



Jacobson's theorem on derivations of primitive rings with nonzero socle: a proof and applications

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Received: 22 February 2022 / Revised: 28 June 2023 / Accepted: 7 July 2023
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Abstract

We provide a proof of Jacobson's theorem on derivations of primitive rings with nonzero socle. Both Jacobson's theorem and its formulation (in terms of the so-called differential operators on left vector spaces over a division ring) underlie our paper. We apply Jacobson's theorem to describe derivations of standard operator rings on real, complex, or quaternionic left normed spaces. Indeed, when the space is infinite-dimensional, every derivation of such a standard operator ring is of the form $A \rightarrow AB - BA$ for some continuous linear operator B on the space. Our quaternionic approach allows us to generalize Rickart's theorem on representation of primitive complete normed associative complex algebras with nonzero socle to the case of primitive real or complex associative normed Q -algebras with nonzero socle. We prove that additive derivations of the Jordan algebra of a continuous nondegenerate symmetric bilinear form on any infinite-dimensional real or complex Banach space are in a one-to-one natural correspondence with those continuous linear operators on the space which are skew-adjoint relative to the form. Finally we prove that additive derivations of a real or complex (possibly non-associative) H^* -algebra with no nonzero finite-dimensional direct summand are linear and continuous.

Keywords Primitive ring · Differential operator · Additive derivation · Standard operator ring · Quaternionic normed space · H^* -algebra

Mathematics Subject Classification 16W25 · 17C50 · 17C65 · 46H70 · 46K15 · 46L70 · 47B01 · 47B47 · 47L10 · 47S05

The authors have been partially supported by the Junta de Andalucía Grant FQM199.

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1 Introduction

Throughout this paper Δ will denote a division ring and, given a left or right vector space X over Δ , we denote by $L(X)$ the ring of all Δ -linear operators on X .

Let X be a left vector space over Δ . Following [39, Definition IV.14.1], we say that a mapping $T: X \rightarrow X$ is a *differential operator* on X if it is additive and there exists a ring derivation d of Δ such that

$$T(\lambda x) = \lambda Tx + d(\lambda)x \quad \text{for all } \lambda \in \Delta \text{ and } x \in X. \quad (1.1)$$

It follows from (1.1) that every differential operator on X has a unique associated derivation. Examples of differential operators on X are all linear operators on X and all operators in ΔI_X , where ΔI_X denotes the set of mappings of the form $x \rightarrow \lambda x$ where λ runs over Δ . Actually the set $\text{Diff}(X)$ of all differential operators on X is a Lie subring of the ring of all additive operators on X , and contains both $L(X)$ and ΔI_X as ideals. As a consequence, we are provided with the following.

Fact 1.1 *Let X be a left vector space over Δ , and let T be a differential operator on X . Then $[A, T] := AT - TA$ lies in $L(X)$ whenever A does, and the mapping $A \rightarrow [A, T]$ is a ring derivation of $L(X)$.*

Differential operators on right vector spaces over Δ are defined analogously. Indeed, given a right vector space Y over Δ , a mapping $T: Y \rightarrow Y$ is said to be a *differential operator* on Y if it is additive and there exists a ring derivation d of Δ such that

$$T(y\lambda) = (Ty)\lambda + yd(\lambda) \quad \text{for all } \lambda \in \Delta \text{ and } y \in Y.$$

We recall that a *pairing* over Δ is a triple $(X, Y, \langle \cdot, \cdot \rangle)$ where X is a left vector space over Δ , Y is a right vector space over Δ and $\langle \cdot, \cdot \rangle$ is a bilinear form on $X \times Y$ which is non-degenerate; i.e. the conditions $x \in X$ and $\langle x, Y \rangle = 0$ imply $x = 0$, and the conditions $y \in Y$ and $\langle X, y \rangle = 0$ imply $y = 0$.

Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ . Following [39, p. 87], we say that a differential operator T on X with associated derivation d has an *adjoint* in Y relative to $\langle \cdot, \cdot \rangle$ if there exists a mapping $T^\# : Y \rightarrow Y$ such that

$$\langle Tx, y \rangle = \langle x, T^\#y \rangle + d(\langle x, y \rangle) \quad \text{for all } (x, y) \in X \times Y. \quad (1.2)$$

It follows from (1.2) that the mapping $T^\#$ is unique. If T is a differential operator on X with associated derivation d , and if T has an adjoint $T^\#$, then it is immediate to verify that $T^\#$ is a differential operator on Y with associated derivation $-d$. As a consequence, if A belongs to $L(X) \subseteq \text{Diff}(X)$, and if A has an adjoint $A^\#$ relative to $\langle \cdot, \cdot \rangle$, then $A^\#$ belongs to $L(Y)$. The set of those linear operators on X which have adjoints relative to $\langle \cdot, \cdot \rangle$ is a subring of $L(X)$, which is denoted by $L(X, Y, \langle \cdot, \cdot \rangle)$. The subset of $L(X, Y, \langle \cdot, \cdot \rangle)$ consisting of those operators in $L(X, Y, \langle \cdot, \cdot \rangle)$ which have finite-dimensional range is an ideal of $L(X, Y, \langle \cdot, \cdot \rangle)$, which is denoted by $F(X, Y, \langle \cdot, \cdot \rangle)$. By the structure theorem for primitive rings with nonzero socle [39, Section IV.9], up

to isomorphisms, the primitive rings with nonzero socle are precisely the subrings of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$ for some choice of the division ring Δ and some choice of the pairing $(X, Y, \langle \cdot, \cdot \rangle)$ over Δ . For such a ring \mathcal{A} , the socle of \mathcal{A} is $F(X, Y, \langle \cdot, \cdot \rangle)$.

The pioneering and fundamental paper on pairings and their properties is that of Mackey [47]. Sometimes the sets of the form $L(X, Y, \langle \cdot, \cdot \rangle)$ for some pairing $(X, Y, \langle \cdot, \cdot \rangle)$ over Δ (regarded as algebras over the centre of Δ) are called Mackey algebras. See for example the paper of Penkov and Serganova [51]. Recently derivations of some Mackey algebras and related Lie algebras were considered in the paper of Bezushchak [8].

Considering Fact 1.1, it is easily realized that the set $\text{Diff}(X, Y, \langle \cdot, \cdot \rangle)$ of all differential operators on X having an adjoint relative to $\langle \cdot, \cdot \rangle$ is a (Lie) subring of the Lie ring $\text{Diff}(X)$ containing both $L(X, Y, \langle \cdot, \cdot \rangle)$ and ΔI_X as ideals. As a consequence, we have the following.

Fact 1.2 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ , and let T be a differential operator on X having an adjoint relative to $\langle \cdot, \cdot \rangle$. Then $[A, T]$ lies in $L(X, Y, \langle \cdot, \cdot \rangle)$ whenever A does, and the mapping $A \rightarrow [A, T]$ is a ring derivation of $L(X, Y, \langle \cdot, \cdot \rangle)$.*

Conversely, according to [39, Proposition IV.14.1 and Theorem IV.14.3], we are provided with the following.

Theorem 1.3 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ , \mathcal{A} be a subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, and*

$$D: \mathcal{A} \rightarrow L(X, Y, \langle \cdot, \cdot \rangle)$$

be a derivation. Then there exists a differential operator T on X which has an adjoint relative to $\langle \cdot, \cdot \rangle$ and satisfies $D(A) = [A, T]$ for every $A \in \mathcal{A}$.

Here, given a subring \mathcal{A} of a ring \mathcal{B} , by a *derivation* from \mathcal{A} to \mathcal{B} we mean an additive mapping $D: \mathcal{A} \rightarrow \mathcal{B}$ satisfying

$$D(A_1 A_2) = D(A_1) A_2 + A_1 D(A_2) \quad \text{for all } A_1, A_2 \in \mathcal{A}.$$

Theorem 1.3 above is due to Jacobson [39]. As far as we know, Theorem 1.3 has been forgotten, and no application of it has appeared in the literature. In the case that Δ is equal to \mathbb{R} (the field of real numbers) or \mathbb{C} (the field of complex numbers), and that X is a Banach space over Δ naturally paired with its topological dual, Theorem 1.3 has been rediscovered by Šemrl [62, 63] (see also [57, Theorem 1.3]).

In Sect. 2, we provide the reader with a complete proof of Theorem 1.3, and derive some first applications of it.

In Sect. 3, we suppose that $\frac{1}{2} \in \Delta$, and apply Theorem 1.3 to prove that this theorem remains true if \mathcal{A} is merely assumed to be a Jordan subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, and consequently $D: \mathcal{A} \rightarrow L(X, Y, \langle \cdot, \cdot \rangle)$ is assumed to be a derivation in the Jordan sense (Theorem 3.5). Precise definitions of a Jordan subring of a ring, and of a Jordan derivation from such a Jordan subring to the whole ring, will be given in this section.

Throughout the remaining part of this paper \mathbb{H} will denote the division real algebra of Hamilton's quaternions, and almost always \mathbb{F} will stand for either \mathbb{R} , \mathbb{C} or \mathbb{H} .

Let X be a left normed space over \mathbb{F} . We denote by $\mathcal{L}(X)$ the subring of $L(X)$ consisting of all continuous \mathbb{F} -linear operators on X , and by $\mathcal{F}(X)$ the ideal of $\mathcal{L}(X)$ consisting of those operators in $\mathcal{L}(X)$ having finite-dimensional range. *Standard operator rings* (respectively, *standard operator Jordan rings*) on X are defined as those subrings (respectively, Jordan subrings) of $\mathcal{L}(X)$ which contain $\mathcal{F}(X)$. We note that $\mathcal{L}(X)$ is a closed subalgebra of the real normed algebra $\mathcal{L}(X_{\mathbb{R}})$, where $X_{\mathbb{R}}$ denotes the real normed space obtained from X by restricting the scalars. Therefore $\mathcal{L}(X)$ becomes naturally a real normed algebra when it is endowed with the operator norm.

Now, to comment on Sect. 4, let us consider the following.

Theorem 1.4 *Let X be an infinite-dimensional left normed space over \mathbb{F} , let \mathcal{A} be a standard operator Jordan ring on X , and let $D: \mathcal{A} \rightarrow \mathcal{L}(X)$ be a Jordan derivation. Then there exists $B \in \mathcal{L}(X)$ such that $D(A) = [A, B]$ for every $A \in \mathcal{A}$. As a consequence, D is continuous.*

Throughout the remaining part of this paper \mathbb{K} will stand for either \mathbb{R} or \mathbb{C} . All forerunners of Theorem 1.4, known in the literature, deal only with the case that $\mathbb{F} = \mathbb{K}$. But, for any normed space X over \mathbb{K} , $\mathcal{L}(X)$ is an algebra over \mathbb{K} containing $\mathcal{F}(X)$ as an algebra ideal, and hence we can think about the so-called *standard operator algebras* (respectively, *standard operator Jordan algebras*) on X , which are defined as those subalgebras (respectively, Jordan subalgebras) of $\mathcal{L}(X)$ which contain $\mathcal{F}(X)$. Now, ordering chronologically, Theorem 1.4 is known in the following particular cases:

- (i) (Chernoff [20]) $\mathbb{F} = \mathbb{K}$, \mathcal{A} is a standard operator algebra on X , and D is a linear derivation. Clearly, in this case, the conclusion in the theorem remains true if the restriction that X is infinite-dimensional is removed.
- (ii) (Šemrl [62]) $\mathbb{F} = \mathbb{K}$, X is a Hilbert space over \mathbb{K} , \mathcal{A} is a standard operator algebra on X , and D is a ring derivation. Moreover, in this case, the conclusion in the theorem does not remain true if the restriction that X is infinite-dimensional is removed.
- (iii) (Šemrl [63]) $\mathbb{F} = \mathbb{K}$, X is complete, \mathcal{A} is a standard operator algebra on X , and D is a ring derivation.
- (iv) (Han [35]) $\mathbb{F} = \mathbb{K}$, \mathcal{A} is a standard operator algebra on X , and D is a ring derivation.
- (v) (Vukman [68], see also [44]) $\mathbb{F} = \mathbb{K}$, \mathcal{A} is a standard operator algebra on X , and D is linear. Clearly, in this case, the conclusion in the theorem remains true if the restriction that X is infinite-dimensional is removed. (In these papers, completeness of X is assumed but is never applied in the proofs.)
- (vi) (The authors [57]) $\mathbb{F} = \mathbb{K}$ and X is complete.

It is worth mentioning that, roughly speaking, the proof of forerunner (v) (respectively, (vi)) consists of a reduction to forerunner (i) (respectively, (iii)). Most part of Sect. 4 is devoted to prove Theorem 1.4. Our proof does not consist of a reduction to any of the forerunners listed above, but relies on an appropriate adaptation of the argument in the proof of Proposition 3.3 in the Boudi–Marhnine–Zarhouti paper [9]

(see Theorem 4.10). By a *normed pairing* over \mathbb{F} we mean any pairing $(X, Y, \langle \cdot, \cdot \rangle)$, where X is a left normed space over \mathbb{F} , Y is a right normed space over \mathbb{F} , and the form $\langle \cdot, \cdot \rangle: X \times Y \rightarrow \mathbb{F}$ is continuous. Now Theorem 4.10 just quoted asserts that, if $(X, Y, \langle \cdot, \cdot \rangle)$ is a normed pairing over \mathbb{F} , with X infinite-dimensional and complete, then $\text{Diff}(X, Y, \langle \cdot, \cdot \rangle) \subseteq \mathbb{F}I_X + \mathcal{L}(X)$, and hence differential operators on X having an adjoint relative to $\langle \cdot, \cdot \rangle$ are continuous, and moreover they are linear whenever $\mathbb{F} = \mathbb{K}$. As in the case of forerunner (ii) of Theorem 1.4, we realize that, also for $\mathbb{F} = \mathbb{H}$, the conclusion in Theorem 1.4 does not remain true if the restriction that X is infinite-dimensional is removed (Remark 4.14).

In Sect. 5, we generalize Rickart's theorem on representation of primitive complete normed associative complex algebras with nonzero socle [52, Theorem 2.4.12] to the case of primitive real or complex associative normed Q -algebras with nonzero socle (Theorem 5.3). For the formulation and proof of this result, our quaternionic approach to normed pairings becomes crucial.

In Sect. 6, we describe additive derivations of Jordan algebras of a nondegenerate symmetric bilinear form on a vector space over any field of characteristic different from 2, by means of differential operators on the vector parts of such algebras (Proposition 6.2). With the help of Proposition 6.2 and Theorem 4.10, we prove that additive derivations of the Jordan algebra of a continuous nondegenerate symmetric bilinear form on any infinite-dimensional real or complex Banach space are in a one-to-one natural correspondence with those continuous linear operators on the space which are skew-adjoint relative to the form (Theorem 6.3). Moreover, neither the restriction that X is infinite-dimensional nor the one that X is complete can be removed (Remark 6.5 and Example 6.7). In particular, Theorem 6.3 provides us with a description of additive derivations of complete smooth normed commutative real algebras, JB -spin factors, quadratic Jordan H^* -algebras, and quadratic JB^* -algebras, whenever they are infinite-dimensional.

In the concluding section (Sect. 7) we prove that derivations of a (possibly non-associative) algebra \mathcal{A} over an arbitrary field are differential operators on (the vector space underlying) \mathcal{A} as soon as \mathcal{A} has zero annihilator and the centroid of \mathcal{A} reduces to the scalar multiples of the identity operator on \mathcal{A} (Proposition 7.4). Considering this crucial result, we apply Theorem 4.10 once more (see Lemma 7.18), and adapt the argument in the proof of the main result in Villena's paper [67] (see Proposition 7.19), to show that additive derivations of any real or complex (possibly non-associative) H^* -algebra with no nonzero finite-dimensional direct summand are linear and continuous (Corollary 7.24). Actually a better result, in the spirit of Theorem 4.1 in the Johnson–Sinclair paper [41], is obtained (see Theorem 7.22).

2 A proof of Jacobson's theorem

§2.1 Given a pairing $(X, Y, \langle \cdot, \cdot \rangle)$ over Δ and $(x, y) \in X \times Y$, we denote by $x \otimes y$ and $y \otimes x$ the rank-one linear operators on X and Y defined by

$$(x \otimes y)v := \langle v, y \rangle x \quad \text{and} \quad (y \otimes x)w := y \langle x, w \rangle$$

for all $v \in X$ and $w \in Y$. The following properties are of direct verification:

- (i) $(x \otimes y)^\# = y \otimes x$.
- (ii) $A(x \otimes y) = Ax \otimes y$ for every $A \in L(X)$.
- (iii) $(x \otimes y)A = x \otimes A^\#y$ for every $A \in L(X, Y, \langle \cdot, \cdot \rangle)$.

Moreover, we have the following description:

$$F(X, Y, \langle \cdot, \cdot \rangle) = \left\{ \sum_{i=1}^n x_i \otimes y_i : n \in \mathbb{N}, (x_i, y_i) \in X \times Y (1 \leq i \leq n) \right\}.$$

Let \mathcal{A} be a ring, and let X be a faithful irreducible left \mathcal{A} -module. Let $E(X)$ denote the ring of all additive mappings on X . Then the set

$$\{B \in E(X) : [A, B] = 0 \text{ for every } A \in \mathcal{A}\}$$

is a division ring, which is called the *associated division ring of \mathcal{A} relative to X* . It is clear that, if $(X, Y, \langle \cdot, \cdot \rangle)$ is a pairing over Δ , and if \mathcal{A} is a subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, then X is a faithful irreducible left \mathcal{A} -module. Here, by the sake of completeness, we include a proof of the well-known fact that, in this case, ΔI_X is the associated division ring of \mathcal{A} relative to X [39, Theorem II.4.1].

