RESEARCH ARTICLE

Morse–Radó theory for minimal surfaces

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Abstract

For a class of functions (called minimal Radó functions) that arise naturally in minimal surface theory, we bound the number of interior critical points (counting multiplicity) in terms of the boundary data and the Euler characteristic of the domain of the function.

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Contents

1 | INTRODUCTION

Consider the following facts from the classical theory of minimal surfaces.

(1) If M is a compact minimal disk in $\mathbf{R}^n$, if $F : \mathbf{R}^n \to \mathbf{R}$ is linear, and if $F^{-1}(c) \cap M$ contains an interior critical point of $F|M$ with multiplicity k, then $F^{-1}(c) \cap \partial M$ contains at least $2(k+1)$ points (see [12, p. 794,(c)]).
(2) If M is a minimal disk in $\mathbf{R}^n$, if $F : \mathbf{R}^n \to \mathbf{R}$ is linear, and if $F : \partial M \to \mathbf{R}$ has at most k local minima, then $F|M$ has at most $k-1$ interior critical points, counting multiplicity (see [14, Lemma 2]).

These facts are powerful tools in minimal surface theory. For instance, Radó [13] used (1) to prove that if the boundary of minimal disk in $\mathbf{R}^n$ projects homeomorphically to the boundary of a convex region in a plane, then the interior of the disk is a smooth graph over that region. (Actually, Radó stated the theorem only for $n = 3$, but Osserman [11, Theorem 7.2] pointed out that Radó's proof works for any n.) Radó also showed that if the boundary of a minimal disk in $\mathbf{R}^n$ projects homeomorphically onto the boundary of a planar star-shaped region, then the interior of the disk has no branch points [12, p. 794]. Finn and Osserman [2] used an analog of (1) to prove a curvature estimate that implies Bernstein's theorem (an entire solution $u : \mathbf{R}^2 \to \mathbf{R}$ of the minimal surface equation must be a plane.) Schneider [14] used (2) to show that for a minimal disk in Euclidean space, the sum of the orders of the interior branch points is bounded by

$$\frac{\kappa}{2\pi} - 1,$$

where κ is the total curvature of the boundary. More recently, (2) was used in the variational existence proof of genus-one helicoids in $\mathbf{R}^3$ [8].

In this paper, we sharpen (1) and (2) and extend them to minimal surfaces of arbitrary genus in Riemannian manifolds. See Theorem 41 for the generalization of (1). Generalizing (2), we show:

Theorem 1. *Suppose that M is a compact minimal surface with boundary in a Riemannian manifold N. Suppose that $F : N \to \mathbf{R}$ is a continuous function such that*

(1) *if $\dim N = 3$, the level sets of F are minimal surfaces, and*
(2) *if $\dim N > 3$, the level sets of F are totally geodesic.*
(3) *for each t, $\{F = t\}$ is in the closure of $\{F > t\}$ and of $\{F < t\}$.*

Suppose also that F is nonconstant on each connected component of M, and that the set Q of local minima of $F|\partial M$ is finite. Then the number $\mathsf{N}(F|M)$ of interior critical points of $F|M$ (counting multiplicity) and the number $s^\partial(F)$ of boundary saddle points of $F|M$ (counting multiplicity) satisfy

$$\mathsf{N}(F|M) + s^\partial(F) = |Q| - \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M and where $|Q|$ is the number of elements in the set Q.

(Theorem 1 is a special case of Theorem 24; Theorem 8 and Remarks 9 and 10 show that the hypotheses of Theorem 1 imply the hypotheses of Theorem 24. Theorem 1 is also true for branched minimal surfaces; see Section 9.)

A continuous function whose level sets form a foliation and that satisfies hypothesis (3) of Theorem 1 is called a foliation function. If the leaves are minimal, it is called a minimal foliation function, and if the leaves are totally geodesic, it is called a totally geodesic foliation function.

In Theorem 1, "interior critical point of $F|M$" means "interior point p of tangency of M and the level set $\{F = F(p)\}$," and the multiplicity of such a critical point is the order of contact of M and $\{F = F(p)\}$. Boundary saddle points and their multiplicities are defined in Definition 23.

It would be natural in Theorem 1 to assume that F is C^1 (or even smooth) with nowhere vanishing gradient. However, that assumption would be undesirable for the following reason. Consider a minimal foliation $\mathcal{F}$ of a Riemannian 3-manifold. Of course the leaves are smooth. At least locally, the foliation can be given as the level sets of a continuous function F. However, for some minimal foliations, there is no such function that is C^1 with nowhere vanishing gradient. (A simple example from [15, §1] is the minimal foliation of $\{(x, y, z) : x > 0\}$ consisting of the halfplanes $z = sx$ with $s \geqslant 0$ and the halfplanes $z = s$ with $s < 0$. If F is a C^1 function whose level sets are the leaves, then $DF(x, y, 0) = 0$.)

For that reason, throughout the paper we work with functions that are only assumed to be continuous.

Theorem 1 provides an exact formula for $\mathsf{N}(F|M)$. In many situations, a good upper bound for $\mathsf{N}(F|M)$ suffices. Simply dropping the term $s^\partial(F)$ in Theorem 1 gives the bound

$$\mathsf{N}(F|M) \leqslant |Q| - \chi(M),$$

which is often adequate. Indeed, that gives Schneider's bound (2). But one can get a better upper bound as follows. Let A be the set of local maxima and local minima of $F|\partial M$ that are not local maxima or local minima of $F|M$. Then $s^\partial(F) \geqslant |A|$ (where $|A|$ is the number of elements of A), so from Theorem 1, we deduce

Corollary 2. *Under the hypotheses of Theorem 1,*

$$\mathsf{N}(F|M) \leqslant |Q| - \chi(M) - |A|.$$

See Theorem 26, which also specifies when equality holds in Corollary 2.

Remark 3. In practice, one sometimes encounters F and M that satisfy all but one of the hypotheses of Theorem 1, namely the hypothesis that the set of local minima of $F|\partial M$ is finite. In particular, that hypothesis will fail if F is constant on one or more arcs of ∂M. One can handle such examples as follows. Suppose F is not constant on any connected component of ∂M. Let $\tilde{M}$ be obtained from M by identifying each arc of ∂M on which F is constant to a point. Let $\tilde{F}$ be the function on $\tilde{M}$ corresponding to F on M. If $\tilde{F}|\partial \tilde{M}$ has a finite set $\tilde{Q}$ of local minima, then

$$\mathsf{N}(F|M) = \mathsf{N}(\tilde{F}|\tilde{M})$$

$$= |\tilde{Q}| - \chi(M) - s^\partial(\tilde{F})$$

$$\leqslant |\tilde{Q}| - \chi(M) - |\tilde{A}|,$$

where $\tilde{A}$ is the set of local minima and local maxima of $\tilde{F}|\partial\tilde{M}$ that are not local minima or local maxima of $\tilde{F}|M$. These facts follow from Theorems 24, 26, and 48.

Special cases of Theorem 1 have been important tools for analyzing properly embedded translators for mean curvature flow in $\mathbf{R}^3$, in particular for the classification of translating graphs in [4, 5], the classification of semigraphical translators (such as the doubly periodic Scherk-type translators and the Nguyen singly periodic translators) [6, 7], the classification of low entropy translators [3], and for the construction of families of nonrotationally invariant translating annuli (analogs of catenoids). (See a forthcoming paper by Hoffman, Martín, and White.)

There is also a version of Theorem 1 for noncompact M:

Theorem 4. *Let $-\infty \leqslant a < b \leqslant \infty$. In Theorem 1, suppose the hypothesis that M is compact is replaced by the hypotheses that $F : M \to (a, b)$ is proper, that $d_1(M) := \dim H_1(M; \mathbf{Z}_2)$ is finite, and that the limit*

$$\beta := \lim_{t \to a, t > a} |(\partial M) \cap F^{-1}(t)|$$

exists and is finite. Then

$$\mathsf{N}(F|M) + s^\partial(F) = \frac{1}{2}\beta + |Q| - \chi(M),$$

and therefore

$$\mathsf{N}(F|M) \leqslant \frac{1}{2}\beta + |Q| - \chi(M) - |A|.$$

Theorem 4 is a special case of Corollary 31, by virtue of Theorem 8, Remarks 9 and 10), and (for the inequality involving $|A|$) Proposition 25.

Another useful fact about $\mathsf{N}(F|M)$ is that it depends lower semicontinuously on F and on M (even without assuming properness); see Theorem 40.

The paper is organized as follows. We define a class of functions on surfaces that we call Radó functions. Roughly speaking, they are continuous functions whose level sets are locally either isolated points or (qualitatively) like the level sets of harmonic functions. (The isolated points occur at strict local minima and at strict local maxima.) We show that if M is a minimal surface in a smooth Riemannian manifold N and if $F : N \to \mathbf{R}$ is a continuous function satisfying hypotheses (1), (2), and (3) of Theorem 1, then F is a Radó function on the interior of M. Under mild hypotheses, it follows that F is a Radó function on all of M; see Theorem 46. We then prove the various theorems bounding numbers of critical points for arbitrary Radó functions.

2 | RADÓ FUNCTIONS

Definition 5. A continuous, real-valued function on a 2-manifold M is called a **Radó function** provided each point $p \in M$ has a neighborhood U such that

(1) $U \cap \{F = F(p)\}$ consists of a finite collection $C_1, \dots, C_v$ of embedded arcs;
(2) each C_i joins the point p to a point in ∂U;
(3) $C_i \cap C_j = \{p\}$ for $i \neq j$;
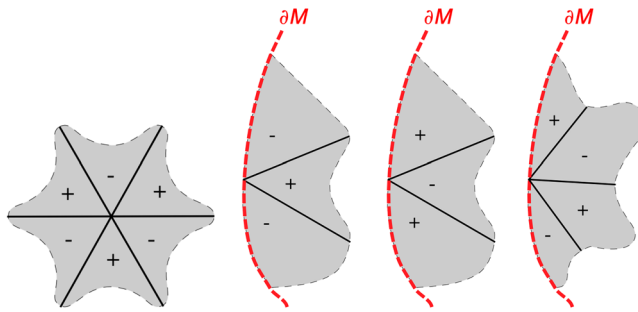(4) each C_i is in the closure of $\{F > F(p)\}$ and in the closure of $\{F < F(p)\}$;

FIGURE 1   From left to right: an interior critical point, a local maximum of $F|\partial M$, a local minimum of $F|\partial M$, and a point which is neither a local maximum nor a local minimum of $F|\partial M$. "+" indicates that $F > F(p)$ in that region and "−" indicates that $F < F(p)$ in that region.

