x)| \colon z^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} x_0^*(x) > 1 - \delta\} \\
&\qquad = \sup\{v_\delta(X^*, x_0^*, T^*z^*) \colon z^* \in S_{Y^*}\} \\
&\qquad \geq \sup\{n(X^*, x_0^*)\|T^*z^*\| \colon z^* \in S_{Y^*}\} \\
&\qquad = n(X^*, x_0^*)\|T^*\| = n(X^*, x_0^*).
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
v_{G,2\delta}(T) &= \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re}(y^*(y_0)x_0^*(x)) > 1 - 2\delta\} \\
&\geq \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(y_0) > 1 - \delta,\, \operatorname{Re} x_0^*(x) > 1 - \delta\} \\
&\geq \sup\{n(Y, y_0)\|Tx\| \colon x \in S_X,\, \operatorname{Re} x_0^*(x) > 1 - \delta\} \geq n(Y, y_0)\, n(X^*, x_0^*),
\end{aligned}$$

which gives the desired inequality $n_G(X, Y) \geq n(X^*, x_0^*) n(Y, y_0)$. ∎

To finish the chapter, we would like to present some results which allow to control the numerical index with respect to operators in terms of the numerical radius of the operators on the domain space or on the range space, which we will profusely use in Chapter 4. They all follow from this easy key lemma.

LEMMA 3.7. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X, Y)$ be such that $\|G\| = 1$. Then*

(a) $v_G(G \circ T) \leq v(T)$ *for every* $T \in \mathcal{L}(X)$,
(b) $v_G(T \circ G) \leq v(T)$ *for every* $T \in \mathcal{L}(Y)$.

Proof. Both statements follow from Lemma 2.4 by considering, respectively, the operator $\mathcal{L}(X) \to \mathcal{L}(X, Y)$ given by $T \mapsto G \circ T$, and the operator $\mathcal{L}(Y) \to \mathcal{L}(X, Y)$ given by $T \mapsto T \circ G$. ∎

As a consequence of this result, we have the following chain of inequalities:

$$\begin{aligned} n_G(X,Y) &\leq \inf\left\{\frac{v(T)}{\|G\circ T\|} : T\in\mathcal{L}(X),\ G\circ T\neq 0\right\} \\ &\leq \sup_{\varepsilon>0}\inf\{v(T) \colon T\in\mathcal{L}(X),\ \|G\circ T\|>1-\varepsilon\} \end{aligned}$$

and, analogously,

$$\begin{aligned} n_G(X,Y) &\leq \inf\left\{\frac{v(T)}{\|T\circ G\|} : T\in\mathcal{L}(Y),\ T\circ G\neq 0\right\} \\ &\leq \sup_{\varepsilon>0}\inf\{v(T) \colon T\in\mathcal{L}(Y),\ \|T\circ G\|>1-\varepsilon\}. \end{aligned}$$

These inequalities immediately imply the following result.

LEMMA 3.8. *Let X, Y be Banach spaces, $G\in\mathcal{L}(X,Y)$ with $\|G\|=1$, and $0\leq\alpha\leq 1$. Then $n_G(X,Y)\leq\alpha$ provided one of the following statements is satisfied:*

(a) *For every $\varepsilon>0$ there exists $T_\varepsilon\in\mathcal{L}(X)$ such that $v(T_\varepsilon)\leq\alpha$ and $\|G\circ T_\varepsilon\|>1-\varepsilon$.*
(b) *For every $\varepsilon>0$ there exists $S_\varepsilon\in\mathcal{L}(Y)$ such that $v(S_\varepsilon)\leq\alpha$ and $\|S_\varepsilon\circ G\|>1-\varepsilon$.*

The previous result gives some important consequences.

PROPOSITION 3.9. *Let X, Y be Banach spaces and let $0\leq\alpha\leq 1$.*

(a) *Let $\mathcal{A}(\alpha)=\{T\in\mathcal{L}(X)\colon \|T\|=1,\ v(T)\leq\alpha\}$. If*

$$B_X=\overline{\operatorname{aconv}}\bigcup_{T\in\mathcal{A}(\alpha)}T(B_X),$$

then $n_G(X,Y)\leq\alpha$ for every norm-one operator $G\in\mathcal{L}(X,Y)$.
(b) *Let $\mathcal{B}(\alpha)=\{T\in\mathcal{L}(Y)\colon \|T\|=1,\ v(T)\leq\alpha\}$. If for every $\varepsilon>0$, the set*

$$\bigcup_{T\in\mathcal{B}(\alpha)}\{y\in S_Y\colon \|Ty\|>1-\varepsilon\}$$

is dense in S_Y, then $n_G(X,Y)\leq\alpha$ for every norm-one operator $G\in\mathcal{L}(X,Y)$.
(c) *In particular, if there exists a surjective isometry $T\in\mathcal{L}(X)$ with $v(T)\leq\alpha$ or there exists a surjective isometry $S\in\mathcal{L}(Y)$ with $v(S)\leq\alpha$, then $n_G(X,Y)\leq\alpha$ for every norm-one operator $G\in\mathcal{L}(X,Y)$.*

Proof. Fix $G\in\mathcal{L}(X,Y)$ with $\|G\|=1$.

(a) For every $\varepsilon>0$, we may use the hypothesis to find $T_\varepsilon\in\mathcal{L}(X)$ with $\|T_\varepsilon\|=1$ and $v(T_\varepsilon)\leq\alpha$ such that $\|G(T_\varepsilon(x))\|>1-\varepsilon$ for some $x\in B_X$. Therefore, $\|G\circ T_\varepsilon\|>1-\varepsilon$ and Lemma 3.8 gives the result.

(b) For every $\varepsilon>0$, we take $x\in S_X$ such that $\|Gx\|>1-\varepsilon/3$. Now, we may use the hypothesis to find $S_\varepsilon\in\mathcal{L}(Y)$ with $\|S_\varepsilon\|=1$ and $v(S_\varepsilon)\leq\alpha$, and $y\in S_Y$ such that $\|S_\varepsilon y\|>1-\varepsilon/3$ and $\big\|y-Gx/\|Gx\|\big\|<\varepsilon/3$. Now, $\|y-Gx\|<2\varepsilon/3$, and so

$$\|S_\varepsilon(Gx)\|\geq\|S_\varepsilon y\|-\|S_\varepsilon(y-Gx)\|>1-\varepsilon/3-2\varepsilon/3=1-\varepsilon.$$

Consequently, $\|S_\varepsilon\circ G\|>1-\varepsilon$ and Lemma 3.8 gives the result.

Finally, (c) clearly follows from (a) and (b). ■

For the special case $\alpha = 0$, the above result can be improved as we do not have to pay attention to the norm of the operators.

PROPOSITION 3.10. *Let X, Y be Banach spaces.*

(a) *Let $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$.*

  (a.1) *If there exists $T \in \mathcal{L}(X)$ with $v(T) = 0$ and $G \circ T \neq 0$, then $n_G(X,Y) = 0$.*
  (a.2) *If there exists $T \in \mathcal{L}(Y)$ with $v(T) = 0$ and $T \circ G \neq 0$, then $n_G(X,Y) = 0$.*

(b) *If*
$$\bigcap_{T \in \mathcal{L}(Y),\, v(T)=0} \ker T = \{0\},$$
then $n_G(X,Y) = 0$ for every norm-one operator $G \in \mathcal{L}(X,Y)$.

(c) *If*
$$\bigcup_{T \in \mathcal{L}(X),\, v(T)=0} T(X)$$
is dense in X, then $n_G(X,Y) = 0$ for every norm-one operator $G \in \mathcal{L}(X,Y)$.

We emphasize the following immediate consequence of the previous result which will be useful.

COROLLARY 3.11. *Let W be a Banach space such that there is an onto isometry $J \in \mathcal{L}(W)$ with $v(J) = 0$. Then*

(a) *$n_G(X,W) = 0$ for every Banach space X and every operator $G \in \mathcal{L}(X,W)$ of norm 1,*
(b) *$n_G(W,Y) = 0$ for every Banach space Y and every operator $G \in \mathcal{L}(W,Y)$ of norm 1.*

4. Set of values of the numerical indices with respect to all operators between two given Banach spaces

We start by showing some general results which can be deduced from the tools implemented in the previous sections. The first result shows that 0 is always a possible value of the numerical index with respect to operators (unless we are in the trivial case of both spaces being one-dimensional). It is a direct consequence of Proposition 2.7.

PROPOSITION 4.1. *Let X, Y be Banach spaces. If* $\dim(X) \geq 2$ *or* $\dim(Y) \geq 2$, *then* $0 \in \mathcal{N}(\mathcal{L}(X,Y))$.

The result above is actually an equivalence, as the following result is immediate.

EXAMPLE 4.2. $\mathcal{N}(\mathcal{L}(\mathbb{K},\mathbb{K})) = \{1\}$.

Next, we particularize Proposition 2.11 to spaces of operators.

PROPOSITION 4.3. *Let X, Y be finite-dimensional real Banach spaces. Then the set of norm-one $G \in \mathcal{L}(X,Y)$ with $n_G(X,Y) > 0$ is countable. In particular, $\mathcal{N}(\mathcal{L}(X,Y))$ is countable.*

Our next result shows that all values of the numerical index are valid for operators between Banach spaces. In the real case, this is clear as the numerical indices of all two-dimensional norms do the job (and they are the numerical index with respect to the corresponding identities). But in the complex case, the values of the numerical indices with respect to the identity are not enough (as they are always greater than or equal to 1/e; see [9, Corollary 2.1.19], for instance).

A first simple way of getting arbitrary values of the numerical indices with respect to operators is given in the following result which follows immediately from Proposition 2.8.

EXAMPLE 4.4. *For every subset $A \subseteq [0,1]$ containing* 0, *there is a Banach space X such that $\mathcal{N}(\mathcal{L}(X,\mathbb{K})) = A$.* Indeed, just take X to be the predual of the space Z provided in Proposition 2.8 (which is a dual Banach space as it is the ℓ_1-sum of finite-dimensional spaces).

Let us also observe that if X is a Banach space of dimension at least 2 whose dual space is smooth, it follows from Lemma 2.3 that $\mathcal{N}(\mathcal{L}(X,\mathbb{K})) = \{0\}$. This result contrasts with the already cited fact that $n(X) \geq 1/\mathrm{e}$ for every complex Banach space X, so $\mathcal{N}(\mathcal{L}(X,X))$ cannot reduce to 0 when X is a complex Banach space. Therefore, it seems more interesting to perform the study of the set of values of the numerical indices with respect to all operators from a Banach space to itself, that is, the set

$$\{n_G(X,X)\colon X \text{ (real or complex) Banach space, } G \in \mathcal{L}(X),\ \|G\| = 1\}.$$

In the real case it is immediate that this set covers $[0,1]$, just using identity operators [16, Theorem 3.6]. In the complex case, using identity operators one can only cover the interval $[1/\mathrm{e},1]$. The result will be stated in Example 6.5. Even more, we will show that there are Banach spaces X such that $\mathcal{N}(\mathcal{L}(X)) = [0,1]$, both in the real and in the complex case, see Theorem 6.4.

For real Banach spaces, the Banach space numerical index may be zero, so there is no obstacle for the set $\mathcal{N}(\mathcal{L}(X))$ to be equal to $\{0\}$. We are going to prove that this happens when X is a real Hilbert space of dimension greater than 1. Actually, we show that zero is the only possible value of the numerical index with respect to operators, when either the domain space or the range space is a real Hilbert space of dimension at least 2.

Theorem 4.5. *Let H be a real Hilbert space of dimension at least 2. Then*

$$\mathcal{N}(\mathcal{L}(X,H)) = \mathcal{N}(\mathcal{L}(H,Y)) = \{0\}$$

for all real Banach spaces X and Y. In particular, $\mathcal{N}(\mathcal{L}(H)) = \{0\}$.

Proof. Observe that for every pair of points $x,y \in S_H$ with $\langle x,y\rangle = 0$, the operator $T \in S_H$ given by $T(z) = \langle z,x\rangle y - \langle z,y\rangle x$ for $z \in H$ satisfies $v(T) = 0$. So, clearly

$$\bigcup_{T\in\mathcal{L}(H),\, v(T)=0} T(H) \ \text{ is dense in } H \quad \text{and} \quad \bigcap_{T\in\mathcal{L}(H),\, v(T)=0} \ker(T) = \{0\}.$$

Now, both assertions are immediate consequences of Proposition 3.10. ■

For every complex Banach space W, its underlying real Banach space $W_{\mathbb{R}}$ also has trivial set of values of the numerical indices with respect to operators. This is an immediate consequence of Corollary 3.11 as multiplication by i is an onto isometry which has numerical radius zero when viewed in $\mathcal{L}(W_{\mathbb{R}})$.

Proposition 4.6. *Let $W_{\mathbb{R}}$ be the real Banach space underlying a complex Banach space W. Then*

$$\mathcal{N}(\mathcal{L}(X,W_{\mathbb{R}})) = \mathcal{N}(\mathcal{L}(W_{\mathbb{R}},Y)) = \{0\}$$

for all real Banach spaces X and Y. In particular, $\mathcal{N}(\mathcal{L}(W_{\mathbb{R}})) = \{0\}$.

Another kind of spaces having trivial set of values of the numerical indices with respect to operators are $\mathcal{L}(H)$ and also $\mathcal{K}(H)$, the space of compact linear operators from H to H.

Theorem 4.7. *Let H be a real Hilbert space of dimension at least 2. Then*

$$\mathcal{N}(\mathcal{L}(X,\mathcal{L}(H))) = \mathcal{N}(\mathcal{L}(X,\mathcal{K}(H))) = \{0\}$$

for every Banach space X. In particular,

$$\mathcal{N}(\mathcal{L}(\mathcal{L}(H))) = \mathcal{N}(\mathcal{L}(\mathcal{K}(H))) = \{0\}.$$

Moreover, if H is infinite-dimensional or has even dimension, then

$$\mathcal{N}(\mathcal{L}(\mathcal{L}(H),Y)) = \mathcal{N}(\mathcal{L}(\mathcal{K}(H),Y)) = \{0\}$$

for every Banach space Y.

Proof. Let us start with the case of $\mathcal{L}(H)$. For $J \in S_{\mathcal{L}(H)}$ we define the operator $\Phi_J\colon \mathcal{L}(H) \to \mathcal{L}(H)$ by $\Phi_J(T) = J\circ T$ for every $T \in \mathcal{L}(H)$. Evidently, $\|\Phi_J\| = \|J\| = 1$

and $\Phi_{\mathrm{Id}_H} = \mathrm{Id}_{\mathcal{L}(H)}$. Therefore,

$$v(\Phi_J) = v(\mathcal{L}(\mathcal{L}(H)), \mathrm{Id}_{\mathcal{L}(H)}, \Phi_J) = v(\mathcal{L}(H), \mathrm{Id}_H, J) = v(J)$$

by Lemma 2.4(b). Let us write

$$\mathcal{B} = \{\Phi_J \colon J \in \mathcal{L}(H),\ \|J\| = 1,\ v(J) = 0\}$$

and observe that the result will follow from Proposition 3.10(b) if we prove the equality

$$\bigcap_{\Phi\in\mathcal{B}} \ker \Phi = \{0\}.$$

To do so, fix $T_0 \in S_{\mathcal{L}(H)}$ and take $x \in S_H$ such that $\|T_0 x\| > 1/2$. Now, define $e_1 = \frac{T_0 x}{\|T_0 x\|}$ and take $e_2 \in S_H$ satisfying $\langle e_1, e_2\rangle = 0$. We define the operator $J \in \mathcal{L}(H)$ given by $Jh = \langle h, e_2\rangle e_1 - \langle h, e_1\rangle e_2$ for $h \in H$, which satisfies $\|J\| = 1$ and $v(J) = 0$, so $\Phi_J \in \mathcal{B}$. Moreover, we can write

$$\|\Phi_J(T_0)\| = \|J \circ T_0\| \geq \|J(T_0 x)\| = \| - \|T_0 x\| e_2\| = \|T_0 x\| > 1/2.$$

Therefore, $T_0 \notin \ker \Phi_J$ and thus $\bigcap_{\Phi\in\mathcal{B}} \ker\Phi = \{0\}$, which finishes the proof for $\mathcal{L}(H)$. For $\mathcal{K}(H)$, it suffices to observe that the same argument is valid since $\Phi_J(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$ and we may repeat the argument considering $\Phi_J \colon \mathcal{K}(H) \to \mathcal{K}(H)$ and getting

$$v(\Phi_J) = v(\mathcal{L}(\mathcal{K}(H)), \mathrm{Id}_{\mathcal{K}(H)}, \Phi_J) = v(\mathcal{L}(H), \mathrm{Id}_H, J) = v(J).$$

The rest of the proof is identical.

To prove the moreover part, observe that when H is infinite-dimensional or has even dimension, then there is an onto isometry $J \in \mathcal{L}(H)$ with $v(J) = 0$. Indeed, in this case we may write $H = [\bigoplus_{\lambda\in\Lambda} \ell_2^2]_{\ell_2}$ for a suitable index set Λ and, defining $A \in \mathcal{L}(\ell_2^2)$ by $A(x, y) = (y, -x)$, the surjective isometry with numerical index zero is given by

$$J[(x_\lambda)_{\lambda\in\Lambda}] = (Ax_\lambda)_{\lambda\in\Lambda} \qquad ((x_\lambda)_{\lambda\in\Lambda} \in H).$$

Now, the operator Φ_J is an onto isometry on $\mathcal{L}(H)$ or $\mathcal{K}(H)$ ($\Phi_{J^{-1}}$ is clearly the inverse of Φ_J) satisfying $v(\Phi_J) = 0$. Then Corollary 3.11 gives the result. ∎

When H has odd dimension, we do not know if the equality $n_G(\mathcal{L}(H), Y) = 0$ holds for every Banach space Y and every operator $G \in \mathcal{L}(\mathcal{L}(H), Y)$.

Another result of the same kind tells us that there are many other spaces of operators having trivial set of values of the numerical indices with respect to operators.

PROPOSITION 4.8. *Let $W_1, \ldots, W_n$ be real Banach spaces, let E be $\mathbb{R}^n$ endowed with an absolute norm, and let $W = [W_1 \oplus \cdots \oplus W_n]_E$. Then the following statements hold:*

(a) *If S_E is smooth at points whose first coordinate is zero and*

$$\bigcap\{\ker(S_1) \colon S_1 \in \mathcal{L}(W_1),\ v(S_1) = 0\} = \{0\},$$

then $\mathcal{N}(\mathcal{L}(X, W)) = \{0\}$ for every Banach space X.

(b) *If S_E is rotund in the direction of the first coordinate, that is, S_E does not contain line segments parallel to $(1, 0, \ldots, 0)$, and $\bigcup\{S_1(W_1) \colon S_1 \in \mathcal{L}(W_1),\ v(S_1) = 0\}$ is dense in W_1, then $\mathcal{N}(\mathcal{L}(W, Y)) = \{0\}$ for every Banach space Y.*

Consequently, if the assumptions of (a) *or* (b) *hold, then $\mathcal{N}(\mathcal{L}(W)) = \{0\}$.*

Proof. (a) Given a Banach space X, a norm-one operator $G \in \mathcal{L}(X, W)$ can be seen as $G = (G_1, \dots, G_n)$ where $G_k \in \mathcal{L}(X, W_k)$ for $k = 1, \dots, n$. We claim $n_G(X, W) = 0$ if $G_1 \neq 0$. Indeed, let $P_1 \in \mathcal{L}(W, W_1)$ denote the natural projection on W_1 and let $I_1 \in \mathcal{L}(W_1, W)$ be the natural inclusion, so $G_1 = P_1 \circ G$. Observe now that for every $S_1 \in \mathcal{L}(W_1)$ with $v(S_1) = 0$, the operator $S \in \mathcal{L}(W)$ given by $S = I_1 \circ S_1 \circ P_1$ clearly satisfies $\|S\| = \|S_1\|$ and $v(S) = 0$. Since

$$P_1 \circ G \neq 0 \quad \text{and} \quad \bigcap_{S_1 \in \mathcal{L}(W_1),\, v(S_1)=0} \ker(S_1) = \{0\},$$

we can find $S_1 \in \mathcal{L}(W_1)$ with $v(S_1) = 0$ such that $S_1 \circ P_1 \circ G \neq 0$ and so $I_1 \circ S_1 \circ P_1 \circ G \neq 0$. As $v(I_1 \circ S_1 \circ P_1) = 0$, we get $n_G(X, W) = 0$ from Proposition 3.10(a.2). Therefore, we may and do assume from now on that $G_1 = 0$. Next we fix $w_0 \in S_{W_1}$ and $x^* \in S_{X^*}$, we consider the norm-one operator $T = x^* \otimes (w_0, 0, \dots, 0) \in \mathcal{L}(X, W)$, and we shall prove $v_G(T) = 0$. To this end, as

$$v_G(T) = \inf_{\delta>0} \sup\{|w^*(Tx)| \colon w^* \in S_{W^*},\ x \in S_X,\ \operatorname{Re} w^*(Gx) > 1 - \delta\}$$

for every $k \in \mathbb{N}$ we can take $w_k^* = (w_{k,1}^*, \dots, w_{k,n}^*) \in S_{W^*}$ and $x_k \in S_X$ satisfying

$$\lim_k \operatorname{Re} w_k^*(Gx_k) = 1 \quad \text{and} \quad \lim_k |w_k^*(Tx_k)| = v_G(T).$$

For each $k \in \mathbb{N}$ define

$$e_k^* = (\|w_{k,1}^*\|, \dots, \|w_{k,n}^*\|) \in S_{E^*} \quad \text{and} \quad e_k = (\|G_1 x_k\|, \dots, \|G_n x_k\|) \in B_E$$

which satisfy $1 = \lim_k \operatorname{Re} w_k^*(Gx_k) \leq \lim_k \langle e_k^*, e_k \rangle \leq 1$, and thus $\lim_k \langle e_k^*, e_k \rangle = 1$. Now, by passing to a subsequence, we may find $y^* = (y_1^*, \dots, y_n^*) \in S_{E^*}$ and $y = (y_1, \dots, y_n) \in S_E$ such that $\lim_{k\to\infty} e_k^* = y^*$ and $\lim_{k\to\infty} e_k = y$. Then it follows that

$$\langle y^*, y \rangle = \lim_k \langle e_k^*, e_k \rangle = 1$$

and y^* is a supporting functional of y. Moreover, we have $y_1 = 0$ as the first coordinate of e_k is equal to $\|G_1 x_k\| = 0$ for every k, so

$$1 = \langle y^*, y \rangle = \sum_{j=1}^n y_j^*(y_j) = \sum_{j=2}^n y_j^*(y_j)$$

and the element $\widetilde{y}^* = (0, y_2^*, \dots, y_n^*) \in B_{E^*}$ is also a supporting functional of y. Therefore, we get $\widetilde{y}^* = y^*$ by the smoothness of S_E at y and so $y_1^* = 0$. Finally, we can write

$$\begin{aligned} v_G(T) &= \lim_k |w_k^*(Tx_k)| = \lim_k |w_k^*(w_0, 0, \dots, 0)|\, |x^*(x_k)| \\ &\leq \lim_k \|w_{k,1}^*\|\, \|w_0\| \leq \lim_k \|w_{k,1}^*\| = y_1^* = 0, \end{aligned}$$

which gives $v_G(T) = 0$ and finishes the proof of (a).

To prove (b) we start by observing that we can assume $G \circ I_1 \circ S_1 \circ P_1 = 0$ for every $S_1 \in \mathcal{L}(W_1)$ with $v(S_1) = 0$. Indeed, if there is $S_1 \in \mathcal{L}(W_1)$ with $v(S_1) = 0$ such that $G \circ I_1 \circ S_1 \circ P_1 \neq 0$, then $S = I_1 \circ S_1 \circ P_1$ satisfies $v(S) = 0$ and $G \circ S \neq 0$. So Proposition 3.10 gives $n_G(W, Y) = 0$. Then $G \circ I_1 \circ S_1 \circ P_1 = 0$ for every $S_1 \in \mathcal{L}(W_1)$

with $v(S_1)=0$. This, together with the fact that the set

$$\bigcup_{S_1\in\mathcal{L}(W_1),\, v(S_1)=0} S_1(W_1)$$

is dense in W_1, implies $G\circ I_1=0$. Next, we fix $y_0\in S_Y$, $w_0^*\in S_{W_1^*}$, we define $w^*=(w_0^*,0,\ldots,0)\in S_{W^*}$ and the rank-one operator $T=w^*\otimes y_0\in S_{\mathcal{L}(W,Y)}$, and we shall prove $v_G(T)=0$. To do so, since

$$v_G(T)=\inf_{\delta>0}\sup\{|y^*(Tw)|\colon y^*\in S_{Y^*},\ w\in S_W,\ \operatorname{Re} y^*(Gw)>1-\delta\}$$

for every $k\in\mathbb{N}$ we can take $w_k=(w_{k,1},\ldots,w_{k,n})\in S_W$ and $y_k^*\in S_{Y^*}$ satisfying

$$\lim_k \operatorname{Re} y_k^*(Gw_k)=1 \quad\text{and}\quad \lim_k |y_k^*(Tw_k)|=v_G(T).$$

By passing to a subsequence, we may assume that $\{\|w_{k,j}\|\}_k$ is convergent for every $j=1,\ldots,n$. So, since the norm in E is absolute, we can define elements

$$\begin{aligned}
e^+ &= \Big(\lim_k\|w_{k,1}\|,\lim_k\|w_{k,2}\|,\ldots,\lim_k\|w_{k,n}\|\Big),\\
e^- &= \Big(-\lim_k\|w_{k,1}\|,\lim_k\|w_{k,2}\|,\ldots,\lim_k\|w_{k,n}\|\Big),\\
\widetilde{e} &= \Big(0,\lim_k\|w_{k,2}\|,\ldots,\lim_k\|w_{k,n}\|\Big)=\frac{1}{2}(e^++e^-),
\end{aligned}$$

which clearly satisfy $\|\widetilde{e}\|\leq\|e^+\|=\|e^-\|\leq 1$. Since $G\circ I_1=0$, we can estimate as follows:

$$\begin{aligned}
1&=\lim_k\operatorname{Re} y_k^*(Gw_k)=\lim_k\operatorname{Re} y_k^*(G(0,w_{k,2},\ldots,w_{k,n}))\\
&\leq\lim_k\|(0,w_{k,2},\ldots,w_{k,n})\|\leq\lim_k\big\|(0,\|w_{k,2}\|,\ldots,\|w_{k,n}\|)\big\|_E=\|\widetilde{e}\|\leq 1
\end{aligned}$$

which gives $\widetilde{e}\in S_E$ and thus $e^\pm\in S_E$. So, we deduce that $\lim_k\|w_{k,1}\|=0$ since S_E is rotund in the direction of the first coordinate. To finish the proof, observe that

$$v_G(T)=\lim_k|y_k^*(Tw_k)|=\lim_k|y_k^*(y_0)|\,|w^*(w_k)|\leq\lim_k\|w_0^*\|\,\|w_{k,1}\|=0.$$

Therefore, we get $v_G(T)=0$ and $n_G(W,Y)=0$. ■

REMARK 4.9. *The smoothness and rotundity hypotheses in Proposition 4.8 cannot be omitted.* Indeed, on the one hand, the rank-one operator $G\in\mathcal{L}(\ell_2^2\oplus_\infty\mathbb{R},\mathbb{R})$ given by $G=(0,0,1)\otimes\mathbb{1}$ is a spear operator by Proposition 3.6 as $\mathbb{1}$ is a spear vector in $\mathbb{R}$ and $(0,0,1)$ is a spear vector in $(\ell_2^2\oplus_\infty\mathbb{R})^*=\ell_2^2\oplus_1\mathbb{R}$. Thus, the assumption of smoothness in Proposition 4.8(a) is essential. On the other hand, the operator $G^*\in\mathcal{L}(\mathbb{R},\ell_2^2\oplus_1\mathbb{R})$ is also a spear operator by the same argument, showing that we cannot omit the rotundity in Proposition 4.8(b).

The next example is even more surprising.

EXAMPLE 4.10. *There exists a Banach space X with $n(X)=0$ such that $\mathcal{L}(X)$ contains a spear operator.* Indeed, consider $X=(\ell_2^2\oplus_\infty\mathbb{R})\oplus_1\mathbb{R}$, which clearly satisfies $n(X)=0$, and $G\in\mathcal{L}((\ell_2^2\oplus_\infty\mathbb{R})\oplus_1\mathbb{R})$ given by $G=(0,0,0,1)\otimes(0,0,0,1)$, which is a spear operator by Proposition 3.6.

Our next result estimates the numerical indices with respect to operators whose domain or range is an ℓ_p-space.

PROPOSITION 4.11. *Let $1 < p < \infty$, $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, let $M_p = \sup\limits_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$, and let Γ be either an infinite set or a finite set with an even number of elements. Then*

$$\mathcal{N}(\mathcal{L}(X, \ell_p(\Gamma))) \subseteq [0, M_p] \quad \text{and} \quad \mathcal{N}(\mathcal{L}(\ell_p(\Gamma), Y)) \subseteq [0, M_p]$$

hold in the real case for all Banach spaces X and Y.

Proof. The argument is very similar to the one given at the end of the proof of Theorem 4.7. By the assumption on the set Γ we may write $\ell_p(\Gamma) = [\bigoplus_{\lambda\in\Lambda} \ell_p^2]_{\ell_p}$ for a suitable index set Λ. Defining $A \in \mathcal{L}(\ell_p^2)$ by $A(x,y) = (y, -x)$, the operator given by

$$J[(x_\lambda)_{\lambda\in\Lambda}] = (Ax_\lambda)_{\lambda\in\Lambda} \quad ((x_\lambda)_{\lambda\in\Lambda} \in \ell_p(\Gamma))$$

is then a surjective isometry. As $v(A) = M_p$ (see the comments after [27, Theorem 1]), we get $v(J) \leq M_p$. Now, Corollary 3.11 gives the result. ■

We now pass to study some results for complex spaces. As a first result, we may calculate the set of values of the numerical indices with respect to operators between two Hilbert spaces.

PROPOSITION 4.12. *Let H_1, H_2 be complex Hilbert spaces with dimension greater than 1. Then $\mathcal{N}(\mathcal{L}(H_1, H_2)) = \{0, 1/2\}$ if H_1 and H_2 are isometrically isomorphic and $\mathcal{N}(\mathcal{L}(H_1, H_2)) = \{0\}$ in the other case.*

Proof. $\mathcal{L}(H_1, H_2)$ is a JB*-triple (see [9, §2.2.27, §4.1.39] for the definition) under the triple product

$$\{xyz\} = \tfrac{1}{2}(xy^*z + zy^*x) \quad (x, y, z \in \mathcal{L}(H_1, H_2)),$$

as it is a closed subtriple of the C^*-algebra $\mathcal{L}(H_1 \oplus_2 H_2)$ (we may use [9, Facts 4.1.40 and 4.1.41]). Now, as $\mathcal{L}(H_1, H_2)$ is not abelian since $\dim(H_1) \geq 2$ and $\dim(H_2) \geq 2$ (see [9, §4.1.47]), it follows from [9, Theorem 4.2.24] that the quantity $n_G(H_1, H_2)$ is equal to 0 or 1/2 for every norm-one operator $G \in \mathcal{L}(H_1, H_2)$.

Next, we take into account that, by [9, Theorem 4.2.24], $J = \mathcal{L}(H_1, H_2)$ contains a geometrically unitary element if and only if J contains a unitary element as Jordan $*$-triple, that is, if there is $U \in J$ such that $\{UUT\} = T$ for every $T \in J$ (see [9, Definition 4.1.53]). This implies that H_1 and H_2 are isometrically isomorphic, as is known to experts, but we give an easy argument. Taking into account the formula for the product in J, we get

$$UU^*T + TU^*U = 2T$$

for every $T \in \mathcal{L}(H_1, H_2)$. Just considering rank-one operators $T \in \mathcal{L}(H_1, H_2)$, we obtain

$$UU^* = \mathrm{Id}_{H_2} \quad \text{and} \quad U^*U = \mathrm{Id}_{H_1},$$

which gives the desired result. ■

Following an argument similar to the one given in Theorem 4.5, we can establish the next result.

Proposition 4.13. *Let H be a complex Hilbert space with* $\dim(H) \geq 2$. *Then*

$$\mathcal{N}(\mathcal{L}(X,H)) \subseteq [0,1/2] \quad \text{and} \quad \mathcal{N}(\mathcal{L}(H,Y)) \subseteq [0,1/2]$$

for all complex Banach spaces X and Y.

Proof. For each $u \in S_H$, let $v \in S_H$ with $\langle u,v\rangle = 0$ and define

$$T\colon H \to H, \qquad T(x) = \langle x,v\rangle\, u,$$

which satisfies $v(T) \leq 1/2$. An application of Proposition 3.9 gives the result. ∎

Our next aim is to study the set $\mathcal{N}(\mathcal{L}(C(K_1),C(K_2)))$, where K_1 and K_2 are compact Hausdorff topological spaces. Recall that, by Lemma 3.1, if $n_G(C(K_1),C(K_2)) > 0$ for some $G \in \mathcal{L}(C(K_1),C(K_2))$, then G is an extreme operator. There is a well studied special kind of extreme operators between $C(K)$ spaces, the nice operators. A norm-one operator $G \in \mathcal{L}(C(K_1),C(K_2))$ is said to be *nice* if

$$G^*(\delta_t) \in \mathbb{T}\{\delta_s\colon s \in K_1\}$$

for every $t \in K_2$ (that is, G^* carries extreme points of $B_{C(K_2)^*}$ to extreme points of $B_{C(K_1)^*}$). It is immediate that nice operators are extreme, but the converse result is not always true (see Remark 4.16 below). We claim that a nice operator G satisfies $n_G(C(K_1),C(K_2)) = 1$. Indeed, this is easy to show by hand using the properties of the δ-functions in the dual of a $C(K)$ space, but also follows directly from [26, Proposition 4.2] and [26, Example 2.12(a)]. Therefore, if for a pair of compact Hausdorff topological spaces (K_1,K_2) it is known that every extreme operator in $\mathcal{L}(C(K_1),C(K_2))$ is nice, then the only possible values of the numerical index of operators in $\mathcal{L}(C(K_1),C(K_2))$ are 0 and 1. This idea leads to a couple of results, one for the real case and another one for the complex case.

Theorem 4.14. *Let K_1, K_2 be compact Hausdorff topological spaces such that at least one of them has more than one point. Then, in the real case, one has*

$$\mathcal{N}(\mathcal{L}(C(K_1),C(K_2))) = \{0,1\}$$

provided at least one of the following assumptions holds:

(1) *K_1 is metrizable,*
(2) *K_1 is Eberlein compact and K_2 is metrizable,*
(3) *K_2 is extremally disconnected,*
(4) *K_1 is scattered.*

Proof. First, as at least one of the spaces $C(K_1)$ and $C(K_2)$ has dimension greater than 1, Proposition 4.1 gives $0 \in \mathcal{N}(\mathcal{L}(C(K_1),C(K_2)))$. By considering the rank-one operator $G = \delta_t \otimes \mathbf{1}$, we immediately obtain $1 \in \mathcal{N}(\mathcal{L}(C(K_1),C(K_2)))$ by Proposition 3.6. Finally, to get the reverse inclusion, by the comments before the statement of the theorem, we just have to check that under the given conditions, every extreme operator in $\mathcal{L}(C(K_1),C(K_2))$ is actually nice. For (1), this is shown in [5, Theorem 1]; for (2) in [1, Theorem 7]; [47, Theorem 4] gives (3); finally, (4) follows from [47, Theorem 5]. ∎

For the complex case, we have a similar result.

Theorem 4.15. *Let K_1, K_2 be compact Hausdorff topological spaces such that at least one of them has more than one point. Then, in the complex case, one has*

$$\mathcal{N}(\mathcal{L}(C(K_1), C(K_2))) = \{0, 1\}$$

provided at least one of the following assumptions holds:

(1) *K_2 is extremally disconnected,*
(2) *K_1 is metrizable and K_2 is basically disconnected* (*i.e. the closure of every F_σ-open is open*),
(3) *K_1 is scattered.*

Proof. We just need to follow the lines of the proof of Theorem 4.14, but here we have to provide references for the fact that, in the complex case, every extreme operator in $\mathcal{L}(C(K_1), C(K_2))$ is actually nice under the presented conditions. For (1), this is shown in [47, Theorem 4]; (2) is proved in [17, Theorem 1.4]; finally, (3) follows from [47, Theorem 5]. ∎

Remark 4.16. There are examples showing that it is not true in general that all extreme operators between spaces of continuous functions are nice [49, 50]. The underlying idea in these examples is to consider for an arbitrary compact Hausdorff space K the canonical inclusion G given by

$$G\colon C(K) \to C(B_{C(K)^*}, w^*)$$

which satisfies

$$G^*(\delta_\mu) = \mu$$

for every $\mu \in B_{C(K)}$ and so it is not nice. Additional hypothesis on the compact space K (e.g. K perfect in the complex case, see [49, Theorem 2.5]) ensure, however, that G is an extreme point. We do not know whether the numerical index with respect to operators G defined as above has to be always 0 or 1.

Remark 4.17. Let us also comment that, in the real case, examples as the ones in the previous remark cannot be compact: for arbitrary compact Hausdorff topological spaces K_1 and K_2, every compact extreme operator $G \in \mathcal{L}(C(K_1), C(K_2))$ is nice [40, Theorem 4.5] (see [54, Theorem 2.4] for an extension of this result). Moreover, if K_2 is separable, every weakly compact extreme operator $G \in \mathcal{L}(C(K_1), C(K_2))$ is nice [14, Proposition 2.8].

As a consequence of Theorems 4.14 and 4.15, we get the following particular case.

Corollary 4.18. *Let K_1 be a compact Hausdorff topological space and let (Ω, Σ, μ) be a σ-finite measure space such that at least one of the spaces $C(K_1)$ or $L_\infty(\mu)$ has dimension at least 2. Then*

$$\mathcal{N}(\mathcal{L}(C(K_1), L_\infty(\mu))) = \{0, 1\}$$

in both the real and the complex case.

Indeed, this is a consequence of the fact that every $L_\infty(\mu)$ space can be identified with a $C(K_\mu)$-space where K_μ is extremally disconnected. With this in mind, the following particular case also holds.

Corollary 4.19. *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, be σ-finite measure spaces such that at least one of the spaces $L_\infty(\mu_i)$, $i = 1, 2$, has dimension at least 2. Then*

$$\mathcal{N}(\mathcal{L}(L_\infty(\mu_1), L_\infty(\mu_2))) = \{0, 1\}$$

in both the real and the complex case.

We get an analogous result for $L_1(\mu)$ spaces.

Corollary 4.20. *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, be σ-finite measure spaces. Then*

$$\mathcal{N}(\mathcal{L}(L_1(\mu_1), L_1(\mu_2))) \subseteq \{0, 1\}$$

in both the real and the complex case.

Proof. Fix a norm-one operator $G \in \mathcal{L}(L_1(\mu_1), L_1(\mu_2))$. If G is not extreme, Lemma 3.1 gives $n_G(L_1(\mu_1), L_1(\mu_2)) = 0$. If, otherwise, G is an extreme operator, then $G^* \in \mathcal{L}(L_\infty(\mu_2), L_\infty(\mu_1))$ is nice by [48, Corollary 2.4], so $n_{G^*}(L_\infty(\mu_2), L_\infty(\mu_1)) = 1$ from the discussion preceding Theorems 4.14 and 4.15. But then $n_G(L_1(\mu_1), L_1(\mu_2)) = 1$ by Lemma 3.5. This shows $\mathcal{N}(\mathcal{L}(L_1(\mu_1), L_1(\mu_2))) \subseteq \{0, 1\}$. ■

Let us show that the set $\mathcal{N}(\mathcal{L}(L_1(\mu_1), L_1(\mu_2)))$ does not always contain the value 1.

Example 4.21. $\mathcal{N}(\mathcal{L}(\ell_1, L_1[0,1])) = \{0\}$.

Indeed, by [26, Proposition 3.3] any norm-one operator $G \in \mathcal{L}(\ell_1, L_1[0,1])$ satisfying $n_G(\ell_1, L_1[0,1]) = 1$ would carry the elements of the basis of ℓ_1 to spear vectors of $L_1[0,1]$ and thus to extreme points of the unit ball of $L_1[0,1]$ [26, Proposition 2.11(b)], so there are no such operators. On the other hand, $0 \in \mathcal{N}(\mathcal{L}(\ell_1, L_1[0,1]))$ by Proposition 4.1.

5. Lipschitz numerical range

We would like to deal now with the Lipschitz numerical range introduced in [51, 52] and show that it can be viewed as a particular case of the numerical range with respect to a linear operator. We need some notation. Let X, Y be Banach spaces. We denote by $\operatorname{Lip}_0(X,Y)$ the set of all Lipschitz maps $F\colon X\to Y$ such that $F(0)=0$. This is a Banach space when endowed with the norm

$$\|F\|_L=\sup\left\{\frac{\|F(x)-F(y)\|}{\|x-y\|}\colon x,y\in X,\, x\neq y\right\}.$$

Following [51, 52], the *Lipschitz numerical range* of $F\in\operatorname{Lip}_0(X,X)$ is

$$W_L(F):=\left\{\frac{\xi^*(F(x)-F(y))}{\|x-y\|}\colon \xi^*\in S_{X^*},\, x,y\in X,\, x\neq y,\, \xi^*(x-y)=\|x-y\|\right\},$$

the *Lipschitz numerical radius* of F is

$$w_L(F):=\sup\{|\lambda|\colon \lambda\in W_L(F)\},$$

and the *Lipschitz numerical index* of X is

$$\begin{aligned} n_L(X) &:= \inf\{w_L(F)\colon F\in\operatorname{Lip}_0(X,X),\, \|F\|_L=1\}\\ &= \max\{k\geq 0\colon k\|F\|_L\leq w_L(F)\ \forall F\in\operatorname{Lip}_0(X,X)\}.\end{aligned}$$

We would like to show that the closed convex hull of the Lipschitz numerical range is equal to the numerical range with respect to a linear operator. To do so, we need to recall the concept of Lipschitz free space. First, observe that we can associate to each $x\in X$ an element $\delta_x\in\operatorname{Lip}_0(X,\mathbb{K})^*$ which is just the evaluation map $\delta_x(F)=F(x)$ for every $F\in\operatorname{Lip}_0(X,\mathbb{K})$. The *Lipschitz free space* over X is defined as

$$\mathcal{F}(X):=\overline{\operatorname{span}}^{\|\cdot\|}\{\delta_x\colon x\in X\}\subseteq\operatorname{Lip}_0(X,\mathbb{K})^*.$$

The space $\mathcal{F}(X)$ is an isometric predual of $\operatorname{Lip}_0(X,\mathbb{K})$. Moreover, the inclusion map $\delta\colon x\rightsquigarrow\delta_x$ establishes an isometric (non-linear) embedding $X\hookrightarrow\mathcal{F}(X)$ since

$$\|\delta_x-\delta_y\|_{\mathcal{F}(X)}=\|x-y\|_X$$

for all $x,y\in X$. The term "Lipschitz free space" comes from [20], but the concept was studied much earlier and it is also known as the Arens–Eells space of X. We refer the reader to the paper [18] and the book [53] for more information and background. The main features of the Lipschitz free space we are going to use here are contained in the following result which is nowadays considered folklore in the theory of Lipschitz maps and can be found in the cited references [18], [20], or [53, Chapter 3].

Lemma 5.1. *Let X, Y be Banach spaces.*

(a) *For every $F \in \mathrm{Lip}_0(X,Y)$ there exists a unique linear operator $T_F\colon \mathcal{F}(X) \to Y$ such that $T_F \circ \delta = F$ and $\|T_F\| = \|F\|_L$. Moreover, $\mathrm{Lip}_0(X,Y)$ is isometrically isomorphic to $\mathcal{L}(\mathcal{F}(X),Y)$. In particular, $\mathrm{Lip}_0(X,\mathbb{K}) = \mathcal{F}(X)^*$.*

(b) *When the above is applied to $\mathrm{Id} \in \mathrm{Lip}_0(X,X)$, we get the operator $\mathcal{G}_X\colon \mathcal{F}(X) \to X$ given by*

$$\mathcal{G}_X\Big(\sum_{x\in X} a_x\delta_x\Big) = \sum_{x\in X} a_x x,$$

which has norm 1 and satisfies $\mathcal{G}_X \circ \delta = \mathrm{Id}_X$.

(c) *The set*

$$\mathcal{B}_X := \left\{\frac{\delta_x - \delta_y}{\|x-y\|} : x,y \in X,\, x \neq y\right\} \subseteq \mathcal{F}(X)$$

is norming for $\mathcal{F}(X)^* = \mathrm{Lip}_0(X,\mathbb{K})$, i.e. $B_{\mathcal{F}(X)} = \overline{\mathrm{aconv}}(\mathcal{B}_X)$.

Our result for Lipschitz numerical ranges is the following.

Theorem 5.2. *Let X be a Banach space. Then*

$$\overline{\mathrm{conv}}(W_L(F)) = V(\mathcal{L}(\mathcal{F}(X),X),\mathcal{G}_X,T_F) = V(\mathrm{Lip}_0(X,X),\mathrm{Id},F)$$

for every $F \in \mathrm{Lip}_0(X,X)$.

The result will be a consequence of two lemmas. The first one follows directly from Proposition 2.14, as the set

$$C = \left\{x^* \otimes \frac{\delta_x - \delta_y}{\|x-y\|} : x,y \in X,\, x \neq y,\, x^* \in S_{X^*}\right\} \subseteq \mathcal{L}(\mathcal{F}(X),X)^*$$

satisfies $B_{\mathcal{L}(\mathcal{F}(X),X)^*} = \overline{\mathrm{conv}}^{w^*}(C)$ by Lemma 5.1(c).

Lemma 5.3. *Let X be a Banach space. Then*

$$V(\mathcal{L}(\mathcal{F}(X),X),\mathcal{G}_X,T) = \\ \overline{\mathrm{conv}\bigcap_{\delta>0}\left\{\frac{x^*(T(\delta_x-\delta_y))}{\|x-y\|} : x,y \in X,\, x\neq y,\, x^*\in S_{X^*},\, \mathrm{Re}\,\frac{x^*(\mathcal{G}_X(\delta_x-\delta_y))}{\|x-y\|} > 1-\delta\right\}}$$

for every $T \in \mathcal{L}(\mathcal{F}(X),X)$. Equivalently,

$$V(\mathrm{Lip}_0(X,X),\mathrm{Id}_X,F) = \\ \overline{\mathrm{conv}\bigcap_{\delta>0}\left\{\frac{x^*(F(x)-F(y))}{\|x-y\|} : x,y \in X,\, x\neq y,\, x^*\in S_{X^*},\, \mathrm{Re}\,\frac{x^*(x-y)}{\|x-y\|} > 1-\delta\right\}}$$

for every $F \in \mathrm{Lip}_0(X,X)$.

The second preliminary result follows from the Bishop–Phelps–Bollobás theorem.

Lemma 5.4. *Let X be a Banach space. Then*

$$\overline{W_L(F)} = \bigcap_{\delta>0}\overline{\left\{\frac{x^*(F(x)-F(y))}{\|x-y\|} : x,y \in X,\, x\neq y,\, x^*\in S_{X^*},\, \mathrm{Re}\,\frac{x^*(x-y)}{\|x-y\|} > 1-\delta\right\}}$$

for every $F \in \mathrm{Lip}_0(X,X)$.

Proof. The inclusion "$\subseteq$" is obvious, so let us prove the reverse one. Fix F in $\mathrm{Lip}_0(X,X)$. For every $\delta>0$, write

$$W_\delta := \left\{ \frac{x^*(F(x)-F(y))}{\|x-y\|} : x,y\in X,\ x\neq y,\ x^*\in S_{X^*},\ \mathrm{Re}\,\frac{x^*(x-y)}{\|x-y\|} > 1-\delta \right\}.$$

It is enough to show that for every $\varepsilon>0$, there is $\delta>0$ such that $W_\delta\subseteq W_L(F)+\varepsilon B_{\mathbb{K}}$. So let us fix $0<\varepsilon<1$ and consider $\delta>0$ such that $2\|F\|_L\sqrt{2\delta}<\varepsilon$. Given $x,y\in X$ with $x\neq y$ and $x^*\in S_{X^*}$ satisfying

$$\mathrm{Re}\, x^*\left(\frac{x-y}{\|x-y\|}\right) > 1-\delta,$$

we can use the Bishop–Phelps–Bollobás theorem (see [13, Corollary 2.4] for this version) to find $u\in S_X$, $0<\rho<\sqrt{2\delta}$ and $z^*\in S_{X^*}$ such that

$$z^*\left(\frac{x-y}{\|x-y\|}+\rho u\right) = \left\|\frac{x-y}{\|x-y\|}+\rho u\right\| = 1 \quad\text{and}\quad \|x^*-z^*\|<\sqrt{2\delta}.$$

Write $x' := x+\rho\|x-y\|u$ and $y' := y$, and observe that

$$\|x'-y'\| = \big\|x-y+\rho\|x-y\|u\big\| = \|x-y\|\left\|\frac{x-y}{\|x-y\|}+\rho u\right\| = \|x-y\|$$

and so

$$z^*\left(\frac{x'-y'}{\|x'-y'\|}\right) = z^*\left(\frac{x-y}{\|x-y\|}+\rho u\right) = 1.$$

Therefore, $\frac{z^*(F(x')-F(y'))}{\|x'-y'\|}\in W_L(F)$. Moreover,

$$\begin{aligned}
&\left\|\frac{z^*(F(x')-F(y'))}{\|x'-y'\|} - \frac{x^*(F(x)-F(y))}{\|x-y\|}\right\| \\
&\qquad\leq \left\|\frac{F(x')-F(y')}{\|x'-y'\|} - \frac{F(x)-F(y)}{\|x-y\|}\right\| + \left\|[z^*-x^*]\frac{F(x)-F(y)}{\|x-y\|}\right\| \\
&\qquad\leq \left\|\frac{F(x')-F(y')-F(x)+F(y)}{\|x-y\|}\right\| + \|F\|_L\,\|z^*-x^*\| \\
&\qquad< \|F\|_L\left\|\frac{x'-x}{\|x-y\|}\right\| + \|F\|_L\sqrt{2\delta} = \|F\|_L(\rho+\sqrt{2\delta}) < 2\|F\|_L\sqrt{2\delta} < \varepsilon.
\end{aligned}$$

We have shown $\frac{x^*(F(x)-F(y))}{\|x-y\|}\in W_L(F)+\varepsilon B_{\mathbb{K}}$, so $W_\delta\subseteq W_L(F)+\varepsilon B_{\mathbb{K}}$ as desired. ■

6. Some stability results

In this chapter we collect some results which show the behaviour of the value of the numerical index when we apply some Banach space operations. We have divided the chapter into several subsections.

6.1. Diagonal operators. The next result allows us to calculate the numerical index with respect to a diagonal operator between c_0-, ℓ_1- and ℓ_∞-sums of Banach spaces.

PROPOSITION 6.1. *Let $\{X_\lambda\colon \lambda\in\Lambda\}$, $\{Y_\lambda\colon \lambda\in\Lambda\}$ be two families of Banach spaces and let $G_\lambda\in\mathcal{L}(X_\lambda,Y_\lambda)$ be a norm-one operator for every $\lambda\in\Lambda$. Let E be one of the Banach spaces c_0, ℓ_∞, or ℓ_1, let $X=[\bigoplus_{\lambda\in\Lambda}X_\lambda]_E$ and $Y=[\bigoplus_{\lambda\in\Lambda}Y_\lambda]_E$, and define the operator $G\colon X\to Y$ by*

$$G[(x_\lambda)_{\lambda\in\Lambda}]=(G_\lambda x_\lambda)_{\lambda\in\Lambda}$$

for every $(x_\lambda)_{\lambda\in\Lambda}\in[\bigoplus_{\lambda\in\Lambda}X_\lambda]_E$. Then

$$n_G(X,Y)=\inf_\lambda n_{G_\lambda}(X_\lambda,Y_\lambda).$$

Proof. We follow the lines of [37, proof of Proposition 1]. Given $\kappa\in\Lambda$, we first have to show $n_G(X,Y)\le n_{G_\kappa}(X_\kappa,Y_\kappa)$. Observe that setting $W=[\bigoplus_{\lambda\neq\kappa}X_\lambda]_E$ and $Z=[\bigoplus_{\lambda\neq\kappa}Y_\lambda]_E$, we can write $X=X_\kappa\oplus_\infty W$ and $Y=Y_\kappa\oplus_\infty Z$ when E is ℓ_∞ or c_0 and $X=X_\kappa\oplus_1 W$ and $Y=Y_\kappa\oplus_1 Z$ when E is ℓ_1. Given $S\in\mathcal{L}(X_\kappa,Y_\kappa)$, define $T\in\mathcal{L}(X,Y)$ by

$$T(x_\kappa,w)=(Sx_\kappa,0)\qquad(x_\kappa\in X_\kappa,\ w\in W)$$

which obviously satisfies $\|T\|=\|S\|$. We claim $v_G(T)=v_{G_\kappa}(S)$. In order to obtain $v_G(T)\le v_{G_\kappa}(S)$, given $\delta>0$, we may suppose $v_{G,\delta}(T)>0$. For our goal, it is sufficient to prove $v_{G,\delta}(T)\le v_{G_\kappa,\hat\delta}(S)$ where $\hat\delta=2\delta/v_{G,\delta}(T)$. For every $0<\varepsilon<v_{G,\delta}(T)/2$, we may find $x=(x_\kappa,w)\in S_X$ and $y^*=(y^*_\kappa,z^*)\in S_{Y^*}$ such that $|y^*(Tx)|>v_{G,\delta}(T)-\varepsilon>v_{G,\delta}(T)/2$ and

$$1-\delta<\operatorname{Re}y^*(Gx)\le\operatorname{Re}y^*_\kappa(G_\kappa x_\kappa)+\|z^*\|\,\|w\|.$$

Moreover,

$$\|y^*_\kappa\|\,\|x_\kappa\|+\|z^*\|\,\|w\|\le\|y^*\|\,\|x\|=1.$$

Consequently,

$$\|y^*_\kappa\|\,\|x_\kappa\|+\|z^*\|\,\|w\|-\delta\le1-\delta<\operatorname{Re}y^*_\kappa(G_\kappa x_\kappa)+\|z^*\|\,\|w\|$$

and so $\operatorname{Re}y^*_\kappa(G_\kappa x_\kappa)>\|y^*_\kappa\|\,\|x_\kappa\|-\delta$. Since

$$\frac{v_{G,\delta}(T)}{2}<|y^*(Tx)|=|y^*_\kappa(Sx_\kappa)|\le\|y^*_\kappa\|\,\|x_\kappa\|,$$

we deduce

$$\operatorname{Re}\frac{y_\kappa^*}{\|y_\kappa^*\|}\left(G_1\frac{x_\kappa}{\|x_\kappa\|}\right) > 1-\frac{\delta}{\|y_\kappa^*\|\,\|x_\kappa\|} > 1-\hat{\delta}.$$

Then

$$v_{G,\delta}(T)-\varepsilon < |y^*(Tx)| = |y_\kappa^*(Sx_\kappa)| \leq \left|\frac{y_\kappa^*}{\|y_\kappa\|}\left(S\frac{x_\kappa}{\|x_\kappa\|}\right)\right| \leq v_{G_\kappa,\hat{\delta}}(S),$$

and hence $v_G(T)\leq v_{G_\kappa}(S)$.

To prove the reverse inequality, we fix $\delta>0$ and $x_\kappa\in S_{X_\kappa}$, $y_\kappa^*\in S_{Y_\kappa^*}$ satisfying $\operatorname{Re} y_\kappa^*(G_\kappa x_\kappa) > 1-\delta$, and define $x=(x_\kappa,0)\in S_X$ and $y^*=(y_\kappa^*,0)\in S_{Y^*}$. We clearly have

$$\operatorname{Re} y^*(Gx) > 1-\delta \quad\text{and}\quad |y_\kappa^*(Sx_\kappa)| = |y^*(Tx)| \leq v_{G,\delta}(T).$$

Consequently, $v_{G_\kappa,\delta}(S)\leq v_{G,\delta}(T)$ and the claim follows by letting $\delta\downarrow 0$.

To sum up, we have proved that given $S\in\mathcal{L}(X_\kappa,Y_\kappa)$ there is $T\in\mathcal{L}(X,Y)$ with $\|T\|=\|S\|$ and $v_G(T)=v_{G_\kappa}(S)$; consequently,

$$n_G(X,Y)\|S\| = n_G(X,Y)\|T\| \leq v_G(T) = v_{G_\kappa}(S)$$

and the arbitrariness of $S\in\mathcal{L}(X_\kappa,Y_\kappa)$ gives $n_G(X,Y)\leq n_{G_\kappa}(X_\kappa,Y_\kappa)$.

We now prove the reverse inequalities when E is c_0 or ℓ_∞. In both cases, an operator $T\in\mathcal{L}(X,Y)$ can be seen as a family $(T_\lambda)_{\lambda\in\Lambda}$, where $T_\lambda\in\mathcal{L}(X,Y_\lambda)$ for every λ, and $\|T\|=\sup\{\|T_\lambda\|\colon \lambda\in\Lambda\}$. Given $\varepsilon>0$, we may find $\kappa\in\Lambda$ such that $\|T_\kappa\|>\|T\|-\varepsilon$, and write $X=X_\kappa\oplus_\infty W$ where $W=[\bigoplus_{\lambda\neq\kappa}X_\lambda]_E$. Since B_X is the convex hull of $S_{X_\kappa}\times S_W$, we may find $x_0\in S_{X_\kappa}$ and $w_0\in S_W$ such that

$$\|T_\kappa(x_0,w_0)\| > \|T\|-\varepsilon.$$

Now, fix $x_0^*\in S_{X_\kappa^*}$ with $x_0^*(x_0)=1$ and define the operator $S\in\mathcal{L}(X_\kappa,Y_\kappa)$ by

$$Sx = T_\kappa(x,0)+x_0^*(x)\,T_\kappa(0,w_0) = T_\kappa(x,x_0^*(x)w_0) \qquad (x\in X_\kappa)$$

which satisfies

$$\|S\|\geq\|Sx_0\| = \|T_\kappa(x_0,x_0^*(x_0)w_0)\| = \|T_\kappa(x_0,w_0)\| > \|T\|-\varepsilon.$$

Given $\delta>0$, we claim that $v_{G_\kappa,\delta}(S)\leq v_{G,\delta}(T)$. Indeed, we may find $u\in S_{X_\kappa}$ and $v^*\in S_{Y_\kappa^*}$ with $\operatorname{Re} v^*(G_{\lambda_0}u) > 1-\delta$. Now, we write

$$x=(u,x_0^*(u)w_0)\in S_X, \qquad y^*=(v^*,0)\in S_{Y^*}$$

which satisfy $\operatorname{Re} y^*(Gx)=\operatorname{Re} v^*(G_\kappa u) > 1-\delta$, hence

$$|v^*(Su)| = |v^*[T_\kappa(u,x_0^*(u)w_0)]| = |y^*(Tx)| \leq v_{G,\delta}(T).$$

Then we deduce $v_{G_\kappa,\delta}(S)\leq v_{G,\delta}(T)$. From this, we get

$$v_G(T)\geq v_{G_\kappa}(S)\geq n_{G_\kappa}(X_\kappa,Y_\kappa)\|S\| \geq n_{G_\kappa}(X_\kappa,Y_\kappa)[\|T\|-\varepsilon].$$

Therefore,

$$v_G(T)\geq \inf_\lambda n_{G_\lambda}(X_\lambda,Y_\lambda)\|T\|$$

and so $n_G(X,Y)\geq \inf_\lambda n_{G_\lambda}(X_\lambda,Y_\lambda)$, as required.

