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Relativistic equations with singular potentials

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Abstract. The first part of this paper concern with the study of the Lorentz force equation

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' = \overrightarrow{E}(t,q) + q' \times \overrightarrow{B}(t,q)$$

in the relevant physical configuration where the electric field $\overrightarrow{E}$ has a singularity in zero. By using Szulkin's critical point theory, we prove the existence of T-periodic solutions provided that T and the electric and magnetic fields interact properly. In the last part, we employ both a variational and a topological argument to prove that the scalar relativistic pendulum-type equation

$$\left(\frac{q'}{\sqrt{1-(q')^2}}\right)' + q = G'(q) + h(t),$$

admits at least a periodic solution when $h \in L^1(0,T)$ and G is singular at zero.

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1. Introduction

The main scope of this paper is to investigate the existence of T-periodic solutions of the relativistic Lorentz force equation

$$\left(\frac{q'}{\sqrt{1-|q'|^2}}\right)' = \overrightarrow{E}(t,q) + q' \times \overrightarrow{B}(t,q). \tag{1.1}$$

Here, $\overrightarrow{E}$ and $\overrightarrow{B}$ denote, respectively, the electric and magnetic fields and are given by

$$\overrightarrow{E} = -\nabla_q V - \frac{\partial W}{\partial t}, \qquad \overrightarrow{B} = \operatorname{curl}_q W, \tag{1.2}$$

where $V : [0,T] \times (\mathbb{R}^3 \backslash \{0\}) \to \mathbb{R}$ and $W : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$. By a solution of Eq. (1.1) we mean a function $q = (q_1, q_2, q_3) \in C^2$ satisfying (1.1) and such that $|q'(t)| < 1$ for all t.

Lorentz force equation (1.1) models the motion, in a relativistic regime, of a slowly accelerated charged particle under the influence of an electromagnetic field. The relativistic nature of Eq. (1.1) turns out in its left-hand side, which involves the relativistic momentum introduced by Poincaré in [11], with the velocity of light in the vacuum and the charge-to-mass ratio normalized to one, for simplicity. Instead, the presence of an electromagnetic field is emphasized by the Lorentz force $\overrightarrow{E}(t,q) + q' \times \overrightarrow{B}(t,q)$ in its right-hand side. It represents one of the most significant equation of Mathematical Physics (see e.g., [9]). Nonetheless, a rigorous mathematical variational approach for its study was developed only recently in

[3,4] (see also [5] for the case $\overrightarrow{B} \equiv 0$) where the maps V and W are assumed to be of class C^1, while the relevant cases including configurations of electric fields coming from the physical models and consisting of singular electric potential, had remained open.

An early result concerning these singular models was achieved recently in [10] via a topological method. In this work, the authors consider the case that the electric field $\overrightarrow{E}$ is a sufficiently large L^1-pertubation of a Coulomb electric potential, or more specifically, $\overrightarrow{E}(t,q) = -\nabla V(q) - h(t)$ with $h \in L^1([0,T],\mathbb{R}^3)$. The potential V is assumed to be singular at zero and, for some $\gamma \geq 1$, $c > 0$, satisfies the inequality $q \cdot \nabla V(q) \leq -c/|q|^\gamma$ if $|q|$ is small enough. On the other hand, the magnetic field $\overrightarrow{B}$ is supposed to be bounded with a singularity at zero of lower order than the singularity $|q|^{-\gamma-1}$. Applying a global continuation theorem, the existence of a T periodic solution is guaranteed only when the mean value $\overline{h}$ of h is greater than the supremum of $C(t) := \limsup_{|q|\to\infty} |\overrightarrow{B}(t,q)|$. In particular, this result clearly fails if e.g., h is identically zero. We emphasize in addition that the approach to the singular problem using variational methods had still maintained open.

The aim of this paper is to fill the observed gaps by developing the variational framework needed to address Eq. (1.1) and other relativistic singular problems as well, establishing the landmark for future investigations related on these topics. To be more precise, we show not only that Eq. (1.1) can be studied using a variational approach even when the electric field $\overrightarrow{E}$ is singular, widening the range of possible choices of V and covering the case untreated in [3,4]; but also, we prove that the topological argument carried on in [10] can be employed in such a way to handle other kinds of relativistic problems in which appears a singular term.