(5) if $p \in \partial M$, we also require that each $C_i \setminus \{p\}$ is contained in the interior of M.

The number $v = v(F, p)$ is called the **valence** of p.

We call these functions Radó functions because Radó observed [13, III.6] that some important properties of harmonic functions on surfaces are shared by functions similar to those in Definition 5 (provided there are no points of valence 0.)

Note that for a Radó function $F : M \to \mathbf{R}$,

(1) the points of valence 0 are the local minima and local maxima of F;
(2) each local maximum (local minimum) of a Radó function is a strict local maximum (local minimum);
(3) for each t, the set $(\partial M) \cap F^{-1}(t)$ is discrete. Indeed, if p and U are as in Definition 5 and if $p \in \partial M$, then $(\partial M) \cap U \cap \{F = F(p)\}$ consists only of the point p.

The following lemma is an immediate consequence of Definition 5; see Figure 1.

Lemma 6. *Suppose that $F : M \to \mathbf{R}$ is a Radó function. If p is an interior point, then $v(F, p)$ is even. If p is a boundary point, then $v(F, p)$ is even if and only if $F|\partial M$ has a local maximum or a local minimum at p.*

Definition 7. Suppose that $F : M \to \mathbf{R}$ is a Radó function. A **Radó critical point** (or **critical point**, for short) of F is an interior point p such that $v(F, p) \neq 2$ or a boundary point p such that $v(F, p) \neq 1$. If p is an interior point of valence $v(F, p) \geqslant 4$, we say that that p is a **saddle of multiplicity** $w(F, p)$, where

$$w(F, p) := \frac{1}{2}v(F, p) - 1.$$

Interior points of valence 2 and boundary points of valence 1 are called **Radó noncritical points** or **Radó regular points**.

Two warnings about Definition 7 are in order. First, in case F is smooth, the notion of Radó critical point is not equivalent to the usual definition of critical point (i.e., a point where DF vanishes). For example, for the Radó function $F(x, y) = y^3$, every point is Radó noncritical, but the points

with $y = 0$ are critical in the usual sense. Second, for a general Radó function, the set of Radó critical points need not be closed. Fortunately, under mild hypotheses, the set of Radó critical points will be locally finite and therefore closed. See Theorem 46. (See also Theorem 33.)

For the rest of the paper, "critical point," "noncritical point," and "regular point" will always mean "Radó critical point," "Radó noncritical point," and "Radó regular point".

The following theorem shows how Radó functions arise naturally in minimal surface theory.

Theorem 8. *Suppose M is an embedded minimal surface in a smooth Riemannian 3-manifold N. Suppose $F : N \to \mathbf{R}$ is a continuous function such that*

(1) *the level sets of F are smooth minimal surfaces;*
(2) *each level set $M[t] := F^{-1}(t)$ is in the closure of $\{F > t\}$ and of $\{F < t\}$.*

Suppose also that F is not constant on any connected component of M. Then the restriction of F to the interior of M is a Radó function without any interior local maxima or interior local minima. The interior saddles of multiplicity n are the points where M makes contact of order n with the level set $\{F = F(p)\}$.

(Condition (2) rules out examples such as $F(x_1, x_2, x_3) = |x_1|$.)

Theorem 8 follows from the well-known way in which two minimal surfaces in a 3-manifold intersect each other. See, for example, [1, Theorem 7.3] and its proof.

Remark 9. Theorem 8 is also true for branched minimal surfaces, and for minimal surfaces in manifolds of arbitrary dimension. (When $\dim N > 3$, the hypothesis that the level sets of F are minimal is replaced by the hypothesis that the level sets are totally geodesic.) See Section 9.

Remark 10. Note that Theorem 8 only asserts that F is Radó on the interior of M. For applications, we generally need to know that F is Radó on all of M. Fortunately, under mild hypotheses, a continuous function that is Radó on the interior of M will indeed be Radó on all of M. In particular, the $F|M$ in Theorem 8 is Radó on all of M provided $F|M$ is proper, $(\partial M) \cap F^{-1}(t)$ is a finite set for each t, and $d_1(M) := \dim H_1(M; \mathbf{Z}_2) < \infty$. See Theorem 46. Although Theorem 46 appears near the end of the paper, its proof does not depend on the intervening sections.

By definition, the level sets of a Radó function consist of isolated points together with curves joining them. For a general Radó function, those curves are merely continuous. But for the functions $F|M$ in Theorem 8, the level sets are nicer: the curves are smooth (because they are transverse intersections of the smooth surface M and the smooth hypersurface $F^{-1}(t)$). Furthermore:

Theorem 11. *Suppose that F and M are as in Theorem 8. Then*

(1) *the set of interior noncritical points of $F|M$ is an open set;*
(2) *at each interior noncritical point p, the level set $M \cap \{F = F(p)\}$ has a tangent line $\mathrm{Tan}(F|M, p)$, and $\mathrm{Tan}(F|M, p)$ depends continuously on p.*

Furthermore, suppose F_n and M_n are a sequence of such examples with F_n converging uniformly to F and M_n converging smoothly to M. If p is a noncritical point of $F|M$ and if $p_n \in M_n$ converges to p, then p_n is noncritical for $F|M_n$ for all sufficiently large n, and $\mathrm{Tan}(F_n|M, p_n)$ converges to $\mathrm{Tan}(F|M, p)$.

We omit the proof, as it follows easily from standard facts about transversality. For example, the last sentence of the statement of Theorem 11 can be reworded as follows: If M intersects $\{F = F(p)\}$ transversely at p, then M_n intersects $\{F_n = F_n(p_n)\}$ transversely at p_n for all sufficiently large n, and the tangent line to

$$M_n \cap \{F_n = F_n(p_n)\}$$

at p_n converges to the tangent line to

$$M \cap \{F = F(p)\}$$

at p.

Theorem 11 is also true for branched minimal surfaces. See Corollary 51.

Radó functions with properties (1) and (2) in Theorem 11 are called **tame**. (See Definition 32.) Tameness implies a number of other nice properties. See Section 7. In particular, we prove an important lower semicontinuity property (Corollary 40). In the context of Theorem 11, it says that the number of interior critical points (i.e., saddles) of $F|M$ (counting multiplicity) is less than or equal to the liminf of the number of interior critical points of $F_n|M_n$ (counting multiplicity).

Remark 12. In Theorem 11, $\mathrm{Tan}(F|M, p)$ is not merely continuous, it is actually locally Lipschitz. (Sketch of proof: let $T(p)$ be tangent plane to $\{F = F(p)\}$ at p. It is not hard to show using the Harnack inequality that $T(\cdot)$ is locally Lipschitz. It follows easily that $\mathrm{Tan}(F|M, \cdot)$ is locally Lipschitz.) The local Lipschitz property does not play a role in this paper.

Remark 13. Suppose in Theorem 11 that F is smooth with nowhere vanishing gradient. Then the function $F|M$ is particularly nice. First, it is smooth. Second, the interior Radó critical points coincide with the usual critical points (i.e., the points where $D(F|M)$ vanishes). Third, the multiplicity of an interior saddle point p is equal to the order of vanishing of $F|M - (F|M)(p)$. These facts are easy to prove, but play no role in this paper.

3 | SURFACES WITHOUT BOUNDARY

Lemma 14. *Let X be a finite network. Then*

$$\chi(X) = \sum_{p \in V} \frac{1}{2}(2 - v(p)),$$

where V is the set of vertices and $v(p)$ is the valence of p. Equivalently,

$$\chi(X) = \sum_n \frac{1}{2}(2 - n)\,|V_n|, \tag{1}$$

where V_n is the set of vertices of valence n.

Here (and throughout the paper), if S is a set, then $|S|$ denotes the number of elements of S.

Proof. A component without vertices is a loop, and both assertions are trivially true for such components. Thus, we can assume that every component contains one or more vertices. Let V be

the set of vertices and E be the set of edges. Note that

$$\sum_{p \in V} v(p) = 2|E|,$$

Thus,

$$\chi = |V| - |E| = \sum_{p \in V} 1 - \frac{1}{2} \sum_{p \in V} v(p) = \sum_{p \in V} \frac{1}{2}(2 - v(p)). \qquad \square$$

Lemma 15. *Let M be a 2-manifold without boundary and of finite topology, and let X be a finite network in M Then $M \setminus X$ has finite topology, and*

$$\chi(M) = \chi(M \setminus X) + \chi(X).$$

Proof. First remove all the vertices of X from M to get an open 2-manifold M' with $\chi(M') = \chi(M) - |V|$, where $|V|$ is the set of vertices of X. Note that removing a properly embedded open arc from a 2-manifold of finite topology increases the Euler characteristic by 1. Thus, removing the components of $X \setminus V$ one at a time from M' produces a 2-manifold M'' with

$$\chi(M'') = \chi(M') + |E| = \chi(M) - |V| + |E| = \chi(M) - \chi(X). \qquad \square$$

Definition 16. If $F : M \to \mathbf{R}$ and $s \in \mathbf{R}$, we let

$$M[s] = M \cap F^{-1}(s).$$

If $I \subset \mathbf{R}$ is an interval, we let $MI := M \cap F^{-1}(I)$. Thus, for example,

$$M[s, t] = M \cap \{s \leqslant F \leqslant t\},$$

$$M(s, t) = M \cap \{s < F < t\}.$$

Lemma 17. *Suppose that $F : M \to \mathbf{R}$ is a Radó function and that*

(1) *there are no critical points in $M(a, b)$, and*
(2) *$M[a', b']$ is compact for $a < a' < b' < b$.*

If I is an interval in (a, b) and if $t \in I$, then $M \cap F^{-1}(I)$ is homeomorphic to $M(t) \times I$.