Suppose now $E=\ell_1$. In this case, we can write every operator $T\in\mathcal{L}(X,Y)$ as a family $(T_\lambda)_{\lambda\in\Lambda}$ of operators where $T_\lambda\in\mathcal{L}(X_\lambda,Y)$ for every $\lambda\in\Lambda$, and satisfying

$\|T\| = \sup\{\|T_\lambda\| \colon \lambda \in \Lambda\}$. Given $\varepsilon > 0$, find $\kappa \in \Lambda$ such that $\|T_\kappa\| > \|T\| - \varepsilon$, and write $X = X_\kappa \oplus_1 W$, $Y = Y_\kappa \oplus_1 Z$, and $T_\kappa = (A, B)$ where $W = [\bigoplus_{\lambda\neq\kappa} X_\lambda]_{\ell_1}$, $Z = [\bigoplus_{\lambda\neq\kappa} Y_\lambda]_{\ell_1}$, $A \in \mathcal{L}(X_\kappa, Y_\kappa)$ and $B \in \mathcal{L}(X_\kappa, Z)$. Now, we choose $x_0 \in S_{X_\kappa}$ such that

$$\|T_\kappa x_0\| = \|Ax_0\| + \|Bx_0\| > \|T\| - \varepsilon,$$

we find $a_0 \in S_{Y_\kappa^*}$ and $z^* \in S_{Z^*}$ satisfying

$$Ax_0 = \|Ax_0\|a_0 \quad \text{and} \quad z^*(Bx_0) = \|Bx_0\|,$$

and define an operator $S \in \mathcal{L}(X_\kappa, Y_\kappa)$ by

$$Sx = Ax + [z^*(Bx)]a_0 \qquad (x \in X_\kappa).$$

Then

$$\|S\| \geq \|Sx_0\| = \|Ax_0 + z^*(Bx_0)a_0\| = \|Ax_0\| + \|Bx_0\| > \|T\| - \varepsilon.$$

Given $\delta > 0$, we prove $v_{G_\kappa,\delta}(S) \leq v_{G,\delta}(T)$. To do so, given $u \in S_{X_\kappa}$ and $v^* \in S_{Y_\kappa^*}$ with $\operatorname{Re} v^*(G_\kappa u) > 1 - \delta$, we define

$$x = (u, x_0^*(u)w_0) \in S_X \quad \text{and} \quad y^* = (v^*, 0) \in S_{Y^*}.$$

Since $\operatorname{Re} y^*(Gx) = \operatorname{Re} v^*(G_\kappa u) > 1 - \delta$, we get

$$|v^*(Su)| = |v^*(Au) + v^*(a_0)z^*(Bu)| = |y^*(T_\kappa u)| = |y^*(Tx)| \leq v_{G,\delta}(T),$$

which gives $v_{G_\kappa,\delta}(S) \leq v_{G,\delta}(T)$ thanks to the arbitrariness of u and v^*. Finally, we can write

$$v_G(T) \geq v_{G_\kappa}(S) \geq n_{G_\kappa}(X_\kappa, Y_\kappa)\|S\| \geq n_{G_\kappa}(X_\kappa, Y_\kappa)[\|T\| - \varepsilon]$$

and so we deduce $v_G(T) \geq \inf_\lambda n_{G_\lambda}(X_\lambda, Y_\lambda)\|T\|$, from which the desired inequality $n_G(X, Y) \geq \inf_\lambda n_{G_\lambda}(X_\lambda, Y_\lambda)$ follows. ∎

Let us observe that the first part of the above proof is valid for general absolute sums.

Proposition 6.2. *Let X_1, X_2, Y_1, Y_2 be Banach spaces and let E be $\mathbb{R}^2$ endowed with an absolute norm. Given norm-one operators $G_i \in \mathcal{L}(X_i, Y_i)$ for $i = 1, 2$, define $G \in \mathcal{L}(X_1 \oplus_E X_2, Y_1 \oplus_E Y_2)$ by*

$$G(x_1, x_2) = (G_1x_1, G_2x_2) \in Y_1 \oplus_E Y_2$$

for every $(x_1, x_2) \in X_1 \oplus_E X_2$. Then

$$n_G(X_1 \oplus_E X_2, Y_1 \oplus_E Y_2) \leq \min\{n_{G_1}(X_1, Y_1), n_{G_2}(X_2, Y_2)\}.$$

The associativity of ℓ_p-sums allows us to get the following result from the above one.

Corollary 6.3. *Let $\{X_\lambda \colon \lambda \in \Lambda\}$, $\{Y_\lambda \colon \lambda \in \Lambda\}$ be two families of Banach spaces, let $G_\lambda \in \mathcal{L}(X_\lambda, Y_\lambda)$ be a norm-one operator for every $\lambda \in \Lambda$, let $1 < p < \infty$, and let $X = [\bigoplus_{\lambda\in\Lambda} X_\lambda]_{\ell_p}$ and $Y = [\bigoplus_{\lambda\in\Lambda} Y_\lambda]_{\ell_p}$. Define the operator $G \colon X \to Y$ by*

$$G[(x_\lambda)_{\lambda\in\Lambda}] = (G_\lambda x_\lambda)_{\lambda\in\Lambda}$$

for every $(x_\lambda)_{\lambda\in\Lambda} \in [\bigoplus_{\lambda\in\Lambda} X_\lambda]_{\ell_p}$. Then

$$n_G(X, Y) \leq \inf_\lambda n_{G_\lambda}(X_\lambda, Y_\lambda).$$

The main application of Proposition 6.1 is the following important example.

THEOREM 6.4. *In both the real and the complex case, there exist Banach spaces X such that*

$$\mathcal{N}(\mathcal{L}(X)) = [0,1].$$

The proof will follow immediately from Proposition 6.1 and the next example.

EXAMPLE 6.5. *For every $\gamma \in [0,1]$ there exist a real or complex Banach space Y_γ and norm-one operators $G_{\gamma,1}, G_{\gamma,2} \in \mathcal{L}(Y_\gamma)$ with $n_{G_{\gamma,1}}(Y_\gamma, Y_\gamma) = \gamma$ and $n_{G_{\gamma,2}}(Y_\gamma, Y_\gamma) = 1$.*

Proof. We start by showing the existence of a real or complex space Z_γ such that there exists a norm-one operator $G \in \mathcal{L}(Z_\gamma)$ satisfying $n_G(Z_\gamma, Z_\gamma) = \gamma$. For $\gamma \in [1/2, 1]$, it is enough to use the fact that the set $\{n(W) \colon W \text{ two-dimensional space}\}$ covers the interval $[0,1]$ in the real case and $[1/\mathrm{e}, 1]$ in the complex case [16, Theorems 3.5 and 3.6]. So, for $\gamma \in [1/2, 1]$ there is a two-dimensional (real or complex) space Z_γ satisfying $n(Z_\gamma) = \gamma$ and it suffices to take $G = \mathrm{Id}_{Z_\gamma}$.

For $\gamma \in [0, 1/2]$, let $X_\gamma = \mathbb{K}^2$ endowed with the norm

$$\|(x_1, x_2)\|_\gamma = \max\{|x_2|, |x_1| + (1-\gamma)|x_2|\} \qquad ((x_1, x_2) \in \mathbb{K}^2),$$

let $Z = \ell_\infty^2$, and let $Z_\gamma = X_\gamma \oplus_\infty Z$. Take $x_0^* = (0,1) \in S_{X_\gamma^*}$, $z_0 = (1,1) \in S_Z$, $x_0 = (1,0) \in S_{X_\gamma}$, $z_0^* = (1,0) \in S_{Z^*}$, and define $J_1 = x_0^* \otimes z_0$, $J_2 = z_0^* \otimes x_0$, and $G = (J_1, J_2)$. Let us prove the equality $n_G(Z_\gamma, Z_\gamma) = \gamma$.

Observe first that X_γ^* is $\mathbb{K}^2$ endowed with the norm

$$\|(x_1^*, x_2^*)\| = \max\{|x_1^*|, \gamma|x_1^*| + |x_2^*|\} \qquad ((x_1, x_2) \in \mathbb{K}^2).$$

Since $\|J_1\| = \|J_2\| = 1$ and $z_0 \in Z$, $z_0^* \in Z^*$ are spear vectors, by Propositions 6.1 and 3.6 we have

$$\begin{aligned} n_G(Z_\gamma, Z_\gamma) &= \min\{n_{J_1}(X_\gamma, Z), n_{J_2}(Z, X_\gamma)\} \\ &= \min\{n(X_\gamma^*, x_0^*) n(Z, z_0), n(Z^*, z_0^*) n(X_\gamma, x_0)\} = \min\{n(X_\gamma^*, x_0^*), n(X_\gamma, x_0)\}. \end{aligned}$$

So it suffices to show $n(X_\gamma^*, x_0^*) = \gamma$ and $n(X_\gamma, x_0) \geq 1 - \gamma$. To do so, we fix $x^* = (x_1^*, x_2^*) \in S_{X_\gamma^*}$ and we compute $v(X_\gamma^*, x_0^*, x^*)$. The points $x \in S_{X_\gamma}$ satisfying $x_0^*(x) = 1$ are of the form $x = (t\theta, 1)$ with $t \in [0, \gamma]$ and $\theta \in \mathbb{T}$. Thus we have

$$v(X_\gamma^*, x_0^*, x^*) = \sup\{|t\theta x_1^* + x_2^*| \colon t \in [0,\gamma], \theta \in \mathbb{T}\} = \gamma|x_1^*| + |x_2^*| \geq \gamma\|x^*\|,$$

which implies $n(X_\gamma^*, x_0^*) \geq \gamma$. Finally, $v(X_\gamma^*, x_0^*, x^*) = \gamma$ for $x^* = (1,0) \in S_{X_\gamma^*}$ and so $n(X_\gamma^*, x_0^*) = \gamma$ as desired.

To prove $n(X_\gamma, x_0) \geq 1 - \gamma$, we fix $x = (x_1, x_2) \in S_{X_\gamma}$ and we estimate $v(X_\gamma, x_0, x)$. The points $x^* \in S_{X_\gamma^*}$ satisfying $x^*(x_0) = 1$ are of the form $x^* = (1, t\theta)$ with $t \in [0, 1-\gamma]$ and $\theta \in \mathbb{T}$. Thus we have

$$v(X_\gamma, x_0, x) = \sup\{|x_1 + t\theta x_2| \colon t \in [0, 1-\gamma],\ \theta \in \mathbb{T}\} = |x_1| + (1-\gamma)|x_2| \geq (1-\gamma)\|x\|,$$

which implies $n(X_\gamma, x_0) \geq 1 - \gamma$. This finishes the proof of the existence of Z_γ.

Now, for each $\gamma \in [0,1]$, we take $Y_\gamma = (Z_\gamma \oplus_\infty \mathbb{K}) \oplus_1 \mathbb{K}$. On the one hand, define $G_{\gamma,1} \in \mathcal{L}(Y_\gamma)$ by

$$G_{\gamma,1}(z,\alpha,\beta) = (Gz,\alpha,\beta) \quad (z \in Z_\gamma, \alpha,\beta \in \mathbb{K}),$$

which satisfies $n_{G_{\gamma,1}}(Y_\gamma, Y_\gamma) = n_G(Z_\gamma, Z_\gamma) = \gamma$ by Proposition 6.1. On the other hand, observe that $Y_\gamma^* = (Z_\gamma^* \oplus_1 \mathbb{K}) \oplus_\infty \mathbb{K}$, so the elements $y = (0,0,1) \in S_{Y_\gamma}$ and $y^* = (0,1,1) \in S_{Y_\gamma^*}$ are spear vectors in Y_γ and Y_γ^* respectively. Therefore, the norm-one operator $G_{\gamma,2} = y^* \otimes y \in \mathcal{L}(Y_\gamma)$ satisfies $n_{G_{\gamma,2}}(Y_\gamma, Y_\gamma) = 1$ by Proposition 3.6. ∎

We are now able to provide the pending proof.

Proof of Theorem 6.4. For each $\gamma \in [0,1]$, consider the space Y_γ given in Example 6.5 and consider the norm-one operators $G_{\gamma,1}, G_{\gamma,2} \in \mathcal{L}(Y_\gamma)$ satisfying $n_{G_{\gamma,1}}(Y_\gamma, Y_\gamma) = \gamma$ and $n_{G_{\gamma,2}}(Y_\gamma, Y_\gamma) = 1$. Now, let $X = [\bigoplus_{\gamma\in[0,1]} Y_\gamma]_{c_0}$, and for every $\xi \in [0,1]$, consider the norm-one operator $G^\xi \in \mathcal{L}(X)$ to be the diagonal operator given by $[G^\xi]_\gamma = G_{\gamma,2}$ if $\gamma \neq \xi$ and $[G^\xi]_\xi = G_{\xi,1}$. By Proposition 6.1, $n_{G^\xi}(X,X) = \xi$, finishing the proof. ∎

6.2. Composition operators on vector-valued function spaces. The first result here gives the numerical index with respect to composition operators between spaces of vector-valued continuous functions.

PROPOSITION 6.6. *Let X, Y be Banach spaces, let K be a compact Hausdorff topological space and $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Consider the norm-one composition operator $\widetilde{G}\colon C(K,X) \to C(K,Y)$ given by $\widetilde{G}(f) = G \circ f$ for every $f \in C(K,X)$. Then*

$$n_{\widetilde{G}}(C(K,X), C(K,Y)) = n_G(X,Y).$$

Proof. We follow the lines of [37, proof of Theorem 5]. To show

$$n_{\widetilde{G}}(C(K,X), C(K,Y)) \geq n_G(X,Y),$$

we fix $T \in \mathcal{L}(C(K,X), C(K,Y))$ with $\|T\| = 1$ and prove the inequality $v_{\widetilde{G}}(T) \geq n_G(X,Y)$. Given $\varepsilon > 0$, we may find $f_0 \in C(K,X)$ with $\|f_0\| = 1$ and $t_0 \in K$ such that

$$\|[Tf_0](t_0)\| > 1 - \varepsilon. \tag{6.1}$$

Define $z_0 = f_0(t_0)$ and find a continuous function $\varphi\colon K \to [0,1]$ such that $\varphi(t_0) = 1$ and $\varphi(t) = 0$ if $\|f_0(t) - z_0\| \geq \varepsilon$. Now write $z_0 = (1-\lambda)x_1 + \lambda x_2$ with $0 \leq \lambda \leq 1$, $x_1, x_2 \in S_X$, and consider the functions

$$f_j = (1-\varphi)f_0 + \varphi x_j \in C(K,X) \quad (j = 1,2).$$

Then $\|\varphi f_0 - \varphi z_0\| < \varepsilon$ meaning that

$$\|f_0 - ((1-\lambda)f_1 + \lambda f_2)\| < \varepsilon,$$

and, by (6.1), we must have

$$\|[Tf_1](t_0)\| > 1 - 2\varepsilon \quad \text{or} \quad \|[Tf_2](t_0)\| > 1 - 2\varepsilon.$$

By making the right choice of $x_0 = x_1$ or $x_0 = x_2$, we get $x_0 \in S_X$ such that

$$\|[T((1-\varphi)f_0 + \varphi x_0)](t_0)\| > 1 - 2\varepsilon. \tag{6.2}$$

Next, we fix $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = 1$, denote

$$\Phi(x) = x_0^*(x)(1-\varphi)f_0 + \varphi x \in C(K,X) \quad (x \in X),$$

and consider the operator $S \in \mathcal{L}(X,Y)$ given by

$$Sx = [T(\Phi(x))](t_0) \quad (x \in X)$$

which, by (6.2), obviously satisfies $\|S\| \geq \|Sx_0\| > 1 - 2\varepsilon$.

Now, given $\delta > 0$, and $x \in S_X$, $y^* \in S_{Y^*}$ such that $\operatorname{Re} y^*(Gx) > 1 - \delta$, we define $f \in S_{C(K,X)}$ by $f = \Phi(x)$, and consider the functional $g^* \in S_{C(K,Y)^*}$ given by

$$g^*(h) = [y^* \otimes \delta_{t_0}](h) = y^*(h(t_0)) \quad (h \in C(K,Y)).$$

Since $f(t_0) = x$, we have $\operatorname{Re} g^*(\widetilde{G}f) > 1 - \delta$ and

$$|y^*(Sx)| = \left|y^*([T(\Phi(x))](t_0))\right| = |g^*(Tf)| \leq v_{\widetilde{G},\delta}(T),$$

hence $v_{G,\delta}(S) \leq v_{\widetilde{G},\delta}(T)$. Therefore,

$$v_{\widetilde{G}}(T) \geq v_G(S) \geq n_G(X,Y)\|S\| \geq (1-2\varepsilon)n_G(X,Y),$$

and the arbitrariness of $\varepsilon > 0$ gives $v_{\widetilde{G}}(T) \geq n_G(X,Y)$, as desired.

To prove the reverse inequality, we take an operator $S \in \mathcal{L}(X,Y)$ and define the operator $T \in \mathcal{L}(C(K,X), C(K,Y))$ by

$$[T(f)](t) = S(f(t)) \quad (t \in K,\ f \in C(K,X)).$$

Clearly, $\|T\| = \|S\|$. To estimate the value of $v_{\widetilde{G}}(T)$ we use Lemma 3.4 considering $A = S_{C(K,X)}$ and $B = \{y^* \otimes \delta_t \colon y^* \in S_{Y^*},\ t \in K\}$, where $(y^* \otimes \delta_t)(g) = y^*(g(t))$ for every $g \in C(K,Y)$ (as these subsets satisfy $\overline{\operatorname{conv}}(A) = B_{C(K,X)}$ and $\overline{\operatorname{conv}}^{w^*}(B) = B_{C(K,Y)^*}$). Now, for every $\delta > 0$, $f \in S_{C(K,X)}$, $t \in K$, and $y^* \in S_{Y^*}$ satisfying $\operatorname{Re} y^*(G(f(t))) > 1-\delta$, we set $x = f(t) \in S_X$ and observe that $\operatorname{Re} y^*(Gx) > 1 - \delta$ and

$$|y^*([Tf](t))| = |y^*(S(f(t)))| = |y^*(Sx)| \leq v_{G,\delta}(S).$$

Consequently, $v_{\widetilde{G},\delta}(T) \leq v_{G,\delta}(S)$ and

$$v_G(S) \geq v_{\widetilde{G}}(T) \geq n_{\widetilde{G}}(C(K,X), C(K,Y))\|T\| = n_{\widetilde{G}}(C(K,X), C(K,Y))\|S\|,$$

so $n_G(X,Y) \geq n_{\widetilde{G}}(C(K,X), C(K,Y))$, as desired. ∎

We next deal with Köthe–Bochner vector-valued function spaces, for which we need to introduce some terminology.

Let (Ω, Σ, μ) be a complete σ-finite measure space. We denote by $L_0(\mu)$ the vector space of all (equivalent classes modulo equality a.e. of) Σ-measurable locally integrable real-valued functions on Ω. A *Köthe function space* is a linear subspace E of $L_0(\mu)$ endowed with a complete norm $\|\cdot\|_E$ satisfying the following conditions:

(i) If $|f| \leq |g|$ a.e. on Ω, $g \in E$ and $f \in L_0(\mu)$, then $f \in E$ and $\|f\|_E \leq \|g\|_E$.
(ii) For every $A \in \Sigma$ with $0 < \mu(A) < \infty$, the characteristic function $\mathbb{1}_A$ belongs to E.

We refer the reader to the classical book by J. Lindenstrauss and L. Tzafriri [31] for more information and background on Köthe function spaces. Let us recall some useful facts about these spaces. First, E is a Banach lattice in the pointwise order. The *Köthe dual* E' of E is the function space defined as

$$E' = \left\{ g \in L_0(\mu) \colon \|g\|_{E'} := \sup_{f \in B_E} \int_\Omega |fg| \, d\mu < \infty \right\},$$

which is again a Köthe space on (Ω, Σ, μ). Every element $g \in E'$ defines naturally a continuous linear functional on E by the formula

$$f \mapsto \int_\Omega fg \, d\mu \qquad (f \in E),$$

so we have $E' \subseteq E^*$ and this inclusion is isometric.

Let E be a Köthe space on a complete σ-finite measure space (Ω, Σ, μ) and let X be a real or complex Banach space. A function $f \colon \Omega \to X$ is said to be *simple* if $f = \sum_{i=1}^n x_i \mathbb{1}_{A_i}$ for some $x_1, \dots, x_n \in X$ and some $A_1, \dots, A_n \in \Sigma$. The function f is said to be *strongly measurable* if there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ of simple functions such that $\lim \|f_n(t) - f(t)\|_X = 0$ for almost all $t \in \Omega$. Given a strongly measurable function $f \colon \Omega \to X$ we use the notation $|f|$ for the function $|f|(\cdot) = \|f(\cdot)\|_X$. We write $E(X)$ to denote the space of (classes of) strongly measurable functions $f \colon \Omega \to X$ such that $|f| \in E$ and we endow $E(X)$ with the norm

$$\|f\|_{E(X)} = \big\||f|\big\|_E.$$

Then $E(X)$ is a real or complex (depending on X) Banach space and it is called a *Köthe–Bochner function space*. We refer the reader to the book [30] for background. For an element $f \in E(X)$ we consider a strongly measurable function $\widetilde{f} \colon \Omega \to S_X$ such that $f = |f| \, \widetilde{f}$ a.e.

Our result for composition operators between Köthe–Bochner function spaces is the following inequality.

Proposition 6.7. *Let X, Y be Banach spaces, let (Ω, Σ, μ) be a σ-finite measure space, let E be a Köthe space on (Ω, Σ, μ) such that E' is norming for E, and let $G \in \mathcal{L}(X, Y)$ be a norm-one operator. Consider the norm-one composition operator $\widetilde{G} \colon E(X) \to E(Y)$ given by $\widetilde{G}(f) = G \circ f$ for every $f \in E(X)$. Then*

$$n_{\widetilde{G}}(E(X), E(Y)) \leq n_G(X, Y).$$

We need a preliminary lemma which is considered folklore in the theory of Köthe–Bochner spaces. As we have not found direct references, we will include a short sketch of its proof. Let us introduce some notation. Let E be a Köthe space on a σ-finite measure space (Ω, Σ, μ) and let Y be a Banach space. If $\Phi \colon \Omega \to Y^*$ belongs to $E'(Y^*)$, then the *integral* functional on $E(Y)$ defined by Φ is given by

$$\langle \Phi, f \rangle = \int_\Omega \langle \Phi(t), f(t) \rangle \, d\mu(t) \qquad (f \in E(Y)). \tag{6.3}$$

We keep the notations $|\Phi| = \|\Phi(\cdot)\|_{Y^*} \in E'$ and $\widetilde{\Phi} \colon \Omega \to S_{Y^*}$, which satisfy $\Phi = |\Phi| \widetilde{\Phi}$ a.e. It is possible to define integral functionals as in (6.3) for functions satisfying weaker requirements but, actually, here we are only interested in those integral functionals coming from functions Φ in $E'(Y^*)$ having countably many values.

LEMMA 6.8. *Let E be a Köthe space on a σ-finite measure space (Ω, Σ, μ) and let Y be a Banach space.*

(a) *The set of measurable functions from Ω to Y having countably many values is dense in $E(Y)$.*
(b) *If $\Phi \in E'(Y^*)$ has countably many values, then the integral functional defined as in (6.3) belongs to $E(Y)^*$ and satisfies $\|\Phi\|_{E(Y)^*} = \|\Phi\|_{E'(Y^*)} = \big\||\Phi|\big\|_{E'}$.*
(c) *If E' is norming for E, then the set $\mathcal{B}$ of all integral functionals defined by norm-one functions in $E'(Y^*)$ having countably many values satisfies $\overline{\mathrm{conv}}^{w^*}(\mathcal{B}) = B_{E(Y)^*}$.*

Sketch of the proof. (a) Fix $f \in E(Y)$ and $\varepsilon > 0$. We consider a partition of Ω into countably many pairwise disjoint measurable sets $\Omega = \bigcup_{n\in\mathbb{N}\cup\{0\}} \Omega_n$ with $\mu(\Omega_0) = 0$, $0 < \mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$, and such that $f(\Omega_n)$ is separable for all $n \in \mathbb{N}$. Now, for every $n \in \mathbb{N}$ we use the Bochner measurability of $f\mathbb{1}_{\Omega_n}$ to find a measurable function $g_n \colon \Omega \to Y$ with $g_n(\Omega \setminus \Omega_n) = \{0\}$, having countably many values and satisfying

$$\|f(t) - g_n(t)\| \leq \frac{\varepsilon}{2^n\|\mathbb{1}_{\Omega_n}\|} \qquad (t \in \Omega_n)$$

(see [15, Corollary 3, p. 42], for instance). We have

$$|f\mathbb{1}_{\Omega_n} - g_n| \leq \frac{\varepsilon\mathbb{1}_{\Omega_n}}{2^n\|\mathbb{1}_{\Omega_n}\|},$$

so $g_n \in E(Y)$ and $\|f\mathbb{1}_{\Omega_n} - g_n\| \leq \frac{\varepsilon}{2^n}$. It is now clear that the sum g of the (formal) series $\sum_{n\geq 1} g_n$ belongs to $E(Y)$, has countably many values, and satisfies $\|f - g\| \leq \varepsilon$.

(b) Our Φ is of the form

$$\Phi(t) = \sum_{n=1}^{\infty} y_n^*\mathbb{1}_{A_n}(t) \qquad (t \in \Omega)$$

for suitable sequences $\{y_n^*\}_{n\in\mathbb{N}}$ of elements of Y^* and $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint elements of Σ such that the scalar function

$$t \mapsto \sum_{n=1}^{\infty} \|y_n^*\|\mathbb{1}_{A_n}(t)$$

belongs to E'. Then the action of Φ on $E(Y)$ is given by

$$\langle \Phi, f\rangle = \int_\Omega \langle \Phi(t), f(t)\rangle\, d\mu(t) = \sum_{n=1}^{\infty} \int_{A_n} y_n^*(f(t))\, d\mu(t) \qquad (f \in E(Y)).$$

It is now routine to show $\Phi \in E(Y)^*$ and $\|\Phi\|_{E(Y)^*} = \big\||\Phi|\big\|_{E'}$.

Assertion (c) follows routinely from the density in $E(Y)$ of the set of countably-valued functions, from the fact that E' is norming for E, and from the density in E' of the set of countably-valued functions. ■

Proof of Proposition 6.7. We follow the lines of [36, proof of Theorem 4.1]. Take an operator $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$, and define $T \in \mathcal{L}(E(X), E(Y))$ by

$$[T(f)](t) = S(f(t)) = |f|(t)\, S(\widetilde{f}(t)) \qquad (t \in \Omega,\ f \in E(X)).$$

We claim that T is well defined and $\|T\| = 1$. Indeed, for $f \in E(X)$, $T(f)$ is strongly measurable and

$$\|[T(f)](t)\|_Y = |f|(t)\|S(\widetilde{f}(t))\| \leq |f|(t) \quad (t \in \Omega),$$

so $T(f) \in E(Y)$ with $\|T(f)\|_{E(Y)} \leq \big\||f|\big\|_E = \|f\|_E(X)$. This gives $\|T\| \leq 1$. Conversely, fix $A \in \Sigma$ with $0 < \mu(A) < \infty$ and for each $x \in S_X$ consider $f = \|\mathbb{1}_A\|_E^{-1} x \mathbb{1}_A \in S_{E(X)}$. Then $\|f\| = 1$ and

$$\|[T(f)](t)\|_Y = \frac{\mathbb{1}_A(t)\|S(x)\|_Y}{\|\mathbb{1}_A\|_E},$$

so

$$\|T\| \geq \|T(f)\|_{E(Y)} = \left\| \frac{\mathbb{1}_A \|S(x)\|_Y}{\|\mathbb{1}_A\|_E} \right\|_E \geq \|S(x)\|_Y.$$

Taking supremum over $x \in S_X$, we get $\|T\| \geq \|S\| = 1$ as desired.

Next, we fix $0 < \delta < 1$, $f = |f|\,\widetilde{f} \in S_{E(X)}$ and $\Phi = |\Phi|\widetilde{\Phi} \in \mathcal{B}$ satisfying the condition $\operatorname{Re}\langle \Phi, \widetilde{G}(f)\rangle > 1 - \delta$, where $\mathcal{B} \subset E(Y)^*$ is the set given in Lemma 6.8(c). Let $0 < \alpha < 1$ be such that $1 - \alpha = \sqrt{\delta}$ and write

$$\Omega_1 = \{t \in \Omega\colon \operatorname{Re}\langle \widetilde{\Phi}(t), G(\widetilde{f}(t))\rangle \leq \alpha\} \quad \text{and} \quad \Omega_2 = \{t \in \Omega\colon \operatorname{Re}\langle \widetilde{\Phi}(t), G(\widetilde{f}(t))\rangle > \alpha\}.$$

Then

$$\begin{aligned}
1 - \delta < \operatorname{Re}\langle \Phi, \widetilde{G}(f)\rangle &= \operatorname{Re} \int_\Omega |\Phi|(t)\,|f|(t)\,\langle \widetilde{\Phi}(t), G(\widetilde{f}(t))\rangle\,d\mu(t) \\
&= \operatorname{Re}\int_{\Omega_1} |\Phi|(t)|f|(t)\langle \widetilde{\Phi}(t), G(\widetilde{f}(t))\rangle\,d\mu(t) + \operatorname{Re}\int_{\Omega_2} |\Phi|(t)|f|(t)\langle \widetilde{\Phi}(t), G(\widetilde{f}(t))\rangle\,d\mu(t) \\
&\leq \alpha \int_{\Omega_1} |\Phi|(t)|f|(t)\,d\mu(t) + \int_{\Omega_2} |\Phi|(t)\,|f|(t)\,d\mu(t) \\
&\leq \alpha \int_{\Omega_1} |\Phi|(t), |f|(t)\,d\mu(t) + 1 - \int_{\Omega_1} |\Phi|(t)\,|f|(t)\,d\mu(t),
\end{aligned}$$

hence $\int_{\Omega_1} |\Phi|(t)|f|(t)\,d\mu(t) < \frac{\delta}{1-\alpha}$. Moreover,

$$\begin{aligned}
|\langle \Phi, Tf\rangle| &= \left| \int_\Omega |\Phi|(t)\,|f|(t)\,\langle \widetilde{\Phi}(t), S(\widetilde{f}(t))\rangle\,d\mu(t)\right| \\
&\leq \int_{\Omega_2} |\Phi|(t)|f|(t)\,v_{G,1-\alpha}(S)\,d\mu(t) + \int_{\Omega_1} |\Phi|(t)\,|f|(t)|\langle \widetilde{\Phi}(t), S(\widetilde{f}(t))\rangle|\,d\mu(t) \\
&\leq v_{G,1-\alpha}(S) + \frac{\delta}{1-\alpha} = v_{G,\sqrt{\delta}}(S) + \sqrt{\delta}.
\end{aligned}$$

Thus, we get $v_{\widetilde{G},\delta}(T) \leq v_{G,\sqrt{\delta}}(S) + \sqrt{\delta}$ by Lemmas 3.4 and 6.8(c). So, taking infimum over $0 < \delta < 1$, we obtain $n_{\widetilde{G}}(E(X), E(Y)) \leq v_{\widetilde{G}}(T) \leq v_G(S)$ and the desired inequality follows. ∎

For $G = \mathrm{Id}_X$, the above result improves [36, Theorem 4.1]:

Corollary 6.9 (Extension of [36, Theorem 4.1]). *Let X be a Banach space, let (Ω, Σ, μ) be a σ-finite measure space, and let E be a Köthe space on (Ω, Σ, μ) such that E' is norming for E. Then*

$$n(E(X)) \leq n(X).$$

There are Köthe spaces which do not satisfy the norming requirement of Proposition 6.7 (see [31, Remark 1, p. 30] for instance). We next present some particular cases in which the previous proposition applies. First, we deal with order continuous spaces. We say that a Köthe space E is *order continuous* if $0 \leq x_\alpha \downarrow 0$ and $x_\alpha \in E$ imply $\lim \|x_\alpha\| = 0$ (this is known to be equivalent to the fact that E does not contain an isomorphic copy of ℓ_∞). If E is order continuous, then $E' = E^*$ (see [30, p. 169] or [31, p. 29]).

Corollary 6.10. *Let X, Y be Banach spaces, let (Ω, Σ, μ) be a probability space, let E be an order continuous Köthe space on (Ω, Σ, μ), and let $G \in \mathcal{L}(X, Y)$ be a norm-one operator. Consider the norm-one composition operator $\widetilde{G}\colon E(X) \to E(Y)$ given by $\widetilde{G}(f) = G \circ f$ for every $f \in E(X)$. Then*

$$n_{\widetilde{G}}(E(X), E(Y)) \leq n_G(X, Y).$$

For $1 \leq p < \infty$, L_p-spaces over σ-finite measures are order continuous Köthe spaces; for $p = \infty$, this is no longer true, but $L_\infty(\mu)'$ is norming for $L_\infty(\mu)$ (see [31, Remark 1, p. 30] for instance). Therefore, we get the following consequence:

Corollary 6.11. *Let X, Y be Banach spaces, let (Ω, Σ, μ) be a σ-finite measure space, let $1 \leq p \leq \infty$, and let $G \in \mathcal{L}(X, Y)$ be a norm-one operator. Consider the norm-one composition operator $\widetilde{G}\colon L_p(\mu, X) \to L_p(\mu, Y)$ given by $\widetilde{G}(f) = G \circ f$ for every $f \in L_p(\mu, X)$. Then*

$$n_{\widetilde{G}}(L_p(\mu, X), L_p(\mu, Y)) \leq n_G(X, Y).$$

Equality does not hold in general, since for $p \neq 1, \infty$ we have $n(\ell_p^2) < 1$. On the other hand, we will show that equality holds for $p = 1$ and $p = \infty$.

We start by dealing with spaces of Bochner integrable functions.

Proposition 6.12. *Let X, Y be Banach spaces, let (Ω, Σ, μ) be a σ-finite measure space, and let $G \in \mathcal{L}(X, Y)$ be a norm-one operator. Consider the norm-one composition operator $\widetilde{G}\colon L_1(\mu, X) \to L_1(\mu, Y)$ given by $\widetilde{G}(f) = G \circ f$ for every $f \in L_1(\mu, X)$. Then*

$$n_{\widetilde{G}}(L_1(\mu, X), L_1(\mu, Y)) = n_G(X, Y).$$

Proof. We follow the lines of [37, proof of Theorem 8]. Without loss of generality, (Ω, Σ, μ) can be considered a probability space, as vector-valued L_1-spaces associated to σ-finite measures are (up to an isometric isomorphism) vector-valued L_1-spaces associated to probability measures (see [12, Proposition 1.6.1] for instance).

In order to prove $n_{\widetilde{G}}(L_1(\mu, X), L_1(\mu, Y)) \geq n_G(X, Y)$, we need to introduce some notation. If (Ω, Σ, μ) is a probability space, we write $\Sigma^+ := \{B \in \Sigma\colon \mu(B) > 0\}$. Given Banach spaces X and Y, the set

$$\mathcal{B} := \Big\{ \sum_{B \in \pi} y_B^* \mathbb{1}_B \colon \pi \subseteq \Sigma^+ \text{ finite partition of } \Omega,\ y_B^* \in S_{Y^*} \Big\} \subseteq S_{L_\infty(\mu, Y^*)}$$

satisfies

$$B_{L_1(\mu, Y)^*} = \overline{\operatorname{conv}}^{w^*}(\mathcal{B}) \tag{6.4}$$

since $\mathbb{T}\mathcal{B} = \mathcal{B}$ and it is clearly norming for the simple functions of $L_1(\mu, Y)$. On the other hand, we will write

$$\mathcal{A} := \left\{ x\, \frac{\mathbb{1}_A}{\mu(A)} : x \in S_X,\ A \in \Sigma^+ \right\},$$

which satisfies

$$B_{L_1(\mu,X)} = \overline{\operatorname{conv}}(\mathcal{A}). \tag{6.5}$$

Indeed, it is enough to notice that every simple function $f \in S_{L_1(\mu,X)}$ belongs to the convex hull of $\mathcal{A}$: such an f can be written as $f = \sum_{A\in\pi} x_A \mathbb{1}_A$, where $\pi \subseteq \Sigma^+$ is a finite family of pairwise disjoint sets of Ω and $x_A \in X \setminus \{0\}$ for each $A \in \pi$. Then

$$\|f\| = \sum_{A\in\pi} \|x_A\|\mu(A) = 1,$$

and hence

$$f = \sum_{A\in\pi} \|x_A\|\mu(A) \frac{x_A}{\|x_A\|} \frac{\mathbb{1}_A}{\mu(A)} \in \operatorname{conv}(\mathcal{A}).$$

Now, fix $T \in \mathcal{L}(L_1(\mu, X), L_1(\mu, Y))$ with $\|T\| = 1$ and $\varepsilon > 0$. We may find by (6.5) elements $x_0 \in S_X$ and $A \in \Sigma^+$ such that

$$\left\| T\left(x_0 \frac{\mathbb{1}_A}{\mu(A)} \right) \right\| > 1 - \varepsilon.$$

By (6.4), there exists $f^* = \sum_{B\in\pi} y_B^* \mathbb{1}_B$, where π is a finite partition of Ω into sets of Σ^+ and $y_B^* \in S_{Y^*}$ for each $B \in \pi$, satisfying

$$\operatorname{Re} f^*\left(T\left(x_0 \frac{\mathbb{1}_A}{\mu(A)} \right)\right) = \operatorname{Re} \sum_{B\in\pi} y_B^* \left(\int_B T\left(x_0 \frac{\mathbb{1}_A}{\mu(A)} \right) d\mu \right) > 1 - \varepsilon. \tag{6.6}$$

Then we can write

$$T\left(x_0 \frac{\mathbb{1}_A}{\mu(A)} \right) = \sum_{\substack{B\in\pi \\ \mu(A\cap B)\neq 0}} \frac{\mu(A\cap B)}{\mu(A)} T\left(x_0 \frac{\mathbb{1}_{A\cap B}}{\mu(A\cap B)} \right)$$

so, by a standard convexity argument, we can assume that there is $B_0 \in \pi$ such that, if we take the set $A \cap B_0$ in the role of new A, the inequality (6.6) remains true. After this modification of A, we additionally obtain $A \subseteq B_0$. By the density of norm-attaining functionals, we can assume that every y_B^* is norm-attaining, so there is $y_{B_0} \in S_Y$ such that $y_{B_0}^*(y_{B_0}) = 1$. Define the operator $S\colon X \to Y$ by

$$S(x) = \int_{B_0} T\left(x \frac{\mathbb{1}_A}{\mu(A)} \right) d\mu + \left[\sum_{B\in\pi\setminus\{B_0\}} y_B^* \left(\int_B T\left(x \frac{\mathbb{1}_A}{\mu(A)} \right) d\mu \right) \right] y_{B_0} \qquad (x \in X).$$

It is easy to check that $\|S\| \leq 1$, and moreover $\|S\| > 1 - \varepsilon$ since, as a consequence of (6.6), we obtain

$$\|S(x_0)\| \geq |y_{B_0}^*(Sx_0)| = \left| f^*\left(T\left(x_0 \frac{\mathbb{1}_A}{\mu(A)} \right)\right) \right| > 1 - \varepsilon.$$

Now, fixed $\delta > 0$, we consider $x \in S_X$ and $y^* \in S_{Y^*}$ with $\operatorname{Re} y^*(Gx) > 1 - \delta$. Take $f \in \mathcal{A}$ defined by

$$f = x \frac{\mathbb{1}_A}{\mu(A)}$$

and $g^* \in \mathcal{B}$ by

$$g^*(h) = y^*\Big(\int_{B_0} h\,d\mu\Big) + \sum_{B\in\pi\setminus\{B_0\}} y_B^*\Big(\int_B h\,d\mu\Big) y^*(y_{B_0}) \qquad (h \in L_1(\mu, Y)).$$

We have

$$\widetilde{G}f = \widetilde{G}\Big(x\,\frac{\mathbb{1}_A}{\mu(A)}\Big) = G(x)\,\frac{\mathbb{1}_A}{\mu(A)},$$

and, since $A \subseteq B_0$ and a partition is a family of pairwise disjoint sets, we deduce

$$\begin{aligned}\operatorname{Re} g^*(\widetilde{G}f) &= \operatorname{Re}\bigg(y^*\Big(\int_{B_0} G(x)\frac{\mathbb{1}_A}{\mu(A)}\,d\mu\Big) + \Big[\sum_{B\in\pi\setminus\{B_0\}} y_B^*\Big(\int_B G(x)\frac{\mathbb{1}_A}{\mu(A)}\,d\mu\Big)\Big] y^*(y_{B_0})\bigg)\\ &= \operatorname{Re} y^*(Gx) > 1-\delta.\end{aligned}$$

Moreover,

$$\begin{aligned}|y^*(Sx)| &= \bigg|y^*\Big(\int_{B_0} T\Big(x\frac{\mathbb{1}_A}{\mu(A)}\Big)\,d\mu\Big) + \Big[\sum_{B\in\pi\setminus\{B_0\}} y_B^*\Big(\int_B T\Big(x\,\frac{\mathbb{1}_A}{\mu(A)}\Big)d\mu\Big)\Big] y^*(y_{B_0})\bigg|\\ &= |g^*(Tf)| \le v_{\widetilde{G},\delta}(T).\end{aligned}$$

So, $v_{G,\delta}(S) \le v_{\widetilde{G},\delta}(T)$ and hence

$$v_{\widetilde{G}}(T) \ge v_G(S) \ge n_G(X,Y)\|S\| \ge (1-\varepsilon) n_G(X,Y).$$

Taking $\varepsilon \downarrow 0$, we get $v_{\widetilde{G}}(T) \ge n_G(X,Y)$, and the arbitrariness of T gives the desired inequality.

The reverse inequality $n_{\widetilde{G}}(L_1(\mu,X), L_1(\mu,Y)) \le n_G(X,Y)$ follows directly from Corollary 6.11. ■

The last result on composition operators on vector-valued function spaces deals with spaces of essentially bounded vector-valued functions.

PROPOSITION 6.13. *Let X, Y be Banach spaces, let (Ω,Σ,μ) be a σ-finite measure space, and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Consider the norm-one composition operator $\widetilde{G}\colon L_\infty(\mu,X) \to L_\infty(\mu,Y)$ given by $\widetilde{G}(f) = G\circ f$ for every $f \in L_\infty(\mu,X)$. Then*

$$n_{\widetilde{G}}(L_\infty(\mu,X), L_\infty(\mu,Y)) = n_G(X,Y).$$

The proof of this result borrows ideas from [38, proof of Theorem 2.3]. We also borrow from [38] two preliminary lemmas that we state for the convenience of the reader.

LEMMA 6.14 ([38, Lemma 2.1]). *Let $f \in L_\infty(\mu,X)$ with $\|f(t)\| > \lambda$ a.e. Then there exists $B \in \Sigma$ with $0 < \mu(B) < \infty$ such that*

$$\bigg\|\frac{1}{\mu(B)}\int_B f(t)\,d\mu(t)\bigg\| > \lambda.$$

LEMMA 6.15 ([38, Lemma 2.2]). *Let $f \in L_\infty(\mu,X)$, $C \in \Sigma$ with positive measure, and $\varepsilon > 0$. Then there exist $x \in X$ and $A \subseteq C$ with $0 < \mu(A) < \infty$ such that $\|x\| = \|f\mathbb{1}_C\|$ and $\|(f-x)\mathbb{1}_A\| < \varepsilon$. Accordingly, the set*

$$\{x\mathbb{1}_A + f\mathbb{1}_{\Omega\setminus A}\colon x \in S_X,\ f \in B_{L_\infty(\mu,X)},\ A \in \Sigma \text{ with } 0 < \mu(A) < \infty\}$$

is dense in $S_{L_\infty(\mu,X)}$.

Proof of Proposition 6.13. In order to show $n_{\widetilde{G}}(L_\infty(\mu,X),L_\infty(\mu,Y)) \geq n_G(X,Y)$, we fix an operator $T \in \mathcal{L}(L_\infty(\mu,X),L_\infty(\mu,Y))$ with $\|T\|=1$. Given $\varepsilon>0$, we may find $f_0 \in S_{L_\infty(\mu,X)}$ and $C \subseteq \Omega$ with $\mu(C)>0$ such that

$$\|[Tf_0](t)\| > 1-\varepsilon \qquad (t \in C). \tag{6.7}$$

On account of Lemma 6.15, there exist $y_0 \in B_X$ and $A \subseteq C$ with $0<\mu(A)<\infty$ such that $\|(f_0-y_0)\mathbb{1}_A\|<\varepsilon$. Now, write $y_0=(1-\lambda)x_1+\lambda x_2$ with $0\leq\lambda\leq 1$, $x_1,x_2 \in S_X$, and consider the functions

$$f_j = x_j\mathbb{1}_A + f_0\mathbb{1}_{\Omega\setminus A} \in L_\infty(\mu,X) \qquad (j=1,2).$$

which clearly satisfy $\|f_0-((1-\lambda)f_1+\lambda f_2)\|<\varepsilon$. Since $A\subseteq C$, by (6.7) we have

$$\|[Tf_1](t)\| > 1-2\varepsilon \quad \text{ or } \quad \|[Tf_2](t)\| > 1-2\varepsilon$$

for every $t \in A$. Now, we choose $i \in \{1,2\}$ such that

$$A_i = \{t \in A\colon \|[Tf_i](t)\| > 1-2\varepsilon\}$$

has positive measure, we write $x_0 = x_i$, and we finally use Lemma 6.14 to get $B \subseteq A_i \subseteq A$ with $0<\mu(B)<\infty$ such that

$$\left\|\frac{1}{\mu(B)}\int_B T(x_0\mathbb{1}_A + f_0\mathbb{1}_{\Omega\setminus A})\,d\mu\right\| > 1-2\varepsilon. \tag{6.8}$$

Next, we fix $x_0^* \in S_{X^*}$ with $x_0^*(x_0)=1$, we write

$$\Phi(x) = x\mathbb{1}_A + x_0^*(x)\,f_0\mathbb{1}_{\Omega\setminus A} \in L_\infty(\mu,X) \qquad (x\in X).$$

and we define the operator $S \in \mathcal{L}(X,Y)$ by

$$Sx = \frac{1}{\mu(B)}\int_B T(\Phi(x))\,d\mu \qquad (x \in X)$$

which, by (6.8), satisfies $\|S\| \geq \|Sx_0\| > 1-2\varepsilon$.

Given $\delta>0$, we fix $x \in S_X$ and $y^* \in S_{Y^*}$ with $\operatorname{Re} y^*(Gx) > 1-\delta$. Define $f \in S_{L_\infty(\mu,X)}$ by

$$f = \Phi(x) = x\mathbb{1}_A + x_0^*(x)f_0\mathbb{1}_{\Omega\setminus A}$$

and $g^* \in S_{L_\infty(\mu,Y)^*}$ by

$$g^*(h) = y^*\left(\frac{1}{\mu(B)}\int_B h\,d\mu\right) \qquad (h \in L_\infty(\mu,Y)).$$

Since $B\subseteq A$, we have

$$\begin{aligned}
\operatorname{Re} g^*(\widetilde{G}f) &= \operatorname{Re} y^*\left(\frac{1}{\mu(B)}\int_B G(f(t))\,d\mu(t)\right)\\
&= \operatorname{Re} y^*\left(\frac{1}{\mu(B)}\int_B G(x\mathbb{1}_A(t) + x_0^*(x)\,f_0(t)\mathbb{1}_{\Omega\setminus A}(t)\right)d\mu(t))\\
&= \operatorname{Re} y^*\left(\frac{1}{\mu(B)}\int_B G(x)\mathbb{1}_B(t)\,d\mu(t)\right) = \operatorname{Re} y^*(Gx) > 1-\delta.
\end{aligned}$$

Moreover,

$$|y^*(Sx)| = \left|y^*\left(\frac{1}{\mu(B)}\int_B T(\Phi(x))\,d\mu\right)\right| = |g^*(Tf)| \leq v_{\widetilde{G},\delta}(T)$$

so $v_{G,\delta}(S) \leq v_{\widetilde{G},\delta}(T)$ and hence

$$v_{\widetilde{G}}(T) \geq v_G(S) \geq n_G(X,Y)\|S\| \geq (1-2\varepsilon)n_G(X,Y).$$

Taking $\varepsilon \downarrow 0$, we get $v_{\widetilde{G}}(T) \geq n_G(X,Y)$, and the arbitrariness of T gives the desired inequality.

The reverse inequality is a consequence of Corollary 6.11. ∎

6.3. Adjoint operators. As shown in Lemma 3.5, the numerical index with respect to an operator always dominates the numerical index with respect to its adjoint. Our aim here is to give some particular cases in which the two indices coincide. First, we have to recall that this is not always the case, as there are Banach spaces X for which $n(X^*) < n(X)$ (see [27, §2] for instance). We also provide an easier example which does not use the identity operator.

Example 6.16. *The inclusion $G\colon c_0 \to c$ satisfies $n_G(c_0,c) = 1$, whereas its adjoint $G^*\colon \ell_1 \oplus_1 \mathbb{K} \to \ell_1$, given by $(x,\lambda) \mapsto x$, is not even a vertex of $\mathcal{L}(c^*, c_0^*)$ and so it satisfies $n_{G^*}(c^*, c_0^*) = 0$.*

Indeed, G is a spear operator by, for instance, [26, Proposition 4.2], so $n_G(c_0,c) = 1$. To prove G^* is not a vertex, consider the operator $T\colon \ell_1 \oplus_1 \mathbb{K} \to \ell_1$ given by $T(x,\lambda) = \lambda e_1^*$ for $x \in \ell_1$ and $\lambda \in \mathbb{K}$. Then we have

$$\begin{aligned}\|G^*(x,\lambda) + \theta T(x,\lambda)\| &= \Big\|(x(1) + \theta\lambda)e_1^* + \sum_{k=2}^{\infty} x(k)e_k^*\Big\| \\ &= |x(1)| + |\lambda| + \sum_{k=2}^{\infty} |x(k)| = \|x\| + |\lambda| = \|(x,\lambda)\|\end{aligned}$$

for every $\theta \in \mathbb{T}$, every $x \in \ell_1$, and every $\lambda \in \mathbb{K}$. This shows $\|G^* + \theta T\| \leq 1$ and so G^* is not an extreme operator. Therefore, G^* is not a vertex by Lemma 2.3.

If X and Y are both reflexive spaces, the numerical index with respect to every norm-one operator $G \in \mathcal{L}(X,Y)$ coincides with the numerical index with respect to G^*. Indeed, the inequality

$$n_{G^{**}}(X^{**}, Y^{**}) \leq n_{G^*}(Y^*, X^*) \leq n_G(X,Y)$$

gives the result. Actually, it is enough that Y is reflexive, or even a much weaker hypothesis: we show that the numerical index with respect to an operator coincides with the one with respect to its adjoint when the range space is L-embedded. Recall that a Banach space Y is *L-embedded* if $Y^{**} = J_Y(Y) \oplus_1 Y_s$ for suitable closed subspace Y_s of Y^{**}. We refer to the monograph [23] for background. Examples of L-embedded spaces are reflexive spaces (trivial), preduals of von Neumann algebras, in particular $L_1(\mu)$ spaces, the Lorentz spaces $d(w,1)$ and $L^{p,1}$, the Hardy space H_0^1, and the dual of the disk algebra $A(\mathbb{D})$ (see [23, Examples IV.1.1 and III.1.4]).

Proposition 6.17. *Let X be a Banach space, let Y be an L-embedded space, and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then $n_G(X,Y) = n_{G^*}(Y^*, X^*)$.*

Proof. We follow the lines of [26, proof of Proposition 5.21]. Write $Y^{**} = J_Y(Y) \oplus_1 Y_s$ and let $P_Y\colon Y^{**} \to J_Y(Y)$ be the natural projection. For a fixed $T \in \mathcal{L}(Y^*, X^*)$ consider the operators

$$A := P_Y \circ T^* \circ J_X : X \to J_Y(Y), \qquad B := [\mathrm{Id} - P_Y] \circ T^* \circ J_X : X \to Y_s.$$

Then $T^* \circ J_X = A \oplus B$. Given $\varepsilon > 0$, since $J_X(B_X)$ is dense in $B_{X^{**}}$ by the Goldstine Theorem and T^* is weak*-to-weak* continuous, we may find $x_0 \in S_X$ such that

$$\|T^* J_X(x_0)\| = \|Ax_0\| + \|Bx_0\| > \|T^*\| - \varepsilon.$$

Now, we may find $y_0 \in S_Y$ and $y_s^* \in S_{Y_s^*}$ such that

$$\|Ax_0\| y_0 = Ax_0 \quad \text{and} \quad y_s^*(Bx_0) = \|Bx_0\|.$$

Define $S\colon X \to Y$ by

$$S(x) = Ax + y_s^*(Bx) y_0 \qquad (x \in X).$$

For this operator

$$\|S\| \geq \|Sx_0\| = \|Ax_0 + y_s^*(Bx_0)y_0\| = \|Ax_0\| + \|Bx_0\| > \|T^*\| - \varepsilon.$$

Given $\delta > 0$, we take $x \in S_X$ and $y^* \in S_{Y^*}$ with $\operatorname{Re} y^*(Gx) > 1 - \delta$, and consider

$$z = J_X(x) \in S_{X^{**}} \quad \text{and} \quad z^* = (J_{Y^*}(y^*), y^*(y_0)y_s^*) \in S_{Y^{***}}$$

as $Y^{***} = J_{Y^*}(Y^*) \oplus_\infty Y_s^*$. Now, $\operatorname{Re} z^*(G^{**}z) = \operatorname{Re} y^*(Gx) > 1 - \delta$ since $G^{**} \circ J_X = J_Y \circ G$.

Moreover,

$$|z^*(T^*z)| = |J_{Y^*}(y^*)(Ax + y^*(y_0)y_s^*(Bx))| = |y^*(Sx)|,$$

hence $|y^*(Sx)| = |z^*(T^*z)| \leq v_{G^{**},\delta}(T^*)$ and, taking supremum, $v_{G,\delta}(S) \leq v_{G^{**},\delta}(T^*)$. Therefore,

$$v_{G^*}(T) = v_{G^{**}}(T^*) \geq v_G(S) \geq n_G(X,Y)\|S\| > n_G(X,Y)[\|T\| - \varepsilon].$$

The arbitrariness of $\varepsilon > 0$ and of $T \in \mathcal{L}(Y^*, X^*)$ gives $n_G(X,Y) \leq n_{G^*}(Y^*, X^*)$, and the other inequality is always true. ■

Particular cases of the above result are the following.

Corollary 6.18. *Let X be a Banach space and let Y be a reflexive space. Then*

$$n_G(X,Y) = n_{G^*}(Y^*, X^*)$$

for every norm-one $G \in \mathcal{L}(X,Y)$.

Corollary 6.19. *Let X be a Banach space and let μ be a positive measure. Then $n_G(X, L_1(\mu)) = n_{G^*}(L_1(\mu)^*, X^*)$ for every norm-one $G \in \mathcal{L}(X, L_1(\mu))$.*

Finally, we show that, for rank-one operators, the numerical index is preserved by passing to the adjoint.

Proposition 6.20. *Let X, Y be Banach spaces, and let $G \in \mathcal{L}(X,Y)$ be a rank-one operator of norm 1. Then $n_G(X,Y) = n_{G^*}(Y^*, X^*)$ and so the same happens to all the successive adjoints of G.*

Proof. We can write $G = x_0^* \otimes y_0$ for some $x_0^* \in S_{X^*}$ and $y_0 \in S_Y$, so

$$n_G(X,Y) = n(X^*, x_0^*) n(Y, y_0)$$

by Proposition 3.6. Furthermore, as $G^* = J_Y(y_0) \otimes x_0^*$, we have

$$n_{G^*}(Y^*, X^*) = n(Y^{**}, J_Y(y_0))n(X^*, x_0^*)$$

again by Proposition 3.6. But $n(Y^{**}, J_Y(y_0)) = n(Y, y_0)$ by Lemma 2.6 and we are done. ∎

6.4. Composition of operators. The next result allows us to control the numerical index with respect to the composition of two operators in two particular cases.

LEMMA 6.21. *Let X, Y, Z be Banach spaces and let $G_1 \in \mathcal{L}(X, Y)$ and $G_2 \in \mathcal{L}(Y, Z)$ be norm-one operators.*

(a) *If G_2 is an isometric embedding, then $n_{G_2 \circ G_1}(X, Z) \leq n_{G_1}(X, Y)$.*
(b) *If $\overline{G_1(B_X)} = B_Y$, then $n_{G_2 \circ G_1}(X, Z) \leq n_{G_2}(Y, Z)$.*

Proof. Both (a) and (b) follow from Lemma 2.4. In the first case, it is enough to see that the map $T \mapsto G_2 \circ T$ from $\mathcal{L}(X, Y)$ to $\mathcal{L}(X, Z)$ is an isometric embedding by the hypothesis on G_2. For (b), we see that $S \mapsto S \circ G_1$ from $\mathcal{L}(Y, Z)$ to $\mathcal{L}(X, Z)$ is an isometric embedding by the hypothesis on G_1. ∎

We now collect some consequences of this result.

The first immediate consequence is that the restriction of the codomain of an operator cannot decrease the numerical index.

PROPOSITION 6.22. *Let X, Y be Banach spaces, let $G \in \mathcal{L}(X, Y)$ be a norm-one operator, and let Z be a closed subspace of Y with $G(X) \subseteq Z$. Consider the operator $\overline{G}\colon X \to Z$ given by $\overline{G}x = Gx$ for every $x \in X$. Then $n_G(X, Y) \leq n_{\overline{G}}(X, Z)$.*

Proof. This follows from Lemma 6.21(a) as $G = I \circ \overline{G}$ where $I\colon Z \to Y$ denotes the inclusion. ∎

The inequality in the above result can be strict:

EXAMPLE 6.23. *The operator $G\colon \mathbb{K} \to \mathbb{K} \oplus_\infty \mathbb{K}$ given by $G(x) = (x, 0)$ satisfies $n_G(\mathbb{K}, \mathbb{K} \oplus_\infty \mathbb{K}) = 0$, whereas $\overline{G}\colon \mathbb{K} \to \mathbb{K}$ satisfies $n_{\overline{G}}(\mathbb{K}, \mathbb{K}) = 1$.*

Another consequence of Lemma 6.21 is that the numerical index with respect to the injectivization of an operator is an upper bound for the numerical index with respect to the original operator.

PROPOSITION 6.24. *Let X, Y be Banach spaces, let $G \in \mathcal{L}(X, Y)$ be a norm-one operator, and let $q\colon X \to X/\ker G$ be the quotient map. Consider the injectivization $\widehat{G} \in \mathcal{L}(X/\ker G, Y)$ satisfying $\widehat{G} \circ q = G$. Then*

$$n_G(X, Y) \leq n_{\widehat{G}}(X/\ker G, Y).$$

Proof. This follows from Lemma 6.21(b) as $\widehat{G} \circ q = G$ and $\overline{q(B_X)} = B_{X/\ker G}$. ∎

In the particular case when $n_G(X, Y) = 1$, we obtain the following result which gives a partial answer to [26, Problem 9.14].

