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On the notion of fuzzy dispersion measure and its application to triangular fuzzy numbers

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ABSTRACT

In this paper, based on the analysis of the most widely used dispersion measure in the real context (namely, the variance), we introduce the notion of *fuzzy dispersion measure* associated to a finite set of data given by fuzzy numbers. This measure is implemented as a fuzzy number, so there is no loss of information caused by any defuzzification. The proposed concept satisfies the usual properties in a genuinely fuzzy sense and it avoids limitations in terms of its geometric shape or its analytical properties: under this conception, it could have a piece of its support in the negative part of the real line. This novel notion can be interpreted as a way of fusing the information included in a fuzzy data set in order to make a decision based on its dispersion. To illustrate the main characteristics of this approach, we present an example of a fuzzy dispersion measure that allows to conclude that this new way to deal this problem is coherent, at least, from the point of view of human intuition.

1. Introduction

Dispersion of data is defined as the degree to which the data approaches to an average value. This information plays a very important role in Statistics for Data Analysis and Data Science [1,2]. For real data, a dispersion measure is performed by determining a central position measurement (usually the arithmetic mean) and calculating the average distance (or semi-distances) from the data to such value. For instance, the variance is worked out as the average squared distance to the mean. This measure is characterized by some general properties: it is invariant by translations and, if all inputs are multiplied by the same scalar, then the variance is multiplied by the square of such real number. Furthermore, this dispersion measure is naturally a non negative real number, and its lowest value (which is 0) characterizes the lowest dispersion (there is none dispersion when all data are equal). In finance, it is common to associate some dispersion measure (usually, the variance) to risk. Low variance corresponds to lower risk and a lower return. Among two investments with the same expected return, it is usual to consider the one with the higher variance to be riskier.

Beyond the real case, the *fuzzy set theory*, introduced by Zadeh [3] in 1965, is a powerful tool to describe and model situations in which the data are imprecise or vague. Due to the uncertainty in the real world, fuzzy numbers have been successfully applied in many different research areas (see [4–13]).

The studies about the computation of the variance involving fuzzy numbers started in the 80s and 90s of the 20th century. Those definitions of variance of fuzzy numbers can be in turn divided into two main blocks, which respectively correspond to the two main interpretations of fuzzy sets, namely the "ontic interpretation" and the "epistemic interpretation". According to the "*ontic interpretation*" of fuzzy sets, the variance of fuzzy numbers is defined as a crisp number representing the mean of the squared (crisp) distances between the corresponding fuzzy numbers and their arithmetic mean (see [14–22]). According to the "*epistemic interpretation*", the (fuzzy) variance of a collection of n fuzzy numbers is defined as a fuzzy set. Each α-cut corresponds to the set of all the variances associated to all the possible collections of n crisp numbers $\{x_1, \ldots, x_n\}$, each of them (x_i) included in the α-level set $(\mathsf{A}_i)_\alpha$ of the corresponding fuzzy number A_i. This definition was given by Kruse [23], and Kruse and Meyer [24], in the 1980s (see also [25] for a discussion on different definitions, and [26–30] to consider some optimization problems for the computation of the extreme points of the α-cuts of the fuzzy variance).

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In trying to extend the concept of real variance to the fuzzy setting, several difficulties naturally arise. First of all, it is worth mentioning the arithmetic. To compute the variance of some real numbers, the four basic arithmetic operations are involved: sum, subtraction, product (the square) and division. Although there is an arithmetic with fuzzy numbers that extends and generalizes the usual arithmetic with real numbers (see [31–34]), it does not preserve some properties of the arithmetic with real numbers. For instance, it is possible to take two distinct non zero fuzzy numbers A and B such that A − B = B − A. Furthermore, the product of two triangular fuzzy numbers is not, in general, a fuzzy number of the same class: it is rather an *LR*-fuzzy number (see [9]).

Secondly, if we desire to compare fuzzy dispersions, we need to consider a ranking methodology in the set of all possible fuzzy quantities that can be obtained. This ranking procedure will play a key role in this topic. The family of all fuzzy numbers is not endowed with a globally accepted partial order that extends the real order. In this line of study, several ranking methodologies have been introduced in the last fifty years (see [35,36] and references therein).

Thirdly, under the epistemic interpretation in the fuzzy setting, the non-negativity of the real variance has been translated into the condition that the support of a fuzzy variance must necessarily be included in the non-negative part of the real line. From our point of view, this property has not been sufficiently discussed in the literature and this paper tries to address this issue. On the contrary, such property of the real variance could be rather interpreted as an extraordinary (but desired) chance: it comes from the fact that the square of any real number is non-negative, which is a property that does not hold when handling fuzzy numbers. In the presence of a ranking methodology, non-negative fuzzy numbers are defined by the simple comparison with the crisp null fuzzy number. This condition says nothing about the level sets (or the support) of the fuzzy number: it is usual to have a piece in the positive part of the real line, but it is not impossible for a positive fuzzy number to have a piece in its negative part. It basically depends on the ranking methodology. The deep analysis of the structure of the interval of all possible real variances (that is, $[0,+\infty)$) leads to that of an ordered convex cone with an absolute minimum: closed for sums and products by positive real numbers, and endowed with an absolute minimum. That is why, in the fuzzy setting, once the fuzzy ranking methodology to be applied has been chosen, it seems reasonable to impose that the values obtained as fuzzy dispersion measures form a convex cone with respect to the chosen fuzzy binary relation.

Finally, in spite of the amount of studies about fuzzy variance, it is difficult to find a research that quantifies the dispersion avoiding the lost of information. On the contrary, the involved procedures often employ defuzzifications or real computations on level sets. Such processes enjoys two main advantages: they are full of significance and they inherit the main properties of the variance in the real context. However, they also lead to some inconsistency with the fuzzy point of view: such proposals take fuzzy numbers, they immediately move to a real scenario, they all time perform real operations (for instance, the computation of the real variance), and they only come back to the fuzzy framework at the end of the process in order to interpret the obtained result.

In this context, the main aim of this work is to introduce the minimal theoretical framework in which a *fuzzy dispersion measure* among fuzzy quantities (especially, when the input data are fuzzy numbers) can be defined. As a result, this paper deals with two shocking ideas: on the one hand, we definitively search for an approach that avoids the lost of information, that is, completely fuzzy-based; on the other hand, we search for a notion in which the fuzzy dispersion measure could have a piece of its support in the negative part of the real line. Under this view-point, this algebraic structure is characterized by five basic properties (namely symmetry, crisp-invariance, homogeneity, normality and positivity with respect to a binary relation), but other reasonable conditions that a fuzzy dispersion measure could satisfy are also described. We also highlight the importance of fixing at the beginning of the study the classes of considered fuzzy quantities (inputs and outputs) and the fuzzy binary relation. To support this proposal, examples of such measures are introduced but, mainly, it is shown that the translation to the fuzzy setting of classical notion of variance of triangular fuzzy numbers permit to consider a *fuzzy dispersion space* with respect to the binary relation introduced in [36] that, additionally, is 2-homogeneous, conservative and of variance-type.

To develop these ideas, this paper is organized as follows. In Section 2 the necessary preliminaries to understand the contents of the paper are introduced. Section 3 contains the notion of *fuzzy dispersion measure* and some of the reasonable properties that it can satisfy. Later, Section 4 is devoted to describe and study a canonical example of fuzzy dispersion measure, and to show some of its main properties. In fact, we compare its results with those obtained by employing the Kruse fuzzy variance. In Section 5 we compute and compare the dispersion (in a fuzzy sense) of two distinct sets of four triangular fuzzy numbers of the real line. It is also highlighted that the canonical example of fuzzy dispersion measure is non-negative when the ranking methodology presented in [36] is applied. Finally, some conclusions and prospect works in this line of study are commented.

2. Preliminaries

Through this manuscript, $\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of all positive integers, and $\mathbb{R}$ will stand for the family of all real numbers. Given $n \in \mathbb{N}$, we use $[n]$ for the indices set $\{1, 2, \ldots, n\}$. To fix the notation, given a set $T = \{t_1, t_2, \ldots, t_n\}$ of n real numbers, the *variance* $\mathrm{var}(T)$ of such set is the real number

$$\mathrm{var}(T) = \frac{1}{n}\sum_{i=1}^{n}(t_i - \overline{T})^2,$$

where $\overline{T} = \frac{1}{n}\sum_{i=1}^{n} t_i$ is the *mean* of the real numbers $t_1, t_2, \ldots, t_n$.

Given a non-empty subset $D \subseteq \mathbb{R}$, a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is: *increasing* (or *non-decreasing*) if $f(t) \leq f(s)$ for all $t, s \in D$ such that $t \leq s$; *strictly increasing* if $f(t) < f(s)$ for all $t, s \in D$ such that $t < s$; *decreasing* (or *non-increasing*) if $f(t) \geq f(s)$ for all $t, s \in D$ such that $t \leq s$; *strictly decreasing* if $f(t) > f(s)$ for all $t, s \in D$ such that $t < s$; *strictly monotone* if f is either strictly increasing or strictly decreasing.

Given two real closed intervals $[a, b]$ and $[c, d]$, we will say that a function $f : [a, b] \to [c, d]$ is a *parametrization from* $[a, b]$ *onto* $[c, d]$ if f is continuous and bijective from $[a, b]$ onto $[c, d]$ (so it is also strictly monotone). A parametrization is *direct* if it is strictly increasing, and *inverse* if it is strictly decreasing.

From now on, let X be a non-empty set. Given $n \in \mathbb{N}$, we denote by X^n to the Cartesian product $X \times X \times \overset{(n)}{\cdots} \times X$. A *binary relation on* X is a non-empty subset $\mathcal{R}$ of the Cartesian product $X \times X$. For simplicity, if $(x, y) \in \mathcal{R}$, we denote it by $x \preceq y$, and we will say that $\preceq$ is the binary relation on X. A binary relation $\preceq$ is *total* (or *linear*) if $x \preceq y$ or $y \preceq x$ whatever $x, y \in X$ (see [37]). Each total binary relation is *reflexive* (that is, $x \preceq x$ for each $x \in \mathbb{R}$). A set X is a *singleton* if it contains a unique element.

2.1. Background on fuzzy sets and fuzzy numbers

Given a set X, a *fuzzy set* A *on* X is a family of pairs $\{\langle x, \eta_A(x)\rangle : x \in X\}$ such that $\eta_A(x) \in [0, 1]$ for each $x \in X$. Each value $\eta_A(x)$ represent the membership degree of x to the fuzzy set A, and the function $\eta_A : X \to [0, 1]$ is called the *membership function* of the fuzzy set A. For the sake of simplicity, we will identify each fuzzy set A with its corresponding membership function, and we will say that a *fuzzy set on* X is a mapping $A : X \to [0, 1]$.

For each $\alpha \in (0, 1]$, the *α-level set* (or *α-cut*) of A is the set $A_\alpha = \{x \in X : A(x) \geq \alpha\}$. The *kernel* (or *core*) *of* A is $\ker(A) = A_1$ and the *support of* A is the set

$$\mathrm{supp}(A) = \{x \in X : A(x) > 0\} = \bigcup_{\alpha \in (0,1]} A_\alpha \subseteq X.$$

$$\mathsf{A}(t)=\begin{cases}\dfrac{t-a}{b-a}, & \text{if } a<t<b,\\ 1, & \text{if } t=b,\\ \dfrac{c-t}{c-b}, & \text{if } b<t<c,\\ 0, & \text{otherwise.}\end{cases}$$

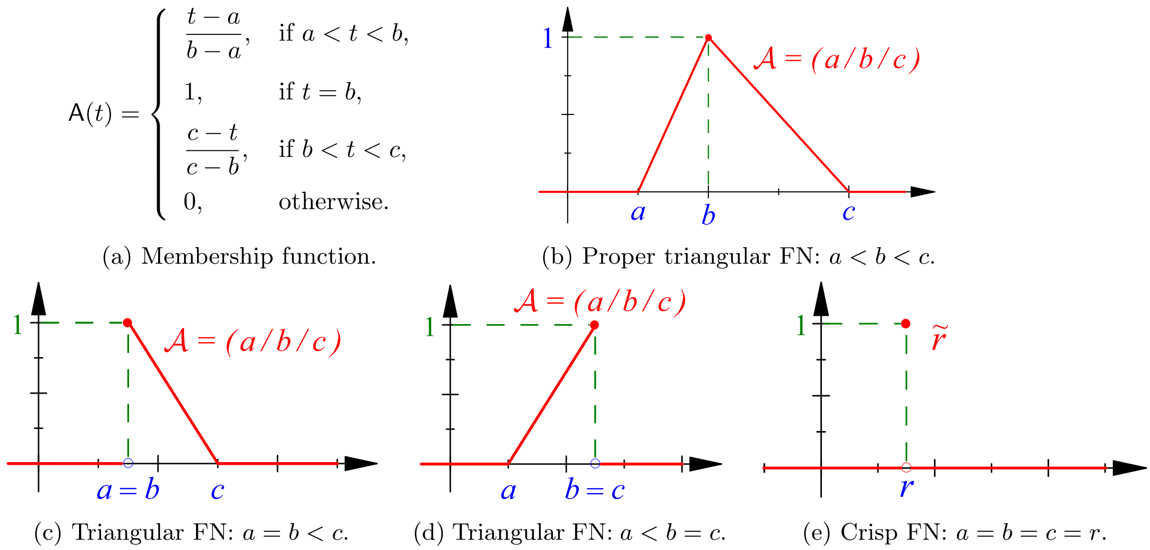


Fig. 1. Membership functions and some classes of generalized triangular fuzzy numbers.