As a first step to study (1.1) variationally, we derive a new version of the Mountain Pass Theorem, which has its own interest and allows one to identify critical points of functionals which possess singularities (see Theorem 2.1). Our abstract result relies upon the idea developed in [1] to address the study of a relativistic spherical pendulum. In particular, it will be essential to impose for the action functional $\mathcal{I}$ that $\mathcal{I}(q_n)$ blows up when $\{q_n\}$ converges uniformly to a function which *"touches"* the singular set of $\mathcal{I}$ (see (2.1) below and compare with Lemma 5.1 in [1]). On this regard, we thank the anonymous referee who brought to our attention the paper [6], in which this condition is also used to provide the existence of solutions for another type of relativistic singular problem, namely, a relativistic Keplerian problem in the plane. Thus, by using our abstract result, we derive the existence of a T-periodic solution for Eq. (1.1). To be more precise, we assume that V is dominated by the function $-c/|q|$ (c is a positive constant) when q is located in a neighborhood of the origin, while the sum of the magnitudes of $V(t,q)$ and $\nabla_q V(t,q)$ tend to 0 uniformly in $t \in [0,T]$ when q approaches infinity. Also, we suppose that W is bounded, its modulus and the sum of the magnitudes of the components of its gradient at q converge to 0 uniformly in t when q goes to infinity. Thus, if also there exists $c_0 > 0$ such that

$$\frac{\pi^2}{2T^2}|q|^2 - |W(t,q)| - V(t,q) \geq c_0 \quad \forall q \in \mathbb{R}^3 \setminus \{0\},$$

we prove that Eq. (1.1) admits a T-periodic solution. Observe that the periodic solution provided by the former result could be trivial provided that V and W depends only on the variable q , V is of class C^2 in $\mathbb{R}^3 \setminus \{0\}$ and there exists $\xi \in \mathbb{R}^3 \setminus \{0\}$ such that $V(\xi) < 0$, $\nabla V(\xi) = 0$. In this case, we also prove that if the matrix

$$\left(2\frac{\partial W_j}{\partial q_i}(\xi) - \frac{\partial^2 V}{\partial q_i \partial q_j}(\xi)\right)_{i,j=1,2,3}$$

is positive definite, then Eq. (1.1) has a periodic solution which is different from the constant solution ξ. Anyway, in order to not weigh this introduction down with too many details, we prefer to specify each hypothesis and to state our main results in Sect. 3.

The remaining part of the paper is motivated by the study of the spherical pendulum in [1]. We study the existence of periodic Lipschitz solutions $q(t) \in \mathbb{R}$ of the scalar relativistic pendulum-type equation

$$\left(\frac{q'}{\sqrt{1-(q')^2}}\right)' + q = G'(q) + h(t), \tag{1.3}$$

where $h \in L^1(0,T)$, the singular function G dominates the function $1/|q|$ as $q \sim 0$ and its first derivative is bounded when q is far away from the singularity at 0. It is worth noting that, unlike (1.1), Eq. (1.3) identifies a scalar problem. Nonetheless, as Eqs. (1.1) and (1.3) share the same relativistic part, we employ again Theorem 2.1 to derive the existence of a T-periodic solution of (1.3). To be more precise, we improve the results in [1] and derive new existence results both in the case $h \not\equiv 0$ (see Theorem 4.6) and $h \equiv 0$ (see Theorem 4.8). Finally, we use the global continuation theorem to address Eq. (1.3) but assuming that there exists $c_0 > 0$ such that $G'(q)q$ is bounded from below by the function $c_0/|q|$ when q is in a neighborhood of the origin. In fact, establishing a suitable homotopic system that drives the original problem into an autonomous system, we compute the Brouwer degree of the vector field and use [7, Theorem 2] to infer the existence of a T-periodic solution for the Eq. (1.3). We point out that, in contrast with [10, Theorem 1], our existence result does not require any hypothesis on the mean value of the function h.