COROLLARY 6.25. *Let X, Y be Banach spaces, let $G \in \mathcal{L}(X, Y)$ be a norm-one operator. Then, under the notation of Proposition 6.24, if G is a spear operator, then so is its injectivization $\widehat{G}$.*

Again, the inequality in Proposition 6.24 may be strict, as the following example shows. It also proves that Corollary 6.25 is not an equivalence.

EXAMPLE 6.26. *The operator* $G\colon \ell_1 \oplus_1 \mathbb{K} \to \ell_1$ *given by* $G(x,\lambda) = x$ *satisfies the condition* $n_G(\ell_1 \oplus_1 \mathbb{K}, \ell_1) = 0$ (*as proved in Example* 6.16), *whereas the injectivization* $\widehat{G}$ *is the identity operator in* ℓ_1 *and so it satisfies* $n_{\widehat{G}}(\ell_1, \ell_1) = n(\ell_1) = 1$.

With the aid of all of these examples and some others from previous sections, we may prove the following assertion.

REMARK 6.27. There is no general function $\Upsilon\colon [0,1]\times[0,1] \to [0,1]$ such that the equality

$$n_{G_2 \circ G_1}(X,Z) = \Upsilon(n_{G_2}(Y,Z), n_{G_1}(X,Y))$$

holds for all Banach spaces X, Y, Z and for all norm-one operators $G_1 \in \mathcal{L}(X,Y)$ and $G_2 \in \mathcal{L}(Y,Z)$.

Indeed, suppose that such a function Υ exists. In Remark 4.9 an example is given of a real Banach space Z with $n(Z) = 0$ and a norm-one operator $G \in \mathcal{L}(Z,\mathbb{R})$ such that $n_G(Z,\mathbb{R}) = 1$. As $G = G \circ \mathrm{Id}_Z$, it follows that $1 = \Upsilon(1,0)$. Moreover, there is a similar example in Remark 4.9 showing $1 = \Upsilon(0,1)$. On the other hand, if X, Y are two-dimensional Banach spaces, we may always find $G \in \mathcal{L}(X,Y)$ with $n_G(X,Y) = 0$ by Proposition 4.1. As $G = G \circ \mathrm{Id}_X = \mathrm{Id}_Y \circ G$, it follows that $0 = \Upsilon(0, n(X)) = \Upsilon(n(Y), 0)$. It is enough to consider $X = Y = \ell_\infty^2$ to get a contradiction.

Now, we may wonder whether a further relationship with the composition is valid in general. We answer this question in the negative giving some counterexamples.

Example 4.10 shows that, in general, there is no inequality

$$n_{G_2 \circ G_1}(X,Z) \leq \max\{n_{G_1}(X,Y), n_{G_2}(Y,Z)\},$$

with G playing the role of G_1 and the identity operator playing the role of G_2.

Example 6.23 also shows the absence, in general, of the inequality

$$n_{G_2 \circ G_1}(X,Z) \geq \max\{n_{G_1}(X,Y), n_{G_2}(Y,Z)\}.$$

Actually, it is possible that the inequality $n_{G_2 \circ G_1}(X,Z) \geq \min\{n_{G_1}(X,Y), n_{G_2}(Y,Z)\}$ fails, as the following example shows, since $n(\ell_p) > 0$ for $p \neq 2$ by [35].

EXAMPLE 6.28. *Let* $1 \leq p < q < \infty$. *The canonical inclusion* $G\colon \ell_p \to \ell_q$ *satisfies* $n_G(\ell_p, \ell_q) = 0$.

Proof. Consider the norm-one operator $T \in \mathcal{L}(\ell_p, \ell_q)$ given by $T = e_2^* \otimes e_1$. Given a scalar $0 < \varepsilon < 1/4$, our goal is to prove $v_G(T) \leq \max\{\varepsilon^{1/p}, (1-(1-2\varepsilon)^q)^{1/q}\}$. To do so we need the following claim.

CLAIM. *Let* $0 < \delta < 1/2$ *be such that* $(1-\delta)^{p/(q-p)} > 1-\varepsilon$. *Given* $x \in S_{\ell_p}$ *such that* $\|x\|_q > (1-2\delta^2)^{1/q}$, *there exists a unique* $k_0 \in \mathbb{N}$ *satisfying* $|x(k_0)|^p > 1-\varepsilon$.

Indeed, the uniqueness of k_0 is clear because $|x(k_0)|^p > 1-\varepsilon$, $\varepsilon < 1/4$, and $\|x\|_p = 1$. Let us show the existence of k_0. Since $1 - 2\delta^2 < \|x\|_q^q$, there is $n \in \mathbb{N}$ satisfying

$1-\delta^2 < \sum_{k=1}^n |x(k)|^q$, and thus

$$\sum_{k=1}^n |x(k)|^p - \delta^2 \leq 1-\delta^2 < \sum_{k=1}^n |x(k)|^q = \sum_{k=1}^n |x(k)|^p |x(k)|^{q-p}.$$

Let $I=\{k\in\{1,\dots,n\}\colon |x(k)|^{q-p} > 1-\delta\}$. Using [26, Lemma 8.14] with $\lambda_k = |x(k)|^p$, $\beta_k = 1$ and $\alpha_k = |x(k)|^{q-p}$, we get $\sum_{k\notin I} |x(k)|^p < \delta$. So we can write

$$1-\delta^2 < \sum_{k\notin I}^n |x(k)|^q + \sum_{k\in I}^n |x(k)|^q \leq \sum_{k\notin I}^n |x(k)|^p + \sum_{k\in I}^n |x(k)|^q < \delta + \sum_{k\in I}^n |x(k)|^q,$$

which gives $\sum_{k\in I} |x(k)|^q > 1-\delta^2-\delta > 0$ and therefore $I\neq\emptyset$. For $k_0\in I$, we have

$$|x(k_0)|^p > (1-\delta)^{\frac{p}{q-p}} > 1-\varepsilon,$$

finishing the proof of the claim.

To estimate the numerical radius of T, let $0<\widetilde{\delta}<\varepsilon$ be such that $1-\widetilde{\delta} > (1-2\delta^2)^{1/q}$ and take $x\in S_{\ell_p}$ and $y^*\in S_{\ell_q^*}$ satisfying $\operatorname{Re} y^*(x) > 1-\widetilde{\delta}$, which implies

$$\|x\|_q > \operatorname{Re} y^*(x) > 1-\widetilde{\delta} > (1-2\delta^2)^{1/q}.$$

The claim tells us that there is $k_0\in\mathbb{N}$ such that $|x(k_0)|^p > 1-\varepsilon$ and so $\sum_{k\neq k_0}^\infty |x(k)|^p < \varepsilon$. Now, we can estimate $|y^*(Tx)| = |y^*(1)|\,|x(2)|$ depending on the value of k_0. If $k_0\neq 2$ then $|x(2)| < \varepsilon^{1/p}$ and $|y^*(Tx)| \leq |x(2)| < \varepsilon^{1/p}$. Suppose, otherwise, $k_0=2$. Then, as

$$\begin{aligned} 1-\widetilde{\delta} < \operatorname{Re} y^*(x) &= |y^*(2)|\,|x(2)| + \sum_{k\neq 2}^\infty |y^*(k)|\,|x(k)| \\ &\leq |y^*(2)| + \|y^*\|_q \sum_{k\neq 2}^\infty |x(k)|^p \leq |y^*(2)| + \varepsilon, \end{aligned}$$

we get $|y^*(2)| > 1-\widetilde{\delta}-\varepsilon > 1-2\varepsilon$. Therefore,

$$|y^*(2)|^q > (1-2\varepsilon)^q \quad \text{and} \quad |y^*(Tx)| \leq |y^*(1)| < (1-(1-2\varepsilon)^q)^{1/q}.$$

Hence, in any case,

$$v_G(T) \leq v_{G,\widetilde{\delta}}(T) \leq \max\bigl\{\varepsilon^{1/p}, (1-(1-2\varepsilon)^q)^{1/q}\bigr\}$$

and the arbitrariness of ε gives $v_G(T)=0$ and so, $n_G(\ell_p,\ell_q)=0$. ■

6.5. Extending the domain and the codomain. Our final aim in this chapter is to study ways of extending the domain and the codomain of an operator maintaining the same numerical index. For the domain, we have the following result.

Proposition 6.29. *Let X,Y,Z be Banach spaces, let $G\in\mathcal{L}(X,Y)$ be a norm-one operator, and consider the norm-one operator $\widetilde{G}\colon X\oplus_\infty Z\to Y$ given by $\widetilde{G}(x,z)=G(x)$ for every $(x,z)\in X\oplus_\infty Z$. Then*

$$n_{\widetilde{G}}(X\oplus_\infty Z, Y) = n_G(X,Y).$$

Proof. Fix $T\in\mathcal{L}(X\oplus_\infty Z,Y)$ with $\|T\|>0$ and $0<\varepsilon<\|T\|$. We may find $x_0\in S_X$ and $z_0\in S_Z$ satisfying $\|T(x_0,z_0)\| > \|T\|-\varepsilon$. Now take $x_0^*\in S_{X^*}$ with $x_0^*(x_0)=1$ and define the operator $S \in \mathcal{L}(X,Y)$ by

$$S(x) = T(x, x_0^*(x)z_0) \quad (x \in X),$$

which satisfies $\|S\| \geq \|Sx_0\| = \|T(x_0, z_0)\| > \|T\| - \varepsilon$.

Now, given $\delta > 0$, $x \in S_X$, and $y^* \in S_{Y^*}$ with $\operatorname{Re} y^*(Gx) > 1 - \delta$, we consider $(x, x_0^*(x)z_0) \in S_{X \oplus_\infty Z}$. Clearly, $\operatorname{Re} y^*(\widetilde{G}(x, x_0^*(x)z_0)) = \operatorname{Re} y^*(Gx) > 1 - \delta$. Moreover,

$$|y^*(Sx)| = |y^*(T(x, x_0^*(x)z_0))| \leq v_{\widetilde{G},\delta}(T),$$

hence $v_{G,\delta}(S) \leq v_{\widetilde{G},\delta}(T)$. Therefore,

$$v_{\widetilde{G}}(T) \geq v_G(S) \geq n_G(X,Y)\|S\| > n_G(X,Y)[\|T\| - \varepsilon].$$

The arbitrariness of $\varepsilon > 0$ and $T \in \mathcal{L}(X \oplus_\infty Z, Y)$ gives $n_{\widetilde{G}}(X \oplus_\infty Z, Y) \geq n_G(X,Y)$.

The reverse inequality follows immediately from Lemma 6.21(b) as $\widetilde{G} = G \circ P$ where $P\colon X \oplus_\infty Z \to X$ denotes the natural projection. ∎

For the range space, the result is the following.

Proposition 6.30. *Let X, Y, Z be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator, and consider the norm-one operator $\widetilde{G}\colon X \to Y \oplus_1 Z$ given by $\widetilde{G}x = (Gx, 0)$ for every $x \in X$. Then*

$$n_{\widetilde{G}}(X, Y \oplus_1 Z) = n_G(X,Y).$$

Proof. Fix $T \in \mathcal{L}(X, Y \oplus_1 Z)$ with $\|T\| > 0$, $\|T\| > \varepsilon > 0$, and $x_0 \in S_X$ such that $\|Tx_0\| > \|T\| - \varepsilon$. Denote by P_Y and P_Z the projections from $Y \oplus_1 Z$ to Y and Z, respectively. Take $y_0 \in S_Y$ so that $P_Y T x_0 = \|P_Y T x_0\| y_0$ and $z_0^* \in S_{Z^*}$ satisfying $z_0^*(P_Z T x_0) = \|P_Z T x_0\|$. Now define $S \in \mathcal{L}(X,Y)$ by

$$Sx = P_Y T x + z_0^*(P_Z T x) y_0 \quad (x \in X),$$

which satisfies

$$\|S\| \geq \|Sx_0\| = \|P_Y T x_0 + \|P_Z T x_0\| y_0\| = \|P_Y T x_0\| + \|P_Z T x_0\| > \|T\| - \varepsilon.$$

Given $\delta > 0$, $x \in S_X$, and $y^* \in S_{Y^*}$ with $\operatorname{Re} y^*(Gx) > 1 - \delta$, we consider $(y^*, y^*(y_0)z_0^*) \in S_{(Y \oplus_1 Z)^*}$ as $(Y \oplus_1 Z)^* = Y^* \oplus_\infty Z^*$. Clearly,

$$\operatorname{Re}(y^*, y^*(y_0)z_0^*)(\widetilde{G}x) = \operatorname{Re} y^*(Gx) > 1 - \delta.$$

Moreover,

$$|y^*(Sx)| = |y^*(P_Y T x + z_0^*(P_Z T x) y_0)| = |(y^*, y^*(y_0)z_0^*)(Tx)| \leq v_{\widetilde{G},\delta}(T),$$

and then $v_{G,\delta}(S) \leq v_{\widetilde{G},\delta}(T)$. Therefore,

$$v_{\widetilde{G}}(T) \geq v_G(S) \geq n_G(X,Y)\|S\| > n_G(X,Y)[\|T\| - \varepsilon].$$

The arbitrariness of ε and T gives $n_{\widetilde{G}}(X, Y \oplus_1 Z) \geq n_G(X,Y)$.

The reverse inequality is an immediate consequence of Lemma 6.21(a) as $\widetilde{G} = I \circ G$ where $I\colon Y \to Y \oplus_1 Z$ denotes the natural inclusion. ∎

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Chapter II

Numerical index and Daugavet property of operator ideals and tensor products

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NUMERICAL INDEX AND DAUGAVET PROPERTY OF OPERATOR IDEALS AND TENSOR PRODUCTS

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Abstract. We show that the numerical index of any operator ideal is less than or equal to the minimum of the numerical indices of the domain space and the range space. Further, we show that the numerical index of the ideal of compact operators or the ideal of weakly compact operators is less than or equal to the numerical index of the dual of the domain space, and this result provides interesting examples. We also show that the numerical index of a projective or injective tensor product of Banach spaces is less than or equal to the numerical index of any of the factors. Finally, we show that if a projective tensor product of two Banach spaces has the Daugavet property and the unit ball of one of the factor is slicely countably determined or its dual contains a point of Fréchet differentiability of the norm, then the other factor inherits the Daugavet property. If an injective tensor product of two Banach spaces has the Daugavet property and one of the factors contains a point of Fréchet differentiability of the norm, then the other factor has the Daugavet property.

1. Introduction

The numerical index of a Banach space is a constant that relates the numerical radius and the norm of bounded linear operators on the space. It was introduced by G. Lumer in 1968 (see [8]). Let us present the needed definitions and notation. Given a Banach space X, we write S_X and B_X to denote, respectively, the unit sphere and the closed unit ball of the space. By X^* we denote the topological dual of X and $\mathcal{L}(X)$ will denote the Banach space of all bounded linear operators on X. The *numerical range* of an operator $T \in \mathcal{L}(X)$ is the set of scalars given by

$$V(T) := \{x^*(Tx) \colon x \in S_X,\, x^* \in S_{X^*}, x^*(x) = 1\},$$

and the *numerical radius* of T is then given by

$$v(T) := \sup\{|\lambda| \colon \lambda \in V(T)\}.$$

It is clear that the numerical radius is a seminorm on $\mathcal{L}(X)$ which is not greater than the operator norm. Very often, the numerical radius is actually an equivalent norm on $\mathcal{L}(X)$ and to quantify this fact it is used the *numerical index* of the space X:

$$\begin{aligned} n(X) &:= \inf\{v(T) \colon T \in \mathcal{L}(X),\, \|T\| = 1\} \\ &= \max\{k \geqslant 0 \colon k\|T\| \leqslant v(T)\, \forall T \in \mathcal{L}(X)\}. \end{aligned}$$

It is clear that $0 \leqslant n(X) \leqslant 1$; the value $n(X) = 1$ means that the numerical radius and the norm coincide, while $n(X) = 0$ when the numerical radius is not an equivalent norm on $\mathcal{L}(X)$. We refer the reader to the expositive paper [12], to Chapter 1 of the recent book [10], and to Subsection 1.1

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of the very recent paper [14]. Some results on numerical index which we would like to emphasize are the following. For every Banach space X, $n(X^*) \leqslant n(X)$ and the inequality can be strict; $n(c_0) = n(\ell_1) = n(\ell_\infty) = 1$, a result which is also valid for all L- and M-spaces, the disk algebra, and H^∞. The numerical index behaves differently when dealing with real or complex Banach spaces. For instance, Hilbert spaces of dimension greater than or equal to two have numerical index 0 in the real case and 1/2 in the complex case. In general, if X is a complex Banach space, then $n(X) \geqslant 1/\mathrm{e}$ and all the values in the interval $[1/\mathrm{e}, 1]$ are valid; for real Banach spaces, there is no restriction and all the values of the interval $[0, 1]$ are possible. The numerical index of L_p spaces for $1 < p < \infty$, $p \neq 2$, is still unknown, but it is known that $n(L_p(\mu)) > 0$ in the real case for $p \neq 2$. All these results can be found in the cited papers [10, 12, 14]. Some recent results can be found in [18], where the exact value of some two-dimensional ℓ_p spaces is calculated, and in [1, 2, 22], for instance. Different extensions of the concept of numerical index appear in [11] and [25].

There is a property somehow related to the numerical index called Daugavet property. A Banach space X has the *Daugavet property* [13] if the norm equality

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{DE}$$

holds for all rank-one operators $T \in \mathcal{L}(X)$ and, in this case, the same happens for all weakly compact operators on X. Examples of Banach spaces satisfying this property are $L_1(\mu, Y)$ when the positive measure μ is atomless and Y is arbitrary, $C(K, Y)$ when the compact space K is perfect and Y is arbitrary, or the disk algebra. Let us say that there is a relation between the Daugavet property and the numerical range of operators: an operator T satisfies (DE) if and only if $\sup \operatorname{Re} V(T) = \|T\|$ (see [8] for instance). Classical references for Daugavet property include [13, 24, 26]. For very recent results, we refer the reader to [5, 19], for instance.

To state the results of the paper, we need to introduce some definitions and notation. Given Banach spaces X and Y, we write $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\mathcal{W}(X,Y)$, and $\mathcal{A}(X,Y)$ to denote, respectively, the space of (bounded linear) operators, compact operators, weakly compact operators, and approximable operators (i.e. norm limits of finite rank operators), all of them endowed with the operator norm. Finally, we consider the space of all nuclear operators: an operator $T\colon X \longrightarrow Y$ between Banach spaces is called *nuclear* if there exist $x_n^* \in X$ and $y_n \in Y$ for every $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} \|x_n^*\| \, \|y_n\| < \infty$ and

$$Tx = \sum_{n=1}^{\infty} x_n^*(x) y_n \qquad (x \in X).$$

The space of all nuclear operators, denoted by $\mathcal{N}(X,Y)$, is a Banach space endowed with the norm

$$N(T) = \inf \left\{ \sum_{n=1}^{\infty} \|x_n^*\| \, \|y_n\| \colon Tx = \sum_{n=1}^{\infty} x_n^*(x) y_n \right\},$$

where the infimum is taken over all the representations of T as above. The *projective tensor product* of X and Y, denoted by $X \hat{\otimes}_\pi Y$, is the completion of $X \otimes Y$ under the norm given by

$$\|u\|_\pi = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \, \|y_i\| \colon u = \sum_{i=1}^{n} x_i \otimes y_i \right\},$$

where the infimum is taken over all the representations of $u = \sum_{i=1}^{n} x_i \otimes y_i$. It follows from the definition that $B_{X\hat{\otimes}_\pi Y} = \overline{\operatorname{conv}}(B_X \otimes B_Y)$. The projective tensor product of two operators $S \in \mathcal{L}(X, W)$ and $T \in \mathcal{L}(Y, Z)$ between Banach spaces, denoted by $S \otimes_\pi T$, is the unique operator between $X \hat{\otimes}_\pi Y$ and $W \hat{\otimes}_\pi Z$ such that $(S \otimes_\pi T)(x \otimes y) = Sx \otimes Ty$ for every $x \in X$ and $y \in Y$, which also satisfies that

$\|S \otimes_\pi T\| = \|S\|\,\|T\|$. The *injective tensor product* of X and Y, denoted by $X \hat{\otimes}_\varepsilon Y$, is the completion of $X \otimes Y$ under the norm given by

$$\|u\|_\varepsilon = \sup\left\{ \left| \sum_{i=1}^{n} x^*(x_i) y^*(y_i) \right| : x^* \in B_{X^*},\ y^* \in B_{Y^*} \right\},$$

where $\sum_{i=1}^n x_i \otimes y_i$ is any representation of u. The injective tensor product of two operators $S \in \mathcal{L}(X, W)$ and $T \in \mathcal{L}(Y, Z)$ between Banach spaces, denoted by $S \otimes_\varepsilon T$, is the unique operator between $X \hat{\otimes}_\varepsilon Y$ and $W \hat{\otimes}_\varepsilon Z$ such that $(S \otimes_\varepsilon T)(x \otimes y) = Sx \otimes Ty$ for every $x \in X$ and $y \in Y$, which also satisfies that $\|S \otimes_\varepsilon T\| = \|S\|\,\|T\|$. We refer the reader to [7] and [21] for more information and background about ideals of operators and tensor products of Banach spaces.

For ideals of operators, we show in Section 2 that for every operator ideal $\mathcal{Z}$ of $\mathcal{L}(X,Y)$ endowed with the operator norm we have that $n(\mathcal{Z}) \leqslant \min\{n(X), n(Y)$. In the case of compact and weakly compact operators, we may improve this inequality to

$$n(\mathcal{K}(X,Y)) \leqslant \min\{n(X^*), n(Y)\}, \qquad n(\mathcal{W}(X,Y)) \leqslant \min\{n(X^*), n(Y)\}.$$

This result allows us to present some interesting examples as the existence of a real Banach space X such that $n(X) = 1$ while $n(\mathcal{K}(X,Y)) = n(\mathcal{W}(X,Y)) = 0$ for every Banach space Y. In particular, $n(X) = 1$ while $n(\mathcal{K}(X,X)) = n(\mathcal{W}(X,X)) = 0$.

For tensor products of Banach spaces, we prove in Section 3 that the numerical indices of $X \hat{\otimes}_\pi Y$ and $X \hat{\otimes}_\varepsilon Y$ are less than or equal to the minimum of $n(X)$ and $n(Y)$. As a consequence, and just using representation theorems, we get some consequences for the space of approximable operators and for the space of nuclear operators:

$$n(\mathcal{A}(X,Y)) \leqslant \min\{n(X^*), n(Y)\}$$

and, in the case where X^* or Y has the approximation property,

$$n(\mathcal{N}(X,Y)) \leqslant \min\{n(X^*), n(Y)\}.$$

Finally, we study in Section 4 the Daugavet property of tensor products of Banach spaces. We show that when $X \hat{\otimes}_\pi Y$ has the Daugavet property and B_Y is a slicely countably determined set (see the definition at the beginning of the section), then X has the Daugavet property. We also provide with the analogous result in the case where the space Y^* has a point of Fréchet differentiability of the norm. For injective tensor products, we do not know if the result with the hypothesis of slicely countably determined unit ball is true or not, but there is a positive result when the space Y has a point of Fréchet differentiability of the norm.

2. Numerical index of some operator ideals of $\mathcal{L}(X,Y)$

Given two Banach spaces X and Y, we first study the relationship between the numerical index of subspaces of $\mathcal{L}(X,Y)$ which are ideals and the numerical indices of the spaces X and Y. Recall that, according to Pietsch, an *operator ideal* $\mathcal{Z}$ is a "rule" (formally a subclass of the class of all continuous linear operators between Banach spaces) assigning to every pair of Banach spaces X and Y a linear subspace $\mathcal{Z}(X,Y)$ of $\mathcal{L}(X,Y)$ (called a *component* of $\mathcal{Z}$) which contains the finite rank operators and satisfies that

$$\mathcal{L}(F,Y) \circ \mathcal{Z}(E,F) \circ \mathcal{L}(X,E) \subseteq \mathcal{Z}(X,Y)$$

for all Banach spaces E, F, X, Y. We refer the reader to the monograph [7] for background. Here, we will only consider ideals whose components are closed subspaces.

Proposition 2.1. *Let X, Y be Banach spaces, then $n\big(\mathcal{L}(X,Y)\big) \leqslant \min\{n(X), n(Y)\}$. Moreover, the same happens to every operator ideal $\mathcal{Z}(X,Y) \leqslant \mathcal{L}(X,Y)$ endowed with the operator norm, that is, $n(\mathcal{Z}(X,Y)) \leqslant \min\{n(X), n(Y)\}$.*

To give the proof of the proposition, we need the following lemma which is well known and can be deduced, for instance, from [6, Corollary 2.1.2].

Lemma 2.2. *Let X_1, X_2 be Banach spaces and suppose that there is an isometric embedding $\Phi\colon \mathcal{L}(X_1) \longrightarrow \mathcal{L}(X_2)$ satisfying $\Phi(\mathrm{Id}_{X_1}) = \mathrm{Id}_{X_2}$. Then, $n(X_2) \leqslant n(X_1)$.*

Proof of Proposition 2.1. We first show that $n\big(\mathcal{L}(X,Y)\big) \leqslant n(X)$. Fixed $J \in \mathcal{L}(X)$, we define the map $\Phi_J\colon \mathcal{L}(X,Y) \longrightarrow \mathcal{L}(X,Y)$ by $\Phi_J(T) = T \circ J$ for every $T \in \mathcal{L}(X,Y)$ and observe that $\|\Phi_J\| = \|J\|$. Indeed, the inequality $\|\Phi_J\| \leqslant \|J\|$ is evident. To prove the reverse one, given $\varepsilon > 0$, we find $x_\varepsilon \in S_X$ satisfying $\|Jx_\varepsilon\| > \|J\| - \varepsilon$ and then we take $x_\varepsilon^* \in S_{X^*}$ such that $x_\varepsilon^*(Jx_\varepsilon) = \|Jx_\varepsilon\| > \|J\| - \varepsilon$. We fix $y_0 \in S_Y$ and define the rank-one operator $T_\varepsilon \in \mathcal{L}(X,Y)$ by $T_\varepsilon(x) = x_\varepsilon^*(x)y_0$ for every $x \in X$, which satisfies $\|T_\varepsilon\| = 1$ and

$$\|\Phi_J(T_\varepsilon)\| = \|T_\varepsilon \circ J\| \geqslant \|[T_\varepsilon \circ J](x_\varepsilon)\| = \|x_\varepsilon^*(Jx_\varepsilon)y_0\| > \|J\| - \varepsilon.$$

Therefore $\|\Phi_J\| \geqslant \|J\|$, and hence the mapping $J \longmapsto \Phi_J$ is an isometric embedding from $\mathcal{L}(X)$ to $\mathcal{L}\big(\mathcal{L}(X,Y)\big)$ carrying Id_X to $\mathrm{Id}_{\mathcal{L}(X,Y)}$, so the inequality $n\big(\mathcal{L}(X,Y)\big) \leqslant n(X)$ follows by Lemma 2.2. The inequality $n\big(\mathcal{L}(X,Y)\big) \leqslant n(Y)$ can be proved analogously, using $\Psi_S(T) = S \circ T$ instead of Φ_J.

To prove the moreover part it suffices to observe that if $T \in \mathcal{Z}(X,Y) \subset \mathcal{L}(X,Y)$ and $J \in \mathcal{L}(X)$, then $\Phi_J(T) = T \circ J$ belongs to $\mathcal{Z}(X,Y)$ for every $T \in \mathcal{Z}(X,Y)$ by the ideal property. So the map $J \longmapsto \Phi_J$ is an isometric embedding from $\mathcal{L}(X)$ to $\mathcal{L}(\mathcal{Z}(X,Y))$ carrying Id_X to $\mathrm{Id}_{\mathcal{Z}(X,Y)}$, and the result follows again by Lemma 2.2. For the inequality involving $n(Y)$, the argument is analogous, considering now that $\Psi_S(T) = S \circ T \in \mathcal{Z}(X,Y)$ for every $T \in \mathcal{Z}(X,Y)$ and so the map $S \longmapsto \Psi_S$ is an isometric embedding from $\mathcal{L}(Y)$ to $\mathcal{L}(\mathcal{Z}(X,Y))$ carrying Id_Y to $\mathrm{Id}_{\mathcal{Z}(X,Y)}$. □

We can get a stronger result for the numerical indices of $\mathcal{K}(X,Y)$ and $\mathcal{W}(X,Y)$. To do so, we recall that $\mathcal{K}_{w^*}(X^*,Y)$ denotes the space of compact operators that are weak*-weakly continuous from X^* into Y endowed with the usual operator norm. This space was originally introduced by L. Schwartz [23] as the ε-product of the spaces X and Y. It is well-known that $\mathcal{K}_{w^*}(X^*,Y) \equiv \mathcal{K}_{w^*}(Y^*,X)$ and that $\mathcal{K}(X,Y)$ can be identified with $\mathcal{K}_{w^*}(X^{**},Y)$ using the mapping $T \longmapsto T^{**}$. Analogously, $\mathcal{L}_{w^*}(X^*,Y)$ denotes the space of operators that are weak*-weakly continuous from X^* into Y. Finally, we recall that $\mathcal{W}(X,Y)$ can be identified with $\mathcal{L}_{w^*}(X^{**},Y)$. We refer the reader to [20, 23] for background on this type of spaces.

Theorem 2.3. *Let X, Y be Banach spaces, then the following hold:*

(a) $n\big(\mathcal{L}_{w^*}(X^*,Y)\big) \leqslant \min\{n(X), n(Y)\}$.
(b) $n\big(\mathcal{K}_{w^*}(X^*,Y)\big) \leqslant \min\{n(X), n(Y)\}$.
(c) $n\big(\mathcal{W}(X,Y)\big) \leqslant \min\{n(X^*), n(Y)\}$.
(d) $n\big(\mathcal{K}(X,Y)\big) \leqslant \min\{n(X^*), n(Y)\}$.

Proof. To prove (a), for $J \in \mathcal{L}(X)$ we define the operator $\Psi_J\colon \mathcal{L}_{w^*}(X^*,Y) \longrightarrow \mathcal{L}_{w^*}(X^*,Y)$ given by $\Psi_J(T) = T \circ J^*$ for every $T \in \mathcal{L}_{w^*}(X^*,Y)$. Observe that it is well-defined because J^* is weak*-weak* continuous. Moreover, reasoning as in the proof of Proposition 2.1 we get $\|\Psi_J\| = \|J\|$. Therefore, the mapping $J \longmapsto \Psi_J$ is an isometric embedding from $\mathcal{L}(X)$ to $\mathcal{L}\big(\mathcal{L}_{w^*}(X^*,Y)\big)$ carrying Id_X to $\mathrm{Id}_{\mathcal{L}_{w^*}(X^*,Y)}$ so the inequality $n\big(\mathcal{L}_{w^*}(X^*,Y)\big) \leqslant n(X)$ follows from Lemma 2.2.

For the proof of $n\big(\mathcal{L}_{w^*}(X^*,Y)\big) \leqslant n(Y)$, just note that $\mathcal{L}_{w^*}(X^*,Y) \equiv \mathcal{L}_{w^*}(Y^*,X)$.

Let us prove (b). To show that $n\big(\mathcal{K}_{w^*}(X^*,Y)\big) \leqslant n(X)$ it suffices to observe that $\Psi_J|_{\mathcal{K}_{w^*}(X^*,Y)}$, the restriction of Ψ_J to $\mathcal{K}_{w^*}(X^*,Y)$, lies in $\mathcal{L}\big(\mathcal{K}_{w^*}(X^*,Y)\big)$ and satisfies $\big\|\Psi_{J|\mathcal{K}_{w^*}(X^*,Y)}\big\| = \|J\|$. Therefore, the mapping $J \longmapsto \Psi_{J|\mathcal{K}_{w^*}(X^*,Y)}$ is an isometric embedding from $\mathcal{L}(Y)$ to $\mathcal{L}\big(\mathcal{K}_{w^*}(X^*,Y)\big)$ carrying Id_X to $\mathrm{Id}_{\mathcal{K}_{w^*}(X^*,Y)}$ and Lemma 2.2 gives the result. To prove $n\big(\mathcal{K}_{w^*}(X^*,Y)\big) \leqslant n(Y)$, we use what we just proved and the identification $\mathcal{K}_{w^*}(X^*,Y) \equiv \mathcal{K}_{w^*}(Y^*,X)$.

(c) follows from (a) using the identification $\mathcal{W}(X,Y) \equiv \mathcal{L}_{w^*}(X^{**},Y)$.

(d) follows from (b) using the identification $\mathcal{K}(X,Y) \equiv \mathcal{K}_{w^*}(X^{**},Y)$. □

As a consequence of [4, Examples 3.3] and Theorem 2.3 we have the following interesting examples.

Examples 2.4.

(a) *There exists a real Banach space X with $n(X) = 1$ and $n\big(\mathcal{K}(X,Y)\big) = n\big(\mathcal{W}(X,Y)\big) = 0$ for every Banach space Y. In particular, $n(X) = 1$ and $n(\mathcal{K}(X,X)) = n(\mathcal{W}(X,X)) = 0$.* Indeed, the real space X given in [4, Examples 3.3.a] satisfies $n(X) = 1$ and $n(X^*) = 0$ so $n\big(\mathcal{K}(X,Y)\big) = n\big(\mathcal{W}(X,Y)\big) = 0$ for every Y by Theorem 2.3.

(b) *There exists a complex Banach space X with $n(X) = 1$ and $n\big(\mathcal{K}(X,Y)\big) = n\big(\mathcal{W}(X,Y)\big) = 1/\mathrm{e}$ for every Banach space Y. In particular, $n(X) = 1$ and $n(\mathcal{K}(X,X)) = n(\mathcal{W}(X,X)) = 1/\mathrm{e}$.* The complex space X given in [4, Examples 3.3.b] satisfies $n(X) = 1$ and $n(X^*) = 1/\mathrm{e}$, so it works by Theorem 2.3 and the fact that every complex Banach space has numerical index less than or equal to $1/\mathrm{e}$.

To obtain the analogue of Theorem 2.3 for the numerical index of the space of approximable operators and also to get an analogous result for nuclear operators, we will use their representation as suitable tensor products in the next section.

We emphasize a consequence of the results for the case when the ideal spaces have numerical index one.

Corollary 2.5. *Let X, Y be Banach spaces.*

(1) *If $n(\mathcal{L}(X,Y)) = 1$, then $n(X) = n(Y) = 1$.*
(2) *If $n(\mathcal{K}(X,Y)) = 1$, then $n(X^*) = n(Y) = 1$.*
(3) *If $n(\mathcal{W}(X,Y)) = 1$, then $n(X^*) = n(Y) = 1$.*

One may wonder whether the inequalities obtained for the numerical indices of operator ideals are equalities in general. The following example shows that this is not the case, even for finite-dimensional spaces.

Example 2.6. *There exist finite-dimensional Banach spaces X and Y with $n(X^*) = n(Y) = 1$ and $n\big(\mathcal{L}(X,Y)\big) = n\big(\mathcal{F}(X,Y)\big) = n\big(\mathcal{K}(X,Y)\big) = n\big(\mathcal{W}(X,Y)\big) < 1$.* Indeed, consider $X = \ell_\infty^4$ and $Y = \ell_1^4$, which have numerical index 1, and observe that $n\big(\mathcal{L}(X,Y)\big) < 1$ by [15, Proposition 2.4, Lemma 3.2].

However there are cases in which the equality holds for the spaces of compact and weakly compact operators.

Remark 2.7. *Let K be a compact Hausdorff space, and let X be a Banach space. Then,*

$$n\big(\mathcal{K}(X, C(K))\big) = n\big(\mathcal{W}(X, C(K))\big) = n(X^*).$$

Indeed, the space $\mathcal{K}(X,C(K))$ can be identified with $C(K,X^*)$ (see [9, Theorem VI.7.1]) and we have $n\big(C(K,X^*)\big) = n(X^*)$ by [17, Theorem 5]. The equality $n\big(\mathcal{W}(X,C(K))\big) = n(X^*)$ holds by [16, Corollary 3].

In the next result we give other conditions for which the equality is satisfied for the space of compact operators.

Proposition 2.8. *Let X be a Banach space such that $n(X^{***}) = 1$ and let Z be an isometric predual of ℓ_1. Then, the space $\mathcal{K}(X,Z)^{**}$ has numerical index one. Therefore, so do $\mathcal{K}(X,Z)^*$ and $\mathcal{K}(X,Z)$. In particular, $n\big(\mathcal{K}(c_0)\big) = n\big(\mathcal{K}(c_0)^*\big) = n\big(\mathcal{K}(c_0)^{**}\big) = n\big(\mathcal{L}(\ell_\infty)\big) = 1$ and $n\big(\mathcal{K}(\ell_1, c_0)^{**}\big) = 1$.*

Proof. Since Z has the approximation property, $\mathcal{K}(X,Z) \equiv X^* \hat{\otimes}_\varepsilon Z$. Since Z^* has the approximation property and the Radon-Nikodým property, we can apply [7, Theorem 16.6] to obtain $\mathcal{K}(X,Z)^* \equiv (X^* \hat{\otimes}_\varepsilon Z)^* \equiv X^{**} \hat{\otimes}_\pi \ell_1$. Therefore, $\mathcal{K}(X,Z)^{**} \equiv (X^{**} \hat{\otimes}_\pi \ell_1)^* \equiv \mathcal{L}(X^{**}, \ell_\infty)$. Now, by using the identification between ℓ_∞ and $C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ is the Stone–Čech compactification of $\mathbb{N}$, and the one between $C_{w^*}(\beta\mathbb{N}, X^{***})$ and $\mathcal{L}\big(X^{**}, C(\beta\mathbb{N})\big)$ (see [9, Theorem VI.7.1]), we obtain that

$$n\big(\mathcal{L}(X^{**}, \ell_\infty)\big) = n\big(C_{w^*}(\beta\mathbb{N}, X^{***})\big) \geqslant n(X^{***}) = 1,$$

where the inequality is given by [16, Proposition 7]. Then, $n\big(\mathcal{K}(X,Z)^{**}\big) = 1$ as desired. The other statements follow straightforwardly. $\square$

3. Numerical index of tensor products

Our goal here is to study the numerical index of projective and injective tensor products of Banach spaces. It is known that $n(X \hat{\otimes}_\varepsilon Y)$ and $n(X \hat{\otimes}_\pi Y)$ cannot be computed as a function of $n(X)$ and $n(Y)$. Indeed, it is shown in [17, Example 10] that there exist Banach spaces X and Y with $n(X) = n(Y) = 1$ and such that $n(X \hat{\otimes}_\varepsilon X) < 1$, $n(Y \hat{\otimes}_\pi Y) < 1$, and $n(X \hat{\otimes}_\pi X) = n(Y \hat{\otimes}_\varepsilon Y) = 1$. Therefore, our results will be inequalities, as in the previous section.

Our first result on tensor products follows immediately by Proposition 2.1 and the identifications $(X \hat{\otimes}_\pi Y)^* \equiv \mathcal{L}(X, Y^*) \equiv \mathcal{L}(Y, X^*)$ (see [7, Proposition 3.2], for instance).

Corollary 3.1. *Let X, Y be Banach spaces. Then, $n\big((X \hat{\otimes}_\pi Y)^*\big) \leqslant \min\{n(X^*), n(Y^*)\}$.*

Our main result in this section is the following pair of inequalities.

Theorem 3.2. *Let X, Y be Banach spaces. Then, the following hold:*
(a) $n(X \hat{\otimes}_\pi Y) \leqslant \min\{n(X), n(Y)\}$,
(b) $n(X \hat{\otimes}_\varepsilon Y) \leqslant \min\{n(X), n(Y)\}$.

We introduce some notation in order to present an interesting tool to calculate numerical radii which we will use in the proof of the theorem. Given a Banach space X, $\delta > 0$, and $T \in \mathcal{L}(X)$, we write

$$v_\delta(T) := \sup\big\{|x^*(Tx)| \colon x \in B_X,\ x^* \in B_{X^*},\ \operatorname{Re} x^*(x) > 1 - \delta\big\}.$$

Lemma 3.3 ([11, Lemma 3.4])**.** *Let X be a Banach space. For $T \in \mathcal{L}(X)$, we have that*

$$v(T) = \inf_{\delta > 0} v_\delta(T).$$

Moreover, if $A \subset B_X$ satisfies that $\overline{\operatorname{conv}}(A) = B_X$ and $B \subset B_{X^*}$ satisfies that $\overline{\operatorname{conv}}^{w^*}(B) = B_{X^*}$, then the same equality holds if we replace B_X and B_{X^*} by A and B respectively in the definition of $v_\delta(T)$, that is,

$$v(T) = \inf_{\delta > 0} \sup\big\{|x^*(Tx)| \colon x \in A,\ x^* \in B,\ \operatorname{Re} x^*(x) > 1 - \delta\big\}.$$

Proof of the Theorem 3.2. (a). We prove first $n(X \hat{\otimes}_\pi Y) \leqslant n(X)$. Given $S \in \mathcal{L}(X)$ with $\|S\| = 1$, we consider the operator $T = S \otimes_\pi \mathrm{Id}_Y \in \mathcal{L}(X \hat{\otimes}_\pi Y)$ which satisfies that $\|T\| = \|S\| \|\mathrm{Id}_Y\| = 1$. Since $B_{X \hat{\otimes}_\pi Y} = \overline{\operatorname{conv}}\,(B_X \otimes B_Y)$ and $(X \hat{\otimes}_\pi Y)^* = \mathcal{L}(Y, X^*)$, by Lemma 3.3 we can estimate the numerical radius of T as

$$v(T) = \inf_{\delta > 0} \tilde{v}_\delta(T),$$

where for $\delta > 0$,

$$\tilde{v}_\delta(T) := \sup\left\{|\langle \Phi, Tz\rangle| : z \in B_X \otimes B_Y,\ \Phi \in B_{\mathcal{L}(Y,X^*)},\ \mathrm{Re}\langle \Phi, z\rangle > 1-\delta\right\}.$$

Fixed $\delta > 0$, we claim that $\tilde{v}_\delta(T) \leqslant v_\delta(S)$. Indeed, fix $z = x \otimes y \in B_X \otimes B_Y$ and $\Phi \in B_{\mathcal{L}(Y,X^*)}$ such that $\mathrm{Re}\langle \Phi, z\rangle = \mathrm{Re}\langle \Phi(y), x\rangle > 1-\delta$, define $x^* = \Phi(y) \in B_{X^*}$, and observe that $\mathrm{Re}\, x^*(x) = \mathrm{Re}\langle \Phi, z\rangle > 1-\delta$. Then,

$$|\langle \Phi, Tz\rangle| = |\langle \Phi, Sx \otimes y\rangle| = |\langle \Phi(y), Sx\rangle| = |x^*(Sx)| \leqslant v_\delta(S)$$

which gives $\tilde{v}_\delta(T) \leqslant v_\delta(S)$. Then we get that $v(T) \leqslant v(S)$ and, as $\|T\| = \|S\| = 1$, we deduce that $n(X\hat{\otimes}_\pi Y) \leqslant n(X)$. By repeating this process using this time the identification $(X\hat{\otimes}_\pi Y)^* \equiv \mathcal{L}(X, Y^*)$, we also obtain that $n(X\hat{\otimes}_\pi Y) \leqslant n(Y)$.

(b). We prove $n(X\hat{\otimes}_\varepsilon Y) \leqslant n(X)$. Given $S \in \mathcal{L}(X)$ with $\|S\| = 1$, we consider $T = S \otimes_\varepsilon \mathrm{Id}_Y \in \mathcal{L}(X\hat{\otimes}_\varepsilon Y)$ which satisfies that $\|T\| = \|S\|\|\mathrm{Id}_Y\| = 1$. Since $B_{(X\hat{\otimes}_\varepsilon Y)^*} = \overline{\mathrm{conv}}^{w^*}(B_{X^*} \otimes B_{Y^*})$ and

$$B_{X\hat{\otimes}_\varepsilon Y} = \overline{\{z \in X \otimes Y : \|z\|_\varepsilon \leqslant 1\}},$$

we use the following to estimate the numerical radius of T (again by by Lemma 3.3):

$$v(T) = \inf_{\delta>0} \bar{v}_\delta(T)$$

where

$$\bar{v}_\delta := \sup\left\{|z^*(Tz)| : z^* \in B_{X^*} \otimes B_{Y^*},\ z \in X \otimes Y \text{ with } \|z\|_\varepsilon \leqslant 1,\ \mathrm{Re}\, z^*(z) > 1-\delta\right\}.$$

Given $\delta > 0$, we claim that $\bar{v}_\delta(T) \leqslant v_\delta(S)$. Indeed, fixed $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ with $\|z\|_\varepsilon \leqslant 1$ and $z^* = x_0^* \otimes y_0^* \in B_{X^*} \otimes B_{Y^*}$ with $\mathrm{Re}\, z^*(z) = \mathrm{Re} \sum_{i=1}^n x_0^*(x_i) y_0^*(y_i) > 1-\delta$, we consider $x = \sum_{i=1}^n y_0^*(y_i) x_i \in B_X$ which satisfies

$$\|x\| = \left\|\sum_{i=1}^n y_0^*(y_i) x_i\right\| \leqslant \sup\left\{\left|\sum_{i=1}^n y_0^*(y_i) x^*(x_i)\right| : x^* \in B_{X^*}\right\} \leqslant \|z\|_\varepsilon$$

and $\mathrm{Re}\, x_0^*(x) = \mathrm{Re} \sum_{i=1}^n x_0^*(x_i) y_0^*(y_i) > 1-\delta$. Hence we can write

$$|z^*(Tz)| = \left|\langle x_0^* \otimes y_0^*, \sum_{i=1}^n Sx_i \otimes y_i\rangle\right| = \left|\sum_{i=1}^n x_0^*(Sx_i) y_0^*(y_i)\right| = |x_0^*(Sx)| \leqslant v_\delta(S).$$

Then, we deduce that $\bar{v}_\delta(T) \leqslant v_\delta(S)$ as claimed. From this, we get that $v(S) \geqslant v(T) \geqslant n(X\hat{\otimes}_\varepsilon Y)$. Therefore $n(X\hat{\otimes}_\varepsilon Y) \leqslant n(X)$. The inequality $n(X\hat{\otimes}_\varepsilon Y) \leqslant n(Y)$ follows by symmetry. □

Let us observe that it is not possible to improve Theorem 3.2 to get the numerical index of the dual of the factors in the right-hand side.

Example 3.4. *Let $X_1 = C[0,1]$, $X_2 = L_1[0,1]$ and let Y be a Banach space with $n(Y) = 1$ and $n(Y^*) < 1$ (use* [4, Examples 3.3] *for instance). Then, $X_1\hat{\otimes}_\varepsilon Y \equiv C([0,1], Y)$, so $n(X_1\hat{\otimes}_\varepsilon Y) = 1$ by* [17, Theorem 5]*, while $n(Y^*) < 1$. On the other hand, $X_2\hat{\otimes}_\pi Y \equiv L_1([0,1], Y)$, so $n(X_1\hat{\otimes}_\pi Y) = 1$ by* [17, Theorem 8]*, while $n(Y^*) < 1$.*

Nevertheless, the next inequality for the numerical index of the dual of an injective tensor product holds.

Corollary 3.5. *Let X, Y be Banach spaces. If X^* or Y^* has the approximation property and X or Y has the Radon-Nikodým property, then*

$$n\big((X\hat{\otimes}_\varepsilon Y)^*\big) \leqslant \min\{n(X^*), n(Y^*)\}.$$

Proof. The result is an immediate consequence of Theorem 3.2 as the identification $(X\hat{\otimes}_\varepsilon Y)^* \equiv X^*\hat{\otimes}_\pi Y^*$ holds under the hypotheses (see [7, Theorem 16.6]). □

The next consequence is an inequality for the numerical index of spaces of approximable operators similar to the one given in Theorem 2.3 for compact and weakly compact operators.

Corollary 3.6. *Let X, Y be Banach spaces. Then,*

$$n\big(\mathcal{A}(X,Y)\big) \leqslant \min\{n(X^*), n(Y)\}.$$

Proof. It follows from Theorem 3.2.b as $\mathcal{A}(X,Y) \equiv X^*\hat{\otimes}_\varepsilon Y$ (see [7, Examples 4.2]). □

For the space of nuclear operators we may also give some interesting inequalities.

Corollary 3.7. *Let X, Y be Banach spaces. If either X^* or Y has the approximation property, then the following hold:*

(a) $n\big(\mathcal{N}(X,Y)\big) \leqslant \min\{n(X^*), n(Y)\}$.
(b) $n\big(\mathcal{N}(X,Y)^*\big) \leqslant \min\{n(X^{**}), n(Y^*)\}$.

Proof. (a). Since X^* or Y has the approximation property, we have that $\mathcal{N}(X,Y) \equiv X^*\hat{\otimes}_\pi Y$ (see [7, Corollary 5.7.1]) and the result follows from Theorem 3.2.a.

(b). Corollary 3.1 gives the result using the equality $\mathcal{N}(X,Y)^* = \big(X^*\hat{\otimes}_\pi Y\big)^*$. □

Finally, we may give a result analogous to Corollary 2.5 for the results of this section.

Corollary 3.8. *Let X, Y be Banach spaces.*

(1) *If $n\big((X\hat{\otimes}_\pi Y)^*\big) = 1$, then $n(X^*) = n(Y^*) = 1$.*
(2) *If $n(X\hat{\otimes}_\varepsilon Y) = 1$, then $n(X) = n(Y) = 1$.*
(3) *If $n(X\hat{\otimes}_\pi Y) = 1$, then $n(X) = n(Y) = 1$.*
(4) *If $n\big(\mathcal{A}(X,Y)\big) = 1$, then $n(X^*) = n(Y) = 1$.*
(5) *If $n\big((X\hat{\otimes}_\varepsilon Y)^*\big) = 1$, then $n(X^*) = n(Y^*) = 1$.*
(6) *If $n\big(\mathcal{N}(X,Y)\big) = 1$, then $n(X^*) = n(Y) = 1$.*
(7) *If $n\big(\mathcal{N}(X,Y)^*\big) = 1$, then $n(X^{**}) = n(Y^*) = 1$.*

4. Daugavet property and tensor products

In this section we study the relationship between the Daugavet property and tensor products. A glance at Corollary 3.8 may lead to think that an analogous result can be true for the Daugavet property, that is, if $X\hat{\otimes}_\pi Y$ or $X\hat{\otimes}_\varepsilon Y$ has the Daugavet property, do X and Y inherit this property? The answer is negative in general since, for instance, $L_1([0,1],Y) = L_1[0,1]\hat{\otimes}_\pi Y$ and $C([0,1],Y) = C[0,1]\hat{\otimes}_\varepsilon Y$ have the Daugavet property for every Banach space Y, regardless that Y has the Daugavet property or not. Our goal here is to show some cases in which the Daugavet property of a tensor product passes to one of the factors. To state our results, we need the definition and basic properties of the concept of slicely countably determined sets introduced in [3], where we refer for background. Let A be a bounded subset of a Banach space X. A countable family $\{V_n\colon n\in\mathbb{N}\}$ of subsets of A is called *determining* for A if the inclusion $A \subseteq \overline{\mathrm{conv}}(B)$ holds for every subset $B \subseteq A$ intersecting all the sets V_n. Recall that a *slice* of A is a nonempty intersection of A with an open half space, and for $x^*\in X^*$ and $\delta>0$, we write

$$\mathrm{Slice}(A,x^*,\delta) := \{x\in A\colon\ \mathrm{Re}\, x^*(x) > \sup \mathrm{Re}\, x^*(A) - \delta\}.$$

The set A is said to be *slicely countably determined* (*SCD* in short) if there exists a countable family of slices which is determining for A. Examples of SCD sets are the Radon-Nikodým set and those sets not containing basic sequences equivalent to the basis of ℓ_1 [3]. A bounded linear operator $T\colon X \longrightarrow Y$ between two Banach spaces X and Y is an *SCD-operator* if $T(B_X)$ is an SCD set, so examples of SCD-operators are the strong Radon-Nikodým ones and those not fixing copies of ℓ_1 [3]. Finally, let us comment that a set A is SCD if and only if $\overline{\text{conv}}(A)$ is SCD [10, Proposition 7.17]. Consequently, if A is SCD then so is every set C satisfying $A \subset \overline{C} \subset \overline{\text{conv}}(A)$.

The main result of this section is the following one which deals with projective tensor products.

Theorem 4.1. *Let X, Y be Banach spaces. Suppose that B_Y is an SCD set and $X\hat{\otimes}_\pi Y$ has the Daugavet property. Then, X has the Daugavet property.*

We need the following preliminary result which shows that the projective tensor product of an SCD-operator and a rank-one operator is again an SCD-operator on a projective tensor product.

Lemma 4.2. *Let X, Y be Banach spaces, let $S \in \mathcal{L}(X)$ be a rank-one operator and let $T \in \mathcal{L}(Y)$ be an SCD-operator. Then $S \otimes_\pi T \in \mathcal{L}(X\hat{\otimes}_\pi Y)$ is an SCD-operator.*

Proof. We may and do assume that $\|S\| = \|T\| = 1$. In order to prove that $[S \otimes_\pi T](B_{X\hat{\otimes}_\pi Y})$ is SCD it is enough to prove that $S(B_X) \otimes T(B_Y)$ is SCD as

$$S(B_X) \otimes T(B_Y) \subset [S \otimes_\pi T](B_{X\hat{\otimes}_\pi Y}) = [S \otimes_\pi T](\overline{\text{conv}}(B_X \otimes B_Y)) \subset \overline{\text{conv}}\left(S(B_X) \otimes T(B_Y)\right).$$

Since S is a rank-one operator, there exist $x_0 \in S_X$ and $\Gamma \subset \mathbb{K}$ such that $S(B_X) = \Gamma\{x_0\}$ (Γ equals either $B_{\mathbb{K}}$ or its interior). So we can write $S(B_X) \otimes T(B_Y) = \{x_0\} \otimes \Gamma T(B_Y)$. Observe that $\Gamma T(B_Y)$ is SCD since $T(B_Y)$ is SCD and

$$T(B_Y) \subset \overline{\Gamma T(B_Y)} \subset \overline{T(B_Y)}.$$

Therefore, for each $n \in \mathbb{N}$ we can find $V_n = \text{Slice}(\Gamma T(B_Y), y_n^*, \varepsilon_n)$ such that the sequence $\{V_n \colon n \in \mathbb{N}\}$ is determining for $\Gamma T(B_Y)$. Now fix $x_0^* \in S_{X^*}$ satisfying $\operatorname{Re} x_0^*(x_0) = 1$ and, for each $n \in \mathbb{N}$, define $\varphi_n \in (X\hat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$ by $\varphi_n(x) = x_0^*(x)y_n^*$ for every $x \in X$. Let us prove that the slices

$$S_n = \{x_0\} \otimes V_n = \text{Slice}\left(\{x_0\} \otimes \Gamma T(B_Y), \varphi_n, \varepsilon_n\right) \qquad (n \in \mathbb{N})$$

form a determining sequence for $\{x_0\} \otimes \Gamma T(B_Y)$. Indeed, if $B \subseteq \{x_0\} \otimes \Gamma T(B_Y)$ intersects all the S_n, then B must be of the form $\{x_0\} \otimes B_2$ with $B_2 \subset \Gamma T(B_Y)$ satisfying $B_2 \cap V_n \neq \emptyset$ for every $n \in \mathbb{N}$. Since V_n is determining for $\Gamma T(B_Y)$, this implies that $\Gamma T(B_Y) \subset \overline{\text{conv}}(B_2)$ and thus

$$\{x_0\} \otimes \Gamma T(B_Y) \subset \{x_0\} \otimes \overline{\text{conv}}(B_2) \subset \overline{\text{conv}}(B)$$

which shows that the sequence $\{S_n\}$ is determining for $\{x_0\} \otimes \Gamma T(B_Y) = S(B_X) \otimes T(B_Y)$. □

We are ready to show that the Daugavet property passes from the projective tensor product to one of the factors if the other one is SCD.

Proof of Theorem 4.1. Fix a rank-one operator $S \in \mathcal{L}(X)$ and consider $T = S \otimes_\pi \text{Id}_Y \in \mathcal{L}(X\hat{\otimes}_\pi Y)$ which satisfies $\|T\| = \|S\|$ and is an SCD-operator by Lemma 4.2. Since $X\hat{\otimes}_\pi Y$ has the Daugavet property, T satifisfies the Daugavet equation by [3, Corollary 5.9]:

$$\left\|\text{Id}_{X\hat{\otimes}_\pi Y} + T\right\| = 1 + \|T\| = 1 + \|S\|.$$

By the definition of T we have

$$\left\|\text{Id}_{X\hat{\otimes}_\pi Y} + T\right\| = \|(\text{Id}_X + S) \otimes_\pi \text{Id}_Y\| = \|\text{Id}_X + S\|$$

and so $\|\text{Id}_X + S\| = 1 + \|S\|$, as desired. □

We do not know whether the corresponding result for the injective tensor product is true or not. But we have the following positive result in the same line.

Proposition 4.3. *Let X, Y be Banach spaces such that $X\hat{\otimes}_\varepsilon Y$ has the Daugavet property. Suppose that the norm of Y is Fréchet differentiable at a point $y_0 \in S_Y$. Then, X has the Daugavet property.*

We need the following characterization of the Daugavet property which appears in the seminal paper [13].

Lemma 4.4 ([13, Lemma 2.2])**.** *Let X be a Banach space. Then, the following assertions are equivalent:*

(i) *X has the Daugavet property;*
(ii) *for every $x \in S_X$, $x^* \in S_{X^*}$ and $\varepsilon > 0$, there is $y \in \operatorname{Slice}(S_X, x^*, \varepsilon)$ such that $\|x + y\| > 2 - \varepsilon$;*
(iii) *for every $x \in S_X$, $x^* \in S_{X^*}$ and $\varepsilon > 0$, there is $y^* \in \operatorname{Slice}(S_{X^*}, x, \varepsilon)$ such that $\|x^*+y^*\| > 2-\varepsilon$.*

Proof of Proposition 4.3. Since the norm of Y is Fréchet differentiable at $y_0 \in S_Y$, there is a unique $y_0^* \in S_{Y^*}$ which is strongly exposed in B_{Y^*} by y_0, that is,

$$\forall\varepsilon > 0\ \exists\delta > 0\colon y^* \in B_{Y^*},\ \operatorname{Re} y^*(y_0) > 1 - \delta \Longrightarrow \|y_0^* - y^*\| < \varepsilon. \tag{4.1}$$

Given $x_0^* \in S_{X^*}$ and $x_0 \in B_X$, we consider $u_0 = x_0 \otimes y_0 \in B_{X\hat{\otimes}_\varepsilon Y}$ and $\varphi_0 = x_0^* \otimes y_0^* \in S_{(X\hat{\otimes}_\varepsilon Y)^*}$. Since $X\hat{\otimes}_\varepsilon Y$ has the Daugavet property, by Lemma 4.4, fixed $\varepsilon > 0$, we may find $\varphi \in \operatorname{Slice}\left(B_{(X\hat{\otimes}_\varepsilon Y)^*}, u_0, \delta\right)$ such that $\|\varphi_0 + \varphi\| > 2 - \varepsilon$. As $B_{(X\hat{\otimes}_\varepsilon Y)^*} = \overline{\operatorname{conv}}^{w^*}(B_{X^*} \otimes B_{Y^*})$, we may suppose that $\varphi = x^* \otimes y^*$ with $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$. On the one hand, from $\varphi \in \operatorname{Slice}\left(B_{(X\hat{\otimes}_\varepsilon Y)^*}, u_0, \delta\right)$ it follows that $x^* \in \operatorname{Slice}(B_{X^*}, x_0, \delta)$ and $y^* \in \operatorname{Slice}(B_{Y^*}, y_0, \delta)$. On the other hand, we can write

$$2 - \varepsilon < \|\varphi_0 + \varphi\| \leqslant \|x_0^* \otimes y_0^* + x^* \otimes y_0^*\| + \|x^* \otimes y_0^* - x^* \otimes y^*\| \leqslant \|x_0^* + x^*\| + \|y_0^* - y^*\|.$$

But $\|y_0^* - y^*\| < \varepsilon$ by (4.1), so we deduce that $\|x_0^* + x^*\| > 2 - 2\varepsilon$. Now, X has the Daugavet property by Lemma 4.4. □

We can obtain a result similar to the previous one for the projective tensor product which does not follow from Theorem 4.1.