Lemma 2.2 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ and \mathcal{A} be a subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$. Then the associated division ring of \mathcal{A} relative to X is equal to ΔI_X .*

Proof Since $\mathcal{A} \subseteq L(X)$, for any $\lambda \in \Delta$, we have that $[A, \lambda I_X] = 0$ for every $A \in \mathcal{A}$. In order to prove the converse inclusion, let us fix $(x_0, y_0) \in X \times Y$ such that $\langle x_0, y_0 \rangle = 1$. If $B \in E(X)$ satisfies the condition $[A, B] = 0$ for every $A \in \mathcal{A}$, then for each $x \in X$ we have

$$\begin{aligned} 0 &= [x \otimes y_0, B]x_0 = (x \otimes y_0)Bx_0 - B(x \otimes y_0)x_0 \\ &= \langle B(x_0), y_0 \rangle x - B(\langle x_0, y_0 \rangle x) = \langle B(x_0), y_0 \rangle x - Bx, \end{aligned}$$

and hence $B = \langle B(x_0), y_0 \rangle I_X \in \Delta I_X$. □

We reformulate the above lemma as follows.

Corollary 2.3 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ , let \mathcal{A} be as in Lemma 2.2, and let S be an additive operator on X such that $[A, S] = 0$ for every $A \in \mathcal{A}$. Then $S \in \Delta I_X$.*

Now, following some ideas in Šemrl’s papers [62, 63], our proof of Theorem 1.3 goes as follows.

Proof of Theorem 1.3 Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ , let \mathcal{A} be a subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, and let

$$D: \mathcal{A} \rightarrow L(X, Y, \langle \cdot, \cdot \rangle)$$

be a derivation. Fix $(x_0, y_0) \in X \times Y$ such that $\langle x_0, y_0 \rangle = 1$, and consider the mapping $T : X \rightarrow X$ given by

$$Tx = -D(x \otimes y_0)x_0.$$

It is clear that T is additive. On the other hand, by § 2.1 (ii), for each $A \in \mathcal{A}$ and $x \in X$ we have

$$\begin{aligned} D(A)x \otimes y_0 &= D(A)(x \otimes y_0) = -AD(x \otimes y_0) + D(A(x \otimes y_0)) \\ &= -AD(x \otimes y_0) + D(Ax \otimes y_0). \end{aligned}$$

Therefore, considering the definition of T , it is enough to evaluate at x_0 to realize that $D(A)x = ATx - TAx$. Therefore $D(A) = [A, T]$ (hence $[A, T] \in L(X, Y, \langle \cdot, \cdot \rangle)$) for all $A \in \mathcal{A}$.

Let A and B be in \mathcal{A} and ΔI_X , respectively. Since both A and $[A, T]$ lie in $L(X, Y, \langle \cdot, \cdot \rangle)$, and $[L(X, Y, \langle \cdot, \cdot \rangle), B] = 0$, we obtain

$$0 = [[A, B], T] = [[A, T], B] + [A, [B, T]] = [A, [B, T]].$$

Therefore, since A is arbitrary in \mathcal{A} , it follows from Lemma 2.2 that $[B, T]$ lies in ΔI_X . Thus $[\Delta I_X, T] \subseteq \Delta I_X$ because B is arbitrary in ΔI_X .

Now consider the mapping $d : \Delta \rightarrow \Delta$ determined by the condition

$$d(\lambda)I_X = -[\lambda I_X, T]$$

for every $\lambda \in \Delta$. Then, clearly, d is a ring derivation of Δ . Moreover, for each $\lambda \in \Delta$ and $x \in X$ we have

$$\begin{aligned} T(\lambda x) &= -D(\lambda x \otimes y_0)x_0 = -[\lambda x \otimes y_0, T]x_0 \\ &= -[(\lambda I_X)(x \otimes y_0), T]x_0 = -[\lambda I_X, T](x \otimes y_0)x_0 - (\lambda I_X)[x \otimes y_0, T]x_0 \\ &= -[\lambda I_X, T]x - (\lambda I_X)D(x \otimes y_0)x_0 = d(\lambda)I_X x - \lambda D(x \otimes y_0)x_0 \\ &= d(\lambda)x + \lambda Tx. \end{aligned}$$

It follows that T is a differential operator on X .

Finally, note that for each $y \in Y$ we have

$$[x_0 \otimes y, T] = D(x_0 \otimes y) \in L(X, Y, \langle \cdot, \cdot \rangle),$$

and hence we may consider the mapping $y \rightarrow Sy := [x_0 \otimes y, T]^\# y_0$ from Y to Y . Note also that $E_0 = x_0 \otimes y_0$ is an idempotent satisfying $E_0 x_0 = x_0$, and hence

$$E_0 D(E_0)x_0 = D(E_0^2)x_0 - D(E_0)E_0 x_0 = D(E_0)x_0 - D(E_0)x_0 = 0.$$

Therefore

$$\langle D(x_0 \otimes y_0)x_0, y_0 \rangle x_0 = E_0 D(E_0)x_0 = 0,$$

hence $\langle D(x_0 \otimes y_0)x_0, y_0 \rangle = 0$, and so $\langle Tx_0, y_0 \rangle = 0$. Now, for any $(x, y) \in X \times Y$, we see that

$$\begin{aligned} \langle x, Sy \rangle &= \langle x, [x_0 \otimes y, T]^\# y_0 \rangle = \langle [x_0 \otimes y, T]x, y_0 \rangle \\ &= \langle (x_0 \otimes y)Tx, y_0 \rangle - \langle T(x_0 \otimes y)x, y_0 \rangle \\ &= \langle \langle Tx, y \rangle x_0, y_0 \rangle - \langle T(\langle x, y \rangle x_0), y_0 \rangle \\ &= \langle Tx, y \rangle \langle x_0, y_0 \rangle - \langle \langle x, y \rangle Tx_0 + d(\langle x, y \rangle)x_0, y_0 \rangle \\ &= \langle Tx, y \rangle - \langle x, y \rangle \langle Tx_0, y_0 \rangle - d(\langle x, y \rangle) \\ &= \langle Tx, y \rangle - d(\langle x, y \rangle). \end{aligned}$$

Hence S is an adjoint of T . □

Remark 2.4 Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ , let \mathcal{A} be a subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, let $D: \mathcal{A} \rightarrow L(X, Y, \langle \cdot, \cdot \rangle)$ be a derivation, and let T be the differential operator on X given by Theorem 1.3. It follows from Corollary 2.3 that, ‘essentially’, T is uniquely determined by D . Moreover, regarding the proof of Theorem 1.3 we have given, we realize that, for each $(x_0, y_0) \in X \times Y$ with $\langle x_0, y_0 \rangle = 1$, the mapping $x \rightarrow -D(x \otimes y_0)x_0$ is a representative of T , and that (for such a choice of T) $[x_0 \otimes y, T]$ lies in $L(X, Y, \langle \cdot, \cdot \rangle)$ for every $y \in Y$, allowing to determine $T^\#$ as the mapping $y \rightarrow [x_0 \otimes y, T]^\# y_0$.

Let X be a left vector space over Δ . Following [39, Definition IV.4.1], the (algebraic) dual X^* of X is defined as the set of all linear mappings from X to Δ . X^* has a natural structure of right vector space over Δ for the operations

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (f\lambda)(x) = f(x)\lambda$$

for all $x \in X, f, g \in X^*$, and $\lambda \in \Delta$. It is clear that, by defining

$$\langle x, f \rangle = f(x) \quad \text{for all } x \in X \text{ and } f \in X^*,$$

$(X, X^*, \langle \cdot, \cdot \rangle)$ becomes a pairing over Δ . Let A be in $L(X)$. Then for $f \in X^*$ we have $fA \in X^*$ and

$$\langle Ax, f \rangle = f(Ax) = \langle x, fA \rangle \quad \text{for every } x \in X.$$

It follows that the mapping $f \rightarrow fA$ from X^* to X^* is an adjoint of A relative to $\langle \cdot, \cdot \rangle$. Hence $L(X) = L(X, X^*, \langle \cdot, \cdot \rangle)$. Therefore it is enough to apply Theorem 1.3 to obtain the converse of Fact 1.1 given by the first conclusion in Corollary 2.5 immediately below. The second conclusion in that corollary is straightforwardly checked by applying Corollary 2.3 to the left vector space X over Δ , naturally paired with its algebraic dual. For any ring \mathcal{A} , we denote by $\text{Der}(\mathcal{A})$ the Lie ring of all derivations of \mathcal{A} .

Corollary 2.5 *Let X be a left vector space over Δ . Then every ring derivation of $L(X)$ is of the form $A \rightarrow [A, T]$ for some differential operator T on X . More precisely:*

for each differential operator T on X , the mapping $D_T: A \rightarrow [A, T]$ becomes a ring derivation of $L(X)$, and the mapping $T \rightarrow D_T$ is a surjective Lie anti-homomorphism from $\text{Diff}(X)$ to $\text{Der}(L(X))$ with kernel ΔI_X .

Arguing similarly, with Fact 1.2 instead of Fact 1.1, we obtain the following.

Corollary 2.6 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ . Then, for each $T \in \text{Diff}(X, Y, \langle \cdot, \cdot \rangle)$, the mapping $D_T: A \rightarrow [A, T]$ becomes a ring derivation of $L(X, Y, \langle \cdot, \cdot \rangle)$. Moreover, the mapping $T \rightarrow D_T$ is a surjective Lie anti-homomorphism from $\text{Diff}(X, Y, \langle \cdot, \cdot \rangle)$ to $\text{Der}(L(X, Y, \langle \cdot, \cdot \rangle))$ with kernel ΔI_X .*

Lemma 2.7 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ , let \mathcal{A} be a subring of $L(X)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, and let $D: \mathcal{A} \rightarrow L(X)$ be a derivation vanishing on $F(X, Y, \langle \cdot, \cdot \rangle)$. Then $D = 0$.*

Proof Let A be in \mathcal{A} , and let (x, y) be in $X \times Y$. Then, by §2.1 (ii), $A(x \otimes y)$ lies in $F(X, Y, \langle \cdot, \cdot \rangle)$. Therefore, since D vanishes on $F(X, Y, \langle \cdot, \cdot \rangle)$,

$$D(A)(x \otimes y) = D(A(x \otimes y)) - AD(x \otimes y) = 0.$$

Since (x, y) is arbitrary in $X \times Y$, it is enough to apply §2.1 (ii) again (with $D(A)$ instead of A) to conclude that $D(A) = 0$. Hence $D = 0$ because of the arbitrariness of $A \in \mathcal{A}$. □

Proposition 2.8 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ , \mathcal{A} be a subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, and*

$$D: \mathcal{A} \rightarrow L(X, Y, \langle \cdot, \cdot \rangle)$$

be a derivation. Then there exists a unique derivation $\widehat{D}: L(X) \rightarrow L(X)$ extending D . Moreover

$$\widehat{D}(L(X, Y, \langle \cdot, \cdot \rangle)) \subseteq L(X, Y, \langle \cdot, \cdot \rangle). \tag{2.1}$$

Proof By Theorem 1.3, there is a differential operator T on X having an adjoint relative to $\langle \cdot, \cdot \rangle$ and satisfying $D(A) = [A, T]$ for every $A \in \mathcal{A}$. Since T is a differential operator on X , it follows from Fact 1.1 that the mapping $\widehat{D}: B \rightarrow [B, T]$ is a derivation of $L(X)$ extending D . But, since T has an adjoint relative to $\langle \cdot, \cdot \rangle$, it follows from Fact 1.2 that the inclusion (2.1) holds. Finally, if \overline{D} is any derivation of $L(X)$ extending D , then $\widehat{D} - \overline{D}$ is a derivation of $L(X)$ vanishing on $F(X, Y, \langle \cdot, \cdot \rangle)$, hence $\widehat{D} = \overline{D}$ thanks to Lemma 2.7. □

Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ , and let \mathcal{A} be a subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$. Then the symmetric ring of quotients of \mathcal{A} is $L(X, Y, \langle \cdot, \cdot \rangle)$, the maximal left ring of quotients of \mathcal{A} is $L(X)$, and the extended centroid of \mathcal{A} is isomorphic to the center of Δ [5, Theorems 4.3.7 and 4.3.8]. Therefore Proposition 2.8 above also follows easily from the left version of [5, Proposition 2.5.1].

3 Considering Jordan derivations

By a *non-associative ring* we mean an additive group $(\mathcal{R}, +)$ endowed with a mapping $(A, B) \rightarrow AB$ from $\mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ which is additive in each of its variables. We recall that, given an element A in a non-associative ring \mathcal{R} , the operator U_A on \mathcal{R} is defined by

$$U_A(B) = A(AB + BA) - A^2B \quad \text{for all } B \in \mathcal{R}.$$

In the case that \mathcal{R} is in fact associative we see that $U_A(B) = ABA$. Now let \mathcal{R} be an (associative) ring such that $\frac{1}{2} \in \mathcal{R}$, and let \mathcal{A} be a *Jordan subring* of \mathcal{R} (i.e. an additive subgroup of \mathcal{R} such that

$$A_1 \bullet A_2 := \frac{1}{2} (A_1A_2 + A_2A_1)$$

lies in \mathcal{A} whenever A_1 and A_2 belong to \mathcal{A}). Then, for each $A \in \mathcal{A}$, the operator U_A on the Jordan ring (\mathcal{A}, \bullet) satisfies that $U_A(B) = ABA$ for every $B \in \mathcal{A}$, where juxtaposition means the associative product of \mathcal{R} .

Throughout this section we assume that $\frac{1}{2} \in \Delta$, and $(X, Y, \langle \cdot, \cdot \rangle)$ will stand for a pairing over Δ . Then, since 2 is a central element of Δ , so is $\frac{1}{2}$, and therefore $L(X, Y, \langle \cdot, \cdot \rangle)$, and $F(X, Y, \langle \cdot, \cdot \rangle)$ are Jordan subrings of $L(X)$.

Lemma 3.1 *Let A be in $L(X)$ such that $U_B(A) = 0$ for every $B \in F(X, Y, \langle \cdot, \cdot \rangle)$. Then $A = 0$.*

Proof By §2.1 (ii), for every $(x, y) \in X \times Y$ we have

$$0 = U_{x \otimes y}(A) = (x \otimes y)A(x \otimes y) = (x \otimes y)(Ax \otimes y) = (\langle Ax, y \rangle x) \otimes y,$$

hence $\langle Ax, y \rangle = 0$, and so $A = 0$. □

The following corollary will not be applied in what follows, but has its own interest.

Corollary 3.2 *Let A be in $L(X)$ such that $F(X, Y, \langle \cdot, \cdot \rangle) \bullet A = 0$. Then $A = 0$.*

Proof For each $B \in F(X, Y, \langle \cdot, \cdot \rangle)$ we have

$$U_B(A) = 2B \bullet (B \bullet A) - B^2 \bullet A = 0,$$

and so $A = 0$ thanks to Lemma 3.1. □

Let \mathcal{B} be a ring. Given $A, B, C \in \mathcal{B}$, we set

$$(A, B, C)^+ := (A \bullet B) \bullet C - A \bullet (B \bullet C),$$

and we straightforwardly realize that

$$[B, [A, C]] = 4(A, B, C)^+. \tag{3.1}$$

Now let \mathcal{A} be a Jordan subring of \mathcal{B} . A mapping $D: \mathcal{A} \rightarrow \mathcal{B}$ is a *Jordan derivation* from \mathcal{A} to \mathcal{B} if it is additive and the equality

$$D(A_1 \bullet A_2) = D(A_1) \bullet A_2 + A_1 \bullet D(A_2)$$

holds for all $A_1, A_2 \in \mathcal{A}$.

Lemma 3.3 *Let \mathcal{A} be a Jordan subring of $L(X)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, and let $D: \mathcal{A} \rightarrow L(X)$ be a Jordan derivation vanishing on $F(X, Y, \langle \cdot, \cdot \rangle)$. Then $D = 0$.*

Proof Let A be in \mathcal{A} , and let B be in $F(X, Y, \langle \cdot, \cdot \rangle)$. Then

$$D(B \bullet A) = B \bullet D(A).$$

But, by § 2.1 (ii), we have

$$[B, A] = BA - AB = 2B \bullet A - 2AB \in \mathcal{A} + F(X, Y, \langle \cdot, \cdot \rangle) \subseteq \mathcal{A},$$

and then

$$D([B, A]) = 2B \bullet D(A). \tag{3.2}$$

Therefore, by applying (3.2) twice,

$$D([B, [B, A]]) = 2B \bullet D([B, A]) = 4B \bullet (B \bullet D(A)). \tag{3.3}$$

On the other hand, since

$$D((B, B, A)^+) = (\widehat{D}(B), B, A)^+ + (B, D(B), A)^+ + (B, B, D(A))^+$$

and $D(B) = 0$, it follows from (3.1) that

$$D([B, [B, A]]) = D(4(B, B, A)^+) = 4(B, B, D(A))^+. \tag{3.4}$$

Now, using (3.3) and (3.4) we see that

$$U_B(D(A)) = 2B \bullet (B \bullet D(A)) - B^2 \bullet D(A) = B \bullet (B \bullet D(A)) - (B, B, D(A))^+ = 0.$$

Since B is arbitrary in $F(X, Y, \langle \cdot, \cdot \rangle)$, it is enough to apply Lemma 3.1 to obtain that $D(A) = 0$. But A is arbitrary in \mathcal{A} . □

Proposition 3.4 *Let \mathcal{A} be a Jordan subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, and let $D: \mathcal{A} \rightarrow L(X, Y, \langle \cdot, \cdot \rangle)$ be a Jordan derivation. Then there exists a unique derivation $\widehat{D}: L(X) \rightarrow L(X)$ extending D . Moreover*

$$\widehat{D}(L(X, Y, \langle \cdot, \cdot \rangle)) \subseteq L(X, Y, \langle \cdot, \cdot \rangle). \tag{3.5}$$

Proof Given $A \in F(X, Y, \langle \cdot, \cdot \rangle)$, it follows from Litoff's Theorem [5, Theorem 4.3.11] that there exists an idempotent $P \in F(X, Y, \langle \cdot, \cdot \rangle)$ such that $AP = PA = A$, hence

$$D(A) = D(P \bullet A) = D(P) \bullet A + P \bullet D(A) \in F(X, Y, \langle \cdot, \cdot \rangle).$$

Since A is arbitrary in $F(X, Y, \langle \cdot, \cdot \rangle)$, it follows that

$$D(F(X, Y, \langle \cdot, \cdot \rangle)) \subseteq F(X, Y, \langle \cdot, \cdot \rangle).$$

Therefore, by Herstein's theorem [37] (see also [38, Theorem 3.3]), D is a derivation of $F(X, Y, \langle \cdot, \cdot \rangle)$. Then, by Proposition 2.8, there exists a unique derivation $\widehat{D}: L(X) \rightarrow L(X)$ such that $\widehat{D}(A) = D(A)$ for every $A \in F(X, Y, \langle \cdot, \cdot \rangle)$, and moreover the inclusion (3.5) holds. Now the mapping $A \rightarrow \widehat{D}(A) - D(A)$ is a Jordan derivation from \mathcal{A} to $L(X)$ vanishing on $F(X, Y, \langle \cdot, \cdot \rangle)$. Therefore, by Lemma 3.3, \widehat{D} is the unique derivation of $L(X)$ which extends D . \square

Now we can conclude the proof of the main result in this section, namely the following.