Proof. If M has no boundary, this is Corollary A4 in the Appendix. The general case follows by doubling M. $\qquad \square$

Lemma 18. *Suppose that M is a compact 2-manifold without boundary, that $F : M \to \mathbf{R}$ is a Radó function, and that the set Q of local maxima and local minima of F is finite. Let $\mathcal{T} \subset \mathbf{R}$ be a finite set that includes $F(Q)$. Let $X = \cup_{t \in \mathcal{T}} M[t]$. Then*

$$\chi(M) \leqslant \chi(X) = \sum_{p \in X} \frac{1}{2}(2 - \upsilon(F, p))$$

$$= |Q| + \sum_{p \in X,\, \upsilon(F,p) > 2} \frac{1}{2}(2 - \upsilon(F, p)), \tag{2}$$

with equality if and only if each component of $M \setminus X$ is an annulus. In particular, if X contains all the critical points of F, then

$$\chi(M) = \sum_{p \in M} \frac{1}{2}(2 - \upsilon(F, p)). \tag{3}$$

Furthermore, if $M(a, b)$ contains no critical points, then $M(a, b)$ is homeomorphic to $M[t] \times (a, b)$ for each $t \in (a, b)$.

Proof. By Lemma 15,

$$\chi(M) = \chi(X) + \chi(M \setminus X). \tag{4}$$

Let W be a component of $M \setminus X$. Then W is a component of $M(a, b)$ for two successive elements a, b in $\mathcal{T}$.

By Lemma 15, W is an open manifold of finite topology. Thus, it is homeomorphic to a closed surface with finitely many points removed. As W has no local maxima,

$$\sup_W F = b.$$

As W has no local minima,

$$\inf_W F = a.$$

Thus, W is homeomorphic to closed surface with at least two points removed. It follows that

$$\chi(W) \leqslant 0,$$

with equality if and only if W is an annulus. Hence, (by (4)) the inequality (2) holds, with equality if and only if each W is an annulus.

The last assertion is a special case of Lemma 17. □

Theorem 19. *Suppose that M is a compact 2-manifold without boundary and that $F : M \to \mathbf{R}$ is a Radó function with a finite set Q of local maxima and local minima. Then there are only finitely many points p with $\upsilon(F, p) \neq 2$, and*

$$\chi(M) = \sum_{p \in M} \frac{1}{2}(2 - \upsilon(F, p)). \tag{5}$$

Equivalently,

$$\chi(M) = \sum_k (1 - k)\, |V_{2k}|,$$

where V_n is the set of points p such that $v(F, p) = n$.

Proof. Note that Q is the set of points of valence 0. Let $\mathcal{T} \subset \mathbf{R}$ be a finite set that includes $F(Q)$. Let $X = \cup_{t \in \mathcal{T}} M[t]$. By Lemma 18,

$$\chi(M) \leqslant |Q| + \sum_{p \in X,\, v(F,p) > 2} \frac{1}{2}(2 - v(F, p))$$

$$\leqslant |Q| - |\{p \in X : v(F, p) > 2\}|$$

$$= 2|Q| - |\{p \in X : v(F, p) \neq 2\}|$$

$$= 2|Q| - |C|,$$

where C is the set of critical points of F in X. Thus, X has at most

$$2|Q| - \chi(M)$$

critical points. As this bound holds for every such $\mathcal{T}$, we see that M has at most $2|Q| - \chi(M)$ critical points. Equation (5) now follows from (3) in Lemma 18 by letting

$$\mathcal{T} = \{F(p) : v(F, p) \neq 2\}. \qquad \square$$

Corollary 20. *The number of saddle points, counting multiplicity, is equal to the number of local maxima and local minima minus the Euler characteristic:*

$$\sum_{w(F,p) > 0} w(F, p) = |Q| - \chi(M).$$

Proof. Each point of valence 0 (i.e., each point of Q) contributes 1 to the sum in (5), each point of valence 2 contributes 0, and there are no points of odd valence. Thus, (5) becomes

$$\chi(M) = |Q| + \sum_{v(F,p) \geqslant 4} \frac{1}{2}(2 - v(F, p))$$

$$= |Q| - \sum_{w(F,p) > 0} w(F, p). \qquad \square$$

A version of Corollary 20 in the case of the 2-sphere occurs in an 1870 paper [9] by the physicist Maxwell. In particular, Maxwell does allow saddles with multiplicity.

4 │ SURFACES WITH BOUNDARY

Theorem 21. *Suppose that M is a compact 2-manifold with boundary, that $F : M \to \mathbf{R}$ is a Radó function, and that there are only finitely many points of valence 0. Then*

$$\chi(M) = \sum_{p \in M \setminus \partial M} \frac{1}{2}(2 - \upsilon(F, p)) + \sum_{p \in \partial M} \frac{1}{2}(1 - \upsilon(F, p)).$$

Equivalently,

$$\chi(M) = \sum_{k=0}^{\infty} \left((1 - k)|V_{2k}^{\text{int}}| + \frac{1}{2}(1 - k)|V_k^{\partial}| \right),$$

where V_n^{int} is the set of interior points $p \in M \setminus \partial M$ with $\upsilon(F, p) = n$, and V_n^{∂} is the set of boundary points $p \in \partial M$ with $\upsilon(F, p) = n$.

Proof. Let $\tilde{M}$ be the closed manifold obtained by doubling M. That is, we take two copies of M and attach them along their boundary. Let $\tilde{F}$ be the obvious extension of F to $\tilde{M}$. As $F : M \to \mathbf{R}$ is Radó, it follows easily that $\tilde{F} : \tilde{M} \to \mathbf{R}$ is also Radó. Let $\tilde{V}_n$ be the set of points $p \in \tilde{M}$ with $\upsilon(\tilde{F}, p) = n$. Then

$$|\tilde{V}_{2k}| = 2|V_{2k}^{\text{int}}| + |V_k^{\partial}|.$$

Applying Theorem 19 to $\tilde{M}$ gives

$$\begin{aligned}
\chi(M) &= \frac{1}{2}\chi(\tilde{M}) \\
&= \frac{1}{2}\sum_k (1 - k)\,|\tilde{V}_{2k}| \\
&= \frac{1}{2}\sum_k (1 - k)(2|V_{2k}^{\text{int}}| + |V_k^{\partial}|) \\
&= \sum_k \left((1 - k)|V_{2k}^{\text{int}}| + \frac{1}{2}(1 - k)|V_k^{\partial}| \right). \qquad \square
\end{aligned}$$

The statement of Theorem 21 is fairly simple. However, the theorem can be rewritten in a way that makes it easier to use.

Theorem 22. *Suppose that $F : M \to \mathbf{R}$ is a Radó function on a compact 2-manifold with boundary. Let Q be*

 (i) *the set of interior local maxima and interior local minima of F, together with*
(ii) *the set of local **minima** of $F|\partial M$.*

Suppose that Q is a finite set. Then

$$\sum_{n \geq 2}(n - 1)\,|V_{2n}^{\text{int}}| + \sum_{n \geq 1} n\big(|V_{2n}^{\partial}| + |V_{2n+1}^{\partial}|\big) = |Q| - \chi(M). \tag{6}$$

This way of rewriting Theorem 21 is very useful for the following reason. Think of M and $F|\partial M$ as given, and the function F as unknown. In many situations (such as for minimal surfaces in Theorem 8), we know that there are no interior local maxima or minima. In that case, Q is the set of local minima of $F|\partial M$, which we regard as known. Thus, the right-hand side is known, and the terms on the left are all positive.

Recall that interior points of valence $v > 2$ are called interior saddle points of multiplicity $w(F, p) := (v/2) - 1$.

Definition 23. A **boundary saddle point** of a Radó function F is a boundary point of valence > 1. The **multiplicity** of a boundary saddle point is

$$w(F, p) := \begin{cases} v/2 & \text{if } v \text{ is even,} \\ (v-1)/2 & \text{if } v \text{ is odd.} \end{cases}$$

Using this definition, Theorem 22 can be restated as follows:

Theorem 24. *Under the hypotheses of Theorem 22,*

$$\sum_{w>0} w(F, p) = |Q| - \chi(M). \tag{7}$$

Note that the left-hand side is the total number of saddles, interior and boundary, counting multiplicity.

Proof of Theorem 22. Write

$$Q = Q^{\text{int}} + Q^{\partial},$$

where $Q^{\text{int}} := Q \setminus \partial M$ is the set of interior local maxima and interior local minima of F, and where $Q^{\partial} := Q \cap \partial M$ is the set of local minima of $F|\partial M$.

Note that the points of valence 0 are the points of Q together with the local maxima of $F|\partial M$. As the number of local maxima of $F|\partial M$ is equal to the number $|Q^{\partial}|$ of local minima of $F|\partial M$, we see that there are only finitely many points of valence 0.

Recall from Theorem 21 that

$$\chi(M) = \sum_k (1-k)|V_{2k}^{\text{int}}| + \frac{1}{2} \sum_k (1-k)|V_k^{\partial}|. \tag{8}$$

Now

$$\sum_k (1-k)|V_{2k}^{\text{int}}| = |Q^{\text{int}}| + 0 - \sum_{k \geqslant 2} (k-1)|V_{2k}^{\text{int}}|. \tag{9}$$

Also

$$\sum_k (1-k)|V_k^{\partial}| = \sum_n ((1-2n)|V_{2n}| + (-2n)|V_{2n+1}|)$$
$$= \sum_n |V_{2n}^{\partial}| - \sum_n (2n)(|V_{2n}| + |V_{2n+1}|). \tag{10}$$

By Lemma 6,

$$\sum_n |V_{2n}^\partial|$$

is the number of local minima and local maxima of $F|\partial M$. The number of local maxima of $F|\partial M$ is equal to the number of local minima of $F|\partial M$ (namely $|Q^\partial|$), so

$$\sum_n |V_{2n}^\partial| = 2|Q^\partial|.$$

Thus, we can rewrite (10) as

$$\frac{1}{2} \sum_k (1-k)|V_k^\partial| = |Q^\partial| - \sum_n n \left(|V_{2n}| + |V_{2n+1}|\right). \tag{11}$$

Combining (8), (9), and (11) gives (6). □

5 | A REMARK ABOUT INEQUALITIES

Various theorems in this paper, such as Theorem 24, give formulae for the total number of saddles, interior and boundary, in some region, counting multiplicity. For many applications, simpler inequalities suffice.

The following proposition describes how the exact formulae imply the simpler inequalities.

