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Solving inverse problems for mixed-variational equations on perforated domains

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Abstract
The aim of this work is to analyze some conditions for the existence of solution of a perturbed mixed variational system and that of an associated inverse problem related to the collage-based approach, both on perforated domains or domains with holes. In addition, we study the influence of the size of the holes and state some convergence results. Finally, we conduct a computational study for solving some of those inverse problems.

Keywords Perturbed mixed variational equations · Perforated domains · Parameter estimation · Inverse problems

Mathematics Subject Classification 65L10, 49J40, 65L09

1 Introduction

The systematic study of mixed variational problems goes back more than 50 years (Babuška, 1971; Brezzi 1974) and since then it has been revealed as a powerful technique for the study of partial differential equations. Moreover, its associated finite element methods, the mixed ones, constitute a fundamental tool for the numerical study of these problems (Boffi 2008; Garralda-Guillem and Ruiz Galán 2019). In this article we consider a variant of a system of mixed variational equations, when we introduce a certain perturbation of one of the equations and also allow the domain to contain holes, that is, the domain of the problem

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is perforated, the latter situation motivated by its enormous applications. The problem posed admits both a direct and an inverse approach, and we deal with both here. For the first of them, we design a Galerkin scheme, and for the second we establish a generalization of the classical collage theorem (Barnsley 1989) that, in this context, allows numerically approximating some considered inverse problems. Thus, for estimating some parameters in the model problem from known data (in practice, observations) we use a target element in a Banach space associated with the perturbed mixed problem and use the stability in a sense of the direct problem. In particular, we generalize previous works along these lines for ordinary and partial differential equations over solid and perforated domains (Berenguer et al. 2016; Kunze et al. 2004; Kunze and La Torre 2018, 2017, 2016, 2015; Kunze et al. 2015, 2010, 2009; Kunze and Vrscay 1999).

The paper is structured around 5 sections. In Sect. 2 we describe the Mixed Variational Equation considered and the stability conditions that will allow us to deal with a suitable inverse problem. In Sect. 3 we introduce the perforated domains considered and in Sects. 4 and 5 we analyze the relationship between the solutions of the direct and inverse problems on solid domains and on perforated domains, when the holes are small enough in a certain sense. We also illustrate the results with a numerical example. Finally, in Sect. 6 we include some conclusions.

2 Collage-type inverse problems for mixed variational equations

We discuss here a more general version of the classical system of mixed variational equations corresponding to the mixed variational formulation of a differential problem which includes a kind of perturbation. The perturbation term is modelled by means of a new bilinear form, that has to be interpreted to be small in some sense.

Suppose that E and F are real Hilbert spaces, $a : E \times E \longrightarrow \mathbb{R}$, $b : E \times F \longrightarrow \mathbb{R}$ and $c : F \times F \longrightarrow \mathbb{R}$ are bounded and bilinear and $x^* : E \to \mathbb{R}$ and $y^* : F \to \mathbb{R}$ are bounded and linear. Our problem reads as follows: Find $(x_0, y_0) \in E \times F$ such that

$$\begin{cases} a(x_0,\cdot) + b(\cdot, y_0) = x^*(\cdot) \\ b(x_0,\cdot) + c(y_0,\cdot) = y^*(\cdot) \end{cases} . \tag{P}$$

We use the following general result for a family of such problems that include a stability property, (2.1), which will be essential for our purposes, since it will allow us to deal with a suitable inverse problem. Furthermore, such a stability condition (2.1) is a *Generalized Collage Theorem* that extends those in Berenguer et al. (2016) and Kunze et al. (2009) in the Hilbertian framework.

Theorem 2.1 *Let E and F be real Hilbert spaces, Λ be a nonempty set and for each $\lambda \in \Lambda$, let $a_\lambda : E \times E \longrightarrow \mathbb{R}$, $b_\lambda : E \times F \longrightarrow \mathbb{R}$ and $c_\lambda : F \times F \longrightarrow \mathbb{R}$ be bounded and bilinear and let $K_\lambda := \{x \in E : \ b_\lambda(x,\cdot) = 0\}$ in such a way that*

(i) $x \in K_\lambda \ \wedge \ a_\lambda(x,\cdot)_{|K_\lambda} = 0 \Rightarrow \ x = 0$

and for some $\alpha_\lambda, \beta_\lambda > 0$ there hold

(ii) $x \in K_\lambda \ \Rightarrow \ \alpha_\lambda \|x\| \le \|a_\lambda(\cdot, x)_{|K_\lambda}\|$ *and*
(iii) $y \in F \ \Rightarrow \ \beta_\lambda \|y\| \le \|b_\lambda(\cdot, y)\|$.

If

$$\rho_\lambda := \max\left\{\frac{1}{\alpha_\lambda}, \frac{1}{\beta_\lambda}\left(1 + \frac{\|a_\lambda\|}{\alpha_\lambda}\right), \frac{1}{\beta_\lambda^2}\|a_\lambda\|\left(1 + \frac{\|a_\lambda\|}{\alpha_\lambda}\right)\right\}$$

and in addition

(iv) $\|c_\lambda\| < \dfrac{1}{\rho_\lambda}$,

then for each $\lambda \in \Lambda$ *and* $(x_\lambda^*, y_\lambda^*) \in E^* \times F^*$ *there exists a unique* $(x_\lambda, y_\lambda) \in E \times F$ *such that*

$$\begin{cases} a_\lambda(x_\lambda, \cdot) + b_\lambda(\cdot, y_\lambda) = x_\lambda^* \\ b_\lambda(x_\lambda, \cdot) + c_\lambda(y_\lambda, \cdot) = y_\lambda^* \end{cases}. \tag{P_λ}$$