If $\alpha,\beta\in(0,1]$ are such that $\alpha\le\beta$, then $\ker(A)=A_1\subseteq A_\beta\subseteq A_\alpha\subseteq \mathrm{supp}(A)$.

Definition 2.1. A *fuzzy number* A *of the real line* is a fuzzy set $\mathsf{A}:\mathbb{R}\to[0,1]$ satisfying the following properties:

(FN_1) A is *normal*, that is, there is $t_0\in\mathbb{R}$ such that $\mathsf{A}(t_0)=1$;

(FN_2) A is *fuzzy convex* (i.e., $\mathsf{A}(\lambda t+(1-\lambda)s)\ge\min\{\mathsf{A}(t),\mathsf{A}(s)\}$ for all $t,s\in\mathbb{R}$ and all $\lambda\in[0,1]$);

(FN_3) A is *upper semicontinuous at every* $t_0\in\mathbb{R}$ (i.e., for all $\varepsilon>0$, there exists $\delta>0$ such that $\mathsf{A}(t)-\mathsf{A}(t_0)<\varepsilon$, whenever $|t-t_0|<\delta$);

(FN_4) the support of A is bounded.

We will denote by $\mathrm{FN}(\mathbb{R})$ to the family of all fuzzy numbers of $\mathbb{R}$.

Some authors omit the condition (FN_4), so they can consider fuzzy numbers with non-bounded supports. However, for our purposes, we will only consider fuzzy numbers with bounded support.

In general, the family $\mathrm{FN}(\mathbb{R})$ is very extensive. Each real number $r\in\mathbb{R}$ is identified with the fuzzy number $\tilde{r}:\mathbb{R}\to[0,1]$ (that will also be denoted by $\mathrm{cr}(r)$ when necessary) defined by $\tilde{r}(t)=1$, if $t=r$, and $\tilde{r}(t)=0$, if $t\neq r$ (see Fig. 1.e). These fuzzy numbers are known as *crisp*. We denote by $\tilde{\mathbb{R}}$ the set of all crisp fuzzy numbers, that is, $\tilde{\mathbb{R}}=\{\tilde{r}:r\in\mathbb{R}\}\subset\mathrm{FN}(\mathbb{R})$. A fuzzy number A on $\mathbb{R}$ is a *generalized triangular* fuzzy number if there are three real numbers (called the *corners* of the triangular fuzzy number) $a,b,c\in\mathbb{R}$, with $a\le b\le c$, such that the membership function of A is given as in Fig. 1.a. In such a case, we denote $\mathsf{A}=(a/b/c)$. We will stand $\mathrm{TFN}(\mathbb{R})$ for the family of all generalized triangular fuzzy numbers of $\mathbb{R}$.

Depending on the vertices, there are four classes of generalized triangular fuzzy numbers, represented in Fig. 1.b–e. The only case in which a triangular fuzzy number is a continuous function on $\mathbb{R}$, and its graphic representation recalls a triangle of basis $[a,c]$ and vertex at $t=b$, occurs when $a<b<c$. In this case we will say that the triangular fuzzy number is *proper*. The case $a=b<c$ corresponds to a function that is not continuous at $t=b$ from the left; if $a<b=c$, then the fuzzy number is not continuous at $t=b$ from the right; and if $a=b=c$, we obtain a crisp fuzzy number. The name "*generalized*" advices about cases $a=b$ and/or $b=c$ in where the membership functions are not continuous at $t=b$, but they have an important advantage: a crisp fuzzy number is not a proper triangular fuzzy number, but it is a generalized triangular fuzzy number (that is, $\tilde{\mathbb{R}}\subset\mathrm{TFN}(\mathbb{R})$). For simplicity, from now on, we will call about "triangular fuzzy numbers" agreeing that they are "generalized triangular fuzzy numbers".

The support of a triangular fuzzy number is an interval, but it can be closed, open, or none of them. Considering the four above-commented cases, the support of $(a/b/c)$ is: $\{b\}$, if $a=b=c$; $[b,c)$, if $a=b<c$; $(a,b]$, if $a<b=c$; and (a,c), if $a<b<c$. What is common in all cases is its closure (in the Euclidean topology of $\mathbb{R}$), which is always the interval $[a,c]$. In general, associated to any fuzzy number, we denote

$$\overline{\mathrm{supp}}(\mathsf{A})=\overline{\{t\in\mathbb{R}:\mathsf{A}(t)>0\}}$$

which, in the case of triangular fuzzy numbers, leads to $\overline{\mathrm{supp}}(a/b/c)=[a,c]$. Although the α-level sets of a fuzzy set are only defined for $\alpha\in(0,1]$, it is usual to consider that $\overline{\mathrm{supp}}(A)$ is its 0-level set. This is according to the fact that all α-level sets of a fuzzy number are non-empty, closed and bounded intervals, as it is described in the following result.

Lemma 2.2. *[(cf. [38–40])] A fuzzy set of the real line $A:\mathbb{R}\to[0,1]$ with compact support is a fuzzy number if, and only if, its level sets are determined by:*

$$A_\alpha=[\,\mathsf{A}_L(\alpha),\,\mathsf{A}_R(\alpha)\,]\qquad \textit{for all } \alpha\in(0,1].$$

where $\mathsf{A}_L:(0,1]\to\mathbb{R}$ is an increasing and left-continuous function and $\mathsf{A}_R:(0,1]\to\mathbb{R}$ is a decreasing and left continuous function. In such a case, $\mathsf{A}_L(\alpha)\le\mathsf{A}_L(1)\le\mathsf{A}_R(1)\le\mathsf{A}_R(\alpha)$ for all $\alpha\in(0,1]$ and the functions A_L and A_R are bounded.

Given $\alpha\in(0,1]$, the α-level set of $\mathsf{A}=(a/b/c)$ is $\mathsf{A}_\alpha=[\,a+(b-a)\alpha,\ c-(c-b)\alpha\,]$. Therefore, the extremes of these closed intervals are, for each $\alpha\in(0,1]$:

$$\mathsf{A}_L(\alpha)=a+(b-a)\alpha\quad\text{and}\quad\mathsf{A}_R(\alpha)=c-(c-b)\,\alpha.\tag{1}$$

These expressions are also valid for $\alpha=0$ when we agree that the 0-level set is $A_0=\overline{\mathrm{supp}}(A)=[a,c]$ so, henceforth, we will agree that the functions A_L and A_R are defined on the whole interval $[0,1]$. This extension can be done for all fuzzy numbers of $\mathbb{R}$, where $\mathsf{A}_0=\overline{\mathrm{supp}}(\mathsf{A})=[\,\mathsf{A}_L(0),\,\mathsf{A}_R(0)\,]$.

Proposition 2.3. *If $\mathsf{A}=(a/b/c)$ is a triangular fuzzy number, then the following properties hold.*

1. *The kernel of A is $\{b\}$.*
2. *The functions $\mathsf{A}_L,\mathsf{A}_R:[0,1]\to\mathbb{R}$, given by (1), are continuous on $[0,1]$ and they satisfy*

$$\begin{aligned}
&(\mathsf{A}+\mathsf{B})_L(\alpha)=\mathsf{A}_L(\alpha)+\mathsf{B}_L(\alpha), \quad (\mathsf{A}+\mathsf{B})_R(\alpha)=\mathsf{A}_R(\alpha)+\mathsf{B}_R(\alpha),\\
&(\mathsf{A}-\mathsf{B})_L(\alpha)=\mathsf{A}_L(\alpha)-\mathsf{B}_R(\alpha), \quad (\mathsf{A}-\mathsf{B})_R(\alpha)=\mathsf{A}_R(\alpha)-\mathsf{B}_L(\alpha),\\
&(\mathsf{A}\cdot\mathsf{B})_L(\alpha)=\min\left(\Delta_{\mathsf{A},\mathsf{B}}(\alpha)\right), \quad (\mathsf{A}\cdot\mathsf{B})_R(\alpha)=\max\left(\Delta_{\mathsf{A},\mathsf{B}}(\alpha)\right),\\
&\quad\text{where}\quad \Delta_{\mathsf{A},\mathsf{B}}(\alpha)=\left\{\,\mathsf{A}_L(\alpha)\mathsf{B}_L(\alpha),\ \mathsf{A}_L(\alpha)\mathsf{B}_R(\alpha),\ \mathsf{A}_R(\alpha)\mathsf{B}_L(\alpha),\ \mathsf{A}_R(\alpha)\mathsf{B}_R(\alpha)\,\right\}.
\end{aligned}$$

Box I.

$$\mathsf{A}+\mathsf{B}=(\,a+a'\,/\,b+b'\,/\,c+c'\,),\qquad \widetilde{r}\mathsf{A}=\begin{cases}(\,ra\,/\,rb\,/\,rc\,), & \text{if } r\geq 0,\\ (\,rc\,/\,rb\,/\,ra\,), & \text{if } r<0.\end{cases}$$
$$\mathsf{A}-\mathsf{B}=(\,a-c'\,/\,b-b'\,/\,c-a'\,),$$

Box II.

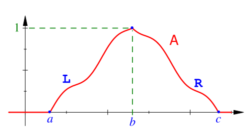
$$a=\mathsf{A}_L(0)\leq\mathsf{A}_L(\alpha)\leq\mathsf{A}_L(\beta)\leq\mathsf{A}_L(1)=b=\mathsf{A}_R(1)$$
$$\leq\mathsf{A}_R(\alpha)\leq\mathsf{A}_R(\beta)\leq\mathsf{A}_R(0)=c$$

for each $\alpha,\beta\in[0,1]$ such that $\alpha\leq\beta$.

3. *If $a=b$, then A_L is constantly "a" in $[0,1]$, and if $a<b$, then $\mathsf{A}_L:[0,1]\to[a,b]$ is a direct parametrization.*
4. *If $b=c$, then A_R is constantly "c" in $[0,1]$, and if $b<c$, then $\mathsf{A}_R:[0,1]\to[b,c]$ is an inverse parametrization.*

Triangular fuzzy numbers are particular cases of *LR*-fuzzy numbers, which form the most general kind of fuzzy numbers that we will use in this study. An *LR-fuzzy number* (see [31]) is a fuzzy number $\mathsf{A}:\mathbb{R}\to[0,1]$ defined by:

$$\mathsf{A}(t)=\begin{cases}\mathtt{L}(t), & \text{if } a<t<b,\\ 1, & \text{if } t=b,\\ \mathtt{R}(t), & \text{if } b<t<c,\\ 0, & \text{otherwise,}\end{cases}$$



where $a,b,c\in\mathbb{R}$ are three real numbers satisfying $a\leq b\leq c$ are such that, if $a<b$, $\mathtt{L}:[a,b]\to[0,1]$ is a direct parametrization, and if $b<c$, $\mathtt{R}:[b,c]\to[0,1]$ is an inverse parametrization (in particular, $\mathtt{L}(a)=\mathtt{R}(c)=0$ and $\mathtt{L}(b)=\mathtt{R}(b)=1$ when $a<b<c$). Notice that if $a=b$, the function $\mathtt{L}$ does not play a role, and if $b=c$, then $\mathtt{R}$ can be omitted. In this study we will handle *LR*-fuzzy numbers obtained as squares of triangular fuzzy numbers. Finally, we remark that *trapezoidal* fuzzy numbers [31,38] and *finite* fuzzy numbers [41] are other kinds of often employed in scientific studies fuzzy numbers.

To finish this subsection, we comment that a *defuzzification* is a mapping $\mathcal{D}:\mathcal{C}\to\mathbb{R}$ defined on a subset $\mathcal{C}\subseteq\mathrm{FN}(\mathbb{R})$ of fuzzy numbers that associates a unique real number to any fuzzy number of $\mathcal{C}$. We will denote by $\mathcal{D}_c:\mathrm{FN}(\mathbb{R})\to\mathbb{R}$ to the defuzzification that associates to each fuzzy number $\mathsf{A}\in\mathrm{FN}(\mathbb{R})$ the midpoint of its kernel, that is, $\mathcal{D}_c(\mathsf{A})=(\mathsf{A}_L(1)+\mathsf{A}_R(1))/2$.