Our paper is organized as follows. In Sect. 2 we present the required version of Mountain Pass Theorem for non-smooth functionals involving singularities. In Sect. 3 we introduce the variational setting and apply our abstract result to the Lorentz force equation with a singular electric field, providing the claimed existence theorems. Finally, in Sect. 4 we study the relativistic pendulum type equation and derive our main results both using variational methods (Sect. 4.1) and topological techniques (Sect. 4.2).

2. Local mountain pass for singular non-smooth functionals

In [3] a Mountain Pass Theorem without compactness conditions is given for the Szulkin critical point theory [13]. We give here a generalization of it which will be useful to handle functionals having singularities.

Theorem 2.1. *Assume that Λ is an open subset of a Banach space E and that the functional $\mathcal{I}$ is the sum of two functionals $\mathcal{I} = \Psi + \mathcal{F}$ where*

(i) $\Psi : E \to (-\infty, +\infty]$ is a convex and proper functional with a closed domain $Dom\,\Psi := \{v \in E : \Psi(v) < \infty\}$ in E and Ψ is continuous in $Dom\,\Psi$.

(ii) $\mathcal{F} : \Lambda \to \mathbb{R}$ is a C^1-functional.

Assume also that

$$\lim_{n\to\infty} \mathcal{I}(q_n) = +\infty, \tag{2.1}$$

for every sequence $\{q_n\} \subset \Lambda$ whose distance $dist(q_n, E\backslash\Lambda)$ is converging to zero.

Let also K be a compact metric space, $K_0 \subset K$ a closed subset and $\gamma_0 : K_0 \to \Lambda$ a continuous map. Consider the set

$$\Gamma_\Lambda = \{\gamma : K \to \Lambda\, \gamma \text{ is continuous and } \gamma|_{K_0} = \gamma_0\}.$$

If

$$c_1 := \sup_{t\in K_0} \mathcal{I}(\gamma_0(t)) < c := \inf_{\gamma\in\Gamma_\Lambda} \sup_{t\in K} \mathcal{I}(\gamma(t)) < \infty, \tag{2.2}$$

then, for every $\varepsilon > 0$ and $\gamma \in \Gamma_\Lambda$ such that

$$c \le \max_{t\in K} \mathcal{I}(\gamma(t)) \le c + \frac{\varepsilon}{2}, \tag{2.3}$$

there exist $\overline{\gamma}_\varepsilon \in \Gamma_\Lambda$ *and* $q_\varepsilon \in \overline{\gamma}_\varepsilon(K) \subset E$ *satisfying*

$$c \le \max_{t\in K} \mathcal{I}(\overline{\gamma}_\varepsilon(t)) \le \max_{t\in K} \mathcal{I}(\gamma(t)) \le c + \frac{\varepsilon}{2},$$
$$\max_{t\in K} \|\overline{\gamma}_\varepsilon(t) - \gamma(t)\| \le \sqrt{\varepsilon},$$
$$c - \varepsilon \le \mathcal{I}(q_\varepsilon) \le c + \frac{\varepsilon}{2},$$

and

$$\Psi(\varphi) - \Psi(q_\varepsilon) + \mathcal{F}'(q_\varepsilon)[\varphi - q_\varepsilon] \ge -\sqrt{\varepsilon}\|\varphi - q_\varepsilon\| \quad \textit{for all } \varphi \in E.$$

Remark 2.2. Notice that the continuity of Ψ in its closed domain implies that Ψ is lower semicontinuous in E.

Proof. Let Γ be defined as

$$\Gamma = \{\gamma : K \to E \ \gamma \text{ is continuous and } \gamma|_{K_0} = \gamma_0\},$$

which is a complete metric space endowed with the uniform distance

$$\mathrm{d}_\Gamma(\gamma_1, \gamma_2) = \max_{t\in K} \|\gamma_1(t) - \gamma_2(t)\|, \qquad (\gamma_1, \gamma_2 \in \Gamma).$$