Furthermore, if $(x, y) \in E \times F$, *then*

$$\max\{\|x_\lambda - x\|, \|y_\lambda - y\|\} \le \frac{\rho_\lambda}{1 - \rho_\lambda\|c_\lambda\|}\left(\|x_\lambda^* - a_\lambda(x, \cdot) - b_\lambda(\cdot, y)\| + \|y_\lambda^* - b_\lambda(x, \cdot) - c_\lambda(y, \cdot)\|\right). \tag{2.1}$$

Proof Let $\lambda \in \Lambda$. The existence and uniqueness of solution for problem (P_λ) is a well-known fact (see, for instance Boffi 2008, Proposition 4.3.2), but we give a sketch of the proof in order to derive also the control of the norms in (2.1) in a precise way. So, let us endow the product space $E \times F$ with the norm

$$\|(x, y)\| := \max\{\|x\|, \|y\|\}, \qquad (x \in E,\ y \in F)$$

and its dual space $E^* \times F^*$ with the corresponding dual norm, that is,

$$\|(x^*, y^*)\| := \|x^*\| + \|y^*\|, \qquad (x^* \in E^*,\ y^* \in F^*).$$

According to conditions (i), (ii) and (iii) and to Gatica (2014, Theorem 2.1), the bounded and linear operator $S_\lambda : E \times F \longrightarrow E^* \times F^*$ defined at each $(x, y) \in E \times F$ as

$$S_\lambda(x, y) := (a_\lambda(x, \cdot) + b_\lambda(\cdot, y), b_\lambda(x, \cdot))$$

is an isomorphism. But, in view of Atkinson and Han (2009, Theorem 2.3.5), in order to state the existence of a unique solution for the perturbed mixed system (P_λ) it is enough to show that

$$\|S_\lambda^{-1}\| < \frac{1}{\|c_\lambda\|}, \tag{2.2}$$

inequality which is valid, since in view of Garralda-Guillem and Ruiz Galán (2014, Theorem 3.6) and (iv), we have that

$$\begin{aligned}
\|S_\lambda^{-1}\| &= \sup_{\|x^*\|+\|y^*\|\le 1} \|S_\lambda^{-1}(x^*, y^*)\| \\
&\le \sup_{\|x^*\|+\|y^*\|\le 1} \max\left\{\frac{\|x^*\|}{\alpha_\lambda} + \frac{1}{\beta_\lambda}\left(1 + \frac{\|a_\lambda\|}{\alpha_\lambda}\right)\|y^*\|, \frac{1}{\beta_\lambda}\left(1 + \frac{\|a_\lambda\|}{\alpha_\lambda}\right)\left(\|x^*\| + \frac{\|a_\lambda\|}{\beta_\lambda}\|y^*\|\right)\right\} \\
&\le \sup_{\|x^*\|+\|y^*\|\le 1} \max\left\{\frac{1}{\alpha_\lambda}, \frac{1}{\beta_\lambda}\left(1 + \frac{\|a_\lambda\|}{\alpha_\lambda}\right), \frac{1}{\beta_\lambda}\left(1 + \frac{\|a_\lambda\|}{\alpha_\lambda}\right), \frac{\|a_\lambda\|}{\beta_\lambda^2}\left(1 + \frac{\|a_\lambda\|}{\alpha_\lambda}\right)\right\} \\
&\quad (\|x^*\| + \|y^*\|) \\
&\le \rho_\lambda \\
&< \frac{1}{\|c_\lambda\|}.
\end{aligned}$$

Furthermore, according to (2.2) and Atkinson and Han (2009, Theorem 2.3.5) or Garralda-Guillem and Ruiz Galán (2014, Theorem 3.6) once again, we arrive at

$$\max\{\|x_\lambda\|, \|y_\lambda\|\} \le \frac{\rho_\lambda}{1 - \rho_\lambda\|c_\lambda\|}\left(\|x^*\| + \|y^*\|\right), \tag{2.3}$$

where $(x_\lambda, y_\lambda) \in E \times F$ is the unique solution of (P_λ). To conclude, given $(\hat{x}_\lambda, \hat{y}_\lambda) \in E \times F$, since $(x_\lambda - \hat{x}_\lambda, y_\lambda - \hat{y}_\lambda)$ is the unique solution of the perturbed mixed problem

$$\begin{cases} a_\lambda(x_\lambda - \hat{x}_\lambda, \cdot) + b_\lambda(\cdot, y_\lambda - \hat{y}_\lambda) = x_\lambda^* - a_\lambda(\hat{x}_\lambda, \cdot) - b_\lambda(\cdot, \hat{y}_\lambda) \\ b_\lambda(x_\lambda - \hat{x}_\lambda, \cdot) + c_\lambda(y_\lambda - \hat{y}_\lambda, \cdot) = y_\lambda^* - b_\lambda(\hat{x}_\lambda, \cdot) - c_\lambda(\hat{y}_\lambda, \cdot) \end{cases},$$

then, according to inequality (2.3),

$$\max\{\|x_\lambda - \hat{x}_\lambda\|, \|y_\lambda - \hat{y}_\lambda\|\} \leq \frac{\rho_\lambda}{1 - \rho_\lambda \|c_\lambda\|} \Big(\|x_\lambda^* - a_\lambda(\hat{x}_\lambda, \cdot) \\ - b_\lambda(\cdot, \hat{y}_\lambda)\| + \|y_\lambda^* - b_\lambda(\hat{x}_\lambda, \cdot) - c_\lambda(\hat{y}_\lambda, \cdot)\| \Big).$$