2.2. Arithmetic with triangular fuzzy numbers

There is an arithmetic with fuzzy numbers that extends the usual arithmetic with real numbers (see [31–34]). The most usual way to operate with fuzzy numbers is throughout the interval arithmetic with the α-level sets [3] and the functions A_L and A_R. Thus, for every $\mathsf{A},\mathsf{B}\in\mathrm{FN}(\mathbb{R})$, the sum $\mathsf{A}+\mathsf{B}$, the difference $\mathsf{A}-\mathsf{B}$ and the product $\mathsf{A}\cdot\mathsf{B}$ are defined through their level sets as follows in Box I:

The previous general rules let to deduce that if A and B are fuzzy numbers, then $\mathsf{A}+\mathsf{B}$, $\mathsf{A}-\mathsf{B}$ and $\mathsf{A}\cdot\mathsf{B}$ are so. When A and B are triangular, the sum, the difference and the product by real scalars can be described in terms of the corners in the following way: if $\mathsf{A}=(a/b/c)$, $\mathsf{B}=(a'/b'/c')\in\mathrm{TFN}(\mathbb{R})$ and $r\in\mathbb{R}$, then (see Box II)

Proposition 2.4. *If A and B are triangular and $\widetilde{r}$ is crisp, then $(\mathsf{A}+\widetilde{r})-(\mathsf{B}+\widetilde{r})=\mathsf{A}-\mathsf{B}$.*

2.3. The Roldán López de Hierro et al.'s binary relation $\preccurlyeq$

In recent times, fuzzy binary relations have been employed for ranking fuzzy numbers. In [36], Roldán López de Hierro et al. introduced the following binary relation on $\mathrm{FN}(\mathbb{R})$, that we will denote on this manuscript by $\preccurlyeq$. Let μ denote the Euclidean measure of subsets of $\mathbb{R}$. Given two FNs $\mathsf{A},\mathsf{B}\in\mathrm{FN}(\mathbb{R})$, let consider the subsets:

$$\mathbb{I}_{\mathsf{A},\mathsf{B}}=\left\{\,\alpha\in[0,1]\,:\,\mathsf{A}_L(\alpha)\leq\mathsf{B}_L(\alpha)\text{ and }\mathsf{A}_R(\alpha)\leq\mathsf{B}_R(\alpha)\,\right\}\qquad\text{and}$$
$$\mathbb{I}_{\mathsf{B},\mathsf{A}}=\left\{\,\alpha\in[0,1]\,:\,\mathsf{B}_L(\alpha)\leq\mathsf{A}_L(\alpha)\text{ and }\mathsf{B}_R(\alpha)\leq\mathsf{A}_R(\alpha)\,\right\}.$$

Definition 2.5 (*Roldán López de Hierro et al. [36], Definition 4*)**.** Given $\mathsf{A},\mathsf{B}\in\mathrm{FN}(\mathbb{R})$, we will write $\mathsf{A}\preccurlyeq\mathsf{B}$ if (see Box III)

The main advantages of the binary relation $\preccurlyeq$ are: (1) it is a genuinely way to ranking fuzzy numbers, that is, it is not based on any ranking index; (2) in most of cases, it is according to human intuition; (3) it satisfies a great list of reasonable properties; (4) it serves to ranking arbitrary FNs; and (5) it is a total binary relation on $\mathrm{FN}(\mathbb{R})$ (for more details, see [36,42]).

2.4. The Kruse fuzzy variance

The Kruse fuzzy variance of the fuzzy numbers $\mathsf{A}_1,\mathsf{A}_2,\ldots,\mathsf{A}_n$, denoted by $\mathrm{Var}(\mathsf{A}_1,\mathsf{A}_2,\ldots,\mathsf{A}_n)$, is the fuzzy number whose level sets are determined by:

$$\mathrm{Var}(\mathsf{A}_1,\mathsf{A}_2,\ldots,\mathsf{A}_n)_L(\alpha)$$
$$=\min\left\{\,\mathrm{var}(t_1,t_2,\ldots,t_n)\,:\,t_i\in(\mathsf{A}_i)_\alpha\text{ for all }i\in[n]\,\right\}\quad\text{and}$$
$$\mathrm{Var}(\mathsf{A}_1,\mathsf{A}_2,\ldots,\mathsf{A}_n)_R(\alpha)$$
$$=\max\left\{\,\mathrm{var}(t_1,t_2,\ldots,t_n)\,:\,t_i\in(\mathsf{A}_i)_\alpha\text{ for all }i\in[n]\,\right\}.$$

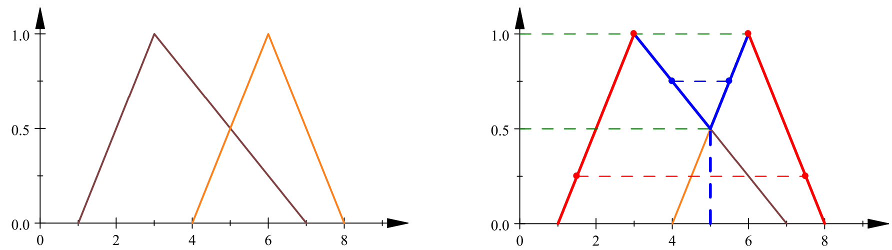
Example 2.6. Let $\mathsf{A}_1=(1/3/7)$ and $\mathsf{A}_2=(4/6/8)$ (see Fig. 2.a). Then $(\mathsf{A}_1)_\alpha=[1+2\alpha,7-4\alpha]$ and $(\mathsf{A}_2)_\alpha=[4+2\alpha,8-2\alpha]$ for each $\alpha\in[0,1]$. On the one hand, the furthest points of the respective α-cuts are $1+2\alpha\in(\mathsf{A}_1)_\alpha$ and $8-2\alpha\in(\mathsf{A}_2)_\alpha$ (plotted in red color in Fig. 2.b), so their variance is:

$$\mathrm{Var}(\mathsf{A}_1,\mathsf{A}_2)_R(\alpha)=\mathrm{var}(1+2\alpha,8-2\alpha)$$
$$=\frac{(1+2\alpha)^2+(8-2\alpha)^2}{2}-\left(\frac{9}{2}\right)^2=\left(\frac{4\alpha-7}{2}\right)^2.$$

On the other hand, if $\alpha\in[0,0.5]$, then $5\in(\mathsf{A}_1)_\alpha\cap(\mathsf{A}_2)_\alpha$ so there is no distance between $(\mathsf{A}_1)_\alpha$ and $(\mathsf{A}_2)_\alpha$ and $\mathrm{Var}(\mathsf{A}_1,\mathsf{A}_2)_L(\alpha)=0$, but if $\alpha\in[0.5,1]$, the closest points of the respective α-cuts are $7-4\alpha\in(\mathsf{A}_1)_\alpha$

$$\begin{cases} \textit{either} & \mu\left(\mathbb{I}_{\mathsf{A},\mathsf{B}}\right) \geq \mu\left(\mathbb{I}_{\mathsf{B},\mathsf{A}}\right) \textit{ and } \mu\left(\mathbb{I}_{\mathsf{A},\mathsf{B}}\right) > 0, \\ \textit{or} & \mu\left(\mathbb{I}_{\mathsf{A},\mathsf{B}}\right) = \mu\left(\mathbb{I}_{\mathsf{B},\mathsf{A}}\right) = 0 \quad \textit{and} \\ & \quad \mathsf{A}_L(0) + \mathsf{A}_L(1) + \mathsf{A}_R(1) + \mathsf{A}_R(0) \leq \mathsf{B}_L(0) + \mathsf{B}_L(1) + \mathsf{B}_R(1) + \mathsf{B}_R(0). \end{cases}$$

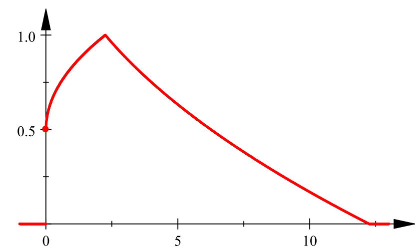
Box III.



(a) Triangular FNs. (b) Furthest and closest points in α-cuts.

$$\mathrm{Var}(\mathsf{A}_1,\mathsf{A}_2)(t) = \begin{cases} \dfrac{3+2\sqrt{t}}{6}, & \textit{if } 0 < t \leq \dfrac{9}{4}, \\ \dfrac{7-2\sqrt{t}}{4}, & \textit{if } \dfrac{9}{4} < t < \dfrac{49}{4}, \\ 0, & \textit{otherwise.} \end{cases}$$

(c) Membership function.



(d) Kruse fuzzy variance.

Fig. 2. Triangular fuzzy numbers and their Kruse fuzzy variance (Example 2.6).

and $4+2\alpha \in (\mathsf{A}_2)_\alpha$ (plotted in blue color in Fig. 2.b), so, in this case, if $\alpha \in [0.5, 1]$,

$$\begin{aligned} \mathrm{Var}(\mathsf{A}_1,\mathsf{A}_2)_L(\alpha) &= \mathrm{var}(7-4\alpha, 4+2\alpha) \\ &= \frac{(7-4\alpha)^2+(4+2\alpha)^2}{2} - \left(\frac{11}{2}-\alpha\right)^2 = \left(\frac{3(2\alpha-1)}{2}\right)^2. \end{aligned}$$

Taking into account these extremes for the level sets of $\mathrm{Var}(\mathsf{A}_1,\mathsf{A}_2)$, we conclude that this fuzzy variance is given as in Fig. 2.c, and plotted in Fig. 2.d.

3. Dispersion measures on fuzzy numbers

In this section we present, at a theoretical level, the essential tools to be able to work out the dispersion associated to a finite set of fuzzy numbers within a specific class. With this aim, we reflect on the need to handle both two classes of fuzzy numbers (those that generate the dispersion and those that model the dispersion) and a fuzzy ranking methodology (which serves to discern when a result is greater than or equal to another one). To face the open problem of introducing an appropriate notion of fuzzy dispersion measure, we will take into account the properties that the real variance satisfies: symmetry, invariance, homogeneity, normality and non-negativity.

Our intuition leads us to consider, as a first tentative, associated to a finite set $\mathcal{H} = \{\mathsf{A}_1,\mathsf{A}_2,\ldots,\mathsf{A}_n\}$ of fuzzy numbers, the algebraic combination:

$$\mathcal{V}_p(\mathcal{H}) = \frac{1}{n}\sum_{i=1}^{n}(\mathsf{A}_i - \overline{\mathsf{A}})^p, \tag{2}$$

where $p \in \mathbb{N}$ and $\overline{\mathsf{A}} = \frac{1}{n}\sum_{i=1}^{n}\mathsf{A}_i$ is the fuzzy *mean*. However, when $p=2$, some researchers can consider that this expression is not a true fuzzy variance in a strict sense: on the one hand, it can produce fuzzy numbers whose level sets include a piece in the negative part of the real line (which, in the fuzzy setting, could cause a lack of interpretability), and, on the other hand, due to the previous fact, it can seem less informative than the real variance. Notice that the case $p=1$ is not trivial in the fuzzy context: although in the real case it leads to zero, in the fuzzy framework it is a fuzzy number whose kernel includes the number 0 (but it could have a piece in $(0,+\infty)$ and another piece on $(-\infty,0)$).

Throughout this section, let $\mathcal{C},\mathcal{C}' \subseteq \mathrm{FN}(\mathbb{R})$ be two subsets of fuzzy numbers of the real line and let $\precsim$ be a binary relation on $\mathcal{C}'$. For the sake of clarity, we denote by $\prec$ to the binary relation on $\mathcal{C}'$ such that $\mathsf{A} \prec \mathsf{B}$ when $\mathsf{A} \precsim \mathsf{B}$ holds but $\mathsf{B} \precsim \mathsf{A}$ is false, and by $\sim$ to the binary relation on $\mathcal{C}'$ such that $\mathsf{A} \sim \mathsf{B}$ when $\mathsf{A} \precsim \mathsf{B}$ and $\mathsf{B} \precsim \mathsf{A}$ at the same time.

In the next definition we will consider a mapping $\mathcal{V} : \bigcup_{n\in\mathbb{N}} \mathcal{C}^n \to \mathcal{C}'$. For simplicity, we will denote the fuzzy number $\mathcal{V}|_{\mathcal{C}^n}(\mathsf{A}_1,\mathsf{A}_2,\ldots,\mathsf{A}_n)$ by $\mathcal{V}(\mathsf{A}_1,\mathsf{A}_2,\ldots,\mathsf{A}_n)$ or simply by $\mathcal{V}(\mathcal{H})$ where $\mathcal{H} = \{\mathsf{A}_1,\mathsf{A}_2,\ldots,\mathsf{A}_n\} \subseteq \mathcal{C}$.