Theorem 3.5 *Let \mathcal{A} be a Jordan subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, and let $D: \mathcal{A} \rightarrow L(X, Y, \langle \cdot, \cdot \rangle)$ be a Jordan derivation. Then there exists an essentially unique differential operator T on X such that $D(A) = [A, T]$ for every $A \in \mathcal{A}$. Moreover, for each $(x_0, y_0) \in X \times Y$ with $\langle x_0, y_0 \rangle = 1$, the mapping $x \rightarrow -D(x \otimes y_0)x_0$ is a representative of T , and (for such a choice of T) $[x_0 \otimes y, T]$ lies in $L(X, Y, \langle \cdot, \cdot \rangle)$ for every $y \in Y$, allowing to determine $T^\#$ as the mapping $y \rightarrow [x_0 \otimes y, T]^\# y_0$.*

Proof By Proposition 3.4, there exists a unique derivation

$$\widehat{D}: L(X, Y, \langle \cdot, \cdot \rangle) \rightarrow L(X, Y, \langle \cdot, \cdot \rangle)$$

extending D . Therefore the proof is concluded by applying Theorem 1.3 and Remark 2.4 to \widehat{D} . \square

As a straightforward consequence of the above theorem (or of Proposition 3.4), we obtain the following.

Corollary 3.6 *Let \mathcal{A} be a subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, and let $D: \mathcal{A} \rightarrow L(X, Y, \langle \cdot, \cdot \rangle)$ be a Jordan derivation. Then D is a derivation.*

The above corollary is also a direct consequence of the recent paper by Lin [45], where a complete characterization of weak Jordan derivations of a prime ring into its maximal left ring of quotients is provided.

4 Derivations of standard operator rings on real, complex, or quaternionic normed spaces

4.1 Algebraic preliminaries

The following lemma is straightforward.

Lemma 4.1 *Let \mathcal{R} be an associative ring, let Z denote the centre of \mathcal{R} , and let d be a derivation of \mathcal{R} . Then Z is invariant under d . Therefore d induces a derivation of Z (namely, the restriction of d to Z , regarded as a mapping from Z to Z) which will be denoted by d_Z .*

As always in this paper, Δ denotes a division ring. As in the above lemma, we denote by Z the centre of Δ , and note that, in the current case, Z is a field.

Proposition 4.2 *Let X be a left vector space over Δ , and let T be a differential operator on X . Suppose that Δ has characteristic zero and is finite-dimensional over Z , and that T is Z -linear. Then there exist $\mu \in \Delta$ and $B \in L(X)$ such that $T = \mu I_X + B$.*

Proof Let d be the ring derivation of Δ associated to T . Since T is Z -linear, it follows that the ring derivation d_Z of Z in Lemma 4.1 is zero. This implies that d is Z -linear. Therefore, since Δ has characteristic zero and is finite-dimensional over Z , it follows from [58, Theorem 2.5] that d is inner, say of the form $d_\mu : \lambda \rightarrow [\mu, \lambda]$ for some fixed $\mu \in \Delta$. Now set $B := T - \mu I_X$. Then for all $\lambda \in \Delta$ and $x \in X$ we have

$$\begin{aligned} B(\lambda x) &= T(\lambda x) - \mu \lambda x = \lambda T x + d(\lambda) x - \mu \lambda x \\ &= \lambda T x + d_\mu(\lambda) x - \mu \lambda x = \lambda T x - \lambda \mu x = \lambda B x, \end{aligned}$$

hence $B \in L(X)$. But clearly $T = \mu I_X + B$. □

Lemma 4.3 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ . If X is infinite-dimensional, then there are sequences x_n and y_n in X and Y , respectively, such that $\langle x_i, y_j \rangle = \delta_{ij}$ for all $i, j \in \mathbb{N}$.*

Proof For $y \in Y$ we set $\ker(y) := \{u \in X : \langle u, y \rangle = 0\}$. Starting with any couple $(x_1, y_1) \in X \times Y$ such that $\langle x_1, y_1 \rangle = 1$, assume that the first n terms of the desired sequences have been constructed. Then $P := \sum_{i=1}^n x_i \otimes y_i$ and $Q := \sum_{i=1}^n y_i \otimes x_i$ are linear projections on X and Y , respectively, such that $\ker(P) = \bigcap_{1 \leq i \leq n} \ker(y_i)$ and $Q(Y) = \text{lin}\{y_1, \dots, y_n\}$, which implies $\langle \ker(P), Q(Y) \rangle = 0$. Since $P(X)$ is a finite-dimensional subspace of X , and X is infinite dimensional, there exists $0 \neq x_{n+1} \in \ker(P)$. Assume that $\langle x_{n+1}, \ker(Q) \rangle = 0$. Then, since $\langle x_{n+1}, Q(Y) \rangle = 0$, we have $\langle x_{n+1}, Y \rangle = \langle x_{n+1}, Q(Y) + \ker(Q) \rangle = 0$, and hence $x_{n+1} = 0$, which is not possible. Therefore there exists $y_{n+1} \in \ker(Q)$ such that $\langle x_{n+1}, y_{n+1} \rangle = 1$. Now we have $\langle x_i, y_j \rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, n + 1\}$. In this way, the $(n + 1)$ -th terms of the desired sequences have been constructed. □

To conclude this subsection, let us explain with more precision some facts on pairings, sketched in Sect. 1. Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a pairing over Δ . Let us say that a

differential operator S on Y with associated derivation d has an adjoint in X relative to $\langle \cdot, \cdot \rangle$ if there exists a (unique) mapping $S^\# : X \rightarrow X$ such that $\langle x, Sy \rangle = \langle S^\#x, y \rangle + d(\langle x, y \rangle)$ for all $(x, y) \in X \times Y$. Then the set $\text{Diff}(Y, X, \langle \cdot, \cdot \rangle)$ of those differential operators on Y having an adjoint in X relative to $\langle \cdot, \cdot \rangle$ is a Lie subring of the ring of all additive operators on Y , and the mapping $\Phi : T \rightarrow T^\#$ from $\text{Diff}(X, Y, \langle \cdot, \cdot \rangle)$ to $\text{Diff}(Y, X, \langle \cdot, \cdot \rangle)$ is a bijective Lie anti-homomorphism with inverse mapping the one $S \rightarrow S^\#$ from $\text{Diff}(Y, X, \langle \cdot, \cdot \rangle)$ to $\text{Diff}(X, Y, \langle \cdot, \cdot \rangle)$. Moreover, the inclusion $\Delta_{I_X} \subseteq \text{Diff}(X, Y, \langle \cdot, \cdot \rangle)$ holds, and for $\lambda \in \Delta$ we have $(\lambda I_X)^\# = I_Y \lambda$, where $I_Y \lambda$ denotes the mapping $y \rightarrow y\lambda$ from Y to Y . Now set

$$L(Y, X, \langle \cdot, \cdot \rangle) := \text{Diff}(Y, X, \langle \cdot, \cdot \rangle) \cap L(Y).$$

Then $L(Y, X, \langle \cdot, \cdot \rangle)$ is a subring of $L(Y)$, and the restriction of the mapping Φ above to $L(X, Y, \langle \cdot, \cdot \rangle)$ becomes a bijective ring anti-homomorphism from $L(X, Y, \langle \cdot, \cdot \rangle)$ to $L(Y, X, \langle \cdot, \cdot \rangle)$. These facts will be applied without notice in the proofs of Corollary 4.11, Proposition 4.12, and Theorem 5.3.

4.2 The main results

As we already said in Sect. 1, we denote by \mathbb{H} the division real algebra of Hamilton’s quaternions (see for example [18, pp. 176–177 and Theorem 2.6.21] and [31, Chapter 7]). Then the centre of \mathbb{H} is equal to $\mathbb{R}1$, and consequently identifies with \mathbb{R} . Let $*$ stand for the standard involution of \mathbb{H} , and let λ be in \mathbb{H} . Then both $\lambda + \lambda^*$ and $\lambda^* \lambda$ lie in \mathbb{R} . Actually $\lambda^* \lambda$ is a nonnegative real number, and the mapping $\lambda \rightarrow |\lambda| := \sqrt{\lambda^* \lambda}$ is an absolute value on \mathbb{H} . Given $\lambda \in \mathbb{H}$ we set $\text{Re}(\lambda) := \frac{1}{2}(\lambda + \lambda^*)$.

By a *left quaternionic normed space* we mean a left vector space X over \mathbb{H} endowed with a subadditive mapping $\|\cdot\| : X \rightarrow \mathbb{R}$ such that $\|x\| = 0 \Leftrightarrow x = 0$, and $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{H}$ and $x \in X$. Analogously *right quaternionic normed spaces* are defined. Since Soukhomlinoff’s pioneering paper [64], quaternionic normed spaces have been fully discussed in the literature (see [46, 50] and references therein), and appear naturally in the theory of complex Hilbert spaces [36, 7.5.5]. We note that, by restricting the scalars, every left or right quaternionic normed space becomes a normed space over \mathbb{R} . Let X be a left quaternionic normed space. Then the set X' of all continuous \mathbb{H} -linear mappings from X to \mathbb{H} is a vector subspace of the right quaternionic vector space X^* (the algebraic dual of X), and becomes a right quaternionic normed space under the norm

$$\|f\| := \sup\{|f(x)| : \|x\| \leq 1\}.$$

The right quaternionic normed space X' just introduced is called the (topological) *dual* of X .

Proposition 4.4 *Let X be a left quaternionic normed space. Then the mapping $\Phi : f \rightarrow \text{Re} \circ f$ becomes a surjective \mathbb{R} -linear isometry from the dual X' of X to the dual $(X_{\mathbb{R}})'$ of $X_{\mathbb{R}}$.*

Proof The facts that $\text{Re} \circ f$ lies in $(X_{\mathbb{R}})'$ whenever f is in X' and that Φ is \mathbb{R} -linear are clear.

Let x and f be in X and X' , respectively. Then we have

$$\begin{aligned} |f(x)|^2 &= f(x)^* f(x) = \text{Re}(f(x)^* f(x)) = \text{Re}(f(f(x)^* x)) = \Phi(f)(f(x)^* x) \\ &\leq \|\Phi(f)\| \|f(x)^* x\| = \|\Phi(f)\| \|f(x)\| \|x\|, \end{aligned}$$

and hence $|f(x)| \leq \|\Phi(f)\| \|x\|$. Since x is arbitrary in X , we deduce that $\|f\| \leq \|\Phi(f)\|$. But the converse inequality is clear. Therefore, since f is arbitrary in X' , we conclude that Φ is an isometry.

Now let ϕ be in $(X_{\mathbb{R}})'$. Take a canonical basis $\{1, i, j, k\}$ of \mathbb{H} , so that $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, and $ki = j = -ik$. Define a mapping $f : X \rightarrow \mathbb{H}$ by $f(x) := \phi(x) - \phi(ix)i - \phi(jx)j - \phi(kx)k$. Then

$$\begin{aligned} f(ix) &= \phi(ix) + \phi(x)i + \phi(kx)j - \phi(jx)k, \\ if(x) &= \phi(x)i + \phi(ix) - \phi(jx)k + \phi(kx)j, \end{aligned}$$

and hence $f(ix) = if(x)$. Analogously $f(jx) = jf(x)$ and $f(kx) = kf(x)$. Therefore, since clearly f is \mathbb{R} -linear and continuous, it follows that f lies in X' . Finally, since clearly $\text{Re} \circ f = \phi$, and ϕ is arbitrary in $(X_{\mathbb{R}})'$, we conclude that Φ is surjective. □

By a *left* (respectively, *right*) *quaternionic Banach space* we mean a left (respectively, right) quaternionic normed space which is a complete metric space relative to the distance $\|u - v\|$. We note that, by restricting the scalars, every left or right quaternionic normed space becomes a normed space over \mathbb{R} . Therefore, as a byproduct of Proposition 4.4, we are provided with the following.

Corollary 4.5 *The dual of any left quaternionic normed space is a right quaternionic Banach space in a natural way.*

Lemma 4.6 *Let X be a left normed space over \mathbb{F} , and let x be in X . Then there exists $f \in X'$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

Proof We may suppose that $\mathbb{F} = \mathbb{H}$. Then, by the Hahn–Banach theorem applied to $(X_{\mathbb{R}})'$, there exists $g \in (X_{\mathbb{R}})'$ such that $\|g\| = 1$ and $g(x) = \|x\|$. Therefore, since the mapping $\Phi : X' \rightarrow (X_{\mathbb{R}})'$ in Proposition 4.4 is a surjective \mathbb{R} -linear isometry, it is enough to apply that proposition to realize that $f := \Phi^{-1}(g) \in X'$ satisfies the conditions in the conclusion of the lemma. □

We note that Lemma 4.6 above follows from the main result in [64], as well as that the main result in [64] follows from Proposition 4.4.

By a *normed pairing* over \mathbb{F} we mean any pairing $(X, Y, \langle \cdot, \cdot \rangle)$, where X is a left normed space over \mathbb{F} , Y is a right normed space over \mathbb{F} , and the form $\langle \cdot, \cdot \rangle$ is continuous. If X (respectively, Y) is complete, then we say that $(X, Y, \langle \cdot, \cdot \rangle)$ is *complete on the left* (respectively, *complete on the right*). If $(X, Y, \langle \cdot, \cdot \rangle)$ is both complete on the left and complete on the right, then we simply say that it is a *Banach pairing*.

Lemma 4.7 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a normed pairing over \mathbb{F} , and let T be in $\text{Diff}(X, Y, \langle \cdot, \cdot \rangle)$. Then there are mappings $g: X \rightarrow \mathbb{R}_0^+$ and $h: Y \rightarrow \mathbb{R}_0^+$ such that $\|T(\langle x, y \rangle x)\| \leq g(x)h(y)$ for all $x \in X$ and $y \in Y$.*

Proof Let d denote the ring derivation of \mathbb{F} associated to T , and set $M := \|\langle \cdot, \cdot \rangle\|$. Let (x, y) be in $X \times Y$. Then, by (1.2), we have

$$|d(\langle x, y \rangle)| \leq |\langle Tx, y \rangle| + |\langle x, T^\#y \rangle| \leq M(\|Tx\|\|y\| + \|x\|\|T^\#y\|).$$

Therefore, considering (1.1), we obtain

$$\begin{aligned} \|T(\langle x, y \rangle x)\| &\leq |\langle x, y \rangle| \|Tx\| + |d(\langle x, y \rangle)| \|x\| \\ &\leq M(\|x\|\|y\|\|Tx\| + (\|Tx\|\|y\| + \|x\|\|T^\#y\|)\|x\|) \\ &= M\|x\|(2\|Tx\|\|y\| + \|x\|\|T^\#y\|) \\ &\leq M\|x\|(2\|Tx\| + \|x\|) \max\{\|y\|, \|T^\#y\|\}. \quad \square \end{aligned}$$

Let X be a left normed space over \mathbb{F} . As we already said in Sect. 1, we denote by $\mathcal{L}(X)$ the normed subalgebra of the \mathbb{R} -algebra $L(X)$ consisting of those operators in $L(X)$ which are continuous, and by $\mathcal{F}(X)$ the ideal of $\mathcal{L}(X)$ consisting of those operators in $\mathcal{L}(X)$ whose range is finite-dimensional. The next lemma follows straightforwardly from the closed graph theorem.

Lemma 4.8 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a normed pairing over \mathbb{F} complete on the left. Then $L(X, Y, \langle \cdot, \cdot \rangle) \subseteq \mathcal{L}(X)$.*

Remark 4.9 (a) Every additive operator on a vector space X over \mathbb{F} is \mathbb{Q} -linear, so such an operator T is \mathbb{R} -linear if X is normed and T is continuous. Moreover, additive operators on a normed space X over \mathbb{F} are continuous if (and only if) they are bounded on some neighborhood of zero. Indeed, since we can regard X as a normed space over \mathbb{R} , it is enough to consider the case that $\mathbb{F} = \mathbb{R}$, and then to consult [1, pp. 36–37].