Finally, the arbitrariness of $\lambda \in \Lambda$ yields (2.1). □

It is worth mentioning that if

$$\alpha := \inf_{\lambda \in \Lambda} \alpha_\lambda > 0, \quad \beta := \inf_{\lambda \in \Lambda} \beta_\lambda > 0, \quad \delta := \sup_{\lambda \in \Lambda} \|a_\lambda\|, \quad \gamma := \inf_{\lambda \in \Lambda} \|c_\lambda\| > 0$$

and

$$\rho := \max \left\{ \frac{1}{\alpha}, \frac{1}{\beta}\left(1 + \frac{\delta}{\alpha}\right), \frac{\delta}{\beta^2}\left(1 + \frac{\delta}{\alpha}\right) \right\},$$

then

$$\inf_{\lambda \in \Lambda} \max\{\|x_\lambda - x\|, \|y_\lambda - y\|\} \leq \frac{\rho}{1 - \rho\gamma} (\|x_\lambda^* - a_\lambda(x, \cdot) \\ - b_\lambda(\cdot, y)\| + \|y_\lambda^* - b_\lambda(x, \cdot) - c_\lambda(y, \cdot)\|).$$

Therefore, in order to approximate the solution of the corresponding inverse problem we solve the optimization problem

$$\min_{\lambda \in \Lambda} (\|x_\lambda^* - a_\lambda(x, \cdot) - b_\lambda(\cdot, y)\| + \|y_\lambda^* - b_\lambda(y, \cdot) - c_\lambda(y, \cdot)\|). \tag{2.4}$$

3 Perforated domains

We address next a modification of the problem (P) that tries to model situations from different engineering or material sciences in which perforated domains appear. We will understand by perforated domains, those in which holes appear. We illustrate this type of problem with the following example.

Example 3.1 Let $\Omega = (0, 1)^2$, $\Gamma = \partial\Omega$, $\delta \in \mathbb{R}$ and $f \in H_0^1(\Omega)$, and let us consider the boundary value problem: Find $\psi \in H^2(\Omega)$ such that

$$\begin{cases} \Delta^2\psi + \delta\psi = f \text{ in } \Omega \\ \psi|_\Gamma = 0 \\ \Delta\psi|_\Gamma = 0 \end{cases}. \tag{3.1}$$

Now, we study the same type of problem in a perforated domain described as follows. Let us denote by Ω_B a collection of circular holes $\cup_{j=1}^m B(x_j, \rho_j)$ where $x_j \in \Omega$, $\rho_j > 0$ and the holes $B(x_j, \rho_j)$ are nonoverlapping and lie strictly inside Ω. We will consider $\varepsilon = \max_j \rho_j$ and denote by Ω_ε the closure of the set $\Omega \backslash \Omega_B$.

Let Ω_ε, $\Gamma_\varepsilon = \partial\Omega_\varepsilon$, $\delta \in \mathbb{R}$ and $f \in H_0^1(\Omega_\varepsilon)$, and let us consider the boundary value problem: Find $\psi \in H^2(\Omega_\varepsilon)$ such that

$$\begin{cases} \Delta^2\psi + \delta\psi = f \text{ in } \Omega_\varepsilon \\ \psi|_{\Gamma_\varepsilon} = 0 \\ \Delta\psi|_{\Gamma_\varepsilon} = 0 \end{cases} . \tag{3.2}$$

Following classical passages, this problem can be written as follows: Find $(x_{0\varepsilon}, y_{0\varepsilon}) \in E_\varepsilon \times F_\varepsilon$ such that

$$\begin{cases} a_\varepsilon(x_{0\varepsilon}, \cdot) + b_\varepsilon(\cdot, y_{0\varepsilon}) = x_\varepsilon^*(\cdot) \\ b_\varepsilon(x_{0\varepsilon}, \cdot) + c_\varepsilon(y_{0\varepsilon}, \cdot) = y_\varepsilon^*(\cdot) \end{cases} . \tag{P_ε}$$

This system adopts the form of (P_λ) with $\mathrm{card}(\Lambda) = 1$, the real Hilbert spaces $E_\varepsilon = F_\varepsilon := H_0^1(\Omega_\varepsilon)$, the continuous bilinear forms $a_\varepsilon : E_\varepsilon \times E_\varepsilon \longrightarrow \mathbb{R}$, $b_\varepsilon : E_\varepsilon \times F_\varepsilon \longrightarrow \mathbb{R}$ and $c_\varepsilon : F_\varepsilon \times F_\varepsilon \longrightarrow \mathbb{R}$ defined for each $x_1, x_2 \in E_\varepsilon$, and $y_1, y_2 \in F_\varepsilon$, as

$$a_\varepsilon(x_1, x_2) := \int_{\Omega_\epsilon} x_1 x_2,$$

$$b_\varepsilon(x_1, y_1) := -\int_{\Omega_\epsilon} \nabla x_1 \nabla y_1,$$

and

$$c_\varepsilon(y_1, y_2) := -\delta \int_{\Omega_\epsilon} y_1 y_2,$$

and the continuous linear functionals $x_\varepsilon^* := 0 \in E_\varepsilon^*$ and $y_\varepsilon^* \in F_\varepsilon^*$ given by

$$y_\varepsilon^*(y) := -\int_{\Omega_\epsilon} f y, \qquad (y \in F_\varepsilon).$$

The next two sections are devoted to study the relations between the solutions of problems (P) and (P_ε) and the corresponding inverse problems when such problems are close in a certain sense.

4 Mixed variational problems on perforated domains: the direct problem

We introduce an abstract formulation of the problem above, considering two sequences of spaces $\{E_{\varepsilon_n}\}_{n\in\mathbb{N}}$, $\{F_{\varepsilon_n}\}_{n\in\mathbb{N}}$ which we note $\{E_n\}_{n\in\mathbb{N}}$ and $\{F_n\}_{n\in\mathbb{N}}$ respectively.