Definition 3.1. A *fuzzy dispersion space* is a quadruple $(\mathcal{C},\mathcal{C}',\mathcal{V},\precsim)$ where $\mathcal{C},\mathcal{C}' \subseteq \mathrm{FN}(\mathbb{R})$ are two non-empty subsets of fuzzy numbers such that $\widetilde{0} \in \mathcal{C}'$, $\precsim$ is a total binary relation on $\mathcal{C}'$ and $\mathcal{V} : \bigcup_{n\in\mathbb{N}} \mathcal{C}^n \to \mathcal{C}'$ is a mapping satisfying the following properties for each $n \in \mathbb{N}$ and each $\mathsf{A}_1,\mathsf{A}_2,\ldots,\mathsf{A}_n \in \mathcal{C}$:

(*a*) *Symmetry* (or *commutativity*): for each permutation $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$,

$$\mathcal{V}(\mathsf{A}_{\sigma(1)},\mathsf{A}_{\sigma(2)},\ldots,\mathsf{A}_{\sigma(n)}) = \mathcal{V}(\mathsf{A}_1,\mathsf{A}_2,\ldots,\mathsf{A}_n);$$

(*b*) *crisp-invariance*: $\mathcal{C}$ is closed under sums with crisp fuzzy numbers and, for all $\widetilde{r} \in \widetilde{\mathbb{R}}$,

$$\mathcal{V}(\mathsf{A}_1+\widetilde{r},\, \mathsf{A}_2+\widetilde{r},\ldots,\, \mathsf{A}_n+\widetilde{r}) = \mathcal{V}(\mathsf{A}_1,\mathsf{A}_2,\ldots,\mathsf{A}_n);$$

(c) *m-homogeneity*: there is $m \in \mathbb{N}$ such that, for all $\widetilde{r} \in \widetilde{\mathbb{R}}$,

$$\mathcal{V}(\widetilde{r}\,\mathsf{A}_1, \widetilde{r}\,\mathsf{A}_2, \ldots, \widetilde{r}\,\mathsf{A}_n) = \widetilde{|r|}^{\,m}\, \mathcal{V}(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n);$$

(d) *normality*:

$$\mathcal{V}(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n) = \widetilde{0} \quad \Leftrightarrow \quad \mathsf{A}_1 = \mathsf{A}_2 = \cdots = \mathsf{A}_n \in \widetilde{\mathbb{R}};$$

(e) *$\precsim$-positivity*:

$$\widetilde{0} \precsim \mathcal{V}(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n).$$

In this case, the mapping $\mathcal{V}$ is called the *fuzzy dispersion measure* of $(\mathcal{C}, \mathcal{C}', \mathcal{V}, \precsim)$ and the fuzzy number $\mathcal{V}(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n) \in \mathcal{C}'$ is called the *fuzzy dispersion of the set* $\{\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n\}$ *w.r.t. the fuzzy dispersion space* $(\mathcal{C}, \mathcal{C}', \mathcal{V}, \precsim)$.

For short, we will write *FDS* rather than "*Fuzzy Dispersion Space*" and *FDM* rather than "*Fuzzy Dispersion Measure*". A FDM can also satisfy another reasonable properties that we now comment.

Definition 3.2. We will say that a mapping $\mathcal{V} : \underset{n\in\mathbb{N}}{\cup} \mathcal{C}^n \to \mathcal{C}'$ is:

(f) *positive* if $\mathrm{supp}(\mathcal{V}(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n)) \subseteq [0, +\infty)$;

(g) *conservative* if the midpoint of the kernel of $\mathcal{V}(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n)$ is the real variance of the midpoints of the kernels of $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$, that is,

$$\mathcal{D}_c(\mathcal{V}(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n)) = \mathrm{var}(\mathcal{D}_c(\mathsf{A}_1), \mathcal{D}_c(\mathsf{A}_2), \ldots, \mathcal{D}_c(\mathsf{A}_n));$$

(h) of *variance-type* if $\widetilde{\mathbb{R}} \subseteq \mathcal{C}$ and:

$$\mathcal{V}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) = \mathrm{cr}\left(\mathrm{var}(r_1, r_2, \ldots, r_n)\right);$$

(i) *quasi-normal* if:

$$\mathsf{A}_1 = \mathsf{A}_2 = \cdots = \mathsf{A}_n \in \widetilde{\mathbb{R}} \quad \Rightarrow \quad \mathcal{V}(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n) = \widetilde{0} \quad \Rightarrow \quad \mathsf{A}_1 = \mathsf{A}_2 = \cdots = \mathsf{A}_n.$$

Remark 3.3.

1. In Definition 3.1.e we have set that a FDM must provide fuzzy numbers that are greater than or equal to $\widetilde{0}$ w.r.t. the binary relation $\precsim$. This condition directly depends on $\precsim$, but it says nothing about the geometrical shape of the obtained FDM or about the sign of the numbers on its support. In fact, it is usual that a fuzzy number could be greater than zero even having a support with a piece included in the negative part of the set of all real numbers. For instance, most of ranking methodologies consider that the symmetric triangular fuzzy number $(-\delta/0/\delta)$ are equivalent to $\widetilde{0}$.
2. The crisp fuzzy number $\widetilde{0}$ could be replaced in items (d) and (e) of Definition 3.1 by any other fuzzy number. Anyway, we have decided to maintain this fuzzy number for simplicity.
3. The class $\mathcal{C}$ is a *convex cone* if it is closed under sums (if $\mathsf{A}, \mathsf{B} \in \mathcal{C}$, then $\mathsf{A} + \mathsf{B} \in \mathcal{C}$) and products by positive scalars (if $\mathsf{A} \in \mathcal{C}$ and $r > 0$, then $\widetilde{r}\,\mathsf{A} \in \mathcal{C}$). Notice that some of the properties associated to a FDM that we have introduced could require that $\mathcal{C}$ is a convex cone.
4. If in properties (g) and (h) of Definition 3.2 we replace the real variance by the standard deviation (or any other real dispersion measure such as the coefficient of variation), and we accordingly modify other items, then we could call about a FDM of *standard-deviation-type*.

Example 3.4. Let $\mathcal{C} = \mathrm{FN}(\mathbb{R})$, let $\mathcal{C}' = \widetilde{\mathbb{R}}$, let $\precsim_0$ the binary relation on $\widetilde{\mathbb{R}}$ such that $\widetilde{r} \precsim_0 \widetilde{s} \Leftrightarrow r \leq s$, and let $\mathcal{V}_0$ the mapping defined as:

$$\mathcal{V}_0(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n) = \mathrm{cr}(\mathrm{var}(\mathcal{D}_c(\mathsf{A}_1), \mathcal{D}_c(\mathsf{A}_2), \ldots, \mathcal{D}_c(\mathsf{A}_n))). \tag{3}$$

Then $\mathcal{V}_0$ is symmetric, crisp-invariant, 2-homogeneous, $\preccurlyeq$-positive, positive, shift-invariant, conservative and of variance type. However, it is not a FDM because it is neither normal nor quasi-normal.

Example 3.5. If in the previous example we take $\mathcal{C} = \widetilde{\mathbb{R}}$ and we define $\mathcal{V}_0$ as in (3), then $(\mathcal{C}, \mathcal{C}', \mathcal{V}_0, \precsim_0)$ is a FDS, and the FDM $\mathcal{V}_0$ satisfies all the above-mentioned properties. In a wide sense, $\mathcal{V}_0$ is the crisp translation to $\widetilde{\mathbb{R}}$ of the notion of real variance.

Example 3.6. Let $\mathcal{C} = \mathcal{C}' = \mathrm{TFN}(\mathbb{R})$ and let $\preceq$ be a total binary relation on $\mathrm{TFN}(\mathbb{R})$ such that $\widetilde{0} \preceq \mathsf{A}$ when $\mathrm{supp}(\mathsf{A}) \subseteq [0, +\infty)$ (see [37]). Given $\mathcal{H} = \{\,\mathsf{A}_i = (a_i/b_i/c_i)\,\}_{i=1}^n \subset \mathrm{TFN}(\mathbb{R})$, let $W_{\mathcal{H}} = \{\mathrm{var}\{a_i\}, \mathrm{var}\{b_i\}, \mathrm{var}\{c_i\}\}$. Although the ordering of the elements of $W_{\mathcal{H}}$ is not known (*cf.* [43]), we can define:

$$\mathcal{V}_{\mathrm{T}}(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n) = \left(\min W_{\mathcal{H}} \,/\, \mathrm{med}\, W_{\mathcal{H}} \,/\, \max W_{\mathcal{H}}\right),$$

where $\mathrm{med}\, W_{\mathcal{H}}$ is the *median* of the set $W_{\mathcal{H}}$ (the central element). Then $\mathcal{V}_{\mathrm{T}}$ is symmetric, crisp-invariant, 2-homogeneous, $\preceq$-positive and positive. However, it is not a FDM because it is quasi-normal rather than normal.

The previous examples show that, in general, the above-mentioned properties are not interesting when they are isolatedly considered, but we must consider a combination of such properties properly adapted to the study in progress.

Theorem 3.7. *The Kruse fuzzy variance is a positive FDM on* $\mathrm{TFN}(\mathbb{R})$ *w.r.t. the Roldán López de Hierro et al.'s binary relation* $\preccurlyeq$.

The following sections are dedicated to study the main properties of the measure described in (2) in the case in which $p = 2$. One of the main aims of this work is to prove that it defines a FDM when it is considered associated to the fuzzy binary relation $\preccurlyeq$ introduced in [36]. Before that, we develop a complete study about the geometrical shapes and analytical properties that a fuzzy number obtained by (2) must satisfy, at least in the case that all input data are triangular fuzzy numbers. In this line, notice that

$$\frac{1}{n}\sum_{i=1}^{n}(\mathsf{A}_i - \overline{\mathsf{A}})^2 = \frac{1}{n}\sum_{i=1}^{n}(\mathsf{A}_i - \overline{\mathsf{A}})\cdot(\mathsf{A}_i - \overline{\mathsf{A}})$$

employs the four basic arithmetic operations with fuzzy numbers (sum, subtraction, product and division), but the product is reduced to compute the square of a fuzzy number.

4. The canonical fuzzy dispersion measure

In this section we describe some of the main properties of the mapping $\mathcal{V}_2 : \underset{n\in\mathbb{N}}{\cup} \mathrm{TFN}(\mathbb{R})^n \to \mathrm{FN}(\mathbb{R})$ defined, for each $n \in \mathbb{N}$ and each set $\mathcal{H} = \{\,\mathsf{A}_i = (a_i/b_i/c_i)\,\}_{i=1}^n$ of n triangular fuzzy numbers (such that $a_i \leq b_i \leq c_i$ for all $i \in [n]$) as:

$$\mathcal{V}_2(\mathcal{H}) = \frac{1}{n}\sum_{i=1}^{n}(\mathsf{A}_i - \overline{\mathsf{A}})^2. \tag{4}$$

One of the main aims of this work is to prove that $\mathcal{V}_2$ is a FDM on $\mathrm{TFN}(\mathbb{R})$. In fact, it is 2-homogeneity, conservativity and it is of variance-type (see Theorem 4.6). To do it, we previously need to introduce some auxiliary results.

Proposition 4.1. *If the kernels of two fuzzy numbers* $\mathsf{A}, \mathsf{B} \in \mathrm{FN}(\mathbb{R})$ *are singleton, then the kernels of* $\mathsf{A} + \mathsf{B}$, $\mathsf{A} - \mathsf{B}$, $\widetilde{r}\,\mathsf{A}$ *and* A^2 *are also singleton. In fact, if* $\ker \mathsf{A} = \{a\}$ *and* $\ker \mathsf{B} = \{b\}$, *then* $\ker(\mathsf{A} + \mathsf{B}) = \{a + b\}$, $\ker(\mathsf{A} - \mathsf{B}) = \{a - b\}$, $\ker(\widetilde{r}\,\mathsf{A}) = \{ra\}$ *and* $\ker(\mathsf{A}^2) = \{a^2\}$.

For convenience, we will introduce the notation we will use throughout the following sections. From now on, unless otherwise is stated, $\mathcal{H} = \{\,\mathsf{A}_i = (a_i/b_i/c_i)\,\}_{i=1}^n$ will denote a set of n triangular fuzzy numbers ($a_i \leq b_i \leq c_i$ for all $i \in [n]$) and $\overline{\mathsf{A}} = (1/n)(\mathsf{A}_1 + \mathsf{A}_2 + \cdots + \mathsf{A}_n) = (\,\overline{a}\,/\,\overline{b}\,/\,\overline{c}\,)$ will denote its mean, where $\overline{a} = (\Sigma a_i)/n$, $\overline{b} = (\Sigma b_i)/n$ and $\overline{c} = (\Sigma c_i)/n$. Clearly $\overline{a} \leq \overline{b} \leq \overline{c}$. We also use the notation $\mathsf{B}_i = \mathsf{A}_i - \overline{\mathsf{A}} = (\,a_i - \overline{c}\,/\,b_i - \overline{b}\,/\,c_i - \overline{a}\,) \in \mathrm{TFN}(\mathbb{R})$ for each $i \in [n]$ and

$$\Delta_i = \left\{\,\left(a_i - \overline{c}\right)^2,\ \left(a_i - \overline{c}\right)\left(c_i - \overline{a}\right),\ \left(c_i - \overline{a}\right)^2\,\right\} \quad \text{and}$$

$\Delta'_i = \{ (a_i - \overline{c})^2, (c_i - \overline{a})^2 \}$

for each $i \in [n]$. To simplify the notation, we will sometimes denote $\mathsf{M} = \mathcal{V}_2(\mathcal{H})$ defined by (4).