Since Λ is open in E and K is compact, the set $\Gamma_\Lambda = \{\gamma \in \Gamma : \gamma(t) \in \Lambda\}$ is open in Γ. Consider the functional $\Upsilon : \Gamma_\Lambda \to (-\infty, +\infty]$ given by

$$\Upsilon(\gamma) = \sup_{t\in K} \mathcal{I}(\gamma(t)), \qquad (\gamma \in \Gamma_\Lambda).$$

Observe that every γ in the domain of Υ, that is, verifying $\Upsilon(\gamma) < +\infty$, satisfies that $\gamma(t) \in \mathrm{Dom}\,\Psi$ for every $t \in K$. Hence, the continuity of Ψ in its closed domain implies that $\mathcal{I} \circ \gamma$ is continuous in the compact K and we have

$$\Upsilon(\gamma) = \max_{t\in K} \mathcal{I}(\gamma(t)).$$

By (2.2) the functional Υ is proper and bounded from below by c_1. Fix $\varepsilon > 0$ that, without loss of generality, can be assumed less than $c - c_1$. Let $\gamma \in \Gamma_\Lambda$ satisfying (2.3). For $0 < \mu < \mu_0 := \mathrm{d}_\Gamma(\gamma, \Gamma \backslash \Gamma_\Lambda)$, we consider the set

$$\mathcal{N}_\mu := \{\eta \in \Gamma : \ \mathrm{d}_\Gamma(\eta, \Gamma\backslash\Gamma_\Lambda) \ge \mu\},$$

which clearly contains γ. By (2.2) and (2.3) we have

$$c \le c_{\delta_0} := \inf_{\overline{\gamma}\in\mathcal{N}_\mu} \sup_{t\in K} \mathcal{I}(\overline{\gamma}(t)) \le \max_{t\in K} \mathcal{I}(\gamma(t)) \le c + \frac{\varepsilon}{2},$$

i.e.,

$$c \le c_{\delta_0} := \inf_{\overline{\gamma}\in\mathcal{N}_\mu} \Upsilon(\overline{\gamma}) \le \Upsilon(\gamma) \le c + \frac{\varepsilon}{2},$$

Taking into account that $\mathcal{N}_\mu$ is closed in the complete metric space Γ, we obtain that it is also a complete metric space. In addition, Υ is lower semicontinuous in $\mathcal{N}_\mu$ by [13, Lemma 3.1]. Applying the Ekeland variational principle [8], we deduce from the above inequality that there exists $\overline{\gamma}_{\varepsilon,\mu} \in \mathcal{N}_\mu$ satisfying

$$c \le c_{\delta_0} \le \Upsilon(\overline{\gamma}_{\varepsilon,\mu}) \le \Upsilon(\gamma) \le c + \frac{\varepsilon}{2},$$
$$\mathrm{d}_\Gamma(\overline{\gamma}_{\varepsilon,\mu}, \gamma) = \max_{t\in K} \|\overline{\gamma}_{\varepsilon,\mu}(t) - \gamma(t)\| \le \sqrt{\varepsilon}, \tag{2.4}$$

and

$$\Upsilon(\overline{\gamma}_{\varepsilon,\mu}) < \Upsilon(\vartheta) + \sqrt{\varepsilon}\,\mathrm{d}_\Gamma(\overline{\gamma}_{\varepsilon,\mu}, \vartheta) \quad \text{for all } \vartheta \in \mathcal{N}_\mu. \tag{2.5}$$

We claim that there exists $\mu \in (0, \mu_0)$ such that $\overline{\gamma}_{\varepsilon,\mu} \in \overset{\circ}{\mathcal{N}}_\mu$ (i.e., $\mathrm{d}_\Gamma(\overline{\gamma}_{\varepsilon,\mu}, \Gamma\backslash\Gamma_\Lambda) < \mu$). Indeed, assume by contradiction that

$$\mathrm{d}_\Gamma(\overline{\gamma}_{\varepsilon,\mu}, \Gamma\backslash\Gamma_\Lambda) = \mu, \qquad \forall \mu \in (0, \mu_0).$$