Proposition 25. *Suppose that $F : M \to \mathbf{R}$ is a Radó function and that K is a region in M. Suppose also that*

$$\sum_{K \cap \{w > 0\}} w(F, p) = \mathcal{W}.$$

Then

$$\sum_{(K \setminus \partial M) \cap \{w > 0\}} w(F, p) \leqslant \mathcal{W} - |A|, \tag{12}$$

where A is the set of points p in $K \cap \partial M$ such that p is a local minimum or local maximum of $F|\partial M$ but is not a local minimum or local maximum of F.

Furthermore, equality holds if and only if $K \cap \partial M$ contains no point p with valence $v(F, p) > 2$. In particular, if F is C^2, if $F|M$ is a Morse function, and if DF does not vanish at any point of ∂M, then equality holds in (12).

Proof. Note that

$$\sum_{K \cap \{w > 0\}} w(F, p) = \sum_{(K \setminus \partial M) \cap \{w > 0\}} w(F, p) + \sum_{(K \cap \partial M) \cap \{w > 0\}} w(F, p),$$

and that

$$\sum_{(K\cap\partial M)\cap\{w>0\}} w(F,p) = \sum_{(K\cap\partial M)\cap\{v\geqslant 2\}} w(F,p)$$

$$\geqslant \sum_{(K\cap\partial M)\cap\{v\geqslant 2,\ v\text{ even}\}} w(F,p)$$

$$= \sum_{(K\cap\partial M)\cap\{v\geqslant 2,\ v\text{ even}\}} \frac{1}{2}v(F,p)$$

$$\geqslant \sum_{(K\cap\partial M)\cap\{v\geqslant 2,\ v\text{ even}\}} 1$$

$$= |A|$$

with equality if and only $K \cap \partial M$ has no points of valence > 2.

This proves the proposition, except for the assertion about the case when F is C^2. Note that if F is C^2 and if $F|\partial M$ is a Morse function, then at each point of ∂M that is not a critical point of $F|\partial M$, the valence is 1, and at each critical point of $F|\partial M$, the valence is either 0 or 2. $\qquad\square$

Thus, for example, from Theorem 24, we get the following inequality:

Theorem 26. *Suppose that $F : M \to \mathbf{R}$ is a Radó function on a compact 2-manifold with boundary. Let Q be the set of interior local maxima and interior local minima of F, together with the local minima of $F|\partial M$. Suppose that Q is a finite set. Then*

$$\sum_{(M\setminus\partial M)\cap\{w>0\}} w(F,p) \leqslant |Q| - \chi(M) - |A|, \tag{13}$$

where A is the set of local maxima and local minima of $F|\partial M$ that are not local maxima or local minima of F. Equality holds if and only if there are no boundary points p of valence $v(F,p) > 2$.

In particular, if F is C^2, if $F|\partial M$ is a Morse function, and if DF does not vanish at any point of $F|\partial M$, then equality holds.

6 | PORTIONS OF SURFACES WITH BOUNDARY

Theorem 27. *Suppose that $F : M \to \mathbf{R}$ is a Radó function, that $a < b$ are regular values of F, and that $M[a,b]$ is compact. Then*

$$\sum_{M(a,b)\cap\{w>0\}} w(F,p) = |Q(a,b)| + \frac{1}{2}\beta(a) - \chi(M(a,b)), \tag{14}$$

provided $|Q(a,b)|$ is finite, where

(1) *Q is the set consisting of the interior local maxima and the interior local minima of F, together with the local minima of $F|\partial M$,*
(2) *$Q(a,b) = Q \cap M(a,b)$, and*
(3) *$\beta(a)$ is the number of points in $(\partial M) \cap \{F = a\}$.*

Proof. Let $\tilde{M}$ be obtained from $M[a, b]$ by identifying each connected component of $M[a]$ to a a point and each connected component of $M[b]$ to a point. Let $\tilde{F}$ be the function on $\tilde{F}$ corresponding to F on $M[a, b]$.

Thus, each closed curve component of $M[a]$ becomes an interior point of $\tilde{M}$, and each non-closed curve component of $M[a]$ becomes a single boundary point of $\tilde{M}$. In both cases, the point is a global minimum of $\tilde{F}$.

Likewise, each closed curve component of $M[b]$ becomes an interior point of $\tilde{M}$, and each non-closed curve component of $M[b]$ becomes a single boundary point of $\tilde{M}$. In both cases, the point is a global maximum of $\tilde{F}$.

Let $\tilde{Q}$ be the set of all interior local maxima and interior local minima of $\tilde{F}$, together will all local minimal of $\tilde{F}|\partial\tilde{M}$.

Let n be the number of nonclosed-curve components of $M[a]$, and let c be the number of closed curved components of $M[a] \cup M[b]$.

Note that

$$\chi(\tilde{M}) = \chi(M(a, b)) + c,$$

$$|\tilde{Q}| = |Q(a, b)| + n + c, \text{ and}$$

$$n = \frac{1}{2}\beta(a).$$

Thus,

$$|\tilde{Q}| - \chi(\tilde{M}) = |Q(a, b)| - \chi(M(a, b)) + \frac{1}{2}\beta(a).$$

Consequently, by Theorem 24,

$$\sum_{\tilde{M}\cap\{w>0\}} w(\tilde{F}, p) = |\tilde{Q}| - \chi(\tilde{M})$$

$$= |Q(a, b)| + \frac{1}{2}\beta(a) - \chi(M(a, b)). \tag{15}$$

The points in $\tilde{M}$ with $\tilde{F} = a$ or $\tilde{F} = b$ all have valence 0, so

$$\sum_{\tilde{M}\cap\{w>0\}} w(\tilde{F}, p) = \sum_{M(a,b)\cap\{w>0\}} w(F, p). \tag{16}$$

Combining (15) and (16) gives (14). □

We now relax the requirement in Theorem 27 that a and b are finite, noncritical values of F. We begin with a lemma.

Lemma 28. *Suppose that M is a surface and that $F : M \to \mathbf{R}$ is a continuous function. Suppose also that F has no interior local minima with $F < a$, and no interior local maxima with $F > b$. Then the inclusion of $M[a, b]$ into M induces a monomorphism on $H_1(-; \mathbf{Z}_2)$.*

Likewise, if F has no interior local minima with $F \leqslant a$, and no interior local maxima with $F \geqslant b$, then inclusion of $M(a, b)$ into M induces a monomorphism on $H_1(-; \mathbf{Z}_2)$.

Proof. We prove the first statement. Let C be a 1-cycle in $M[a, b]$ that is homologically trivial in $H_1(M; \mathbf{Z}_2)$. Then C bounds a region K in M. Let p be a point where $F|K$ attains its maximum. If $p \in C$, then $F(p) \leqslant b$ because $C \subset M[a, b]$. If $p \in K \setminus C$, then p is an interior local maximum of F and hence $F(p) \leqslant b$. Either way, $\max_K F \leqslant b$. Likewise, $\min_K F \geqslant a$. Hence, K lies in $M[a, b]$, so C is homologically trivial in $H_1(M[a, b]; \mathbf{Z}_2)$. $\square$

For the next two theorems, we make the following hypotheses.

(h1) $F : M \to \mathbf{R}$ is a Radó function and $-\infty \leqslant a < b \leqslant \infty$.
(h2) $d_1(M) := \dim H_1(M; \mathbf{Z}_2)$ is finite.
(h3) The set Q is finite, where Q consists of the interior local minima and maxima of F together with the local minima of $F|\partial M$.

Theorem 29. *Under the hypotheses (h1)–(h3), if $M[a, t]$ is compact for all $a \leqslant t < b$ and if a is a regular value of F, then*

$$\sum_{M(a,b) \cap \{w>0\}} w(F, p) = |Q(a, b)| + \frac{1}{2}\beta(a) - \chi(M(a, b)),$$

where $\beta(t)$ is the number of points in $(\partial M) \cap \{F = t\}$.

Proof. Let Z be the set of interior local maxima and minima of F and let $Z^*(a, b) = Z \cap \{F \notin (a, b)\}$. Note that $Z \subset Q$ and that

$$M(s, t) \subset M \setminus Z^*(s, t)$$

induces a monomorphism of first homology (see Lemma 28), so

$$d_1(M(s, t)) \leqslant d_1(M \setminus Z^*(s, t))$$

$$= d_1(M) + |Z^*(s, t)|,$$

$$\leqslant d_1(M) + |Q|.$$

Therefore,

$$-\chi(M(s, t)) \leqslant d_1(M(s, t)) \leqslant d_1(M) + |Q|.$$

If $t \in (a, b)$ is a regular value of F, then by Theorem 27,

$$\sum_{p \in M(a,t), \, w>0} w(F, p) = |Q(a, t)| + \frac{1}{2}\beta(a) - \chi(M(a, t))$$

$$\leqslant \frac{1}{2}\beta(a) + d_1(M) + |Q|.$$

Note that this final expression is indepent of t. By elementary topology (see Corollary A6), there are at most countably many critical points and hence at most countably many critical values. Thus, (letting $t \to b$ among regular values t),

$$\sum_{p \in M(a,b), \, w>0} w(F, p) \leqslant \frac{1}{2}\beta(a) + d_1(M) + |Q| < \infty.$$

Hence, the set $S := \{p \in M(a, b) : w(F, p) > 0\}$ is finite. The set $Q(a, b)$ is also finite, so we can choose a regular value b' of F in (a, b) such that

$$S \cup Q(a, b) \subset M(a, b').$$

Now $S \cup Q(a, b)$ contains all the critical points of $F|M(a, b)$. Thus, there are no critical points in $M[b', b]$. Consequently, $M(a, b)$ is homotopy equivalent to $M(a, b')$ (see Lemma 17), so

$$\chi(M(a, b')) = \chi(M(a, b)).$$

By Theorem 27,

$$\sum_{p \in M(a,b),\, w>0} w(F, p) = \sum_{p \in M(a,b'),\, w>0} w(F, p)$$

$$= |Q(a, b')| + \frac{1}{2}\beta(a) - \chi(M(a, b'))$$

$$= |Q(a, b)| + \frac{1}{2}\beta(a) - \chi(M(a, b)). \qquad \square$$

Theorem 30. *Under the hypotheses (h1)–(h3), if $M[s, t]$ is compact for all $a < s < t < b$, and if the limit*

$$\beta(a+) = \lim_{t \to a,\, t>a} \beta(t)$$

exists and is finite, then

$$\sum_{M(a,b) \cap \{w>0\}} w(F, p) = |Q(a, b)| + \frac{1}{2}\beta(a+) - \chi(M(a, b)).$$