Lemma 4.2. *Given a set of n triangular fuzzy numbers $\mathcal{H} = \{ \mathsf{A}_i = (a_i/b_i/c_i) \}_{i=1}^n$, the following properties hold.*

1. $\ker(\mathcal{V}_2(\mathcal{H})) = \{ \operatorname{var}(b_1, b_2, \ldots, b_n) \}$.
2. *The functions $\mathcal{V}_2(\mathcal{H})_L, \mathcal{V}_2(\mathcal{H})_R : [0,1] \to \mathbb{R}$ are continuous on $[0,1]$ and they are given by:*

$$\mathcal{V}_2(\mathcal{H})_L = \frac{1}{n}\sum_{i=1}^n [(\mathsf{A}_i - \overline{\mathsf{A}})^2]_L \quad \textit{and} \quad \mathcal{V}_2(\mathcal{H})_R = \frac{1}{n}\sum_{i=1}^n [(\mathsf{A}_i - \overline{\mathsf{A}})^2]_R. \tag{5}$$

3. *The 0-level set of $\mathcal{V}_2(\mathcal{H})$ is the interval*

$$\overline{\operatorname{supp}}(\mathcal{V}_2(\mathcal{H})) = \left[\frac{1}{n}\sum_{i=1}^n \min \Delta_i, \ \frac{1}{n}\sum_{i=1}^n \max \Delta'_i \right].$$

4. *Furthermore, $[\mathcal{V}_2(\mathcal{H})]_L(0) + [\mathcal{V}_2(\mathcal{H})]_R(0) \geq 0$.*
5. *In particular,*

$$\mathcal{V}_2(\mathcal{H})_R(0) = \frac{1}{n}\sum_{i=1}^n \max\{ (a_i - \overline{c})^2, (c_i - \overline{a})^2 \} \geq 0,$$

and the equality holds if, and only if, $\mathsf{A}_1 = \mathsf{A}_2 = \cdots = \mathsf{A}_n \in \widetilde{\mathbb{R}}$.

Proof. Item (1) Since A_i and $\overline{\mathsf{A}}$ are triangular fuzzy numbers, their kernels are singletons (recall item 1 of Proposition 2.3); concretely, $\ker \mathsf{A}_i = \{b_i\}$ and $\ker \overline{\mathsf{A}} = \{\overline{b}\}$. Using Proposition 4.1 we deduce that:

$$\begin{aligned}
&\ker \mathsf{A}_i = \{b_i\}, \ \ker \overline{\mathsf{A}} = \{\overline{b}\} \quad \Rightarrow \quad \ker(\mathsf{A}_i - \overline{\mathsf{A}}) = \{b_i - \overline{b}\} \\
&\Rightarrow \quad \ker\left((\mathsf{A}_i - \overline{\mathsf{A}})^2 \right) = \{(b_i - \overline{b})^2\} \\
&\Rightarrow \quad \ker\left(\sum_{i=1}^n (\mathsf{A}_i - \overline{\mathsf{A}})^2 \right) = \left\{ \sum_{i=1}^n (b_i - \overline{b})^2 \right\} \\
&\Rightarrow \quad \ker(\mathcal{V}_2(\mathcal{H})) = \left\{ \frac{1}{n}\sum_{i=1}^n (b_i - \overline{b})^2 \right\} = \{ \operatorname{var}(b_1, b_2, \ldots, b_n) \}.
\end{aligned}$$

Item (2) Since, for each $i \in [n]$, $\mathsf{B}_i = \mathsf{A}_i - \overline{\mathsf{A}} = (a_i - \overline{c}/b_i - \overline{b}/c_i - \overline{a}) \in \operatorname{TFN}(\mathbb{R})$ is a triangular fuzzy number, the functions $(\mathsf{B}_i)_L, (\mathsf{B}_i)_R : [0,1] \to \mathbb{R}$ are continuous in $[0,1]$. Therefore, for each $\alpha \in (0,1]$, as

$$\begin{aligned}
&[(\mathsf{A}_i - \overline{\mathsf{A}})^2]_L(\alpha) \\
&= [(\mathsf{B}_i)^2]_L(\alpha) = \min\{ ((\mathsf{B}_i)_L(\alpha))^2, (\mathsf{B}_i)_L(\alpha) \cdot (\mathsf{B}_i)_R(\alpha), ((\mathsf{B}_i)_R(\alpha))^2 \} \quad \text{and}
\end{aligned} \tag{6}$$

$$\begin{aligned}
&[(\mathsf{A}_i - \overline{\mathsf{A}})^2]_R(\alpha) \\
&= [(\mathsf{B}_i)^2]_R(\alpha) = \max\{ ((\mathsf{B}_i)_L)^2(\alpha), (\mathsf{B}_i)_L(\alpha) \cdot (\mathsf{B}_i)_R(\alpha), ((\mathsf{B}_i)_R(\alpha))^2 \},
\end{aligned} \tag{7}$$

then the functions $[(\mathsf{A}_i - \overline{\mathsf{A}})^2]_L$ and $[(\mathsf{A}_i - \overline{\mathsf{A}})^2]_R$ are also continuous in $(0,1]$. Letting $\alpha \to 0^+$ and using the continuity of the involved functions, this means that (5) holds in $[0,1]$ and such functions are continuous in $[0,1]$.

Item (3) Since $\mathsf{B}_i = \mathsf{A}_i - \overline{\mathsf{A}} = (a_i - \overline{c}/b_i - \overline{b}/c_i - \overline{a})$, then $(\mathsf{B}_i)_L(0) = a_i - \overline{c}$ and $(\mathsf{B}_i)_R(0) = c_i - \overline{a}$. Using (6)–(7), we deduce that

$$\begin{aligned}
&[(\mathsf{A}_i - \overline{\mathsf{A}})^2]_L(0) \\
&= [(\mathsf{B}_i)^2]_L(0) = \min\{ [(\mathsf{B}_i)_L(0)]^2, (\mathsf{B}_i)_L(0) \cdot (\mathsf{B}_i)_R(0), [(\mathsf{B}_i)_R(0)]^2 \} \\
&= \min\{ (a_i - \overline{c})^2, (a_i - \overline{c})(c_i - \overline{a}), (c_i - \overline{a})^2 \} = \min \Delta_i,
\end{aligned}$$

and similarly $[(\mathsf{A}_i - \overline{\mathsf{A}})^2]_R(0) = \max \Delta_i$. Taking into account that if $t, s \in \mathbb{R}$, then

$$ts + \max\{t^2, s^2\} \geq 0 \tag{8}$$

(because $ts \leq |ts| \leq \max\{t^2, s^2\}$) and applying this property to $t = a_i - \overline{c}$ and $s = c_i - \overline{a}$, we deduce that $(a_i - \overline{c})(c_i - \overline{a}) \leq \max\{ (a_i - \overline{c})^2, (c_i - \overline{a})^2 \}$. As a result,

$$\begin{aligned}
[(\mathsf{A}_i - \overline{\mathsf{A}})^2]_R(0) &= \max \Delta_i = \max\{ (a_i - \overline{c})^2, (a_i - \overline{c})(c_i - \overline{a}), (c_i - \overline{a})^2 \} \\
&= \max\{ (a_i - \overline{c})^2, (c_i - \overline{a})^2 \} = \max \Delta'_i.
\end{aligned}$$

Finally, applying (5), we derive that

$$\begin{aligned}
\mathcal{V}_2(\mathcal{H})_L(0) &= \frac{1}{n}\sum_{i=1}^n [(\mathsf{A}_i - \overline{\mathsf{A}})^2]_L(0) = \frac{1}{n}\sum_{i=1}^n \min \Delta_i \quad \text{and} \\
\mathcal{V}_2(\mathcal{H})_R(0) &= \frac{1}{n}\sum_{i=1}^n [(\mathsf{A}_i - \overline{\mathsf{A}})^2]_R(0) = \frac{1}{n}\sum_{i=1}^n \max \Delta'_i.
\end{aligned} \tag{9}$$

Item (4) Also applying inequality (8) to $t = a_i - \overline{c}$ and $s = c_i - \overline{a}$, we deduce that

$$(a_i - \overline{c})(c_i - \overline{a}) + \max\{ (a_i - \overline{c})^2, (c_i - \overline{a})^2 \} \geq 0.$$

Hence

$$\left\{ \begin{array}{l}
(a_i - \overline{c})^2 + \max\{ (a_i - \overline{c})^2, (c_i - \overline{a})^2 \} \geq 0, \\
(a_i - \overline{c})(c_i - \overline{a}) + \max\{ (a_i - \overline{c})^2, (c_i - \overline{a})^2 \} \geq 0, \\
(c_i - \overline{a})^2 + \max\{ (a_i - \overline{c})^2, (c_i - \overline{a})^2 \} \geq 0,
\end{array} \right.$$

so:

$$\begin{aligned}
\min \Delta_i + \max \Delta'_i = \min\{ &(a_i - \overline{c})^2, (a_i - \overline{c})(c_i - \overline{a}), (c_i - \overline{a})^2 \} \\
&+ \max\{ (a_i - \overline{c})^2, (c_i - \overline{a})^2 \} \geq 0.
\end{aligned}$$

As a result,

$$\mathcal{V}_2(\mathcal{H})_L(0) + \mathcal{V}_2(\mathcal{H})_R(0) = \frac{1}{n}\sum_{i=1}^n \left(\min \Delta_i + \max \Delta'_i \right) \geq 0.$$

Item (5). By (9),

$$\mathcal{V}_2(\mathcal{H})_R(0) = \frac{1}{n}\sum_{i=1}^n \max \Delta'_i = \frac{1}{n}\sum_{i=1}^n \max\{ (a_i - \overline{c})^2, (c_i - \overline{a})^2 \} \geq 0.$$

As all the maximums are non-negative, the equality to zero is equivalent to say that $(a_i - \overline{c})^2 = (c_i - \overline{a})^2 = 0$ for each $i \in [n]$. Then $a_i = \overline{c}$ and $c_i = \overline{a}$ for all $i \in [n]$, which implies that

$$\overline{c} = a_i \leq b_i \leq c_i = \overline{a} \leq \overline{c},$$

so $a_i = b_i = c_i = r \in \mathbb{R}$ for each $i \in [n]$, that is, $\mathsf{A}_1 = \mathsf{A}_2 = \cdots = \mathsf{A}_n = \widetilde{r} \in \widetilde{\mathbb{R}}$. ■

Lemma 4.3. *If $\mathcal{H} = \{ \mathsf{A}_i = (a_i/b_i/c_i) \}_{i=1}^n$ is a set of n triangular fuzzy numbers such that $b_1 = b_2 = \cdots = b_n$, then, for each $\alpha \in [0,1]$,*

$$\mathcal{V}_2(\mathcal{H})_L(\alpha) \leq 0 \leq \mathcal{V}_2(\mathcal{H})_R(\alpha) \quad \textit{and} \quad 0 \leq \mathcal{V}_2(\mathcal{H})_L(\alpha) + \mathcal{V}_2(\mathcal{H})_R(\alpha).$$

Proof. Let denote $\mathsf{M} = \mathcal{V}_2(\mathcal{H})$. If $\mathsf{M} = \widetilde{0}$, then $\mathsf{M}_L(\alpha) = \mathsf{M}_R(\alpha) = 0$ for each $\alpha \in [0,1]$, and the announced inequalities are, in fact, equalities. Next, suppose that $\mathsf{M} \neq \widetilde{0}$. By item 1 of Lemma 4.2, $[\mathsf{M}_L(1), \mathsf{M}_R(1)] = \ker(\mathsf{M}) = \{ \operatorname{var}(b_1, b_2, \ldots, b_n) \} = \{0\}$ because $b_1 = b_2 = \cdots = b_n$. Therefore, for each $\alpha \in [0,1]$,

$$\mathsf{M}_L(\alpha) \leq \mathsf{M}_L(1) = 0 = \mathsf{M}_R(1) \leq \mathsf{M}_R(\alpha),$$

which proves the first inequality. Next, let compute $\mathsf{M}_L(\alpha)$ and $\mathsf{M}_R(\alpha)$ for each $\alpha \in (0,1]$. Let $i \in [n]$ and $\alpha \in (0,1]$ be arbitrary. Since $b_1 = b_2 = \cdots = b_n$, then $\mathsf{B}_i = \mathsf{A}_i - \overline{\mathsf{A}} = (a_i - \overline{c}/b_i - \overline{b}/c_i - \overline{a}) = (a_i - \overline{c}/0/c_i - \overline{a}) \in \operatorname{TFN}(\mathbb{R})$, which means that $a_i - \overline{c} \leq 0 \leq c_i - \overline{a}$ and

$$\begin{aligned}
&(\mathsf{B}_i)_L(\alpha) = (a_i - \overline{c}) + [0 - (a_i - \overline{c})]\alpha = (a_i - \overline{c})(1 - \alpha) \quad \text{and} \\
&(\mathsf{B}_i)_R(\alpha) = (c_i - \overline{a}) + [(c_i - \overline{a}) - 0]\alpha = (c_i - \overline{a})(1 - \alpha).
\end{aligned}$$