Proposition 4.5. *Let X, Y be Banach spaces such that $X\hat{\otimes}_\pi Y$ has the Daugavet property. Suppose that the norm of Y^* is Fréchet differentiable at a point $y_0^* \in S_{Y^*}$. Then, X has the Daugavet property.*

Proof. Since $y_0^* \in S_{Y^*}$ is a point of Fréchet differentiability, there is a unique $y_0 \in S_Y$ satisfying:

$$\forall\varepsilon > 0\ \exists\delta > 0\colon y \in B_Y,\ \operatorname{Re} y_0^*(y) > 1 - \delta \Longrightarrow \|y_0 - y\| < \varepsilon. \tag{4.2}$$

Given $x_0 \in S_X$ and $x_0^* \in B_{X^*}$, we consider $u_0 = x_0 \otimes y_0 \in S_{X\hat{\otimes}_\pi Y}$ and $\varphi_0 = x_0^* \otimes y_0^* \in B_{(X\hat{\otimes}_\pi Y)^*}$. Since $X\hat{\otimes}_\pi Y$ has the Daugavet property and $B_{X\hat{\otimes}_\pi Y} = \overline{\operatorname{conv}}(B_X \otimes B_Y)$, fixed $\varepsilon > 0$, we may find $u \in \operatorname{Slice}\left(B_{X\hat{\otimes}_\pi Y}, \varphi_0, \delta\right)$ of the form $u = x \otimes y$ with $x \in B_X$ and $y \in B_Y$ such that $\|u_0 + u\| > 2 - \varepsilon$. On the one hand, from $u \in \operatorname{Slice}\left(B_{X\hat{\otimes}_\pi Y}, \varphi_0, \delta\right)$ it follows that $x \in \operatorname{Slice}(B_X, x_0^*, \delta)$ and $y \in \operatorname{Slice}(B_Y, y_0^*, \delta)$. On the other hand, we have

$$2 - \varepsilon < \|u_0 + u\| \leqslant \|x_0 \otimes y_0 + x \otimes y_0\| + \|x \otimes y - x \otimes y_0\| \leqslant \|x_0 + x\| + \|y - y_0\|.$$

But $\|y - y_0\| < \varepsilon$ by (4.2), so $\|x_0 + x\| > 2 - 2\varepsilon$. Now, X has the Daugavet property by Lemma 4.4. □

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Chapter III

On the numerical index of absolute symmetric norms on the plane

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ON THE NUMERICAL INDEX OF ABSOLUTE SYMMETRIC NORMS ON THE PLANE

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Abstract. We give a lower bound for the numerical index of two-dimensional real spaces with absolute and symmetric norm. This allows us to compute the numerical index of the two-dimensional real L_p-space for $3/2 \leqslant p \leqslant 3$.

1. Introduction

The numerical index of a Banach space is a constant relating the norm and the numerical range of bounded linear operators on the space. Let us recall the relevant definitions. Given a Banach space X, we will write X^* for its topological dual and $\mathcal{L}(X)$ for the Banach algebra of all (bounded linear) operators on X. For an operator $T \in \mathcal{L}(X)$, its *numerical range* is defined as

$$V(T) := \{x^*(Tx) \colon x^* \in X^*,\ x \in X,\ \|x^*\| = \|x\| = x^*(x) = 1\},$$

and its *numerical radius* is

$$v(T) := \sup\{|\lambda| \colon \lambda \in V(T)\}.$$

Clearly, v is a seminorm on $\mathcal{L}(X)$ satisfying $v(T) \leqslant \|T\|$ for every $T \in \mathcal{L}(X)$. The *numerical index* of X is the constant given by

$$n(X) := \inf\{v(T) \colon T \in \mathcal{L}(X),\ \|T\| = 1\}$$

or, equivalently, $n(X)$ is the greatest constant $k \geqslant 0$ satisfying $k\,\|T\| \leqslant v(T)$ for every $T \in \mathcal{L}(X)$. Classical references on numerical index are the paper [3] and the monographs by F.F. Bonsall and J. Duncan [1, 2] from the seventies. There has been a deep development of this field of study with the contribution of several authors. The reader will find the state of the art on the subject in the survey paper [7] and references therein.

In the following we recall some results concerning the numerical index which will be relevant to our discussion. It is clear that $0 \leqslant n(X) \leqslant 1$ for every Banach space X. In the real case, all values in $[0, 1]$ are possible for the numerical index. In the complex case, one has $1/\mathrm{e} \leqslant n(X) \leqslant 1$ and all of these values are possible. Let us also mention that $v(T^*) = v(T)$ for every $T \in \mathcal{L}(X)$, where T^* is the adjoint operator of T (see [1, § 9]), so it clearly follows that $n(X^*) \leqslant n(X)$. Although the equality does not always hold, when X is a reflexive space, one clearly gets $n(X) = n(X^*)$. There are some

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classical Banach spaces for which the numerical index has been calculated. If H is a Hilbert space of dimension greater than one, then $n(H)=0$ in the real case and $n(H)=1/2$ in the complex case. Besides, $n(L_1(\mu))=1$ and the same happens to all its isometric preduals. In particular, it follows that $n\big(C(K)\big)=1$ for every compact K.

The problem of computing the numerical index of the L_p-spaces has been latent since the beginning of the theory [3]. In order to present the known results on this matter we need to fix some notation. For $1<p<\infty$, we write ℓ_p^m for the m-dimensional L_p-space, $q=p/(p-1)$ for the conjugate exponent to p, and

$$M_p := \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p} = \max_{t\geqslant 1} \frac{|t^{p-1}-t|}{1+t^p},$$

which is the numerical radius of the operator represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ defined on the real space ℓ_p^2. This can be found in [9, Lemma 2], where it is also observed that $M_q=M_p$. Although it is known that $\big\{n(\ell_p^2)\colon 1<p<\infty\big\}=[0,1[$ in the real case (see [3, p. 488]), the exact computation of $n(\ell_p^2)$ has not been achieved for $p\neq 2$, all the more of $n(\ell_p)$. However, some results have been obtained on the numerical index of the L_p-spaces [4, 5, 6, 9, 10], we summarize them in the following list.

(a) The sequence $\big(n(\ell_p^m)\big)_{m\in\mathbb{N}}$ is decreasing.
(b) $n\big(L_p(\mu)\big)=\inf\{n(\ell_p^m)\colon m\in\mathbb{N}\}$ for every measure μ such that $\dim\big(L_p(\mu)\big)=\infty$.
(c) In the real case, $n(L_p[0,1])\geqslant \frac{M_p}{12}$.
(d) In the real case, $\max\left\{\dfrac{1}{2^{1/p}},\ \dfrac{1}{2^{1/q}}\right\} M_p \leqslant n(\ell_p^2)\leqslant M_p$.

The presence of the numerical radius of the operator represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the value of the numerical index of L_p-spaces is not a coincidence. Although there are not too many examples of Banach spaces for which the numerical index has been computed, for those two-dimensional real spaces with absolute and symmetric norm whose numerical index is known, it coincides with the numerical radius of the mentioned operator. This happens, for instance, to a family of octagonal norms and to the spaces whose unit ball is a regular polygon, see [8, Theorem 2 and Theorem 5]. The aim of this paper is to show that the same happens for many absolute and symmetric norms on $\mathbb{R}^2$, this is the content of Theorem 2.2. We say that a norm $\|\cdot\|\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is *absolute* if $\|(1,0)\|=\|(0,1)\|=1$ and

$$\|(a,b)\| = \|(|a|,|b|)\|$$

for every $a,b\in\mathbb{R}$, and that the norm is *symmetric* if $\|(b,a)\|=\|(a,b)\|$ for every $a,b\in\mathbb{R}$. Some of the most important examples of absolute and symmetric norms are ℓ_p-norms on $\mathbb{R}^2$. As a major consequence of Theorem 2.2 we show that $n(\ell_p^2)=M_p$ for $3/2\leqslant p\leqslant 3$, which improves partially [9, Theorem 1] and throws some light to the long standing problem of computing the numerical index of L_p-spaces.

To finish the introduction, we recall some facts about numerical radius and about optimization of linear functions on convex sets that will be useful in our arguments. Let X be a Banach space, and suppose that $S\in\mathcal{L}(X)$ is an onto isometry. Then, for every operator $T\in\mathcal{L}(X)$, it is easy to check that

$$v(T)=v(\pm S^{-1}TS).$$

This becomes particularly useful when X is $\mathbb{R}^2$ endowed with an absolute and symmetric norm, as we can find a basis of the space of operators $\mathcal{L}(X)$ formed by onto isometries:

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad I_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For a convex set A, $\operatorname{ext}(A)$ stands for the set of its extreme points, that is, those points which are not the mid point of any non-trivial segment contained in A. By Minkowski's Theorem (see [11, Corollary 1.13] for instance) a nonempty compact convex subset of $\mathbb{R}^n$ is equal to the convex hull of its extreme points. Therefore, every linear function on a compact convex set attains its minimum (and its maximum) at an extreme point of the set.

2. The results

We start with an easy lemma showing that, for two dimensional real spaces with absolute and symmetric norm, the elements in the numerical range of I_4 are smaller than those of I_j for $j = 1, 2, 3$.

Lemma 2.1. *Let X be $\mathbb{R}^2$ endowed with an absolute and symmetric norm. Then*

$$|x^*(I_j x)| \geqslant |x^*(I_4 x)| \qquad (j = 1, 2, 3)$$

for every $x \in S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$.

Proof. Fixed $x = (a, b) \in S_X$ and $x^* = (\alpha, \beta) \in S_{X^*}$ with $x^*(x) = \alpha a + \beta b = 1$, it is obvious that $1 = |x^*(I_1 x)| \geqslant |x^*(I_4 x)|$. To prove $|x^*(I_3 x)| \geqslant |x^*(I_4 x)|$ observe that

$$1 = \alpha a + \beta b \leqslant |\alpha|\,|a| + |\beta|\,|b| \leqslant \|(|\alpha|, |\beta|)\|\,\|(|a|, |b|)\| = \|x^*\|\,\|x\| = 1 \tag{1}$$

which clearly implies $\alpha a = |\alpha|\,|a|$ and $\beta b = |\beta|\,|b|$. Moreover, we deduce that αb and βa have the same sign as $\alpha a\,\beta b \geqslant 0$ and, therefore,

$$|x^*(I_3 x)| = |\alpha b + \beta a| = |\alpha|\,|b| + |\beta|\,|a| \geqslant |\alpha b - \beta a| = |x^*(I_4 x)|.$$

To prove $|x^*(I_2 x)| \geqslant |x^*(I_4 x)|$ observe that

$$|x^*(I_2 x)| = |\alpha a - \beta b| = \big||\alpha|\,|a| - |\beta|\,|b|\big| \qquad \text{and} \qquad |x^*(I_4 x)| = |\alpha b - \beta a| = \big||\alpha|\,|b| - |\beta|\,|a|\big|.$$

So, when $|a| = |b|$, it is evident that $|x^*(I_2 x)| = |x^*(I_4 x)|$. When $|a| \neq |b|$ we need the following claim.

Claim: $|a| > |b|$ implies $|\alpha| \geqslant |\beta|$ and $|b| > |a|$ implies $|\beta| \geqslant |\alpha|$.

We only show the first implication, as the second one is analogous. Using the symmetry of the norm and (1) we can write

$$\|(|\beta|, |\alpha|)\|\,\|(|a|, |b|)\| = \|(|\alpha|, |\beta|)\|\,\|(|a|, |b|)\| = |\alpha|\,|a| + |\beta|\,|b|.$$

On the other hand, writing $y^* = (|\beta|, |\alpha|)$ and $y = (|a|, |b|)$,it is clear that

$$\|(|\beta|, |\alpha|)\|\,\|(|a|, |b|)\| \geqslant y^*(y) = |\beta|\,|a| + |\alpha|\,|b|.$$

Therefore, we get $|\beta|\,|a| + |\alpha|\,|b| \leqslant |\alpha|\,|a| + |\beta|\,|b|$, and so $|\beta|(|a| - |b|) \leqslant |\alpha|(|a| - |b|)$. Since $|a| > |b|$, it follows that $|\alpha| \geqslant |\beta|$ and the claim is proved.

Let us finish the proof of $|x^*(I_2 x)| \geqslant |x^*(I_4 x)|$. If $|a| > |b|$, we get $|\alpha| \geqslant |\beta|$ by the claim and, moreover, $|\alpha|\,|a| \geqslant |\alpha|\,|b| \geqslant |\beta|\,|b|$ and $|\alpha|\,|a| \geqslant |\beta|\,|a| \geqslant |\beta|\,|b|$ hold, which clearly imply

$$|x^*(I_2 x)| = \big||\alpha|\,|a| - |\beta|\,|b|\big| \geqslant \big||\alpha|\,|b| - |\beta|\,|a|\big| = |x^*(I_4 x)|.$$

The remaining case $|b| > |a|$ is completely analogous. $\square$

We are ready to state and prove the first main result of the paper.

Theorem 2.2. *Let X be $\mathbb{R}^2$ endowed with an absolute and symmetric norm. Let $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ be such that $|x_0^*(I_4 x_0)| = v(I_4)$ and write $c_j = |x_0^*(I_j x_0)|$ for every $j = 1, \ldots, 4$. If $c_4 = 0$, then $n(X) = 0$. If otherwise $c_4 > 0$, then*

$$n(X) \geqslant \min \left\{ c_4, \frac{2}{1 + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}} \right\}.$$

Moreover, if the inequality $c_4 \left(1 + \frac{1}{c_2} + \frac{1}{c_3}\right) \leqslant 1$ holds, then

$$n(X) = v(I_4).$$

Proof. Observe first that $n(X) \leqslant v(I_4)$ since $\|I_4\| = 1$. So $n(X) = 0$ holds when $c_4 = 0$. Thus we assume that $c_4 > 0$ which, by Lemma 2.1, implies $c_j > 0$ for $j = 2, 3$.

Fixed a non-zero operator $T \in \mathcal{L}(X)$ our aim is to estimate $\frac{v(T)}{\|T\|}$. To do so, observe that there exist $A_j \in \mathbb{R}$ for $j = 1, \ldots, 4$ satisfying $T = \sum_{k=1}^4 A_k I_k$, as the onto isometries $I_1, \ldots, I_4$ form a basis of $\mathcal{L}(X)$. Observe next that

$$\begin{aligned}
I_1^{-1} T I_1 &= A_1 I_1 + A_2 I_2 + A_3 I_3 + A_4 I_4 \\
I_2^{-1} T I_2 &= A_1 I_1 + A_2 I_2 - A_3 I_3 - A_4 I_4 \\
I_3^{-1} T I_3 &= A_1 I_1 - A_2 I_2 + A_3 I_3 - A_4 I_4 \\
I_4^{-1} T I_4 &= A_1 I_1 - A_2 I_2 - A_3 I_3 + A_4 I_4
\end{aligned}$$

so, using that $v(T) = v(\pm I_j^{-1} T I_j)$ for every $j = 1, \ldots, 4$, we can write

$$\begin{aligned}
v(T) &= \max \left\{ \left| \pm v(I_j^{-1} T I_j) \right| : j = 1, \ldots, 4 \right\} \\
&\geqslant \max \left\{ \left| \pm x_0^*(I_j^{-1} T I_j x_0) \right| : j = 1, \ldots, 4 \right\} \\
&= \max\{ \left| \pm (A_1 x_0^*(I_1 x_0) + A_2 x_0^*(I_2 x_0) + A_3 x_0^*(I_3 x_0) + A_4 x_0^*(I_4 x_0)) \right|, \\
&\qquad \left| \pm (A_1 x_0^*(I_1 x_0) + A_2 x_0^*(I_2 x_0) - A_3 x_0^*(I_3 x_0) - A_4 x_0^*(I_4 x_0)) \right|, \\
&\qquad \left| \pm (A_1 x_0^*(I_1 x_0) - A_2 x_0^*(I_2 x_0) + A_3 x_0^*(I_3 x_0) - A_4 x_0^*(I_4 x_0)) \right|, \\
&\qquad \left| \pm (A_1 x_0^*(I_1 x_0) - A_2 x_0^*(I_2 x_0) - A_3 x_0^*(I_3 x_0) + A_4 x_0^*(I_4 x_0)) \right| \}.
\end{aligned}$$

The combination of signs in the last expression allows us to deduce

$$v(T) \geqslant \max \left\{ \sum_{\substack{k=1 \\ k \neq j}}^{4} |A_k| c_k - |A_j| c_j \colon j = 1, \ldots, 4 \right\}.$$

Now, writing $\|T\|_+ = \sum_{k=1}^4 |A_k|$, we get $\|T\| = \left\| \sum_{k=1}^4 A_k I_k \right\| \leqslant \|T\|_+$. Besides, calling $\alpha_j = \frac{|A_j|}{\|T\|_+}$ for $j = 1, \ldots, 4$, we can estimate $n(X)$ as follows:

$$\begin{aligned}
n(X) &= \inf \left\{ \frac{v(T)}{\|T\|} \colon T \in \mathcal{L}(X),\ T \neq 0 \right\} \geqslant \inf \left\{ \frac{v(T)}{\|T\|_+} \colon T \in \mathcal{L}(X),\ T \neq 0 \right\} \\
&\geqslant \min_{\substack{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1 \\ \alpha_j \geqslant 0}} \max \left\{ \sum_{\substack{k=1 \\ k \neq j}}^{4} \alpha_k c_k - \alpha_j c_j \colon j = 1, \ldots, 4 \right\}.
\end{aligned}$$

So, defining the function

$$f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\max\left\{\sum_{\substack{k=1\\k\neq j}}^{4}\alpha_k c_k-\alpha_j c_j\colon j=1,\dots,4\right\}\qquad\big((\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\mathbb{R}^4\big)$$

and the compact set

$$K=\left\{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\mathbb{R}^4\colon \sum_{k=1}^{4}\alpha_k=1,\ \alpha_j\geqslant 0,\ j=1,\dots,4\right\},$$

we have that

$$n(X)\geqslant \min_K f.$$

Our goal now is to compute this minimum. As f is the maximum of linear functions, following a typical strategy of linear programming, we can transform this minimization problem into a linear optimization one: we have to minimize the function

$$g(\alpha_1,\alpha_2,\alpha_3,\alpha_4,z)=z\qquad\big((\alpha_1,\alpha_2,\alpha_3,\alpha_4,z)\in\mathbb{R}^5\big)$$

on the compact convex set

$$K'=\left\{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,z)\in\mathbb{R}^5\colon \sum_{k=1}^{4}\alpha_k=1,\ z\leqslant 2,\ \alpha_j\geqslant 0,\ z\geqslant\sum_{\substack{k=1\\k\neq j}}^{4}\alpha_k c_k-\alpha_j c_j,\ j=1,\dots,4\right\}.$$

In fact, it is easy to check that

$$\min_K f=\min_{K'} g.$$

Indeed, if $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in K$ is such that $\min\limits_K f=f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, then we clearly have that

$$\begin{aligned}&\big(\alpha_1,\alpha_2,\alpha_3,\alpha_4,f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\big)\in K'\qquad\text{and}\\&g\big(\alpha_1,\alpha_2,\alpha_3,\alpha_4,f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\big)=f(\alpha_1,\alpha_2,\alpha_3,\alpha_4).\end{aligned}$$

Therefore, we have $\min\limits_K f\geqslant\min\limits_{K'} g$. To prove the reverse inequality take $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,z)\in K'$ satisfying $\min\limits_{K'} g=g(\alpha_1,\alpha_2,\alpha_3,\alpha_4,z)=z$ and observe that $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in K$ and $f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\leqslant z$. So we get $\min\limits_K f\leqslant\min\limits_{K'} g$.

To finish the proof we just have to compute $\min\limits_{K'} g$. Since K' is a compact convex set, the linear function g attains its minimum on K' at an extreme point of K'. Fixed $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,z)\in\operatorname{ext}(K')$, as $K'\subset\mathbb{R}^5$, it must happen that at least five of the ten restrictions that define K' become equalities. We calculate $g(\alpha_1,\alpha_2,\alpha_3,\alpha_4,z)$ depending on which equalities occur. If there exists $j_0\in\{1,\dots,4\}$ such that $\alpha_{j_0}=0$, then

$$g(\alpha_1,\alpha_2,\alpha_3,\alpha_4,z)=z\geqslant\sum_{\substack{k=1\\k\neq j_0}}\alpha_k c_k\geqslant c_4\sum_{\substack{k=1\\k\neq j_0}}\alpha_k=c_4,$$

where we have used that $c_j\geqslant c_4$ for every $j\in\{1,2,3\}$ by Lemma 2.1.

If otherwise $\alpha_j > 0$ for every $j \in \{1,\dots,4\}$, we have that $z = \sum_{\substack{k=1\\k\neq j}}^4 \alpha_k c_k - \alpha_j c_j$ for every $j \in \{1,\dots,4\}$, as $z < 2$ whenever $z = \sum_{\substack{k=1\\k\neq j}}^4 \alpha_k c_k - \alpha_j c_j$ for any j. Hence

$$\sum_{k=2}^{4} \alpha_k c_k - \alpha_1 c_1 = \sum_{\substack{k=1\\k\neq 2}}^{4} \alpha_k c_k - \alpha_2 c_2 = \sum_{\substack{k=1\\k\neq 3}}^{4} \alpha_k c_k - \alpha_3 c_3 = \sum_{k=1}^{3} \alpha_k c_k - \alpha_4 c_4,$$

and so $\alpha_1 c_1 = \alpha_2 c_2 = \alpha_3 c_3 = \alpha_4 c_4$. Since $c_1 = 1$, we get

$$\alpha_2 = \frac{\alpha_1}{c_2}, \qquad \alpha_3 = \frac{\alpha_1}{c_3}, \qquad \alpha_4 = \frac{\alpha_1}{c_4},$$

and it follows from $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$ that

$$\alpha_1 = \frac{1}{1 + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}}.$$

Therefore,

$$g(\alpha_1, \alpha_2, \alpha_3, \alpha_4, z) = z = 2\alpha_1 = \frac{2}{1 + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}}.$$

So, for every $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, z) \in \operatorname{ext}(K')$ we have shown that either

$$g(\alpha_1, \alpha_2, \alpha_3, \alpha_4, z) \geqslant \frac{2}{1 + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}}$$

or $g(\alpha_1, \alpha_2, \alpha_3, \alpha_4, z) \geqslant c_4$. Thus, we can write

$$n(X) \geqslant \min_{K'} g \geqslant \min\left\{ c_4, \frac{2}{1 + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}} \right\}$$

which finishes the first part of the proof. Finally, to prove the moreover part, it suffices to observe that if $c_4\left(1 + \frac{1}{c_2} + \frac{1}{c_3}\right) \leqslant 1$, then

$$c_4 \leqslant \frac{2}{1 + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}},$$

and hence, we get $n(X) = c_4 = v(I_4)$. □

Using the preceding result we can obtain the numerical index of two-dimensional L_p-spaces for some values of p. In order to use Theorem 2.2, we need to find one pair $x \in S_{\ell_p^2}$, $x^* \in S_{\ell_q^2}$ satisfying $x^*(x) = 1$, at which I_4 attains its numerical radius. However, this seems to be a rather tricky problem for arbitrary p. We can avoid this by showing that condition $c_4\left(1 + \frac{1}{c_2} + \frac{1}{c_3}\right) \leqslant 1$ in the statement of Theorem 2.2 holds not only for a particular choice of $x \in S_{\ell_p^2}$, $x^* \in S_{\ell_q^*}$ satisfying $x^*(x) = 1$ but for all of them.

Theorem 2.3. *Let $p \in \left[\frac{3}{2}, 3\right]$. Then,*

$$n(\ell_p^2) = M_p = \sup_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p}.$$

Proof. It is known that $n(\ell_2^2) = 0$. Besides, for $p \in]2,3]$ we get $q \in [3/2, 2[$ so, using the fact that $n(\ell_p^2) = n(\ell_q^2)$, the result will be proved if we compute $n(\ell_p^2)$ for $p \in [3/2, 2[$. So we fix $p \in [3/2, 2[$ and use the parametrization of the duality mapping for absolute norms on $\mathbb{R}^2$ given in [3, Lemma 3.2]. Indeed, for $t \in [0,1]$ consider

$$x_t = \frac{1}{(1+t^p)^{1/p}}(1,t) \qquad \text{and} \qquad x_t^* = \frac{1}{(1+t^p)^{\frac{p-1}{p}}}(1,t^{p-1})$$

which satisfy $x_t \in S_{\ell_p^2}$, $x_t^* \in S_{\ell_q^2}$, and $x_t^*(x_t) = 1$. We next define the functions

$$c_1(t) = x_t^*(I_1 x_t) = 1, \qquad c_2(t) = x_t^*(I_2 x_t) = \frac{1-t^p}{1+t^p},$$
$$c_3(t) = x_t^*(I_3 x_t) = \frac{t^{p-1}+t}{1+t^p}, \qquad c_4(t) = x_t^*(I_4 x_t) = \frac{t^{p-1}-t}{1+t^p} \qquad \big(t \in [0,1]\big).$$

Since the maximum defining $v(I_4) = \max_{t\in[0,1]} \frac{t^{p-1}-t}{1+t^p}$ is obviously attained at some $t_0 \in]0,1[$, if we show that $c_4(t)\left(1 + \frac{1}{c_2(t)} + \frac{1}{c_3(t)}\right) \leqslant 1$ for every $t \in]0,1[$, then we will have $n(\ell_p^2) = v(I_4) = M_p$ by Theorem 2.2. So, for fixed $t \in]0,1[$, observe that

$$c_4(t)\left(1+\frac{1}{c_2(t)}+\frac{1}{c_3(t)}\right) = \frac{t^{p-1}-t}{1+t^p}\left(1+\frac{1+t^p}{1-t^p}+\frac{1+t^p}{t^{p-1}+t}\right) = \frac{2t^{p-1}-2t}{1-t^{2p}} + \frac{t^{p-1}-t}{t^{p-1}+t}$$

and, therefore,

$$\begin{aligned} c_4(t)\left(1+\frac{1}{c_2(t)}+\frac{1}{c_3(t)}\right) \leqslant 1 &\iff \frac{2t^{p-1}-2t}{1-t^{2p}} + \frac{t^{p-1}-t}{t^{p-1}+t} \leqslant 1 \\ &\iff \frac{2(t^{p-1}-t)}{1-t^{2p}} \leqslant \frac{2t}{t^{p-1}+t} \\ &\iff 0 \leqslant t - t^{2p-2} + t^2 - t^{2p+1} \\ &\iff 0 \leqslant t(1-t^{2p-3}) + t^2(1-t^{2p-1}). \end{aligned}$$

Since the last inequality holds for $3/2 \leqslant p < 2$ and $t \in]0,1[$, Theorem 2.2 applies and finishes the proof. $\square$

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Chapter IV

On the numerical index of the real two-dimensional L_p space

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ON THE NUMERICAL INDEX OF THE REAL TWO-DIMENSIONAL L_p SPACE

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Abstract. We compute the numerical index of the two-dimensional real L_p space for $\frac{6}{5} \leqslant p \leqslant 1+\alpha_0$ and $\alpha_1 \leqslant p \leqslant 6$, where α_0 is the root of $f(x) = 1+x^{-2}-(x^{-\frac{1}{x}}+x^{\frac{1}{x}})$ and $\frac{1}{1+\alpha_0}+\frac{1}{\alpha_1} = 1$. This, together with the previous results in [Merí and Quero, On the numerical index of absolute symmetric norms on the plane, Linear Multilinear Algebra **69** (2021), no. 5, 971–979] and [Monika and Zheng, The numerical index of ℓ_p^2, Linear Multilinear Algebra (2021), published online, DOI: 10.1080/03081087.2022.2043818], gives the numerical index of the two-dimensional real L_p space for $\frac{6}{5} \leqslant p \leqslant 6$.

1. Introduction

The numerical index of a Banach space is a constant relating the norm and the numerical radius of bounded linear operators on the space. Let us recall the relevant definitions. Given a Banach space X, we will write X^* for its topological dual and $\mathcal{L}(X)$ for the Banach algebra of all bounded linear operators on X. For an operator $T \in \mathcal{L}(X)$, its *numerical radius* is defined as

$$v(T) := \{|x^*(Tx)| \colon x^* \in X^*,\ x \in X,\ \|x^*\| = \|x\| = x^*(x) = 1\}$$

which is a seminorm on $\mathcal{L}(X)$ satisfying $v(T) \leqslant \|T\|$ for every $T \in \mathcal{L}(X)$. The *numerical index* of X is the constant given by

$$n(X) := \inf\{v(T) \colon T \in \mathcal{L}(X),\ \|T\| = 1\}$$

or, equivalently, $n(X)$ is the greatest constant $k \geqslant 0$ satisfying $k\,\|T\| \leqslant v(T)$ for every $T \in \mathcal{L}(X)$. Classical references on numerical index are the paper [3] and the monographs by F.F. Bonsall and J. Duncan [1, 2] from the seventies. In the last decades this field of study has grown in various directions with the contribution of several authors. The reader will find the state of the art on the subject in the survey paper [8] and a more recent account in the first chapter of the book [7].

In the following we recall some results concerning the numerical index which will be relevant to our discussion. It is clear that $0 \leqslant n(X) \leqslant 1$ for every Banach space X. In the real case, the numerical index can take any value in $[0, 1]$. In the complex case, one has $1/\mathrm{e} \leqslant n(X) \leqslant 1$ and all of these values are possible. Let us also mention that $v(T^*) = v(T)$ for every $T \in \mathcal{L}(X)$, where T^* is the adjoint operator of T (see [1, § 9]), so it clearly follows that $n(X^*) \leqslant n(X)$. Although the equality

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does not always hold, when X is a reflexive space, one clearly gets $n(X) = n(X^*)$. There are some classical Banach spaces for which the numerical index has been calculated. If H is a Hilbert space of dimension greater than one, then $n(H) = 0$ in the real case and $n(H) = 1/2$ in the complex case. Besides, $n(L_1(\mu)) = 1$ and the same happens to all its isometric preduals. In particular, it follows that $n\big(C(K)\big) = 1$ for every compact K and the same is true for all finite-codimensional subspaces of $C[0,1]$.

The exact computation of the numerical index of concrete spaces is usually a difficult task. However it has been achieved for some polyhedral Banach spaces [9, 14]. The computation of the numerical index of L_p-spaces when $p \neq 1, 2, \infty$ remains as an important open problem in the theory of numerical index since it started. Let us present the known results on the matter. For $1 < p < \infty$, we write ℓ_p^m for the m-dimensional L_p-space, $q = p/(p-1)$ for the conjugate exponent of p, and

$$M_p := \max_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p} = \max_{t\geqslant 1} \frac{|t^{p-1} - t|}{1 + t^p},$$

which is the numerical radius of the operator represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ defined on the real space ℓ_p^2. This is stated in [10, Lemma 2], where it is also observed that $M_q = M_p$. It is known that the sequence $\big(n(\ell_p^m)\big)_{m\in\mathbb{N}}$ is decreasing and that $n\big(L_p(\mu)\big) = \inf\{n(\ell_p^m) \colon m \in \mathbb{N}\}$ for every measure μ such that $\dim\big(L_p(\mu)\big) = \infty$, this can be found in [4, 5, 6]. Moreover, in the real case, the inequality $n(L_p[0,1]) \geqslant \frac{M_p}{12}$ holds for every $1 < p < \infty$ [11]. Also in the real case, one has $\max\left\{\frac{1}{2^{1/p}}, \frac{1}{2^{1/q}}\right\} M_p \leqslant n(\ell_p^2) \leqslant M_p$ [10].

Very recently, it has been proved [12] that the numerical index is attained at the operator $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for many absolute and symmetric norms on $\mathbb{R}^2$ and, as a major consequence, it is shown that $n(\ell_p^2) = M_p$ for $\frac{3}{2} \leqslant p \leqslant 3$. In [13] the authors polished skilfully the arguments of [12] to provide a slight improvement: the equality $n(\ell_p^2) = M_p$ is proved for $1 + \alpha_0 \leqslant p \leqslant \alpha_1$ where α_0 is the root of $f(x) = 1 + x^{-2} - (x^{-\frac{1}{x}} + x^{\frac{1}{x}})$ and $\frac{1}{1+\alpha_0} + \frac{1}{\alpha_1} = 1$ ($\alpha_0 \approx 0.4547$). The aim of this paper is to show that $n(\ell_p^2) = M_p$ holds for $p \in [\frac{6}{5}, 6]$. The main difference with previous works is the use of Riesz–Thorin interpolation theorem (see [15, Theorem 2.1 in chapter 2], for instance) to estimate the norm of operators on ℓ_p^2. More precisely, we will use that the inequality

$$\|T\| \leqslant \|T\|_1^{1/p} \|T\|_\infty^{1/q}$$

holds for every operator $T \in \mathcal{L}(\ell_p^2)$.

To finish the introduction, we recall two facts about numerical radius that we will need in our discussion. Let X be a Banach space, and suppose that $S \in \mathcal{L}(X)$ is an onto isometry. Then, for every operator $T \in \mathcal{L}(X)$, it is easy to check that

$$v(T) = v(\pm S^{-1} T S).$$

The following result, which can be deduced from [3, Lemma 3.2], will be useful to compute the numerical radius of operators in $\mathcal{L}(\ell_p^2)$.

Lemma 1.1. *Let $1 < p < \infty$ and $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an operator in $\mathcal{L}(\ell_p^2)$. Then*

$$v(T) = \max\left\{ \max_{t\in[0,1]} \frac{|a + d\,t^p| + |b\,t + c\,t^{p-1}|}{1 + t^p}, \max_{t\in[0,1]} \frac{|d + a\,t^p| + |c\,t + b\,t^{p-1}|}{1 + t^p} \right\}.$$

In particular, $M_p = v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max\limits_{t\in[0,1]} \dfrac{\left|t^{p-1}-t\right|}{1+t^p}$.

2. The results

We start our discussion giving some information about the point t_0 where the operator $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{L}(\ell_p^2)$ attains its numerical radius which will be of help in the proof of the main theorem. The exact computation of t_0 for arbitrary p seems to be a rather tricky problem. However, obtaining estimations of its value has allowed to get fruitful information on $n(\ell_p^2)$ as it is done in [13]. The next result arises with the same purpose.

Lemma 2.1. *Let* $t_0 \in]0,1[$ *be such that* $M_p = \max\limits_{t\in[0,1]} \dfrac{|t^{p-1}-t|}{1+t^p} = \dfrac{|t_0^{p-1}-t_0|}{1+t_0^p}$. *The inequalities*

$$\left(\frac{2p-2}{4-p}\right)^{\frac{1}{2-p}} \leqslant t_0 \leqslant \left(\frac{p-1}{2p+1}\right)^{\frac{1}{p}} \qquad \text{and} \qquad t_0^{2p-3} \leqslant \frac{q}{p}$$

hold for every $p \in \left[\frac{6}{5}, \frac{3}{2}\right]$.

Proof. We start showing that $\left(\frac{2p-2}{4-p}\right)^{\frac{1}{2-p}} \leqslant t_0 \leqslant \left(\frac{p-1}{2p+1}\right)^{\frac{1}{p}}$. To do so, take $\xi = \dfrac{p-1}{3}$, define the functions

$$f(t) = \frac{t^\xi}{1+t^p} \qquad \text{and} \qquad g(t) = t^{p-1-\xi} - t^{1-\xi} \qquad (t \in [0,1]),$$

and observe that $\frac{t^{p-1}-t}{1+t^p} = f(t)g(t)$ for $t \in [0,1]$. It is easy to see that f increases until $t_1 = \left(\frac{\xi}{p-\xi}\right)^{\frac{1}{p}} = \left(\frac{p-1}{2p+1}\right)^{\frac{1}{p}}$ and then decreases since

$$f'(t) = \frac{(\xi + (\xi - p)t^p)}{t^{1-\xi}\,(1+t^p)^2} \qquad (0 < t < 1).$$

Similarly, g increases until $t_2 = \left(\dfrac{p-1-\xi}{1-\xi}\right)^{\frac{1}{2-p}} = \left(\dfrac{2p-2}{4-p}\right)^{\frac{1}{2-p}}$ and then decreases as

$$g'(t) = \frac{(p-1-\xi) - (1-\xi)t^{2-p}}{t^{2-p+\xi}} \qquad (0 < t < 1).$$

Therefore, the function $\frac{t^{p-1}-t}{1+t^p} = f(t)g(t)$ increases in the interval $]0, \min\{t_1,t_2\}[$ and decreases in the interval $]\max\{t_1,t_2\},1[$, so we deduce that $\min\{t_1,t_2\} \leqslant t_0 \leqslant \max\{t_1,t_2\}$. Let us show now that $t_2 \leqslant t_1$. To do so, observe that

$$\begin{aligned}
\left(\frac{2p-2}{4-p}\right)^{\frac{1}{2-p}} &\leqslant \left(\frac{p-1}{2p+1}\right)^{\frac{1}{p}} \\
&\Longleftrightarrow \frac{1}{2-p}\log\left(\frac{2p-2}{4-p}\right) \leqslant \frac{1}{p}\log\left(\frac{p-1}{2p+1}\right) \\
&\Longleftrightarrow p\left(\log(2) + \log(p-1) - \log(4-p)\right) \leqslant (2-p)\big(\log(p-1) - \log(2p+1)\big) \\
&\Longleftrightarrow p\log(2) + (2p-2)\log(p-1) + (2-p)\log(2p+1) \leqslant p\log(4-p)
\end{aligned}$$

and define the functions $h,k\colon \left[\frac{6}{5},\frac{3}{2}\right]\longrightarrow\mathbb{R}$ by

$$h(p)=p\log(2)+(2p-2)\log(p-1)+(2-p)\log(2p+1)\qquad\text{and}\qquad k(p)=p\log\left(4-p\right).$$

Since

$$k'(p)=\log(4-p)-\frac{p}{4-p},$$
$$k''(p)=-\frac{1}{4-p}-\frac{4}{(4-p)^2}<0,$$

we get that k' is decreasing which, together with $k'\left(\frac{3}{2}\right)>0$, tells us that k is increasing. Besides, we have that

$$\begin{aligned}h'(p)&=\log(2)+2\log(p-1)+2-\log(2p+1)+\frac{4-2p}{2p+1},\\ h''(p)&=\frac{2}{p-1}-\frac{2}{2p+1}-\frac{10}{(2p+1)^2}\\ &=\frac{4p^2+14}{(p-1)(2p+1)^2}>0,\end{aligned}$$

and so $h'(p)$ is increasing. This, together with $h'\left(\frac{6}{5}\right)<0$ and $h'\left(\frac{3}{2}\right)>0$, tells us that

$$\max_{p\in\left[\frac{6}{5},\frac{3}{2}\right]}h(p)=\max\left\{h\left(\frac{6}{5}\right),h\left(\frac{3}{2}\right)\right\}=h\left(\frac{6}{5}\right).$$

So $\max_{p\in\left[\frac{6}{5},\frac{3}{2}\right]}h(p)=h\left(\frac{6}{5}\right)<k\left(\frac{6}{5}\right)<\min_{p\in\left[\frac{6}{5},\frac{3}{2}\right]}k(p)$.

The inequality $t_0^{2p-3}\leqslant\frac{q}{p}=\frac{1}{p-1}$ is equivalent to $1\leqslant\frac{1}{p-1}t_0^{3-2p}$. As we already know that $\left(\frac{2p-2}{4-p}\right)^{\frac{1}{2-p}}\leqslant t_0$, the required inequality will follow if we prove that

$$1\leqslant\frac{1}{p-1}\left(\frac{2p-2}{4-p}\right)^{\frac{3-2p}{2-p}},$$

which is equivalent to show that

$$\left(\frac{4-p}{2}\right)^{3-2p}\leqslant\frac{1}{(p-1)^{p-1}}.$$

To see this, consider the functions $\phi,\psi\colon\left[\frac{6}{5},\frac{3}{2}\right]\longrightarrow\mathbb{R}$ given by

$$\phi(p)=\left(\frac{4-p}{2}\right)^{3-2p}\qquad\text{and}\qquad\psi(p)=\frac{1}{(p-1)^{p-1}},$$

and observe that ϕ is decreasing since

$$\phi'(p)=\left(\frac{4-p}{2}\right)^{3-2p}\left(-2\log\left(\frac{4-p}{2}\right)-\frac{3-2p}{4-p}\right)<0$$

for $\frac{6}{5}\leqslant p<\frac{3}{2}$. Besides, we have that

$$\psi'(p)=(p-1)^{(1-p)}\left(-\log(p-1)-1\right)$$

so ψ increases in $\left]\frac{6}{5},1+\frac{1}{\mathrm{e}}\right[$ and decreases in $\left]1+\frac{1}{\mathrm{e}},\frac{3}{2}\right[$. Therefore, we deduce that

$$\min_{p\in\left[\frac{6}{5},\frac{3}{2}\right]}\psi(p)=\min\left\{\psi\left(\frac{6}{5}\right),\psi\left(\frac{3}{2}\right)\right\}=\psi\left(\frac{6}{5}\right)>\phi\left(\frac{6}{5}\right)=\max_{p\in\left[\frac{6}{5},\frac{3}{2}\right]}\phi(p)$$

which finishes the proof. $\square$

We are ready to present and prove the main result of the paper.

Theorem 2.2. *Let $p \in \left[\frac{6}{5}, 6\right]$. Then,*

$$n(\ell_p^2) = M_p = \max_{t\in[0,1]} \frac{|t^{p-1} - t|}{1+t^p}.$$

Proof. Using that $n(\ell_p^2) = n(\ell_q^2)$ and that the result is already known for $p \in \left[\frac{3}{2}, 2\right]$ we only need to work for $1 < p \leqslant \frac{3}{2}$. We divide the proof into three claims, the first two are valid for all the values of $1 < p \leqslant \frac{3}{2}$. Observe that

$$\begin{aligned} n(\ell_p^2) &= \inf\left\{\frac{v(T)}{\|T\|} : 0 \neq T \in \mathcal{L}(\ell_p^2)\right\} \\ &= \min\left\{\inf\left\{\frac{v(T)}{\|T\|} : 0 \neq T \in \mathcal{L}(\ell_p^2),\ \|T\|_\infty \leqslant \|T\|_1\right\},\right. \\ &\qquad \left.\inf\left\{\frac{v(T)}{\|T\|} : 0 \neq T \in \mathcal{L}(\ell_p^2),\ \|T\|_1 \leqslant \|T\|_\infty\right\}\right\}. \end{aligned}$$

By [10, Remark 3], if $T \in \mathcal{L}(\ell_p^2)$ is such that $\|T\|_\infty \leqslant \|T\|_1$, then $M_p = \max\limits_{t\in[0,1]} \dfrac{t^{p-1} - t}{1+t^p} \leqslant \dfrac{v(T)}{\|T\|}$. Therefore, it is enough to prove that

$$\inf\left\{\frac{v(T)}{\|T\|} : 0 \neq T \in \mathcal{L}(\ell_p^2),\ \|T\|_1 \leqslant \|T\|_\infty\right\} \geqslant M_p. \tag{1}$$

To give a lower estimation for $v(T)$, we make some observations. First, we may suppose that $\|T\|_1 = \max\{|a| + |c|, |b| + |d|\} = |a| + |c|$. Indeed, the operator

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

satisfies that $v(S) = v(T)$ and $\|S\| = \|T\|$ since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an isometry and $S^{-1} = S$.

In addition, we may assume that $T = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$ with $a, b, c, d \geqslant 0$. Indeed, if $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$, we may consider the operator $S = \begin{pmatrix} |a| & |b| \\ -|c| & -|d| \end{pmatrix}$ which clearly satisfies that $\|S\| \geqslant \|T\|$ and $v(S) \leqslant v(T)$.

So, from now on we consider operators of the form $T = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$ with $a, b, c, d \geqslant 0$ and satisfying $\|T\|_1 = a + c \leqslant \|T\|_\infty$. For this class of operators, using Lemma 1.1, we have that

$$\begin{aligned} v(T) &= \max\left\{\max_{t\in[0,1]} \frac{|a - d\,t^p| + |b\,t - c\,t^{p-1}|}{1+t^p}, \max_{t\in[0,1]} \frac{|d - a\,t^p| + |c\,t - b\,t^{p-1}|}{1+t^p}\right\} \\ &\geqslant \max\left\{\frac{|a - d\,t_0^p| + |b\,t_0 - c\,t_0^{p-1}|}{1+t_0^p}, \frac{|d - a\,t_0^p| + |c\,t_0 - b\,t_0^{p-1}|}{1+t_0^p}\right\} \end{aligned} \tag{2}$$

where t_0 is taken as in Lemma 2.1. Let us write

$$F(T) = \frac{|a - d\,t_0^p| + |b\,t_0 - c\,t_0^{p-1}|}{1+t_0^p} \qquad \text{and} \qquad G(T) = \frac{|d - a\,t_0^p| + |c\,t_0 - b\,t_0^{p-1}|}{1+t_0^p},$$

and recall that by Riesz–Thorin theorem we have that $\|T\| \leqslant \|T\|_1^{1/p}\|T\|_\infty^{1/q}$. Hence, using (2), it is clear that

$$\begin{aligned}
&\inf\left\{\frac{v(T)}{\|T\|} : 0 \neq T \in \mathcal{L}(\ell_p^2),\ \|T\|_1 \leqslant \|T\|_\infty\right\} \\
&\geqslant \inf\left\{\frac{\max\{F(T), G(T)\}}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}} : 0 \neq T \in \mathcal{L}(\ell_p^2),\ T = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix},\right. \\
&\qquad\qquad \left. a, b, c, d \geqslant 0,\ \|T\|_1 = a + c \leqslant \|T\|_\infty \right\} := (\alpha).
\end{aligned}$$

In view of (1), to prove the theorem it is enough to show that

$$(\alpha) \geqslant \frac{t_0^{p-1} - t_0}{1 + t_0^p}.$$

To do so, we distinguish three cases:

- $\|T\|_1 = a + c \leqslant a + b = \|T\|_\infty$.
- $\|T\|_1 = a + c \leqslant c + d = \|T\|_\infty$ and $c + a - d \leqslant c\, t_0^{2-p}$.
- $\|T\|_1 = a + c \leqslant c + d = \|T\|_\infty$ and $c\, t_0^{2-p} \leqslant c + a - d$.

Claim 1. *Let $1 < p \leqslant \frac{3}{2}$ and let $T = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$ be a non-zero operator in $\mathcal{L}(\ell_p^2)$ with $a, b, c, d \geqslant 0$ and $\|T\|_1 = a + c \leqslant a + b = \|T\|_\infty$. Then,*

$$(\alpha) \geqslant \frac{t_0^{p-1} - t_0}{1 + t_0^p}.$$

Observe that it suffices to prove that $(\alpha)_F \geqslant \frac{t_0^{p-1} - t_0}{1 + t_0^p}$ where

$$\begin{aligned}
(\alpha)_F := \inf&\left\{\frac{F(T)}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}} : 0 \neq T \in \mathcal{L}(\ell_p^2),\ T = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix},\right. \\
&\qquad \left. a, b, c, d \geqslant 0,\ \|T\|_1 = a + c \leqslant a + b = \|T\|_\infty \right\}.
\end{aligned}$$

Note that the restriction $\|T\|_1 = a + c \leqslant a + b = \|T\|_\infty$ is equivalent to impose $c \leqslant b$ and $b + d \leqslant a + c$, which clearly implies $d \leqslant a$ and so

$$F(T) = \frac{a - d\, t_0^p + |b\, t_0 - c\, t_0^{p-1}|}{1 + t_0^p}.$$

In order to estimate $(\alpha)_F$ we may suppose that $b\, t_0^{2-p} \leqslant c$ (equivalently, $b\, t_0 \leqslant c\, t_0^{p-1}$). Indeed, if otherwise $b\, t_0^{2-p} > c$, we consider the operator $S = \begin{pmatrix} a & b \\ -b\, t_0^{2-p} & -d \end{pmatrix}$ which satisfies the hypotheses of Claim 1 since

$$\|S\|_1 = a + b\, t_0^{2-p} > a + c \geqslant b + d \qquad \text{and} \qquad \|S\|_\infty = a + b > d + b\, t_0^{2-p}.$$

Moreover, $\|S\|_1 > \|T\|_1$, $\|S\|_\infty = \|T\|_\infty$, and $F(S) = \dfrac{a - d\,t_0^p}{1+t_0^p} < F(T)$, so

$$\frac{F(S)}{\|S\|_1^{1/p}\|S\|_\infty^{1/q}} < \frac{F(T)}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}}.$$

Thus, we can write

$$\begin{aligned}(\alpha)_F &= \inf\left\{\frac{F(T)}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}} : 0 \neq T \in \mathcal{L}(\ell_p^2),\ T = \begin{pmatrix} a & b \\ -c & -d\end{pmatrix},\ a,b,c,d \geqslant 0,\ c \leqslant b,\ b+d \leqslant a+c\right\}\\ &\geqslant \frac{1}{1+t_0^p}\inf_{(a,b,c,d)\in A_1}\frac{a - d\,t_0^p + c\,t_0^{p-1} - b\,t_0}{(a+c)^{1/p}(a+b)^{1/q}}\end{aligned}$$

where

$$A_1 = \left\{(a,b,c,d) \in \mathbb{R}^4\backslash\{0\} : a,b,c,d \geqslant 0,\ b\,t_0^{2-p} \leqslant c \leqslant b,\ b+d \leqslant a+c\right\}.$$

From the restriction $d \leqslant a+c-b$, it follows that

$$a - d\,t_0^p + c\,t_0^{p-1} - b\,t_0 \geqslant a(1-t_0^p) + b(t_0^p - t_0) + c(t_0^{p-1} - t_0^p)$$

so

$$\inf_{(a,b,c,d)\in A_1}\frac{a - d\,t_0^p + c\,t_0^{p-1} - b\,t_0}{(a+c)^{1/p}(a+b)^{1/q}} \geqslant \inf_{(a,b,c)\in A_2}\frac{a(1-t_0^p) + b(t_0^p - t_0) + c(t_0^{p-1} - t_0^p)}{(a+c)^{1/p}(a+b)^{1/q}},$$

where

$$A_2 = \left\{(a,b,c) \in \mathbb{R}^3\backslash\{0\} : a,b,c \geqslant 0,\ b\,t_0^{2-p} \leqslant c \leqslant b \leqslant a+c\right\}.$$

We define the function

$$f(a,b,c) = \frac{a(1-t_0^p) + b(t_0^p - t_0) + c(t_0^{p-1} - t_0^p)}{(a+c)^{1/p}(a+b)^{1/q}} \qquad \big((a,b,c) \in \mathbb{R}^3\big)$$

and our goal is to show that

$$\inf_{(a,b,c)\in A_2} f(a,b,c) \geqslant t_0^{p-1} - t_0.$$

Observe that f decreases in the variable b since $t_0^p - t_0 < 0$, so using that $b \leqslant a+c$, it is clear that $f(a,b,c) \geqslant f(a,a+c,c)$ for every $(a,b,c) \in A_2$. Therefore, we have to minimize the two-variable function given by

$$g(a,c) = f(a,a+c,c) = \frac{a(1-t_0) + c(t_0^{p-1} - t_0)}{(a+c)^{1/p}(2a+c)^{1/q}} \qquad \big((a,c) \in \mathbb{R}^2\big)$$

on the set $A_3 = \left\{(a,b) \in \mathbb{R}^2\backslash\{0\} : a,c \geqslant 0,\ a\,t_0^{2-p} \leqslant c(1-t_0^{2-p})\right\}$.

To get the inequality $g(a,c) \geqslant t_0^{p-1} - t_0$ it suffices to show that g is decreasing in c, since in such a case it follows that $g(a,c) \geqslant \lim_{c\to\infty} g(a,c) = t_0^{p-1} - t_0$ for every $(a,c) \in A_3$ as desired. So let us prove that $\dfrac{\partial g}{\partial c}(a,c) \leqslant 0$ for every $(a,c) \in A_3$:

$$\begin{aligned}&\frac{\partial g}{\partial c}(a,c)\,(a+c)^{1/p+1}(2a+c)^{1/q+1}\\ &\qquad = (t_0^{p-1} - t_0)(a+c)(2a+c) - \Big(a(1-t_0) + c(t_0^{p-1} - t_0)\Big)\left(\frac{1}{p}(2a+c) + \frac{1}{q}(a+c)\right)\\ &\qquad = \left(2t_0^{p-1} - \frac{1}{q}t_0 - 1 - \frac{1}{p}\right)a^2 + \left(\left(1+\frac{1}{q}\right)t_0^{p-1} - \frac{1}{q}t_0 - 1\right)ac.\end{aligned}$$

Observe that

$$2t_0^{p-1} - \frac{1}{q}t_0 - 1 - \frac{1}{p} = \left(1+\frac{1}{q}\right)t_0^{p-1} - \frac{1}{q}t_0 - 1 - \frac{1}{p}(1-t_0^{p-1}) \leqslant \left(1+\frac{1}{q}\right)t_0^{p-1} - \frac{1}{q}t_0 - 1,$$

so to finish the proof of the claim it is enough to show that $\left(1+\frac{1}{q}\right)t_0^{p-1} - \frac{1}{q}t_0 - 1 \leqslant 0$. To do so, define the function

$$u(t) = \left(1+\frac{1}{q}\right)t^{p-1} - \frac{1}{q}t - 1 \qquad (t \in [0,1])$$

and note that $u(1)=0$. Thus, the inequality $u(t) \leqslant 0$ will hold for every $t \in [0,1]$ if we prove that u is an increasing function. This is easy to check as

$$u'(t) = \left(1+\frac{1}{q}\right)(p-1)t^{p-2} - \frac{1}{q}$$

and

$$u''(t) = \left(1+\frac{1}{q}\right)(p-1)(p-2)t^{p-3} \leqslant 0$$

for every $t \in]0,1[$, hence u' is decreasing. Since $u'(1) = \left(1+\frac{1}{q}\right)(p-1) - \frac{1}{q} = \frac{2}{q}(p-1) \geqslant 0$, it follows that $u'(t) \geqslant 0$ for every $t \in]0,1[$ and u is an increasing function as desired. Therefore, Claim 1 is proved.

Now we consider the operators satisfying $\|T\|_1 = a+c \leqslant c+d = \|T\|_\infty$. Observe that this restriction is equivalent to impose $a \leqslant d$ and $b+d \leqslant a+c$ (and also gives $b \leqslant c$), so from now on we have

$$F(T) = \frac{|a - d\,t_0^p| + c\,t_0^{p-1} - b\,t_0}{1+t_0^p} \qquad \text{and} \qquad G(T) = \frac{d - a\,t_0^p + |c\,t_0 - b\,t_0^{p-1}|}{1+t_0^p}.$$

Claim 2. *Let $1 < p \leqslant \frac{3}{2}$ and let $T = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$ be a non-zero operator in $\mathcal{L}(\ell_p^2)$ with $a,b,c,d \geqslant 0$, $\|T\|_1 = a+c \leqslant c+d = \|T\|_\infty$, and $c+a-d \leqslant c\,t_0^{2-p}$. Then,*

$$(\alpha) \geqslant \frac{t_0^{p-1} - t_0}{1+t_0^p}.$$

In this case we will use only $G(T)$ to estimate (α). Define

$$B_1 = \left\{(a,b,c,d) \in \mathbb{R}^4\backslash\{0\} \colon a,b,c,d \geqslant 0,\ a \leqslant d,\ b \leqslant c+a-d \leqslant c\,t_0^{2-p}\right\}$$

and observe that

$$(\alpha) \geqslant \frac{1}{1+t_0^p} \inf_{(a,b,c,d)\in B_1} \frac{d - a\,t_0^p + c\,t_0 - b\,t_0^{p-1}}{(a+c)^{1/p}(c+d)^{1/q}}.$$

Using that $b \leqslant c+a-d$, we obtain

$$\inf_{(a,b,c,d)\in B_1} \frac{d - a\,t_0^p + c\,t_0 - b\,t_0^{p-1}}{(a+c)^{1/p}(c+d)^{1/q}} \geqslant \inf_{(a,c,d)\in B_2} \frac{-a(t_0^{p-1}+t_0^p) - c(t_0^{p-1}-t_0) + d(1+t_0^{p-1})}{(a+c)^{1/p}(c+d)^{1/q}},$$

where $B_2 = \left\{(a,c,d) \in \mathbb{R}^3\backslash\{0\} \colon a,c,d \geqslant 0,\ a \leqslant d - c(1-t_0^{2-p})\right\}$. Therefore, defining

$$f(a,c,d) = \frac{-a(t_0^{p-1}+t_0^p) - c(t_0^{p-1}-t_0) + d(1+t_0^{p-1})}{(a+c)^{1/p}(c+d)^{1/q}} \qquad \left((a,c,d) \in \mathbb{R}^3\right),$$

our problem is to show that

$$\inf_{(a,c,d)\in B_2} f(a,c,d) \geqslant t_0^{p-1} - t_0.$$

It is clear that f is decreasing in a, therefore $f(a,c,d) \geqslant f(d-c(1-t_0^{2-p}),c,d)$ for every $(a,c,d)\in B_2$, so we have to minimize

$$g(c,d) = f(d-c(1-t_0^{2-p}),c,d) = \frac{c(t_0^p - t_0^2) + d(1-t_0^p)}{(d+c\,t_0^{2-p})^{1/p}(c+d)^{1/q}} \qquad \big((c,d)\in\mathbb{R}^2\big)$$

on the set $B_3 = \Big\{(c,d)\in\mathbb{R}^2\backslash\{0\} \colon c,d\geqslant 0,\ c(1-t_0^{2-p})\leqslant d\Big\}$. Observe that g is increasing in d since

$$\begin{aligned}
&\frac{\partial g}{\partial d}(c,d)\,(d+c\,t_0^{2-p})^{1/p+1}(c+d)^{1/q+1}\\
&\quad = (1-t_0^p)(d+c\,t_0^{2-p})(c+d) - \big(c(t_0^p-t_0^2)+d(1-t_0^p)\big)\left(\frac{1}{p}(c+d)+\frac{1}{q}(d+c\,t_0^{2-p})\right)\\
&\quad = \frac{1}{q}(1-t_0^p)(c+d)d - \frac{1}{q}(d+c\,t_0^{2-p})\big(c(t_0^p-t_0^2)+d(1-t_0^p)\big) + \left(t_0^{2-p}-\frac{1}{q}t_0^2-\frac{1}{p}t_0^p\right)(c+d)c\\
&\quad \geqslant \frac{1}{q}(1-t_0^p)(c+d)d - \frac{1}{q}(c+d)\big(c(t_0^p-t_0^2)+d(1-t_0^p)\big) + \left(t_0^{2-p}-\frac{1}{q}t_0^2-\frac{1}{p}t_0^p\right)(c+d)c\\
&\quad = (t_0^{2-p}-t_0^p)(c+d)c \geqslant 0.
\end{aligned}$$

Therefore, for every $(c,d)\in B_3$, we have that

$$g(c,d) \geqslant g(c,c(1-t_0^{2-p})) = \frac{1-t_0^{2-p}}{(2-t_0^{2-p})^{1/q}} = \frac{t_0^{p-1}-t_0}{(2t_0^p-t_0^2)^{1/q}} \geqslant t_0^{p-1}-t_0,$$

where we have used that

$$t_0^p \leqslant \frac{p-1}{2p+1} \leqslant p-1$$

by Lemma 2.1 and so, $2t_0^p - t_0^2 \leqslant 2t_0^p \leqslant 2p-2 \leqslant 1$.