(b) If \mathcal{A} is a (possibly non-associative) complex algebra with zero annihilator, and if D is an additive derivation of \mathcal{A} , then, by [41, Remark 4.2], the equality $D(iA) = iD(A)$ holds for all $A \in \mathcal{A}$. Consequently, \mathbb{R} -linear derivations of any complex algebra with zero annihilator are \mathbb{C} -linear. Therefore, by part (a) of the current remark, continuous additive derivations of any normed algebra over \mathbb{K} with zero annihilator are linear. As a consequence, if d is a continuous ring derivation of \mathbb{K} , then $d = 0$. It is well known that nonzero derivations of \mathbb{R} do exist [71, Chapter II, Section 17].

Now we adapt the argument in the proof of [9, Proposition 3.3] to prove the following.

Theorem 4.10 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a normed pairing over \mathbb{F} complete on the left, with X infinite-dimensional, and let T be in $\text{Diff}(X, Y, \langle \cdot, \cdot \rangle)$. Then there exist $\mu \in \mathbb{F}$ and $A \in L(X, Y, \langle \cdot, \cdot \rangle)$ such that $T = \mu I_X + A$. As a consequence, T is continuous and, when $\mathbb{F} = \mathbb{K} := \mathbb{R}$ or \mathbb{C} , T is linear.*

Proof Let d denote the ring derivation of \mathbb{F} associated to T and, to derive a contradiction, assume that d is discontinuous. Then, by Remark 4.9 (a), d is unbounded on every neighborhood of zero. Let x_n and y_n be the sequences in X and Y , respectively, given by Lemma 4.3, and note that, replacing x_n with $\|x_n\|^{-1}x_n$ and y_n with $y_n\|x_n\|$ if necessary, we may suppose that $\|x_n\| = 1$ for every $n \in \mathbb{N}$. Let $g : X \rightarrow \mathbb{R}_0^+$ and $h : Y \rightarrow \mathbb{R}_0^+$ be the mappings given by Lemma 4.7, and for $n \in \mathbb{N}$ set $\alpha_n := \max\{1, h(y_n)\}$. Then there is $\lambda_n \in \mathbb{F}$ such that $|\lambda_n| \leq \frac{1}{2^n\alpha_n}$ and $|d(\lambda_n)| > 2^n\alpha_n$. Set $x := \sum_{k=1}^\infty \lambda_k x_k$. Then we have

$$\sum_n \frac{1}{2^n\alpha_n} T(\lambda_n x) = \sum_n \frac{1}{2^n\alpha_n} \lambda_n T x + \sum_n \frac{1}{2^n\alpha_n} d(\lambda_n) x.$$

Therefore, since the series $\sum_n \frac{1}{2^n\alpha_n} \lambda_n T x$ converges, and the series $\sum_n \frac{1}{2^n\alpha_n} d(\lambda_n) x$ diverges, it follows that the series $\sum_n \frac{1}{2^n\alpha_n} T(\lambda_n x)$ diverges. But this is not possible because, since $\lambda_n = \langle x, y_n \rangle$, we have $T(\lambda_n x) = T(\langle x, y_n \rangle x)$, so $\|T(\lambda_n x)\| \leq g(x)h(y_n) \leq g(x)\alpha_n$ (by Lemma 4.7), and so the series $\sum_n \frac{1}{2^n\alpha_n} T(\lambda_n x)$ is absolutely convergent.

Therefore d is continuous. If $\mathbb{F} = \mathbb{K}$, then $d = 0$ (by Remark 4.9 (b)), so T is linear, hence T lies in $L(X, Y, \langle \cdot, \cdot \rangle)$ (as it has an adjoint relative to $\langle \cdot, \cdot \rangle$), and the continuity of T follows from Lemma 4.8. Suppose that $\mathbb{F} = \mathbb{H}$. Then, with the notation in Lemma 4.1, $d_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, so $d_{\mathbb{R}} = 0$ (by Remark 4.9 (b) again), and so T is \mathbb{R} -linear. Therefore, by Proposition 4.2, there exist $\mu \in \mathbb{F}$ and $A \in L(X)$ such that $T = \mu I_X + A$, with $A = T - \mu I_X \in L(X, Y, \langle \cdot, \cdot \rangle)$ (as A has an adjoint relative to $\langle \cdot, \cdot \rangle$), and the continuity of T follows by applying Lemma 4.8 again. \square

Corollary 4.11 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a normed pairing over \mathbb{F} complete on the right, with X infinite-dimensional, let \mathcal{A} be a Jordan subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, and let*

$$D : \mathcal{A} \rightarrow L(X, Y, \langle \cdot, \cdot \rangle)$$

be a Jordan derivation. Then there exists $B \in L(X, Y, \langle \cdot, \cdot \rangle)$ such that $D(A) = [A, B]$ for every $A \in \mathcal{A}$. As a consequence, if in fact $(X, Y, \langle \cdot, \cdot \rangle)$ is a Banach pairing, then D is continuous.

Proof By Theorem 3.5, there exists a differential operator T on X having an adjoint $T^\#$ relative to $\langle \cdot, \cdot \rangle$ and satisfying $D(A) = [A, T]$ for every $A \in \mathcal{A}$. But $T^\#$ is a differential operator on Y having an adjoint (namely T) relative to $\langle \cdot, \cdot \rangle$. Therefore, by the complete-on-the-right version of Theorem 4.10, there exist $\mu \in \mathbb{F}$ and $C \in L(Y, X, \langle \cdot, \cdot \rangle)$ such that $T^\# = I_Y\mu + C$. Now $T = (T^\#)^\# = (I_Y\mu + C)^\# = \mu I_X + B$, where $B := C^\#$ lies in $L(X, Y, \langle \cdot, \cdot \rangle)$. Finally, since $D(A) = [A, T]$ for every $A \in \mathcal{A}$, we conclude that $D(A) = [A, B]$ for every $A \in \mathcal{A}$.

The consequence follows from Lemma 4.8. \square

In the case that \mathbb{F} is equal to \mathbb{R} or \mathbb{C} , the next proposition is known in [47, Theorem II-4].

Proposition 4.12 *Let X be a left normed space over \mathbb{F} , and for $(x, f) \in X \times X'$ set $\langle x, f \rangle := f(x)$. Then we have:*

- (i) $(X, X', \langle \cdot, \cdot \rangle)$ is a normed pairing over \mathbb{F} complete on the right.
- (ii) $\mathcal{L}(X) = L(X, X', \langle \cdot, \cdot \rangle)$.
- (iii) $\mathcal{F}(X) = F(X, X', \langle \cdot, \cdot \rangle)$.

Proof If $\mathbb{F} = \mathbb{K}$, then assertion (i) is well known. Suppose that $\mathbb{F} = \mathbb{H}$. Then, by Corollary 4.5, X' is a right Banach space over \mathbb{H} . Therefore to conclude the proof of (i) it is enough to show that the form $\langle \cdot, \cdot \rangle$ is nondegenerate. Let f be in X' such that $\langle X, f \rangle = 0$. Then clearly $f = 0$. Now let x be in X such that $\langle x, X' \rangle = 0$. Then, as a byproduct of Lemma 4.6, we have $x = 0$. Thus (i) has been proved.

Let F be in $\mathcal{L}(X)$. Then $F \in L(X)$, and the topological transpose F' , where $F'(f) := f \circ F$ for every $f \in X'$, is an adjoint of F relative to $\langle \cdot, \cdot \rangle$. Therefore F lies in $L(X, X', \langle \cdot, \cdot \rangle)$. Conversely, let F be in $L(X, X', \langle \cdot, \cdot \rangle)$. Then, by (i) and the complete-on-the-right version of Lemma 4.8, $F^\#$ is continuous. Therefore, for all $x \in X$ and $f \in X'$ we have

$$|f(Fx)| = |\langle Fx, f \rangle| = |\langle x, F^\# f \rangle| \leq \|F^\#\| \|f\| \|x\|. \tag{4.1}$$

Let x be in X . Then, by Lemma 4.6, there exists $f \in X'$ such that $\|f\| = 1$ and $f(Fx) = \|Fx\|$. Therefore, by (4.1), $\|Fx\| \leq \|F^\#\| \|f\| \|x\| = \|F^\#\| \|x\|$. Since x is arbitrary in X , we conclude that F lies in $\mathcal{L}(X)$. Thus the proof of (ii) has been concluded.

Assertion (iii) follows straightforwardly from (ii). □

Proof of Theorem 1.4 Combine Corollary 4.11 and Proposition 4.12. □

For the reader interested only in associative notions, we emphasize the straightforward consequence of Theorem 1.4 given by the following.

Corollary 4.13 *Let X be an infinite-dimensional left normed space over \mathbb{F} , let \mathcal{A} be a standard operator ring on X , and let $D: \mathcal{A} \rightarrow \mathcal{L}(X)$ be a derivation. Then there exists $B \in \mathcal{L}(X)$ such that $D(A) = [A, B]$ for every $A \in \mathcal{A}$. As a consequence, D is continuous.*

Remark 4.14 Let d be a (ring) derivation of \mathbb{R} . Then the mapping $d_{\mathbb{C}}: a + ib \rightarrow d(a) + id(b)$ ($a, b \in \mathbb{R}$) is a derivation of \mathbb{C} . Similarly, if $\{1, i, j, k\}$ is a canonical basis of \mathbb{H} , then the mapping

$$d_{\mathbb{H}}: \alpha + \beta i + \gamma j + \delta k \rightarrow d(\alpha) + d(\beta)i + d(\gamma)j + d(\delta)k$$

($\alpha, \beta, \gamma, \delta \in \mathbb{R}$) is a derivation of \mathbb{H} . Therefore, keeping in mind the existence of discontinuous derivations of \mathbb{R} (Remark 4.9(b)), the existence of discontinuous derivations of $\mathbb{F} := \mathbb{R}, \mathbb{C}$, or \mathbb{H} is assured.

Now let X be a finite-dimensional left normed space over \mathbb{F} . Then $X = \mathbb{F}^m$ for some positive integer m , and $\mathcal{L}(X) = M_m(\mathbb{F})$ (the ring of all $m \times m$ matrices over \mathbb{F}) is the unique standard operator Jordan ring on X . Moreover, given any derivation

d of \mathbb{F} , the mapping $d_m : (a_{i,j}) \rightarrow (d(a_{i,j}))$ ($(a_{i,j}) \in M_m(\mathbb{F})$) is a derivation of $M_m(\mathbb{F})$. It follows from the above paragraph that the restriction in Theorem 1.4 and Corollary 4.13 that X is infinite-dimensional cannot be removed.

To conclude this remark, let us incidentally note that derivations of $M_m(\mathbb{F})$ can be described in terms of derivations of \mathbb{R} . Indeed, according to [48, Example 19], each derivation of $M_m(\mathbb{F})$ decomposes as a sum of an inner derivation of $M_m(\mathbb{F})$ and a derivation of the form d_m for some derivation d of \mathbb{F} . (Even, in this result, \mathbb{F} can be replaced with any unital ring; note also that, in the very particular case that $\mathbb{F} = \mathbb{K}$, the result we are reviewing was rediscovered in [62, Theorem 2.2].) Moreover, it is easily realized that every derivation d of \mathbb{C} can be written as $d = (d_1)_{\mathbb{C}} + i(d_2)_{\mathbb{C}}$ for suitable derivations d_1, d_2 of \mathbb{R} . Finally, according to [33, Theorem 3.1] or [34, Theorem 2.1], each derivation of \mathbb{H} decomposes as a sum of an inner derivation of \mathbb{H} and a derivation of the form $d_{\mathbb{H}}$ for some derivation d of \mathbb{R} .

Definition 4.15 Let X be a nonzero left normed space over \mathbb{F} , and let $\|\cdot\|$ denote the norm of X as well as the dual norm on X' and the operator norm on the real algebra $\mathcal{L}(X)$. Arguing as in the proof of [18, Proposition 1.4.32], we realize that $\mathcal{F}(X)$ is the smallest nonzero ring ideal of $\mathcal{L}(X)$. Generalizing a notion of Schatten [59, Definition V.1.3], by a *norm ideal of operators* on X we mean any nonzero \mathbb{R} -algebra ideal \mathcal{A} of $\mathcal{L}(X)$ endowed with a norm $\|\cdot\|$ satisfying the following conditions:

- (i) $\|x \otimes f\| = \|x\| \|f\|$ for all $x \in X$ and $f \in X'$ (here $x \otimes f$ has the meaning given in § 2.1 particularized to the case that $\Delta = \mathbb{F}$, $Y = X'$, and X is naturally paired with X').
- (ii) $\|FAG\| \leq \|F\| \|A\| \|G\|$ for all $A \in \mathcal{A}$ and $F, G \in \mathcal{L}(X)$.

It follows from these conditions that, for $A \in \mathcal{A}$, $x \in X$, and $f \in X'$ with $\|x\| = \|f\| = 1$, we have

$$\|Ax\| = \|Ax \otimes f\| = \|A(x \otimes f)\| \leq \|x \otimes f\| \|A\| = \|A\|.$$

Hence $\|A\| \leq \|A\|$ for every $A \in \mathcal{A}$. Therefore, keeping in mind (ii) again, we realize that $(\mathcal{A}, \|\cdot\|)$ is a real normed algebra.

Corollary 4.16 *Let $(X, \|\cdot\|)$ be an infinite-dimensional left normed space over \mathbb{F} , let $(\mathcal{A}, \|\cdot\|)$ be a norm ideal of operators on X , and let D be a ring derivation of \mathcal{A} . Then D is \mathbb{R} -linear and $\|\cdot\|$ -continuous.*

Proof Clearly \mathcal{A} is a standard ring of operators on X . Therefore, by Corollary 4.13, there exists $B \in \mathcal{L}(X)$ such that $D(A) = [A, B]$ for every $A \in \mathcal{A}$. As a first consequence, D is \mathbb{R} -linear. Moreover, by requirement (ii) in Definition 4.15, for $A \in \mathcal{A}$ we have $\|D(A)\| = \|AB - BA\| \leq 2\|B\| \|A\|$, and hence D is continuous (with $\|D\| \leq 2\|B\|$). □

Remark 4.17 With the notation and requirements in the above corollary, suppose that actually $\mathbb{F} = \mathbb{C}$ and that in addition \mathcal{A} is a complex subspace of $\mathcal{L}(X)$. Then the above proof shows that D is \mathbb{C} -linear.

5 Primitive associative normed Q -algebras with nonzero socle

This section is a homage to Rickart, who knew relevant forerunners of Lemmas 5.1 and 5.2, and of Theorem 5.3 below (see [52, Lemma 2.4.4, and Theorems 1.3.3 and 2.4.12]).

Lemma 5.1 *Let \mathcal{A} be a ring, let X be a faithful irreducible left \mathcal{A} -module, let Δ denote the associated division ring of \mathcal{A} relative to X , and let us regard X as a left vector space over Δ . Suppose that actually \mathcal{A} is (the ring underlying) an algebra over a field \mathfrak{F} . Then X becomes naturally a vector space over \mathfrak{F} , and \mathfrak{F} is contained in the centre of Δ . Therefore Δ is an algebra over \mathfrak{F} , and ΔI_X consists only of \mathfrak{F} -linear operators on X .*

Proof Since X is an irreducible left module over the ring \mathcal{A} , and \mathcal{A} is an algebra over \mathfrak{F} , it follows from [39, Proposition I.9.2] that X can be regarded in one and only one way as an *irreducible left module over the algebra \mathcal{A}* , i.e. X is an irreducible left module over the ring \mathcal{A} , and there is a unique multiplication $(\lambda, x) \rightarrow \lambda x$ from $\mathfrak{F} \times X$ to X converting $(X, +)$ into a vector space over \mathfrak{F} , and satisfying the bilinearity condition

$$\lambda(Ax) = A(\lambda x) = (\lambda A)x \quad \text{for all } \lambda \in \mathfrak{F}, A \in \mathcal{A}, \text{ and } x \in X \quad (5.1)$$

(compare [39, Definitions I.9.1 and I.9.3]). Now the first equality in (5.1) reads as that \mathfrak{F} is included in Δ .

Let x be arbitrary in X . Then, by the irreducibility of X , there exists $A \in \mathcal{A}$ such that $Ax = x$. Now let (λ, μ) be in $\mathfrak{F} \times \Delta$. Then, since λA lies in \mathcal{A} , and $B(\mu x) = \mu Bx$ for every $B \in \mathcal{A}$, we have

$$\mu \lambda x = \mu \lambda Ax = \mu((\lambda A)x) = (\lambda A)(\mu x) = \lambda A(\mu x) = \lambda \mu Ax = \lambda \mu x.$$

Therefore, since x is arbitrary in X , we obtain that $\mu \lambda = \lambda \mu$. Finally, since (λ, μ) is arbitrary in $\mathfrak{F} \times \Delta$, we conclude that \mathfrak{F} is contained in the centre of Δ . \square

Lemma 5.2 *Let \mathbb{F} denote either \mathbb{C} or \mathbb{H} , let X be a left vector space over \mathbb{F} , and let $\|\cdot\|$ be a norm on X converting X into a real normed space and satisfying $\|\lambda x\| \leq \|\lambda\| \|x\|$ for some \mathbb{R} -algebra norm $\|\cdot\|$ on \mathbb{F} and all $\lambda \in \mathbb{F}$ and $x \in X$. Then there exists an equivalent norm on X converting X into a normed space over \mathbb{F} .*

Proof For $x \in X$, set $\|x\| = \sup\{\|\mu x\| : \mu \in \mathbb{F} \text{ such that } |\mu| = 1\}$. Then it is easily checked that $\|\cdot\|$ is an equivalent norm on X converting X into a normed space over \mathbb{F} . \square

By an *associative normed Q -algebra* over \mathbb{K} we mean any normed associative algebra over \mathbb{K} such that the set of quasi-invertible elements is open. Associative normed Q -algebras have become a classical topic in the theory of associative normed algebras, whose study goes back to Kaplansky [42] (see [18, Section 3.6.61] for additional information). Complete normed associative algebras are associative normed Q -algebras [18, Example 3.6.42], but the converse is not true. Thus, for example, every (possibly non closed) one-sided ideal of a complete normed associative algebra is an associative normed Q -algebra.