Then

$[(\mathsf{A}_i - \overline{\mathsf{A}})^2]_L(\alpha)$

$$= [(\mathsf{B}_i)^2]_L(\alpha) = \min\{\, [(\mathsf{B}_i)_L(\alpha)]^2,\ (\mathsf{B}_i)_L(\alpha)\cdot(\mathsf{B}_i)_R(\alpha),\ [(\mathsf{B}_i)_R(\alpha)]^2 \,\}$$
$$= \min\{\, \left(a_i-\overline{c}\right)^2 (1-\alpha)^2,\ \left(a_i-\overline{c}\right)\left(c_i-\overline{a}\right)(1-\alpha)^2,\ \left(c_i-\overline{a}\right)^2 (1-\alpha)^2 \,\}$$
$$= (1-\alpha)^2 \min\{\, \left(a_i-\overline{c}\right)^2,\ \left(a_i-\overline{c}\right)\left(c_i-\overline{a}\right),\ \left(c_i-\overline{a}\right)^2 \,\} = (1-\alpha)^2 \min \Delta_i$$

and similarly $[\,(\mathsf{A}_i-\overline{\mathsf{A}}\,)^2]_R(\alpha) = (1-\alpha)^2 \max \Delta_i$. Since $a_i-\overline{c} \le 0 \le c_i-\overline{a}$, then $\left(a_i-\overline{c}\right)\left(c_i-\overline{a}\right) \le 0$, so

$$[\,(\mathsf{A}_i-\overline{\mathsf{A}}\,)^2]_L(\alpha) = (1-\alpha)^2\left(a_i-\overline{c}\right)\left(c_i-\overline{a}\right) \le 0 \quad \text{and}$$
$$[\,(\mathsf{A}_i-\overline{\mathsf{A}}\,)^2]_R(\alpha) = (1-\alpha)^2 \max \Delta_i'.$$

Therefore

$$\mathsf{M}_L(\alpha) = \frac{1}{n}\sum_{i=1}^{n} [\,(\mathsf{A}_i-\overline{\mathsf{A}}\,)^2]_L(\alpha)$$
$$= (1-\alpha)^2 \frac{1}{n}\sum_{i=1}^{n} \left(a_i-\overline{c}\right)\left(c_i-\overline{a}\right) \le 0 \quad \text{and} \tag{10}$$
$$\mathsf{M}_R(\alpha) = \frac{1}{n}\sum_{i=1}^{n} [\,(\mathsf{A}_i-\overline{\mathsf{A}}\,)^2]_R(\alpha)$$
$$= (1-\alpha)^2 \frac{1}{n}\sum_{i=1}^{n} \max\{\, \left(a_i-\overline{c}\right)^2,\ \left(c_i-\overline{a}\right)^2 \,\} \ge 0. \tag{11}$$

Using inequality (8) with $t = a_i-\overline{c}$ and $s = c_i-\overline{a}$, we deduce that

$$\left(a_i-\overline{c}\right)\left(c_i-\overline{a}\right) + \max\{\, \left(a_i-\overline{c}\right)^2,\ \left(c_i-\overline{a}\right)^2 \} \ge 0$$

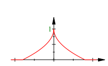
so, in particular,

$$\mathsf{M}_L(\alpha) + \mathsf{M}_R(\alpha)$$
$$= (1-\alpha)^2 \frac{1}{n}\sum_{i=1}^{n} \left[\left(a_i-\overline{c}\right)\left(c_i-\overline{a}\right) + \max\{\, \left(a_i-\overline{c}\right)^2,\ \left(c_i-\overline{a}\right)^2 \,\} \right] \ge 0.$$

The previous inequality has just been demonstrated for each $\alpha \in (0,1]$, but as M_L and M_R are continuous on $[0,1]$, then it also holds for $\alpha = 0$. ■

Corollary 4.4. *If* $\mathsf{A}_1 = \mathsf{A}_2 = \cdots = \mathsf{A}_n = (a/b/c)$, *then* $\mathcal{V}_2(\mathcal{H}) = \mathrm{cr}[(c-a)^2]\,(-1/0/1)^2$*, whose level sets are, for each* $\alpha \in [0,1]$*:*

$$\mathcal{V}_2(\mathcal{H})_\alpha = [\, -(1-\alpha)^2\ (c-a)^2\ ,\ (1-\alpha)^2\ (c-a)^2 \,].$$



Proof. It follows from $\mathsf{A}_i - \overline{\mathsf{A}} = (a/b/c) - (a/b/c) = (a-c/0/c-a) = \mathrm{cr}[c-a](-1/0/1)$, so

$$(\,\mathsf{A}_i-\overline{\mathsf{A}}\,)^2 = (\mathrm{cr}[c-a](-1/0/1))^2 = \mathrm{cr}[c-a](-1/0/1)\,\mathrm{cr}[c-a](-1/0/1) =$$
$$= \mathrm{cr}[c-a]\,\mathrm{cr}[c-a](-1/0/1)(-1/0/1) = \mathrm{cr}[(c-a)^2](-1/0/1)^2.$$

Hence

$$\mathcal{V}_2(\mathcal{H})$$
$$= \frac{1}{n}\sum_{i=1}^{n}(\,\mathsf{A}_i-\overline{\mathsf{A}}\,)^2 = \frac{1}{n}\sum_{i=1}^{n} \mathrm{cr}[(c-a)^2](-1/0/1)^2 = \mathrm{cr}[(c-a)^2](-1/0/1)^2.$$

The second part follows from equations (10)–(11) taking into account that $a_i = a = \overline{a}$ and $c_i = c = \overline{c}$ for all $i \in [n]$. ■

Lemma 4.5. $\mathcal{V}_2$ *is of variance-type.*

Proof. Clearly $\widetilde{\mathbb{R}} \subset \mathrm{TFN}(\mathbb{R})$. Given $\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n \in \widetilde{\mathbb{R}}$, their mean is

$$\overline{\mathsf{r}} = \frac{\widetilde{r}_1 + \widetilde{r}_2 + \cdots + \widetilde{r}_n}{n} = \mathrm{cr}\left(\frac{r_1+r_2+\cdots+r_n}{n}\right) = \mathrm{cr}\left(\overline{r}\right)$$

and each subtraction and square in the crisp case work as in the real case:

$$\left(\widetilde{r}_i - \overline{\mathsf{r}}\right)^2 = \left(\widetilde{r}_i - \mathrm{cr}\left(\overline{r}\right)\right)^2 = \left(\mathrm{cr}\left(r_i - \overline{r}\right)\right)^2 = \mathrm{cr}\left(\left(r_i-\overline{r}\right)^2\right),$$

so:

$$\mathcal{V}_2(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) = \frac{1}{n}\sum_{i=1}^{n}(\widetilde{r}_i - \overline{\mathsf{r}})^2 = \frac{1}{n}\sum_{i=1}^{n} \mathrm{cr}\left(\left(r_i-\overline{r}\right)^2\right) =$$
$$= \mathrm{cr}\left(\frac{1}{n}\sum_{i=1}^{n}\left(r_i-\overline{r}\right)^2\right) = \mathrm{cr}\left(\mathrm{var}(r_1, r_2, \ldots, r_n)\right). \quad ■$$

The main result in this paper is the following one.

Theorem 4.6. *The mapping* $\mathcal{V}_2$ *defined in* (4) *is a FDM w.r.t. the fuzzy binary relation* $\preccurlyeq$ *introduced by Roldán López de Hierro et al. in* [36] *and, consequently, the quadruple* $(\mathcal{C} = \mathrm{TFN}(\mathbb{R}), \mathcal{C}' = \mathrm{FN}(\mathbb{R}), \mathcal{V}_2, \preccurlyeq)$ *is a FDS.*

Furthermore, the FDM with $m = 2$, $\mathcal{V}_2$ *also satisfies the following properties:* 2*-homogeneity, conservativity and it is of variance-type.*

Proof. We check all properties. The symmetry is immediate because the sum of fuzzy numbers is commutative. Also notice that, by Lemma 4.5, $\mathcal{V}_2$ is of variance-type.

- *Crisp-invariance*.
  The class $\mathcal{C} = \mathrm{TFN}(\mathbb{R})$ is closed under sums with crisp fuzzy numbers and

  $$(a/b/c) + \widetilde{r} = (a/b/c) + (r/r/r) = (\,a+r\,/\,b+r\,/\,c+r\,) \in \mathrm{TFN}(\mathbb{R}).$$

  Then

  $$\overline{\mathsf{A}'} = \mathrm{mean}(\,\mathsf{A}_1 + \widetilde{r},\ \mathsf{A}_2 + \widetilde{r}, \ldots,\ \mathsf{A}_n + \widetilde{r}\,)$$
  $$= \frac{1}{n}\sum_{i=1}^{n}(\,\mathsf{A}_i + \widetilde{r}\,) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathsf{A}_i\right) + \widetilde{r} = \overline{\mathsf{A}} + \widetilde{r}.$$

  As a result, by Proposition 2.4,

  $$(\,\mathsf{A}_i + \widetilde{r}\,) - \overline{\mathsf{A}'} = (\,\mathsf{A}_i + \widetilde{r}\,) - (\,\overline{\mathsf{A}} + \widetilde{r}\,) = \mathsf{A}_i - \overline{\mathsf{A}},$$

  so

  $$\mathcal{V}_2(\mathsf{A}_1 + \widetilde{r}, \mathsf{A}_2 + \widetilde{r}, \ldots, \mathsf{A}_n + \widetilde{r}) = \frac{1}{n}\sum_{i=1}^{n}\left((\,\mathsf{A}_i + \widetilde{r}\,) - \overline{\mathsf{A}'}\right)^2$$
  $$= \frac{1}{n}\sum_{i=1}^{n}(\,\mathsf{A}_i - \overline{\mathsf{A}}\,)^2 = \mathcal{V}_2(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n).$$

- 2*-homogeneity*.
  Let $\widetilde{r} \in \widetilde{\mathbb{R}}$. If $\widetilde{r} = \widetilde{0}$, then $\widetilde{0}\,\mathsf{A}_i = \widetilde{0}$, so $\mathcal{V}_2(\widetilde{r}\,\mathsf{A}_1, \widetilde{r}\,\mathsf{A}_2, \ldots, \widetilde{r}\,\mathsf{A}_n) = \mathcal{V}(\widetilde{0}, \widetilde{0}, \ldots, \widetilde{0}) = \widetilde{0} = \widetilde{0}^2\,\mathcal{V}_2(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n)$. Suppose that $r > 0$. Then $\widetilde{r}\,\mathsf{A}_i = \widetilde{r}(a_i/b_i/c_i) = (ra_i/rb_i/rc_i)$ for each $i \in [n]$, and its mean is

  $$\overline{\mathsf{A}'} = \mathrm{mean}(\,\widetilde{r}\,\mathsf{A}_1, \widetilde{r}\,\mathsf{A}_2, \ldots, \widetilde{r}\,\mathsf{A}_n\,)$$
  $$= \frac{1}{n}\sum_{i=1}^{n}(ra_i/rb_i/rc_i) = \widetilde{r}\,\frac{1}{n}\sum_{i=1}^{n}(a_i/b_i/c_i) = \widetilde{r}\,\overline{\mathsf{A}}.$$

  Therefore,

  $$\left(\widetilde{r}\,\mathsf{A}_i - \overline{\mathsf{A}'}\right)^2$$
  $$= \left(\widetilde{r}\,\mathsf{A}_i - \widetilde{r}\,\overline{\mathsf{A}}\right)^2 = \left[\widetilde{r}\left(\mathsf{A}_i - \overline{\mathsf{A}}\right)\right]^2 = \widetilde{r}\left(\mathsf{A}_i - \overline{\mathsf{A}}\right)\widetilde{r}\left(\mathsf{A}_i - \overline{\mathsf{A}}\right)$$
  $$= \widetilde{r}\,\widetilde{r}\left(\mathsf{A}_i - \overline{\mathsf{A}}\right)\left(\mathsf{A}_i - \overline{\mathsf{A}}\right) = \widetilde{r^2}\left(\mathsf{A}_i - \overline{\mathsf{A}}\right)^2 = \widetilde{r}^2\left(\mathsf{A}_i - \overline{\mathsf{A}}\right)^2.$$

  As a result,

  $$\mathcal{V}_2(\widetilde{r}\,\mathsf{A}_1, \widetilde{r}\,\mathsf{A}_2, \ldots, \widetilde{r}\,\mathsf{A}_n) = \frac{1}{n}\sum_{i=1}^{n}\left(\widetilde{r}\,\mathsf{A}_i - \overline{\mathsf{A}'}\right)^2 = \frac{1}{n}\sum_{i=1}^{n}\widetilde{r}^2\left(\mathsf{A}_i - \overline{\mathsf{A}}\right)^2$$
  $$= \widetilde{r}^2\,\frac{1}{n}\sum_{i=1}^{n}\left(\mathsf{A}_i - \overline{\mathsf{A}}\right)^2 = \widetilde{r}^2\,\mathcal{V}_2(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n).$$

  If $r < 0$, then $\widetilde{r}\,\mathsf{A}_i = \widetilde{r}(a_i/b_i/c_i) = (rc_i/rb_i/ra_i)$, but the proof is similar taking into account that for each $\mathsf{B} \in \mathrm{TFN}(\mathbb{R})$, $(-\mathsf{B})^2 = (-\mathsf{B})(-\mathsf{B}) = (-1)\mathsf{B}(-1)\mathsf{B} = (-1)^2\mathsf{B}^2 = \mathsf{B}^2$.