We consider now the remaining case.

Claim 3. *Let $\frac{6}{5}\leqslant p\leqslant\frac{3}{2}$ and let $T = \begin{pmatrix} a & b\\ -c & -d\end{pmatrix}$ be a non-zero operator in $\mathcal{L}(\ell_p^2)$ with $a,b,c,d\geqslant 0$, $\|T\|_1 = a+c\leqslant c+d = \|T\|_\infty$, and $c\,t_0^{2-p}\leqslant c+a-d$. Then,*

$$(\alpha) \geqslant \frac{t_0^{p-1}-t_0}{1+t_0^p}.$$

First, in order to estimate (α), observe that we may suppose that $a\geqslant d\,t_0^p$. Indeed, if otherwise $a<d\,t_0^p$, we consider the operator $S = \begin{pmatrix} d\,t_0^p & b\\ -c & -d\end{pmatrix}$ which satisfies the hypotheses of Claim 3 since $c\,t_0^{2-p}\leqslant c+a-d < c+d\,t_0^p - d$,

$$\|S\|_1 = d\,t_0^p + c > a+c \geqslant b+d, \qquad \text{and} \qquad \|S\|_\infty = c+d\geqslant b+d > b+d\,t_0^p.$$

Moreover, $\|S\|_1 > \|T\|_1$, $\|S\|_\infty = \|T\|_\infty$,

$$F(S) = \frac{c\,t_0^{p-1} - b\,t_0}{1+t_0^p} < F(T), \qquad \text{and} \qquad G(S) = \frac{d - d\,t_0^{2p} + |b\,t_0^{p-1} - c\,t_0|}{1+t_0^p} < G(T),$$

so
$$\frac{\max\{F(T),G(T)\}}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}} \geqslant \frac{\max\{F(S),G(S)\}}{\|S\|_1^{1/p}\|S\|_\infty^{1/q}}.$$
Additionally, we may assume that $c\,t_0^{2-p} \leqslant b$. Indeed, if otherwise $c\,t_0^{2-p} > b$, we consider the operator $S = \begin{pmatrix} a & c\,t_0^{2-p} \\ -c & -d \end{pmatrix}$ which satisfies the hypotheses of Claim 3 since $c\,t_0^{2-p} \leqslant c+a-d$,
$$\|S\|_1 = a+c \geqslant c\,t_0^{2-p}+d, \qquad \text{and} \qquad \|S\|_\infty = c+d > c\,t_0^{2-p}+a.$$
Furthermore, $\|S\|_1 = \|T\|_1$, $\|S\|_\infty = \|T\|_\infty$, and, using that $c\,t_0^{2-p} > b$, it is is clear that
$$F(S) = \frac{a-d\,t_0^p + c\,t_0^{p-1} - c\,t_0^{2-p}\,t_0}{1+t_0^p} < F(T) \qquad \text{and} \qquad G(S) = \frac{d-a\,t_0^p}{1+t_0^p} < G(T),$$
so
$$\frac{\max\{F(T),G(T)\}}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}} \geqslant \frac{\max\{F(S),G(S)\}}{\|S\|_1^{1/p}\|S\|_\infty^{1/q}}.$$
Therefore, we assume from now on that $a,b,c,d \geqslant 0$, $d\,t_0^p \leqslant a \leqslant d$, and $c\,t_0^{2-p} \leqslant b \leqslant c+a-d$. Under such restrictions, we have
$$F(T) = \frac{a-d\,t_0^p + c\,t_0^{p-1} - b\,t_0}{1+t_0^p} \qquad \text{and} \qquad G(T) = \frac{d-a\,t_0^p + b\,t_0^{p-1} - c\,t_0}{1+t_0^p}$$
and so our goal is to give a lower bound of
$$\inf_{(a,b,c,d)\in C_1} \frac{\max\{F(T),G(T)\}}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}}$$
where $C_1 = \Big\{(a,b,c,d)\in\mathbb{R}^4\backslash\{0\}\colon a,b,c,d\geqslant 0,\ d\,t_0^p \leqslant a \leqslant d,\ c\,t_0^{2-p}\leqslant b \leqslant c+a-d\Big\}$. To do so, note that
$$\begin{aligned} F(T)\leqslant G(T) &\iff a-d\,t_0^p + c\,t_0^{p-1} - b\,t_0 \leqslant d-a\,t_0^p + b\,t_0^{p-1} - c\,t_0 \\ &\iff b \geqslant c-(d-a)\frac{1+t_0^p}{t_0^{p-1}+t_0}, \end{aligned} \tag{3}$$
and the equality holds if and only if $b = c-(d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0}$. Our next step is to observe that we may compute the infimum using only operators satisfying $b = c-(d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0}$, but first we need to show that
$$ct_0^{2-p} \leqslant c-(d-a)\frac{1+t_0^p}{t_0^{p-1}+t_0} \leqslant c+a-d.$$

On the one hand, we claim that $1+t_0^p \geqslant t_0^{p-1}+t_0$ and, consequently, $c-(d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0} \leqslant c+a-d$. Consider the function $u\colon[0,1]\longrightarrow\mathbb{R}$ given by
$$u(t) = 1+t^p - t^{p-1} - t \qquad (t\in[0,1])$$
and observe that $u(1)=0$. Hence, the inequality $1+t_0^p \geqslant t_0^{p-1}+t_0$ will follow immediately if we prove that u decreases in t. Indeed,
$$u'(t) = p\,t^{p-1} - (p-1)\,t^{p-2} - 1$$

and

$$u''(t) = p(p-1)\,t^{p-2} - (p-1)(p-2)\,t^{p-3} \geqslant 0$$

for every $t \in]0,1[$, thus u' is increasing. Since $u'(1) = 0$, it follows that $u'(t) \leqslant 0$ for every $t \in]0,1[$ as desired.

On the other hand, we may and do assume that $c\,t_0^{2-p} \leqslant c - (d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0}$. Indeed, suppose that for an operator T given by $(a,b,c,d) \in C_1$ we have $c - (d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0} < c\,t_0^{2-p}$. Then it follows that $F(T) < G(T)$ by (3) and $c < (d-a)\dfrac{1+t_0^p}{(t_0^{p-1}+t_0)(1-t_0^{2-p})}$. We consider the operator $S = \begin{pmatrix} a & b' \\ -c' & -d \end{pmatrix}$ where

$$c' = (d-a)\frac{1+t_0^p}{(t_0^{p-1}+t_0)(1-t_0^{2-p})} \qquad \text{and} \qquad b' = c' t_0^{2-p} = (d-a)\frac{(1+t_0^p)\,t_0^{2-p}}{(t_0^{p-1}+t_0)(1-t_0^{2-p})}.$$

As $c' - b' = (d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0} \geqslant d-a$, the operator S satisfies the conditions in C_1. Besides, we have $c' > c$ so $\|S\|_1 > \|T\|_1$ and $\|S\|_\infty > \|T\|_\infty$. Moreover, since $b' = c' t_0^{2-p} = c' - (d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0}$, it follows from (3) that

$$F(S) = G(S) = \frac{d - a\,t_0^p}{1+t_0^p} \leqslant G(T) = \frac{d - a\,t_0^p + b\,t_0^{p-1} - c\,t_0}{1+t_0^p}$$

as $c\,t_0^{2-p} \leqslant b$. Therefore, we get $\max\{F(S), G(S)\} \leqslant \max\{F(T), G(T)\}$ and, consequently,

$$\inf_{(a,b,c,d)\in C_1} \frac{\max\{F(T),G(T)\}}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}} \geqslant \inf_{(a,b,c,d)\in C_2} \frac{\max\{F(T),G(T)\}}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}}$$

where

$$C_2 = \Big\{(a,b,c,d) \in \mathbb{R}^4\backslash\{0\} \colon a,b,c,d \geqslant 0,\ d\,t_0^p \leqslant a \leqslant d,\ c\,t_0^{2-p} \leqslant b \leqslant c+a-d,$$
$$c\,t_0^{2-p} \leqslant c - (d-a)\tfrac{1+t_0^p}{t_0^{p-1}+t_0}\Big\}$$

as $(a,b',c',d) \in C_2$.

We are ready to observe that the infimum

$$\inf_{(a,b,c,d)\in C_2} \frac{\max\{F(T),G(T)\}}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}}$$

can be computed using only operators such that $F(T) = G(T)$, that is, satisfying $b = c - (d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0}$. Indeed, if T is an operator satisfying the conditions in C_2, consider the operator

$$S = \begin{pmatrix} a & c - (d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0} \\ -c & -d \end{pmatrix}$$

which clearly satisfies the conditions in C_2, $F(S)=G(S)$, and $\|S\|_1^{1/p}\|S\|_\infty^{1/q}=\|T\|_1^{1/p}\|T\|_\infty^{1/q}$. Observe that if $b\leqslant c-(d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0}$, then $\max\{F(T),G(T)\}=F(T)\geqslant F(S)$. If otherwise $b\geqslant c-(d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0}$, then $\max\{F(T),G(T)\}=G(T)\geqslant G(S)$. So in either case we have $F(S)=G(S)\leqslant\max\{F(T),G(T)\}$.

For operators satisfying satisfying $b=c-(d-a)\dfrac{1+t_0^p}{t_0^{p-1}+t_0}$ we have that

$$
\begin{aligned}
F(T)=G(T)&=\frac{a\left(1-t_0\dfrac{1+t_0^p}{t_0^{p-1}+t_0}\right)+d\left(t_0\dfrac{1+t_0^p}{t_0^{p-1}+t_0}-t_0^p\right)+c(t_0^{p-1}-t_0)}{1+t_0^p}\\
&=\frac{a\dfrac{t_0^{p-1}-t_0^{p+1}}{t_0^{p-1}+t_0}+d\dfrac{t_0-t_0^{2p-1}}{t_0^{p-1}+t_0}+c(t_0^{p-1}-t_0)}{1+t_0^p}.
\end{aligned}
$$

Hence, defining the function

$$
f(a,c,d)=\frac{a\dfrac{t_0^{p-1}-t_0^{p+1}}{t_0^{p-1}+t_0}+d\dfrac{t_0-t_0^{2p-1}}{t_0^{p-1}+t_0}+c(t_0^{p-1}-t_0)}{(a+c)^{1/p}(c+d)^{1/q}}\qquad\big((a,c,d)\in\mathbb{R}^3\big)
$$

we have that

$$
\inf_{(a,b,c,d)\in C_2}\frac{\max\{F(T),G(T)\}}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}}\geqslant\frac{1}{1+t_0^p}\inf_{(a,c,d)\in C_3}f(a,c,d)
$$

where

$$
C_3=\left\{(a,c,d)\in\mathbb{R}^3\backslash\{0\}\colon a,c,d\geqslant 0,\ d\,t_0^p\leqslant a\leqslant d,\ c\,t_0^{2-p}\leqslant c-(d-a)\tfrac{1+t_0^p}{t_0^{p-1}+t_0}\right\}.
$$

Our aim is to prove that f decreases in c for $\frac{6}{5}\leqslant p\leqslant\frac{3}{2}$. In such a case, it is clear that

$$
f(a,c,d)\geqslant\lim_{c\to\infty}f(a,c,d)=t_0^{p-1}-t_0
$$

for every $(a,c,d)\in C_3$ and, as a consequence, $(\alpha)\geqslant\dfrac{t_0^{p-1}-t_0}{1+t_0^p}$ as desired.

So, let us show that $\dfrac{\partial f}{\partial c}(a,c,d)\leqslant 0$ for every $(a,c,d)\in C_3$. Calling $K=a\dfrac{t_0^{p-1}-t_0^{p+1}}{t_0^{p-1}+t_0}+d\dfrac{t_0-t_0^{2p-1}}{t_0^{p-1}+t_0}$, we can write

$$
\begin{aligned}
&\frac{\partial f}{\partial c}(a,c,d)\,(a+c)^{1/p+1}(c+d)^{1/q+1}\\
&\qquad=(t_0^{p-1}-t_0)(a+c)(c+d)-\Big(c(t_0^{p-1}-t_0)+K\Big)\left(\frac{1}{p}(c+d)+\frac{1}{q}(a+c)\right)\\
&\qquad=\left((t_0^{p-1}-t_0)\left(\frac{a}{p}+\frac{d}{q}\right)-K\right)c+(t_0^{p-1}-t_0)ad-K\left(\frac{a}{q}+\frac{d}{p}\right)\\
&\qquad\leqslant\left((t_0^{p-1}-t_0)\left(\frac{a}{p}+\frac{d}{q}\right)-K\right)\left(c+\frac{a}{q}+\frac{d}{p}\right),
\end{aligned}
$$

where we have used that

$$\left(\frac{a}{p}+\frac{d}{q}\right)\left(\frac{a}{q}+\frac{d}{p}\right)=\frac{a^2+d^2}{pq}+\left(\frac{1}{p^2}+\frac{1}{q^2}\right)ad=\frac{(a-d)^2}{pq}+ad\geqslant ad.$$

Now, observe that

$$\begin{aligned}(t_0^{p-1}-t_0)\left(\frac{a}{p}+\frac{d}{q}\right)-K\leqslant 0 &\iff (t_0^{2p-2}-t_0^2)\left(\frac{a}{p}+\frac{d}{q}\right)\leqslant a(t_0^{p-1}-t_0^{p+1})+d(t_0-t_0^{2p-1})\\ &\iff a\left(t_0^{p-1}-t_0^{p+1}-\frac{1}{p}t_0^{2p-2}+\frac{1}{p}t_0^2\right)\geqslant d\left(\frac{1}{q}t_0^{2p-2}-\frac{1}{q}t_0^2-t_0+t_0^{2p-1}\right).\end{aligned}$$

To prove the last inequality, using that $a\geqslant d\,t_0^p$, it is enough to prove that

$$t_0^{2p-1}-t_0^{2p+1}-\frac{1}{p}t_0^{3p-2}+\frac{1}{p}t_0^{p+2}\geqslant\frac{1}{q}t_0^{2p-2}-\frac{1}{q}t_0^2-t_0+t_0^{2p-1}$$

which, decomposing $t_0=\frac{1}{p}t_0+\frac{1}{q}t_0$ and $t_0^{2p+1}=\frac{1}{p}t_0^{2p+1}+\frac{1}{q}t_0^{2p+1}$, is equivalent to

$$\begin{aligned}\frac{1}{p}\left(t_0+t_0^{p+2}-t_0^{2p+1}-t_0^{3p-2}\right)&\geqslant\frac{1}{q}\left(t_0^{2p-2}+t_0^{2p+1}-t_0^2-t_0\right)\\ &=\frac{1}{q}t_0^{2p-3}\left(t_0+t_0^4-t_0^{5-2p}-t_0^{4-2p}\right).\end{aligned}$$

Note that $t_0+t_0^{p+2}-t_0^{2p+1}-t_0^{3p-2}\geqslant t_0+t_0^4-t_0^{5-2p}-t_0^{4-2p}$ for $\frac{6}{5}\leqslant p\leqslant\frac{3}{2}$ as

$$t_0^{p+2}-t_0^4+t_0^{5-2p}-t_0^{2p+1}+t_0^{4-2p}-t_0^{3p-2}=t_0^{p+2}\left(1-t_0^{2-p}\right)+t_0^{5-2p}\left(1-t_0^{4p-4}\right)+t_0^{4-2p}\left(1-t_0^{5p-6}\right)\geqslant 0,$$

therefore, it suffices to show that $t_0^{2p-3}\leqslant\frac{q}{p}$. But this inequality holds for $p\in\left[\frac{6}{5},\frac{3}{2}\right]$ thanks to Lemma 2.1 and so Claim 3 is proved. $\square$

Remark 2.3. The only use of the restriction $\frac{6}{5}\leqslant p$ in the above proof was to guarantee that the inequality

$$t_0^{p+2}-t_0^4+t_0^{5-2p}-t_0^{2p+1}+t_0^{4-2p}-t_0^{3p-2}=t_0^{p+2}\left(1-t_0^{2-p}\right)+t_0^{5-2p}\left(1-t_0^{4p-4}\right)+t_0^{4-2p}\left(1-t_0^{5p-6}\right)\geqslant 0$$

holds. This inequality remains true for some values of p smaller than $\frac{6}{5}$ but close to it. The same happens with Lemma 2.1, so our procedure can give the equality $n(\ell_p^2)=M_p$ for a little wider range of values of p. However it seems that it does not work for p close to 1. Indeed, for $p=1.16$, numerical computations give $t_0\approx 0.073924$ and $M_p\approx 0.558064$. Besides, the operator $T=\begin{pmatrix} a & b\\ -c & -d\end{pmatrix}$ with

$$a=0.0487295,\qquad b=13.639181,\qquad c=15,\qquad\text{and}\qquad d=1$$

satisfies

$$\frac{\max\{F(T),G(T)\}}{\|T\|_1^{1/p}\|T\|_\infty^{1/q}}=\frac{1}{1+t_0^p}\frac{a-d\,t_0^p+c\,t_0^{p-1}-b\,t_0}{(a+c)^{1/p}(c+d)^{1/q}}\approx 0.557895<M_p.$$

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Chapter V

Generating operators between Banach spaces

[22] V. Kadets, M. Martín, J. Merí, and A. Quero, Generating operators between Banach spaces, *Preprint* (2023).

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GENERATING OPERATORS BETWEEN BANACH SPACES

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Abstract. We introduce and study the notion of generating operators as those norm-one operators $G\colon X \longrightarrow Y$ such that for every $0 < \delta < 1$, the set $\{x \in X\colon \|x\| \leqslant 1,\ \|Gx\| > 1 - \delta\}$ generates the unit ball of X by closed convex hull. This class of operators includes isometric embeddings, spear operators (actually, operators with the alternative Daugavet property), and other examples like the natural inclusions of ℓ_1 into c_0 and of $L_\infty[0,1]$ into $L_1[0,1]$. We first present a characterization in terms of the adjoint operator, make a discussion on the behaviour of diagonal generating operators on c_0-, ℓ_1-, and ℓ_∞-sums, and present examples in some classical Banach spaces. Even though rank-one generating operators always attain their norm, there are generating operators, even of rank-two, which do not attain their norm. We discuss when a Banach space can be the domain of a generating operator which does not attain its norm in terms of the behaviour of some spear sets of the dual space. Finally, we study when the set of all generating operators between two Banach spaces X and Y generates all non-expansive operators by closed convex hull. We show that this is the case when $X = L_1(\mu)$ and Y has the Radon-Nikodým property with respect to μ. Therefore, when $X = \ell_1(\Gamma)$, this is the case for every target space Y. Conversely, we also show that a real finite-dimensional space X satisfies that generating operators from X to Y generate all non-expansive operators by closed convex hull only in the case that X is an ℓ_1-space.

1. Introduction

Let X and Y be Banach spaces over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). We denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from X to Y and write $X^* = \mathcal{L}(X,\mathbb{K})$ to denote the dual space. By B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively, and we write $\mathbb{T}$ for the set of modulus one scalars. Some more notation and definitions (which are standard) are included in Subsection 1.1 at the end of this introduction.

The concept of spear operator was introduced in [1] and deeply studied in the book [7]. A norm-one operator $G \in \mathcal{L}(X,Y)$ is said to be an *spear operator* if the norm equality

$$\max_{\theta\in\mathbb{T}} \|G + \theta T\| = 1 + \|T\|$$

holds for all $T \in \mathcal{L}(X,Y)$. This concept extends the properties of the identity operator in those Banach spaces having numerical index one and it is satisfied, for instance, by the Fourier transform on L_1. There are isometric and isomorphic consequences on the domain and range spaces of a spear operator as, for instance, in the real case, the dual of the domain of a spear operator with infinite rank has to contain a copy of ℓ_1. For more information and background, we refer the interested reader to the already cited book [7]. Even though the definition of spear operator given above does not need numerical ranges, it is well known that spear operators are exactly those operators such that the numerical radius with respect to them coincides with the operator norm. Let us introduce the

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relevant definitions. Fixed a norm-one operator $G \in \mathcal{L}(X,Y)$, the *numerical radius with respect to* G is the seminorm defined as

$$\begin{aligned} v_G(T) &:= \sup\{|\phi(T)| \colon \phi \in \mathcal{L}(X,Y)^*,\ \phi(G) = 1\} \\ &= \inf_{\delta>0} \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\ x \in S_X,\ \operatorname{Re} y^*(Gx) > 1-\delta\} \end{aligned}$$

for every $T \in \mathcal{L}(X,Y)$ (the equality above was proved in [14, Theorem 2.1]). Observe that $v_G(\cdot)$ is a seminorm in $\mathcal{L}(X,Y)$ which clearly satisfies

$$v_G(T) \leqslant \|T\| \qquad (T \in \mathcal{L}(X,Y)). \tag{1}$$

Then, G is a spear operator if and only if $v_G(T) = \|T\|$ for every $T \in \mathcal{L}(X,Y)$ (see [7, Proposition 3.2]).

Our discussion here starts with the observation that it is possible to introduce a natural seminorm between $v_G(T)$ and $\|T\|$ in Eq. (1): the (semi-)norm relative to G. Let us introduce the needed notation and definitions. Let X, Y, Z be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. For $\delta > 0$, we write $\operatorname{att}(G,\delta)$ to denote the δ-*attainment set* of G, that is,

$$\operatorname{att}(G,\delta) := \{x \in S_X \colon \|Gx\| > 1-\delta\}.$$

If there exists $x \in S_X$ such that $\|Gx\| = 1$, we say that G *attains its norm* and we denote by $\operatorname{att}(G)$ the attainment set of G:

$$\operatorname{att}(G) := \{x \in S_X \colon \|Gx\| = 1\}.$$

We consider the parametric family of norms on $\mathcal{L}(X,Z)$ defined by

$$\|T\|_{G,\delta} := \sup\{\|Tx\| \colon x \in \operatorname{att}(G,\delta)\} \qquad (T \in \mathcal{L}(X,Z))$$

which are equivalent to the usual norm on $\mathcal{L}(X,Z)$ (this is so since $\operatorname{att}(G,\delta)$ has nonempty interior). We are interested in the (semi-)norm obtained taking infimum on this parametric family.

Definition 1.1. Let X, Y and Z be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. For $T \in \mathcal{L}(X,Z)$, we define the *(semi-)norm of* T *relative to* G by

$$\|T\|_G := \inf_{\delta>0} \|T\|_{G,\delta}.$$

When $Z = Y$, we clearly have that

$$v_G(T) \leqslant \|T\|_G \leqslant \|T\| \qquad (T \in \mathcal{L}(X,Y))$$

and so this $\|\cdot\|_G$ is the promised seminorm to extend Eq. (1). We may study the possible equality between $v_G(\cdot)$ and $\|\cdot\|_G$ and between $\|\cdot\|_G$ and the usual operator norm. We left the first relation for a subsequent paper which is still in process [9]. The main aim in this manuscript is to study when the norm equality

$$\|T\|_G = \|T\| \tag{2}$$

holds true.

Definition 1.2. Let X, Y be Banach spaces. We say that $G \in \mathcal{L}(X,Y)$ with norm-one is *generating* (or a *generating operator*) if equality (2) holds true for all $T \in \mathcal{L}(X,Y)$. We denote by $\operatorname{Gen}(X,Y)$ the set of all generating operators from X to Y.

Observe that both $\|\cdot\|_G$ and the operator norm can be defined for operators with domain X and arbitrary range, so one may wonder if there are different definitions of generating requiring that Eq. (2) holds replacing Y for other range spaces. This is not the case, as we will show in Section 2 that a generating operator G satisfies that $\|T\|_G = \|T\|$ for every $T \in \mathcal{L}(X,Z)$ and every Banach space Z

(see Corollary 2.3). This is so thanks to a characterization of generating operators in terms of the sets $\operatorname{att}(G,\delta)$: G is generating (if and) only if $\overline{\operatorname{conv}}(\operatorname{att}(G,\delta)) = B_X$ for every $\delta > 0$, see Corollary 2.3 again. When the dimension of X is finite, this is clearly equivalent to the fact that $\operatorname{conv}(\operatorname{att}(G)) = B_X$ (actually, the same happens for compact operators defined on reflexive spaces, see Proposition 2.5). For some infinite-dimensional X, there are generating operators from X which do not attain their norm, even of rank-two (see Example 3.2); but there are even generating operators attaining the norm such that $\overline{\operatorname{conv}}(\operatorname{att}(G))$ has empty interior (see Example 3.4).

There is another characterization which involves the geometry of the dual space. We need some definitions. A subset F of the unit ball of a Banach space Z is said to be a *spear set* of Z [7, Definition 2.3] if

$$\max_{\theta\in\mathbb{T}} \sup_{z\in F} \|z+\theta x\| = 1 + \|x\| \qquad (x \in Z).$$

If $z \in S_Z$ satisfies that $F = \{z\}$ is a spear set, we just say that z is a *spear vector* and we write $\operatorname{Spear}(Z)$ for the set of spear vectors of Z. We refer the reader to [7, Chapter 2] for more information and background. We will show that a norm-one operator $G \in \mathcal{L}(X,Y)$ is generating if and only if $G^*(B_{Y^*})$ is a spear set of X^*, see Corollary 2.17. These characterizations appear in Section 2, together with a discussion on the behaviour of diagonal generating operators on c_0-, ℓ_1-, and ℓ_∞-sums, and examples in some classical Banach spaces.

We next discuss in Section 3 the relationship between generating operators and norm attainment. On the one hand, we show that rank-one generating operators attain their norm (see Corollary 3.1) and, clearly, the same happens with isometric embeddings (which are generating), or with generating operators whose domain has the RNP (see Corollary 2.12), as every generating operator attains its norm on denting points (see Lemma 2.8). But, on the other hand, there are generating operators, even of rank two, which do not attain their norm (see Example 3.2). We further discuss the possibility for a Banach space X to be the domain of a generating operator which does not attain its norm in terms of the behaviour of some spear sets of X^* (see Theorem 3.5).

Finally, Section 4 is devoted to the study of the set $\operatorname{Gen}(X,Y)$. We show that it is closed (see Proposition 4.1), and show that for every Banach space Y, there is a Banach space X such that $\operatorname{Gen}(X,Y) = \emptyset$ (see Proposition 4.2), but this result is not true for $Y = C[0,1]$ if we restrict the space X to be separable (Example 4.5). We next study properties of $\operatorname{Gen}(X,Y)$ when X is fixed. We first show that $\operatorname{Gen}(X,Y) \neq \emptyset$ for every Y if and only if $\operatorname{Spear}(X^*) \neq \emptyset$ (see Corollary 4.6) and that the only case in which there is Y such that $\operatorname{Gen}(X,Y) = S_{\mathcal{L}(X,Y)}$ is when X is one-dimensional (see Corollary 4.7). We then study the possibility that the set $\operatorname{Gen}(X,Y)$ generates the unit ball of $\mathcal{L}(X,Y)$ by closed convex hull, showing first that this is the case when $X = L_1(\mu)$ and Y has the RNP (Theorem 4.10) and when $X = \ell_1(\Gamma)$ and Y is arbitrary (see Proposition 4.12) and that this is the only possibility for *real* finite-dimensional spaces (see Proposition 4.14).

1.1. **A bit of notation.** Let X, Y be Banach spaces. We write $J_X\colon X \longrightarrow X^{**}$ to denote the natural inclusion of X into its bidual space. A subset $\mathcal{A} \subseteq B_{X^*}$ is *r-norming for* X ($0 < r \leqslant 1$) if $rB_{X^*} \subseteq \overline{\operatorname{aconv}}^{w^*}(\mathcal{A})$ or, equivalently, if $r\|x\| \leqslant \sup_{x^*\in\mathcal{A}} |x^*(x)|$ for every $x \in X$. The most interesting case is $r = 1$: $\mathcal{A}$ is *one-norming for* X if $B_{X^*} = \overline{\operatorname{aconv}}^{w^*}(\mathcal{A})$ or, equivalently, if $\|x\| = \sup_{x^*\in\mathcal{A}} |x^*(x)|$ for every $x \in X$. A *slice* of a closed convex bounded set $C \subset X$ is a nonempty intersection of C with an open half-space. We write

$$\operatorname{Slice}(C,f,\alpha) := \left\{ x \in C\colon\ \operatorname{Re} f(x) > \sup_C \operatorname{Re} f - \alpha \right\}$$

where $f \in X^*$ and $\alpha > 0$, and observe that every slice of C is of the above form.

For $A \subset X$, $\operatorname{conv}(A)$ and $\operatorname{aconv}(A)$ are, respectively, the convex hull and the absolutely convex hull of A; $\overline{\operatorname{conv}}(A)$ and $\overline{\operatorname{aconv}}(A)$ are, respectively, the closures of these sets. For $B \subset X$ convex, $\operatorname{ext}(B)$ denotes the set of extreme points of B.

2. Characterizations, first results, and some examples

Our first result gives different characterizations for the equivalence of $\|\cdot\|$ and $\|\cdot\|_G$ on $\mathcal{L}(X,Z)$. As one may have expected, this does not depend on the range space Z.

Proposition 2.1. *Let X, Y be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator, and let $r \in (0,1]$. Then, the following are equivalent:*

(i) *$\|T\|_G \geqslant r\|T\|$ for every Banach space Z and every $T \in \mathcal{L}(X,Z)$.*
(ii) *There is a (non null) Banach space Z such that $\|T\|_G \geqslant r\|T\|$ for every $T \in \mathcal{L}(X,Z)$.*
(iii) *There is a (non null) Banach space Z such that $\|T\|_G \geqslant r\|T\|$ for every rank-one operator $T \in \mathcal{L}(X,Z)$.*
(iv) *$\|x^*\|_G \geqslant r\|x^*\|$ for every $x^* \in X^*$.*
(v) *$\|x^*\|_{G,\delta} \geqslant r\|x^*\|$ for every $x^* \in X^*$ and every $\delta > 0$.*
(vi) *$\overline{\operatorname{conv}}(\operatorname{att}(G,\delta)) \supseteq rB_X$ for every $\delta > 0$.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$, $(iv) \Leftrightarrow (v)$, and $(vi) \Rightarrow (i)$ are evident.

$(iii) \Rightarrow (iv)$. Fix $z \in S_Z$ and, given $x^* \in X^*$, consider $T = x^* \otimes z \in \mathcal{L}(X,Z)$ which obviously satisfies $\|T\| = \|x^*\|$ and $\|T\|_G = \|x^*\|_G$.

The remaining implication $(v) \Rightarrow (vi)$ follows from the Bipolar theorem. Indeed, for $\delta > 0$, take $x \in rB_X$, we have to prove that $J_X(x)$ belongs to $\operatorname{att}(G,\delta)^{\circ\circ}$. For $x^* \in \operatorname{att}(G,\delta)^{\circ}$,

$$|J_X(x)(x^*)| = |x^*(x)| \leqslant r\|x^*\| \leqslant \|x^*\|_G \leqslant \sup\{|x^*(x)|\colon x \in \operatorname{att}(G,\delta)\} \leqslant 1,$$

where the second inequality follows from (iv) and the last one from the fact that $x^* \in \operatorname{att}(G,\delta)^{\circ}$. Therefore $J_X(x) \in \operatorname{att}(G,\delta)^{\circ\circ} = \overline{\operatorname{conv}}^{w^*}(\operatorname{att}(G,\delta))$. □

Observe that item (vi) in the previous result just means that, for every $\delta \in (0,1)$, the set $\operatorname{att}(G,\delta)$ is r-norming for X^*. This leads to the following concept which extends the one of generating operator.

Definition 2.2. Let X, Y be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator and let $r \in (0,1]$. We say that G is *r-generating* if $\overline{\operatorname{conv}}(\operatorname{att}(G,\delta)) \supseteq rB_X$ for every $\delta > 0$.

Of course, the case $r = 1$ coincides with the generating operators introduced in the introduction. For them, the following characterization deserves to be emphasized.

Corollary 2.3. *Let X, Y be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then, the following are equivalent:*

(i) *G is generating.*
(ii) *$\|T\|_G = \|T\|$ for every $T \in \mathcal{L}(X,Z)$ and every Banach space Z.*
(iii) *There is a (non null) Banach space Z such that $\|T\|_G = \|T\|$ for every rank-one operator $T \in \mathcal{L}(X,Z)$.*
(iv) *$B_X = \overline{\operatorname{conv}}(\operatorname{att}(G,\delta))$ for every $\delta > 0$.*

In particular, if there exists $A \subseteq B_X$ which satisfies $\overline{\operatorname{aconv}}(A) = B_X$ and $A \subseteq \operatorname{att}(G,\delta)$ for every $\delta > 0$, then G is generating.

In the next list we give the first easy examples of generating operators.

Examples 2.4.

(1) *The identity operator on every Banach space is generating.*
(2) *Actually, all isometric embeddings are generating.*
(3) *Spear operators are generating since, in this case,* $v_G(T) = \|T\|$ *for every* $T \in \mathcal{L}(X,Y)$*.*
(4) *Actually, operators with the* alternative Daugavet property *(i.e. those* $G \in \mathcal{L}(X,Y)$ *such that* $v_G(T) = \|T\|$ *for every* $T \in \mathcal{L}(X,Y)$ *with rank-one, cf.* [7, Section 3.2]*) are also generating by using Corollary 2.3 with* $Z = Y$ *in item* (iii)*.*
(5) *The natural embedding* G *of* ℓ_1 *into* c_0 *is a generating operator.*
Indeed, for every $\delta > 0$, we have that
$$\operatorname{att}(G,\delta) = \{x \in S_{\ell_1} \colon \|Gx\|_\infty > 1-\delta\} \supset \mathbb{T}\{e_n \colon n \in \mathbb{N}\},$$
so $\overline{\operatorname{conv}}(\operatorname{att}(G,\delta)) = B_{\ell_1}$.
(6) *The natural embedding* G *of* $L_\infty[0,1]$ *into* $L_1[0,1]$ *is a generating operator.*
Indeed, for every $\delta > 0$, notice that $B_{L_\infty[0,1]} = \overline{\operatorname{conv}}\left(\{f \in L_\infty[0,1] \colon |f(t)| = 1 \text{ a.e.}\}\right)$ (this should be well known, but in any case it follows from Lemma 4.11 which includes the vector-valued case). Observe then that, for every $f \in L_\infty[0,1]$ satisfying $|f(t)| = 1$ a.e., it follows $\|f\|_\infty = \|G(f)\|_1 = 1$. So $\|G(f)\|_1 = 1$ and $f \in \operatorname{att}(G,\delta)$.

We will provide some more examples in classical Banach spaces in Subsection 2.2.

The next result deals with compact operators defined on a reflexive Banach space.

Proposition 2.5. *Let* X *be a reflexive Banach space, let* Y *be a Banach space, and let* $G \in \mathcal{L}(X,Y)$ *be a compact operator with* $\|G\| = 1$*. Then,*
$$\bigcap_{\delta>0} \overline{\operatorname{conv}}(\operatorname{att}(G,\delta)) = \overline{\operatorname{conv}}(\operatorname{att}(G)).$$
Consequently, G *is* r*-generating if and only if* $rB_X \subseteq \overline{\operatorname{conv}(\operatorname{att}(G))}$*.*

Proof. Let $x_0 \in \bigcap_{\delta>0} \overline{\operatorname{conv}}(\operatorname{att}(G,\delta))$ and suppose that $x_0 \notin \overline{\operatorname{conv}}(\operatorname{att}(G))$. Then there exist $x_0^* \in X^*$ and $\alpha > 0$ such that
$$\sup_{x \in \overline{\operatorname{conv}}(\operatorname{att}(G))} \operatorname{Re} x_0^*(x) < \alpha \leqslant \operatorname{Re} x_0^*(x_0). \tag{3}$$
Fix $\varepsilon > 0$. Given $n \in \mathbb{N}$, since $x_0 \in \overline{\operatorname{conv}}\left(\operatorname{att}\left(G, \frac{1}{n}\right)\right)$, we may find $m \in \mathbb{N}$, $y_1, \ldots, y_m \in \operatorname{att}\left(G, \frac{1}{n}\right)$, and $\lambda_1, \ldots, \lambda_m \in [0,1]$ with $\sum_{k=1}^m \lambda_k = 1$ such that
$$\left\| x_0 - \sum_{k=1}^m \lambda_k y_k \right\| < \varepsilon,$$
hence
$$\alpha - \varepsilon \leqslant \operatorname{Re} x_0^*(x_0) - \varepsilon < \sum_{k=1}^m \lambda_k \operatorname{Re} x_0^*(y_k).$$
By convexity, there is $k_0 \in \{1, \ldots, m\}$ such that $\operatorname{Re} x_0^*(y_{k_0}) \geqslant \alpha - \varepsilon$. Repeating this argument for each $n \in \mathbb{N}$, we obtain a sequence $\{y_n\}$ in B_X such that $\operatorname{Re} x_0^*(y_n) > \alpha - \varepsilon$ and $\|Gy_n\| > 1 - \frac{1}{n}$ for every $n \in \mathbb{N}$. Now, using that B_X is weakly compact by Dieudonné's theorem, we obtain a subsequence $\{y_{\sigma(n)}\}$ of $\{y_n\}$ which is weakly convergent to some $y_0 \in B_X$. Then, by the arbitrariness of ε and the compactness of G we have that
$$\operatorname{Re} x_0^*(y_0) \geqslant \alpha \qquad \text{and} \qquad \|Gy_0\| = 1,$$

which contradicts (3). □

Clearly, the previous result applies when X is finite-dimensional.

Corollary 2.6. *Let X be a finite-dimensional space, let Y be a Banach space, and let $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. Then,*

$$\bigcap_{\delta>0} \overline{\operatorname{conv}}(\operatorname{att}(G,\delta)) = \operatorname{conv}(\operatorname{att}(G)).$$

Consequently, G is r-generating if and only if $rB_X \subseteq \operatorname{conv}(\operatorname{att}(G))$.

The next result characterizes those operators acting from a finite-dimensional space which are r-generating for some $0 < r \leqslant 1$.

Proposition 2.7. *Let X be a Banach space with $\dim(X) = n$, let Y be a Banach space, and let $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. The following are equivalent:*

- *(i) G is r-generating for some $r \in (0,1]$.*
- *(ii) The set $\operatorname{att}(G)$ contains n linearly independent elements.*

Proof. $(i) \Rightarrow (ii)$. By Corollary 2.6, we have that $rB_X \subseteq \operatorname{conv}(\operatorname{att}(G))$. Therefore, $\operatorname{att}(G)$ contains n linearly independent elements.

$(ii) \Rightarrow (i)$. We start proving that the set $\operatorname{conv}(\operatorname{att}(G))$ is absorbing. Indeed, let $\{x_1, \ldots, x_n\}$ be a linearly independent subset of $\operatorname{att}(G)$. Then, fixed $0 \neq x \in X$, there are $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that $x = \sum_{j=1}^n \lambda_j x_j$. Calling $0 < \rho = \sum_{j=1}^n |\lambda_j|$ we can write

$$x = \sum_{k=1}^n \lambda_j x_j = \sum_{\lambda_j \neq 0} |\lambda_j| \frac{\lambda_j}{|\lambda_j|} x_j = \rho \sum_{\lambda_j \neq 0} \frac{|\lambda_j|}{\rho} \frac{\lambda_j}{|\lambda_j|} x_j \in \rho \operatorname{conv}(\operatorname{att}(G))$$

where we used that $\frac{\lambda_j}{|\lambda_j|} x_j \in \operatorname{att}(G)$ as this set is balanced. Hence, the set $\operatorname{conv}(\operatorname{att}(G))$ is absorbing. Besides, $\operatorname{conv}(\operatorname{att}(G))$ is clearly balanced, convex, and compact. So its Minkowski functional defines a norm on X which must be equivalent to the original one. Then, there is $r > 0$ such that $rB_X \subseteq \operatorname{conv}(\operatorname{att}(G))$ and, therefore, G is r-generating by Corollary 2.6. □

We next would like to present the relationship of generating operators with denting points (and so with the Radon-Nikodým property, RNP in short). We need some notation. Let A be a bounded closed convex set. Recall that $x_0 \in A$ is a *denting point* if for every $\delta > 0$ $x_0 \notin \overline{\operatorname{conv}}(A \setminus B(x_0, \delta))$ or, equivalently, if x_0 belongs to slices of B_X of arbitrarily small diameter. We write $\operatorname{dent}(A)$ to denote the set of denting points of A. A closed convex subset C of X has the *Radon-Nikodým property* (*RNP* in short), if all of its closed convex bounded subsets contain denting points or, equivalently, if all of its closed convex bounded subsets are equal to the closed convex hull of their denting points. In particular, the whole space X may also have this property.

The following result tells us that generating operators must attain their norms on every denting point.

Lemma 2.8. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a (norm-one) generating operator. If $x_0 \in \operatorname{dent}(B_X)$, then $\|Gx_0\| = 1$.*

Proof. Given $\delta > 0$, observe that $x_0 \in \overline{\mathrm{att}(G,\delta)}$. Otherwise, there would exist $r > 0$ such that $B(x_0, r) \cap \mathrm{att}(G,\delta) = \emptyset$, so $\mathrm{att}(G,\delta) \subseteq B_X \setminus B(x_0, r)$ and

$$x_0 \in B_X = \overline{\mathrm{conv}}(\mathrm{att}(G,\delta)) \subseteq \overline{\mathrm{conv}}(B_X \setminus B(x_0, r))$$

which contradicts x_0 being a denting point of B_X. Consequently, $\|Gx_0\| \geqslant 1 - \delta$ and the arbitrariness of δ finishes the proof. □

The above result can be slightly improved by using the following definition.

Definition 2.9. Let $x_0 \in S_X$. We say that x_0 is a point of *sliced fragmentability* if for every $\delta > 0$ there is a slice S_δ of B_X such that $S_\delta \subset x_0 + \delta B_X$.

Observe that this notion is weaker than that of denting point (for instance, points in the closure of the set of denting points are of sliced fragmentability but they do not need to be denting, even in the finite-dimensional case).

Lemma 2.10. *Let X, Y be Banach spaces, let $G \in S_{\mathcal{L}(X,Y)}$ be a generating operator, and let $x_0 \in S_X$ be a point of sliced fragmentability, then $\|Gx_0\| = 1$.*

Proof. Fixed $\delta > 0$, by our assumption, $\overline{\mathrm{conv}}(\mathrm{att}(G,\delta)) = B_X$ for every $\delta > 0$. This implies that, fixed $\delta > 0$, the set $\mathrm{att}(G,\delta)$ intersects every slice of B_X. Applying this to the slice S_δ from Definition 2.9, we obtain that there is a point $x_\delta \in S_\delta \cap \mathrm{att}(G,\delta)$. For this x_δ, we have $\|x_0 - x_\delta\| < \delta$ and $\|Gx_\delta\| > 1 - \delta$. Consequently,

$$\|Gx_0\| \geqslant \|Gx_\delta\| - \|G(x_0 - x_\delta)\| \geqslant 1 - 2\delta$$

and the arbitrariness of δ finishes the proof. □

We do not know if Lemma 2.10 is a characterization, but in Proposition 3.6 we will characterize those points on which every generating operator attains its norm.

Proposition 2.11. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Suppose that $B_X = \overline{\mathrm{conv}}(\mathrm{dent}(B_X))$. Then, G is generating if and only if $\|Gx\| = 1$ for every $x \in \mathrm{dent}(B_X)$.*

Proof. If $\|Gx\| = 1$ for every $x \in \mathrm{dent}(B_X)$, then $\mathrm{dent}(B_X) \subset \mathrm{att}(G,\delta)$ for every $\delta > 0$ and, therefore, G is generating by Corollary 2.3.iv as $B_X = \overline{\mathrm{conv}}(\mathrm{dent}(B_X))$. The converse result follows from Lemma 2.8. □

Corollary 2.12. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Suppose that X has the Radon-Nikodým property. Then, G is generating if and only if $\|Gx\| = 1$ for every $x \in \mathrm{dent}(B_X)$.*

In the finite-dimensional case, the RNP is for free and denting points and extreme points coincide. Therefore, the following particular case holds.

Corollary 2.13. *Let X be a finite-dimensional space, let Y be a Banach space, and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then, G is generating if and only if $\|Gx\| = 1$ for every $x \in \mathrm{ext}(B_X)$.*

The following particular case of Corollary 2.12 is especially interesting.

Example 2.14. *Let Y be a Banach space and let $G \in \mathcal{L}(\ell_1, Y)$ be a norm-one operator. Then, G is generating if and only if $\|Ge_n\| = 1$ for every $n \in \mathbb{N}$.*

When every point of the unit sphere of the domain is a denting point, Proposition 2.11 tells us that generating operators are isometric embeddings. Spaces with such property of the unit sphere are average locally uniformly rotund (ALUR for short) spaces. They were introduced in [20] and it can be deduced from [13, Theorem] that a Banach space is ALUR if and only if every point of the unit sphere is a denting point.

Corollary 2.15. *Let X, Y be Banach spaces and suppose that X is ALUR. Then, every generating operator $G \in \mathcal{L}(X,Y)$ is an isometric embedding.*

The next result gives another useful characterization of r-generating operators.

Theorem 2.16. *Let X, Y be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator, let $r \in (0,1]$, and let $\mathcal{A} \subset B_{Y^*}$ such that $\overline{\operatorname{aconv}}^{w^*}(\mathcal{A}) = B_{Y^*}$. Then, G is r-generating if and only if $\max\limits_{\theta\in\mathbb{T}} \sup\limits_{y^*\in\mathcal{A}} \|G^*(y^*)+\theta x^*\| \geqslant 1 + r\|x^*\|$ for every $x^* \in X^*$.*

Proof. If G is r-generating, fixed $x^* \in X^*$ and $\delta > 0$, we can write

$$\begin{aligned}\max_{\theta\in\mathbb{T}} \sup_{y^*\in\mathcal{A}} \|G^*(y^*)+\theta x^*\| &= \max_{\theta\in\mathbb{T}} \sup_{y^*\in\mathcal{A}} \sup_{x\in B_X} |(G^*y^*)(x)+\theta x^*(x)| \\ &= \sup_{x\in B_X} \sup_{y^*\in\mathcal{A}} (|y^*(Gx)|+|x^*(x)|) \\ &= \sup_{x\in B_X} (\|Gx\|+|x^*(x)|) \geqslant \sup_{x\in\operatorname{att}(G,\delta)} (\|Gx\|+|x^*(x)|) \\ &\geqslant \sup_{x\in\operatorname{att}(G,\delta)} (1-\delta+|x^*(x)|) \geqslant 1-\delta+r\|x^*\|\end{aligned}$$

where the last inequality holds by Proposition 2.1. The arbitrariness of δ gives the desired inequality.

To prove the converse, fixed $x^* \in S_{X^*}$ and $\delta > 0$, it suffices to show that $\|x^*\|_{G,\delta} \geqslant r$ by Proposition 2.1. We use the hypothesis for $\frac{\delta}{2}x^*$ to get that

$$\max_{\theta\in\mathbb{T}} \sup_{y^*\in\mathcal{A}} \left\|G^*(y^*)+\theta\frac{\delta}{2}x^*\right\| \geqslant 1 + r\frac{\delta}{2}.$$

So, given $0 < \varepsilon < \frac{\delta}{2}$, there are $y^* \in \mathcal{A}$, $\theta \in \mathbb{T}$, and $x \in B_X$ such that

$$\|Gx\| + \frac{\delta}{2}|x^*(x)| \geqslant \left|y^*(Gx)+\theta\frac{\delta}{2}x^*(x)\right| > 1 + r\frac{\delta}{2} - \varepsilon$$

which implies that

$$\frac{\delta}{2}|x^*(x)| > r\frac{\delta}{2} - \varepsilon \quad \text{and} \quad \|Gx\| > 1 + (r-1)\frac{\delta}{2} - \varepsilon \geqslant 1 - \delta.$$

The arbitrariness of ε gives $\|x^*\|_{G,\delta} \geqslant r$ as desired. □

Of course, one can always use $\mathcal{A} = B_{Y^*}$ in Theorem 2.16 if no other interesting choice for $\mathcal{A}$ is available and still one obtains a useful characterization of r-generating operators.

In the case of generating operators, we emphasize the following result.

Corollary 2.17. *Let X, Y be Banach spaces, let $\mathcal{A} \subset B_{Y^*}$ be one-norming for Y, and let $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. Then, the following are equivalent:*

(i) *G is generating.*
(ii) *$G^*(B_{Y^*})$ is a spear set of X^*.*
(iii) *$G^*(\mathcal{A})$ is a spear set of X^*.*

(*iv*) $\max_{\theta\in\mathbb{T}} \sup_{y^*\in B_{Y^*}} \|G^*(y^*)+\theta x^*\| = 2$ *for every* $x^*\in S_{X^*}$.

Only item (*iv*) is new, and follows immediately from the following remark.

Remark 2.18. *Let Z be a Banach space and $F\subset B_Z$. Then, F is a spear set if and only if $\max_{\theta\in\mathbb{T}}\sup_{z\in F}\|z+\theta z_0\|=2$ for every $z_0\in S_Z$.*

Indeed, to prove the sufficiency, fixed $0\neq z_1\in X$, observe that

$$\max_{\theta\in\mathbb{T}}\sup_{z\in F}\left\|z+\theta\frac{z_1}{\|z_1\|}\right\|=2$$

implies that $\max_{\theta\in\mathbb{T}}\sup_{z\in F}\big\|\|z_1\|z+\theta z_1\big\|=2\|z_1\|$. So, if $\|z_1\|\geqslant 1$, the triangle inequality allows to write

$$\max_{\theta\in\mathbb{T}}\sup_{z\in F}\|z+\theta z_1\|\geqslant\max_{\theta\in\mathbb{T}}\sup_{z\in F}\big\|\|z_1\|z+\theta z_1\big\|-(\|z_1\|-1)=1+\|z_1\|.$$

If otherwise $\|z_1\|<1$, just observe that

$$\max_{\theta\in\mathbb{T}}\sup_{z\in F}\|z+\theta z_1\|\geqslant\max_{\theta\in\mathbb{T}}\sup_{z\in F}\left\|z+\theta\frac{z_1}{\|z_1\|}\right\|-(1-\|z_1\|)=1+\|z_1\|. \qquad\square$$

What we have shown is that it suffices to use elements $x^*\in S_{X^*}$ in Theorem 2.16 when $r=1$. However, the following example shows that this is not the case for any other value of $0<r<1$.

Example 2.19. *Let $0<r<1$ be fixed, let X be the real two-dimensional Hilbert space, $\{e_1,e_2\}$ be its orthonormal basis with $\{e_1^*,e_2^*\}$ being the corresponding coordinate functionals. The norm-one operator $G\in\mathcal{L}(X)$ given by $G=r\,\mathrm{Id}+(1-r)e_1^*\otimes e_1$ is not r-generating but satisfies*

$$\max_{\theta\in\mathbb{T}}\sup_{x^*\in B_{X^*}}\|G^*(x^*)+\theta x^*\|\geqslant 1+r\|x^*\|$$

for every $x^*\in S_{X^*}$.

Indeed, it is clear that $\|G\|=1$ and $G^*=r\,\mathrm{Id}+(1-r)e_1\otimes e_1^*$. So, given $x^*\in S_{X^*}$, we have that

$$\begin{aligned}\max_{\theta\in\mathbb{T}}\sup_{x^*\in B_{X^*}}\|G^*(x^*)+\theta x^*\| &\geqslant \|G^*(x^*)+x^*\| = \|(1+r)x^*+(1-r)x^*(e_1)e_1^*\|\\ &= \|2x^*(e_1)e_1^*+(1+r)x^*(e_2)e_2^*\|\geqslant 1+r.\end{aligned}$$

Observe that G attains its norm only at $\pm e_1$ so Proposition 2.7 tells us that G is not r-generating (in fact, it is not s-generating for any $0<s\leqslant 1$). $\square$

If we are able to guarantee that $G^*(B_{Y^*})$ is a spear set of X^*, Corollary 2.17 shows that G is generating. The most naive way to do so is to require $G^*(B_{Y^*})=B_{X^*}$ but observe that, as $\|G^*\|=1$, this implies that G^* is surjective and G is an isometry.

The other extreme possibility is $G^*(B_{Y^*})=\{\lambda x_0^*\colon \lambda\in\mathbb{K},\,|\lambda|\leqslant 1\}$ for some $x_0^*\in S_{X^*}$. This obviously means that G is a rank one operator; in this case, $G^*(B_{Y^*})$ is a spear set of X^* if and only if x_0^* is a spear vector of X^*. In this particular case, Corollary 2.17 reads as follows.

Corollary 2.20. *Let X, Y be Banach spaces, $x_0^*\in S_{X^*}$, and $y_0\in S_Y$. Then, the rank-one operator $G=x_0^*\otimes y_0$ is generating if and only if $x_0^*\in\mathrm{Spear}(X^*)$.*

Observe the similarity with [7, Corollary 5.9] which states that $G=x_0^*\otimes y_0$ is spear if and only if x_0^* is a spear functional and y_0 is a spear vector. Here the condition is easier to satisfy, of course.

2.1. **Some stability results.** The following result shows that the property of being generating is stable by c_0-, ℓ_1-, and ℓ_∞-sums of Banach spaces.

Proposition 2.21. *Let $\{X_\lambda\colon \lambda\in\Lambda\}$, $\{Y_\lambda\colon \lambda\in\Lambda\}$ be two families of Banach spaces and let $G_\lambda\in\mathcal{L}(X_\lambda,Y_\lambda)$ be a norm-one operator for every $\lambda\in\Lambda$. Let E be one of the Banach spaces c_0, ℓ_∞, or ℓ_1, let $X=\left[\bigoplus_{\lambda\in\Lambda}X_\lambda\right]_E$ and $Y=\left[\bigoplus_{\lambda\in\Lambda}Y_\lambda\right]_E$, and define the operator $G\colon X\longrightarrow Y$ by*

$$G\left[(x_\lambda)_{\lambda\in\Lambda}\right]=(G_\lambda x_\lambda)_{\lambda\in\Lambda}$$

for every $(x_\lambda)_{\lambda\in\Lambda}\in\left[\bigoplus_{\lambda\in\Lambda}X_\lambda\right]_E$. Then, G is generating if and only if G_λ is generating for every $\lambda\in\Lambda$.

Proof. Suppose first that G is generating and, fixed $\kappa\in\Lambda$, let us show that G_κ is generating. Observe that calling $W=\left[\bigoplus_{\lambda\neq\kappa}X_\lambda\right]_E$ and $Z=\left[\bigoplus_{\lambda\neq\kappa}Y_\lambda\right]_E$, we can write $X=X_\kappa\oplus_\infty W$ and $Y=Y_\kappa\oplus_\infty Z$ when E is ℓ_∞ or c_0 and $X=X_\kappa\oplus_1 W$ and $Y=Y_\kappa\oplus_1 Z$ when E is ℓ_1. Given $T_\kappa\in\mathcal{L}(X_\kappa,Y_\kappa)$, define $T\in\mathcal{L}(X,Y)$ by

$$T(x_\kappa,w)=(T_\kappa x_\kappa,0)\qquad (x_\kappa\in X_\kappa,\ w\in W)$$

which satisfies $\|T\|=\|T_\kappa\|$ and $\|T\|_G=\|T\|$ as G is generating. Moreover,

$$\|T\|_G=\inf_{\delta>0}\sup\{\|T(x_\kappa,w)\|\colon (x_\kappa,w)\in\operatorname{att}(G,\delta)\}=\inf_{\delta>0}\sup\{\|T_\kappa x_\kappa\|\colon x_\kappa\in\operatorname{att}(G_\kappa,\delta)\}=\|T_\kappa\|_{G_\kappa},$$

thus $\|T_\kappa\|=\|T_\kappa\|_{G_\kappa}$. The arbitrariness of T_κ gives that G_κ is generating.

To prove the sufficiency when E is c_0 or ℓ_∞, given $T\in\mathcal{L}(X,Y)$, it is enough to show that $\|T\|_G\geqslant\|T\|$. Fixed $\varepsilon>0$, we may find $\kappa\in\Lambda$ such that $\|P_\kappa T\|>\|T\|-\varepsilon$, where P_κ denotes the projection from Y onto Y_κ. Now, writing $X=X_\kappa\oplus_\infty W$ where $W=\left[\bigoplus_{\lambda\neq\kappa}X_\lambda\right]_E$, we have that $B_X=\operatorname{conv}(S_{X_\kappa}\times S_W)$ and so we may find $x_0\in S_{X_\kappa}$ and $w_0\in S_W$ such that

$$\|P_\kappa T(x_0,w_0)\|>\|T\|-\varepsilon.$$

Take $x_0^*\in S_{X_\kappa{}^*}$ with $x_0^*(x_0)=1$ and define the operator $S\in\mathcal{L}(X_\kappa,Y_\kappa)$ by

$$S(x)=P_\kappa T(x,x_0^*(x)w_0)\qquad (x\in X_\kappa)$$

which satisfies $\|S\|\geqslant\|Sx_0\|=\|P_\kappa T(x_0,w_0)>\|T\|-\varepsilon$ and $\|S\|_{G_\kappa}=\|S\|$ since G_κ is generating. Moreover, fixed $\delta>0$,

$$\begin{aligned}\|T\|_{G,\delta}&=\sup\{\|Tx\|\colon x\in S_X,\ \|Gx\|>1-\delta\}\\&\geqslant\sup\{\|T(x,x_0^*(x)w_0)\|\colon x\in X_\kappa,\ (x,x_0^*(x)w_0)\in S_X,\ \|G(x,x_0^*(x)w_0)\|>1-\delta\}\\&\geqslant\sup\{\|P_\kappa T(x,x_0^*(x)w_0)\|\colon x\in S_{X_\kappa},\ \|G_\kappa x\|>1-\delta\}=\|S\|_{G_\kappa,\delta}.\end{aligned}$$

Therefore, $\|T\|_G\geqslant\|S\|_{G_\kappa}=\|S\|>\|T\|-\varepsilon$ and the arbitrariness of ε gives that $\|T\|_G\geqslant\|T\|$ as desired.

In the case when $E=\ell_1$, fixed $\delta>0$, consider the set

$$A_\delta:=\bigcup_{\lambda\in\Lambda}\{x\in X\colon x_\lambda\in\operatorname{att}(G_\lambda,\delta),\ x_\kappa=0\text{ if }\kappa\neq\lambda\},$$

which satisfies that $A_\delta\subseteq\operatorname{att}(G,\delta)$ and

$$\begin{aligned}\overline{\operatorname{conv}}(A_\delta)&\supseteq\bigcup_{\lambda\in\Lambda}\overline{\operatorname{conv}}\left(\{x\in X\colon x_\lambda\in\operatorname{att}(G_\lambda,\delta),\ x_\kappa=0\text{ if }\kappa\neq\lambda\}\right)\\&=\bigcup_{\lambda\in\Lambda}\{x\in X\colon x_\lambda\in B_{X_\lambda},\ x_\kappa=0\text{ if }\kappa\neq\lambda\},\end{aligned}$$

where in the last equality we have used Corollary 2.3.iv as G_λ is generating for every $\lambda \in \Lambda$. Therefore, $B_X = \overline{\operatorname{conv}}\left(\bigcup_{\lambda\in\Lambda}\{x \in X : x_\lambda \in B_{X_\lambda},\ x_\kappa = 0 \text{ if } \kappa \neq \lambda\}\right) \subseteq \overline{\operatorname{conv}}(A_\delta) \subseteq \overline{\operatorname{conv}}(\operatorname{att}(G,\delta))$ and the arbitrariness of δ gives that G is generating by Corollary 2.3.iv. $\square$

We next discuss the relationship of being generating with the operation of taking the adjoint. We show next that if the second adjoint is r-generating then the operator itself is r-generating.