Proof of Theorem 30. As $\beta(t)$ is integer-valued, there is an $a' \in (a, b)$ such that

$$\beta(t) = \beta(a+) \quad \text{for } a < t \leqslant a'.$$

Now let $s \in (a, a']$ be a regular value of F. Then (by Theorem 29)

$$\sum_{M(s,b) \cap \{w>0\}} w(F, p) = |Q(s, b)| + \frac{1}{2}\beta(t) - \chi(M)$$

$$\leqslant |Q(a, b)| + \frac{1}{2}\beta(a+) - d_1(M).$$

As this last expression is finite and independent of s, letting $s \to a$ (among regular values s) gives

$$\sum_{p \in M(a,b),\, w>0} w(F, p) < \infty.$$

Thus, the set S of points in $M(a, b)$ where $w > 0$ is finite. The set $Q(a, b)$ is also finite. By replacing a' by smaller noncritical value in (a, b), we can assume that

$$S \cup Q(a, b) \subset M(a', b).$$

Now $S \cup Q(a,b)$ contains all the critical points of F in $M(a,b)$. Thus, there are no critical points in $M(a,a')$, so $M(a,b)$ is homotopy equivalent to $M(a',b)$ (by Lemma 17). Consequently,

$$\chi(M(a,b)) = \chi(M(a',b)).$$

Thus,

$$\sum_{p \in M(a,b),\, w>0} w(F,p) = \sum_{p \in M(a',b),\, w>0} w(F,p)$$

$$= |Q(a',b)| + \frac{1}{2}\beta(a') - \chi(M(a',b))$$

$$= |Q(a,b)| + \frac{1}{2}\beta(a+) - \chi(M(a,b)). \qquad \square$$

Corollary 31. *In Theorem 30, if* $F : M \to (a,b)$ *is proper, then*

$$\sum_{M \cap \{w>0\}} w(F,p) = |Q| + \frac{1}{2}\beta(a+) - \chi(M).$$

7 | TAME RADÓ FUNCTIONS

Definition 32. Suppose that $F : M \to \mathbf{R}$ is a Radó function. If p is an interior regular point of F, we let $\mathrm{Tan}(F,p)$ be the tangent line to $\{F = F(p)\}$ at p, if the tangent line exists. We say that F is **tame** provided:

(1) the set of interior regular points (i.e., the set of interior points of valence 2) is open,
(2) $\mathrm{Tan}(F,p)$ exists at each interior regular point, and $\mathrm{Tan}(F,\cdot)$ is a continuous function on the set of interior regular points.

By Theorem 11, the Radó functions that arise in minimal surface theory are tame.

Theorem 33. *Suppose that* $F : M \to \mathbf{R}$ *is a tame Radó function such that the set* Q^{int} *of interior local minima and interior local maxima is closed and discrete. Then the set of interior critical points is closed and discrete. In other words, each interior point* p *has a neighborhood* U *such that* $U \setminus \{p\}$ *contains no critical points.*

Proof. Let p be an interior critical point. Thus, $v(F,p)$ is an even number $\neq 2$.

Case 1. $v(F,p) = 0$. Then p is a local maximum or local minimum. We may assume that it is a local minimum. Let K be a compact set such that p is the interior of K, such that $\min_{\partial K} F > F(p)$, and such that $K \setminus \{p\}$ contains no local minima or local maxima of F. Choose t with

$$F(p) < t < \min_{\partial K} F.$$

Let $D := K \cap \{F \leqslant t\}$. Then D is a disk, so if we identify ∂D to a point, we get a topological sphere Σ on which F is a well-defined Radó function. Note that $F|\Sigma$ has exactly one local maximum and
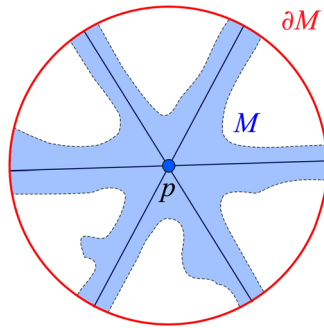
FIGURE 2 As Case 2 is local, we can assume that M is a disk, that M has no interior local maxima or local minima, and that $\{F = 0\}) \setminus \{p\}$ consists of $2k$ disjoint, embedded C_1 curves, each joining p to a point in ∂M.

one local minimum. Thus, by Corollary 20, $F|\Sigma$ has no saddle points. Therefore, F has no critical points on $D \setminus \{p\}$. This completes the proof in Case 1.

Case 2. $\upsilon(F, p) = 2k > 0$.

We may assume that $F(p) = 0$. As the result is local, we can assume that M is a disk, that M has no interior local maxima or local minima, and that

$$\{F = 0\}) \setminus \{p\}$$

consists of $2k$ disjoint, embedded C_1 curves, each joining p to a point in ∂M (see Figure 2.)

By applying a homeomorphism from M into $\mathbf{R}^2$ that is C^1 in $M \setminus \{p\}$, we can assume that

$$M = \{q \in \mathbf{R}^2 : |q| < 2\},$$

$$p = 0,$$

and that

$$F(r \cos\theta, r \sin\theta) = 0 \iff \theta \text{ is an integral multiple of } \pi/k.$$

Let Δ be the region given in polar coordinates by $0 < r \leqslant 1$ and $0 < \theta < \pi/k$. Note that Δ is one of the components of

$$\mathbf{B}(0, 1) \setminus \{F = 0\}.$$

We may assume that $F > 0$ on Δ. By tameness, the unit circle is transverse to $\mathrm{Tan}(F, \cdot)$ near the points $(\cos(j\pi/k), \sin(j\pi/k))$.

In particular, we can choose $\epsilon > 0$ so that the unit circle is transverse to $\mathrm{Tan}(F, \cdot)$ at $(\cos\theta, \sin\theta)$ for $\theta \in [0, \epsilon]$ and for $\theta \in [(\pi/k) - \epsilon, (\pi/k)]$. Thus, $F(\cos\theta, \sin\theta)$ is strictly increasing for $0 \leqslant \theta \leqslant \epsilon$ and is strictly decreasing for $(\pi/k) - \epsilon \leqslant \theta \leqslant \pi/k$.

Let

$$\eta = \min\{F(\cos\theta, \sin\theta) : \epsilon \leqslant \theta \leqslant (\pi/k) - \epsilon\}.$$

Hence, for $0 < t < \eta$, there are exactly two points q and q' in ∂D at which $F = t$.

Claim 34. $\Gamma(t) := \Delta \cap \{F = t\}$ consists of a curve of noncritical points joining q to q'.

Proof of Claim 34. This is the well-known argument of Radó. (See Theorem 41 for a very general form of Claim 34.) We know that $\Gamma(t)$ is a network. As it is contained in a compact subset of the interior of M, it is a finite network. It cannot contain a closed curve, as if it did, that closed curve would bound a disk in M (as M is simply connected), and F on that disk would its minimum and/or its maximum at an interior point, which is impossible because we are assuming that there are no interior local maxima or minima. Thus, $\Gamma(t)$ is a tree. As the tree has at most two points of valence $\leqslant 1$ (namely q and q'), it is a curve joining q to q'. □

We have shown: for small enough η, there are no critical points in $\overline{D}$ with $F < \eta$. The same argument in the other components of $M \setminus \{F = 0\}$ shows that there is an $\eta' > 0$ such there are no critical points in $\overline{D}$ with $0 < |F| < \eta'$. Also, all the points on $\{F = 0\} \setminus \{p\}$ are regular. □

Remark 35. Without the hypothesis of tameness, Theorem 33 is false. Consider the harmonic function $h(x, y) = y - (\cosh x)(\sin y)$, and let

$$F(x, y) = \begin{cases} h(x, y^{-1})^{-1} & \text{if } y \neq 0, \text{ and} \\ 0 & \text{if } y = 0. \end{cases}$$

Note that the class of Radó functions is closed under composition with homeomorphisms of the domain and of $\mathbf{R}$. As h is harmonic, it is Radó, and thus F is Radó on $\mathbf{R}^2 \setminus \{y = 0\}$. We leave it to the reader to check that F is Radó on all of $\mathbf{R}^2$. Thus, F is a proper, Radó function on $[-1, 1] \times \mathbf{R}$. The interior critical points of F are the points $(0, (2\pi n)^{-1})$ where n is an integer. Thus, the noncritical point $(0,0)$ is a limit of critical points.

Theorem 36. *Suppose that* $F : M \to \mathbf{R}$ *is a tame Radó function and that* p *is an interior point. Then*

$$\text{Hopf}(F, p) = \left(1 - \frac{v(F, p)}{2}\right) = -w(F, p),$$

where $\text{Hopf}(F, p)$ *is the Hopf index of* $\text{Tan}(F, \cdot)$ *at* p.

Proof. We use the notation in the proof of Theorem 33. In that proof, we can modify F near the unit circle $\partial \mathbf{B}$ so that the level sets of F cross the circle orthogonally when $|F| \leqslant \delta$ for some small $\delta > 0$. By the proof of Theorem 33, we can choose δ small enough so that

$$N := \{q \in \mathbf{B}(0, 1) : |F(q)| \leqslant \delta\}$$

contains no critical points other than 0. Note that N is bounded by $4k$ arcs, $2k$ in the circle ∂D and the other $2k$ in the interior of the unit disk. Now we can invert N in the unit circle to get a 2-manifold with boundary $\tilde{N}$ in $\mathbf{R}^2 \cup \{\infty\}$. We extend F and $\text{Tan}(F, \cdot)$ to $\tilde{N}$ by inversion. Note that $\text{Tan}(F, \cdot)$ is tangent to $\partial \tilde{N}$, so by the Poincare–Hopf Index Theorem,

$$\chi(\tilde{N}) = \text{Hopf}(F, 0) + \text{Hopf}(F, \infty) = 2\,\text{Hopf}(F, 0).$$

Now $\tilde{N}$ is S^2 with $2k$ disjoint open disks removed, so $\chi(\tilde{N}) = 2 - 2k$. Thus,

$$2 - 2k = 2\,\text{Hopf}(F, 0). \qquad \square$$

Suppose that M is a locally compact space, that U_n and U are open subsets of M, and that $\phi_n : U_n \to V$ and $\phi : U \to V$ are continuous maps to a metrizable space V. We say that ϕ_n converges locally uniformly to ϕ provided the following holds: if $p \in \text{domain}(\phi)$ and if $p_n \to p$, then $p_n \in \text{domain}(\phi_n)$ for all sufficiently large n and $\phi_n(p_n) \to \phi(p)$. It follows that if ϕ_n converges locally uniformly to ϕ and if K is a compact subset of $\text{domain}(\phi)$, then $K \subset \text{domain}(\phi_n)$ for all sufficiently large n, and $\phi_n|K$ converges uniformly to $\phi|K$.