- *Conservativity*.
From item 1 of Lemma 4.2 we have that $\ker\left(\mathcal{V}_2(\mathcal{H})\right) = \{\, \mathrm{var}(b_1, b_2, \ldots, b_n)\,\}$, so
$$\mathcal{D}_c(\mathcal{V}_2(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n)) = \mathrm{var}(b_1, b_2, \ldots, b_n) = \mathrm{var}(\mathcal{D}_c(\mathsf{A}_1), \mathcal{D}_c(\mathsf{A}_2), \ldots, \mathcal{D}_c(\mathsf{A}_n)).$$
- *Normality*.
If $\mathsf{A}_1 = \mathsf{A}_2 = \cdots = \mathsf{A}_n \in \widetilde{\mathbb{R}}$, then there is $\widetilde{r} \in \widetilde{\mathbb{R}}$ such that $\mathsf{A}_1 = \mathsf{A}_2 = \cdots = \mathsf{A}_n = \widetilde{r}$, so Lemma 4.5 guarantees that
$$\mathcal{V}_2(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n) = \mathcal{V}_2(\widetilde{r}, \widetilde{r}, \ldots, \widetilde{r}) = \mathrm{cr}\,(\mathrm{var}(r, r, \ldots, r)) = \widetilde{0}.$$
Conversely, suppose that $\mathcal{V}_2(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n) = \mathcal{V}_2(\mathcal{H}) = \widetilde{0}$. Therefore $[\mathcal{V}_2(\mathcal{H})]_R(0) = \underset{\sim}{0}$, so item 5 of Lemma 4.2 concludes that $\mathsf{A}_1 = \mathsf{A}_2 = \cdots = \mathsf{A}_n \in \widetilde{\mathbb{R}}$.
- *$\preccurlyeq$-positivity*.
Let denote $\mathsf{M} = \mathcal{V}_2(\mathcal{H})$ and $v_0 = \mathrm{var}(b_1, b_2, \ldots, b_n) \geq 0$. By item 1 of Lemma 4.2, $\left[\mathsf{M}_L(1), \mathsf{M}_R(1)\right] = \ker(\mathsf{M}) = \{v_0\}$. Therefore, for each $\alpha \in [0,1]$, $0 \leq v_0 = \mathsf{M}_R(1) \leq \mathsf{M}_R(\alpha)$. We consider the following three cases.

  - *Case (1) Suppose that* $\mu(\mathbb{I}_{\widetilde{0},\mathsf{V}}) = \mu(\mathbb{I}_{\mathsf{V},\widetilde{0}}) = 0$.

  In this case, by item 4 of Lemma 4.2,
  $$\begin{aligned}\mathsf{M}_L(0) + \mathsf{M}_L(1) + \mathsf{M}_R(1) + \mathsf{M}_R(0) &= 2v_0 + [\mathcal{V}_2(\mathcal{H})]_L(0) \\ &\quad + [\mathcal{V}_2(\mathcal{H})]_R(0) \geq 0 \\ &= \widetilde{0}_L(0) + \widetilde{0}_L(1) + \widetilde{0}_R(1) + \widetilde{0}_R(0).\end{aligned}$$
  By the definition of the binary relation $\preccurlyeq$, we conclude that $\widetilde{0} \preccurlyeq \mathsf{M}$.

  - *Case (2) Suppose that* $v_0 > 0$.

  If $v_0 > 0$, then $\mathsf{M}_R(\alpha) \geq v_0 > 0 = \widetilde{0}_R(\alpha)$ for each $\alpha \in [0,1]$, which means that
  $$\mathbb{I}_{\mathsf{M},\widetilde{0}} = \{\, \alpha \in [0,1] : \mathsf{M}_L(\alpha) \leq 0 \text{ and } \mathsf{M}_R(\alpha) \leq 0 \,\} = \varnothing.$$
  In this case, $\mu(\mathbb{I}_{\mathsf{M},\widetilde{0}}) = 0 \leq \mu(\mathbb{I}_{\widetilde{0},\mathsf{M}})$. If $\mu(\mathbb{I}_{\widetilde{0},\mathsf{M}}) > 0$, then we can conclude that $\widetilde{0} \prec \mathsf{M}$, and if $\mu(\mathbb{I}_{\widetilde{0},\mathsf{M}}) = 0$, then $\mu(\mathbb{I}_{\widetilde{0},\mathsf{M}}) = \mu(\mathbb{I}_{\mathsf{M},\widetilde{0}}) = 0$ as in the previous case.

  - *Case (3) Suppose that* $v_0 = 0$.

  Since $v_0 = \mathrm{var}(b_1, b_2, \ldots, b_n) = 0$, then $b_1 = b_2 = \cdots = b_n$. In this case, Lemma 4.3 guarantees that $\mathsf{M}_L(\alpha) \leq 0 \leq \mathsf{M}_R(\alpha)$ and $\mathsf{M}_L(\alpha) + \mathsf{M}_R(\alpha) \geq 0$ for each $\alpha \in [0,1]$. We claim that $\mathbb{I}_{\mathsf{M},\widetilde{0}} \subseteq \mathbb{I}_{\widetilde{0},\mathsf{M}}$. Let $\alpha \in \mathbb{I}_{\mathsf{M},\widetilde{0}}$ be arbitrary. Then $\mathsf{M}_L(\alpha) \leq 0$ and $\mathsf{M}_R(\alpha) \leq 0$. Hence $\mathsf{M}_R(\alpha) \leq 0 \leq \mathsf{M}_R(\alpha)$, so $\mathsf{M}_R(\alpha) = 0$. Furthermore,
  $$\mathsf{M}_L(\alpha) \leq 0 \leq \mathsf{M}_L(\alpha) + \mathsf{M}_R(\alpha) = \mathsf{M}_L(\alpha),$$
  so $\mathsf{M}_L(\alpha) = 0$. Since $\mathsf{M}_L(\alpha) = \mathsf{M}_R(\alpha) = 0$, then $\alpha \in \mathbb{I}_{\widetilde{0},\mathsf{M}}$. This proves that $\mathbb{I}_{\mathsf{M},\widetilde{0}} \subseteq \mathbb{I}_{\widetilde{0},\mathsf{M}}$. Therefore $0 \leq \mu(\mathbb{I}_{\mathsf{M},\widetilde{0}}) \leq \mu(\mathbb{I}_{\widetilde{0},\mathsf{M}})$. If $\mu(\mathbb{I}_{\widetilde{0},\mathsf{M}}) > 0$ then $\widetilde{0} \prec \mathsf{M}$ by definition. And if $\mu(\mathbb{I}_{\widetilde{0},\mathsf{M}}) = 0$, then $\mu(\mathbb{I}_{\mathsf{M},\widetilde{0}}) = \mu(\mathbb{I}_{\widetilde{0},\mathsf{M}}) = 0$ and we can apply the first case. ■

Corollary 4.7. *Given $\lambda > 0$, let $\mathcal{V}_2^\lambda : \bigcup_{n\in\mathbb{N}} \mathrm{TFN}(\mathbb{R})^n \to \mathrm{FN}(\mathbb{R})$ defined, for each $n \in \mathbb{N}$ and each set $\mathcal{H} = \{\, \mathsf{A}_i = (a_i/b_i/c_i)\,\}_{i=1}^n$ of n triangular fuzzy numbers, as:*
$$\mathcal{V}_2^\lambda(\mathcal{H}) = \frac{\lambda}{n} \sum_{i=1}^n (\,\mathsf{A}_i - \overline{\mathsf{A}}\,)^2.$$
Then the quadruple $(\mathcal{C} = \mathrm{TFN}(\mathbb{R}), \mathcal{C}' = \mathrm{FN}(\mathbb{R}), \mathcal{V}_2^\lambda, \preccurlyeq)$ is a FDS. Furthermore, the FDM $\mathcal{V}_2^\lambda$ is conservative and 2-homogeneous.

Proof. The properties of $\mathcal{V}_2^\lambda$ can be directly derived from the properties of $\mathcal{V}_2$. For the homogeneity, notice that
$$\begin{aligned}&\mathcal{V}_2^\lambda(\,\widetilde{r}\,\mathsf{A}_1, \widetilde{r}\,\mathsf{A}_2, \ldots, \widetilde{r}\,\mathsf{A}_n\,) \\ &= \frac{\lambda}{n} \sum_{i=1}^n (\,\mathsf{A}_i - \overline{\mathsf{A}}\,)^2 = \lambda\, \frac{1}{n} \sum_{i=1}^n (\,\mathsf{A}_i - \overline{\mathsf{A}}\,)^2 = \lambda\, \mathcal{V}_2(\,\widetilde{r}\,\mathsf{A}_1, \widetilde{r}\,\mathsf{A}_2, \ldots, \widetilde{r}\,\mathsf{A}_n\,) \\ &= \lambda\, \widetilde{|r|}^{\,2}\, \mathcal{V}_2(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n) \\ &= \widetilde{|r|}^{\,2}\, \lambda \mathcal{V}_2(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n) = \widetilde{|r|}^{\,2}\, \mathcal{V}_2^\lambda(\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n). \quad ■\end{aligned}$$

Remark 4.8. The mapping $\mathcal{V}_2' : \bigcup_{n\in\mathbb{N}} \mathrm{TFN}(\mathbb{R})^n \to \mathrm{FN}(\mathbb{R})$ be defined, for each $n \in \mathbb{N}$ and each set $\mathcal{H} = \{\, \mathsf{A}_i = (a_i/b_i/c_i)\,\}_{i=1}^n \subset \mathrm{TFN}(\mathbb{R})$, as:
$$\mathcal{V}_2'(\mathcal{H}) = \frac{1}{n-1} \sum_{i=1}^n (\,\mathsf{A}_i - \overline{\mathsf{A}}\,)^2,$$
satisfies the same properties than $\mathcal{V}_2$, so it is a FDM. It can be called the *canonical sampled FDM*.

In the following example we compute and compare the above-mentioned fuzzy dispersions of the set considered in Example 2.6.

Example 4.9. Let $\mathcal{H} = \{\, \mathsf{A}_1 = (1/3/7), \mathsf{A}_2 = (4/6/8)\,\}$ be as in Example 2.6 (see Fig. 2.a). The Kruse fuzzy variance $\mathrm{Var}(\mathcal{H})$ was given in Fig. 2.c and plotted in Fig. 2.d. After carrying out the necessary accounts, it can be checked that the canonical FDM $\mathcal{V}_2(\mathcal{H})$ is:
$$\mathcal{V}_2(\mathcal{H})(t) = \begin{cases} \dfrac{103 - \sqrt{909 - 400t}}{100}, & \text{if } \dfrac{-97}{4} \leq t \leq \dfrac{-9}{20}, \\ \dfrac{53 - \sqrt{1449 - 160t}}{20}, & \text{if } \dfrac{-9}{20} \leq t \leq \dfrac{1}{32}, \\ \dfrac{89 + \sqrt{488t - 9}}{122}, & \text{if } \dfrac{1}{32} \leq t \leq \dfrac{9}{4}, \\ \dfrac{109 - \sqrt{328t - 9}}{82}, & \text{if } \dfrac{9}{4} \leq t \leq \dfrac{145}{4}, \\ 0, & \text{otherwise.} \end{cases}$$

Fig. 3 shows both *LR*-fuzzy numbers: the Kruse fuzzy variance $\mathrm{Var}(\mathcal{H})$ (in red color) and the canonical FDM $\mathcal{V}_2(\mathcal{H})$ (in blue color). We observe that $\ker(\mathcal{V}_2(\mathcal{H})) = \ker(\mathrm{Var}(\mathcal{H})) = \{\frac{9}{4}\} = \{\mathrm{var}(3,6)\} = \{\mathrm{var}(\mathcal{D}_c\mathsf{A}_1, \mathcal{D}_c\mathsf{A}_2)\}$.