Let $E, F, \{E_n\}_{n\in\mathbb{N}}, \{F_n\}_{n\in\mathbb{N}}$ be real Hilbert spaces, $a : E \times E \longrightarrow \mathbb{R}$, $b : E \times F \longrightarrow \mathbb{R}$ and $c : F \times F \longrightarrow \mathbb{R}$ be bounded bilinear forms, and for $n \in \mathbb{N}$, let $a_n : E_n \times E_n \longrightarrow \mathbb{R}$, $b_n : E_n \times F_n \longrightarrow \mathbb{R}$ and $c_n : F_n \times F_n \longrightarrow \mathbb{R}$ be bounded bilinear forms. Let $x^* : E \longrightarrow \mathbb{R}$ and $y^* : F \longrightarrow \mathbb{R}$ be bounded linear functionals and for $n \in \mathbb{N}$, let $x_n^* : E_n \longrightarrow \mathbb{R}$ and $y_n^* : F_n \longrightarrow \mathbb{R}$ be bounded linear functionals.

We consider the problem (P) and for $n \in \mathbb{N}$, the following problems: find $(x_{0n}, y_{0n}) \in E_n \times F_n$ such that

$$\begin{cases} a_n(x_{0n}, \cdot) + b_n(\cdot, y_{0n}) = x_n^* \\ b_n(x_{0n}, \cdot) + c_n(y_{0n}, \cdot) = y_n^* \end{cases} . \tag{P_n}$$

We write

$$K := \{x \in E : \ b(x, \cdot) = 0\},$$

and for $n \in \mathbb{N}$

$$K_n := \{x \in E_n : \ b_n(x, \cdot) = 0\}.$$

Then, we suppose now that the bounded bilinear forms in problems (P) and (P_n) verify assumption (i) of Theorem 2.1 for cardinal of Λ equal to 1, and assumptions (ii), (iii) and (iv) of the same result in the following way: For some $\alpha, \beta > 0$ and $\alpha_n, \beta_n > 0$, ($n \in \mathbb{N}$), there hold

(ii) • $x \in K \ \Rightarrow \ \alpha\|x\| \leq \|a(\cdot, x)_{|K}\|$,
• $x \in K_n \ \Rightarrow \ \alpha_n\|x\| \leq \|a_n(\cdot, x)_{|K_n}\|$,

(iii) • $y \in F \ \Rightarrow \ \beta\|y\| \leq \|b(\cdot, y)\|$,
• $y \in F_n \ \Rightarrow \ \beta_n\|y\| \leq \|b_n(\cdot, y)\|$.

and noting

$$\rho := \max\left\{\frac{1}{\alpha}, \frac{1}{\beta}\left(1 + \frac{\|a\|}{\alpha}\right), \frac{1}{\beta^2}\|a\|\left(1 + \frac{\|a\|}{\alpha}\right)\right\}$$

and for $n \in \mathbb{N}$

$$\rho_n := \max\left\{\frac{1}{\alpha_n}, \frac{1}{\beta_n}\left(1 + \frac{\|a_n\|}{\alpha_n}\right), \frac{1}{\beta_n^2}\|a_n\|\left(1 + \frac{\|a_n\|}{\alpha_n}\right)\right\},$$

(iv) • $\|c\| < \dfrac{1}{\rho}$,
• for $n \in \mathbb{N}$, $\|c_n\| < \dfrac{1}{\rho_n}$.

In view of Theorem 2.1 these assumptions ensures the existence and uniqueness of solution for problems (P) and (P_n), noted $(x_0, y_0) \in E \times F$ and $(x_{0n}, y_{0n}) \in E_n \times F_n$ respectively. Moreover, we have the following control of the norms:

$$\max\{\|x_0\|_E, \|y_0\|_F\} \leq \frac{\rho}{1 - \rho\|c\|}\left(\|x^*\| + \|y^*\|\right), \tag{4.1}$$

and for $n \in \mathbb{N}$

$$\max\{\|x_{0n}\|_{E_n}, \|y_{0n}\|_{F_n}\} \leq \frac{\rho_n}{1 - \rho_n\|c_n\|}\left(\|x_n^*\| + \|y_n^*\|\right). \tag{4.2}$$

The next result establishes the relation between the solutions of problems (P) and (P_n) when such problems are close in a certain sense:

Theorem 4.1 *With the previous notations and assumptions, let us suppose that*

(a) *The Hilbert spaces* E, F, $\{E_n\}_{n\in\mathbb{N}}$, $\{F_n\}_{n\in\mathbb{N}}$ *verify:*

- *The sequences* $\{E_n\}_{n\in\mathbb{N}}$ *and* $\{F_n\}_{n\in\mathbb{N}}$ *are increasing i.e., if* $n, m \in \mathbb{N}$, $n < m$, *then* $\overline{E_n \subset E_m} \subset E$ *and* $\overline{F_n \subset F_m} \subset F$.
- $\overline{\bigcup_{n\in\mathbb{N}} E_n} = E$ *and* $\overline{\bigcup_{n\in\mathbb{N}} F_n} = F$.
- *There exist* $\gamma_E, \gamma_F > 0$ *such that for each* $n \in \mathbb{N}$, $x \in E_n$, $y \in F_n$,

$$\|x\|_E \leq \gamma_E\|x\|_{E_n}, \quad \textit{and} \quad \|y\|_F \leq \gamma_F\|y\|_{F_n}.$$

(b) *There exist three sequences* $\{\mu_n\}$, $\{\eta_n\}$, $\{\delta_n\}$, *with*

$$\lim_{n\to\infty}\mu_n = \lim_{n\to\infty}\eta_n = \lim_{n\to\infty}\delta_n = 0,$$

such that for $n \in \mathbb{N}$, $x_1, x_2 \in E_n$ *and* $y_1, y_2 \in F_n$ *we have:*

- $|a(x_1, x_2) - a_n(x_1, x_2)| \leq \mu_n \|x_1\|_{E_n} \|x_2\|_{E_n}$,
- $|b(x_1, y_1) - b_n(x_1, y_1)| \leq \eta_n \|x_1\|_{E_n} \|y_1\|_{F_n}$,
- $|c(y_1, y_2) - c_n(y_1, y_2)| \leq \delta_n \|y_1\|_{F_n} \|y_2\|_{F_n}$.