Theorem 37. *Suppose $F_n : M_n \to \mathbf{R}$ and $F : M \to \mathbf{R}$ are tame Radó functions, where M_n is an exhaustion of M, such that $\text{Tan}(F_n, \cdot)$ converges locally uniformly to $\text{Tan}(F, \cdot)$. Suppose that K is a compact region of $M \setminus \partial M$ such that ∂K is contained in the regular set of F. Then for all sufficiently large n,*

$$\sum_{p \in K} \left(1 - \frac{v(F_n, p)}{2} \right) = \sum_{p \in K} \left(1 - \frac{v(F_n, p)}{2} \right). \tag{17}$$

Equivalently,

$$\sum_{p \in K} w(F_n, p) = \sum_{p \in K} w(F, p).$$

Proof. First, consider the case when K is topologically a disk. Then we can choose local coordinates so that K is a disk in $\mathbf{R}^2$. By the Poincare–Hopf Theorem, the right-hand side of (17) is equal to the degree of the map

$$q \in \partial K \mapsto \text{Tan}(F, q) \in \mathbf{RP}^1,$$

and the left side is equal to the degree of

$$q \in \partial K \mapsto \text{Tan}(F_n, q) \in \mathbf{RP}^1,$$

By hypothesis, the two maps are homotopic for all sufficiently large n, and thus the two degrees are equal.

In the general case, let D be a finite union of disjoint closed disks in the interior of K such that the interior U of D contains all the critical points of F in K. Now $K \setminus U$ is contained in $\text{Reg}(F)$, so it is contained in $\text{Reg}(F_n)$ for all sufficiently large n. Thus, by the simply connected case,

$$\sum_{p \in K} \left(1 - \frac{v(F_n, p)}{2} \right) = \sum_{p \in D} \left(1 - \frac{v(F_n, p)}{2} \right)$$

$$= \sum_{p \in D} \left(1 - \frac{v(F, p)}{2} \right)$$

$$= \sum_{p \in K} \left(1 - \frac{v(F, p)}{2} \right),$$

for all sufficiently large n. $\qquad \square$

Definition 38. Suppose that $F : M \to \mathbf{R}$ is a Radó function. We say that F is a **minimal Radó function** provided F has no interior local minima and no interior local maxima.

We call these functions minimal Radó functions because the Radó functions that arise from minimal surfaces as in Theorem 8 have no interior local minima or interior local maxima.

Theorem 39 (Lower Semicontinuity Theorem). *Suppose that $F_n : M_n \to \mathbf{R}$ and $F : M \to \mathbf{R}$ are tame minimal Radó functions, where M_n is an exhaustion of M. Suppose also that $\mathrm{Tan}(F_n, \cdot)$ converges locally uniformly to $\mathrm{Tan}(F, \cdot)$. Then*

$$\sum_{p \in M \setminus \partial M} w(F, p) \leqslant \liminf \sum_{p \in M_n \setminus \partial M_n} w(F_n, p).$$

Proof. It suffices to prove it for M without boundary. (Otherwise replace M by $M \setminus \partial M$ in the following proof.)

Note that as there are no local maxima or minima, $w(F, p)$ and $w(F_n, p)$ are both nonnegative for every p.

Let K be any compact subset of M. Let K' be a compact subset of M such that $K \subset K'$ and such that $\partial K'$ lies in the regular set of F. For all sufficiently large n,

$$\sum_{p \in M_n} w(F_n, p) \geqslant \sum_{p \in K'} w(F_n, p) = \sum_{p \in K'} w(F, p) \geqslant \sum_{p \in K} w(F, p),$$

by Theorem 37. Thus,

$$\liminf_n \sum_{p \in M} w(F_n, p) \geqslant \sum_{p \in K} w(F, p).$$

Now take the supremum over all $K \subset\subset M$. $\qquad\square$

Corollary 40. *Suppose that*

(1) *g_n are smooth Riemannian metrics on a 3-manifold N that converge smoothly to a metric g;*
(2) *$\mathcal{F}_n$ are g_n-minimal foliations of N that converge to a g-minimal foliation $\mathcal{F}$;*
(3) *M_n are g_n-minimal surfaces that converge smoothly to a g-minimal surface M;*
(4) *no connected component of M lies in a leaf of $\mathcal{F}$;*

then

$$\mathsf{N}(\mathcal{F}, M) \leqslant \liminf \mathsf{N}(\mathcal{F}_n, M_n), \tag{18}$$

where $\mathsf{N}(\mathcal{F}, M)$ is the number of interior points of tangency of $\mathcal{F}$ and M, counting multiplicity.

In other words, $\mathsf{N}(\mathcal{F}, M)$ is the sum over the interior points $p \in M$ of the order of contact at p of M and the leaf of $\mathcal{F}$ through p.

Proof. Suppose first that $\mathcal{F}_n$ and $\mathcal{F}$ are given as the level sets of functions F_n and F on N as in Theorem 8. Then $F_n|M_n$ and $F|M$ are tame Radó functions by Theorems 8 and 11, and $\mathrm{Tan}(F_n|M_n, \cdot)$ converges locally uniformly to $\mathrm{Tan}(F|M, \cdot)$ by Theorem 11. Thus, (18) holds by Theorem 39.

In general, $\mathcal{F}_n$ and/or $\mathcal{F}$ might not be expressible as the level sets of functions F_n and F. However, locally that is always possible. Furthermore, even if F_n and F are only defined locally, $\mathrm{Tan}(F_n, \cdot)$ and $\mathrm{Tan}(F, \cdot)$ make sense globally, and the proof of Theorem 39 only really depends on those line fields, and not on the functions themselves. Thus, Corollary 40 holds for arbitrary minimal foliations. □

8 | SLICES

A classical result of Radó states that if a plane in $\mathbf{R}^3$ intersects the boundary of a minimal disk in fewer than four points, then it intersects the disk transversely. In this section, we prove a very general form of Radó's principle.

(When we apply the following theorem to minimal surfaces, the set S and the set of interior local maxima and minima will typically be empty.)

Theorem 41 (Slice Theorem). *Let M be a 2-manifold and $F : M \to \mathbf{R}$ be a continuous function with only finitely many interior local minima and maxima. Suppose that there is a finite set S of interior points such that F is a Radó function on $M \setminus (\partial M \cup S)$. Let $X = F^{-1}(t)$ be a level set of M. Suppose that*

(1) *X is compact;*
(2) *$X \cap \partial M$ is finite;*
(3) *$d_1(M) := \dim H_1(M; \mathbf{Z}_2) < \infty$.*

Then X is a finite network.

Now suppose that F is a Radó function on all of $M \setminus \partial M$ (i.e., that the set S is empty.) Let

$$V_n = \{p \in X : v(F, p) = n\}$$

be the set of nodes of valence n in X. Then

$$\frac{1}{2} \sum_{n \geqslant 3} (n-2)|V_n| + d_0(X \setminus V_0) \leqslant \frac{1}{2}|V_1| + d_1(M) + |S^*|$$

$$\leqslant \frac{1}{2}|J| + d_1(M) + |S^*|, \tag{19}$$

$$\leqslant \frac{1}{2}k + d_1(M) + |S^*|,$$

where J is the set of points in $X \cap \partial M$ that are neither local maxima nor local minima of $F|\partial M$, S^* is the set of interior local maxima and local minima of F in $M \cap \{F \neq t\}$, and k is the number of points p in ∂M where $F(p) = t$.

Note that $d_0(X \setminus V_0)$ is the number of components of X that are not isolated points.

In the examples that arise in minimal surface theory, S^* is empty:

Corollary 42. *If $S^* = \emptyset$, then*

$$\frac{1}{2} \sum_{n \geqslant 3} (n-2)|V_n| + d_0(X \setminus V_0) \leqslant \frac{1}{2}|J| + d_1(M) \leqslant \frac{1}{2}k + d_1(M),$$

Proof of Theorem 41. By Lemma 28, the inclusion of X into $M \setminus S^*$ induces a monomorphism of $H_1(-; \mathbf{Z}_2)$, so

$$d_1(X) \leqslant d_1(M \setminus S^*) = d_1(M) + |S^*| < \infty.$$

Thus, by a general theorem about networks (Theorem 44), X is a finite network. (One lets the set B in Theorem 44 be the union of the following three sets: the set of interior local minima and interior local maxima of F in X, the set S, and the set $X \cap \partial M$.) By a general counting theorem (Theorem 45) for finite graphs,

$$\sum_{n \geqslant 3} \frac{1}{2}(n-2)|V_n| + d_0(X \setminus V_0) = \frac{1}{2}|V_1| + d_1(X) \tag{20}$$

$$\leqslant \frac{1}{2}|V_1| + d_1(M) + |S^*|.$$

Now suppose that S is empty. Recall that for every interior point p, $v(F, p)$ is even. For a boundary point p, $V(F, p)$ is even if and only if p is a local maximum or local minimum of $F|\partial M$. Hence, $V_1 \subset J$. Thus, (19) follows from (20). $\qquad\square$

Remark 43. In the bound (19), we could let S^* be the set consisting of interior local minima of F in $\{F < t\}$ and of interior local maxima of F in $\{F > t\}$. No changes are required in the proof. In the case of minimal Radó functions, there are no interior local maxima or minima.

We now prove the finiteness theorem for general networks that was used in the proof of Theorem 41 (the Slice Theorem).