In this case, it is possible to choose sequences $\{\mu_n\} \subset (0, \mu_0)$ and $\{\eta_n\} \subset \Gamma\backslash\Gamma_\Lambda$ satisfying

$$\mathrm{d}_\Gamma(\overline{\gamma}_{\varepsilon,\mu_n}, \eta_n) = \max_{t\in K} \|\overline{\gamma}_{\varepsilon,\mu_n}(t) - \eta_n(t)\| = \mu_n \overset{(n\to\infty)}{\longrightarrow} 0.$$

Since $\eta_n \in \Gamma\backslash\Gamma_\Lambda$, there exists $t_n \in K$ such that $\eta_n(t_n) \in E\backslash\Lambda$ and using that $\|\overline{\gamma}_{\varepsilon,\mu_n}(t_n) - \eta_n(t_n)\| \leq \mathrm{d}_\Gamma(\overline{\gamma}_{\varepsilon,\mu_n}, \eta_n)$, we infer from the above convergence to zero that

$$\lim_{n\to\infty} \|\overline{\gamma}_{\varepsilon,\mu_n}(t_n) - \eta_n(t_n)\| = 0,$$

and by assumption (2.1) that

$$\lim_{n\to\infty} \mathcal{I}(\overline{\gamma}_{\varepsilon,\mu_n}(t_n)) = +\infty.$$

By the inequality $\mathcal{I}(\overline{\gamma}_{\varepsilon,\mu_n}(t_n)) \leq \max_{t\in K} \mathcal{I}(\overline{\gamma}_{\varepsilon,\mu_n}(t)) = \Upsilon(\overline{\gamma}_{\varepsilon,\mu_n})$, it follows that

$$\lim_{n\to\infty} \Upsilon(\overline{\gamma}_{\varepsilon,\mu_n}) = +\infty,$$

contradicting (2.4). The claim has been proved and thus there exists $\mu \in (0, \mu_0)$ such that $\overline{\gamma}_{\varepsilon,\mu} \in \overset{\circ}{\mathcal{N}}_\mu$. In the sequel we fix this constant μ and we denote $\overline{\gamma}_{\varepsilon,\mu} = \overline{\gamma}_\varepsilon$. We conclude the proof by showing the existence of $t_\varepsilon \in \mathcal{T} := \{t \in K : \mathcal{I}(\overline{\gamma}_\varepsilon(t)) \geq c - \varepsilon\}$ such that, if $q_\varepsilon = \overline{\gamma}_\varepsilon(t_\varepsilon)$, then

$$\Psi(\varphi) - \Psi(q_\varepsilon) + \mathcal{F}'(q_\varepsilon)[\varphi - q_\varepsilon] \geq -\sqrt{\varepsilon}\|\varphi - q_\varepsilon\| \quad \text{for all } \varphi \in E.$$

Indeed, assume by contradiction that for every $t \in \mathcal{T}$ there exists $\varphi_t \in E\backslash\{\overline{\gamma}_\varepsilon(t)\}$ such that

$$\Psi(\varphi_t) - \Psi(\overline{\gamma}_\varepsilon(t)) + \mathcal{F}'(\overline{\gamma}_\varepsilon(t))[\varphi_t - \overline{\gamma}_\varepsilon(t)] < -\sqrt{\varepsilon}\|\varphi_t - \overline{\gamma}_\varepsilon(t)\|.$$

We can repeat the argument in the proof of Theorem 1 in Section 2 of [3] to deduce for every sufficiently small $\delta > 0$ the existence of $\gamma^* \in \Gamma$ such that

$$\mathrm{d}_\Gamma(\gamma^*, \overline{\gamma}_\varepsilon) \leq \delta,$$

and

$$\Upsilon(\gamma^*) < \Upsilon(\overline{\gamma}_\varepsilon) - \delta\sqrt{\varepsilon} \leq \Upsilon(\overline{\gamma}_\varepsilon) - \mathrm{d}_\Gamma(\gamma^*, \overline{\gamma}_\varepsilon)\sqrt{\varepsilon}.$$

The first inequality allows to choose $\delta > 0$ such that $\gamma^* \in \mathcal{N}_\mu$ (remind that $\overline{\gamma}_\varepsilon \in \overset{\circ}{\mathcal{N}}_\mu$) and then the second inequality contradicts (2.5) and completes the proof. □