Proposition 2.22. *Let X, Y be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator, and let $r \in (0,1]$. If G^{**} is r-generating, then G is also r-generating.*

Proof. Fixed $x_0^* \in S_{X^*} \subset X^{***}$, we have that $\|x_0^*\|_{G^{**},\delta} \geqslant r\|x^*\|$ for every $\delta > 0$ by Proposition 2.1. So, fixed $\delta > 0$ and $\varepsilon > 0$, there exists $x^{**} \in \operatorname{att}(G^{**},\delta)$ with $|x^{**}(x_0^*)| > (1-\varepsilon)r$. Now, as $\|G^{**}x^{**}\| > 1-\delta$, there is $y^* \in S_{X^*}$ satisfying $|x^{**}(G^*y^*)| > 1-\delta$. By Goldstine's theorem there is $x \in B_X$ such that

$$|x_0^*(x)| = |J_X(x)(x_0^*)| > (1-\varepsilon)r \qquad \text{and} \qquad \|Gx\| \geqslant |y^*(Gx)| = |J_X(x)(G^*y^*)| > 1-\delta$$

which gives $\|x_0^*\|_{G,\delta} \geqslant r$ since $\varepsilon > 0$ was arbitrary. So G is r-generating by Proposition 2.1. $\square$

We do not know if the converse of the above result holds in general or even for $r = 1$. On the other hand, the following example shows that there is no good behaviour of the property of being generating with respect to taking one adjoint, as the property does not pass from an operator to its adjoint, nor the other way around.

Example 2.23. *Consider the norm-one operator $G\colon c_0 \longrightarrow c_0$ defined by*

$$Gx = \sum_{n=1}^{\infty} \frac{1}{n} x(n) e_n \qquad (x \in c_0).$$

For any $x \in S_{c_0}$ with $x(1) \in \mathbb{T}$ we have that $\|G(x)\| = 1$ and, consequently, $x \in \operatorname{att}(G,\delta)$ for every $\delta > 0$. Since such elements are enough to recover the whole unit ball of c_0 by taking closed convex hull, G is generating by Corollary 2.3.iv.

- *The adjoint operator $G^*\colon \ell_1 \longrightarrow \ell_1$*

$$G^*(x^*) = \sum_{n=1}^{\infty} \frac{1}{n} x^*(n) e_n^* \qquad (x^* \in \ell_1)$$

  is not generating by Example 2.14 since $\|G^*(e_n^*)\| = \frac{1}{n} < 1$ for $n > 1$.
- *The second adjoint $G^{**}\colon \ell_\infty \longrightarrow \ell_\infty$*

$$G^{**}(x^{**}) = \sum_{n=1}^{\infty} \frac{1}{n} x^{**}(n) e_n^{**} \qquad (x^{**} \in \ell_\infty)$$

  is again generating following an analogous argument to the one used for G, using this time elements $x \in S_{\ell_\infty}$ with $x(1) \in \mathbb{T}$.

2.2. **Some examples in classical Banach spaces.** Our aim here is to provide some characterizations of generating operators when the domain space is $L_1(\mu)$ or the range space is $C_0(L)$ by making use of Corollary 2.17.

2.2.1. *Operators acting from $L_1(\mu)$.* Let Y be a Banach space and let (Ω,Σ,μ) be a finite measure space. Recall that an operator $T\in\mathcal{L}(L_1(\mu),Y)$ is *representable* if there exists $g\in L_\infty(\mu,Y)$ such that

$$T(f)=\int_\Omega f(t)g(t)\,d\mu(t)\qquad (f\in L_1(\mu)).$$

In such case, $\|T\|=\|g\|_\infty$. Moreover, its adjoint $T^*\colon Y^*\to L_\infty(\mu)$ is given by

$$[T^*(y^*)](f)=y^*(T(f))=\int_\Omega f(t)y^*(g(t))\,d\mu(t)\qquad (f\in L_1(\mu),\ y^*\in Y^*),$$

then $T^*(y)=y^*\circ g\in L_\infty(\mu)$ for $y^*\in Y^*$.

Weakly compact operators are representable (see [5, p. 65, Theorem 12], for instance). If Y has the RNP, then every operator in $\mathcal{L}(L_1(\mu),Y)$ is representable (see [5, p. 63, Theorem 5], for instance) and so $\mathcal{L}(L_1(\mu),Y)$ identifies with $L_\infty(\mu,Y)$ in this case.

The question of which operators acting from $L_1(\mu)$ are generating leads to study the spear sets in $L_\infty(\mu)$. We do so in the next result which is valid for arbitrary measures.

Proposition 2.24 (Spear sets in $B_{L_\infty(\mu)}$)**.** *Let (Ω,Σ,μ) be a positive measure space and let $F\subset B_{L_\infty(\mu)}$. Then, the following are equivalent:*

(i) F is a spear set.
(ii) For every measurable set $A\in\Sigma$ with $\mu(A)\neq 0$ and every $\varepsilon>0$ there exists $B\in\Sigma$, $B\subset A$ with $\mu(B)\neq 0$ and $f\in F$ such that $|f(t)|>1-\varepsilon$ for every $t\in B$.

Proof. Suppose first that F is a spear set. Given $A\in\Sigma$ with $\mu(A)\neq 0$ and $\varepsilon>0$, since

$$\max_{\theta\in\mathbb{T}}\sup_{f\in F}\|f+\theta\mathbb{1}_A\|_\infty=2,$$

there exists $f_0\in F$ and $\theta_0\in\mathbb{T}$ such that $\|f_0+\theta_0\mathbb{1}_A\|_\infty>2-\varepsilon$ and thus, there exists $B\subset A$ with $\mu(B)\neq 0$ such that $|f(t)|>1-\varepsilon$ for every $t\in B$. To prove the converse implication, given $x\in L_\infty(\mu)$ and $\varepsilon>0$, there is $A\in\Sigma$ with $\mu(A)\neq 0$ such that $|x(t)|\geqslant\|x\|_\infty-\varepsilon$ for every $t\in A$. By the hypothesis, there is a subset B of A with $\mu(B)\neq 0$ and $f\in F$ such that $|f_0(t)|>1-\varepsilon$ for every $t\in B$. Now, thanks to the compactness of $\mathbb{T}$ we can fix an ε-net $\mathbb{T}_\varepsilon$ of $\mathbb{T}$, then we may find $\theta_0\in\mathbb{T}_\varepsilon$ and $C\subset B$ with $\mu(C)\neq 0$ such that $|f_0(t)+\theta_0x(t)|\geqslant|f_0(t)|+|x(t)|(1-\varepsilon)$ for every $t\in C$. Therefore,

$$\begin{aligned}\max_{\theta\in\mathbb{T}}\sup_{f\in F}\|f+\theta x\|_\infty &\geqslant \max_{\theta\in\mathbb{T}}\|f_0+\theta x\|_\infty\geqslant\inf_{t\in C}|f_0(t)+\theta x(t)|\geqslant\inf_{t\in C}|f_0(t)|+|x(t)|(1-\varepsilon)\\ &\geqslant 1-\varepsilon+(\|x\|_\infty-\varepsilon)(1-\varepsilon),\end{aligned}$$

and the arbitrariness of ε gives $\max_{\theta\in\mathbb{T}}\sup_{f\in F}\|f+\theta x\|_\infty\geqslant 1+\|x\|_\infty$. □

As an immediate consequence we get the following characterization of generating representable operators acting on $L_1(\mu)$.

Corollary 2.25. *Let Y be a Banach space, let (Ω,Σ,μ) be a finite measure space, and let $G\in\mathcal{L}(L_1(\mu),Y)$ be a norm-one operator which is representable by $g\in L_\infty(\mu,Y)$. Then, the following are equivalent:*

(i) G is generating.
(ii) $\{y^*\circ g\colon y^*\in B_{Y^*}\}$ is a spear set of $B_{L_\infty(\mu)}$.
(iii) For every measurable set $A\subset\Omega$ with $\mu(A)>0$ and every $\varepsilon>0$ there exists $B\subset A$ with $\mu(B)>0$ such that $\|g(t)\|>1-\varepsilon$ for all $t\in B$.
(iv) $\|g(t)\|=1$ μ-almost everywhere.

Remark 2.26. *The restriction on the measure μ being finite in Corollary 2.25 can be relaxed to being σ-finite.*

Indeed, given a σ-finite measure μ, there is a suitable probability measure ν such that $L_1(\mu) \equiv L_1(\nu)$ and $L_\infty(\mu, Y) \equiv L_\infty(\nu, Y)$, see [3, Proposition 1.6.1] for instance).

Compare Corollary 2.25 the above result with [7, Corollary 4.22] which says that $G \in \mathcal{L}(L_1(\mu), Y)$ of norm-one which is representable by $g \in L_\infty(\mu, Y)$ is a spear operator if and only if it has the alternative Daugavet property if and only if $g(t) \in \operatorname{Spear}(Y)$ for a.e. $t \in \Omega$. It is then easy to construct generating operators from $L_1(\mu)$ which do not have the alternative Daugavet property: for instance, $G \in \mathcal{L}(L_1[0,1], \ell_2^2)$ given by $G(f) = \displaystyle\int_0^1 f(t)(\cos(2\pi t), \sin(2\pi t))\, dt$ for every $f \in L_1[0,1]$.

2.2.2. *Operators arriving to $C_0(L)$.* Let L be a Hausdorff locally compact topological space. It is immediate from the definition of the norm, that the set $\mathcal{A} = \{\delta_t : t \in L\} \subset C_0(L)^*$ is one-norming for $C_0(L)$. Hence, Corollary 2.17 reads in this case as follows.

Proposition 2.27. *Let X be a Banach space, let L be a Hausdorff locally compact topological space, and let $G \in \mathcal{L}(X, C_0(L))$ be a norm-one operator. Then, the following are equivalent:*

(*i*) *G is generating.*
(*ii*) *The set $\{G^*(\delta_t) : t \in L\}$ is a spear set of X^*.*

We would like to compare the result above with [7, Proposition 4.2] where it is proved that $G \in \mathcal{L}(X, C_0(L))$ has the alternative Daugavet property if and only if $\{G^*(\delta_t) : t \in U\}$ is a spear set of X^* for every open subset $U \subset L$. It is then easy to construct examples of generating operators arriving to $C_0(L)$ spaces which do not have the alternative Daugavet property. For instance, consider $G \in \mathcal{L}(c_0, c_0)$ given by

$$[Gx](n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ x(n) & \text{if } n \text{ is even.} \end{cases}$$

3. Generating operators and norm-attainment

We discuss here when generating operators are norm-attaining. On the one hand, it is shown in [7, Theorem 2.9] that every spear $x^* \in X^*$ attains its norm. So rank-one generating operators also attain their norm by Corollary 2.20.

Corollary 3.1. *Let X, Y be Banach spaces and $G \in \operatorname{Gen}(X, Y)$ of rank-one. Then, G attains its norm.*

Besides, if B_X contains denting points, all generating operators with domain X are norm attaining by Lemma 2.8.

On the other hand, operators with the alternative Daugavet property are generating (see Example 2.4.(4)), and there are operators with the alternative Daugavet property which do not attain their norm (see [7, Example 8.7]). The construction of the cited example in [7] is not easy at all, but we may construct easier examples of generating operators which do not attain their norm, even with rank two.

Example 3.2. *Consider* $g\colon [0,1] \longrightarrow \ell_2^2$ *given by* $g(t) = (\cos t, \sin t)$ *and the norm-one operator* $G \in \mathcal{L}(L_1[0,1], \ell_2^2)$ *represented by* g*:*

$$G(x) = \int_0^1 x(t)g(t)\,dt \qquad (x \in L_1[0,1]).$$

Then, G *is generating but does not attain its norm.*

Proof. Observe that G is generating by Corollary 2.25 as $\|g(t)\| = 1$ for every $t \in [0,1]$. To prove that G does not attain its norm, recall that for an integrable complex-valued function f the equality $\left|\int_0^1 f(t)dt\right| = \int_0^1 |f(t)|dt$ holds if and only if there is $\lambda \in \mathbb{T}$ such that $f = \lambda|f|$ except for a set of zero measure. Suppose, to find a contradiction, that there is a non-zero $x \in L_1[0,1]$ satisfying $\|Gx\| = \|x\|$. Then, as xg can be seen as a complex-valued function and we can identify the norm on ℓ_2^2 with the modulus in $\mathbb{C}$, we have that

$$\begin{aligned}\left|\int_0^1 x(t)g(t)dt\right| &= \left\|\int_0^1 x(t)g(t)dt\right\| \\ &= \|Gx\| = \|x\| = \int_0^1 |x(t)|dt = \int_0^1 |x(t)g(t)|dt.\end{aligned}$$

Therefore, there is $\lambda \in \mathbb{T}$ such that $xg = \lambda|xg| = \lambda|x|$ except for a set of zero measure. But this is impossible since x takes real values and g covers a non-trivial arc of the unit circumference. □

Example 3.2 can be generalized for other two-dimensional spaces Y, but we need some assumptions on the shape of S_Y. If S_Y can be expressed as a finite or countable union of segments, then every generating operator $G \in \mathcal{L}(L_1[0,1], Y)$ attains its norm, leading to a complete characterization.

Proposition 3.3. *Let* Y *be a real two-dimensional space. Then, the following are equivalent:*

(*i*) S_Y *is a finite or countable union of segments.*
(*ii*) *Every generating operator* $G \in \mathcal{L}(L_1[0,1], Y)$ *attains its norm.*

Moreover, if the previous assertions hold, we have that $B_{L_1[0,1]} = \overline{\operatorname{conv}}(\operatorname{att}(G))$ *for every generating operator* $G \in \mathcal{L}(L_1[0,1], Y))$.

Proof. (*i*) ⇒ (*ii*) Let $G \in \mathcal{L}(L_1[0,1], Y)$ be a generating operator. Since Y has dimension two, G can be represented by

$$G(x) = \int_0^1 x(t)g(t)\,dt \qquad (x \in L_1[0,1])$$

for a suitable $g \in L_\infty([0,1], Y)$ with $\|g\|_\infty = 1$ and $\|g(t)\| = 1$ almost everywhere by Corollary 2.25. Since S_Y is a finite or countable union of segments, we may find a partition π of $[0,1]$ in measurable subsets of positive measure such that $g(A)$ is contained in a segment of S_Y almost everywhere for every $A \in \pi$. Then, for every $\Delta \in \pi$ and every measurable subset $A \subset \Delta$ of positive measure, consider $x_A = \frac{1}{|A|}\mathbb{1}_A \in S_{L_1[0,1]}$, where $|A|$ denotes the Lebesgue measure of A, and let us show that G attains its norm at x_A. Indeed, as $g(A)$ is contained in a segment of S_Y a.e., there exists $y^* \in S_{Y^*}$ such that $y^*(g(t)) = 1$ a.e. in A, thus

$$\|G(x_A)\| \geqslant y^*(Gx_A) = y^*\left(\int_0^1 \frac{1}{|A|}\mathbb{1}_A(t)g(t)\,dt\right) = \frac{1}{|A|}\int_A y^*(g(t))\,dt = 1,$$

and so $\|G(x_A)\| = 1$ as desired.
Moreover, for this π

$$B_{L_1[0,1]} \subseteq \overline{\text{aconv}}\left(\left\{\frac{1}{|A|}\mathbb{1}_A \colon A \subset \Delta, \Delta \in \pi, |A| > 0\right\}\right) \subseteq \overline{\text{conv}}(\text{att}(G)),$$

hence $B_{L_1[0,1]} = \overline{\text{conv}}(\text{att}(G))$.

To prove $(ii) \Rightarrow (i)$, suppose that S_Y cannot be written as a finite or countable union of segments and let us construct a generating operator $G \in \mathcal{L}(L_1[0,1], Y)$ not attaining its norm. Observe that the number of open maximal segments in S_Y is finite or countable as S_Y is a curve on a two-dimensional space with finite length. Let Δ_n, $n \in \mathbb{N}$, be the open maximal segments in S_Y and denote $D = S_Y \setminus (\cup_{n\in\mathbb{N}}\Delta_n)$. Clearly, D is an uncountable metric compact subset of S_Y, hence it contains a homeomorphic copy of the Cantor set K [11, Chapter I] and so there exists an injective continuous function $\varphi \colon K \longrightarrow D$. Now, let us construct an injection from $[0,1]$ to K. To do so, recall that the Cantor set is the set of numbers of $[0,1]$ that have a triadic representation consisting purely of 0's and 2's, that is,

$$K = \left\{y \in [0,1] \colon y = \sum_{k=1}^{\infty} \frac{\beta_k}{3^k},\ \beta_k = 0, 2\right\}.$$

Every $t \in [0,1]$ has a dyadic representation:

$$t = \sum_{k=1}^{\infty} \frac{\alpha_k(t)}{2^k},$$

where $\alpha_k(t) \in \{0,1\}$. This representation is unique except for a countable subset of $[0,1]$ consisting of those numbers with finite dyadic representation. Consider $\phi \colon [0,1] \longrightarrow K$ given by

$$\phi(t) = \sum_{k=1}^{\infty} \frac{2\alpha_k(t)}{3^k} \qquad (t \in [0,1]),$$

where $\alpha_k(t) \in \{0,1\}$ are the coefficients in the dyadic representation of t. The function ϕ is well-defined almost everywhere on $[0,1]$, injective, measurable, and its image lies on K. Then, the function $g = \varphi \circ \phi \colon [0,1] \longrightarrow D$ is well-defined almost everywhere on $[0,1]$, $g \in L_\infty[0,1]$, and it is injective. Consider the operator $G \colon L_1[0,1] \longrightarrow Y$ defined by

$$G(x) = \int_0^1 x(t)g(t)\,dt \qquad (x \in L_1[0,1]).$$

G is generating by Corollary 2.25 as $\|g(t)\| = 1$ almost everywhere but it does not attain its norm. Indeed, suppose on the contrary that there is a non-zero $x \in L_1[0,1]$ such that

$$\|G(x)\| = \left\|\int_0^1 x(t)g(t)\,dt\right\| = \int_0^1 |x(t)|\,dt = \|x\|.$$

We may find $y_0^* \in S_{Y^*}$ such that

$$\int_0^1 |x(t)|\,dt = \left\|\int_0^1 x(t)g(t)\,dt\right\| = y_0^*\left(\int_0^1 x(t)g(t)\,dt\right) = \int_0^1 x(t)y_0^*(g(t))\,dt.$$

This equality implies the existence of a measurable subset A of $[0,1]$ with positive measure such that $|x(t)| = x(t)y_0^*(g(t))$ for every $t \in A$, thus $y_0^*(g(t)) \in \{1,-1\}$ for every $t \in A$. Note that $g(A) \subseteq \{y \in D \colon y_0^*(y) \in \{1,-1\}\}$. However, this leads to a contradiction. On the one hand, the latter set has at most four elements as D does not contain open segments of S_Y. On the other hand, since

g is injective and A has positive measure, $g(A)$ has infinitely many elements. Thus, G cannot attain its norm. $\square$

The next example shows that, even in the case of norm-attaining operators, the set $\operatorname{att}(G)$ cannot be used to characterize when G is generating outside the case when X is reflexive and G is compact covered by Proposition 2.5.

Example 3.4. *Let $G \in \mathcal{L}(X,Y)$ be a generating operator between two Banach spaces X and Y such that it does not attain its norm. Then, the operator $\widetilde{G}\colon X\oplus_1\mathbb{K} \longrightarrow Y\oplus_1\mathbb{K}$ defined by $\widetilde{G}(x,\lambda) = (Gx,\lambda)$ is generating by Proposition 2.21 and attains its norm, but $\overline{\operatorname{conv}}(\operatorname{att}(\widetilde{G})) = \overline{\operatorname{conv}}(\{(0,\lambda)\colon \lambda\in\mathbb{T}\}) = \{(0,\lambda)\colon \lambda \in B_{\mathbb{K}}\}$ does not contain any ball of $X\oplus_1\mathbb{K}$.*

The following result characterizes the possibility to construct a generating operator not attaining its norm acting from a given Banach space which somehow extend Example 3.2.

Theorem 3.5. *Let X be a Banach space, the following are equivalent:*

(*i*) *There exists a Banach space Y and a norm-one operator $G\in\mathcal{L}(X,Y)$ such that G is generating but $\operatorname{att}(G)=\emptyset$.*
(*ii*) *There exists a spear set $\mathcal{B}\subseteq B_{X^*}$ such that $\sup\limits_{x^*\in\mathcal{B}} |x^*(x)| < 1$ for every $x\in S_X$.*

Proof. (*i*) $\Rightarrow$ (*ii*) Taking $\mathcal{B} = G^*(B_{Y^*})$, since G is generating, we can use Corollary 2.17 to deduce that $\mathcal{B}$ is a spear set. Besides, as G does not attain its norm, we have that

$$1 > \|G(x)\| = \sup_{y^*\in B_{Y^*}} |y^*(Gx)| = \sup_{y^*\in B_{Y^*}} |(G^*y^*)(x)| = \sup_{x^*\in\mathcal{B}} |x^*(x)|$$

for every $x\in S_X$.

(*ii*) $\Rightarrow$ (*i*) Consider $Y=\ell_\infty(\mathcal{B})$ and $G\colon X\longrightarrow \ell_\infty(\mathcal{B})$ defined by

$$(Gx)(x^*) = x^*(x) \qquad (x^*\in X^*,\, x\in X).$$

On the one hand, for $x\in S_X$, we have that

$$\|G(x)\| = \sup_{x^*\in\mathcal{B}} |(Gx)(x^*)| = \sup_{x^*\in\mathcal{B}} |x^*(x)| < 1.$$

On the other hand, using that $\mathcal{B}$ is a spear set, for every $\varepsilon>0$ we may find $x^*\in\mathcal{B}$ with $\|x^*\| > 1-\varepsilon$ and so

$$\|G\| = \sup_{x\in B_X} \|G(x)\| \geqslant \sup_{x\in B_X} |(Gx)(x^*)| = \sup_{x\in B_X} |x^*(x)| = \|x^*\| > 1-\varepsilon.$$

Therefore, $\|G\|=1$ but the norm is not attained.

To show that G is generating, we start claiming that, for every $g\in\ell_1(\mathcal{B})\subset \ell_\infty(\mathcal{B})^*$, we have

$$G^*(g) = \sum_{x^*\in\mathcal{B}} g(x^*)x^* \in X^*.$$

Indeed, given $g\in\ell_1(\mathcal{B})$, observe that

$$g(f) = \sum_{x^*\in\mathcal{B}} g(x^*)f(x^*) \qquad (f\in\ell_\infty(\mathcal{B}))$$

and

$$[G^*(g)](x) = g(Gx) = \sum_{x^*\in\mathcal{B}} g(x^*)(Gx)(x^*) = \sum_{x^*\in\mathcal{B}} g(x^*)x^*(x) \qquad (x\in X),$$

so $G^*(g) = \sum_{x^*\in\mathcal{B}} g(x^*)x^*$. Now, fixed $x_0^* \in \mathcal{B}$, define $g_0 \in \ell_\infty(\mathcal{B})$ by

$$g_0(x^*) = \begin{cases} 1 & \text{if } x^* = x_0^* \\ 0 & \text{if } x^* \neq x_0^* \end{cases}$$

which clearly satisfies $G^*(g_0) = x_0^*$. Therefore, by the arbitrariness of $x_0^* \in \mathcal{B}$, we get $G^*(B_{\ell_\infty(\mathcal{B})^*}) \supset \mathcal{B}$, so $G^*(B_{\ell_\infty(\mathcal{B})^*})$ is a spear set and G is generating by Corollary 2.17. □

The above proof, when read pointwise, allows to give a characterization of those points at which every generating operator attains its norm.

Proposition 3.6. *Let X be a Banach space and $x_0 \in S_X$. Then, the following are equivalent:*

(i) *For every Banach space Y and for every generating operator $G \in S_{\mathcal{L}(X,Y)}$ one has $\|Gx_0\| = 1$.*
(ii) *The equality $\sup\limits_{x^*\in\mathcal{B}} |x^*(x_0)| = 1$ holds for every spear set $\mathcal{B} \subseteq B_{X^*}$.*

Proof. (ii) $\Rightarrow$ (i) Given a generating operator G, $\mathcal{B} = G^*(B_{Y^*})$ is a spear set by Corollary 2.17 so

$$\|Gx_0\| = \sup_{y^*\in B_{Y^*}} |y^*(Gx_0)| = \sup_{x^*\in\mathcal{B}} |x^*(x_0)| = 1.$$

(i) $\Rightarrow$ (ii) Suppose that (ii) does not hold. Then, there is a spear set $\mathcal{B} \subseteq B_{X^*}$ such that $\sup\limits_{x^*\in\mathcal{B}} |x^*(x_0)| < 1$. Now, the operator $G\colon X \longrightarrow \ell_\infty(\mathcal{B})$ defined by

$$(Gx)(x^*) = x^*(x) \qquad (x^* \in X^*,\, x \in X)$$

is generating (as shown in the proof of Theorem 3.5) and satisfies

$$\|G(x_0)\| = \sup_{x^*\in\mathcal{B}} |(Gx_0)(x^*)| = \sup_{x^*\in\mathcal{B}} |x^*(x_0)| < 1.$$

Therefore, (i) does not hold. □

4. The set of all generating operators

Our aim here is to study the set $\operatorname{Gen}(X,Y)$ of all generating operators between the Banach spaces X and Y. Recall, on the one hand, that $\mathrm{Id}_X \in \operatorname{Gen}(X,X)$ for every Banach space X, so $\operatorname{Gen}(X,X) \neq \emptyset$ for every Banach space X. On the other hand, recall that Corollary 2.20 shows that $\operatorname{Gen}(X,\mathbb{K}) = \operatorname{Spear}(X^*)$, so $\operatorname{Gen}(X,\mathbb{K})$ is empty for many Banach spaces X: those for which $\operatorname{Spear}(X^*) = \emptyset$ as uniformly smooth spaces, strictly convex spaces, or real smooth spaces with dimension at least two (see [7, Proposition 2.11]). We will be interested in finding conditions to ensure that $\operatorname{Gen}(X,Y)$ is non-empty and, in those cases, to study how big the set $\operatorname{Gen}(X,Y)$ can be. We start with an easy observation on $\operatorname{Gen}(X,Y)$.

Proposition 4.1. *Let X, Y be Banach spaces. Then,* $\operatorname{Gen}(X,Y)$ *is norm-closed.*

Proof. Fixed $G_0 \in \overline{\operatorname{Gen}(X,Y)}$ and $n \in \mathbb{N}$, there is $G_n \in \operatorname{Gen}(X,Y)$ such that $\|G_0 - G_n\| < 1/n$ and, therefore, $\|G_0^* - G_n^*\| < 1/n$. Observe now that, for $x^* \in X^*$, we have

$$\begin{aligned}\max_{\theta\in\mathbb{T}} \sup_{y^*\in B_{Y^*}} \|G_0^*(y^*) + \theta x^*\| &\geqslant \max_{\theta\in\mathbb{T}} \sup_{y^*\in B_{Y^*}} \|G_n^*(y^*) + \theta x^*\| - \sup_{y^*\in B_{Y^*}} \|(G_0^* - G_n^*)(y^*)\| \\ &= \|G_n^*(B_{Y^*}) + \mathbb{T}x^*\| - \|G_0^* - G_n^*\| > 1 + \|x^*\| - 1/n,\end{aligned}$$

where the last inequality holds by Corollary 2.17 since G_n is generating. Now, it follows again from Corollary 2.17 that $G_0 \in \operatorname{Gen}(X,Y)$. □

Next, we study the problem of finding out whether $\operatorname{Gen}(X,Y)$ is empty or not for the Banach spaces X and Y from two points of view: fixing the space Y and fixing the space X.

4.1. **$\operatorname{Gen}(X,Y)$ when Y is fixed.** We will show that for every Banach space Y there is another Banach space X such that $\operatorname{Gen}(X,Y)=\emptyset$.

Proposition 4.2. *For every Banach space Y there is a Banach space X such that* $\operatorname{Gen}(X,Y)=\emptyset$.

We need the following obstructive result for the existence of generating operators that will serve to our purpose.

Lemma 4.3. *Let X, Y be Banach spaces and let $G\in\operatorname{Gen}(X,Y)$. If the norm of X^* is Fréchet differentiable at $x_0^*\in S_{X^*}$ and x_0^* is strongly exposed, then $x_0^*\in\overline{G^*(B_{Y^*})}$.*

Proof. Suppose that $x_0^*\notin\overline{G^*(B_{Y^*})}$ and let $\alpha=\operatorname{dist}(x_0^*,\overline{G^*(B_{Y^*})})>0$. Since x_0^* is strongly exposed, there are $x\in S_X$ and $\delta>0$ satisfying $\operatorname{Re}x_0^*(x)=1$ and $\operatorname{diam}(\operatorname{Slice}(B_{X^*},x,\delta))<\alpha$. Therefore, we get $\operatorname{Re}x^*(x)\leqslant 1-\delta$ for every $x^*\in\overline{G^*(B_{Y^*})}$ and, as $\overline{G^*(B_{Y^*})}$ is a balanced set, we get in fact that

$$|x^*(x)|\leqslant 1-\delta \qquad \forall\, x^*\in\overline{G^*(B_{Y^*})}. \tag{4}$$

By Corollary 2.17, $\overline{G^*(B_{Y^*})}$ is a spear set, so we can find a sequence $\{x_n^*\}$ in $\overline{G^*(B_{Y^*})}$ and a sequence $\{\theta_n\}$ in $\mathbb{T}$ such that $\|\theta_n x_n^*+x_0^*\|\to 2$. Therefore, there is a sequence $\{x_n\}$ in S_X satisfying

$$\operatorname{Re}x_0^*(x_n)\to 1 \qquad \text{and} \qquad |x_n^*(x_n)|\to 1.$$

Since the norm of X^* is Fréchet differentiable at $x_0^*\in S_{X^*}$, by the Šmulyian's test, we have that $\|x_n-x\|\to 0$. Thus, we get $|x_n^*(x)|\to 1$ which contradicts (4). ☐

We are now able to provide the pending proof. For a Banach space X let $\operatorname{dens}(X)$ denote its density character.

Proof of Proposition 4.2. Take a set Λ with cardinality greater than $\operatorname{dens}(Y^*)$ and set $X=\ell_2(\Lambda)$. If $G\in\operatorname{Gen}(X,Y)$, it follows from Lemma 4.3 that $\overline{G^*(Y^*)}=X^*=\ell_2(\Lambda)$ since every point in S_{X^*} is Fréchet differentiable and strongly exposed. Then, $\operatorname{dens}(X^*)=\operatorname{dens}(\overline{G^*(Y^*)})\leqslant\operatorname{dens}(Y^*)$, which is a contradiction. ☐

The above argument is based on the possibility of considering Banach spaces in the domain with a very big density character. It is then natural to raise the following question.

Question 4.4. Does there exist a Banach space Y with $\operatorname{dens}(Y)=\Gamma$ such that $\operatorname{Gen}(X,Y)\neq\emptyset$ for every Banach space X satisfying $\operatorname{dens}(X)\leqslant\Gamma$?

This question is easily solvable for separable spaces. Indeed, the space $Y=C[0,1]$ contains isometrically every separable Banach space. Since isometric embeddings are generating, we get the following example.

Example 4.5. *The separable Banach space $Y=C[0,1]$ satisfies $\operatorname{Gen}(X,Y)\neq\emptyset$ for every separable Banach space X.*

The question of whether the same trick works for all density characters is involved and depends on the Axiomatic Set Theory. On the one hand, assuming CH, ℓ_∞/c_0 is isometrically universal for all Banach spaces of density character the continuum [17] but, on the other hand, it is consistent that no such a universal space exists [19], even a isomorphically universal space, see [4].

4.2. **Gen(X, Y) when X is fixed.** We start our discussion recalling that, by Corollary 2.20, a rank-one operator $x^* \otimes y \in G(X,Y)$ is generating if and only if $x^* \in \operatorname{Spear}(X^*)$. This, together with the fact that $\operatorname{Gen}(X,\mathbb{K}) = \operatorname{Spear}(X^*)$, gives the following result.

Corollary 4.6. *Let X be a Banach space. Then,*

$$\operatorname{Gen}(X,Y) \neq \emptyset \text{ for every Banach space } Y \quad \Longleftrightarrow \quad \operatorname{Spear}(X^*) \neq \emptyset.$$

For instance, if X has the alternative Daugavet property and B_{X^*} has w^*-denting points, then $\operatorname{Spear}(X^*) \neq \emptyset$ by [7, Proposition 5.1].

Once we know about the existence of Banach spaces for which $\operatorname{Gen}(X,Y) \neq \emptyset$ for every Banach space Y, it is natural to ask about the possible size of the set $\operatorname{Gen}(X,Y)$. The maximal possibility is $\operatorname{Gen}(X,Y) = S_{\mathcal{L}(X,Y)}$, but this forces $X = \mathbb{K}$.

Corollary 4.7. *Let X be a Banach space. Then, there exists a Banach space Y such that $\operatorname{Gen}(X,Y) = S_{\mathcal{L}(X,Y)}$ if and only if $X = \mathbb{K}$. In this case, $\operatorname{Gen}(X,Z) = S_{\mathcal{L}(X,Z)}$ for all Banach spaces Z.*

Proof. If $X = \mathbb{K}$ then $\operatorname{Gen}(X,Y) = S_{\mathcal{L}(X,Y)}$ obviously holds for every Banach space Y. Conversely, suppose that there is a Banach space Y such that $\operatorname{Gen}(X,Y) = S_{\mathcal{L}(X,Y)}$. So, in particular, every rank-one operator in $S_{\mathcal{L}(X,Y)}$ is generating but this means that $\operatorname{Spear}(X^*) = S_{X^*}$ by Corollary 2.20. Therefore, $X = \mathbb{K}$ by [7, Proposition 2.11.(e)] □

It is now natural to wonder if there can be enough generating operators to recover the unit ball of $\mathcal{L}(X,Y)$ by convex (or closed convex) hull. That is, we are looking for Banach spaces X such that $B_{\mathcal{L}(X,Y)} = \operatorname{conv}(\operatorname{Gen}(X,Y))$ or $B_{\mathcal{L}(X,Y)} = \overline{\operatorname{conv}}(\operatorname{Gen}(X,Y))$ for every Banach space Y.

We start our discussion with an observation on lush spaces. Recall that a Banach spaces X is *lush* [2] if for every $x, y \in S_X$ and every $\varepsilon > 0$, there exists $y^* \in S_{Y^*}$ such that $y \in \operatorname{Slice}(B_X, y^*, \varepsilon)$ and $\operatorname{dist}(x, \operatorname{aconv}(\operatorname{Slice}(B_X, y^*, \varepsilon))) < \varepsilon$. Observe that $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{Gen}(X,\mathbb{K})) = \overline{\operatorname{conv}}^{w^*}(\operatorname{Spear}(X^*))$ implies that X is lush by [7, Proposition 3.32]. Conversely, if X is lush and separable, then $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{Gen}(X,\mathbb{K}))$ by [7, Theorem 3.33]. If one replaces the weak-star closed convex hull by the norm closed convex hull, one gets some interesting results on almost CL-spaces. A Banach space X is said to be an *almost CL-space* [12] if B_X is the absolutely closed convex hull of every maximal convex subset of S_X. By Hahn-Banach and Krein-Milman theorems, every maximal convex subset of S_X has the form $\operatorname{Face}(B_X, x^*) := \{x \in S_X : x^*(x) = 1\}$ for suitable $x^* \in \operatorname{ext}(B_{X^*})$. In this case, we say that x^* is a *maximal extreme point*, and write $x^* \in \operatorname{extm}(B_{X^*})$.

Proposition 4.8. *Let X be a Banach space satisfying that $B_{X^*} = \overline{\operatorname{conv}}(\operatorname{Gen}(X,\mathbb{K}))$. Then, X^* is an almost CL-space.*

Proof. Indeed, let $F = \operatorname{Face}(S_{X^*}, x^{**})$ for some $x^{**} \in \operatorname{extm}(B_{X^{**}})$ be a maximal convex subset of S_{X^*}. Then, $B_{X^*} = \overline{\operatorname{conv}}(\mathbb{T}F)$ since $\operatorname{Spear}(X^*) \equiv \operatorname{Gen}(X,\mathbb{K}) \subseteq \mathbb{T}\operatorname{Face}(S_{X^*}, x^{**})$ for all $x^{**} \in \operatorname{ext}(B_{X^{**}})$ by [7, Corollary 2.8.iv]. □

A partial converse of the above result is also true:

Proposition 4.9. *Let X be an almost CL-space. Then, $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{Gen}(X,\mathbb{K}))$. If, moreover, X does not contain ℓ_1, then $B_{X^*} = \overline{\operatorname{conv}}(\operatorname{Gen}(X,\mathbb{K}))$.*

Proof. Being $\operatorname{extm}(B_{X^*})$ norming for X, we always have that

$$B_{X^*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{extm}(B_{X^*})).$$

But when X is an almost CL-space, we have that $|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every $x^* \in \operatorname{extm}(B_{X^*})$ by using [15, Lemma 3]. Then, $\operatorname{extm}(B_{X^*}) \subseteq \operatorname{Spear}(X^*) \equiv \operatorname{Gen}(X, \mathbb{K})$ by [7, Corollary 2.8.iv], and we are done.

For the moreover part, it is enough to see that $\operatorname{extm}(B_{X^*})$ is actually a James boundary for X and so $B_{X^*} = \overline{\operatorname{conv}}(\operatorname{extm}(B_{X^*}))$ by [6, Theorem III.1]. □

Our next aim is to show that the set $\operatorname{Gen}(L_1(\mu), Y)$ is quite big for every finite measure μ and many Banach spaces Y, and that in some cases it allows to recover the unit ball of $\mathcal{L}(L_1(\mu), Y)$ by taking closed convex hull. Given a finite measure space (Ω, Σ, μ) and a Banach space Y we write

$$\mathcal{R}(L_1(\mu), Y) = \{T \in \mathcal{L}(L_1(\mu), Y) \colon \|T\| \leqslant 1,\ T \text{ is representable}\}.$$

Theorem 4.10. *Let (Ω, Σ, μ) be a finite measure space and let Y be a Banach space. Then,*

$$\mathcal{R}(L_1(\mu), Y) \subseteq \overline{\operatorname{conv}}\big(\operatorname{Gen}(L_1(\mu), Y)\big).$$

As a consequence, if Y has the RNP, then

$$B_{\mathcal{L}(L_1(\mu), Y)} = \overline{\operatorname{conv}}\big(\operatorname{Gen}(L_1(\mu), Y)\big).$$

Observe that the restriction on the measure μ to be finite can be relaxed to be σ-finite as in Remark 2.26.

The proof of the theorem follows immediately using Corollary 2.25 and the next lemma, which we do not know whether it is already known.

Lemma 4.11. *Let (Ω, Σ, μ) be a positive measure space and let Y be a Banach space. Then,*

$$B_{L_\infty(\mu, Y)} = \overline{\operatorname{conv}}\big(\{g \in L_\infty(\mu, Y) \colon \|g(t)\| = 1\ \mu\text{-almost everywhere}\}\big).$$

Proof. Calling $\mathcal{B} = \{g \in L_\infty(\mu, Y) \colon \|g(t)\| = 1\ \mu\text{-almost everywhere}\}$, it obviously suffices to show that $S_{L_\infty(\mu, Y)} \subset \overline{\operatorname{conv}}(\mathcal{B})$. We divide the proof into two steps.

Step one. Let $f \in S_{L_\infty(\mu, Y)}$ and suppose that there are $N \in \mathbb{N}$, numbers $\alpha_1 < \cdots < \alpha_N \in [0, 1]$, and pairwise disjoint subsets $B_k \subset \Omega$ with $\mu(B_k) \neq 0$ for $k = 1, \ldots, N$ such that $\bigcup_{k=1}^N B_k = \Omega$ and $\|f(t)\| = \alpha_k$ for every $t \in B_k$ and every $k = 1, \ldots, N$ (observe that $\alpha_N = 1$ as $\|f\| = 1$). Then, f can be written as a convex combination of 2^{N-1} functions in $\mathcal{B}$.

Indeed, we proceed by induction on N: for $N = 1$, the function f belongs to $\mathcal{B}$. The case $N = 2$ gives the flavour of the proof. In this case we have that $\|f(t)\| = \alpha_1$ for every $t \in B_1$ and $\|f(t)\| = 1$ for every $t \in B_2$. So, call $\lambda_1 = \frac{1+\alpha_1}{2}$, $\lambda_2 = \frac{1-\alpha_1}{2} \in [0, 1]$ and define $g_1, g_2 \in L_\infty(\mu, Y)$ by $g_1(t) = g_2(t) = f(t)$ for every $t \in B_2$. Besides, if $\alpha_1 \neq 0$, define

$$g_1(t) = \frac{f(t)}{\|f(t)\|}, \quad \text{and} \quad g_2(t) = -\frac{f(t)}{\|f(t)\|} \quad \forall t \in B_1.$$

If otherwise $\alpha_1 = 0$, fix $y_0 \in S_Y$, and define $g_1(t) = y_0$ and $g_2(t) = -y_0$ for every $t \in B_1$. It is clear that in any case we have $f = \lambda_1 g_1 + \lambda_2 g_2$ and that $g_1, g_2 \in \mathcal{B}$.

Suppose now that the result is true for $N \geqslant 2$ and let us prove it for $N + 1$. So, let $f \in S_{L_\infty(\mu, Y)}$ and suppose that there are numbers $\alpha_1 < \cdots < \alpha_{N+1} \in [0, 1]$ with $\alpha_{N+1} = 1$, and pairwise disjoint subsets $B_k \subset \Omega$ with $\mu(B_k) \neq 0$ for $k = 1, \ldots, N + 1$ such that $\bigcup_{k=1}^{N+1} B_k = \Omega$ and $\|f(t)\| = \alpha_k$ for every $t \in B_k$ and every $k = 1, \ldots, N + 1$. Observe that, as $N \geqslant 2$, we have that $\alpha_N > 0$. Then, we call $\lambda_1 = \frac{1+\alpha_N}{2}$, $\lambda_2 = \frac{1-\alpha_N}{2} \in [0,1]$ and we define $f_1, f_2 \in L_\infty(\mu, Y)$ by

$$f_1(t) = \frac{f(t)}{\|f(t)\|} \quad \text{if } t \in B_N \quad \text{and} \quad f_1(t) = f(t) \quad \text{if } t \in \Omega \setminus B_N,$$
$$f_2(t) = -\frac{f(t)}{\|f(t)\|} \quad \text{if } t \in B_N \quad \text{and} \quad f_2(t) = f(t) \quad \text{if } t \in \Omega \setminus B_N$$

which clearly satisfy $f = \lambda_1 f_1 + \lambda_2 f_2$. Besides, it is also clear that $\|f_1(t)\| = \|f_2(t)\| = 1$ for every $t \in B_N \cup B_{N+1}$. So, we can apply the induction step for f_1 and f_2 to write

$$f_1 = \sum_{k=1}^{2^{N-1}} \mu_k g_k \qquad \text{and} \qquad f_2 = \sum_{k=1}^{2^{N-1}} \beta_k h_k$$

where $g_k, h_k \in \mathcal{B}$, $\mu_k, \beta_k \in [0,1]$ for $k = 1, \ldots, 2^{N-1}$, $\sum_{k=1}^{2^{N-1}} \mu_k = 1$, and $\sum_{k=1}^{2^{N-1}} \beta_k = 1$. Therefore, the convex combination we are looking for is

$$f = \lambda_1 \sum_{k=1}^{2^{N-1}} \mu_k g_k + \lambda_2 \sum_{k=1}^{2^{N-1}} \beta_k h_k$$

which finishes the induction process.

Step two. Every function $f \in S_{L_\infty(\mu,Y)}$ can be approximated by functions of the class described in the first step.

Indeed, fixed $\varepsilon > 0$, we may find a partition of $[0,1] = \bigcup_{k=1}^N A_k$ such that $0 < \operatorname{diam}(A_k) < \varepsilon$ for every $k = 1, \ldots, N$, $0 \in A_1$, and $1 \in A_N$. Next, fix $\alpha_k \in A_k$ for each $k = 1, \ldots, N$ with $\alpha_1 = 0$ and $\alpha_N = 1$, and define $B_k = \{t \in \Omega \colon \|f(t)\| \in A_k\}$ for every $k = 1, \ldots, N$. We assume without loss of generality that $B_1, \ldots, B_N$ are non-empty. Now, consider the function $h \in L_\infty(\mu, Y)$ given by

$$h(t) = \begin{cases} 0 & \text{if } t \in B_1 \\ \alpha_k \dfrac{f(t)}{\|f(t)\|} & \text{if } t \in B_k \text{ with } k \geqslant 2. \end{cases}$$

For $t \in B_1$, we have

$$\|f(t) - h(t)\| = \|f(t)\| \leqslant \operatorname{diam}(A_1) < \varepsilon.$$

Besides, for $t \in B_k$ with $k \geqslant 2$, we have

$$\|f(t) - h(t)\| = \left\| f(t) - \alpha_k \frac{f(t)}{\|f(t)\|} \right\| = \big| \|f(t)\| - \alpha_k \big| \leqslant \operatorname{diam}(A_k) < \varepsilon.$$

Therefore, $\|f - h\| \leqslant \varepsilon$ and the proof is finished. □

Let us now discuss the case of purely atomic measures. When μ is purely atomic and σ-finite (so $L_1(\mu)$ can be easily viewed as $L_1(\nu)$ for a suitable purely atomic and finite measure ν, see [3, Proposition 1.6.1] for instance), every operator in $\mathcal{L}(L_1(\mu), Y)$ is representable for every Banach space Y (see [5, p. 62], for instance). So, Theorem 4.10 gives that $B_{\mathcal{L}(\ell_1(\Gamma),Y)} = \overline{\operatorname{conv}}(\operatorname{Gen}(\ell_1(\Gamma), Y))$ for every Banach space Y and every countable set Γ. Actually, the restriction of countability for the set Γ can be remove and the proof in this case is much more direct.

Proposition 4.12. $B_{\mathcal{L}(\ell_1(\Gamma),Y)} = \overline{\operatorname{conv}}(\operatorname{Gen}(\ell_1(\Gamma), Y))$ *for every Banach space Y and every set Γ.*

Proof. The space $\mathcal{L}(\ell_1(\Gamma), Y)$ can be easily identified with $\left[\bigoplus_{\gamma\in\Gamma} Y\right]_{\ell_\infty}$ using the isometric isomorphism $\Phi \colon \mathcal{L}(\ell_1(\Gamma), Y) \longrightarrow \left[\bigoplus_{\gamma\in\Gamma} Y\right]_{\ell_\infty}$ given by $\Phi(T) = (Te_\gamma)_{\gamma\in\Gamma}$ (see the proof of [18, Lemma 2], for instance). With this identification and Example 2.14, generating operators in $\mathcal{L}(\ell_1(\Gamma), Y)$ are exactly elements in $\left[\bigoplus_{\gamma\in\Gamma} Y\right]_{\ell_\infty}$ with every coordinate having norm one. Therefore, Lemma 4.11 gives the result. $\square$

For finite-dimensional ℓ_1-spaces, we get a better result.

Corollary 4.13. *$B_{\mathcal{L}(\ell_1^n, Y)} = \operatorname{conv}(\operatorname{Gen}(\ell_1^n, Y))$ for every Banach space Y and every $n \in \mathbb{N}$.*

Proof. For $T \in B_{\mathcal{L}(\ell_1^n, Y)}$ consider the finite-dimensional subspace of Y given by $Y_1 = T(\ell_1^n)$ and observe that $\overline{\operatorname{conv}}(\operatorname{Gen}(\ell_1^n, Y_1)) = \operatorname{conv}(\operatorname{Gen}(\ell_1^n, Y_1))$ as $\operatorname{Gen}(\ell_1^n, Y_1)$ is compact. So, Proposition 4.12 tells us that

$$T \in \operatorname{conv}(\operatorname{Gen}(\ell_1^n, Y_1)).$$

Finally, denoting G_1 the inclusion of Y_1 in Y, it is obvious that $G_1 \circ G \in \operatorname{Gen}(\ell_1^n, Y)$ for every $G \in \operatorname{Gen}(\ell_1^n, Y_1)$. So $T \in \operatorname{conv}(\operatorname{Gen}(\ell_1^n, Y))$. $\square$

The next result shows that the only finite-dimensional real spaces with this property are ℓ_1^n for $n \in \mathbb{N}$.

Proposition 4.14. *Let X be a* real *Banach space with $\dim(X) = n$ and such that $B_{\mathcal{L}(X,Y)} = \overline{\operatorname{conv}}(\operatorname{Gen}(X, Y))$ for every Banach space Y. Then, $X = \ell_1^n$.*

Proof. Proposition 4.8 tells us that X^* is an almost CL-space so $n(X^*) = n(X) = 1$. Therefore, as X is real, the set $\operatorname{ext}(B_X)$ is finite by [16, Theorem 3.2]. Our goal is to show that $\operatorname{ext}(B_X)$ contains exactly $2n$ elements as this clearly implies that X is isometrically isomorphic to the real space ℓ_1^n.

We suppose that $\operatorname{ext}(B_X)$ has more than $2n$ elements and we show that, in such a case, there is a Banach space Y ($= X$ with a new norm) such that $B_{\mathcal{L}(X,Y)} \neq \operatorname{conv}(\operatorname{Gen}(X, Y))$. Since $\dim(X) = n$ and $\operatorname{ext}(B_X)$ has more than $2n$ elements, we may find $\{e_1, \ldots, e_n\} \subset \operatorname{ext}(B_X)$ linearly independent and $e_{n+1} \in \operatorname{ext}(B_X)$ satisfying

$$e_{n+1} \notin \{\pm e_j \colon j = 1, \ldots, n\}.$$

For each $j = 1, \ldots, n$, as $\operatorname{ext}(B_X)$ is finite, we can pick $f_j \in X^*$ such that

$$1 = f_j(e_j) > c_j = \max\{f_j(x) \colon x \in \operatorname{ext}(B_X) \setminus \{e_j\}\}.$$

Besides, define $c = \max\{c_j \colon j = 1, \ldots, n\} < 1$, take $\varepsilon > 0$ satisfying $(1 + \varepsilon)c < 1$, and consider the Banach space Y whose unit ball is

$$B_Y = \operatorname{conv}\Big(\operatorname{ext}(B_X) \cup \{\pm(1 + \varepsilon)e_{n+1}\}\Big).$$

Now, observe that $e_1, \ldots, e_n$ are also extreme points of B_Y. Indeed, fixed $j \in \{1, \ldots, n\}$, our choice of c gives

$$f_j(x) \leqslant (1 + \varepsilon)c_j < 1 = f_j(e_j)$$

for every $x \in \operatorname{ext}(B_X) \cup \{\pm(1 + \varepsilon)e_{n+1}\}$ with $x \neq e_j$. So e_j cannot lie in a proper segment of B_Y.

Observe that $\overline{\operatorname{conv}}(\operatorname{Gen}(X, Y)) = \operatorname{conv}(\operatorname{Gen}(X, Y))$, as $\mathcal{L}(X, Y)$ is finite-dimensional and $\operatorname{Gen}(X, Y)$ is norm-closed by Proposition 4.1.

Finally, consider the operator $\operatorname{Id} \in \mathcal{L}(X, Y)$ which is not generating by Corollary 2.13 because $e_{n+1} \in \operatorname{ext}(B_X)$ and $\|\operatorname{Id}(e_{n+1})\|_Y = \|e_{n+1}\|_Y < 1$. If $\operatorname{Id} \in \operatorname{conv}(\operatorname{Gen}(X, Y))$, we may find $M \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_M \geqslant 0$ with $\sum_{i=1}^M \lambda_i = 1$ and $G_1, \ldots, G_M \in \operatorname{Gen}(X, Y)$ such that

$$\operatorname{Id} = \sum_{i=1}^M \lambda_i G_i.$$

Then, for each $j = 1, \ldots, n$, we have that

$$e_j = \mathrm{Id}(e_j) = \sum_{i=1}^{M} \lambda_i G_i(e_j) \quad \Longrightarrow \quad G_i(e_j) = e_j \quad \forall\, i \in \{1, \ldots, M\}$$

as $e_j \in \mathrm{ext}(B_Y)$. Since $\{e_1, \ldots, e_n\}$ is linearly independent and $\dim(X) = n$, it follows that $G_i = \mathrm{Id}$ for all $i = 1, \ldots, M$. Therefore, we have that $\mathrm{Id} \notin \mathrm{conv}(\mathrm{Gen}(X, Y)) = \overline{\mathrm{conv}}(\mathrm{Gen}(X, Y))$ which finishes the proof. □

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Chapter VI

A numerical range approach to Birkhoff-James orthogonality with applications

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A NUMERICAL RANGE APPROACH TO BIRKHOFF-JAMES ORTHOGONALITY WITH APPLICATIONS

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Abstract. The main aim of this paper is to provide characterizations of Birkhoff-James orthogonality (BJ-orthogonality in short) in a number of families of Banach spaces in terms of the elements of significant subsets of the unit ball of their dual spaces, which makes the characterizations more applicable. The tool to do so is a fine study of the abstract numerical range and its relation with the BJ-orthogonality. Among other results, we provide a characterization of BJ-orthogonality for spaces of vector-valued bounded functions in terms of the domain set and the dual of the target space, which is applied to get results for spaces of vector-valued continuous functions, uniform algebras, Lipschitz maps, injective tensor products, bounded linear operators with respect to the operator norm and to the numerical radius, multilinear maps, and polynomials. Next, we study possible extensions of the well-known Bhatia-Šemrl Theorem on BJ-orthogonality of matrices, showing results in spaces of vector-valued continuous functions, compact linear operators on reflexive spaces, and finite Blaschke products. Finally, we find applications of our results to the study of spear vectors and spear operators. We show that no smooth point of a Banach space can be BJ-orthogonal to a spear vector of Z. As a consequence, if X is a Banach space containing strongly exposed points and Y is a smooth Banach space with dimension at least two, then there are no spear operators from X to Y. Particularizing this result to the identity operator, we show that a smooth Banach space containing strongly exposed points has numerical index strictly smaller than one. These latter results partially solve some open problems.

1. Introduction

Let Z be a Banach space over the field $\mathbb{K}$ (which will always be considered as $\mathbb{R}$ or $\mathbb{C}$). Given $x, y \in Z$, we say that x is *Birkhoff-James orthogonal* to y (*BJ-orthogonal* in short), denoted by $x \perp_B y$, if

$$\|x + \lambda y\| \geqslant \|x\| \quad \forall \lambda \in \mathbb{K}.$$

This definition, proposed by Birkhoff [6] in the setting of metric linear spaces, has a natural geometric interpretation: $x \perp_B y$ if and only if the (real or complex) line $\{x+\lambda y \colon \lambda \in \mathbb{K}\}$ is disjoint with the open ball of radius $\|x\|$ centered at the origin. Observe that BJ-orthogonality is homogeneous, i.e., $x \perp_B y$

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implies that $\alpha x \perp_B \beta y$ for every $\alpha, \beta \in \mathbb{K}$. Also, smoothness of the norm of Z at x is equivalent to the right-additivity of $\perp_B$ at x: x is smooth in Z if and only if for any $y, z \in Z$, $x \perp_B y$, $x \perp_B z$ together imply that $x \perp_B (y+z)$. In case the norm is induced by an inner product $\langle\ ,\ \rangle$, it is elementary to notice that BJ-orthogonality is equivalent to the usual orthogonality: $x \perp y$ if and only if $\langle x, y\rangle = 0$ if and only if $x \perp_B y$. This shows that BJ-orthogonality generalizes the concept of usual orthogonality to the framework of norms. Although there exist several non-equivalent notions of orthogonality in Banach spaces, it is commonly accepted that BJ-orthogonality is arguably the most useful one amongst them by virtue of its rich connections with many important concepts in the geometric theory of Banach spaces, including smoothness, operator norm attainment, characterizations of Euclidean and Hilbert spaces among Banach spaces, and best approximations. We refer the interested readers to [36, 37, 38, 40, 41, 42], and the references therein, for more information in this regard.

A general way to study BJ-orthogonality in any Banach space Z was given by R. C. James in terms of the dual space Z^* of Z.

Fact 1.1 ([13, Corollary 2.2])**.** Let Z be a Banach space and let $x, y \in Z$. Then,

$$x \perp_B y \iff \text{there exists } \phi \in Z^* \text{ with } \|\phi\| = 1 \text{ such that } \phi(x) = \|x\| \text{ and } \phi(y) = 0.$$

This characterization of BJ-orthogonality immediately relates it with the norm attainment problem for functionals in the dual space. As a matter of fact, one of the useful ways to reap the benefits out of the concept of BJ-orthogonality for the purpose of understanding the geometric and analytic structures of a Banach space, is to apply James' characterization of BJ-orthogonality in the corresponding dual space. As we will see in this article, it is possible to obtain further refinements of the James characterization in many important cases including the Banach space of bounded linear operators between Banach spaces.

The above result by James also relates BJ-orthogonality with the concept of (abstract) numerical range. Let us introduce the required notations and definitions. Given a Banach space Z, we write B_Z and S_Z to denote, respectively, the closed unit ball and the unit sphere of Z, $\operatorname{Re}(\cdot)$ will denote the real part (which is nothing but the identity if we are dealing with real numbers), and we write $\mathbb{T}$ for the set of modulus-one scalars. If $u \in Z$ is a norm-one element, the *(abstract) numerical range* of $z \in Z$ with respect to (Z, u) is the non-empty compact convex subset of $\mathbb{K}$ given by

$$V(Z, u, z) := \{\phi(z) \colon \phi \in \mathrm{F}(B_{Z^*}, u)\},$$

where $\mathrm{F}(B_{Z^*}, u) := \{\phi \in S_{Z^*} \colon \phi(u) = 1\}$ is the *face* of B_{Z^*} generated by u, also known as the *set of states of Z relative to u.* The concept of abstract numerical range takes its roots in a 1955 paper by Bohnenblust and Karlin [7] and it was introduced in the 1985 paper [23]. We refer the reader to the classical books [8, 9] by Bonsall and Duncan, to Sections 2.1 and 2.9 of the book [10], and to Section 2 of [16] for more information and background.

Observe that, with the definition of numerical range in hands, Fact 1.1 can be easily written in the following way.

Proposition 1.2. *Let Z be a Banach space, let $u \in S_Z$, and let $z \in Z$. Then,*

$$u \perp_B z \iff 0 \in V(Z, u, z).$$

Let us also comment that it is also possible to write the numerical range in terms of the BJ-orthogonality, see Proposition 2.1. It is then clear that the study of BJ-orthogonality and the study of abstract numerical ranges are somehow equivalent.