(c) *The sequences of funcionals* $\{x_n^*\}_{n\in\mathbb{N}}$ *and* $\{y_n^*\}_{n\in\mathbb{N}}$, *converge to* x^* *and* y^*, *respectively, in the* w^**–topology.*

Then, the sequences of solutions $(\{x_{0n}, y_{0n})\}_{n\in\mathbb{N}}$ *of problems* (P_n) *converge in the* w*–topology on* $E \times F$, *except partials, to* (x_0, y_0), *solution of problem* (P).

Proof From assumption c) we have that there exist $M_E, M_F \geq 0$ such that

$$\|x_n^*\| \leq M_E, \quad \|y_n^*\| \leq M_F, \quad (n \in \mathbb{N}). \tag{4.3}$$

We can deduce from this fact and from (4.2) that the boundness of the sequences $\{x_{0n}\}_{n\in\mathbb{N}}$, $\{y_{0n}\}_{n\in\mathbb{N}}$ depends on the boundness of $\{\rho_n\}$ or that of $\{\|a_n\|\}$ and $\{\|c_n\|\}$. But the sequence $\{\|a_n\|\}$ is bounded, since

$$\begin{aligned}\|a_n\| &= \sup_{x_1,x_2\in E_n}\frac{|a_n(x_1,x_2)|}{\|x_1\|_{E_n}\|x_2\|_{E_n}}\\ &\leq \sup_{x_1,x_2\in E_n}\frac{|a_n(x_1,x_2)-a(x_1,x_2)|}{\|x_1\|_{E_n}\|x_2\|_{E_n}} + \sup_{x_1,x_2\in E_n}\frac{|a(x_1,x_2)|}{\|x_1\|_{E_n}\|x_2\|_{E_n}},\end{aligned}$$

and taking into account assumption b) for the first term and assumption a) for the second, we have that the last sum is less or equal that

$$\sup_{x_1,x_2\in E_n}\frac{\mu_n\|x_1\|_{E_n}\|x_2\|_{E_n}}{\|x_1\|_{E_n}\|x_2\|_{E_n}} + \sup_{x_1,x_2\in E_n}\gamma_E\gamma_E\frac{|a(x_1,x_2)|}{\|x_1\|_E\|x_2\|_E} \leq \mu_n + \gamma_E^2\|a\|.$$

Similar arguments show the boundness of $\{\|c_n\|\}$.

We deduce that $\{x_{0n}\}_{n\in\mathbb{N}}$, $\{y_{0n}\}_{n\in\mathbb{N}}$ are bounded an then they have partial subsequences $\{x_{0n_k}\}_{n\in\mathbb{N}}$, $\{y_{0n_k}\}_{n\in\mathbb{N}}$ which converge weakly. We note x_1 and y_1 the limits of such subsequences. We prove finally that $(x_1, y_1) \in E \times F$ is solution of problem (P) and then from the uniqueness of the solution we have the result. For this purpose, for each $x \in E$, according to a), the continuity of the bilinear forms and the density of $\cup_{n\in\mathbb{N}}E_n$ and $\cup_{n\in\mathbb{N}}F_n$, we can suppose that there exists n_k such that $x \in E_{n_k}$. Then,

$$\begin{aligned}|a(x_1,x)+b(x,y_1)-x^*(x)| &\leq |a(x_1,x)-a(x_{0n_k},x)| + |b(x,y_1)-b(x,y_{0n_k})|\\ &\quad +|a(x_{0n_k},x)+b(x,y_{0n_k})-x^*(x)|.\end{aligned}$$

From the weak continuity of a and b in each variable we deduce that the first two terms in the sum converge to 0. For the third one, we have that

$$\begin{aligned}|a(x_{0n_k},x)+b(x,y_{0n_k})-x^*(x)| &\leq |a(x_{0n_k},x)-a_{n_k}(x_{0n_k},x)|\\ &\quad +|a_{n_k}(x_{0n_k},x)+b_{n_k}(x,y_{0n_k})-x^*_{n_k}(x)|\\ &\quad +|b(x,y_{0n_k})-b_{n_k}(x,y_{0n_k})|\\ &\quad +|x^*_{n_k}(x)-x^*(x)|.\end{aligned}$$

In view the assumption (b) we deduce that the first and third terms tend to 0. The second is 0 because (x_{0n_k}, y_{0n_k}) is the solution of the problem (P_{n_k}), and the last term tends to 0 from (c).

A similar reasoning proves that, given $y \in F$, $b(x_1, y) + c(y_1, y) = y^*(y)$, which concludes the proof. □

5 Mixed variational problems on perforated domains: the inverse problem

We deal now with inverse problems associated with problems in perforated domains and we analyse the relationship between the minimizers of an inverse problem defined on a solid domain and the minimizers of an inverse problem defined on a perforated domain when the holes are small enough.

Let Λ a compact set of $\mathbb{R}^n$. Let E, F, $\{E_n\}_{n\in\mathbb{N}}$, $\{F_n\}_{n\in\mathbb{N}}$ be real Hilbert spaces, $a^\lambda : E \times E \longrightarrow \mathbb{R}$, $b^\lambda : E \times F \longrightarrow \mathbb{R}$ and $c^\lambda : F \times F \longrightarrow \mathbb{R}$ be bounded bilinear forms, and for $n \in \mathbb{N}$, let $a_n^\lambda : E_n \times E_n \longrightarrow \mathbb{R}$, $b_n^\lambda : E_n \times F_n \longrightarrow \mathbb{R}$ and $c_n^\lambda : F_n \times F_n \longrightarrow \mathbb{R}$ be bounded bilinear forms. Let $x_\lambda^* : E \longrightarrow \mathbb{R}$ and $y_\lambda^* : F \longrightarrow \mathbb{R}$ be bounded linear functionals, and for $n \in \mathbb{N}$, $x_{\lambda n}^* : E_n \longrightarrow \mathbb{R}$ and $y_{\lambda n}^* : F_n \longrightarrow \mathbb{R}$ be bounded linear functionals.