In the previous example we have computed two distinct ways ($\mathrm{Var}(\mathcal{H})$ and $\mathcal{V}_2(\mathcal{H})$) of measuring the fuzzy dispersion of the same set $\mathcal{H}$ of fuzzy numbers. It has no much sense to compare such fuzzy dispersions because they have been obtained by employing distinct methodologies. This makes us reflect on the fact that the two dispersions provide distinct information: the Kruse fuzzy variance informs us about the real variances that we can obtain by employing real numbers on the respective level sets, and the canonical FDM informs us about a fuzzy number that is defined by employing the usual operations among fuzzy numbers. Both methodologies have their advantages and disadvantages.

- The Kruse fuzzy variance is easy to interpret and generates positive supports, but it is difficult to compute in practice when more than ten fuzzy numbers are involved (the optimization process is far from human capabilities). Furthermore, it suffers the fuzzy incoherence of descending to the real setting in order to compute real variances rather than using fuzzy operations.
- The canonical FDM can be described as a piecewise function that employs fuzzy operations, but it has no a clear interpretability further from the fact of measuring the mean of the squared differences from each term to the global mean.

It is true that the canonical FDM leads to fuzzy numbers with a piece of its support included in the negative part of the real line. However, this is not an incoherence because we are only measuring the dispersion in terms of positive fuzzy numbers w.r.t. the considered ranking methodology. One can believe that the canonical FDM is less informative than the Kruse fuzzy variance, but we believe this is not the case because they represent distinct information. Furthermore, if

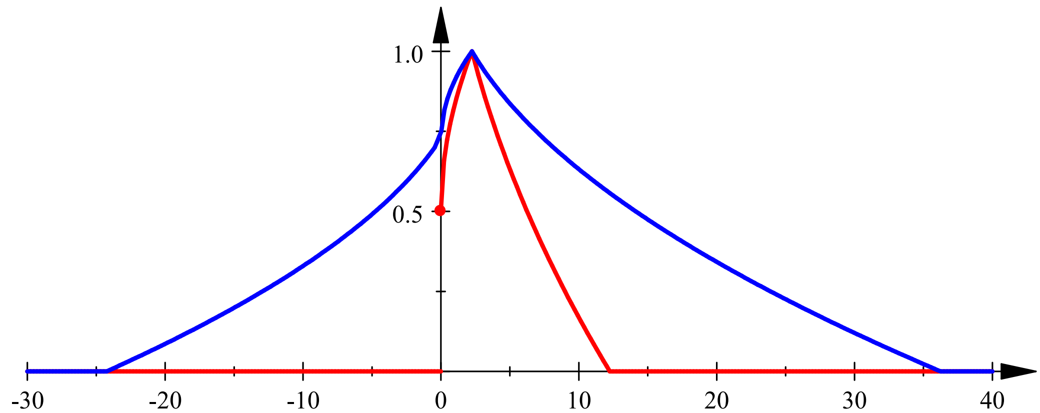


Fig. 3. Kruse fuzzy variance (in red) and canonical FDM (in blue) in Example 4.9.

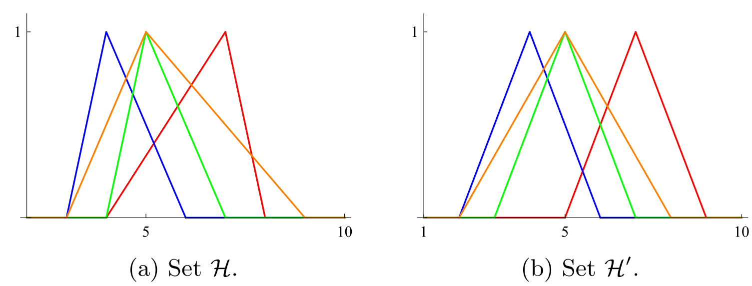


Fig. 4. Graphic representation of the fuzzy numbers in $\mathcal{H}$ and in $\mathcal{H}'$.

we have compared the fuzzy numbers $\mathrm{Var}(\mathcal{H})$ and $\mathcal{V}_2(\mathcal{H})$ obtained in Example 4.9, we would have obtained that $\mathcal{V}_2(\mathcal{H}) \prec \mathrm{Var}(\mathcal{H})$ so, at least in this example, $\mathcal{V}_2$ measures the fuzzy dispersion by employing lower fuzzy numbers than Var.

5. Comparison of fuzzy dispersions

This section is devoted to compute and compare the fuzzy dispersion measures of a pair of sets of triangular fuzzy numbers by employing the canonical FDM. In this process, we will use the fuzzy binary relation $\preccurlyeq$ introduced in [36]. From the theoretical point of view, in a FDS $(\mathcal{C}, \mathcal{C}', \mathcal{V}, \precsim)$, since we assume that the binary relation $\precsim$ is total on $\mathcal{C}'$, the fuzzy dispersions of two sets of fuzzy numbers are always comparable by $\precsim$. The main sense of Definition 3.1 is to set the framework in which such fuzzy dispersions can be compared. This is what we do in the following lines.

Definition 5.1. Given a FDS $(\mathcal{C}, \mathcal{C}', \mathcal{V}, \precsim)$ and two subsets $\mathcal{H} = \{\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n\} \subseteq \mathcal{C}$ and $\mathcal{H}' = \{\mathsf{B}_1, \mathsf{B}_2, \ldots, \mathsf{B}_m\} \subseteq \mathcal{C}$, we will say that the dispersion of $\mathcal{H}$ is:

- *less than (or equal to) the dispersion of* $\mathcal{H}'$ if $\mathcal{V}(\mathcal{H}) \precsim \mathcal{V}(\mathcal{H}')$;
- *less than the dispersion of* $\mathcal{H}'$ if $\mathcal{V}(\mathcal{H}) \prec \mathcal{V}(\mathcal{H}')$;
- *equal* (or *equivalent*) *to the dispersion of* $\mathcal{H}'$ if $\mathcal{V}(\mathcal{H}) \sim \mathcal{V}(\mathcal{H}')$.

Next we compare the canonical FDM of two distinct sets of four triangular fuzzy numbers. In order to show that the comparison of their corresponding FDMs is a deeper methodology than the comparison of their real kernels, we will choose two sets of four triangular fuzzy numbers with the same centers. Let $\mathcal{H} = \{\, \mathsf{A}_i = (a_i/b_i/c_i)\,\}_{i=1}^4$ and $\mathcal{H}' = \{\, \mathsf{A}'_i = (a'_i/b_i/c'_i)\}_{i=1}^4$ be the sets whose fuzzy numbers are, on the one hand, $\mathsf{A}_1 = (4/7/8)$, $\mathsf{A}_2 = (3/4/6)$, $\mathsf{A}_3 = (4/5/7)$ and $\mathsf{A}_4 = (3/5/9)$, and, on the other hand, $\mathsf{A}'_1 = (5/7/9)$, $\mathsf{A}'_2 = (2/4/6)$, $\mathsf{A}'_3 = (3/5/7)$ and $\mathsf{A}'_4 = (2/5/8)$ (see Fig. 4). Notice that all fuzzy numbers in $\mathcal{H}'$ are symmetric, and they have the same kernels than their corresponding fuzzy numbers in $\mathcal{H}$, that is, $\ker \mathsf{X}'_i = y_i = \ker \mathsf{X}_i$ for each $i \in [4]$.

After carrying out all necessary accounts, it can be computed the canonical FDM $\mathcal{V}_2(\mathcal{H})$ of $\mathcal{H}$ and $\mathcal{V}_2(\mathcal{H}')$ of $\mathcal{H}'$, that are jointly plotted in Fig. 5 (it is highlighted the unique points in which such fuzzy numbers are not differentiable). It can be checked that $(\mathcal{V}_2(\mathcal{H}))_R(\alpha) < (\mathcal{V}_2(\mathcal{H}'))_R(\alpha)$ for each $\alpha \in [0,1)$ and $(\mathcal{V}_2(\mathcal{H}))_R(1) = (\mathcal{V}_2(\mathcal{H}'))_R(1)$. Furthermore,

$$\begin{cases} (\mathcal{V}_2(\mathcal{H}))_L(\alpha) < (\mathcal{V}_2(\mathcal{H}'))_L(\alpha), & \text{if } alpha \in \left(\frac{47}{58}, 1\right), \\ (\mathcal{V}_2(\mathcal{H}))_L(\alpha) = (\mathcal{V}_2(\mathcal{H}'))_L(\alpha), & \text{if } \alpha \in \left\{\frac{47}{58}, 1\right\}, \\ (\mathcal{V}_2(\mathcal{H}))_L(\alpha) > (\mathcal{V}_2(\mathcal{H}'))_L(\alpha), & \text{if } \alpha \in \left[0, \frac{47}{58}\right). \end{cases}$$

In particular, for each $\alpha \in \left(\frac{47}{58}, 1\right)$, $(\mathcal{V}_2(\mathcal{H}))_L(\alpha) < (\mathcal{V}_2(\mathcal{H}'))_L(\alpha)$ and $(\mathcal{V}_2(\mathcal{H}))_R(\alpha) < (\mathcal{V}_2(\mathcal{H}'))_R(\alpha)$, which means that $\mathcal{V}_2(\mathcal{H}) \prec \mathcal{V}_2(\mathcal{H}')$ by employing the fuzzy binary relation $\preccurlyeq$. As conclusion, the fuzzy dispersion of the set $\mathcal{H}$ is smaller than the fuzzy dispersion of the set $\mathcal{H}'$ by employing the mentioned ranking methodology.

6. Conclusions and prospect research

In this paper we have introduced the notion of *fuzzy dispersion measure* and have described some of the reasonable properties that a FDM could satisfy. We have reflected about the difficulties that naturally arise when we try to translate the real variance to the fuzzy setting, especially due to the traditional definition of the operations (subtraction and product) among fuzzy numbers. Accordingly to a genuine fuzzy view-point, we have only employed the fuzzy arithmetic operations (avoiding the real framework) and, coherently, we have explored the unknown field where a FDM can have a piece of its support in the negative interval of real numbers.

Prospect research must be done in this line of study. On the one hand, the variance is important in Statistics to compare the dispersion of distinct variables. In this case, we have not only computed the FDM of two distinct set of triangular fuzzy numbers, but we have compared them by using a ranking procedure. Hence, it will be interesting to

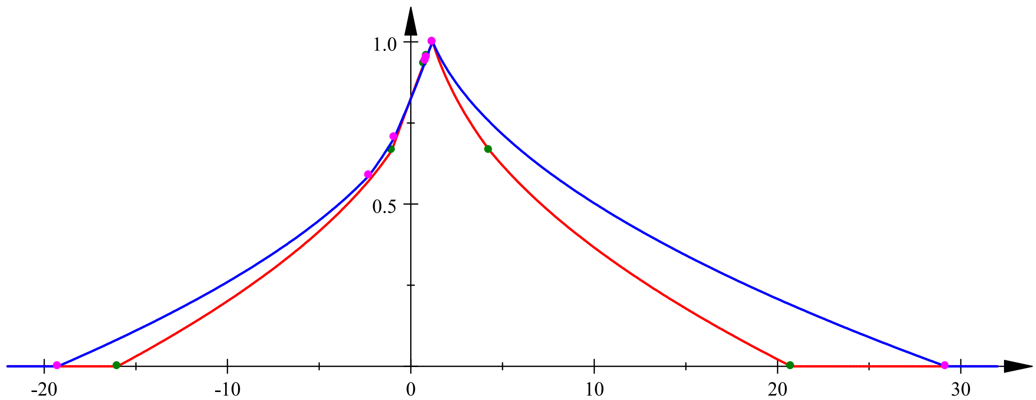


Fig. 5. Graphic representation of the fuzzy variances of $\mathcal{H}$ (red) and $\mathcal{H}'$ (blue).

study how the FDM are ordered by using distinct ranking procedures. On the other hand, as we can see in Fig. 5, the canonical FDM $\mathcal{V}_2$ is a fuzzy number whose support could contain negative numbers. It is not clear the significance of this fact, but is does not mean that such fuzzy number could be negative (it depends on the ranking methodology we are using). Anyway, some researchers could prefer to define $\mathcal{V}(\mathcal{H})(t) = 0$ for each $t < 0$. It remains as an open problem to study the necessity of defining $\mathcal{V}(\mathcal{H})$ as zero on the negative part of the real line, or to take advantage of the knowledge of the general function $\mathcal{V}$ even on negative real numbers, that is, to determine if such membership function could provide more information about the dispersion of the variable.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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