The main disadvantage of Proposition 1.2 (and of Fact 1.1) is that we have to deal with the whole dual of the Banach space Z, and this is difficult in many occasions. For instance, when Z is a space of bounded linear operators, which is the most interesting case for us, the dual space is a wild object that it is not easy to manage. For an easier writing of our discussion, let us introduce the following notation: given Banach spaces X, Y, we write $\mathcal{L}(X,Y)$ to denote the space of all bounded linear operators from X to Y and $\mathcal{K}(X,Y)$ for its subspace consisting of compact operators. When $X = Y$, we just write $\mathcal{L}(X)$ and $\mathcal{K}(X)$. In the case when Z is the space of $n \times n$ matrices (identified with $\mathcal{L}(H)$ where H is an n-dimensional Hilbert space), a celebrated result by Bhatia and Šemrl [5, Theorem 1.1] says that two matrices A, B satisfy that $A \perp_B B$ if and only if there is a norm-one vector x such that $\|Ax\| = \|A\|$ and $\langle Ax, Bx\rangle = 0$ (that is, there is a norm-one vector x at which A attains its norm and such that $Ax \perp Bx$). Observe that it is equivalent to say that, in this case, when $A \perp_B B$, the functional ϕ on the space of $n \times n$ matrices given by Fact 1.1 can be taken of the form $\phi(C) = \langle Cx, y\rangle$ for some norm-one vectors x and y (see the proof of Corollary 4.2). Clearly, this gives much more information than the one provided by Fact 1.1 and avoids to deal with the wild dual of the space of matrices. This result does not extend to general operators on infinite-dimensional Hilbert spaces (as they do not need to attain the norm), but there is a similar result: given two bounded linear operators A and B on a Hilbert space H, $A \perp_B B$ if and only if there is a sequence $\{x_n\}$ in S_H satisfying that $\lim \|Ax_n\| = \|A\|$ and $\lim\langle Ax_n, Bx_n\rangle = 0$ [21, Lemma 2.2], [5, Remark 3.1]. The significance of the result obtained by Bhatia and Šemrl lies in the fact that it allows us to examine the orthogonality of bounded linear operators on a Hilbert space in terms of the usual orthogonality of certain special elements in the ground space. We would like to emphasize here that such a characterization of BJ-orthogonality is certainly more handy than James' characterization, since we do not need to deal with the dual of the operator space. Moreover, as already mentioned in [5], it is natural to speculate about the validity of the above results in case of bounded linear operators on a Banach space. In general, they do not extend to operators between general Banach spaces, even in the finite-dimensional case, as it was shown by Li and Schneider [20, Example 4.3]. However, a related weaker result was proved in the same paper [20, Proposition 4.2]: if X and Y are finite-dimensional Banach spaces and $T, A \in \mathcal{L}(X,Y)$, then

$$T \perp_B A \iff 0 \in \operatorname{conv}\left(\left\{y^*(Ax)\colon x \in \operatorname{ext}(B_X),\ y^* \in \operatorname{ext}(B_{Y^*}),\ y^*(Tx) = \|T\|\right\}\right),$$

where $\operatorname{ext}(C)$ denotes the set of extreme points of a convex set C and $\operatorname{conv}(\cdot)$ is the convex hull. Observe that this result is similar to Bhatia-Šemrl's one up to taking convex hull in $\mathbb{K}$. How did Li and Schneider get this result? Just by characterizing the extreme points of the dual unit ball of $\mathcal{L}(X,Y)$ when X and Y are finite-dimensional and then using a classical result by Singer about best approximation. Let Z be a Banach space, let M be a subspace of Z, and let $x \in Z$. An element $m_0 \in M$ is said to be a *best approximation* of x at M if

$$\|x - m_0\| \leqslant \|x - m\| \quad \forall m \in M.$$

Observe that m_0 is a best approximation to x in M if and only if $x - m_0$ is BJ-orthogonal to M, i.e., $x - m_0 \perp_B m$ for every $m \in M$. Equivalently, given $x, y \in Z$, $x \perp_B y$ if and only if 0 is a best approximation to x in $\operatorname{span}\{y\}$. We refer the interested reader to the classical book [41] by I. Singer for background. Using the relation between best approximation and BJ-orthogonality, the classical result of I. Singer that Li and Schneider used reads as follows.

Fact 1.3 ([41, Theorem II.1.1])**.** Let Z be a Banach space and let $u, z \in Z$. Then,

$$x \perp_B y \iff 0 \in \operatorname{conv}\left(\left\{\phi(z)\colon \phi \in \operatorname{ext}(B_{Z^*}),\ \phi(u) = \|u\|\right\}\right).$$

Observe that this result just says that the functional ϕ in Fact 1.1 can be taken in the convex hull of the set of extreme points of B_{Z^*}. We will provide in Proposition 2.2 a version of Proposition 1.2

using only extreme points with an independent proof. Of course, Fact 1.3 is very interesting in the cases when the extreme points of the dual ball are known and easy to manage: for $Z = C(K)$ or for Z being an isometric predual of an $L_1(\mu)$ space, or even when $Z = \mathcal{L}(X,Y)$ and X and Y are finite-dimensional (as it was done by Li and Schneider, see [20, Proposition 4.2]). Actually, for arbitrary spaces X and Y, the extreme points of the dual ball of $Z = \mathcal{K}(X,Y)$ have been described in [34] as $\operatorname{ext}(B_{X^{**}}) \otimes \operatorname{ext}(B_{Y^*})$. In the particular case when X is reflexive, this provides the following result which covers Li and Schneider's one: let X be a reflexive space, let Y be a Banach space, and let $T, A \in \mathcal{K}(X,Y)$; then

$$T \perp_B A \iff 0 \in \operatorname{conv}\left(\left\{y^*(Ax)\colon x \in \operatorname{ext}(B_X),\ y^* \in \operatorname{ext}(B_{Y^*}),\ y^*(Tx) = \|T\|\right\}\right)$$

(see Corollary 3.11). When we deal with non-compact operators, there is no description of the extreme points of the unit ball of $\mathcal{L}(X,Y)^*$ available, hence Fact 1.3 is not applicable in this case. However, a somehow similar result was proved in [28, Theorem 2.2]: let X, Y be Banach spaces and let $T, A \in \mathcal{L}(X,Y)$; then,

$$T \perp_B A \iff 0 \in \operatorname{conv}\left(\left\{\lim y_n^*(Ax_n)\colon (x_n, y_n^*) \in S_X \times S_{Y^*}\ \forall n \in \mathbb{N},\ \lim y_n^*(Tx_n) = \|T\|\right\}\right).$$

This is, as far as we know, the most general result concerning a characterization of BJ-orthogonality of operators in terms of the domain and range spaces and their duals.

Our main aim in this paper is to provide a very general result characterizing BJ-orthogonality in a Banach space Z in terms of the actions of elements on an arbitrary one-norming subset. Recall that a subset $\Lambda \subset S_{Z^*}$ is said to be *one-norming* for Z if $\|z\| = \sup\{|\phi(z)|\colon \phi \in \Lambda\}$ for all $z \in Z$ (equivalently, if B_{Z^*} equals the absolutely weak-star closed convex hull of Λ). One of the assertions of this general result (see Corollary 2.6) is the following: let Z be a Banach space, $\Lambda \subset S_{Z^*}$ be one-norming for Z; then for $u \in S_Z$ and $z \in Z$,

$$u \perp_B z \iff 0 \in \operatorname{conv}\left(\left\{\lim \psi_n(z)\overline{\psi_n(u)}\colon \psi_n \in \Lambda, \lim |\psi_n(u)| = 1\right\}\right).$$

The way to get the result is to combine Proposition 1.2 with a very general result on numerical ranges, Theorem 2.4, which extends previous characterizations from [16]. This result also allows to characterize smooth points, see Corollary 2.11. There are also nicer versions of these results in the case when instead of a one-norming subset Λ, we have a subset C of S_{Z^*} such that its weak-star closed convex hull is the whole B_{Z^*}, see Theorem 2.3 and Corollaries 2.5 and 2.10. All of this is the content of Section 2 of this manuscript.

Section 3 contains a number of particular cases in which the results of Section 2 apply. It is divided in several subsections, and covers results in a number of spaces. Even though some of the results of this section were previously known, the previous approaches were different and use ad hoc techniques for each of the particular cases, while our present approach generalizes all these techniques. On the other hand, the general result for $\ell_\infty(\Gamma, Y)$ we give in Theorem 3.2 seems to be new, as they are its applications for spaces of vector-valued continuous functions (Corollaries 3.4 and 3.5), uniform algebras (Corollary 3.6), Lipschitz maps (Proposition 3.7), and injective tensor products (Proposition 3.8). For bounded linear operators (Subsection 3.4), most of the results were already known, but there are some improvements of previous results in Proposition 3.10 and Corollary 3.11. Besides, we include a result on smoothness of bounded linear operators which will be used in Section 5. Subsection 3.5 deals with multilinear maps and polynomials and the results seem to be new. Finally, Subsection 3.6 contains results on BJ-orthogonality with respect to the numerical radius of operators which were previously known.

Next, in Section 4 we provide several results related to the Bhatia-Šemrl's theorem (in the sense of removing the convex hull and the limits of the characterization of BJ-orthogonality). The main result

(Theorem 4.3) is for vector-valued continuous functions on a compact Hausdorff space and seems to be completely new. As consequences, we obtain Bhatia-Šemrl's kind of results for compact operators on reflexive Banach spaces, Proposition 4.5 for the real case, Theorem 4.6 in the complex case, and the latter is new for infinite-dimensional spaces. We also obtain a nice characterization of BJ-orthogonality for finite Blaschke products (Corollary 4.8).

Finally, Section 5 contains applications of the results in the paper to the study of spear vectors, spear operators, and Banach spaces with numerical index one. They are consequences of Theorem 5.1 which says that if u is a vertex of a Banach space Z and $z \in Z$ is smooth in (Z, v_u), then z cannot be BJ-orthogonal to u in (Z, v_u). As a consequence, no smooth point of a Banach space Z can be BJ-orthogonal to a spear vector of Z (Corollary 5.3). The particularization of the results to the case $Z = \mathcal{L}(X, Y)$ leads to obstructive results for the existence of spear operators. In particular, we show that if X is a Banach space containing strongly exposed points and Y is a smooth Banach space with dimension at least two, then there are no spear operators in $\mathcal{L}(X, Y)$ (Corollary 5.5) and this result is proved using a sufficient condition for an operator to be smooth (Proposition 3.14). This result somehow extends [15, Proposition 6.5.a] and provides a partial answer to [15, Problem 9.12]. Particularizing this to the identity operator, we get an obstructive condition for a Banach space to have numerical index one: the existence of a smooth point which is BJ-orthogonal to a strongly exposed point (Corollary 5.9). In particular, smooth Banach spaces with dimension at least two containing strongly exposed points do not have numerical index one (Corollary 5.11). This latter result is a partial answer to the question of whether a smooth Banach space of dimension at least two may have numerical index one [14]. Let us comment that the mix of ideas from numerical ranges and from BJ-orthogonality is the key to obtaining these interesting applications which partially solve some open questions. Moreover, the abstract numerical range approach to BJ-orthogonality considered in this article generalizes all of the previously mentioned characterizations to a much broader framework. In view of this, it is reasonable to expect that the methods developed here will cover more particular cases, known and new.

2. The numerical range approach

The aim of this section is to connect BJ-orthogonality and smoothness with the theory of abstract numerical ranges and present different expressions of the abstract numerical range which will be very useful in order to characterize BJ-orthogonality and smoothness in several contexts.

Let us start with a result showing that the abstract numerical range can be expressed in terms of BJ-orthogonality. This result, together with Proposition 1.2, shows that the study of BJ-orthogonality and the study of abstract numerical ranges are somehow equivalent.

Proposition 2.1. *Let Z be a Banach space and let $u \in S_Z$. Then, for every $z \in Z$,*

$$V(Z, u, z) = \{\alpha \in \mathbb{K} \colon u \perp_B (z - \alpha u)\}.$$

Proof. Let $\alpha \in V(Z, u, z)$, then there exists $\phi \in S_{Z^*}$ such that $\phi(u) = 1$ and $\phi(z) = \alpha$. Thus $\phi(z - \alpha u) = 0$ and so $u \perp_B (z - \alpha u)$. Conversely, if $\alpha \in \mathbb{K}$ is such that $u \perp_B (z - \alpha u)$, then there exists $\phi \in S_{Z^*}$ such that $\phi(u) = 1$ and $\phi(z - \alpha u) = \phi(z) - \alpha = 0$, therefore $\alpha = \phi(z) \in V(Z, u, z)$. □

Let Z be a Banach space and let $u \in S_Z$. Our aim here is to show how to describe the abstract numerical range $V(Z, u, \cdot)$ in terms of a fixed one-norming subset $\Lambda \subset S_{Z^*}$ which will allow to get characterizations of BJ-orthogonality and smoothness. In the particular case in which Λ is equal to $\operatorname{ext}(B_{Z^*})$, the characterization of BJ-orthogonality actually follows from Fact 1.3. But we are also able to get a result on abstract numerical ranges as an easy consequence of Bauer Maximum Principle. Observe that Fact 1.3 can be also deduced from the next proposition and Proposition 1.2.

Proposition 2.2. *Let Z be a Banach space and let $u \in S_Z$. Then, for every $z \in Z$,*

$$V(Z,u,z) = \operatorname{conv}\{\phi(z)\colon \phi \in \operatorname{ext}(B_{Z^*}),\ \phi(u)=1\}.$$

Proof. We apply Bauer Maximum Principle (see [1, 7.69], for instance) to the set $\mathrm{F}(B_{Z^*},u)$, which is convex and w^*-compact, and to the function $\phi \longmapsto \operatorname{Re}\phi(z)$ from $\mathrm{F}(B_{Z^*},u)$ to $\mathbb{R}$, which is w^*-continuous and convex. Then, this function attains its maximum at an extreme point of $\mathrm{F}(B_{Z^*},u)$ (which is also an extreme point of B_{Z^*} since $\mathrm{F}(B_{Z^*},u)$ is an extremal subset). That is,

$$\max \operatorname{Re} V(Z,u,z) = \max \operatorname{Re}\{\phi(z)\colon \phi \in \operatorname{ext}(B_{Z^*}),\ \phi(u)=1\}.$$

Now, the result follows using that $V(Z,u,\theta z) = \theta V(Z,u,z)$ and

$$\{\phi(\theta z)\colon \phi \in \operatorname{ext}(B_{Z^*}),\ \phi(u)=1\} = \theta\{\phi(z)\colon \phi \in \operatorname{ext}(B_{Z^*}),\ \phi(u)=1\}$$

for every $\theta \in \mathbb{T}$. □

There are Banach spaces Z for which the set of extreme points of the dual space is not known (for instance, this is the case for $Z = \mathcal{L}(X,Y)$ in general). In those cases, another way to characterize the numerical range is needed. This was done in [16, Propostion 2.14] substituting the set of extreme points of the dual ball by a subset $C \subseteq B_{Z^*}$ such that $B_{Z^*} = \overline{\operatorname{conv}}^{w^*}(C)$. We give next a reformulation of that result which will be useful in applications.

Theorem 2.3. *Let Z be a Banach space, let $u \in S_Z$, and let $C \subseteq B_{Z^*}$ be such that $B_{Z^*} = \overline{\operatorname{conv}}^{w^*}(C)$. Then,*

$$\begin{aligned} V(Z,u,z) &= \operatorname{conv} \bigcap\nolimits_{\delta>0} \overline{\{\phi(z)\colon \phi \in C,\ \operatorname{Re}\phi(u) > 1-\delta\}} \\ &= \operatorname{conv}\left(\left\{\lim \phi_n(z)\colon \phi_n \in C\ \forall n \in \mathbb{N},\ \lim \phi_n(u) = 1\right\}\right) \end{aligned}$$

for every $z \in Z$.

Let us comment that comparing this theorem with Proposition 2.2, we lose information as we have to deal with limits, but we obtain a lot of generality, as there are many situations in which $B_{Z^*} = \overline{\operatorname{conv}}^{w^*}(C)$ holds and C is completely different from $\operatorname{ext}(B_{Z^*})$ (even disjoint). In what follows, and in the rest of the paper, when we write $\lim z_n$ for a *bounded scalar sequence* $\{z_n\}$ we are understanding that the sequence is convergent, but it is also fine if one understands that $\lim z_n$ represents an adherent point of the sequence, which always exists.

Proof of Theorem 2.3. The first equality was already proved in [16, Propostion 2.14], let us prove that

$$V(Z,u,z) = \operatorname{conv}\left(\left\{\lim \phi_n(z)\colon \phi_n \in C\ \forall n \in \mathbb{N},\ \lim \phi_n(u) = 1\right\}\right).$$

For $z \in Z$, we write $W(z) := \{\lim \phi_n(z)\colon \phi_n \in C\ \forall n \in \mathbb{N},\ \lim \phi_n(u) = 1\}$ and we prove first the inclusion $V(Z,u,z) \supseteq \operatorname{conv} W(z)$. Given $\lambda_0 \in W(z)$, for each $n \in \mathbb{N}$ there exists $\phi_n \in C$ such that

$$|\phi_n(z) - \lambda_0| < 1/n \quad \text{and} \quad |\phi_n(u) - 1| < 1/n.$$

Since B_{Z^*} is w^*-compact, there is $\phi_0 \in B_{Z^*}$ a w^*-limiting point of the sequence $\{\phi_n\}_{n\in\mathbb{N}}$. Then, it follows that $\phi_0(z) = \lambda_0$ and $\phi_0(u) = 1$, so $\lambda_0 \in V(Z,u,z)$ and the desired inclusion holds by the convexity of $V(Z,u,z)$.

To prove the reverse inclusion, it is enough to show that the inequality

$$\sup \operatorname{Re} V(Z,u,z) \leqslant \sup \operatorname{Re} W(z) \tag{2.1}$$

holds for every $z \in Z$, as $V(Z,u,\theta z) = \theta V(Z,u,z)$ and $W(\theta z) = \theta W(z)$ for every $\theta \in \mathbb{T}$, and $W(z)$ is closed. So, fixed $z \in Z$ and $\phi_0 \in \mathrm{F}(B_{Z^*}, u)$, we apply [16, Lemma 2.15] for $\delta = 1/n$ to obtain a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ in C such that

$$\operatorname{Re}\phi_n(u) > 1 - 1/n \quad \text{and} \quad \operatorname{Re}\phi_n(z) > \operatorname{Re}\phi_0(z) - 1/n.$$

We may and do suppose (up to taking a subsequence, if needed), that the sequences $\{\phi_n(u)\}$ and $\{\phi_n(z)\}$ are convergent. Therefore, we get $\lim\phi_n(u) = 1$ and $\operatorname{Re}\phi_0(z) \leqslant \operatorname{Re}\lim\phi_n(z) \leqslant \sup\operatorname{Re}W(z)$, and so inequality (2.1) follows. □

We are now able to generalize the previous result to the case of one-norming subsets. We will include more characterizations here as this is the most general result that we have.

Theorem 2.4. *Let Z be a Banach space, let $u \in S_Z$, and let $\Lambda \subset B_{Z^*}$ be one-norming for Z. Then,*

$$\begin{aligned}
V(Z,u,z) &= \operatorname{conv}\left(\left\{\theta_0 \lim\psi_n(z) \colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \lim\psi_n(u) = \overline{\theta_0},\ \theta_0 \in \mathbb{T}\right\}\right)\\
&= \operatorname{conv}\left(\left\{\lim\psi_n(z)\overline{\psi_n(u)} \colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \lim|\psi_n(u)| = 1\right\}\right)\\
&= \operatorname{conv} \bigcap\nolimits_{\delta>0} \overline{\left\{\psi(z)\overline{\psi(u)} \colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}}\\
&= \bigcap\nolimits_{\delta>0} \overline{\operatorname{conv}\left\{\psi(z)\overline{\psi(u)} \colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}}
\end{aligned}$$

for every $z \in Z$.

Proof. We start proving the first three equalities. To do so, we apply Theorem 2.3 for $C = \mathbb{T}\Lambda$ which satisfies $\overline{\operatorname{conv}}^{w^*}(C) = \overline{\operatorname{aconv}}^{w^*}(\Lambda) = B_{Z^*}$ to obtain that

$$V(Z,u,z) = \operatorname{conv}\left(\left\{\lim\phi_n(z) \colon \phi_n \in \mathbb{T}\Lambda\ \forall n \in \mathbb{N},\ \lim\phi_n(u) = 1\right\}\right).$$

So it is enough to show the following chain of inclusions:

$$\begin{aligned}
\left\{\lim\phi_n(z) \colon \phi_n \in \mathbb{T}\Lambda\ \forall n \in \mathbb{N},\ \lim\phi_n(u) = 1\right\} &\subseteq \left\{\theta_0\lim\psi_n(z) \colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \lim\psi_n(u) = \overline{\theta_0}\right\}\\
&\subseteq \left\{\lim\psi_n(z)\overline{\psi_n(u)} \colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \lim|\psi_n(u)| = 1\right\}\\
&\subseteq \bigcap\nolimits_{\delta>0} \overline{\left\{\psi(z)\overline{\psi(u)} \colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}}\\
&\subseteq \left\{\lim\phi_n(z) \colon \phi_n \in \mathbb{T}\Lambda\ \forall n \in \mathbb{N},\ \lim\phi_n(u) = 1\right\}.
\end{aligned}$$

For the first inclusion, take $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathbb{T}\Lambda$ with $\lim\phi_n(u) = 1$, then $\phi_n = \theta_n\psi_n$ for $\theta_n \in \mathbb{T}$ and $\psi_n \in \Lambda$. Let $\theta_0 \in \mathbb{T}$ be a limiting point of $\{\theta_n\}_{n\in\mathbb{N}}$, then

$$\lim\psi_n(u) = \lim\overline{\theta_n}\phi_n(u) = \overline{\theta_0} \quad \text{and} \quad \theta_0\lim\psi_n(z) = \theta_0\lim\overline{\theta_n}\phi_n(z) = \lim\phi_n(z).$$

The second inclusion is evident. For the third one, let $\lambda = \lim\psi_n(z)\overline{\psi_n(u)}$ for some sequence $\{\psi_n\}_{n\in\mathbb{N}} \subseteq \Lambda$ with $\lim|\psi_n(u)| = 1$ and fix $\delta > 0$. There exists $n_0 \in \mathbb{N}$ such that

$$\left|\lambda - \psi_n(z)\overline{\psi_n(u)}\right| < 1/n \quad \text{and} \quad |\psi_n(u)| > 1 - 1/n > 1 - \delta$$

for every $n \geqslant n_0$, therefore $\lambda \in \overline{\left\{\psi(z)\overline{\psi(u)} \colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}}$ and the arbitrariness of δ gives the inclusion.

For the last inclusion, let $\lambda \in \bigcap_{\delta>0} \overline{\left\{\psi(z)\overline{\psi(u)} \colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}}$. For every $n \in \mathbb{N}$ there exists $\psi_n \in \Lambda$ such that

$$\left|\lambda - \psi_n(z)\overline{\psi_n(u)}\right| < 1/n \quad \text{and} \quad |\psi_n(u)| > 1 - 1/n.$$

For each $n \in \mathbb{N}$, take $\theta_n \in \mathbb{T}$ such that $\theta_n \psi_n(u) = |\psi_n(u)|$ and define $\phi_n = \theta_n \psi_n \in \mathbb{T}\Lambda$. Then, we have that

$$\lim \phi_n(u) = \lim \theta_n \psi_n(u) = \lim |\psi_n(u)| = 1$$

and

$$\lambda = \lim \psi_n(z)\overline{\psi_n(u)} = \lim \theta_n \psi_n(z)\overline{\theta_n \psi_n(u)} = \lim \phi_n(z)|\psi_n(u)| = \lim \phi_n(z),$$

which finishes the proof of the chain of inclusions.

Finally, in order to prove the last equality in the lemma, observe that

$$\operatorname{conv} \bigcap_{\delta>0} \overline{\left\{\psi(z)\overline{\psi(u)} \colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}} \subseteq \bigcap_{\delta>0} \overline{\operatorname{conv}\left\{\psi(z)\overline{\psi(u)} \colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}}$$

and let us show that the latter set is contained in $V(Z,u,z)$.
Fixed $\lambda \in \bigcap_{\delta>0} \overline{\operatorname{conv}\left\{\psi(z)\overline{\psi(u)} \colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}}$, it is clear that

$$\lambda \in \overline{\operatorname{conv}\left\{\psi(z)\overline{\psi(u)} \colon \psi \in \Lambda,\ |\psi(u)| > 1-1/n\right\}}$$

for every $n \in \mathbb{N}$, and we apply Carathéodory's Theorem to obtain the existence of $a_n, b_n, c_n \in [0,1]$ with $a_n + b_n + c_n = 1$ and $\phi_n, \psi_n, \xi_n \in \Lambda$ such that

$$|\phi_n(u)| \geqslant 1-1/n, \quad |\psi_n(u)| \geqslant 1-1/n, \quad |\xi_n(u)| \geqslant 1-1/n, \quad \text{and} \tag{2.2}$$

$$\left|\lambda - \left(a_n\phi_n(z)\overline{\phi_n(u)} + b_n\psi_n(z)\overline{\psi_n(u)} + c_n\xi_n(z)\overline{\xi_n(u)}\right)\right| < 1/n \tag{2.3}$$

for every $n \in \mathbb{N}$. We may find $a, b, c \in [0,1]$ and $\{a_{\sigma(n)}\}_{n\in\mathbb{N}}$, $\{b_{\sigma(n)}\}_{n\in\mathbb{N}}$, $\{c_{\sigma(n)}\}_{n\in\mathbb{N}}$ subsequences of $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, $\{c_n\}_{n\in\mathbb{N}}$ respectively such that $\{a_{\sigma(n)}\}_{n\in\mathbb{N}} \to a$, $\{b_{\sigma(n)}\}_{n\in\mathbb{N}} \to b$, $\{c_{\sigma(n)}\}_{n\in\mathbb{N}} \to c$ and $a+b+c=1$. Additionally, by passing to a subsequence, we may assume that $\{\phi_{\sigma(n)}(u)\}_{n\in\mathbb{N}}$, $\{\psi_{\sigma(n)}(u)\}_{n\in\mathbb{N}}$, $\{\xi_{\sigma(n)}(u)\}_{n\in\mathbb{N}}$, $\{\phi_{\sigma(n)}(z)\}_{n\in\mathbb{N}}$, $\{\psi_{\sigma(n)}(z)\}_{n\in\mathbb{N}}$, and $\{\xi_{\sigma(n)}(z)\}_{n\in\mathbb{N}}$ are convergent. Since B_{Z^*} is w^*-compact, let $\phi_0, \psi_0, \xi_0 \in B_{Z^*}$ be w^*-limiting points of the sequences $\{\phi_{\sigma(n)}\}_{n\in\mathbb{N}}$, $\{\psi_{\sigma(n)}\}_{n\in\mathbb{N}}$, $\{\xi_{\sigma(n)}\}_{n\in\mathbb{N}}$ respectively. Then,

$$\begin{aligned} &\lim \phi_{\sigma(n)}(u) = \phi_0(u), && \lim \psi_{\sigma(n)}(u) = \psi_0(u), && \lim \xi_{\sigma(n)}(u) = \xi_0(u),\\ &\lim \phi_{\sigma(n)}(z) = \phi_0(z), && \lim \psi_{\sigma(n)}(z) = \psi_0(z), && \lim \xi_{\sigma(n)}(z) = \xi_0(z). \end{aligned}$$

It follows from (2.2) that $|\phi_0(u)| = |\psi_0(u)| = |\xi_0(u)| = 1$. Define

$$\Phi = a\overline{\phi_0(u)}\phi_0 + b\overline{\psi_0(u)}\psi_0 + c\overline{\xi_0(u)}\xi_0 \in B_{Z^*},$$

then $\Phi(u) = a\,|\phi_0(u)|^2 + b\,|\psi_0(u)|^2 + c\,|\xi_0(u)|^2 = 1$ and

$$\lambda = \lim\left(a_{\sigma(n)}\phi_{\sigma(n)}(z)\overline{\phi_{\sigma(n)}(u)} + b_{\sigma(n)}\psi_{\sigma(n)}(z)\overline{\psi_{\sigma(n)}(u)} + c_{\sigma(n)}\xi_{\sigma(n)}(z)\overline{\xi_{\sigma(n)}(u)}\right) = \Phi(z)$$

by (2.3), which imply that $\lambda \in V(Z,u,z)$. □

Thanks to the different expressions of the numerical range provided in Theorems 2.3 and 2.4, we are able to give new characterizations of BJ-orthogonality .

Corollary 2.5. *Let Z be a Banach space, let $u \in S_Z$, and let $C \subseteq B_{Z^*}$ be such that $B_{Z^*} = \overline{\operatorname{conv}}^{w^*}(C)$. Then, for $z \in Z$, the following are equivalent:*

(i) $u \perp_B z$;
(ii) $0 \in \operatorname{conv} \bigcap_{\delta>0} \overline{\{\phi(z) \colon \phi \in C, \operatorname{Re}\phi(u) > 1-\delta\}}$;
(iii) $0 \in \operatorname{conv}\left(\left\{\lim \phi_n(z) \colon \phi_n \in C\ \forall n \in \mathbb{N},\ \lim \phi_n(u) = 1\right\}\right)$.

Corollary 2.6. *Let Z be a Banach space, let $u \in S_Z$, and let $\Lambda \subset B_{Z^*}$ be one-norming for Z. Then, for $z \in Z$, the following are equivalent:*

(i) $u \perp_B z$;
(ii) $0 \in \operatorname{conv}\left(\left\{\theta_0 \lim \psi_n(z) \colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \theta_0 \in \mathbb{T},\ \lim \psi_n(u) = \overline{\theta_0}\right\}\right)$;
(iii) $0 \in \operatorname{conv}\left(\left\{\lim \psi_n(z)\overline{\psi_n(u)} \colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \lim |\psi_n(u)| = 1\right\}\right)$;
(iv) $0 \in \operatorname{conv} \bigcap_{\delta>0} \overline{\left\{\psi(z)\overline{\psi(u)} \colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}}$;
(v) $0 \in \bigcap_{\delta>0} \overline{\operatorname{conv}\left\{\psi(z)\overline{\psi(u)} \colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\right\}}$.

The next easy result will allow to rephrase the characterizations given above without using the language of convex hull.

Lemma 2.7. *Let A be a non-empty subset of $\mathbb{K}$. Then, $0 \in \operatorname{conv}(A)$ if and only if given any $\mu \in \mathbb{T}$, there exists $a_\mu \in A$ such that $\operatorname{Re} \mu a_\mu \geqslant 0$.*

Moreover, if A is connected, then $0 \in \operatorname{conv}(A)$ if and only if given any $\mu \in \mathbb{T}$, there exists $a_\mu \in A$ such that $\operatorname{Re} \mu a_\mu = 0$.

The real case is obvious. In the complex case, the result follows straightforwardly from the Hahn-Banach separation Theorem. An elementary proof of the sufficiency can be found in [29, Lemma 2.1]. Let us give an elementary argument for the necessity. By Carathéodory's Theorem, there exists $\lambda_j \geqslant 0$ and $a_j \in A$ $(j = 1, 2, 3)$ such that $\sum_{j=1}^3 \lambda_j = 1$ and $\sum_{j=1}^3 \lambda_j a_j = 0$. Consider any $\mu \in \mathbb{T}$. Since $\sum_{j=1}^3 \mu\lambda_j a_j = 0$, there exists $b_1, b_2 \in \{a_1, a_2, a_3\}$ such that $\operatorname{Re} \mu b_1 \geqslant 0$ and $\operatorname{Re} \mu b_2 \leqslant 0$, and we are done. Now, if A is connected, then $\operatorname{Re}\{\mu a \colon a \in A\}$ is an interval in $\mathbb{R}$ containing $\operatorname{Re} \mu b_1$ and $\operatorname{Re} \mu b_2$. Thus, there exists $\widetilde{a} \in A$ such that $\operatorname{Re} \mu \widetilde{a} = 0$. This completes the argument.

Let us now use the same spirit of Corollaries 2.5 and 2.6 to characterize the notion of smoothness in terms of the numerical range. Recall that an *smooth point* z of a Banach space Z (we may also say that Z is smooth at z) is just a point at which the norm of Z is Gateaux differentiable; equivalently, z is a smooth point if $\{\phi \in S_{Z^*} \colon \phi(z) = \|z\|\}$ is a singleton. The following lemma will allow to use the characterizations of BJ-orthogonality to describe smooth points. Although the proof of the lemma is immediate, we record it for the sake of completeness.

Lemma 2.8. *Let Z be a Banach space and let $u \in S_Z$. Then, u is a smooth point if and only if $V(Z, u, z)$ is a singleton set for every $z \in Z$.*

Proof. If u is smooth, then $\mathrm{F}(B_{Z^*}, u)$ is a singleton and then so are the sets $V(Z, u, z)$ for all $z \in Z$. Conversely, suppose that there exist $\phi_1, \phi_2 \in S_{Z^*}$ such that $\phi_1(u) = \phi_2(u) = 1$ and $\phi_1 \neq \phi_2$; then we may find $z \in S_Z$ such that $\phi_1(z) \neq \phi_2(z)$ and so $V(Z, u, z)$ is not a singleton set. □

The above lemma allows to characterize smoothness using Proposition 2.2, Theorem 2.3, and Theorem 2.4.

Corollary 2.9. *Let Z be a Banach space and let $u \in S_Z$. Then, u is a smooth point if and only if $\{\phi(z) \colon \phi \in \operatorname{ext}(B_{Z^*}),\ \phi(u) = 1\}$ is a singleton set for every $z \in Z$.*

Corollary 2.10. *Let Z be a Banach space, let $u \in S_Z$, and let $C \subset B_{Z^*}$ be such that $B_{Z^*} = \overline{\operatorname{conv}}^{w^*}(C)$. Then, the following are equivalent:*

(i) *u is a smooth point;*
(ii) *$\left\{\lim \phi_n(z) \colon \phi_n \in C\ \forall n \in \mathbb{N},\ \lim \phi_n(u) = 1\right\}$ is a singleton set for every $z \in Z$.*
(iii) *$\bigcap_{\delta>0} \overline{\left\{\phi(z) \colon \phi \in C, \operatorname{Re} \phi(u) > 1-\delta\right\}}$ is a singleton set for every $z \in Z$;*

Corollary 2.11. *Let Z be a Banach space, let $u \in S_Z$, and $\Lambda \subset B_{Z^*}$ be one-norming for Z. Then, the following are equivalent:*

(i) *u is a smooth point;*
(ii) $\{\theta_0 \lim \psi_n(z)\colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \theta_0 \in \mathbb{T},\ \lim \psi_n(u) = \overline{\theta_0}\}$ *is a singleton set for every $z \in Z$;*
(iii) $\Big\{\lim \psi_n(z)\overline{\psi_n(u)}\colon \psi_n \in \Lambda\ \forall n \in \mathbb{N},\ \lim |\psi_n(u)| = 1\Big\}$ *is a singleton set for every $z \in Z$;*
(iv) $\bigcap_{\delta>0} \overline{\{\psi(z)\overline{\psi(u)}\colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\}}$ *is a singleton set for every $z \in Z$;*
(v) $\bigcap_{\delta>0} \overline{\operatorname{conv}\{\psi(z)\overline{\psi(u)}\colon \psi \in \Lambda,\ |\psi(u)| > 1-\delta\}}$ *is a singleton set for every $z \in Z$.*

3. Using the general results in some interesting particular cases

We devote this section to apply the abstract results of the previous section in several settings. We begin with a characterization of BJ-orthogonality in a dual space that extends [39, Theorem 2.7] to the complex case. The proof is immediate from Goldstine's Theorem.

Proposition 3.1. *Let Y be a Banach space, let $C \subset B_Y$ such that $\operatorname{conv}(C)$ is dense in B_Y. For $u^*, z^* \in Y^*$, we have that*

$$u^* \perp_B z^* \iff 0 \in \operatorname{conv}\big(\{\lim z^*(y_n)\colon y_n \in C\ \forall n \in \mathbb{N},\ \lim u^*(y_n) = \|u\|\}\big).$$

Moreover, the norm of Y^* is smooth at u^* if and only if the set

$$\{\lim u^*(y_n)\colon y_n \in C\ \forall n \in \mathbb{N},\ \lim u^*(y_n) = \|u\|\}$$

is a singleton for all $y^* \in Y^*$.

The results in the rest of the section are divided into subsections for clarity of the exposition.

3.1. **Spaces of bounded functions.** Given a non-empty set Γ and a Banach space X, we write $\ell_\infty(\Gamma, X)$ to denote the Banach space of all bounded functions from Γ to Y endowed with the supremum norm. For $\gamma \in \Gamma$, $\delta_\gamma\colon \ell_\infty(\Gamma, Y) \longrightarrow Y$ denotes the evaluation map. We characterize next BJ-orthogonality in $\ell_\infty(\Gamma, Y)$. Fix a subset $C \subset S_{Y^*}$ satisfying that the weak-star closed convex hull of C is the whole B_{Y^*}. Consider the set

$$\mathfrak{C} := \{y^* \otimes \delta_\gamma\colon \gamma \in \Gamma,\ y^* \in C\} \subseteq \ell_\infty(\Gamma, Y)^*,$$

where $[y^* \otimes \delta_\gamma](f) := y^*(f(\gamma))$ for every $f \in \ell_\infty(\Gamma, Y)$. Since, clearly, $B_{\ell_\infty(\Gamma,X)^*}$ is the weak-star closed convex hull of $\mathfrak{C}$, the following result is a consequence of Corollary 2.5.

Theorem 3.2. *Let Γ be a non-empty set, let Y be a Banach space, let $C \subset S_{Y^*}$ be such that $B_{Y^*} = \overline{\operatorname{conv}}^{w^*}(C)$, and let $f, g \in \ell_\infty(\Gamma, Y)$. Then,*

$$f \perp_B g \iff 0 \in \operatorname{conv}\{\lim y_n^*(g(\gamma_n))\colon \gamma_n \in \Gamma,\ y_n^* \in C\ \forall n \in \mathbb{N},\ \lim y_n^*(f(\gamma_n)) = \|f\|\}.$$

Of course, the same characterization is valid in every closed subspace of $\ell_\infty(\Gamma, Y)$, since the BJ-orthogonality only depends on the two-dimensional subspace generated by the involved vectors. Then, as a consequence, we get a characterization of smoothness in any closed subspace $\mathcal{Z} \leqslant \ell_\infty(\Gamma, Y)$.

Corollary 3.3. *Let Γ be a non-empty set, let Y be a Banach space, let $C \subset S_{Y^*}$ such that $B_{Y^*} = \overline{\operatorname{conv}}^{w^*}(C)$, and let $\mathcal{Z} \leqslant \ell_\infty(\Gamma, Y)$ be a closed subspace. Then, for $f \in \mathcal{Z}$ the following are equivalent:*

(i) *f is a smooth point;*
(ii) $\{\lim y_n^*(g(\gamma_n))\colon \gamma_n \in \Gamma,\ y_n^* \in C\ \forall n \in \mathbb{N},\ \lim y_n^*(f(\gamma_n)) = \|f\|\}$ *is a singleton set for every $g \in Z$.*

As far as we know, the above two results are new.

Let us consider some interesting particular cases. Given a Hausdorff topological space Ω and a Banach space Y, we write $C_b(\Omega, Y)$ to denote the Banach space of all bounded continuous functions from Ω to Y, endowed with the supremum norm.

Corollary 3.4. *Let Ω be a Hausdorff topological space, let Y be a Banach space, and let $f, g \in C_b(\Omega, Y)$. Then,*

$$f \perp_B g \Longleftrightarrow 0 \in \operatorname{conv}\left\{\lim y_n^*(g(t_n)) \colon t_n \in \Omega,\ y_n^* \in S_{Y^*}\ \forall n \in \mathbb{N},\ \lim y_n^*(f(t_n)) = \|f\|\right\}.$$

This result extends [19, Corollary 3.1] to the vector-valued case. When Ω is compact, the result can be improved using Fact 1.3 and the description of the dual ball of $C(K, Y) \equiv \mathcal{K}_{w^*}(Y^*, C(K))$ given in [34, Theorem 1.1].

Corollary 3.5. *Let K be a compact Hausdorff topological space, let Y be a Banach space, and let $f, g \in C(K, Y)$. Then,*

$$f \perp_B g \Longleftrightarrow 0 \in \operatorname{conv}\left\{y^*(g(t)) \colon t \in K,\ y^* \in \operatorname{ext}(B_{Y^*}),\ y^*(f(t)) = \|f\|\right\}.$$

Moreover, $f \in C(K, Y)$ is smooth if and only if the set

$$\left\{y^*(g(t)) \colon t \in K,\ y^* \in \operatorname{ext}(B_{Y^*}),\ y^*(f(t)) = \|f\|\right\}$$

is a singleton for every $g \in C(K, Y)$.

The first part of the above corollary improves [32, Theorem 2.1], where the result was given only in the real case, and [29, Theorem 2.2], where it was proved in the case when Y is a finite-dimensional Hilbert space.

Another case in which Theorem 3.2 applies is the one of *unital uniform algebras*: closed subalgebras of a $C(K)$ space separating the points of K and containing the constant functions. Actually, in this case an improved result can be stated. For a unital uniform algebra A on $C(K)$, the *Choquet boundary* of A is the set

$$\partial A := \{s \in K \colon \delta_s|_A \in \operatorname{ext}(B_{A^*})\}$$

endowed with the topology induced by K. We refer to [27, Chap. 6] for background. It is immediate that

$$\operatorname{ext}(B_{A^*}) = \mathbb{T}\{\delta_s|_A \colon s \in \partial A\},$$

hence the next result follows from Fact 1.3 and Corollary 2.9.

Corollary 3.6. *Let A be a unital uniform algebra on $C(K)$ and let $\partial A \subset K$ be its Choquet boundary.*

(a) $f, g \in A$ satisfy $f \perp_B g$ if and only if

$$0 \in \operatorname{conv}\big\{\theta g(s) \colon \theta \in \mathbb{T},\ s \in \partial A,\ f(s) = \overline{\theta}\|f\|\big\}$$

(b) $f \in A$ is a smooth point of A if and only if the set

$$\big\{\theta g(s) \colon \theta \in \mathbb{T},\ s \in \partial A,\ f(s) = \overline{\theta}\|f\|\big\}$$

is a singleton for every $g \in A$.

This result applies, in particular, to the *disk algebra* $\mathbb{A}(\mathbb{D})$ of those continuous functions on the unit disk $\mathbb{D} = \{w \in \mathbb{C} \colon |w| \leqslant 1\}$ which are holomorphic in the interior, whose Choquet boundary is $\mathbb{T}$. We will improve this result in the case of finite Blaschke products in Corollary 4.8.

3.2. **Lipschitz maps.** Next, we give a characterization of BJ-orthogonality in the space of Lipschitz maps. To do so, we present the basic notions and notations. Given a pointed metric space (that is, a metric space M with a distinguished element called 0) and a Banach space Y, we denote by $\mathrm{Lip}_0(M,Y)$ the Banach space of all Lipschitz maps $F\colon M \longrightarrow Y$ such that $F(0)=0$ endowed with the norm

$$\|F\|_L = \sup\left\{\frac{\|F(t)-F(s)\|}{d(t,s)}: t,s\in M,\ t\neq s\right\}.$$

We refer the reader to the book [43] for more information and background. Given $s,t\in M$, $s\neq t$, and $y^*\in Y^*$, we define

$$\left[\widetilde{\delta}_{s,t}\otimes y^*\right](F) := \frac{y^*(F(t)-F(s))}{d(t,s)}$$

for every $F\in \mathrm{Lip}_0(M,Y)$. It is immediate that this formula defines a bounded linear functional on $\mathrm{Lip}_0(M,Y)$ and that, given a one-norming subset $C\subset S_{Y^*}$ for Y, the subset

$$\mathfrak{C} := \left\{\widetilde{\delta}_{s,t}\otimes y^*: s,t\in M,\ s\neq t,\ y^*\in C\right\}$$

is one-norming for $\mathrm{Lip}_0(M,Y)$. Therefore, Corollary 2.6 and Corollary 2.11 give the following result.

Proposition 3.7. *Let M be a pointed metric space, let Y be a Banach space, and let $C\subseteq S_{Y^*}$ be one-norming for Y.*

(a) $F,G\in \mathrm{Lip}_0(M,Y)$ satisfy $F\perp_B G$ if and only if 0 belongs to

$$\mathrm{conv}\left\{\lim \theta_0\frac{y_n^*\big(G(s_n)-G(t_n)\big)}{d(s_n,t_n)}: s_n,t_n\in M,\ s_n\neq t_n,\ y_n^*\in C,\ \theta_0\in\mathbb{T},\ \lim\frac{y_n^*\big(F(s_n)-F(t_n)\big)}{d(s_n,t_n)} = \overline{\theta_0}\|F\|_L\right\}.$$

(b) $F\in \mathrm{Lip}_0(M,Y)$ is a smooth point if and only if the set

$$\left\{\lim \theta_0\frac{y_n^*\big(G(s_n)-G(t_n)\big)}{d(s_n,t_n)}: s_n,t_n\in M,\ s_n\neq t_n,\ y_n^*\in C,\ \theta_0\in\mathbb{T},\ \lim\frac{y_n^*\big(F(s_n)-F(t_n)\big)}{d(s_n,t_n)} = \overline{\theta_0}\|F\|_L\right\}$$

is a singleton for every $G\in \mathrm{Lip}_0(M,Y)$.

Let us comment that there is a result on smoothness in spaces of Lipschitz functions showing that smoothness and Fréchet smoothness are equivalent in $\mathrm{Lip}_0(M,\mathbb{R})$, see [11, Corollary 5.8].

This result also follows from Theorem 3.2 by using a vector-valued version of De Leeuw's map, see [43, §2.4] for instance.

3.3. **Injective tensor products.** Let X, Y be Banach spaces. The *injective tensor product* of X and Y, denoted by $X\hat{\otimes}_\varepsilon Y$, is the completion of $X\otimes Y$ endowed with the norm given by

$$\|u\|_\varepsilon = \sup\left\{\left|\sum_{i=1}^n x^*(x_i)y^*(y_i)\right|: x^*\in B_{X^*},\ y^*\in B_{Y^*}\right\},$$

where $\sum_{i=1}^n x_i\otimes y_i$ is any representation of u. Since $B_{(X\hat{\otimes}_\varepsilon Y)^*} = \overline{\mathrm{conv}}^{w^*}(B_{X^*}\otimes B_{Y^*})$, we obtain the following result as consequence of Corollaries 2.5 and 2.10.

Proposition 3.8. *Let X, Y be Banach spaces. and let $u,z\in X\hat{\otimes}_\varepsilon Y$.*

(a) $u,z\in X\hat{\otimes}_\varepsilon Y$ satisfy $u\perp_B z$ if and only if

$$0\in\mathrm{conv}\big(\big\{\lim(x_n^*\otimes y_n^*)(z): x_n^*\otimes y_n^*\in B_{X^*}\otimes B_{Y^*}\ \forall n\in\mathbb{N},\ \lim(x_n^*\otimes y_n^*)(u)=\|u\|_\varepsilon\big\}\big).$$

(b) $u \in X\hat{\otimes}_\varepsilon Y$ *is smooth if and only if the set*

$$\left\{\lim(x_n^* \otimes y_n^*)(z) \colon x_n^* \otimes y_n^* \in B_{X^*} \otimes B_{Y^*}\ \forall n \in \mathbb{N},\ \lim(x_n^* \otimes y_n^*)(u) = \|u\|_\varepsilon\right\}$$

is a singleton for every $z \in X\hat{\otimes}_\varepsilon Y$.

3.4. **Spaces of operators endowed with the operator norm.** Let X, Y be Banach spaces. Consider $C \subset S_X$ such that $\operatorname{conv}(C)$ is dense in B_X and $D \subset S_{Y^*}$ which is one-norming for Y. Our general characterization of BJ-orthogonality in $\mathcal{L}(X,Y)$ endowed with the usual norm is obtained by using Corollary 2.5 with

$$\mathfrak{C} := \{y^* \otimes x \colon x \in C,\ y^* \in D\}$$

where $y^* \otimes x \in \mathcal{L}(X,Y)^*$ is defined by

$$[y^* \otimes x](T) := y^*(Tx) \qquad (T \in \mathcal{L}(X,Y)).$$

For $C = S_X$ and $D = S_{Y^*}$, the result already appeared in [28, Theorem 2.2], with a different proof.

Proposition 3.9 (Extension of [28, Theorem 2.2])**.** *Let X, Y be Banach spaces, $C \subset S_X$ such that* $\operatorname{conv}(C)$ *is dense in B_X and $D \subset S_{Y^*}$ which is one-norming for Y, and let $T, A \in \mathcal{L}(X,Y)$. Then,*

$$T \perp_B A \iff 0 \in \operatorname{conv}\left(\left\{\lim y_n^*(Ax_n) \colon (x_n, y_n^*) \in C \times D\ \forall n \in \mathbb{N},\ \lim y_n^*(Tx_n) = \|T\|\right\}\right).$$

Observe that the result also follows from Theorem 3.2 as $\mathcal{L}(X,Y)$ can be viewed as a closed subspace of $\ell_\infty(C,Y)$.

When the operators involved are compact we can remove the limits in Proposition 3.9 and also we can use extreme points of $B_{X^{**}}$ and of B_{Y^*}. For $y^* \in Y^*$ and $x^{**} \in X^{**}$, we consider $[x^{**} \otimes y^*](T) := x^{**}(T^*y^*)$ for every $T \in \mathcal{K}(X,Y)$.

Proposition 3.10. *Let X, Y be Banach spaces, and let $T, A \in \mathcal{K}(X,Y)$. Then,*

$$T \perp_B A \iff 0 \in \operatorname{conv}\left(\left\{x^{**}(A^*(y^*)) \colon x^{**} \in \operatorname{ext}(B_{X^{**}}),\ y^* \in \operatorname{ext}(B_{Y^*}),\ x^{**}(T^*(y^*)) = \|T\|\right\}\right).$$

The proof of this result follows from Fact 1.3 as the set

$$C = \{x^{**} \otimes y^* \colon x^{**} \in \operatorname{ext}(B_{X^{**}}),\ y^* \in \operatorname{ext}(B_{Y^*})\}$$

coincides with the set of extreme points of the unit ball of $\mathcal{K}(X,Y)^*$, see [34, Theorem 1.3]. In the case when X is reflexive, the above result has a nicer form. Let us remark here that a special case of the following result was obtained in Theorem 2.1 of [39], where X is assumed to be a real reflexive Banach space.

Corollary 3.11. *Let X be a reflexive Banach space, let Y be a Banach space, and let $T, A \in \mathcal{K}(X,Y)$. Then,*

$$T \perp_B A \iff 0 \in \operatorname{conv}\left(\left\{y^*(Ax) \colon x \in \operatorname{ext}(B_X),\ y^* \in \operatorname{ext}(B_{Y^*}),\ y^*(Tx) = \|T\|\right\}\right).$$

Of course, the previous result applies when X is finite-dimensional.

Corollary 3.12 ([20, Proposition 4.2])**.** *Let X be a finite-dimensional space, let Y be a Banach space, and let $T, A \in \mathcal{L}(X,Y)$. Then*

$$T \perp_B A \iff 0 \in \operatorname{conv}\left(\left\{y^*(Ax) \colon x \in \operatorname{ext}(B_X),\ y^* \in \operatorname{ext}(B_{Y^*}),\ y^*(Tx) = \|T\|\right\}\right).$$

We finish this subsection on the operator norm by presenting a characterization of smooth operators which follows directly from Corollary 2.10.

Proposition 3.13. *Let X, Y be Banach spaces, $C \subset S_X$ such that* $\operatorname{conv}(C)$ *is dense in B_X and $D \subset S_{Y^*}$ which is one-norming for Y, and let $0 \neq T \in \mathcal{L}(X,Y)$. Then, T is a smooth operator if and only if*

$$\left\{\lim y_n^*(Ax_n)\colon (x_n, y_n^*) \in C \times D \ \forall n \in \mathbb{N},\ \lim y_n^*(Tx_n) = \|T\|\right\}$$

is a singleton set for every $A \in \mathcal{L}(X,Y)$.

As an easy consequence of this proposition and Corollary 2.10, we obtain a result that gives the existence of smooth operators under reasonable restrictions. In fact, they are quite similar to those used by Heinrich in [12, Theorem 3.1] to characterize Fréchet smooth operators in $\mathcal{L}(X,Y)$ (but there is no characterization of smoothness of operators outside $\mathcal{K}(X,Y)$ in [12]). The result extends [40, Theorem 3.4] to the complex case. It will be used in Subsection 5.2.

Proposition 3.14. *Let X, Y be Banach spaces. Let $0 \neq T \in \mathcal{L}(X,Y)$ be such that there is $x_0 \in S_X$ satisfying the following conditions:*

1. *Tx_0 is a smooth point in Y;*
2. *every sequence $\{x_n\} \subset B_X$ satisfying $\lim \|Tx_n\| = \|T\|$ has a subsequence converging to αx_0 for some $\alpha \in \mathbb{T}$.*

Then, T is smooth.

Proof. Using Proposition 3.13 it suffices to show that, for every $A \in \mathcal{L}(X,Y)$, the set

$$\left\{\lim y_n^*(Ax_n)\colon (x_n, y_n^*) \in S_X \times S_{Y^*} \ \forall n \in \mathbb{N},\ \lim y_n^*(Tx_n) = \|T\|\right\}$$

is a singleton. To do so, fix an arbitrary $\lambda = \lim y_n^*(Ax_n)$ and observe that $\lim y_n^*(Tx_n) = \|T\|$ implies $\lim \|Tx_n\| = \|T\|$. So, using (2), there are $\alpha \in \mathbb{T}$ and a subsequence $\{x_{\sigma(n)}\}$ with $\lim x_{\sigma(n)} = \alpha x_0$. Now, it is clear that

$$\lim(\alpha y_{\sigma(n)}^*)(Tx_0) = \lim y_{\sigma(n)}^*(Tx_{\sigma(n)}) = \lim y_n^*(Tx_n) = \|T\| = \|Tx_0\|$$

and

$$\lim(\alpha y_{\sigma(n)}^*)(Ax_0) = \lim y_{\sigma(n)}^*(Ax_{\sigma(n)}) = \lambda.$$

Therefore, we get that

$$\lambda \in \left\{\lim z_n^*(Ax_0)\colon z_n^* \in S_{Y^*} \ \forall n \in \mathbb{N},\ \lim z_n^*(Tx_0) = \|Tx_0\|\right\}$$

and the latter set is a singleton by Corollary 2.10 as Tx_0 is a smooth point of Y by (1). ☐

3.5. **Multilinear maps and polynomials.** In an analogous way that we deal with bounded operators, it is possible to describe the BJ-orthogonality of multilinear maps and polynomials.

Let $X_1, \ldots, X_k$ and Y be Banach spaces. The set of all bounded k-linear maps from $X_1 \times \cdots \times X_k$ to Y will be denoted by $\mathcal{L}(X_1, \ldots, X_k; Y)$. As usual, we define the norm of $A \in \mathcal{L}(X_1, \ldots, X_k; Y)$ by

$$\|A\| = \sup\left\{\|A(x_1, \ldots, x_k)\|\colon (x_1, \ldots, x_k) \in S_{X_1} \times \cdots \times S_{X_k}\right\}.$$

It is then immediate that

$$\mathcal{L}(X_1, \ldots, X_k; Y) \subset \ell_\infty(\Gamma, Y)$$

where $\Gamma = S_{X_1} \times \cdots \times S_{X_k}$. Therefore, the following result follows immediately from Theorem 3.2 and Corollary 3.3. It was proved in [28].

Proposition 3.15 ([28, Theorem 2.2 and Theorem 3.1])**.** *Let $X_1, \ldots, X_k$ and Y be Banach spaces and let $C \subset S_{Y^*}$ be such that $B_{Y^*} = \overline{\operatorname{conv}}^{w^*}(C)$.*

(a) For $T, A \in \mathcal{L}(X_1, \ldots, X_k; Y)$ we have that $T \perp_B A$ if and only if 0 belongs to the convex hull of

$$\left\{\lim_n y_n^*(A(x_1^n, \ldots, x_k^n)) \colon (x_1^n, \ldots, x_k^n) \in S_{X_1} \times \cdots \times S_{X_k},\, y_n^* \in C,\, \lim_n y_n^*(T(x_1^n, \ldots, x_k^n)) = \|T\|\right\}.$$

(b) $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ is a smooth point if and only if the set

$$\left\{\lim_n y_n^*(A(x_1^n, \ldots, x_k^n)) \colon (x_1^n, \ldots, x_k^n) \in S_{X_1} \times \cdots \times S_{X_k},\, y_n^* \in C,\, \lim_n y_n^*(T(x_1^n, \ldots, x_k^n)) = \|T\|\right\}$$

is a singleton for every $A \in \mathcal{L}(X_1, \ldots, X_k; Y)$.

We now deal with polynomials between Banach spaces. Let X and Y be Banach spaces. A (continuous) *N-homogeneous polynomial* P from X to Y is a mapping $P\colon X \longrightarrow Y$ for which we can find a multilinear operator $T \in \mathcal{L}(X \times \overset{N}{\overbrace{\cdots}} \times X; Y)$ (continuous) which is symmetric (i.e., $T(x_1, \ldots, x_N) = T(x_{\sigma(1)}, \ldots, x_{\sigma(N)})$ for every permutation σ of the set $\{1, \ldots, N\}$) and satisfying $P(x) = T(x, \ldots, x)$ for every $x \in X$. A (general) polynomial from X to Y is a mapping $P\colon X \longrightarrow Y$ which can be written as a finite sum of homogeneous polynomials. We write $\mathcal{P}(X, Y)$ for the space of all polynomials from X to Y. It is immediate that $\mathcal{P}(X, Y)$ is a subspace of $\ell_\infty(B_X, Y)$, so the next result follows again from Theorem 3.2 and Corollary 3.3.

Proposition 3.16. *Let X, Y be Banach spaces and let $C \subset S_{Y^*}$ be such that $B_{Y^*} = \overline{\operatorname{conv}}^{w^*}(C)$.*

(a) Given $P, Q \in \mathcal{P}(X, Y)$, we have that $P \perp_B Q$ if and only if

$$0 \in \operatorname{conv}\left\{\lim y_n^*(P(x_n)) \colon x_n \in B_X,\, y_n^* \in C,\, \lim y_n^*(Q(x_n)) = \|Q\|\right\}.$$

(b) $P \in \mathcal{P}(X, Y)$ is a smooth point if and only if the set

$$\left\{\lim y_n^*(P(x_n)) \colon x_n \in B_X,\, y_n^* \in C,\, \lim y_n^*(Q(x_n)) = \|Q\|\right\}$$

is a singleton for every $Q \in \mathcal{P}(X, Y)$.

3.6. **Spaces of operators endowed with the numerical radius as norm.** Let X be a Banach space. We deal here with the space $\mathcal{L}(X)$ endowed with the numerical radius. Let us recall the necessary definitions. Write $\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} \colon x^*(x) = 1\}$. The *numerical radius* of $T \in \mathcal{L}(X)$ is

$$v(T) := \sup\{|x^*(Tx)| \colon (x, x^*) \in \Pi(X)\}.$$

It is a well-known fact that

$$v(T) = \sup\big\{|\lambda| \colon \lambda \in V(\mathcal{L}(X), \mathrm{Id}, T)\big\}$$

for every $T \in \mathcal{L}(X)$ (see [10, Proposition 2.1.31], for instance). We refer the interested reader to the classical books [8, 9] and to Sections 2.1 and 2.9 of the book [10] for more information and background. It is clear that the numerical radius is a seminorm on $\mathcal{L}(X)$ and $v(T) \leqslant \|T\|$ for every $T \in \mathcal{L}(X)$. We would like to remark here that although BJ-orthogonality is defined in the framework of norms, it may also be considered in exactly the same way in any seminormed space. Of course, when the seminorm is a norm, we return to the original setting.