Let p be a point in a topological space X and k be a nonnegative integer. Suppose p has a neighborhood U such that $X \cap U$ is the union of k embedded curves, where each curve joins p to a point in ∂U and where the curves intersect each other only at p. Then we say that X has **valence** k at p and write $v(p) = V(X, p) = k$. If there is no such k and U, then $v(X, p)$ is undefined.

Theorem 44 (Finiteness Theorem). *Let X be a compact Hausdorff space. Suppose that $B \subset X$ is a finite set with the following properties.*

(1) *Each point p in $X \setminus B$ has a well-defined valence $v(p)$ that is $\geqslant 2$.*
(2) $d_1(X) := \dim H_1(X; \mathbf{Z}_2)$ *is finite.*

Then X is a finite network.

Proof. Define the set $V = V(X, B)$ of vertices by

$$V := B \cup \{p \in X \setminus B : v(p) \neq 2\},$$

and let $\mathcal{E} = \mathcal{E}(X, Q)$ be the set of connected components of $X \setminus V$. Note that each element E of $\mathcal{E}$ is an embedded curve and that $\overline{E} \setminus E \subset V$. Elements of $\mathcal{E}$ are called **edges**. The assertion of the theorem is that V and $\mathcal{E}$ are finite sets. We prove the theorem by induction on $d_1(X)$.

Suppose first that $d_1(X) = 0$. Let T be a connected component of $X \setminus B$. Then T is a connected network with no closed loops. Thus, T is a tree. As $d_1(X) = 0$, $d_1(\overline{T}) = 0$, and thus no two ends of

T can limit to the same point in B. Thus, T has at most $|B|$ ends. It follows that $\overline{T}$ is a finite tree. Now $\overline{T} \setminus T$ is a subset of the finite set B. Thus, if $X \setminus B$ had infinitely many components, then it would have two components T and T' with

$$\overline{T} \setminus T = \overline{T'} \setminus T'.$$

But then $T \cup T'$ would contain a loop, which is impossible because $d_1(X) = 0$. This completes the proof in the case $d_1(X) = 0$.

Now suppose that $d_1(X) > 0$. Then X contains a closed loop. Let $E \in \mathcal{E}$ be an edge in that loop. Then

$$d_1(X \setminus E) < d_1(X).$$

Let

$$B' = B \cup \partial E,$$

$$X' = X \setminus E.$$

Then X' and B' satisfy the hypotheses of the theorem and $d_1(X') < d_1(X)$, so (by induction), $V(X', B')$ and $\mathcal{E}(X', B')$ are finite. Consequently, $V(X, B)$ and $\mathcal{E}(X, B)$ are also finite. $\qquad\square$

The following counting theorem for arbitrary finite networks was used in the proof of Theorem 41 (the Slice Theorem).

Theorem 45 (Counting Theorem). *Let X be a finite network, and V_n be the set of vertices of valence n. Then*

$$\sum_{n \geqslant 3} \frac{1}{2}(n-2)|V_n| = \frac{1}{2}|V_1| + d_1(X) - d_0(X \setminus V_0),$$

where $d_i(\cdot) := \dim H_i(\cdot; \mathbf{Z}_2)$.

(Note that $d_0(X \setminus V_0)$ is the number of connected components of X that are not isolated points.)

Proof. Note that

$$\chi(X) = \sum_n \frac{1}{2}(2-n)|V_n| = |V_0| + \frac{1}{2}|V_1| + \frac{1}{2}\sum_{n \geqslant 3}(2-n)|V_n|$$

(by Lemma 14) and

$$\chi(X) = d_0(X) - d_1(X) = d_0(X \setminus V_0) + |V_0| - d_1(X).$$

The assertion follows immediately. $\qquad\square$

The Slice Theorem (Theorem 41) has the following important consequence:

Theorem 46. *Suppose that $I \subset \mathbf{R}$ is an open interval (possibly all of $\mathbf{R}$) and that $F : M \to I$ is a proper continuous function such that F is Radó on the interior of M. Suppose also that*

(1) $d_1(M) < \infty$,
(2) *for each t, $(\partial M) \cap \{F = t\}$ is finite,*
(3) *the set of interior local maxima and interior local minima of F is finite.*

Then F is a Radó function on all of M.

The Slice Theorem also implies an interesting removal-of-singularities theorem:

Theorem 47. *Suppose that $I \subset \mathbf{R}$ is an open interval (possibly all of $\mathbf{R}$), that $F : M \to I$ is a proper continuous function, and that $S \subset M \setminus \partial M$ is a finite set such that F is Radó on $M \setminus (S \cup \partial M)$. Suppose also that*

(1) $d_1(M) < \infty$,
(2) *for each t, $(\partial M) \cap \{F = t\}$ is finite,*
(3) *the set of interior local maxima and interior local minima of F is finite.*

Then F is a Radó function on all of M.

In minimal surface theory, one sometimes encounters functions that are Radó on the interior of the surface, but that are constant on some arcs and/or some connected components of the boundary. No such function can be Radó on the whole surface. The following theorem lets one get around that difficulty in many situations.

Theorem 48. *Suppose that $I \subset \mathbf{R}$ is an open interval (possibly all of $\mathbf{R}$) and that $F : M \to \mathbf{R}$ is a proper function that is Radó on the interior of M. Suppose also that*

(1) $d_1(M) < \infty$;
(2) *the set of interior local minima and interior local maxima of F is finite;*
(3) *there are only finitely many connected components of ∂M on which F is constant;*
(4) *for each t, $(\partial M) \cap F^{-1}(t)$ is the union of finitely many connected components.*

Define an equivalent relation $\sim$ on M as follows: $p \sim q$ if and only if $p = q$ or p and q belong to a connected subset of ∂M on which F is constant. Let $\tilde{M}$ be the quotient $M / \sim$ and let $\tilde{F}$ be the function on $\tilde{M}$ corresponding to F.

Then $\tilde{F} : \tilde{M} \to \mathbf{R}$ is a proper Radó function.

Proof. Let Γ be a connected component of $(\partial M) \cap F^{-1}(t)$. If Γ is a closed curve, then Γ becomes an interior point p of $\tilde{M}$. If Γ is not a closed curve, then Γ becomes a boundary point p of $\tilde{M}$. In either case, $\tilde{F}(p) = t$.

Now let S be the set of interior points in $\tilde{M}$ that correspond to closed curves in ∂M along which F is constant.

Then $\tilde{F} : \tilde{M} \to I$ and S satisfy the hypotheses of Theorem 47, so $\tilde{F}$ is a Radó function. □

9 | BRANCHED MINIMAL SURFACES

Theorem 49. *Suppose that N is a smooth Riemannian manifold and that $F : M \to \mathbf{R}$ is a continuous function such that*

(1) *each level set is a smooth minimal surface if $\dim N = 3$;*

(2) *each level set is totally geodesic if* $\dim N > 3$;

(3) *for each t, the level set $F^{-1}(t)$ is in the closure of $\{F > t\}$ and of $\{F < t\}$.*

Suppose that M is a connected surface without boundary, that $u : M \to N$ is a branched minimal immersion, and that $F \circ u$ is not constant. Then $F \circ u$ is a Radó function without local minima or local maxima.

Of course if M is minimal surface with boundary in N, then we can apply Theorem 49 to conclude that F is Radó on the interior of M, and then we can conclude from Theorem 46 that $F \circ u$ is Radó on all of M, provided the hypotheses of Theorem 46 are satisfied.

Proof for $n = 3$. As the result is local, it suffices to consider the case

$$u : \mathbf{B} \subset \mathbf{C} \to \mathbf{R}^3,$$

$$u(0) = 0,$$

where $\mathbf{R}^3$ is endowed with a Riemannian metric g such that $g_{ij}(0) = \delta_{ij}$ and $Dg_{ij}(0) = 0$. By rotating, we can assume that, after a nonconformal reparameterization,

$$u(z) = (z^Q, f(z)) \in \mathbf{C} \times \mathbf{R} \cong \mathbf{R}^3, \tag{21}$$

for some positive integer Q, where $f(z) = O(|z|^{Q+1})$. See [10, Theorem 1.4].

Now let Σ be the level set of F passing through the point $0 = u(0)$. If $\mathrm{Tan}(\Sigma, 0)$ is not the horizontal plane $\mathbf{R}^2 \times \{0\}$, then the desired behavior follows easily from (21). Indeed, in this case, minimality of Σ is not even needed. Note that in this case the valence of the point is $v(F \circ u, 0) = 2Q$ and thus $w(F \circ u, 0) = Q - 1$.

Now suppose that $\mathrm{Tan}(\Sigma, 0)$ is the horizontal plane $\mathbf{R}^2 \times \{0\}$. Then, near the origin, Σ is the graph of a function $\phi : \Omega \subset \mathbf{R}^2 \to \mathbf{R}$ with $\phi(0) = 0$ and $D\phi(0) = 0$.

Note that

$$z \mapsto (z^Q, \phi(z^Q))$$

is a nonconformal reparameterization of a branched minimal immersion. (If $Q > 1$, then 0 is a false branch point of order $Q - 1$).

Now consider the map

$$z \in D \mapsto f(z) - \phi(z^Q). \tag{22}$$

If this were identically 0, then $F \circ u$ would be constant on a neighborhood of 0 and thus on all of M by unique continuation, contrary to the hypotheses of the theorem. Thus, the function (22) is not constant. By [10, 1.6], there is a nonzero homogeneous polynomal h of degree $d \geqslant Q$ such that

$$f(z) - \phi(z^Q) = h(z) + o(|z|^d),$$

$$D(f(z) - \phi(z^Q)) = Dh(z) + o(|z|^{d-1}).$$

The desired behavior near 0 of the level set of $F \circ u$ through 0 follows immediately. Note that in this case, $v(F \circ u, 0) = 2d \geqslant 2Q$ and hence $w(F \circ u, 0) \geqslant Q - 1$. $\qquad \square$

Proof for $n > 3$. The result is local, so we may assume that the branched immersion is

$$u : D \subset \mathbf{R}^2 \to N.$$

We wish to prove that the level set of $F \circ u$ through the origin has the behavior specified in the definition of Radó function. Let Σ be the level set of F through $p = u(0)$. Choose Fermi-type local coordinates on N as follows. First, let $(y^1, \ldots, y^{n-1})$ be normal coordinates on Σ at p. For q in N near p, let $y^n(q)$ be the signed distance from q to Σ, and for $i < n$, let $y^i(q) = y^i(q')$ where q' is the point in Σ closest to q.