For $\lambda \in \Lambda$ we consider the family of problems (P_λ) described on Theorem 2.1 and for each $n \in \mathbb{N}$ the family of problems: Find $(x_{\lambda n}, y_{\lambda n}) \in E_n \times F_n$ such that

$$\begin{cases} a_n^\lambda(x_{\lambda n}, \cdot) + b_n^\lambda(\cdot, y_{\lambda n}) = x_{\lambda n}^* \\ b_n^\lambda(x_{\lambda n}, \cdot) + c_n^\lambda(y_{\lambda n}, \cdot) = y_{\lambda n}^* \end{cases} . \tag{P_n^λ}$$

If we suppose that all the bilinear forms verify assumptions (i), (ii), (iii) and (iv) of Theorem 2.1, it follows that for each $\lambda \in \Lambda$ and for each $n \in \mathbb{N}$, problems (P_λ) and (P_n^λ) have a unique solution (x_λ, y_λ) and $(x_{\lambda n}, y_{\lambda n})$ repectively. Moreover, given a target element $(x, y) \in E \times F$, Theorem 2.1 states that

$$\begin{aligned} &\inf_{\lambda\in\Lambda} \max\{\|x_\lambda - x\|_E, \|y_\lambda - y\|_F\} \\ &\quad \le \inf_{\lambda\in\Lambda} \frac{\rho_\lambda}{1-\rho_\lambda\gamma}\left(\|x_\lambda^* - a^\lambda(x,\cdot) - b^\lambda(\cdot, y)\| + \|y_\lambda^* - b^\lambda(x,\cdot) - c^\lambda(y,\cdot)\|\right) \end{aligned} \tag{5.1}$$

with $\gamma := \inf_{\lambda\in\Lambda}\|c^\lambda\| > 0$. Then, in order to solve the inverse problem, we must solve the optimization problem

$$\min_{\lambda\in\Lambda}\left(G^\lambda(x, y) + S^\lambda(x, y)\right),$$

where, $G^\lambda(x, y) = \|x_\lambda^* - a^\lambda(x,\cdot) - b^\lambda(\cdot, y)\|$ and $S^\lambda(x, y) = \|y_\lambda^* - b^\lambda(x,\cdot) - c^\lambda(y,\cdot)\|$, for a given $(x, y) \in E \times F$. With the same arguments as above, given a target element $(x_n, y_n) \in E_n \times F_n$ in order to approximate the solution of the inverse problem (P_n^λ) we must minimize the collage distance, that is, solve the optimization problem

$$\min_{\lambda\in\Lambda}\left(G_n^\lambda(x_n, y_n) + S_n^\lambda(x_n, y_n)\right),$$

where $G_n^\lambda(x_n, y_n) = \|x_{\lambda n}^* - a_n^\lambda(x,\cdot) - b_n^\lambda(\cdot, y)\|$ and $S_n^\lambda(x_n, y_n) = \|y_{\lambda n}^* - b_n^\lambda(x,\cdot) - c_n^\lambda(y,\cdot)\|$.

Our goal is to show that solutions of inverse problems (P_λ) and (P_n^λ) are arbitrary closed when problems are closed enough in the sense established in the next result.

Theorem 5.1 *With the above notation, suppose that the Hilbert spaces E, F, $\{E_n\}_{n\in\mathbb{N}}$, $\{F_n\}_{n\in\mathbb{N}}$ verify:*

(i) *The sequences $\{E_n\}_{n\in\mathbb{N}}$ and $\{F_n\}_{n\in\mathbb{N}}$ are increasing sequences, i.e. if $n, m \in \mathbb{N}$, $n < m$, then $E_n \subset E_m \subset E$ and $F_n \subset F_m \subset F$.*

(ii) *There exist two sequences of projections $\pi_n : E \longrightarrow E_n$, and $Q_n : F \longrightarrow F_n$, such that for $x \in E$ and $y \in F$,*

$$\lim_{n\to\infty} \|x - \pi_n(x)\| = \lim_{n\to\infty} \|y - Q_n(y)\| = 0.$$

(iii) *The bilinear forms and functionals are given by $a_n^\lambda = a^\lambda_{|E_n\times E_n}$, $b_n^\lambda = b^\lambda_{|E_n\times F_n}$, $c_n^\lambda = c^\lambda_{|F_n\times F_n}$, $x^*_{\lambda n} = x^*_{\lambda_{|E_n}}$, $y^*_{\lambda n} = y^*_{\lambda_{|F_n}}$.*

(iv) *For all $n \in \mathbb{N}$, $x \in E$, $y \in F$, $x_n \in E_n$ and $y_n \in F_n$, the functions $G^\lambda(x, y)$, $G_n^\lambda(x_n, y_n)$, $S^\lambda(x, y)$ and $S_n^\lambda(x_n, y_n) : \Lambda \to \mathbb{R}^+$ are continuous.*

Let $\{\lambda_n\}$ a sequence of minimizers of $G_n^\lambda(\pi_n(x), Q_n(y)) + S_n^\lambda(\pi_n(x), Q_n(y))$ over Λ. Then there exists $\lambda^* \in \Lambda$ and a partial subsequence of $\{\lambda_n\}$, which we will note $\{\lambda_n\}$ as well, in order to simplify the notation, such that $\{\lambda_n\} \to \lambda^*$, with λ^* a minimizer of $G^\lambda(x, y) + S^\lambda(x, y)$ over Λ.