Theorem 5.3 *Let $(\mathcal{A}, \|\cdot\|)$ be a primitive associative normed Q -algebra over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} having a nonzero socle. Let \mathbb{F} stand for \mathbb{C} if $\mathbb{K} = \mathbb{C}$, and for \mathbb{R}, \mathbb{C} , or \mathbb{H} if $\mathbb{K} = \mathbb{R}$. Then there exists a normed pairing $((X, \|\cdot\|), (Y, \|\cdot\|), \langle \cdot, \cdot \rangle)$ over \mathbb{F} in such a way that \mathcal{A} is a \mathbb{K} -subalgebra of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$, operators in \mathcal{A} are $\|\cdot\|$ -continuous, and the mapping $A \rightarrow A$ from $(\mathcal{A}, \|\cdot\|)$ to $(\mathcal{A}, \|\cdot\|)$ is continuous. Moreover, if the associative normed Q -algebra $(\mathcal{A}, \|\cdot\|)$ is in fact a complete normed algebra, then the pairing $((X, \|\cdot\|), (Y, \|\cdot\|), \langle \cdot, \cdot \rangle)$ above is a Banach pairing.*

Proof By [39, Section IV.9], there exist a division ring Δ and a pairing $(X, Y, \langle \cdot, \cdot \rangle)$ over Δ such that \mathcal{A} is a subring of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$. Then, since \mathcal{A} is an algebra over \mathbb{K} , it follows from Lemmas 2.2 and 5.1 that X is a vector space over \mathbb{K} , that Δ is an algebra over \mathbb{K} , and that ΔI_X consists only of \mathbb{K} -linear operators on X . Now, by [18, Corollary 3.6.44 (i) and its proof], there exists a norm $\|\cdot\|$ on X converting X into a normed space (occasionally, a Banach space if we are dealing with the complete normed case) over \mathbb{K} , and with the properties that every operator $A \in \mathcal{A}$ is $\|\cdot\|$ -continuous with $\|A\| \leq \|A\|$, and that ΔI_X consists only of $\|\cdot\|$ -continuous \mathbb{K} -linear operators on X . Moreover, a choice of such a norm $\|\cdot\|$ on X can be constructed by taking $0 \neq x_0 \in X$, and defining

$$\|x\| = \inf \{ \|A\| : A \in \mathcal{A} \text{ such that } Ax_0 = x \}. \tag{5.2}$$

On the other hand, since ΔI_X consists only of $\|\cdot\|$ -continuous \mathbb{K} -linear operators on X , we can convert Δ into a normed algebra over \mathbb{K} (under the operator norm corresponding to $\|\cdot\|$), and apply the Gelfand–Mazur theorem [18, Proposition 2.5.40 and Corollary 1.1.43] to obtain that $\Delta = \mathbb{F}$, with \mathbb{F} as in the statement of the theorem. Now, if $\mathbb{K} = \mathbb{C}$, or if $\mathbb{K} = \mathbb{R}$ and $\mathbb{F} = \mathbb{R}$, then $(X, \|\cdot\|)$ is a left normed (occasionally, Banach) space over \mathbb{F} and the inequality $\|A\| \leq \|A\|$ holds for every $A \in \mathcal{A}$. In the remaining cases (that $\mathbb{K} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ or \mathbb{H}), it is enough to apply Lemma 5.2 to realize that, up to an equivalent renorming (we remain calling $\|\cdot\|$ the new norm), $(X, \|\cdot\|)$ becomes a left normed (occasionally, Banach) space over \mathbb{F} and there is $M > 0$ such that the inequality $\|A\| \leq M\|A\|$ holds for every $A \in \mathcal{A}$. Therefore, in any case, $(X, \|\cdot\|)$ is a left normed (occasionally, Banach) space over \mathbb{F} and there is $K > 0$ such that the inequality $\|A\| \leq K\|A\|$ holds for every $A \in \mathcal{A}$. Thus the mapping $A \rightarrow A$ from $(\mathcal{A}, \|\cdot\|)$ to $(\mathcal{A}, \|\cdot\|)$ is continuous. We note that the change of the norm $\|\cdot\|$ we have done in some cases could affect equality (5.2), which should be replaced with a double inequality of the form

$$K_1 \|x\| \leq \inf \{ \|A\| : A \in \mathcal{A} \text{ such that } Ax_0 = x \} \leq K_2 \|x\|$$

for some positive constants K_1, K_2 . In what follows we codify this incidence by writing \simeq instead of $=$ in (5.2) and similar places.

In principle, $\mathcal{A}^\#$ has no any natural norm. Nevertheless, since $\mathcal{A}^\#$ is antiisomorphic to \mathcal{A} in a natural way, we can move the norm of \mathcal{A} to $\mathcal{A}^\#$, and define $\|A^\#\| := \|A\|$ ($A \in \mathcal{A}$), to realize that $(\mathcal{A}^\#, \|\cdot\|)$ becomes an associative normed Q -algebra (occasionally, a

complete normed associative algebra) over \mathbb{K} .¹ Therefore, replacing in the above paragraph X with Y , $(\mathcal{A}, \|\cdot\|)$ with $(\mathcal{A}^\#, \|\cdot\|)$, and Lemmas 2.2, 5.1, and 5.2 with their ‘right’ versions, we realize that there exists a norm $\|\cdot\|$ on Y converting Y into a right normed (occasionally, Banach) space over \mathbb{F} , and satisfying

$$\|y\| \simeq \inf \{ \|A\| : A \in \mathcal{A} \text{ such that } A^\# y_0 = y \} \tag{5.3}$$

for every $y \in Y$, where y_0 is a prefixed nonzero element of Y .

Now, to conclude the proof of the theorem, it only remains to show that the form $\langle \cdot, \cdot \rangle$ is $\|\cdot\|$ -continuous. But this can be done by arguing as in the proof of [52, Lemma 2.4.11]. Indeed, let (x, y) be in $X \times Y$. Then $(x \otimes y)^2 = \langle x, y \rangle x \otimes y$, so $|\langle x, y \rangle| \|x \otimes y\| = \|(x \otimes y)^2\| \leq \|(x \otimes y)\|^2$, and so $|\langle x, y \rangle| \leq \|x \otimes y\|$. Now let $A, B \in \mathcal{A}$ be such that $Ax_0 = x$ and $B^\# y_0 = y$. Then $x \otimes y = A(x_0 \otimes y_0)B$, and hence $\|x \otimes y\| \leq \|x_0 \otimes y_0\| \|A\| \|B\|$. Taking the infimum over all such A and B , it follows from (5.2) (with \simeq instead of $=$) and (5.3) that there exists a positive constant L (independent of (x, y)) such that $\|x \otimes y\| \leq L \|x\| \|y\|$. It follows that $|\langle x, y \rangle| \leq L \|x\| \|y\|$. \square

A normed algebra $(\mathcal{A}, \|\cdot\|)$ over \mathbb{K} is said to have *minimality of norm topology* if every continuous algebra norm on \mathcal{A} is equivalent to $\|\cdot\|$. The next corollary follows straightforwardly from Theorem 5.3.

Corollary 5.4 *Let \mathcal{A} be a primitive associative normed Q -algebra over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} having a nonzero socle and minimality of norm topology. Let \mathbb{F} stand for \mathbb{C} if $\mathbb{K} = \mathbb{C}$, and for \mathbb{R}, \mathbb{C} , or \mathbb{H} if $\mathbb{K} = \mathbb{R}$. Then there exist a normed pairing $(X, Y, \langle \cdot, \cdot \rangle)$ over \mathbb{F} , and a \mathbb{K} -subalgebra \mathcal{B} of $L(X, Y, \langle \cdot, \cdot \rangle)$ containing $F(X, Y, \langle \cdot, \cdot \rangle)$ and contained in $\mathcal{L}(X)$, such that \mathcal{A} is bicontinuously isomorphic to \mathcal{B} endowed with the operator norm. Moreover, if the associative normed Q -algebra \mathcal{A} is in fact a complete normed algebra, then the pairing $(X, Y, \langle \cdot, \cdot \rangle)$ above is a Banach pairing.*

The next corollary follows from [41, Theorem 4.1]. Nevertheless, the proof we are going to provide, by combining Corollary 4.11 and Theorem 5.3, has its own interest.

Corollary 5.5 *Let $(\mathcal{A}, \|\cdot\|)$ be an infinite-dimensional complete normed primitive associative real or complex algebra with nonzero socle. Then ring derivations of \mathcal{A} are linear and continuous.*

Proof Let D be a ring derivation of \mathcal{A} . Then representing \mathcal{A} as in Theorem 5.3, and applying Corollary 4.11, we realize that D is linear and $\|\cdot\|$ -continuous. Now the $\|\cdot\|$ -continuity of D follows from a closed graph argument. Indeed, let A_n be a sequence in \mathcal{A} such that $\|\cdot\|$ - $\lim_n A_n = 0$ and $\|\cdot\|$ - $\lim_n D(A_n) = B \in \mathcal{A}$. Then, since the mapping $A \rightarrow A$ from $(\mathcal{A}, \|\cdot\|)$ to $(\mathcal{A}, \|\cdot\|)$ is continuous, we have $\|\cdot\|$ - $\lim_n A_n = 0$ and $\|\cdot\|$ - $\lim_n D(A_n) = B$. But, since D is $\|\cdot\|$ -continuous, this implies $B = 0$. \square

¹ This nice idea is taken from the paragraph immediately before [52, Lemma 2.4.11].

6 Additive derivations of Jordan algebras of a bilinear form

Let \mathcal{R} be a non-associative ring. The *centre* of \mathcal{R} is defined as the subset of \mathcal{R} consisting of those elements $A \in \mathcal{R}$ such that

$$[A, \mathcal{R}] = (A, \mathcal{R}, \mathcal{R}) = (\mathcal{R}, A, \mathcal{R}) = (\mathcal{R}, \mathcal{R}, A) = 0,$$

where, for $A, B, C \in \mathcal{R}$, $[A, B] := AB - BA$ and

$$(A, B, C) := (AB)C - A(BC).$$

By a *derivation* of \mathcal{R} we mean any additive mapping $D: \mathcal{R} \rightarrow \mathcal{R}$ satisfying

$$D(AB) = D(A)B + AD(B) \tag{6.1}$$

for all $A, B \in \mathcal{R}$. For such a mapping D , one easily check that the equalities $D([A, B]) = [D(A), B] + [A, D(B)]$ and

$$D((A, B, C)) = (D(A), B, C) + (A, D(B), C) + (A, B, D(C))$$

hold for all $A, B, C \in \mathcal{R}$. As a consequence, we are provided with the following more general version of Lemma 4.1.

Lemma 6.1 *The centre of a non-associative ring \mathcal{R} is invariant under any derivation of \mathcal{R} .*

From now on \mathfrak{F} will denote a field of characteristic different from 2. Let X be a vector space over \mathfrak{F} , and let $\langle \cdot, \cdot \rangle$ be a symmetric bilinear form on X . Then the direct sum $\mathfrak{F} \oplus X$ becomes a unital Jordan algebra over \mathfrak{F} under de product:

$$(\lambda + x)(\mu + y) := (\lambda\mu + \langle x, y \rangle) + (\lambda y + \mu x). \tag{6.2}$$

Such a Jordan algebra will be denoted by $J(X, \langle \cdot, \cdot \rangle)$. It is well known that, if $\dim(X) \geq 2$, and if $\langle \cdot, \cdot \rangle$ is nondegenerate, then $J(X, \langle \cdot, \cdot \rangle)$ is simple, and its centre is equal to \mathfrak{F} (see for example [10, Satz VII.3.5]).

Proposition 6.2 *Let X be a vector space over \mathfrak{F} , and let $\langle \cdot, \cdot \rangle$ be a nondegenerate symmetric bilinear form on X . We have:*

- (i) *If T is a differential operator on X with associated derivation $d: \mathfrak{F} \rightarrow \mathfrak{F}$, and if $-T$ is an adjoint of T relative to the pairing $(X, X, \langle \cdot, \cdot \rangle)$, then the mapping*

$$D_{(d,T)}: \lambda + x \rightarrow d(\lambda) + Tx$$

is an additive derivation of $J(X, \langle \cdot, \cdot \rangle)$.

- (ii) *If $\dim(X) \geq 2$, then every additive derivation of $J(X, \langle \cdot, \cdot \rangle)$ is of the form $D_{(d,T)}$ for some couple (d, T) as in assertion (i).*

Proof The proof of assertion (i) is left to the reader.

Suppose that $\dim(X) \geq 2$, and let D be an additive derivation of $J(X, \langle \cdot, \cdot \rangle)$. We claim that D diagonalizes relative to the direct sum $J(X, \langle \cdot, \cdot \rangle) = \mathfrak{F} \oplus X$. Indeed, \mathfrak{F} is invariant under D , as \mathfrak{F} is the centre of $J(X, \langle \cdot, \cdot \rangle)$ and Lemma 6.1 applies. Now note that, if $x, y \in X$, then $xy \in \mathfrak{F}$. Let x be in X . Then $x^2 \in \mathfrak{F}$, hence $D(x^2) \in \mathfrak{F}$. Write $D(x) = \lambda + y$ with $\lambda \in \mathfrak{F}$ and $y \in X$. Then we have $D(x^2) = 2D(x)x = 2(\lambda x + yx)$. Therefore, since both $D(x^2)$ and yx lie in \mathfrak{F} , and $x \in X$, we conclude that $\lambda x = 0$, and this implies that $D(x) \in X$. Thus, since x is arbitrary in X , we have shown that X is invariant under D . Now that the claim has been proved, let d denote the restriction of D to \mathfrak{F} , regarded as a mapping from \mathfrak{F} to \mathfrak{F} , and let T denote the restriction of D to X , regarded as a mapping from X to X . Then, with the help of (6.1) and (6.2), it is routine to verify that d is a ring derivation of \mathfrak{F} , that T is a differential operator on X with associated derivation d , that $-T$ is an adjoint of T relative to the pairing $(X, X, \langle \cdot, \cdot \rangle)$, and that $D = D_{(d,T)}$ in the sense of assertion (i). In this way, assertion (ii) has been proved. \square

Now the main result in this section reads as follows.

Theorem 6.3 *Let $\langle \cdot, \cdot \rangle$ be a continuous nondegenerate symmetric bilinear form on an infinite-dimensional Banach space X over \mathbb{K} . Then the additive derivations of the complete normed Jordan algebra $J(X, \langle \cdot, \cdot \rangle) = \mathbb{K} \oplus X$ are precisely the mappings of the form $D_A: \lambda + x \rightarrow Ax$ for some $A \in \mathcal{L}(X)$ such that*

$$\langle Ax, y \rangle = -\langle x, Ay \rangle \text{ for all } x, y \in X. \tag{6.3}$$

Proof Since linear operators on X are precisely those differential operators on X whose associated derivation is zero, it follows from Proposition 6.2(i) that, for each $A \in \mathcal{L}(X)$ satisfying (6.3), D_A is an additive derivation of $J(X, \langle \cdot, \cdot \rangle)$.

Let D be any additive derivation of $J(X, \langle \cdot, \cdot \rangle)$. Then, by Proposition 6.2(ii), $D = D_{(d,T)}$ for some couple (d, T) as in Proposition 6.2(i). But T is a differential operator on X having an adjoint relative to $\langle \cdot, \cdot \rangle$. Therefore, by Theorem 4.10, T is linear and continuous. Now the proof that $D = D_A$, for some $A \in \mathcal{L}(X)$ satisfying (6.3), is concluded by taking $A = T$. \square

Remark 6.4 Let $\langle \cdot, \cdot \rangle$ be a continuous nondegenerate symmetric bilinear form on an infinite-dimensional Banach space X over \mathbb{K} . Then, as a byproduct of Theorem 6.3, additive derivations of $J(X, \langle \cdot, \cdot \rangle)$ are linear and continuous. But, considering Remark 4.9(b), this is well known [9, Proposition 3.3].