Thus,

$$g_{ni} = \delta_{ni} \quad (i \leqslant n),$$

$$g_{ij}(0) = \delta_{ij},$$

$$Dg_{ij}(0) = 0.$$

Now u is a conformal harmonic map. Harmonicity means that

$$\frac{\partial}{\partial x^k}\left(g_{i\alpha}(u(x))\frac{\partial u^i}{\partial x^k}\right) - (D_\alpha g_{ij}(u(x))\frac{\partial u^i}{\partial x^k}\frac{\partial u^j}{\partial x^k} = 0$$

for each $\alpha = 1, \ldots, n$. Here, k is summed from 1 to 2 and the other repeated indices from 1 to n. In particular, this holds for $\alpha = n$:

$$\begin{aligned}
0 &= \frac{\partial}{\partial x^k}\left(g_{in}(u(x))\frac{\partial u^i}{\partial x^k}\right) - (D_n g_{ij}(u(x))\frac{\partial u^i}{\partial x^k}\frac{\partial u^j}{\partial x^k} \\
&= \Delta u^n + (D_n g_{ij}(u(x))\frac{\partial u^i}{\partial x^k}\frac{\partial u^j}{\partial x^k},
\end{aligned} \tag{23}$$

as $g_{ni} \equiv \delta_{ni}$.

If i and/or j is n, then g_{ij} is constant, so $D_n g_{ij} \equiv 0$. On the other hand, if i and j are less than n, then

$$D_n g_{ij}(y) = 0 \quad \text{when } y^n = 0$$

as Σ is totally geodesic, and therefore

$$|D_n g_{ij}(y)| \leqslant c\,|y^n|$$

for $|y| \leqslant r$ and for some constant $c = c_r$. Thus, from (23), we see that

$$|\Delta u^n| \leqslant K|u^n|. \tag{24}$$

By hypothesis, $F \circ u$ is not constant. Thus, by (24) and the Hartman–Wintner Theorem (as formulated in [10, Theorem 1.1]), there is a nonzero homogeneous harmonic polynomial h of degree

$d \geqslant 1$ such that

$$u^n(z) = h(z) + o(|z|^d),$$

$$Du^n(z) = Dh(z) + o(|z|^{d-1}).$$

The assertion follows immediately.

Note that the valence $v(F \circ u, 0)$ is $2d$ and thus $w(F \circ u, 0) = d - 1$. We remark that if u has branch point of order $Q - 1$ at the origin, then

$$\lim_{z \to 0} \frac{|u(z)|}{|z|^Q}$$

exists and is nonzero. Thus, $d \leqslant Q$. Note that $d > Q$ if only if $u^n = 0(|u|) = o(|z|^Q)$, that is, if and only if the tangent plane to $u(D)$ at 0 is contained in $\mathrm{Tan}(\Sigma, u(0))$. $\qquad \square$

Here we summarize the facts about saddle-multiplicity and branch point order that were established in proving Theorem 49:

Theorem 50. *Suppose that $u : M \to N$ and $F : N \to \mathbf{R}$ are as in Theorem 49. If p is not a branch point, then $w(F \circ u, p)$ is the order of contact of $u(M)$ and $\{F = F(p)\}$ at $u(p)$ Now suppose that p is a branch point of order m. Then*

$$w(F \circ u, p) \geqslant m. \tag{25}$$

Equality holds if the tangent plane to $u(M)$ at $u(p)$ and the tangent plane to $\{F = F(u(p))\}$ at $u(p)$ are transverse.

In case the level sets of F are totally geodesic (as they are if $n > 3$), equality holds in (25) if and only if the tangent planes are transverse.

Corollary 51. *The function $F \circ u$ is a tame Radó function.*

Proof. Tameness is a local property of the regular points of a Radó function. As all branch points are critical points, tameness of $F \circ u$ follows from the embedded case (Theorem 8). $\qquad \square$

APPENDIX A: SOME BASIC TOPOLOGICAL FACTS

Lemma A1. *Suppose that C_i, $i = 1, \dots, k$ are disjoint simple closed curves in an open annulus U and that each C_i is homotopically nontrivial in U. Then each component of $U \setminus (\cup_i C_i)$ is an annulus.*

The proof is a simple induction.

Recall that if $F : \Sigma \to \mathbf{R}$ and if $s < t$, we let $\Sigma[s] = F^{-1}(s)$ and $\Sigma[s, t] = F^{-1}([s, t])$.

Proposition A2. *Let Σ be a compact, connected 2-manifold with boundary and let*

$$F : \Sigma \to [a, b]$$

be a continuous function such that

(1) *each level set $\Sigma[t]$ is a finite union of disjoint simple closed curves,*
(2) *F has no interior local maxima or minima,*
(3) *$\partial\Sigma = \Sigma[a] \cup \Sigma[b]$.*

Then Σ is an annulus.

Proof. Note that

$$\text{No collection of curves in } \Sigma[t] \text{ can bound a region in } \Sigma. \tag{*}$$

For if there were such a region D, then $F|\overline{D}$ would attain its maximum and/or its minimum at an interior point of D, violating [2].

Claim A3. Let $a \leqslant s < t \leqslant b$. Let K be a connected component of $\Sigma[s,t]$. If K lies in an annular region U of Σ, then K is an annulus, with one boundary component in $\Sigma[s]$ and one in $\Sigma[t]$.

Proof. Note that ∂K is contained in $\Sigma[s] \cup \Sigma[t]$. By (*), it must have at least one component in $\Sigma[s]$ and at least one component in $\Sigma[t]$. If C is a component of ∂K, it cannot bound a region in Σ by (*). In particular, it does not bound a disk in U. Thus, each component of ∂K is homotopically nontrivial in U. By Lemma A1, K is an annulus. Thus, we have proved Claim A3. □

Note for each $T \in [a,b]$, there is a relatively open subset U_T of Σ containing $\Sigma[T]$ such that U contains $\Sigma[T]$ and such that each component of U is an annulus. Now $F(\Sigma \setminus U)$ is a compact subset of $[a,b]$ that does not contain T. Thus, there is an open interval $I_T \subset \mathbf{R}$ containing T and disjoint from $F(\Sigma \setminus U_T)$.

As the $\{I_T : T \in [a,b]\}$ form an open cover of I, there exists $a_0 = a < a_1 < a_2 < \cdots < a_n = b$ such that each $[a_{i-1}, a_i]$ belongs to some I_T.

By Claim A3, each component of $\Sigma[a_{i-1}, a_i]$ is an annulus with one component in $\Sigma[a_{i-1}]$ and one component in $\Sigma[a_i]$. Proposition A2 follows immediately. □

Corollary A4. *Suppose that*

(1) *Σ is an open surface;*
(2) *$F : \Sigma \to (a,b)$ is a continuous function with no local maxima or local minima;*
(3) *for each $t \in (a,b)$, $\Sigma[t]$ is a union of finitely many disjoint simple closed curves;*
(4) *if $a < s < t < b$, then $\Sigma[s,t]$ is compact and*

$$\partial\Sigma[s,t] = \Sigma[s] \cup \Sigma[t].$$

Then each connected component of Σ is an annulus.

Proof. Let a_i, $i \in \mathbf{Z}$ be a strictly increasing sequence with $\lim_{i \to -\infty} a_i = a$ and $\lim_{i \to \infty} a_i = b$. By Proposition A2, each component of $\Sigma[a_{i-1}, a_i]$ is an annulus with one component in $\Sigma[a_{i-1}]$ and one component in $\Sigma[a_i]$. Corollary A4 follows immediately. □

Let $n \geqslant 3$. We define an n-**ad** K to be a closed set consisting of $(n + 1)$ points $p_0, p_1, \dots, p_n$, together with n embedded arcs $\gamma_1, \dots, \gamma_n$, where each γ_i joins p_0 to p_i, and where $\gamma_i \cap \gamma_j = \{p_0\}$ for $i \neq j$. We say that p_0 is the **center** of the n-ad, and that the points $p_1, \dots, p_n$ are the **boundary** ∂K of the n-ad.

Lemma A5. *Let M be a 2-manifold and let $n \geqslant 3$. Suppose C is a collection of disjoint subsets of M, each of which contains an n-ad. Then C is countable.*

Proof. Suppose to the contrary that there is an uncountable collection C. We can assume that each set in C is an n-ad. (Otherwise, replace each set in C by an n-ad that it contains.) Consider a countable collection $\mathcal{D}$ of open disks $D \subset M$ such that $\overline{D}$ is a closed disk and such that $\mathcal{D}$ is a basis for the topology of M. For $D \in \mathcal{D}$, let C_D be the collection of $K \in C$ such that the center of K is in D and such that the boundary points of K are not in D. Note that there is a $D \in \mathcal{D}$ for which C_D is uncountable. For each $K \in C_D$, let K' be the closure of $K \cap D$. Then K' is an n-ad with center in D and with boundary in ∂D. Let

$$C_D' = \{K' \,:\, K \in C_D\}.$$

Then C_D' is uncountable. Define $\Phi : C_D' \to (\partial D)^n$ by

$$\Phi(K) = (p_1, \dots, p_n),$$

where $p_1, \dots, p_n$ are the boundary points of K. (We choose an ordering of the endpoints.) As C_D' is uncountable and as $(\partial D)^n$ is separable, there exist K_i ($i \in N$) and K in C_D' such that $\Phi(K_i) \to \Phi(K)$. But that is impossible because each K_i lies in one of the connected components of $\overline{D} \setminus K$.

(If the last sentence is not clear, note that ∂K_i must lie in an arc A of $(\partial D) \setminus \partial K$. As ∂K has $n \geqslant 3$ points, it has a point p that is not in $\overline{A}$. As $\partial K_i \subset A$, the points of ∂K_i are bounded away from p.) $\qquad\square$

Corollary A6. *Suppose $F : M \to \mathbf{R}$ is a Rado function. Then there are only countably many critical points, and hence only countably many critical values.*

Proof. By doubling, it suffices to prove it for M without boundary. The points of valence 0 are strict local maxima or minima, and hence there are only countably many of them. The other critical points are points of valence $\geqslant 4$. By Lemma A5, for each $v \geqslant 4$, there are only countably many points of valence v. $\qquad\square$

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