Proof Let M, N, R, μ and ν given by

$$M = \sup_{\lambda\in\Lambda} \left\{\|a^\lambda\|\right\}, \quad N = \sup_{\lambda\in\Lambda} \left\{\|b^\lambda\|\right\}, \quad R = \sup_{\lambda\in\Lambda} \left\{\|c^\lambda\|\right\}, \quad \mu = \sup_{\lambda\in\Lambda} \left\{\|x^*_\lambda\|\right\},$$
$$\nu = \sup_{\lambda\in\Lambda} \left\{\|y^*_\lambda\|\right\}.$$

Then on the one hand, given $(x, y) \in E \times F$,

$$\begin{aligned}
G_n^\lambda(\pi_n x, Q_n y) + S_n^\lambda(\pi_n x, Q_n y) &\leq G^\lambda(\pi_n x, Q_n y) + S^\lambda(\pi_n x, Q_n y)) \\
&= G^\lambda(\pi_n x - x + x, Q_n y - y + y) \\
&\quad + S^\lambda(\pi_n x - x + x, Q_n y - y + y) \\
&\leq \|\phi^\lambda - a^\lambda(\pi_n x - x + x, \cdot) - b^\lambda(\cdot, Q_n y - y + y)\| \\
&\quad + \|\psi^\lambda - b^\lambda(\pi_n x - x + x, \cdot) - c^\lambda(Q_n y - y + y, \cdot)\| \\
&\leq \|\phi^\lambda - a^\lambda(x, \cdot) - b^\lambda(\cdot, y)\| + \|\psi^\lambda - b^\lambda(x, \cdot) - c^\lambda(y, \cdot)\| \\
&\quad + \|a^\lambda(\pi_n x - x, \cdot) - b^\lambda(\cdot, Q_n y - y)\| \\
&\quad + \|b^\lambda(\pi_n x - x, \cdot) - c^\lambda(Q_n y - y, \cdot)\| \\
&\leq G^\lambda(x, y) + S^\lambda(x, y) + \|a^\lambda\|\|\pi_n x - x\| + \|b^\lambda\|\|Q_n y - y\| \\
&\quad + \|b^\lambda\|\|\pi_n x - x\| + \|c^\lambda\|\|Q_n y - y\| \\
&\leq G^\lambda(x, y) + S^\lambda(x, y) \\
&\quad + \max\{\|\pi_n x - x\|_E, \|Q_n y - y\|_F\}(M + 2N + R).
\end{aligned} \tag{5.2}$$

And, on the other hand,

$$\begin{aligned}
G^\lambda(\pi_n x, Q_n y) &= \|\phi^\lambda - a^\lambda(\pi_n x, \cdot) - b^\lambda(\cdot, Q_n y)\| \\
&\leq \|\phi^\lambda \circ \pi_n - a^\lambda(\pi_n x, \pi_n(\cdot)) - b^\lambda(\pi_n(\cdot), Q_n y)\| \\
&\quad + \|a^\lambda(\pi_n x, \pi_n(\cdot)) - a^\lambda(\pi_n x, \cdot)\| + \|b^\lambda(\pi_n(\cdot), Q_n y) - b^\lambda(\cdot, Q_n y)\| \\
&\quad + \|\phi^\lambda - \phi^\lambda \circ \pi_n\| \\
&\leq G_n^\lambda(\pi_n x, Q_n y) \\
&\quad + M\|\pi_n x\| \sup_{v\in E, \|v\|_E=1} \|v - \pi_n v\|_E + N\|Q_n y\| \sup_{v\in E, \|v\|_E=1} \|v - \pi_n v\|_E \\
&\quad + \mu \sup_{v\in E, \|v\|_E=1} \|v - \pi_n v\|_E \\
&= G_n^\lambda(\pi_n x, Q_n y) + (M\|\pi_n x\| + N\|Q_n y\| + \mu) \sup_{v\in E, \|v\|_E=1} \|v - \pi_n v\|_E,
\end{aligned}$$

and

$$\begin{aligned}
S^\lambda(\pi_n x, Q_n y) &= \|\psi^\lambda - b^\lambda(\pi_n x, \cdot) - c^\lambda(Q_n y, \cdot)\| \\
&\le \|\psi^\lambda \circ Q_n - b^\lambda(\pi_n x, Q_n(\cdot)) - c^\lambda(Q_n y, Q_n(\cdot))\| + \|b^\lambda(\pi_n x, Q_n(\cdot)) - b^\lambda(\pi_n x, \cdot)\| \\
&\quad + \|c^\lambda(Q_n y, Q_n(\cdot)) - c^\lambda(Q_n y, \cdot)\| + \|\psi^\lambda - \psi^\lambda \circ Q_n\| \\
&\le S_n^\lambda(\pi_n x, Q_n y) + N\|\pi_n x\| \sup_{w\in F, \|w\|_F=1} \|w - Q_n w\|_F \\
&\quad + R\|Q_n y\| \sup_{w\in F, \|w\|_F=1} \|w - Q_n w\|_F + \nu \sup_{w\in F, \|w\|_F=1} \|w - Q_n w\|_F \\
&= S_n^\lambda(\pi_n x, Q_n y) + (N\|\pi_n x\| + R\|Q_n y\| + \nu) \sup_{w\in F, \|w\|_F=1} \|w - Q_n w\|_F.
\end{aligned}$$

Then,

$$\begin{aligned}
G^\lambda(\pi_n x, Q_n y) + S^\lambda(\pi_n x, Q_n y) &\le G_n^\lambda(\pi_n x, Q_n y) + S_n^\lambda(\pi_n x, Q_n y) \\
&\quad + (M\|\pi_n x\| + N\|Q_n y\| + \mu) \sup_{v\in E, \|v\|_E=1} \|v - \pi_n v\|_E \\
&\quad + (N\|\pi_n x\| + R\|Q_n y\| + \nu) \sup_{w\in F, \|w\|_F=1} \|w - Q_n w\|_F.
\end{aligned} \tag{5.3}$$