We particularize Corollary 2.6 to the space of operators with the numerical radius, taking

$$\Lambda := \{x^* \otimes x \colon (x, x^*) \in \Pi(X)\} \subset (\mathcal{L}(X), v)^*$$

which is clearly one-norming for $(\mathcal{L}(X), v)$. The following result appeared in [22, Theorem 3.4].

Proposition 3.17 ([22, Theorem 3.4])**.** *Let X be a Banach space and let $T, A \in \mathcal{L}(X)$. Then,*

$$T \perp_B^v A \iff 0 \in \operatorname{conv}\left(\left\{\lim x_n^*(Ax_n)\overline{x_n^*(Tx_n)} \colon (x_n, x_n^*) \in \Pi(X)\ \forall n \in \mathbb{N},\ \lim |x_n^*(Tx_n)| = v(T)\right\}\right).$$

In the case of compact operators defined on a reflexive space, it is straightforward to show that the limits can be removed.

Corollary 3.18. *Let X be a reflexive Banach space and let $T, A \in \mathcal{K}(X)$. Then,*

$$T \perp_B^v A \iff 0 \in \operatorname{conv}\left(\left\{x^*(Ax)\overline{x^*(Tx)} \colon (x,x^*) \in \Pi(X),\ |x^*(Tx)| = v(T)\right\}\right).$$

A characterization in the particular case when X has finite dimension has been recently proved by Roy and Sain [31, Theorem 2.3].

Corollary 3.19 ([31, Theorem 2.3])**.** *Let X be a finite-dimensional space and let $T, A \in \mathcal{L}(X)$. Then*

$$T \perp_B^v A \iff 0 \in \operatorname{conv}\left(\left\{x^*(Ax)\overline{x^*(Tx)} \colon (x,x^*) \in \Pi(X),\ |x^*(Tx)| = v(T)\right\}\right).$$

We may state the following characterization of smoothness in $(\mathcal{L}(X), v)$, as a consequence of the previous observations and Corollary 2.11. We say that $T \in \mathcal{L}(X)$ is a *smooth operator for the numerical radius* if T is a smooth point of $(\mathcal{L}(X), v)$.

Proposition 3.20. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then, T is a smooth operator for the numerical radius if and only if*

$$\left\{\lim x_n^*(Ax_n)\overline{x_n^*(Tx_n)} \colon (x_n, x_n^*) \in \Pi(X)\ \forall n \in \mathbb{N},\ \lim |x_n^*(Tx_n)| = v(T)\right\}$$

is a singleton set for every $A \in \mathcal{L}(X)$.

As far as we could check, the above characterization of smoothness for the numerical radius has not appeared previously in its most general form.

4. Bhatia-Šemrl's kind of results

In the particular case of operators on Hilbert spaces, the results of the Subsection 3.4 can be improved as there is no need of taking convex hull. The first characterization in this line was obtained by Stampfli [42, Theorem 2] in the special case when one of the operators is the identity. Later, Magajna [21, Lemma 2.2] observed that Stampfli's result holds for any pair of operators, leading to a complete characterization of BJ-orthogonality in $\mathcal{L}(H)$. The same characterization was obtained by Bhatia and Šemrl [5, Remark 3.1], and also by Kečkić [18, Corollary 3.1] with different approaches.

Here we present an alternative proof which follows from our Proposition 3.9 and [24, Theorem 2].

Corollary 4.1 ([21, Lemma 2.2], [5, Remark 3.1], [18, Corollary 3.1])**.** *Let H be a Hilbert space and let $T, A \in \mathcal{L}(H)$. Then $T \perp_B A$ if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in S_H such that $\|Tx_n\| \to \|T\|$ and $\langle Tx_n, Ax_n\rangle \to 0$.*

Proof. It follows from Proposition 3.9 that

$$T \perp_B A \iff 0 \in \operatorname{conv}\left(\left\{\lim\langle Ax_n, y_n\rangle \colon x_n, y_n \in S_H\ \forall n \in \mathbb{N},\ \lim\langle Tx_n, y_n\rangle = \|T\|\right\}\right).$$

Observe that

$$\begin{aligned}\left\{\lim\langle Ax_n, y_n\rangle \colon x_n, y_n \in S_H\ \forall n \in \mathbb{N},\ \lim\langle Tx_n, y_n\rangle = \|T\|\right\}\\ = \left\{\lim\langle Ax_n, Tx_n\rangle \colon x_n \in S_H\ \forall n \in \mathbb{N},\ \lim \|Tx_n\| = \|T\|\right\}\end{aligned}$$

and that the latter set is convex (this was first stated without proof in [21, Lemma 2.1], see [24, Theorem 2] for a proof). □

In the particular case when H is finite-dimensional, Bhatia and Šemrl were the first to write down the characterization of BJ-orthogonality of two matrices in terms of the elements of H [5, Theorem 1.1]. An alternative proof of this characterization was given by Roy, Bagchi, and Sain in [30]. We obtain this result as a consequence of Corollary 3.12.

Corollary 4.2 (Bhatia-Šemrl theorem, [5, Theorem 1.1])**.** *Let H be a finite-dimensional Hilbert space and let $T, A \in \mathcal{L}(H)$. Then $T \perp_B A$ if and only if there exists $x \in S_H$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$.*

Proof. It follows from Corollary 3.12 that

$$T \perp_B A \Longleftrightarrow 0 \in \operatorname{conv}\left(\left\{\langle Ax, y\rangle \colon x, y \in S_H,\ \langle Tx, y\rangle = \|T\|\right\}\right).$$

Now, observe that

$$\left\{\langle Ax, y\rangle \colon x, y \in S_H,\ \langle Tx, y\rangle = \|T\|\right\} = \left\{\langle Ax, Tx\rangle \colon x \in S_H,\ \langle Tx, Tx\rangle = \|T\|\right\}.$$

The result follows since the latter set is convex ([21, Lemma 2.1], [24, Theorem 2]). □

It has been shown by Li and Schneider that Bhatia-Šemrl theorem cannot be extended in general to arbitrary finite-dimensional Banach spaces [20, Example 4.3]. Actually, the validity of Bhatia-Šemrl theorem for all operators characterizes Hilbert spaces among finite-dimensional Banach spaces, see [4]. However, it is natural to study for which operators T it is possible to have a Bhatia-Šemrl theorem for all operators A: conditions on T such that whenever $T \perp_B A$, one has that there is a norm-one x such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$ (that is, whether we may remove the convex hull in Corollary 3.11). This has been done in [25, 35, 38] for the real case and in [26, 30] for the complex case. Our aim in what follows is to give a unified approach that allows to recover some of these results and to obtain an improvement in the complex setting. Actually, we will work in the more general framework of vector-valued continuous functions on a compact Hausdorff space. To deal with both the real and the complex case, we need to introduce the notion of directional orthogonality from [26]. Given x, y elements of a Banach space Z, we say that x is *orthogonal* to y *in the direction of* $\gamma \in \mathbb{T}$, which we denote by $x \perp_\gamma y$, if $\|x + t\gamma y\| \geqslant \|x\|$ for every $t \in \mathbb{R}$. Obviously, $x \perp_B y$ if and only if $x \perp_\gamma y$ for every $\gamma \in \mathbb{T}$. In the real case, it is obvious that $x \perp_B y$ if and only if $x \perp_1 y$ if and only if $x \perp_{-1} y$. In the complex case, there are easy examples showing that $x \not\perp_B y$ while $x \perp_\gamma y$ for some $\gamma \in \mathbb{T}$ is possible, see [30, Example 1]. It is shown in [30, Theorem 4] that

$$x \perp_\gamma y \Longleftrightarrow \exists\, x^* \in S_{X^*} \text{ with } x^*(x) = \gamma\|x\| \text{ and } \operatorname{Re} x^*(y) = 0 \tag{4.1}$$

(indeed, this result is immediate as $x \perp_\gamma y$ if and only if $x \perp_B \gamma y$ in the real space $X_\mathbb{R}$ underlying X and $(X_\mathbb{R})^* = \{\operatorname{Re} x^* \colon x^* \in X^*\}$).

For a Hausdorff compact topological space K and a Banach space Y, the norm attainment set of $f \in C(K, Y)$ is the (non-empty) set

$$\mathcal{M}_f = \{t \in K \colon \|f(t)\| = \|f\|\}.$$

Our main result in $C(K, Y)$ is a Bhatia-Šemrl's type result when $\mathcal{M}_f$ is connected.

Theorem 4.3. *Let K be a compact Hausdorff topological space and let Y be a Banach space. Let $f, g \in C(K, Y)$ be such that $\mathcal{M}_f$ is connected. Then,*

$$f \perp_B g \iff \forall \mu \in \mathbb{T}\ \exists t \in \mathcal{M}_f \text{ such that } f(t) \perp_\mu g(t).$$

In the real case, we actually have

$$f \perp_B g \iff \exists t \in \mathcal{M}_f \text{ such that } f(t) \perp_B g(t).$$

The most technical part of the proof is contained in the next lemma, which is actually valid in $C_b(\Omega, Y)$. We still use the notation $\mathcal{M}_f$ for the (maybe empty) norm attainment set of a function $f \in C_b(\Omega, Y)$.

Lemma 4.4. *Let Ω be a Hausdorff topological space, let Y be a Banach space, and let $f \in C_b(\Omega, Y)$. Suppose that there exists a closed connected subset D of Ω such that $D \subseteq \mathcal{M}_f$. Then, for every $g \in C_b(\Omega, Y)$, the set*

$$\{y^*(g(t)) \colon t \in D,\ y^* \in S_{Y^*},\ y^*(f(t)) = \|f\|\}$$

is a connected subset of $\mathbb{C}$.

Proof. We follow the lines of the proof of the spatial numerical range of operators being connected given in [8, Section 11]. Consider the product $\Omega \times B_{Y^*}$, topologized by the product of the topology of Ω and the w^* topology of B_{Y^*}. For any fixed $h \in C_b(\Omega, Y)$, define $\Theta_h \colon \Omega \times B_{Y^*} \longrightarrow \mathbb{C}$ by

$$\Theta_h(t, y^*) = y^*(h(t)) \qquad \big((t, y^*) \in \Omega \times B_{Y^*}\big).$$

Observe that

$$\begin{aligned} |\Theta_h(t, y^*) - \Theta_h(s, z^*)| &= |y^*(h(t)) - z^*(h(s))| \\ &\leqslant |y^*(h(t)) - y^*(h(s))| + |J_Y(h(s))(y^*) - J_Y(h(s))(z^*)| \\ &\leqslant \|h(t) - h(s)\| + |J_Y(h(s))(y^*) - J_Y(h(s))(z^*)|, \end{aligned}$$

where $J_Y \colon Y \longrightarrow Y^{**}$ denotes the canonical embedding. It follows from the continuity of h and the w^*-continuity of $J_Y(h(s))$ that the map Θ_h is continuous.

Thus to prove our assertion, it is enough to show that

$$\mathcal{A} = \{(t, y^*) \in D \times S_{Y^*} \colon y^*(f(t)) = \|f\|\}$$

is connected. Suppose by contradiction that $\mathcal{A} = F_1 \cup F_2$, where F_1, F_2 are non-empty and closed in $\mathcal{A}$ with $\mathrm{F}_1 \cap F_2 = \emptyset$. The projections $\pi_1(F_1)$ and $\pi_1(F_2)$ are closed subsets of Ω. Indeed, consider any net (t_τ) in $\pi_1(F_1)$ such that $t_\tau \to t_0$ in Ω. Evidently, $\pi_1(F_1) \subseteq D$ and D is closed. Thus, $t_0 \in D$. For each τ, consider $y_\tau^* \in S_{Y^*}$ such that $(t_\tau, y_\tau^*) \in F_1$. The net (y_τ^*) has an adherent point y_0^* in B_{Y^*}, since B_{Y^*} is w^*-compact. Thus, (t_0, y_0^*) is an adherent point of the net $((t_\tau, y_\tau^*))$. Moreover, it follows from the continuity of Θ_f that $y_0^*(f(t_0)) = \|f\|$. Thus, $y_0^* \in S_{Y^*}$ and we have $(t_0, y_0^*) \in \mathcal{A}$. Since F_1 is closed in $\mathcal{A}$, we have $(t_0, y_0^*) \in F_1$. Therefore, $t_0 \in \pi_1(F_1)$ and $\pi_1(F_1)$ is a closed subset of Ω. Similarly, $\pi_1(F_2)$ is a closed subset of Ω. Note that $D = \pi_1(F_1) \cup \pi_1(F_2)$. It follows from the connectedness of D that there exists $\widetilde{t} \in \pi_1(F_1) \cap \pi_1(F_2)$. Therefore, we may find y_1^* and y_2^* in S_{Y^*} such that $(\widetilde{t}, y_1^*) \in F_1$ and $(\widetilde{t}, y_2^*) \in F_2$. Then,

$$\mathcal{B} := \big\{\big(\widetilde{t}, (\lambda y_1^* + (1 - \lambda) y_2^*)\big) : \lambda \in [0, 1]\big\}$$

is a connected subset and it is contained in $\mathcal{A}$. However, $(\mathcal{B} \cap F_1)$ and $(\mathcal{B} \cap F_2)$ are non-empty, closed in $\mathcal{B}$ and form a separation of $\mathcal{B}$. This contradicts the connectedness of $\mathcal{B}$. $\square$

We are now ready to give the pending proof of the theorem.

Proof of Theorem 4.3. We only prove the necessity as the sufficiency is straightforward. Suppose that $f \perp_B g$ and consider

$$\begin{aligned} \mathcal{A}_1 &:= \{y^*(g(t)) \colon t \in \mathcal{M}_f,\ y^* \in \mathrm{ext}(B_{Y^*}),\ y^*(f(t)) = \|f\|\}, \\ \mathcal{A}_2 &:= \{y^*(g(t)) \colon t \in \mathcal{M}_f,\ y^* \in S_{Y^*},\ y^*(f(t)) = \|f\|\}. \end{aligned}$$

Observe that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and that $0 \in \mathrm{conv}(\mathcal{A}_1)$ by Corollary 3.5, hence $0 \in \mathrm{conv}(\mathcal{A}_2)$. Now, by Lemma 4.4, $\mathcal{A}_2$ is connected. Therefore, by Lemma 2.7, for every $\mu \in \mathbb{T}$ there exists $(t, y^*) \in \mathcal{M}_f \times S_{Y^*}$

such that $y^*(f(t)) = \|f(t)\| = \|f\|$ and $\operatorname{Re} \mu y^*(g(t)) = 0$. Hence, (4.1) shows that $f(t) \perp_\mu g(t)$, as desired. $\square$

Our next aim is to apply Theorem 4.3 to spaces of operators. Given Banach spaces X, Y and $T \in \mathcal{L}(X,Y)$, let M_T denote the (maybe empty) norm attainment set of T, that is,

$$M_T := \{x \in S_X : \|Tx\| = \|T\|\}.$$

In the real case, the result we get is the following one, which appeared in [25].

Proposition 4.5 ([25, Theorem 2.1])**.** *Let X be a* real *reflexive Banach space, let Y be a* real *Banach space, and let $T, A \in \mathcal{K}(X,Y)$. Suppose that $M_T = D \cup (-D)$ for a connected subset D of S_X. Then, $T \perp_B A$ if and only if there exists $x \in D$ such that $Tx \perp_B Ax$.*

Proof. We only prove the necessity as sufficiency is obvious. Suppose that $T \perp_B A$ and consider

$$\begin{aligned}\mathcal{A}_1 &:= \{y^*(Ax) : x \in \operatorname{ext} B_X,\ y^* \in \operatorname{ext} B_{Y^*},\ y^*(Tx) = \|T\|\};\\ \mathcal{A}_2 &:= \{y^*(Ax) : x \in D,\ y^* \in S_{Y^*},\ y^*(Tx) = \|T\|\}.\end{aligned}$$

Let us show that $\mathcal{A}_1 \subseteq \mathcal{A}_2$. Indeed,

$$\begin{aligned}\mathcal{A}_1 &\subseteq \{y^*(Ax) : x \in S_X,\ y^* \in S_{Y^*},\ y^*(Tx) = \|T\|\}\\ &= \{y^*(Ax) : x \in M_T,\ y^* \in S_{Y^*},\ y^*(Tx) = \|T\|\} = \mathcal{A}_2.\end{aligned}$$

The first inclusion is obvious and the second equality is clear since $y^*(Tx) = \|T\|$ implies $x \in M_T$. For the third one, given $x \in M_T$, there exist $\theta \in \{-1, 1\}$ and $z \in D$ with $x = \theta z$. If $y^* \in S_{Y^*}$ satisfies $y^*(Tx) = \|T\|$, then we have that

$$(\theta y^*)(Tz) = y^*(Tx) = \|T\| \qquad \text{and} \qquad (\theta y^*)(Az) = y^*(Ax),$$

and we deduce the desired equality. Now, B_X equipped with the weak topology is a compact Hausdorff topological space. Consider the Banach space $C\big((B_X, w), Y\big)$. The identification $T \longmapsto \widetilde{T}$ where $\widetilde{T} = T|_{B_X}$, is an isometric embedding of $\mathcal{K}(X,Y)$ into $C\big((B_X, w), Y\big)$. Thus, by virtue of this identification, we have that the set

$$\mathcal{A}_3 := \{y^*(\widetilde{A}x) : x \in D,\ y^* \in S_{Y^*},\ y^*(\widetilde{T}x) = \|\widetilde{T}\|\}$$

coincides with $\mathcal{A}_2$ and is connected by Lemma 4.4. It follows from Corollary 3.11 that $0 \in \operatorname{conv}(\mathcal{A}_1)$ and so $0 \in \operatorname{conv}(\mathcal{A}_3)$. Hence, Lemma 2.7 gives that for every $\mu \in \{-1, 1\}$ there exists $x_\mu \in D$ such that $\widetilde{T}x_\mu \perp_\mu \widetilde{A}x_\mu$. Therefore, there exists $x \in D$ such that $Tx \perp_B Ax$ as desired. $\square$

The complex case can be treated similarly using the notion of directional orthogonality. Our main result extends [25, Theorem 2.1] to the complex case and [30, Theorem 7] and [26, Theorem 2.6] to the infinite-dimensional case. Observe that the connectedness of M_T *in the complex case* is equivalent to requiring that $M_T = \bigcup_{\theta \in \mathbb{T}} \theta D$ for a connected set D. In the real case, the second condition is weaker.

Theorem 4.6. *Let X be a* complex *reflexive Banach space, let Y be a* complex *Banach space, and let $T, A \in \mathcal{K}(X,Y)$. Suppose that M_T is connected. Then, $T \perp_B A$ if and only if for each $\gamma \in \mathbb{T}$ there exists $x \in M_T$ such that $Tx \perp_\gamma Ax$.*

This result can be proved following a completely analogous argument to the one for Proposition 4.5, or alternatively, it can be established as a direct consequence of Theorem 4.3 since M_T is connected in this case.

When X is finite-dimensional, the previous two results clearly apply.

Corollary 4.7 ([30, Theorem 7] and [26, Theorem 2.6]). *Let X be a finite-dimensional space, let Y be a Banach space, and let $T, A \in \mathcal{L}(X,Y)$. In the real case, suppose that $M_T = D \cup -D$ for a connected set D; in the complex case, suppose that M_T is connected. Then, $T \perp_B A$ if and only if for each $\gamma \in \mathbb{T}$ there exists $x \in M_T$ such that $Tx \perp_\gamma Ax$.*

We finally give another Bhatia-Šemrl's type result which improves Corollary 3.6 for a class of inner functions of the disk algebra $\mathbb{A}(\mathbb{D})$, known as finite Blaschke products. A *Blaschke product* of degree n is defined by

$$B_n(z) := z^k \prod_{j=1}^{n} \frac{|a_j|}{a_j} \frac{z - a_j}{1 - \overline{a_j} z} \qquad (z \in \mathbb{D})$$

where k is an integer, $k \geqslant 0$, and $0 < |a_j| < 1$, $1 \leqslant j \leqslant n$. Observe that $|B_n(z)| = 1$ for $z \in \mathbb{T}$. We refer the reader to [33, page 310] for more information and background.

Corollary 4.8. *Let B_m and B_n be two Blaschke products of degree m and n, respectively, viewed as elements of $\mathbb{A}(\mathbb{D})$. Then,*

$$B_n \perp_B B_m \iff \forall \mu \in \mathbb{T}\ \exists z_0 \in \mathbb{T} \text{ such that } \mu\overline{B_n(z_0)}B_m(z_0) \in \{i, -i\}.$$

Proof. Observe that Corollary 3.6 gives that

$$B_n \perp_B B_m \iff 0 \in \operatorname{conv}\{\overline{B_n(z)}B_m(z) \colon z \in \mathbb{T}\}$$

since $|B_n(z)| = |B_m(z)| = 1$ for every $z \in \mathbb{T}$. Using that the set $\{\overline{B_n(z)}B_m(z) \colon z \in \mathbb{T}\}$ is connected and Lemma 2.7, we have that

$$\begin{aligned} B_n \perp_B B_m &\iff \forall \mu \in \mathbb{T}\ \exists z_0 \in \mathbb{T} \text{ such that } \operatorname{Re} \mu\overline{B_n(z_0)}B_m(z_0) = 0 \\ &\iff \forall \mu \in \mathbb{T}\ \exists z_0 \in \mathbb{T} \text{ such that } \mu\overline{B_n(z_0)}B_m(z_0) \in \{i, -i\}. \end{aligned}$$ □

Remark 4.9. The same proof also allows to characterize when $f \perp_B g$ for holomorphic functions f and g on the open unit disk either if $M_f \subset \mathbb{T}$ is a connected subset of $\mathbb{T}$ and g has radial limits with modulus one at every $z \in M_f$.

5. Applications: obstructive results for spear vectors, spear operators, and Banach spaces with numerical index one

The aim of this section is to use the results in Section 2 together with the mix of ideas from numerical ranges and BJ-orthogonality to obtain obstructive results for the existence of spear vectors, spear operators and, in particular, for the possibility of having $n(X) = 1$ for a Banach space X. Let us introduce here some notation which will be used along this section. Let Z be a Banach space. We write $\operatorname{Smooth}(Z)$ to denote the set of smooth points of Z. For $z \in Z$, $z^\perp = \{x \in Z \colon z \perp_B x\}$ and ${}^\perp z = \{x \in Z \colon x \perp_B z\}$. Finally, $\operatorname{StrExp}(B_Z)$ denotes the set of *strongly exposed points* of B_Z: $z_0 \in \operatorname{StrExp}(B_Z)$ if there is $f_0 \in S_{Z^*}$ such that whenever $\lim \operatorname{Re} f_0(z_n) = 1$ for $\{z_n\} \subset B_Z$, it follows that $\lim z_n = z_0$ in norm.

5.1. **Spear vectors.** Let us first give some notation. Let Z be a Banach space and let $u \in S_Z$. The *numerical radius* of $z \in Z$ with respect to (Z, u) is

$$v(Z, u, z) := \sup\{|\lambda| \colon \lambda \in V(Z, u, z)\} = \sup\{|\phi(z)| \colon \phi \in \mathrm{F}(B_{Z^*}, u)\},$$

which is a seminorm on Z satisfying $v(Z,u,z) \leqslant \|z\|$ for every $z \in Z$. When $v(Z,u,\cdot)$ is a norm in Z, we say that u is a *vertex*. When $v(Z,u,z) = \|z\|$ for every $z \in Z$, u is said to be a *spear vector*. It is known that u is a spear vector if and only if

$$\max_{\theta\in\mathbb{T}} \|u+\theta z\| = 1 + \|z\| \quad \forall z \in Z.$$

We write $\operatorname{Spear}(Z)$ for the set of spear vectors of Z. A lot of information on spear vectors can be found in Chapter 2 of the book [15].

Consider a Banach space Z and a vertex $u \in S_Z$, and let us consider Z endowed with the norm v_u given by the numerical radius with respect to u:

$$v_u(z) := v(Z,u,z) = \sup\{|\phi(z)| \colon \phi \in \mathrm{F}(B_{Z^*},u)\} \qquad (z \in Z).$$

Then, we can consider its dual space $(Z,v_u)^*$ consisting of the linear functionals $\psi\colon Z \longrightarrow \mathbb{K}$ satisfying

$$\sup\{|\psi(y)| \colon y \in Z,\, v_u(y) \leqslant 1\} < \infty,$$

endowed with the norm

$$v_u^*(\psi) := \sup\{|\psi(y)| \colon y \in Z,\, v_u(y) \leqslant 1\} \qquad (\psi \in (Z,v_u)^*).$$

For $x \in Z$ with $v_u(x) = 1$, the numerical range of $y \in Z$ with respect to the numerical range space $\big((Z,v_u),x\big)$ is

$$V\big((Z,v_u),x,y\big) = \{\psi(y) \colon \psi \in (Z,v_u)^*,\ v_u^*(\psi) = \psi(x) = 1\}.$$

Our obstructive result for spear vectors will follow from the next result.

Theorem 5.1. *Let Z be a Banach space and let $u \in S_Z$ be a vertex of Z. If z is smooth in (Z,v_u), then $z \not\perp_B^{v_u} u$.*

A technical part of the proof is contained in the following lemma which could be of independent interest.

Lemma 5.2. *Let Z be a Banach space, let $u \in S_Z$ be a vertex, and let $z \in Z$ with $v_u(z) = 1$. Then, $V\big((Z,v_u),z,u\big) \cap \mathbb{T} \neq \emptyset$.*

Proof. Since $v_u(z) = 1$, there exists $\phi_0 \in S_{Z^*}$ and $\theta_0 \in \mathbb{T}$ such that $\phi_0(u) = \theta_0$ and $\phi_0(z) = 1$. We claim that $\phi_0 \in (Z,v_u)^*$ and $v_u^*(\phi_0) = 1$. Indeed, fix $y \in Z$ with $v_u(y) \leqslant 1$. As $\overline{\theta_0}\phi_0(u) = 1$, we have that $\overline{\theta_0}\phi_0(y) \in V(Z,u,y)$, hence $|\phi_0(y)| \leqslant v(Z,u,y) = v_u(y) \leqslant 1$. This shows that $\phi_0 \in (Z,v_u)^*$ and

$$v_u^*(\phi_0) = \sup\{|\phi_0(y)| \colon y \in Z,\, v_u(y) \leqslant 1\} \leqslant 1.$$

On the other hand, since $v_u(u) = 1$, we have that $v_u^*(\phi_0) \geqslant |\phi_0(u)| = 1$.

This, together with $\phi_0(z) = 1$, gives that

$$\theta_0 = \phi_0(u) \in V\big((Z,v_u),z,u\big) = \{\psi(u) \colon \psi \in (Z,v_u)^*,\ v_u^*(\psi) = \psi(z) = 1\}. \qquad \square$$

We are now ready to present the pending proof.

Proof of Theorem 5.1. As z is a smooth point, we have that $z \neq 0$ so, being u a vertex, this implies that $v_u(z) \neq 0$. Now, $V\left((Z,v_u), \dfrac{z}{v_u(z)}, u\right)$ is a singleton set by Lemma 2.8 as the norm of (Z,v_u) is smooth at z, hence also at $\dfrac{z}{v_u(z)}$. Moreover, since $v_u\left(\dfrac{z}{v_u(z)}\right) = 1$, it follows from Lemma 5.2 that

$$V\left((Z, v_u), \frac{z}{v_u(z)}, u\right) = \{\theta_0\}$$

for some $\theta_0 \in \mathbb{T}$. Hence, $0 \notin V\left((Z, v_u), \dfrac{z}{v_u(z)}, u\right)$ so Proposition 1.2 gives that $\dfrac{z}{v_u(z)} \not\perp_B^{v_u} u$ and hence $z \not\perp_B^{v_u} u$. □

We are ready to obtain the promised obstructive result for spear vectors.

Corollary 5.3. *Let Z be a Banach space and $u \in S_Z$. If there exists a smooth point z_0 in Z such that $z_0 \perp_B u$, then (Z, v_u) is not isometrically isomorphic to Z. In particular, u is not a spear vector or, in other words,*

$$\left(\bigcup_{z \in \mathrm{Smooth}(Z)} z^{\perp}\right) \bigcap \mathrm{Spear}(Z) = \emptyset \quad \textit{and} \quad \mathrm{Smooth}(Z) \bigcap \left(\bigcup_{z \in \mathrm{Spear}(Z)} {}^{\perp}z\right) = \emptyset.$$

Proof. Suppose on the contrary that (Z, v_u) is isometrically isomorphic to Z. Since z_0 is smooth in Z and $z_0 \perp_B u$, we have that z_0 is smooth in (Z, v_u) and $z_0 \perp_B^{v_u} u$, which contradicts Theorem 5.1. If u is a spear vector, then the identity map $\mathrm{Id}\colon (Z, \|\cdot\|) \longrightarrow (Z, v_u)$ is an isometric isomorphism. □

5.2. **Spear operators.** In the case when $Z = \mathcal{L}(X, Y)$ for some Banach spaces X and Y, spear vectors are called *spear operators*, which were introduced in [2] and have been deeply studied in [15], where we refer for more information and background.

Our aim here is to particularize Corollary 5.3 for the numerical radius with respect to an operator and for spear operators. The results follow directly from the above ones, but we include some particular notation for this case. Given Banach spaces X and Y, and $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$, the *numerical radius of $T \in \mathcal{L}(X, Y)$ with respect to G* is

$$\begin{aligned} v_G(T) := v\big(\mathcal{L}(X, Y), G, T\big) &= \inf_{\delta > 0} \sup\big\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(Gx) > 1 - \delta\big\} \\ &= \sup\big\{\lim |y_n^*(Tx_n)| \colon \{y_n^*\} \subset S_{Y^*},\, \{x_n\} \subset S_X,\, \lim y_n^*(Gx_n) = 1\big\}, \end{aligned}$$

where the second and third equalities hold by [16, Proposition 2.14] and our Theorem 2.3, respectively. We refer to [16] for background on numerical radius with respect to an operator.

The main result of the previous subsection in this setting reads as follows.

Corollary 5.4. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$. If there exists a smooth operator T in $\mathcal{L}(X, Y)$ such that $T \perp_B G$, then $\big(\mathcal{L}(X, Y), v_G\big)$ is not isometrically isomorphic to $\mathcal{L}(X, Y)$. In particular, G is not a spear operator or, in other words,*

$$\left(\bigcup_{T \in \mathrm{Smooth}(\mathcal{L}(X,Y))} T^{\perp}\right) \cap \mathrm{Spear}(\mathcal{L}(X, Y)) = \emptyset \quad \textit{and} \quad \mathrm{Smooth}(\mathcal{L}(X, Y)) \cap \left(\bigcup_{G \in \mathrm{Spear}(\mathcal{L}(X,Y))} {}^{\perp}G\right) = \emptyset.$$

Our next aim is to provide an obstructive result for the existence of spear operators which uses the geometry of the domain and range spaces instead of the geometry of the space of operators and so it would be easier to apply. Other restrictions on the geometry of the domain and range spaces to the existence of spear operators can be found in [15, Ch. 6].

Corollary 5.5. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. Suppose that there is $x_0 \in \mathrm{StrExp}(B_X)$ and $u_0 \in \mathrm{Smooth}(Y)$ satisfying that $u_0 \perp_B Gx_0$. Then, $\big(\mathcal{L}(X,Y), v_G\big)$ is not isometrically isomorphic to $\mathcal{L}(X,Y)$. In particular, G is not a spear operator. As a consequence, if X is a Banach space with $\mathrm{StrExp}(B_X) \neq \emptyset$ and Y is a smooth Banach space with dimension at least two, then there are no spear operators in $\mathcal{L}(X,Y)$.*

Observe that the last assertion of this result extends [15, Proposition 6.5.a] when X contains strongly exposed points (in particular, when X has the RNP) and provides a partial answer to [15, Problem 9.12].

To state the proof of this corollary from Corollary 5.4, we need to construct smooth operators orthogonal to a given one under mild restrictions. This can be done easily as a consequence of our Proposition 3.14.

Lemma 5.6. *Let X, Y be Banach spaces and suppose that $x_0 \in B_X$ is strongly exposed by $x_0^* \in S_{X^*}$. Given $A \in \mathcal{L}(X,Y)$, suppose that there is a smooth point $u_0 \in Y$ satisfying $u_0 \perp_B Ax_0$. Then, the operator $T \in \mathcal{L}(X,Y)$ given by $T(x) = x_0^*(x)u_0$ is smooth and satisfies $T \perp_B A$.*

Proof. Observe that T clearly satisfies the hypotheses of Proposition 3.14 so it is a smooth operator. Besides, using that $u_0 \perp_B Ax_0$, we have that

$$\|T + \lambda A\| \geqslant \|Tx_0 + \lambda Ax_0\| = \|u_0 + \lambda Ax_0\| \geqslant \|u_0\| = \|T\|$$

for every $\lambda \in \mathbb{K}$. Consequently, $T \perp_B A$. □

An immediate consequence of the previous result is the next remark which can be interesting by itself.

Remark 5.7. Let X be a Banach space with $\mathrm{StrExp}(B_X) \neq \emptyset$ and let Y be a smooth Banach space of dimension at least two. Then, for every $A \in \mathcal{L}(X,Y)$ there is a smooth operator $T \in \mathcal{L}(X,Y)$ satisfying $T \perp_B A$.

Proof of Corollary 5.5. The first part follows immediately from Corollary 5.4 by just using Lemma 5.6. The second assertion follows from Corollary 5.4 and Remark 5.7. □

Let us write the first part of Corollary 5.5 in a more suggestive way.

Corollary 5.8. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$ be a spear operator. Then,*

$$\left(\bigcup_{y \in \mathrm{Smooth}(Y)} y^{\perp} \right) \cap G(\mathrm{StrExp}(B_X)) = \emptyset \quad \textit{and} \quad \mathrm{Smooth}(Y) \cap \left(\bigcup_{x \in \mathrm{StrExp}(B_X)} {}^{\perp}(Gx) \right) = \emptyset.$$

5.3. **Banach spaces with numerical index one.** We finally particularize the results of the previous subsection to the case when $X = Y$ and $G = \mathrm{Id}_X$. In this case, we use the usual notation $v(\cdot)$ for the numerical radius (instead of v_{Id}) which was introduced in Subsection 3.6. We need the following notation. The *numerical index* of a Banach space X is defined by

$$n(X) := \inf\{v(T) \colon T \in S_{\mathcal{L}(X)}\}.$$

Equivalently, $n(X)$ is the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for every $T \in \mathcal{L}(X)$. Note that $0 \leqslant n(X) \leqslant 1$ and $n(X) > 0$ if and only if $v(\cdot)$ and $\|\cdot\|$ are equivalent norms on $\mathcal{L}(X)$. The case $n(X) = 1$ is equivalent to the fact that Id_X is a spear operator and we say that X is a

Banach space with numerical index one or that X has *numerical index one*. We refer the reader to the expositive paper [17] and to Chapter 1 of the already cited book [15] for an overview of classical and recent results on Banach spaces with numerical index one. Let us mention that some isomorphic and isometric restrictions on a Banach space X to have numerical index one are known: X^* cannot be smooth nor strictly convex [14, Theorem 2.1] and, in the *real* infinite-dimensional case, X^* contains a copy of ℓ_1 [3, Corollary 4.9]. It is open, as far as we know, whether the latter result extends to the complex case and whether a Banach space with numerical index one can be smooth or strictly convex ([14] or [15, Problem 9.12]). The particularization of the results of the previous subsection to the case of the identity reads as follows.

Corollary 5.9. *Let X be a Banach space. If there is $x_0 \in \operatorname{StrExp}(B_X)$ and $u_0 \in \operatorname{Smooth}(X)$ such that $u_0 \perp_B x_0$, then X does not have numerical index one.*

This result can be written in the following more suggestive way:

Corollary 5.10. *Let X be a Banach space with numerical index one. Then,*

$$\left(\bigcup_{x\in\operatorname{Smooth}(X)} x^{\perp}\right) \cap \operatorname{StrExp}(B_X) = \emptyset \quad \textit{and} \quad \operatorname{Smooth}(X) \cap \left(\bigcup_{x\in\operatorname{StrExp}(B_X)} {}^{\perp}x\right) = \emptyset.$$

The above result provides a necessary condition to have numerical index one for a Banach space in the way that was asked in [17, Problem 11]: *Find necessary and sufficient conditions for a Banach space to have numerical index one which do not involve operators.*

The next is a consequence of Corollary 5.9 which gives a partial answer to the question of whether there is a smooth Banach space with numerical index one.

Corollary 5.11. *Let X be a smooth Banach space of dimension at least two such that* $\operatorname{StrExp}(B_X) \neq \emptyset$. *Then, X does not have numerical index one.*

This applies, in particular, when X has the RNP.

Corollary 5.12. *Let X be a smooth Banach space of dimension at least two having the RNP. Then, X does not have numerical index one.*

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Conclusions and open problems

The aim of **Chapter I** was to deeply study the numerical index with respect to an operator between Banach spaces.

Section I.2 was devoted to presenting some known and new results on abstract numerical index. Among the new results presented, we want to highlight Proposition I.2.11, which shows that the set of points $u \in S_Z$ satisfying $n(Z,u) > 0$ is countable when Z is a finite-dimensional real space. It is natural to wonder whether the same remains true in the complex case.

Problem 1. Let Z be a finite-dimensional complex Banach space. Is the set $\{u \in S_Z \colon n(Z,u) > 0\}$ countable up to rotations?

Furthermore, since in the proof of Proposition I.2.11 we use the separability of the dual space Z^*, we wonder if this is enough to obtain the result.

Problem 2. Let Z be a Banach space with a separable dual. Is the set $\{u \in S_Z \colon n(Z,u) > 0\}$ countable up to rotations?

Additonally, we gave estimations on the sum of $n(Z,u)$ over all elements $u \in S_Z$ and showed the existence of a (real or complex) Banach space Z such that $\{n(Z,u) \colon u \in S_Z\} = A$ for every subset $A \subseteq [0,1]$ containing 0. To finish this section, we presented a new expression of $V(Z,u,z)$ which has proved to be useful to compute numerical radii with respect to operators.

In Section I.3 we provided some tools for studying the numerical index with respect to an operator. The first results were direct translations to the operator space setting of the abstract results in the previous section. Then, we showed that the numerical index with respect to an operator dominates the numerical index with respect to its adjoint, we provided a formula for the numerical index with respect to a rank-one operator, and gave some estimations on the numerical index with respect to an operator in terms of the numerical radii of operators on the domain or on the codomain.

Next, we dedicated Section I.4 to study the set $\mathcal{N}(\mathcal{L}(X,Y))$ of values of the numerical indices with respect to all norm-one operators between two given Banach spaces X and Y. First, we obtained some consequences of the results in the previous sections, namely $0 \in \mathcal{N}(\mathcal{L}(X,Y))$ unless both X and Y are one-dimensional and the set $\mathcal{N}(\mathcal{L}(X,Y))$ is countable when X and Y are finite-dimensional real spaces. Then, we provided several examples of spaces having trivial set of values of the numerical indices with respect to operators. For instance, $\mathcal{N}(\mathcal{L}(X,Y)) = \{0\}$ when X or Y is a real Hilbert space of dimension at least two and also when X or Y is $\mathcal{L}(H)$, where H is an infinite-dimensional real Hilbert space. Moreover, the last part of Theorem I.4.7 states that the set $\mathcal{N}(\mathcal{L}(\mathcal{L}(H),Y))$ is also trivial for every Banach space Y when H is a real Hilbert space with even dimension, so the following question arises naturally.

Problem 3. Let H be a real Hilbert space with a finite odd dimension greater than 1. Is it true that $\mathcal{N}(\mathcal{L}(\mathcal{L}(H),Y)) = \{0\}$ for every Banach space Y?

Let us comment that the dependence on the parity of the dimension is due to the identification $H = \left[\oplus_{\lambda\in\Lambda}\ell_2^2\right]_{\ell_2}$ for a suitable index set Λ used in the proof. We do not know if there is a general argument that works for both cases.

We also gave some inclusions for the set of numerical indices with respect to operators whose domain or codomain is a real ℓ_p-space. More precisely, Proposition I.4.11 shows that

$$\mathcal{N}(\mathcal{L}(X,\ell_p)) \subseteq [0, M_p] \quad \text{and} \quad \mathcal{N}(\mathcal{L}(\ell_p,Y)) \subseteq [0, M_p]$$

for $1 < p < \infty$ and for all real Banach spaces X and Y, where $M_p = \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$.

Problem 4. Let $1 < p < \infty$ and $1 < q < \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$, is it true that

$$\mathcal{N}(\mathcal{L}(X,\ell_p)) \subseteq \left[0, \frac{1}{p^{1/p}q^{1/q}}\right] \quad \text{and} \quad \mathcal{N}(\mathcal{L}(\ell_p,Y)) \subseteq \left[0, \frac{1}{p^{1/p}q^{1/q}}\right]$$

in the complex case for all Banach spaces X and Y?

A similar argument to the one given in the proof of Proposition I.4.11 is not valid in the complex case since the operator $A \in \mathcal{L}(\ell_p^2)$ given by $A(x,y) = (0,x)$, which satisfies that $v(A) = \frac{1}{p^{1/p}q^{1/q}}$, is not a surjective isometry.

For complex Hilbert spaces H_1, H_2 with dimension at least 2, $\mathcal{N}(\mathcal{L}(H_1,H_2)) = \{0, 1/2\}$ if H_1 and H_2 are isometrically isomorphic and $\mathcal{N}(\mathcal{L}(H_1,H_2)) = \{0\}$ otherwise. Moreover, we proved in Proposition I.4.13 that for a complex Hilbert space H with dimension greater than 1, $\mathcal{N}(\mathcal{L}(X,H)) \subseteq [0,1/2]$ and $\mathcal{N}(\mathcal{L}(H,Y)) \subseteq [0,1/2]$ for all complex Banach spaces X and Y. Notice that these inclusions are proved using Proposition I.3.9 and the only values that we are certain that can belong to $\mathcal{N}(\mathcal{L}(X,H))$ and $\mathcal{N}(\mathcal{L}(H,Y))$ are 0 and 1/2. It seems natural to wonder if the rest of values of the interval $[0,1/2]$ can also be contained in $\mathcal{N}(\mathcal{L}(X,H))$ and $\mathcal{N}(\mathcal{L}(H,Y))$.

Problem 5. Does there exist an operator G whose domain or codomain is a complex Hilbert space with dimension greater than 1 such that the numerical index with respect to G is different from 0 and 1/2?

We also studied the set of values of the numerical indices with respect to operators whose domain and codomain are $C(K)$ spaces, and proved that $\mathcal{N}(\mathcal{L}(C(K_1),C(K_2))) = \{0,1\}$ for many families of compact Hausdorff topological spaces K_1 and K_2, both in the real and complex case. As a consequence, we obtained that $\mathcal{N}(\mathcal{L}(L_\infty(\mu_1),L_\infty(\mu_2))) \subseteq \{0,1\}$ and $\mathcal{N}(\mathcal{L}(L_1(\mu_1),L_1(\mu_2))) \subseteq \{0,1\}$ for all σ-finite measures μ_1 and μ_2.

Using the tools presented in Section I.3, we proved in Section I.5 that the concept of Lipschitz numerical range for Lipschitz self-maps of a Banach space is a particular case of numerical range with respect to a linear operator between two different Banach spaces.

The last section of this chapter was dedicated to presenting several results which show the behaviour of the value of the numerical index when we apply some Banach space operations. For instance, we showed that the numerical index of a c_0-, ℓ_1- or ℓ_∞-sum of Banach spaces with respect to a direct sum of norm-one operators in the corresponding spaces coincides with the infimum of the numerical indices of the corresponding summands. As an important consequence, we obtained in Theorem I.6.4 the existence of real and complex Banach spaces X for which $\mathcal{N}(\mathcal{L}(X)) = [0,1]$. We also showed that a composition operator between vector-valued function spaces $C(K,X)$, $L_1(\mu,X)$ and $L_\infty(\mu,X)$ produces the same numerical index as the original operator. Next, we provided conditions ensuring that the numerical index with respect to an operator coincides with the numerical index with respect to its adjoint, namely when the codomain is L-embedded or when the operator has rank-one. Finally, we discussed the numerical index with respect to the composition of two operators and showed how to extend the domain and the codomain of an operator maintaining the value of the numerical index. In particular, these results allowed to solve a part of Problem 9.14 posed in [20].

Chapter II was dedicated to analysing the behaviour of the numerical index of operator ideals and tensor products, and to studying the Daugavet property in tensor products. We began Section II.2 showing that for every operator ideal $\mathcal{Z}(X,Y)$ of $\mathcal{L}(X,Y)$ endowed with the operator norm we have that $n(\mathcal{Z}(X,Y)) \leqslant \min\{n(X), n(Y)\}$. Then, with the help of suitable representations, we were able to give stronger inequalities for the numerical indices of the spaces of compact and weakly compact operators, namely $n(\mathcal{K}(X,Y)) \leqslant \min\{n(X^*), n(Y)\}$ and $n(\mathcal{W}(X,Y)) \leqslant \min\{n(X^*), n(Y)\}$. As a consequence of this result, we presented some interesting examples such as the existence of a real Banach space X with $n(X) = 1$ while $n(\mathcal{K}(X,Y)) = n(\mathcal{W}(X,Y)) = 0$ for every Banach space Y. In particular, $n(X) = 1$ while $n(\mathcal{K}(X,X)) = n(\mathcal{W}(X,X)) = 0$. We also provided an example to show that the previous inequalities can be strict and discuss some cases in which the equality holds.

For tensor products of Banach spaces, we proved in Section II.3 that the numerical indices of $X \hat{\otimes}_\pi Y$ and $X \hat{\otimes}_\varepsilon Y$ are less than or equal to the minimum of $n(X)$ and $n(Y)$. Next, we obtained some consequences for the spaces of approximable and nuclear operators using representation theorems. More specifically, we proved that $n(\mathcal{A}(X,Y)) \leqslant \min\{n(X^*), n(Y)\}$, and, if X^* or Y has the approximation property, $n(\mathcal{N}(X,Y)) \leqslant \min\{n(X^*), n(Y)\}$.

To finish this chapter, we devoted a section to studying when the Daugavet property is transferred from the tensor product to the factors. The main result in this line is Theorem II.4.1, which states that the Daugavet property of a projective tensor product passes to one of the factors if the unit ball of the other one is a slicely countably determined set. However, we do not know if the corresponding result for the injective tensor product is true.

Problem 6. Let X, Y be Banach spaces. Suppose that B_Y is an slicely countably determined set and $X \hat{\otimes}_\varepsilon Y$ has the Daugavet property. Does X have the Daugavet property?

The difficulty in this case is that we do not have a clear representation of the unit ball of $X \hat{\otimes}_\varepsilon Y$ and we were not able to give an analogous result to Lemma II.4.2, which was the key to prove Theorem II.4.1.

Finally, we provided other positive results: for projective tensor products, in the case where the space Y^* has a point of Fréchet differentiability of the norm, and for injective tensor products, when the space Y has a point of Fréchet differentiability of the norm.

The next two chapters were related to the computation of the numerical index of L_p spaces when $p \neq 1, 2, \infty$, which remains as an important open problem since the beginning of the theory.

In **Chapter III**, we addressed the problem of calculating the numerical index of two-dimensional real spaces endowed with an absolute and symmetric norm. More specifically, Theorem III.2.2 gives a lower bound for the numerical index of such spaces and shows that, in many instances, the numerical index is attained at the operator represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. As a major consequence, we proved that $n(\ell_p^2) = M_p = \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$ for $3/2 \leqslant p \leqslant 3$ in the real case (see Theorem III.2.3).

Since this abstract approach did not provide a complete solution for the problem of calculating the numerical index of the real ℓ_p^2 spaces, we explored an alternative method in **Chapter IV**. The main difference there was the use of Riesz–Thorin interpolation theorem to estimate the norm of operators on ℓ_p^2. This allowed us to show that $n(\ell_p^2) = M_p$ for $6/5 \leqslant p \leqslant 3/2$ and $2 \leqslant p \leqslant 6$ in the real case (see Theorem IV.2.2).

Let us comment that our procedures have been completely exploited. On the one hand, it was proved in [37] that condition $c_4 \left(1 + \frac{1}{c_2} + \frac{1}{c_3}\right) \leqslant 1$ in Theorem III.2.2 holds for a wider range of values of p, specifically for $1 + \alpha_0 \leqslant p \leqslant \alpha_1$, where α_0 is the root of $f(x) = 1 + x^{-2} - (x^{-\frac{1}{x}} + x^{\frac{1}{x}})$ and $\frac{1}{1+\alpha_0} + \frac{1}{\alpha_1} = 1$ ($\alpha_0 \approx 0.4547$). However, the range of values of p cannot be enlarged much more. Indeed, let $p = 1.454$ and $t_0 \in]0, 1[$ be such that $M_p = \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p} = \frac{|t_0^{p-1}-t_0|}{1+t_0^p}$, then numerical computations give $t_0 \approx 0.17646$ and

$$c_1 = 1, \quad c_2 = \frac{1-t_0^p}{1+t_0^p} \approx 0.851367, \quad c_3 = \frac{t_0^{p-1}+t_0}{1+t_0^p} \approx 0.584498, \quad c_4 = \frac{t_0^{p-1}-t_0}{1+t_0^p} \approx 0.257807,$$

therefore $c_4 \left(1 + \frac{1}{c_2} + \frac{1}{c_3}\right) \approx 1.0017 > 1$. On the other hand, the techniques used in the proof of Theorem IV.2.2 can give the equality $n(\ell_p^2) = M_p$ for a slightly wider range of values of p, however it does not work for p close to 1 (see Remark IV.2.3). It is for these reasons that we need to adopt a different approach to the problem in order to get a solution for the remaining values of p.

Problem 7. Does the equality $n(\ell_p^2) = M_p = \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$ hold for $1 < p < \frac{6}{5}$ and $p > 6$ in the real case?

It is also worth noting that our arguments heavily rely on the assumption that the scalar field is real, which limits their applicability in the complex case. It was conjectured in [23] that the numerical index of the complex ℓ_p^2 space is attained at the operator $S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{L}(\ell_p^2)$, which satisfies that $v(S) = \frac{1}{p^{1/p}q^{1/q}}$.

Problem 8. Is it true that, in the complex case, $n(\ell_p^2) = \frac{1}{p^{1/p}q^{1/q}}$ for every $1 < p < \infty$?

Let us highlight that we will obtain the numerical index of ℓ_p for $p \neq 1, 2, \infty$ if we solve the following problem.

Problem 9. Compute the numerical index of ℓ_p^m for every $m \geqslant 2$ and every $1 < p < \infty$.

In **Chapter V** we introduced a new seminorm on the space of bounded linear operators, the relative norm, and studied its relation with the usual norm of operators. The special case when they coincide motivated the definition of generating operator. We began Section V.2 with Proposition V.2.1, which established the connection between the relative norm and the operator norm in several ways. As a consequence, we obtained a characterization of generating operators in terms of the sets $\operatorname{att}(G,\delta)$: A norm-one operator $G \in \mathcal{L}(X,Y)$ is generating if and only if $\overline{\operatorname{conv}}(\operatorname{att}(G,\delta)) = B_X$ for every $\delta > 0$ (see Corollary V.2.3). Thanks to this result, it is clear that the property of being generating does not depend on the codomain. When X is reflexive and G is compact, this is also equivalent to the fact that $B_X = \overline{\operatorname{conv}}(\operatorname{att}(G))$. Additionally, we related the concept of generating operator with the one of denting point, proving that any generating operator $G \in \mathcal{L}(X,Y)$ attains its norm at every denting point of B_X. In fact, this necessary condition is also sufficient when B_X is the closed convex hull of its denting points and, in particular, when X has the Radon-Nikodým property. We gave another useful characterization in Corollary V.2.17 which involves the geometry of the dual space of the domain: G is generating if and only if $G^*(B_{Y^*})$ is a spear set of X^*. We also analysed the behaviour of generating operators when applying the operation of taking adjoint and showed that this property does not pass in general from an operator to its adjoint, nor the other way around. Nevertheless, we showed in Proposition V.2.22 that if G^{**} is r-generating, then so is G. We do not know if the converse holds in general or even for $r = 1$.

Problem 10. Let X, Y be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator, and let $r \in (0,1]$. If G is r-generating, does it imply that G^{**} is r-generating?

We finished this section studying the stability of generating operators by taking c_0-, ℓ_1-, and ℓ_∞-sums, and providing some examples in classical Banach spaces.

We devoted Section V.3 to discussing the relationship between generating operators and norm-attainment. We provided an example of a rank-two operator which is generating but does not attain its norm. This differs from the case of rank-one operators, which are always norm-attaining. In addition, we proved in Theorem V.3.5 that, given a Banach space X, we may find a Banach space Y and an operator $G \in \mathcal{L}(X,Y)$ which is generating but not norm-attaining if and only if there exists a spear set $\mathcal{B}$ in X^* such that $\sup\limits_{x^* \in \mathcal{B}} |x^*(x)| < 1$ for every $x \in S_X$.

In Section V.4, we considered the set $\operatorname{Gen}(X,Y)$ of all generating operators between two Banach spaces X and Y. We showed that this set is closed and that for every Banach space Y, there exists a Banach space X such that $\operatorname{Gen}(X,Y) = \emptyset$. However, this result is not longer true if we restrict the space X to be separable. Then, we focused on some properties of $\operatorname{Gen}(X,Y)$ when X is fixed. We proved that $\operatorname{Gen}(X,Y) \neq \emptyset$ for every Y if and only if $\operatorname{Spear}(X^*) \neq \emptyset$ and that only one-dimensional spaces X can satisfy $\operatorname{Gen}(X,Y) = S_{\mathcal{L}(X,Y)}$ for some Banach space Y. Furthermore, we studied when the set $\operatorname{Gen}(X,Y)$ generates the unit ball of $\mathcal{L}(X,Y)$ by closed convex hull. In this sense, Theorem V.4.10 states that every representable operator in the unit ball of $\mathcal{L}(L_1(\mu),Y)$ belongs to the closed convex hull of $\operatorname{Gen}(L_1(\mu),Y)$, where μ is a finite measure and Y is an arbitrary Banach space. Consequently, $B_{\mathcal{L}(L_1(\mu),Y)} = \overline{\operatorname{conv}}\,(\operatorname{Gen}(L_1(\mu),Y))$ if Y has the Radon-Nikodým property. Moreover, $B_{\mathcal{L}(\ell_1(\Gamma),Y)} = \overline{\operatorname{conv}}\,(\operatorname{Gen}(\ell_1(\Gamma),Y))$ for every Banach space Y and the only *real* finite-dimensional spaces with this property are ℓ_1^n for $n \in \mathbb{N}$. We do not know if the same it is true for complex finite-dimensional spaces.

Problem 11. Let X be a *complex* Banach space with $\dim(X) = n$ and such that $B_{\mathcal{L}(X,Y)} = \overline{\text{conv}}(\text{Gen}(X,Y))$ for every Banach space Y. Can we deduce that $X = \ell_1^n$?

The main aim of **Chapter VI** was to provide a widely applicable approach to address Birkhoff-James orthogonality using the following connection between this concept and numerical range: given two elements x and y in a Banach space Z,

$$x \perp_B y \iff 0 \in V(Z, u, z).$$

We began Section VI.2 showing that it is also possible to express the numerical range in terms of Birkhoff-James orthogonality. Then, we provided in Theorem VI.2.4 different expressions of the numerical range which extend a previous result from Chapter I. This was the key to prove Corollary VI.2.6, the main result of this section, which characterizes Birkhoff-James orthogonality in a Banach space in terms of the actions of functionals on an arbitrary one-norming subset. One of the statements in this result is the following: let Z be a Banach space, and $\Lambda \subset B_{Z^*}$ be one-norming for Z; then, for $u \in S_Z$ and $z \in Z$,

$$u \perp_B z \iff 0 \in \text{conv}\left(\left\{\lim \psi_n(z)\overline{\psi_n(u)} \colon \psi_n \in \Lambda, \lim |\psi_n(u)| = 1\right\}\right).$$

Additionally, we gave several characterizations of smooth points following the same spirit in Corollary VI.2.11.

We considered in Section VI.3 a number of particular cases in which the results of Section VI.2 are applicable. Some of the results in this section were already known. Nevertheless, the techniques previously used to prove them depended on the particular case, while our approach was unified for all of them. The new results included general characterizations of Birkhoff-James orthogonality and smoothness in $\ell_\infty(\Gamma, Y)$, where Γ is a non-empty set and Y is an arbitrary Banach space (see Theorem VI.3.2 and Corollary VI.3.3). As consequences, we obtained new applications for spaces of vector-valued continuous functions, uniform algebras, polynomials, Lipschitz maps, and injective tensor products. We also presented several results for the space of bounded linear operators endowed with the operator norm and with the numerical radius, most of them were previously known but there are some improvements for compact operators. It is natural to wonder if it is possible to give similar characterizations of Birkhoff-James orthogonality in the space of bounded linear operators endowed with the numerical radius with respect to an operator or with the relative norm, however such characterizations cannot be deduced immediately from Corollary VI.2.6 since we do not have one-norming subsets for $\mathcal{L}(X,Y)$ endowed with $v_G(\cdot)$ or $\|\cdot\|_G$.

Problem 12. Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. Find characterizations of Birkhoff-James orthogonality in $(\mathcal{L}(X,Y), v_G(\cdot))$ and $(\mathcal{L}(X,Y), \|\cdot\|_G)$ in terms of the elements in the domain, codomain, and their duals.

In Section VI.4, we presented some cases in which it is possible to remove the convex hull and the limits when characterizing Birkhoff-James orthogonality. The main result in this section, Theorem VI.4.3, was a Bhatia-Šemrl's type of result in the space of vector-valued continuous functions on a compact Hausdorff topological space when the norm attaiment set of the function involved is connected. As a consequence, we obtained analogous results for compact operators on reflexive Banach spaces, which was new in the context of complex infinite-dimensional spaces. We finished this section with a nice characterization of Birkhoff-James orthogonality for finite Blaschke products.

Finally, we devoted the last section of Chapter VI to applications to the study of spear vectors, spear operators, and Banach spaces with numerical index one. They all follow from Theorem VI.5.1 which connects the concepts of vertex, smoothness, and Birkhoff-James orthogonality with respect to the abstract numerical radius. As a consequence, we proved in Corollary VI.5.3 that no smooth point of a Banach space Z can be Birkhoff-James orthogonal to a spear vector of Z. Restricting to the case $Z = \mathcal{L}(X, Y)$, we obtained obstructive results for the existence of spear operators. In particular, Corollary VI.5.5 states that if X is a Banach space with strongly exposed points and Y is a smooth Banach space with dimension at least two, then there are no spear operators in $\mathcal{L}(X, Y)$. This result somehow extends [20, Proposition 6.5.a] and partially answers [20, Problem 9.12]. Particularizing this result to the identity operator, we obtained an obstructive condition for a Banach space to have numerical index one: the existence of a smooth point which is Birkhoff-James orthogonal to a strongly exposed point. In particular, smooth Banach spaces with dimension at least two containing strongly exposed points do not have numerical index one. This gives a partial answer to the question of whether a smooth Banach space of dimension at least two may have numerical index one [19, page 166].

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