Remark 6.5 Let m be a natural number, and let X be a vector space over \mathbb{K} of dimension m , and let $\langle \cdot, \cdot \rangle$ be a nondegenerate symmetric bilinear form on X . Then identifying X with \mathbb{K}^m by means of a suitable basis, we have

$$\langle (\lambda_1, \dots, \lambda_m), (\mu_1, \dots, \mu_m) \rangle = \lambda_1\mu_1 + \dots + \lambda_p\mu_p - \lambda_{p+1}\mu_{p+1} - \dots - \lambda_m\mu_m$$

for a suitable $p \in \{0, 1, \dots, m\}$ with $p = m$ in the case that $\mathbb{K} = \mathbb{C}$ (see for example [40, Section 6.3]). Note that, given any ring derivation d of \mathbb{K} , the mapping $\widehat{d}: X \rightarrow X$

defined by

$$\widehat{d}(\lambda_1, \dots, \lambda_m) := (d(\lambda_1), \dots, d(\lambda_m))$$

is a differential operator on X whose associated derivation is d , and that $-\widehat{d}$ is an adjoint of \widehat{d} relative to $\langle \cdot, \cdot \rangle$. Now let T be any differential operator on X . Then, according to [57, Theorem 1.4] (a reformulation of [62, Theorem 2.2] via Fact 1.1), there exist $C \in L(X)$ and a ring derivation d of \mathbb{K} such that $T = C + \widehat{d}$. Therefore, $-T$ is an adjoint of T relative to $\langle \cdot, \cdot \rangle$ if and only if $-C$ is an adjoint of C relative to $\langle \cdot, \cdot \rangle$. In this way we have proved that *the differential operators T on X such that $-T = T^\#$ are precisely the operators of the form $C + \widehat{d}$, where $C \in L(X)$ satisfies $-C = C^\#$, and d is a ring derivation of \mathbb{K}* . As a byproduct, there are discontinuous choices of such operators T , and hence, applying Proposition 6.2 (i), we realize that there are discontinuous ring derivations of $J(X, \langle \cdot, \cdot \rangle)$. It follows that the restriction in Theorem 6.3 that X is infinite-dimensional cannot be removed. Anyway, the italicized assertion above, together with Proposition 6.2, provides us with a precise description of all additive derivations of $J(X, \langle \cdot, \cdot \rangle)$ in the case that $m \geq 2$.

§ 6.6 By a *smooth normed algebra* we mean a norm-unital normed algebra whose closed unit ball has a unique tangent hyperplane at the unit. According to [18, Proposition 2.6.2], \mathbb{C} is the unique smooth normed complex algebra. However, considering that the closed unit ball of every nonzero pre-Hilbert space has a unique tangent hyperplane at any of its norm-one points, it turns out straightforward that, given an arbitrary real pre-Hilbert space $(X, (\cdot | \cdot))$, the Jordan algebra $J(X, -(\cdot | \cdot))$ becomes a smooth normed real algebra under the norm $\|\lambda + x\|_2 := \sqrt{|\lambda|^2 + \|x\|^2}$. But, according to [18, Definition 2.6.4 and Theorem 2.6.9], *every commutative smooth normed real algebra is of the form $(J(X, -(\cdot | \cdot)), \|\cdot\|_2)$ for some real pre-Hilbert space $(X, (\cdot | \cdot))$* . Therefore, in the complete infinite-dimensional case, it follows from Theorem 6.3 that *additive derivations of such an algebra are the mappings of the form $\lambda + x \rightarrow Ax$ for some $A \in \mathcal{L}(X)$ which is skew-adjoint relative to the C^* -algebra involution of $\mathcal{L}(X)$* .

As the next example shows, the restriction of completeness cannot be removed in the above discussion.

Example 6.7 Let I be any infinite set, let X denote the real vector space of all families $(\lambda_i)_{i \in I}$ of real numbers having only a finite number of nonzero terms, and let $(\cdot | \cdot) : X \times X \rightarrow \mathbb{R}$ be defined by $((\lambda_i) | (\mu_i)) := \sum_{i \in I} \lambda_i \mu_i$. Then $(X, (\cdot | \cdot))$ is a real pre-Hilbert space, and hence $J(X, -(\cdot | \cdot))$ is an infinite-dimensional smooth normed commutative real algebra under the norm $\|\lambda + x\|_2 := \sqrt{\lambda^2 + \|x\|^2}$. Now take a nonzero derivation d of \mathbb{R} , and define a mapping $T : X \rightarrow X$ by $T((\lambda_i)) := (d(\lambda_i))$. Then T is a differential operator on X whose associated derivation is d , and $-T$ is an adjoint of T relative to $(\cdot | \cdot)$. Therefore, according to Proposition 6.2 (i), the mapping $D : \lambda + x \rightarrow Tx$ is an additive derivation of $J(X, -(\cdot | \cdot))$. However, since T is not linear, there is no $A \in \mathcal{L}(X)$ such that $D(\lambda + x) = Ax$ for every $(\lambda, x) \in \mathbb{R} \times X$.

§ 6.8 JB -algebras are complete normed Jordan real algebras which, roughly speaking, behave like the self-adjoint parts of C^* -algebras endowed with the Jordan product $A \bullet B = \frac{1}{2}(AB + BA)$. In fact JB -algebras are defined as those complete normed

Jordan real algebras \mathcal{A} satisfying $\|A\|^2 \leq \|A^2 + B^2\|$ for all $A, B \in \mathcal{A}$. JB -algebras enjoy a deep and complete structure theory, which is nicely collected in the book of Hanche–Olsen and Størmer [36]. Additional information on JB -algebras can be found in [18, Section 3.1 and Subsection 3.4.1] and [19, Theorems 5.1.29 (i) and 5.1.38].

In the structure theory of JB -algebras, the so-called JB -spin factors have special relevance. To introduce them, consider any real Hilbert space $(X, (\cdot|\cdot))$ with $\dim(X) \geq 2$. Then, according to [36, Lemma 6.1.3], $J(X, (\cdot|\cdot))$ is a JB -algebra under the norm $\|\lambda + x\|_1 := |\lambda| + \|x\|$. Now, by definition, JB -spin factors are nothing other than the JB -algebras obtained by the procedure just described. Therefore, as in the case of infinite-dimensional complete smooth normed real algebras, discussed in § 6.6, *additive derivations of any infinite-dimensional JB -spin factor are the mappings of the form $\lambda + x \rightarrow Ax$ for some $A \in \mathcal{L}(X)$ which is skew-adjoint relative to the C^* -algebra involution of $\mathcal{L}(X)$* . This result can be also derived from Remark 6.4 and [66, Example 2.3].

§ 6.9 Let \mathcal{A} be an algebra over \mathbb{K} . We say that \mathcal{A} is *quadratic* if it is unital and, for every $a \in \mathcal{A}$, a^2 lies in the linear hull of $\{\mathbf{1}, a\}$. As a consequence of [18, Proposition 2.5.13], the commutative quadratic algebras over \mathbb{K} are precisely those of the form $J(X, \langle \cdot, \cdot \rangle)$ for some vector space X over \mathbb{K} , and some symmetric bilinear form $\langle \cdot, \cdot \rangle$ on X .

JB^* -algebras are defined as those complete normed Jordan complex algebras \mathcal{A} endowed with a conjugate-linear algebra involution $*$ satisfying $\|U_A(A^*)\| = \|A\|^3$ for every $A \in \mathcal{A}$. C^* -algebras, endowed with the Jordan product $A \bullet B = \frac{1}{2}(AB + BA)$, become examples of JB^* -algebras. Thus, in some sense, JB^* -algebras generalize C^* -algebras. Actually, in a very precise sense, JB^* -algebras become the largest possible generalization of C^* -algebras (see [18, Definitions 3.3.1 and 3.5.29, and Proposition 3.5.31] and [19, Theorem 5.9.9 and Corollary 5.9.12]). Since Wright’s pioneering paper [69], JB^* -algebras have been studied in depth. The reader is referred to [18, 19, 56] for a full overview of their theory.

From a purely algebraic point of view, quadratic Jordan complex H^* -algebras (see Definition 7.15 below) and quadratic JB^* -algebras are the same. Indeed, according to [22, Theorem 2(3)] and [18, Corollary 3.5.7], in dimension ≥ 3 they are of the form $J(X, \langle \cdot, \cdot \rangle)$, where X is a complex Hilbert space, $\langle \cdot, \cdot \rangle$ is defined by $\langle x, y \rangle := (x | y^\natural)$ for all $x, y \in X$, \natural denotes the essentially unique *conjugation* (i.e. isometric conjugate-linear involutive operator) on X [36, Lemma 7.5.6], and $(\lambda + x)^* := \bar{\lambda} + x^\natural$ for all $\lambda \in \mathbb{C}$ and $x \in X$. In the H^* -algebra case, the inner product is given by $(\lambda + x | \mu + y) := \lambda\bar{\mu} + (x | y)$, whereas, in the JB^* -algebra case, the norm of the algebra can be suitably reconstructed from \natural and the inner product of X in such a way that its topology extends those of \mathbb{C} and X .

Now let X and \natural be as above, and for $B \in \mathcal{L}(X)$ define $B^\natural \in \mathcal{L}(X)$ by $B^\natural x := (Bx^\natural)^\natural$. Set $\langle x, y \rangle := (x | y^\natural)$ for all $x, y \in X$. Then for $A \in \mathcal{L}(X)$ satisfying (6.3) and all $x, y \in X$ we have

$$(x | A^*y) = (Ax | y) = \langle Ax, y^\natural \rangle = - \langle x, Ay^\natural \rangle = - (x | A^\natural y),$$

and hence $A^* = -A^\natural$. Since $*$ and \natural commute on $\mathcal{L}(X)$, it follows from Theorem 6.3 that, when an infinite-dimensional quadratic Jordan complex H^* -algebra or an infinite-dimensional quadratic JB^* -algebra is regarded as in the above paragraph, the additive derivations of the algebra are precisely the mappings of the form $\lambda + x \rightarrow Ax$, where $A \in \mathcal{L}(X)$ is skew-symmetric relative to the involutive linear $*$ -antiautomorphism τ of $\mathcal{L}(X)$ defined by $\tau(B) := (B^*)^\natural$.

We conclude this section with the following.

Corollary 6.10 *Let Z be an infinite-dimensional reflexive real or complex Banach space. Set $X := Z \oplus Z'$ and $\langle z + f, w + g \rangle := f(w) + g(z)$ for all $z, w \in Z$ and $f, g \in Z'$. Then the additive derivations of $J(X, \langle \cdot, \cdot \rangle)$ are precisely the mappings of the form*

$$\lambda + (z + f) \rightarrow (Az + Bf) + (Cz - A'f)$$

where $A: Z \rightarrow Z, B: Z' \rightarrow Z,$ and $C: Z \rightarrow Z'$ are continuous linear mappings such that $B' = -B$ and $C' = -C$.

Proof It is clear that $\langle \cdot, \cdot \rangle$ is a continuous nondegenerate symmetric bilinear form on X . For the sake of convenience, let us write

$$J(X, \langle \cdot, \cdot \rangle) = \mathbb{K} \oplus X = \left\{ \lambda + \begin{pmatrix} z \\ f \end{pmatrix} : \lambda \in \mathbb{K}, z \in Z, f \in Z' \right\},$$

$\left\langle \begin{pmatrix} z \\ f \end{pmatrix}, \begin{pmatrix} w \\ g \end{pmatrix} \right\rangle = f(w) + g(z)$, and, for $H \in \mathcal{L}(X), H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A: Z \rightarrow Z, B: Z' \rightarrow Z, C: Z \rightarrow Z'$ and $D: Z' \rightarrow Z'$ are continuous linear mappings such that $H \begin{pmatrix} z \\ f \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} z \\ f \end{pmatrix} = \begin{pmatrix} Az + Bf \\ Cz + Df \end{pmatrix}$. By Theorem 6.3, derivations of $J(X, \langle \cdot, \cdot \rangle)$ are of the form

$$\lambda + \begin{pmatrix} z \\ f \end{pmatrix} \rightarrow D_H \left(\lambda + \begin{pmatrix} z \\ f \end{pmatrix} \right) = H \begin{pmatrix} z \\ f \end{pmatrix}$$

for some $H \in \mathcal{L}(X)$ satisfying

$$\left\langle H \begin{pmatrix} z \\ f \end{pmatrix}, \begin{pmatrix} w \\ g \end{pmatrix} \right\rangle = - \left\langle \begin{pmatrix} z \\ f \end{pmatrix}, H \begin{pmatrix} w \\ g \end{pmatrix} \right\rangle.$$

Therefore, writing H as a matrix as above, we have

$$\left\langle \begin{pmatrix} Az + Bf \\ Cz + Df \end{pmatrix}, \begin{pmatrix} w \\ g \end{pmatrix} \right\rangle = - \left\langle \begin{pmatrix} z \\ f \end{pmatrix}, \begin{pmatrix} Aw + Bg \\ Cw + Dg \end{pmatrix} \right\rangle,$$

or equivalently

$$(Cz)(w) + (Df)(w) + g(Az) + g(Bf) = -f(Aw) - f(Bg) - (Cw)(z) - (Dg)(z).$$

Taking $f = 0$ and $w = 0$, we obtain $g(Az) = -(Dg)(z)$, and hence $D = -A'$. Taking $f = g = 0$, we get $(Cz)(w) = -(Cw)(z)$, and hence, identifying Z with Z'' via the canonical injection, we see that $C' = -C$. Analogously, taking $z = w = 0$, we obtain that $B' = -B$. □

7 Additive derivations of non-associative H^* -algebras

7.1 Auxiliary results

In what follows \mathcal{R} will denote a non-associative ring, and $\text{Ad}(\mathcal{R})$ will stand for the associative ring of all additive mappings from \mathcal{R} to \mathcal{R} . The *centroid* of \mathcal{R} is defined as the subring of $\text{Ad}(\mathcal{R})$ consisting of those $\lambda \in \text{Ad}(\mathcal{R})$ such that

$$A\lambda(B) = \lambda(AB) = \lambda(A)B \tag{7.1}$$

for all $A, B \in \mathcal{R}$, and is denoted by $\Gamma(\mathcal{R})$. For $\lambda \in \Gamma(\mathcal{R})$ and $A \in \mathcal{R}$, we write λA instead of $\lambda(A)$.

Proposition 7.1 *Let D be a derivation of \mathcal{R} . Then there exists a derivation d of $\Gamma(\mathcal{R})$ such that $D(\lambda A) = \lambda D(A) + d(\lambda)A$ for every $A \in \mathcal{R}$.*

Proof Note that the equalities (7.1) can be read as that a mapping $\lambda \in \text{Ad}(\mathcal{R})$ lies in $\Gamma(\mathcal{R})$ if and only if $[L_A, \lambda] = 0$ and $[R_A, \lambda] = 0$ for every $A \in \mathcal{R}$, where L_A and R_A denote the operators of left and right multiplication by A . Note also that the equality (6.1) is equivalent to $[D, L_A] = L_{D(A)}$ for every $A \in \mathcal{R}$, or to $[D, R_A] = R_{D(A)}$ for every $A \in \mathcal{R}$. These facts will be applied in what follows without notice.

Let A and λ be in \mathcal{R} and $\Gamma(\mathcal{R})$, respectively. Then

$$\begin{aligned} 0 &= [D, [L_A, \lambda]] = [[D, L_A], \lambda] + [L_A, [D, \lambda]] \\ &= [L_{D(A)}, \lambda] + [L_A, [D, \lambda]] = [L_A, [D, \lambda]], \end{aligned}$$

and analogously $0 = [R_A, [D, \lambda]]$. Therefore, since A is arbitrary in \mathcal{R} , we realize that $[D, \lambda]$ lies in $\Gamma(\mathcal{R})$. Thus $[D, \Gamma(\mathcal{R})] \subseteq \Gamma(\mathcal{R})$ because λ is arbitrary in $\Gamma(\mathcal{R})$. This allows us to consider the mapping $d: \Gamma(\mathcal{R}) \rightarrow \Gamma(\mathcal{R})$ defined by $d(\lambda) := [D, \lambda]$. Clearly, d is a derivation of $\Gamma(\mathcal{R})$. Moreover, for $\lambda \in \Gamma(\mathcal{R})$ and $A \in \mathcal{R}$ we have

$$D(\lambda A) = \lambda D(A) + [D, \lambda](A) = \lambda D(A) + d(\lambda)A. \tag{7.2} \quad \square$$

The *annihilator* of \mathcal{R} is defined as the ideal of \mathcal{R} consisting of those elements $A \in \mathcal{R}$ such that $A\mathcal{R} = 0 = \mathcal{R}A$, and is denoted by $\text{Ann}(\mathcal{R})$. Now, arguing as in the case of algebras [18, Proposition 1.1.11 (i)], we obtain the following.

Fact 7.2 *If \mathcal{R} has zero annihilator, then $\Gamma(\mathcal{R})$ is a commutative ring.*

In what follows \mathfrak{F} will denote a field, and \mathcal{A} will stand for an algebra over \mathfrak{F} . Usually, the *algebra centroid* $\Gamma_{\mathcal{A}}$ of \mathcal{A} is defined as the set of all linear operators λ on

\mathcal{A} satisfying the equalities (7.1) for all $A, B \in \mathcal{A}$. But, regarded \mathcal{A} as a non-associative ring, we may think about $\Gamma(\mathcal{A})$ in the sense defined above. To avoid any confusion, we will say that $\Gamma(\mathcal{A})$ is the *additive centroid* of \mathcal{A} .

Proposition 7.3 *Suppose that \mathcal{A} has zero annihilator. Then the additive centroid of \mathcal{A} coincides with the algebra centroid of \mathcal{A} .*

Proof The inclusions $\mathfrak{F}I_{\mathcal{A}} \subseteq \Gamma_{\mathcal{A}} \subseteq \Gamma(\mathcal{A})$ are clear. Moreover $\Gamma_{\mathcal{A}}$ is the set of those operators in $\Gamma(\mathcal{A})$ which commute with all elements in $\mathfrak{F}I_{\mathcal{A}}$. Therefore, since \mathcal{A} has zero annihilator, it follows from Fact 7.2 that $\Gamma_{\mathcal{A}} = \Gamma(\mathcal{A})$, as desired. \square

A proof of the above proposition, which avoids Fact 7.2, can be found in the argument of [41, Remark 4.2]. Indeed, let T be in $\Gamma(\mathcal{A})$, let A, B be in \mathcal{A} , and let λ be in \mathfrak{F} . Then

$$B(T(\lambda A) - \lambda T(A)) = 0 = (T(\lambda A) - \lambda T(A))B.$$

Therefore $T(\lambda A) - \lambda T(A) = 0$ because B is arbitrary in \mathcal{A} and \mathcal{A} has zero annihilator. The arbitrariness of $A \in \mathcal{A}$ and $\lambda \in \mathfrak{F}$ yields $T \in \Gamma_{\mathcal{A}}$, as desired.