Therefore, according to the compactness of Λ, given a sequence of minimizers $\{\lambda_n\}$ of $G_n^\lambda(\pi_n x, Q_n y) + S_n^\lambda(\pi_n x, Q_n y)$ over Λ, there exists a convergent partial subsequence, also noted $\{\lambda_n\}$, *i.e.*, there exists $\lambda^* \in \Lambda$ such that $\{\lambda_n\} \to \lambda^*$. To see that λ^* is a minimizer of $G^\lambda(x, y) + S^\lambda(x, y)$ over Λ, we compute

$$\begin{aligned}
G^{\lambda^*}(x, y) + S^{\lambda^*}(x, y) &= \lim_{n\to+\infty} \left(G^{\lambda_n}(\pi_n x, Q_n y) + S^{\lambda_n}(\pi_n x, Q_n y)\right) \quad \text{(by (5.12))} \\
&\le \lim_{n\to+\infty} (G_n^{\lambda_n}(\pi_n x, Q_n y) \\
&\quad + S_n^{\lambda_n}(\pi_n x, Q_n y) + (M\|\pi_n x\| + N\|Q_n y\| + \mu) \sup_{v\in E, \|v\|_E=1} \|v - \pi_n v\|_E \\
&\quad + (N\|\pi_{\varepsilon_n} x\| + R\|Q_n y\| + \nu) \sup_{w\in F, \|w\|_F=1} \|w - Q_n w\|_F) \quad \text{(minimizers)} \\
&\le \lim_{n\to+\infty} (G_n^\lambda(\pi_n x, Q_n y) + S_n^\lambda(\pi_n x, Q_n y) \\
&\quad + (M\|\pi_n x\| + N\|Q_n y\| + \mu) \sup_{v\in E, \|v\|_E=1} \|v - \pi_n v\|_E \\
&\quad + (N\|\pi_n x\| + R\|Q_n y\| + \nu) \sup_{w\in F, \|w\|_F=1} \|w - Q_n w\|_F) \quad \text{(by (5.11))} \\
&\le \lim_{n\to+\infty} (G^\lambda(x, y) + S^\lambda(x, y) \\
&\quad + (M + 2N + R) \max\{\|\pi_n x - x\|_E, \|Q_n y - y\|_F\} \\
&\quad + (M\|\pi_n x\| + N\|Q_n y\| + \mu) \sup_{v\in E, \|v\|_E=1} \|v - \pi_n v\|_E \\
&\quad + (N\|\pi_n x\| + R\|Q_n y\| + \nu) \sup_{w\in F, \|w\|_F=1} \|w - Q_n w\|_F) \\
&= G^\lambda(x, y) + S^\lambda(x, y).
\end{aligned}$$

□

Remark 5.2 Condition (ii) is not too restrictive. In fact, if we suppose that the Hilbert spaces $E, F, \{E_n\}_{n\in\mathbb{N}}, \{F_n\}_{n\in\mathbb{N}}$ are separable and verify (i) and $\overline{\bigcup_{n\in\mathbb{N}} E_n} = E$ and $\overline{\bigcup_{n\in\mathbb{N}} F_n} = F$, then condition (ii) is satisfied.

Finally we illustrate the above results considering the following example related to Example 3.1.

Example 5.3 We consider the problem (3.1) with $\delta = -2$ and $f(x, y)$ the function for which the solution $\psi(x, y)$ to (3.1) is $10^3(x(1-x)y(1-y))^4$.

The same problem in a porous domain is (3.2) and if we take $w = -\Delta\psi$ the mentioned problem is equivalent to

Table 1 Results with $n = 10$

	$r = 0.005,\ l = 0.005$ $a = 0.005,\ b = 0.003$	$r = 0.0005,\ l = 0.0005$ $a = 0.0005,\ b = 0.0003$	$r = 0.00005,\ l = 0.00005$ $a = 0.00005,\ b = 0.00003$
A	1.000192021	1.000157458	1.000156459
B	1.003093897	1.000297366	1.000268181
C	−3.670642019	−2.104443730	−2.087745875
Collage distance	$1.964405817 \times 10^{-12}$	$3.957312313 \times 10^{-15}$	$2.088385461 \times 10^{-15}$

Table 2 Results with $n = 20$

	$r = 0.005,\ l = 0.005$ $a = 0.005,\ b = 0.003$	$r = 0.0005,\ l = 0.0005$ $a = 0.0005,\ b = 0.0003$	$r = 0.00005,\ l = 0.00005$ $a = 0.00005,\ b = 0.00003$
A	1.000249733	1.000161039	1.000159465
B	1.003200522	1.000303959	1.000273715
C	−3.841718970	−2.115096071	−2.096689098
Collage distance	$6.683393914 \times 10^{-13}$	$1.533338673 \times 10^{-15}$	$8.256778945 \times 10^{-16}$

$$\begin{cases} \Delta\psi + w = 0 \text{ in } \Omega_\varepsilon \\ -\Delta\psi - 2\psi = f(x, y) \text{ in } \Omega_\varepsilon \\ \psi|_{\Gamma_\varepsilon} = 0 \\ \Delta\psi|_{\Gamma_\varepsilon} = 0 \end{cases}, \tag{5.4}$$

which could be written as (P_ε).

Our purpose is to recover A, B and C in the perturbed mixed system

$$\begin{cases} A\Delta\psi + Bw = 0 \text{ in } \Omega_\varepsilon \\ -A\Delta w + C\psi = f(x, y) \text{ in } \Omega_\varepsilon \end{cases}.$$

Observe that the exact values are $A = B = 1$ and $C = -2$.

We consider four holes which are randomly taken with different shapes (squares, circles, and ellipses). The Tables 1 and 2 show the results after running the collage codding approach over the perforated domains for different sizes, considering n = 10 y n = 20 respectively. We will denote r the circle radius, l the square side and a and b the ellipse major and minor axis respectively.

6 Conclusion

Some conditions for the existence of solution of a perturbed mixed variational system and that of an associated inverse problem have been given. Furthermore, some convergence results related to the impact of the size of the holes have been derived. The numerical results show that as hole diameter decreases, results improve.

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