Following [18, Definition 1.1.10], we say that \mathcal{A} is *central over \mathfrak{F}* if the algebra centroid of \mathcal{A} reduces to $\mathfrak{F}I_{\mathcal{A}}$. Now, combining Propositions 7.1 and 7.3, we obtain the following

Proposition 7.4 *Suppose that \mathcal{A} is central over \mathfrak{F} and has zero annihilator. Let D be an additive derivation of \mathcal{A} . Then D is a differential operator on (the vector space of) \mathcal{A} .*

The subalgebra of $L(\mathcal{A})$ generated by all operators of left and right multiplication on \mathcal{A} by elements of \mathcal{A} is called the *multiplication ideal* of \mathcal{A} , and is denoted by $\mathcal{J}(\mathcal{A})$.

Fact 7.5 *The subring of $\text{Ad}(\mathcal{A})$ generated by all operators of left and right multiplication on \mathcal{A} by elements of \mathcal{A} coincides with $\mathcal{J}(\mathcal{A})$.*

Proof Let \mathfrak{S} denote the subset of $\text{Ad}(\mathcal{A})$ consisting of all finite sums of products of the form $M_{A_1} \cdots M_{A_n}$, where $n \in \mathbb{N}$ and, for $1 \leq i \leq n$, M_{A_i} is equal to either L_{A_i} or R_{A_i} . Then \mathfrak{S} is a subring of $\text{Ad}(\mathcal{A})$ containing all operators of left and right multiplication on \mathcal{A} by elements of \mathcal{A} , and is contained in any subring of $\text{Ad}(\mathcal{A})$ containing such operators. Therefore \mathfrak{S} is the subring of $\text{Ad}(\mathcal{A})$ generated by all operators of left and right multiplication on \mathcal{A} by elements of \mathcal{A} . Since clearly $\mathfrak{S} \subseteq \mathcal{J}(\mathcal{A}) \subseteq L(\mathcal{A})$, and \mathfrak{S} is a subalgebra of $L(\mathcal{A})$ (as $\lambda M_A = M_{\lambda A}$ for all $\lambda \in \mathfrak{F}$ and $A \in \mathcal{A}$), the equality $\mathfrak{S} = \mathcal{J}(\mathcal{A})$ follows. \square

Lemma 7.6 *Let D be an additive derivation of \mathcal{A} , and let B be in $\mathcal{J}(\mathcal{A})$. Then $[B, D]$ lies in $\mathcal{J}(\mathcal{A})$.*

Proof The set $\{C \in \text{Ad}(\mathcal{A}) : [C, D] \in \mathcal{J}(\mathcal{A})\}$ is a subring of $\text{Ad}(\mathcal{A})$ containing the generators of $\mathcal{J}(\mathcal{A})$, and hence, by Fact 7.5, contains $\mathcal{J}(\mathcal{A})$. Therefore $[B, D] \in \mathcal{J}(\mathcal{A})$. \square

Set $\mathcal{M}(\mathcal{A}) := \mathfrak{F}I_{\mathcal{A}} + \mathcal{J}(\mathcal{A})$. Then $\mathcal{M}(\mathcal{A})$ is the so-called *multiplication algebra* of \mathcal{A} , and is indeed the subalgebra of $L(\mathcal{A})$ generated by $I_{\mathcal{A}}$ and all operators of left and right multiplication on \mathcal{A} by elements of \mathcal{A} .

Proposition 7.7 *Suppose that \mathcal{A} is central over \mathfrak{F} and has zero annihilator. Let D be an additive derivation of \mathcal{A} , and let B be in $\mathcal{M}(\mathcal{A})$. Then $[B, D]$ lies in $\mathcal{M}(\mathcal{A})$.*

Proof Write $B = \lambda I_{\mathcal{A}} + C$ with $\lambda \in \mathfrak{F}$ and $C \in \mathcal{J}(\mathcal{A})$. By Lemma 7.6, $[C, D] \in \mathcal{J}(\mathcal{A})$. Moreover, by Proposition 7.4, $[\lambda I_{\mathcal{A}}, D] \in \mathfrak{F}I_{\mathcal{A}}$. Therefore

$$[B, D] = [\lambda I_{\mathcal{A}}, D] + [C, D] \in \mathfrak{F}I_{\mathcal{A}} + \mathcal{J}(\mathcal{A}) = \mathcal{M}(\mathcal{A}). \quad \square$$

Direct summands of \mathcal{A} are defined as those ideals \mathcal{B} of \mathcal{A} that there is another ideal \mathcal{C} of \mathcal{A} satisfying $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$. Now, arguing as in the case where $\mathfrak{F} = \mathbb{K} := \mathbb{R}$ or \mathbb{C} , and derivations are assumed to be linear [19, Lemma 8.1.40], and noticing that both restrictions are unnecessary in the argument, we obtain the following.

Fact 7.8 *Suppose that \mathcal{A} has zero annihilator. Then direct summands of \mathcal{A} are invariant under any additive derivation of \mathcal{A} .*

From now on, we suppose that $\mathfrak{F} = \mathbb{K} := \mathbb{R}$ or \mathbb{C} , and that \mathcal{A} actually is a normed algebra.

Fact 7.9 *The inclusion $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$ holds.*

Proof $\mathcal{L}(\mathcal{A})$ is a subalgebra of $L(\mathcal{A})$ containing the generators of $\mathcal{M}(\mathcal{A})$. Therefore the result follows. \square

Lemma 7.10 *Suppose that \mathcal{A} has zero annihilator. Let D be an additive derivation of \mathcal{A} , and let \mathcal{F} be a family of ideals of \mathcal{A} such that $\sum_{\mathcal{B} \in \mathcal{F}} \mathcal{B}$ is dense in \mathcal{A} . We have:*

- (i) *If D is linear on each $\mathcal{B} \in \mathcal{F}$, then D is linear.*
- (ii) *If actually the normed algebra \mathcal{A} is complete, and if D is linear and continuous on each $\mathcal{B} \in \mathcal{F}$, then D is linear and continuous.*

Proof Suppose that D is linear on each $\mathcal{B} \in \mathcal{F}$. Then for $\lambda \in \mathbb{K}$, $A \in \mathcal{A}$, $\mathcal{B} \in \mathcal{F}$, and $B \in \mathcal{B}$ we have

$$\begin{aligned} (D(\lambda A) - \lambda D(A))B &= D(\lambda AB) - \lambda AD(B) - \lambda D(A)B \\ &= \lambda(D(AB) - AD(B) - D(A)B) = 0 \end{aligned}$$

(as AB lies in \mathcal{B}), and analogously $B(D(\lambda A) - \lambda D(A)) = 0$. Therefore, since \mathcal{B} is arbitrary in \mathcal{F} , and B is arbitrary in \mathcal{B} , and $\sum_{\mathcal{B} \in \mathcal{F}} \mathcal{B}$ is dense in \mathcal{A} , we derive that $D(\lambda A) - \lambda D(A)$ lies in the annihilator of \mathcal{A} . Since \mathcal{A} has zero annihilator, and λ is arbitrary in \mathbb{K} , and A is arbitrary in \mathcal{A} , the proof of (i) is concluded.

Now suppose that \mathcal{A} is complete and that D is linear and continuous on each $\mathcal{B} \in \mathcal{F}$. Then, by assertion (i) just proved, D is linear. Therefore, since \mathcal{A} is complete, to prove the continuity of D it is enough to show that D has closed graph. But this can be done

by arguing as in the last paragraph of the proof of [19, Theorem 8.1.41]. Indeed, let A_n be a sequence in \mathcal{A} such that $\lim A_n = 0$ and $\lim D(A_n) = A$ for some $A \in \mathcal{A}$. Then for $\mathcal{B} \in \mathcal{F}$ and $B \in \mathcal{B}$ we have

$$0 = \lim D(A_n B) = \lim (D(A_n)B + A_n D(B)) = AB,$$

and in the same way $BA = 0$. Therefore, since \mathcal{B} is arbitrary in \mathcal{F} , and B is arbitrary in \mathcal{B} , and $\sum_{\mathcal{B} \in \mathcal{F}} \mathcal{B}$ is dense in \mathcal{A} , we derive that $A = 0$. Thus assertion (ii) has been proved. \square

Lemma 7.11 *Let X be a normed space over \mathbb{K} , let T be a differential operator on X . Suppose that there exists a nonzero linear functional f on X (continuity of f is not required) such that fT is continuous. Then T is linear.*

Proof Let d denote the ring derivation of \mathbb{K} associated to T . Take $x \in X$ such that $f(x) = 1$. Then $d(\lambda) = fT(\lambda x) - \lambda fTx$ for every $\lambda \in \mathbb{K}$, and hence d is continuous. Therefore, by Remark 4.9(b), $d = 0$. \square

Proposition 7.12 *Let \mathcal{A} be a topologically simple complete normed central algebra over \mathbb{K} , and let D be an additive derivation of \mathcal{A} . Suppose that there exists a nonzero continuous linear functional f on \mathcal{A} such that fD is continuous. Then D is linear and continuous.*

Proof By Proposition 7.4 and Lemma 7.11, D is linear. Let $\mathfrak{S}(D)$ denote the separating space of D . Then $\mathfrak{S}(D)$ is a closed subspace of \mathcal{A} [18, Lemma 1.1.57], and $\mathfrak{S}(D) \subseteq \ker(f)$, which implies that $\mathfrak{S}(D) \neq \mathcal{A}$. But it is straightforward that the separating space of any linear derivation of any normed algebra over \mathbb{K} is an ideal of the algebra. Therefore $\mathfrak{S}(D) = 0$, as \mathcal{A} is topologically simple. Then, by [18, Fact 1.1.56], D is continuous. \square

Generalized complemented normed algebras over \mathbb{K} are defined as those normed algebras over \mathbb{K} having zero annihilator and satisfying the property that all their closed ideals are direct summands (compare [19, Fact 5.1.2]). For later application, we note that, as a consequence of [19, Lemma 5.1.1], *generalized complemented normed algebras over \mathbb{K} are semiprime*. Moreover, according to [32] (see also [19, Theorem 8.2.44(ii)]), if \mathcal{A} is a generalized complemented complete normed algebra over \mathbb{K} with zero weak radical, and if $\{\mathcal{A}_i\}_{i \in I}$ stands for the family of its minimal closed ideals, then for each $A \in \mathcal{A}$ there exists a unique summable family $\{A_i\}_{i \in I}$ in \mathcal{A} such that $A_i \in \mathcal{A}_i$ for every $i \in I$, and $A = \sum_{i \in I} A_i$. (For the meaning of the weak radical of an algebra over \mathbb{K} , the reader is referred to [18, Definition 4.4.39].)

Lemma 7.13 *Let \mathcal{A} be a generalized complemented complete normed algebra over \mathbb{K} with zero weak radical, let $\{\mathcal{A}_i\}_{i \in I}$ stand for the family of its minimal closed ideals, let A be in \mathcal{A} , and let $\{A_i\}_{i \in I}$ be the unique summable family in \mathcal{A} such that $A_i \in \mathcal{A}_i$ for every $i \in I$ and $A = \sum_{i \in I} A_i$. Let $D: \mathcal{A} \rightarrow \mathcal{A}$ be an additive derivation. Then the family $\{D(A_i)\}_{i \in I}$ is summable in \mathcal{A} with sum equal to $D(A)$.*

Proof Let $\{B_i\}_{i \in I}$ be the unique summable family in \mathcal{A} such that $B_i \in \mathcal{A}_i$ for every $i \in I$ and $D(A) = \sum_{i \in I} B_i$. It is enough to show that $B_j = D(A_j)$ for every $j \in I$. Let j be in I . Then both families $\{A_i\}_{i \in I \setminus \{j\}}$ and $\{B_i\}_{i \in I \setminus \{j\}}$ are summable in \mathcal{A} , and we have $D(A) = B_j + \sum_{i \in I \setminus \{j\}} B_i$ and

$$D(A) = D(A_j) + D\left(\sum_{i \in I \setminus \{j\}} A_i\right).$$

Therefore, since B_j and $D(A_j)$ belong to \mathcal{A}_j (the latest by Fact 7.8), and $\sum_{i \in I \setminus \{j\}} B_i$ and $D(\sum_{i \in I \setminus \{j\}} A_i)$ belong to $\mathcal{B} := \overline{\sum_{i \in I \setminus \{j\}} \mathcal{A}_i}$ (the latest by Fact 7.8 again), and $\mathcal{A}_j \cap \mathcal{B} = 0$ (by semiprimeness of \mathcal{A} and [19, Fact 6.1.75]), it follows that $B_j = D(A_j)$, as desired. \square

Proposition 7.14 *Let \mathcal{A} be a generalized complemented complete normed algebra over \mathbb{K} with zero weak radical, and let $D: \mathcal{A} \rightarrow \mathcal{A}$ be an additive derivation. Let \mathcal{F} denote the family of those minimal closed ideals \mathcal{B} of \mathcal{A} such that the restriction of D to \mathcal{B} is discontinuous. Then \mathcal{F} is finite.*

Proof Let $\{\mathcal{A}_i\}_{i \in I}$ denote the family of all minimal closed ideals of \mathcal{A} . To derive a contradiction, assume that there exists an infinite sequence i_n of pair-wise different elements of I such that, for every $n \in \mathbb{N}$, the restriction of D to \mathcal{A}_{i_n} is discontinuous. Then, by Remark 4.9 (a), for each $n \in \mathbb{N}$ there is $B_n \in \mathcal{A}_{i_n}$ such that $\|B_n\| \leq \frac{1}{2^n}$ and $\|D(B_n)\| \geq 1$. Now let $\{A_i\}_{i \in I}$ be the family of elements of \mathcal{A} defined by $A_i := 0$ if $i \neq i_n$ for every $n \in \mathbb{N}$, and $A_{i_n} := B_n$ otherwise. Then, by [21, Proposition VII.9.18], the family $\{A_i\}_{i \in I}$ is summable in \mathcal{A} . Set $A := \sum_{i \in I} A_i$, and note that $A_i \in \mathcal{A}_i$ for every $i \in I$. It follows from Lemma 7.13 that the family $\{D(A_i)\}_{i \in I}$ is summable in \mathcal{A} , and hence, by [21, Corollary VII.9.6 and Proposition VII.9.8], the set $\{i \in I : \|D(A_i)\| \geq 1\}$ is finite. But this is not possible because $\|D(A_{i_n})\| \geq 1$ for every $n \in \mathbb{N}$. \square

7.2 The main result

Definition 7.15 We recall that *semi- H^* -algebras* are defined as those real or complex algebras X which are also Hilbert spaces, and are endowed with a conjugate-linear vector space involution $*$ satisfying

$$(xy \mid z) = (y \mid x^*z) = (x \mid zy^*) \quad \text{for all } x, y, z \in X. \quad (7.2)$$

H^ -algebras* are defined as those semi- H^* -algebras whose involution $*$ is an algebra involution. We remark that, up to the multiplication of the inner product by a suitable positive number, semi- H^* -algebras become complete normed algebras [18, Lemma 2.8.12 (i)].

Since the pioneering papers of Ambrose [2] and Kaplansky [43] determining associative complex and real H^* -algebras, respectively, the study of H^* -algebras into the most familiar classes of non-associative algebras (such a Jordan [7, 26, 27, 29, 30,

53, 54], Lie [3, 4, 6, 12, 24, 28, 49, 60, 61, 65], and Malcev algebras [13, 16, 70]), has attained a complete development, getting in particular the determination of those H^* -algebras in such classes. Nevertheless, concerning our current goal, we should pay attention to the approach to (semi-) H^* -algebras from a general non-associative point of view, such as is done in [6, 14–17, 25, 26, 54, 55, 67]. Anyway, for a full discussion of the theory of (semi-) H^* -algebras, the reader is referred to [18, Subsection 2.8.2], [19, Section 8.1], and [23].

From now on, X will denote a semi- H^* -algebra over \mathbb{K} . For $A \in \mathcal{L}(X)$ we denote by A^\bullet the unique operator in $\mathcal{L}(X)$ such that $(Ax \mid y) = (x \mid A^\bullet y)$ for all $x, y \in X$. We adopt this notation because \bullet is a purely Hilbert-space notion, and hence, a priori, it has nothing to do with the involution $*$ of X , and moreover the classical symbol A^* for the adjoint of A relative to $(\cdot \mid \cdot)$ could be confused with the (possibly discontinuous) linear operator on X defined by $A^*x := (Ax^*)^*$. Nevertheless, given $x \in X$, the equalities (7.2) read as that $(L_x)^\bullet = L_{x^*}$ and $(R_x)^\bullet = R_{x^*}$. Therefore, since $\mathcal{M}(X) \subseteq \mathcal{L}(X)$ (by Fact 7.9), and \bullet is a conjugate-linear algebra involution on $\mathcal{L}(X)$ taking the set of generators of $\mathcal{M}(X)$ into such a set, we obtain the following well-known result.

Fact 7.16 $\mathcal{M}(X)$ is a \bullet -invariant subalgebra of $\mathcal{L}(X)$.

Since topologically simple complex semi- H^* -algebras are central over \mathbb{C} [19, Lemma 8.1.29] and have zero annihilator, we can combine Proposition 7.4, Fact 1.1, and Propositions 7.7 and 7.12 (in this order) to obtain the following.

Proposition 7.17 *Suppose that $\mathbb{K} = \mathbb{C}$ and that X is topologically simple. Let T be an additive derivation of X . We have:*

- (i) T is a differential operator on X .
- (ii) The mapping $A \rightarrow [T, A]$ is a ring derivation of $L(X)$ leaving $\mathcal{M}(X)$ invariant.
- (iii) If there exists some nonzero element $y \in X$ such that the mapping $x \rightarrow (Tx \mid y)$ is continuous, then T is linear and continuous.

As for any Hilbert space, given $x, y \in X$, we denote by $x \odot y$ the continuous linear operator on X defined by $(x \odot y)z := (z \mid y)x$ for every $z \in X$.

Lemma 7.18 *Suppose that $\mathbb{K} = \mathbb{C}$, that X is topologically simple and infinite-dimensional, and that there exists an operator in $\mathcal{M}(X)$ whose range is one-dimensional. Then additive derivations of X are linear and continuous.*

Proof By Fact 7.9, the operator in $\mathcal{M}(X)$ whose existence has been assumed should be of the form $x_0 \odot y_0$ for some nonzero $x_0, y_0 \in X$. Then, since $(x_0 \odot y_0)(x_0 \odot y_0)^\bullet = (x_0 \odot y_0)(y_0 \odot x_0) = \|y_0\|^2 x_0 \odot x_0$, it follows from Fact 7.16 that $x_0 \odot x_0$ lies in $\mathcal{M}(X)$. Moreover, we may suppose that $\|x_0\| = 1$. Set $Y := \mathcal{M}(X)x_0$. Then Y is a nonzero ideal of X , so it is dense in X because X is topologically simple. We note that,

$$\text{for every } y \in Y, \text{ the operator } x_0 \odot y \text{ lies in } \mathcal{M}(X). \tag{7.3}$$

Indeed, taking $A \in \mathcal{M}(X)$ such that $y = Ax_0$, it follows from Fact 7.16 that

$$x_0 \odot y = x_0 \odot Ax_0 = (x_0 \odot x_0)A^\bullet \in \mathcal{M}(X).$$

Now let T be any additive derivation of X . Then, by Proposition 7.17 (ii), the mapping $D: A \rightarrow [T, A]$ is a ring derivation of $L(X)$ leaving $\mathcal{M}(X)$ invariant. On the other hand, by Proposition 7.17 (i), T is a differential operator on X . Let d denote the ring derivation of \mathbb{C} associated to T , and set $S := T - (Tx_0 | x_0)I_X$. Then S is a differential operator on X whose associated derivation is d , and satisfies that $(Sx_0 | x_0) = 0$ and that $D(A) = [S, A]$ for every $A \in L(X)$. Let (x, y) be in $X \times Y$. It follows that

$$\begin{aligned} D(x_0 \otimes y)x &= [S, x_0 \otimes y]x = S((x | y)x_0) - (Sx | y)x_0 \\ &= (x | y)Sx_0 + (d((x | y)) - (Sx | y))x_0. \end{aligned}$$

Therefore, taking inner products with x_0 , we obtain

$$(D(x_0 \otimes y)x | x_0) = d((x | y)) - (Sx | y). \quad (7.4)$$

Now note that, considering (7.3) and that $\mathcal{M}(X)$ is D -invariant, we have that $D(x_0 \otimes y) \in \mathcal{M}(X)$, and that then, by Fact 7.16, also $(D(x_0 \otimes y))^\bullet \in \mathcal{M}(X)$. Therefore, $(D(x_0 \otimes y))^\bullet x_0 \in Y$, and (7.4) reads as

$$(Sx | y) = d((x | y)) + (x | S^\bullet y), \quad (7.5)$$

where $S^\bullet: Y \rightarrow Y$ is the mapping defined by $S^\bullet y := -(D(x_0 \otimes y))^\bullet x_0$.

Consider a set copy \widehat{Y} of Y with sum, product by scalars, and norm defined by $\widehat{y}_1 + \widehat{y}_2 := \widehat{y_1 + y_2}$, $\lambda \widehat{y} := \widehat{\lambda y}$, and $\|\widehat{y}\| := \|y\|$, respectively, and for $(x, \widehat{y}) \in X \times \widehat{Y}$, set $\langle x, \widehat{y} \rangle = (x | y)$. Then $(X, \widehat{Y}, \langle \cdot, \cdot \rangle)$ is a normed pairing over \mathbb{C} , complete on the left. (In verifying this, the unique point which could be non clear is the one that $\langle x, \widehat{Y} \rangle = 0$ implies $x = 0$; but this follows from the fact that Y is dense in X .) Moreover (7.5) reads as

$$\langle Sx, \widehat{y} \rangle = d(\langle x, \widehat{y} \rangle) + \langle x, S^\# \widehat{y} \rangle,$$

where $S^\# \widehat{y} := \widehat{S^\bullet y}$. Thus the differential operator S on X has an adjoint relative to $\langle \cdot, \cdot \rangle$, and hence, by Theorem 4.10, is linear and continuous. Finally, since $T = S + (Tx_0 | x_0)I_X$, T is linear and continuous. \square

The proof of the next proposition follows Villena's argument in the proof of the main result in [67] (see also [19, Theorem 8.1.41]), with the appropriate changes.

Proposition 7.19 *Suppose that $\mathbb{K} = \mathbb{C}$ and that X is topologically simple and infinite-dimensional. Let T be an additive derivation of X . Then T is linear and continuous.*

Proof Suppose that there exists some nonzero element $y \in X$ such that the mapping $x \rightarrow (Tx | y)$ is continuous. Then, by Proposition 7.17 (iii), T is linear and continuous. Therefore, to conclude the proof of the proposition it is enough to discuss the case that, for every nonzero $y \in X$, the mapping $x \rightarrow (Tx | y)$ is discontinuous. Actually we are going to show that this case cannot happen.

To derive a contradiction, assume that, for every nonzero $y \in X$, the mapping $x \rightarrow (Tx | y)$ is discontinuous. Then, by Remark 4.9 (a), for every nonzero $y \in X$, the mapping $x \rightarrow (Tx | y)$ is unbounded on any neighborhood of zero. Moreover, clearly T is discontinuous, and hence, by Lemma 7.18, the inequality $\dim A(X) > 1$ holds for every nonzero $A \in \mathcal{M}(X)$. It follows from [19, Corollary 8.1.34 and Proposition 8.1.38] that there exist sequences y_n in X and A_n in $\mathcal{M}(X)$ such that $A_n \cdots A_1 y_n \neq 0$ and $A_{n+1} \cdots A_1 y_n = 0$ for every $n \in \mathbb{N}$, and clearly we may suppose that $\|y_n\| = \|A_n\| = 1$ for every $n \in \mathbb{N}$.

Now, according to Proposition 7.17 (ii), let D denote the ring derivation of $\mathcal{M}(X)$ defined by $D(A) := [T, A]$ for every $A \in \mathcal{M}(X)$. Then, considering Fact 7.16 and the above paragraph, we can construct inductively a sequence x_n in X with the property that for every $n \in \mathbb{N}$ we have $\|x_n\| \leq 2^{-n}$ and

$$|(Tx_n | A_n \cdots A_1 y_n)| \geq n + \left| \sum_{j=1}^{n-1} (TA_1^\bullet \cdots A_j^\bullet x_j | y_n) \right| + \|D(A_1^\bullet \cdots A_n^\bullet)\| + \|D(A_1^\bullet \cdots A_{n+1}^\bullet)\|.$$

Now we consider the element $x \in X$ defined by $x := \sum_{j=1}^\infty A_1^\bullet \cdots A_j^\bullet x_j$, and for $n \in \mathbb{N}$ we write $z_n := x_{n+1} + \sum_{j=n+2}^\infty A_{n+2}^\bullet \cdots A_j^\bullet x_j$. Then we have

$$\begin{aligned} (Tx | y_n) &= \sum_{j=1}^{n-1} (TA_1^\bullet \cdots A_j^\bullet x_j | y_n) + (TA_1^\bullet \cdots A_n^\bullet x_n | y_n) \\ &\quad + \left(T \left(\sum_{j=n+1}^\infty A_1^\bullet \cdots A_j^\bullet x_j \right) | y_n \right) \\ &= \sum_{j=1}^{n-1} (TA_1^\bullet \cdots A_j^\bullet x_j | y_n) + (D(A_1^\bullet \cdots A_n^\bullet)x_n + A_1^\bullet \cdots A_n^\bullet Tx_n | y_n) \\ &\quad + (TA_1^\bullet \cdots A_{n+1}^\bullet z_n | y_n) \\ &= \sum_{j=1}^{n-1} (TA_1^\bullet \cdots A_j^\bullet x_j | y_n) + (D(A_1^\bullet \cdots A_n^\bullet)x_n | y_n) + (Tx_n | A_n \cdots A_1 y_n) \\ &\quad + (D(A_1^\bullet \cdots A_{n+1}^\bullet)z_n | y_n) + (Tz_n | A_{n+1} \cdots A_1 y_n) \\ &= (Tx_n | A_n \cdots A_1 y_n) + \sum_{j=1}^{n-1} (TA_1^\bullet \cdots A_j^\bullet x_j | y_n) \\ &\quad + (D(A_1^\bullet \cdots A_n^\bullet)x_n | y_n) + (D(A_1^\bullet \cdots A_{n+1}^\bullet)z_n | y_n), \end{aligned}$$

where for the last equality we have used that $A_{n+1} \cdots A_1 y_n = 0$. Therefore, since $\|z_n\| \leq 1$, we obtain

$$\begin{aligned} \|Tx\| &\geq |(Tx | y_n)| \geq |(Tx_n | A_n \cdots A_1 y_n)| - \left| \sum_{j=1}^{n-1} (TA_1^\bullet \cdots A_j^\bullet x_j | y_n) \right| \\ &\quad - |(D(A_1^\bullet \cdots A_n^\bullet)x_n | y_n)| \\ &\geq |(Tx_n | A_n \cdots A_1 y_n)| - \left| \sum_{j=1}^{n-1} (TA_1^\bullet \cdots A_j^\bullet x_j | y_n) \right| \\ &\quad - \|D(A_1^\bullet \cdots A_n^\bullet)\| - \|D(A_1^\bullet \cdots A_{n+1}^\bullet)\| \geq n. \end{aligned}$$

Now the impossible fact that $\|Tx\| \geq n$ for every $n \in \mathbb{N}$ is the desired contradiction. \square

Corollary 7.20 *Suppose that X is topologically simple and infinite-dimensional. Let T be an additive derivation of X . Then T is linear and continuous.*

Proof If $\mathbb{K} = \mathbb{C}$, then the result follows from Proposition 7.19.

Suppose that $\mathbb{K} = \mathbb{R}$. Then, by [19, Theorem 8.1.88], either X is a topologically simple complex semi- H^* -algebra, regarded as a real semi- H^* -algebra, or there exists a couple (Z, \natural) , where Z is a topologically simple complex semi- H^* -algebra and \natural is an isometric involutive conjugate-linear algebra $*$ -automorphism of Z , such that $X = \{z \in Z : z^\natural = z\}$. If the first possibility happens, then the result follows by applying Proposition 7.19 again. Suppose that $X = \{z \in Z : z^\natural = z\}$ for (Z, \natural) as above. Then, since $Z = X \oplus iX$, we may define an additive derivation $\bar{T} : Z \rightarrow Z$ by setting $\bar{T}(x_1 + ix_2) := Tx_1 + iTx_2$ ($x_1, x_2 \in X$). By applying Proposition 7.19 once more, we obtain that \bar{T} is \mathbb{C} -linear and continuous. Therefore, since T is the restriction of \bar{T} to X , T is \mathbb{R} -linear and continuous. \square

We recall that *closed ideals of any semi- H^* -algebra over \mathbb{K} are direct summands* [19, Proposition 8.1.13 (i)]. Considering this result, together with [19, Lemma 8.2.18], we obtain the following.

Fact 7.21 *Suppose that X has zero annihilator. Then X is a generalized complemented complete normed algebra over \mathbb{K} with zero weak radical.*

Now the main result in this section reads as follows.

Theorem 7.22 *Suppose that X has zero annihilator. Let T be an additive derivation of X . Then there exist T -invariant closed $*$ -ideals Y and Z of X such that $X = Y \oplus Z$, T is linear and continuous on Y , Z is finite-dimensional, and T is discontinuous on Z .*

Proof Let M be an infinite-dimensional minimal closed ideal of X . Then, by [19, Proposition 8.1.13 (iv)–(v)], M is an infinite-dimensional topologically simple semi- H^* -algebra over \mathbb{K} in a natural way. On the other hand, by Fact 7.8, M is invariant under T , and hence the restriction of T to M can be seen as an additive derivation of M . It follows from Corollary 7.20 that T is continuous on M .

Let $\{M_i\}_{i \in I}$ denote the family of all minimal closed ideals of X , and let J stand for the set of those $i \in I$ such that T is discontinuous on M_i . It follows from

Fact 7.21, Proposition 7.14, and the above paragraph that J is finite, and that M_i is finite-dimensional whenever $i \in J$. Moreover, by Fact 7.8 and Remark 4.9(b), T is linear on M_i whenever i belongs to $I \setminus J$. Now set $Y := \overline{\sum_{i \in I \setminus J} M_i}$ and $Z := \sum_{i \in J} M_i$. Then Y and Z are T -invariant closed $*$ -ideals of X (by Fact 7.8 and [19, Proposition 8.1.13(v)]), T is linear and continuous on Y (by Lemma 7.10(ii)), Z is finite-dimensional, T is discontinuous on Z , and $X = Y \oplus Z$ (by [19, Theorem 8.1.16 and Fact 6.1.75]). \square

Fact 7.23 *Suppose that X has no nonzero finite-dimensional direct summand. Then X has zero annihilator.*

Proof Let Y be any finite-dimensional subspace of $\text{Ann}(X)$. Then Y is a closed ideal of X , hence a finite-dimensional direct summand. Therefore $Y = 0$, as X has no nonzero such a direct summand. Thus $\text{Ann}(X)$ has no nonzero finite-dimensional subspace, and hence $\text{Ann}(X) = 0$. \square

Combining Theorem 7.22 and Fact 7.23, we obtain the following.

Corollary 7.24 *Suppose that X has no nonzero finite-dimensional direct summand. Then additive derivations of X are linear and continuous.*

In relation to the following remark, we note that alternative (so, in particular, associative) or Jordan semi- H^* -algebras with zero annihilator are H^* -algebras [19, Proposition 8.1.23(ii) and Corollary 8.1.80], and that, for associative algebras, semisimplicity and J -semisimplicity mean the same [18, Definition 4.4.12].

Remark 7.25 (a) As far as we know, even the particularization of Theorem 7.22 to the case that X is associative (respectively, Jordan) has not been noticed in the literature. Nevertheless, such a particularization can be easily obtained from previously known results. Indeed, associative (respectively, Jordan) complex H^* -algebras with zero annihilator are J -semisimple [19, Corollary 8.1.146] (respectively, [19, Proposition 8.1.145]). Therefore, by [41, Theorem 4.1] (respectively, [9, Theorem 3.5]), the particularization of Theorem 7.22 to the case that X is complex and associative (respectively, Jordan) follows. Finally, reducing the real case to the complex one by means of [19, Proposition 8.1.77], we obtain the associative (respectively, Jordan) version of Theorem 7.22.

(b) The particularization of Corollary 7.24 to the case that X is alternative follows from the associative version of that corollary because *alternative H^* -algebras over \mathbb{K} with no nonzero finite-dimensional direct summand are associative*. Indeed, let X be an alternative H^* -algebra over \mathbb{K} with no nonzero finite-dimensional direct summand. Then, by Fact 7.23, X has zero annihilator, and hence, as in the proof of Theorem 7.22, each minimal closed ideal of X is an infinite-dimensional topologically simple alternative H^* -algebra over \mathbb{K} in a natural way, and the sum of all minimal closed ideals of X is dense in X . Therefore, since every infinite-dimensional topologically simple alternative H^* -algebra over \mathbb{K} is associative [11, Theorems 8.2 and 8.4] (see also [19, pp. 551–552]), we conclude that X is associative, as desired.

(c) Let m be a natural number. Then $M_m(\mathbb{K})$ becomes an H^* -algebra over \mathbb{K} under the inner product $((a_{i,j} \mid b_{i,j})) := \sum_{i,j} a_{i,j} \overline{b_{i,j}}$ and the involution $(a_{i,j})^* := (\overline{a_{j,i}})$.

It follows from Remark 4.14 that, even in the associative case, the possibility that the direct summand Z in Theorem 7.22 be nonzero can happen.

Theorem 7.22 would follow straightforwardly from Fact 7.21 if the following conjecture were proved.

Conjecture 7.26 *Let X be a generalized complemented complete normed algebra over \mathbb{K} with zero weak radical, and let T be an additive derivation of X . Then there exist T -invariant closed ideals Y and Z of X such that $X = Y \oplus Z$, T is linear and continuous on Y , Z is finite-dimensional, and T is discontinuous on Z .*

Conjecture 7.26 above holds if X is associative and commutative. Indeed, in this case, by [18, Proposition 4.4.65 (ii)], the requirement that X has zero weak radical is equivalent to the semisimplicity of X , and the Johnson–Sinclair theorem [41] applies. Nevertheless we do not know if Conjecture 7.26 holds in the case that X is associative but not commutative (see [18, Remark 4.4.68 (a)]). Anyway, looking at the proof of Theorem 7.22, and considering [19, Theorem 8.2.44], one can realize that (the general case of) Conjecture 7.26 is equivalent to the simpler one which follows.

Conjecture 7.27 *Let X be an infinite-dimensional topologically simple complete normed algebra over \mathbb{K} with zero weak radical. Then additive derivations of X are continuous.*

Acknowledgements The authors would like to thank the referee for careful reading of the paper and useful suggestions.

Funding Funding for open access publishing: Universidad de Granada/CBUA

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