3. The relativistic Lorentz force equation

Consider the relativistic Lorentz force equation (1.1) when the electric and magnetic fields, respectively $\overrightarrow{E}$ and $\overrightarrow{B}$, are given by (1.2), with $V : [0, T] \times (\mathbb{R}^3\backslash\{0\}) \to \mathbb{R}$ and $W : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ two C^1-functions. In order to study it, we denote by $W^{1,\infty}(0, T)$ the space of all Lipschitz functions in $[0, T]$ (or equivalently the absolutely continuous functions in $[0, T]$ with bounded derivatives) and we consider the Banach space

$$W^{1,\infty} = [W^{1,\infty}(0, T)]^3$$

endowed with the norm $\|\cdot\|$ defined from the usual norm $\|\cdot\|_\infty$ of $[L^\infty(0, T)]^3$ as

$$\|q\| = \|q\|_\infty + \|q'\|_\infty \quad (q \in W^{1,\infty}).$$

We consider also the subspace E of all T-periodic vector functions $q \in W^{1,\infty}$ (i.e. $q \in W^{1,\infty}$ such that $q(0) = q(T)$). Let also $\mathcal{K}$ be the convex and closed set given by

$$\mathcal{K} = \{q \in E : \|q'\|_\infty \le 1\},$$

and

$$\Lambda = \{q \in E : q(t) \neq 0, \ \forall t \in [0, T]\}, \quad \mathcal{K}_\Lambda = \mathcal{K} \cap \Lambda.$$

Following [3] the Lagrangian action $\mathcal{I} : \Lambda \to (-\infty, +\infty]$ associated to the problem of the existence of T-periodic solutions of the Lorentz force equation (1.1) is given by

$$\mathcal{I}(q) = \Psi(q) + \mathcal{F}(q), \ q \in \Lambda.$$

where the functionals Ψ and $\mathcal{F}$ are defined by

$$\Psi(q) = \begin{cases} \int_0^T [1 - \sqrt{1 - |q'|^2}]\mathrm{d}t, & \text{if } q \in \mathcal{K}, \\ +\infty, & \text{if } q \in E \backslash \mathcal{K}, \end{cases}$$

$$\mathcal{F}(q) := \int_0^T [q' \cdot W(t, q) - V(t, q)]\mathrm{d}t, \ \forall q \in \Lambda.$$

Since Ψ is a proper convex function which is continuous in its domain $\mathcal{K}$ (similar proof to that of Lemma 2 in Section 3 of [3]) and $\mathcal{F}$ is a function of class C^1 in $\mathcal{K}_\Lambda$, Szulkin's critical point theory from [13] is applicable for $\mathcal{I}$. Recall what is it understood by a critical point in this theory.

Definition 3.1. A function $q \in \mathcal{K}_\Lambda$ is a **critical point** of $\mathcal{I}$ if

$$\Psi(\varphi) - \Psi(q) + \mathcal{F}'(q)[\varphi - q] \ge 0 \quad \text{for all } \varphi \in \mathcal{K}_\Lambda,$$

or, equivalently, for $\mathcal{E} : [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\mathcal{E}(t, q, p) = (p \cdot D_{q_1} W(t, q), p \cdot D_{q_2} W(t, q), p \cdot D_{q_3} W(t, q)), \ \forall (t, q, p) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3,$$

if

$$\int_0^T [\sqrt{1 - |q'|^2} - \sqrt{1 - |\varphi'|^2}]\mathrm{d}t + \int_0^T [\mathcal{E}(t, q, q') - \nabla_q V(t, q)] \cdot (\varphi - q)\mathrm{d}t$$
$$+ \int_0^T W(t, q) \cdot (\varphi' - q')\mathrm{d}t \ge 0, \quad \text{for all } \varphi \in \mathcal{K}_\Lambda.$$

By a similar argument to this one in Theorem 2 of Section 3 in [3], we have that the critical points $q \in \mathcal{K}_\Lambda$ of $\mathcal{I}$ are just the T-periodic solutions of (1.1).

The following lemma will be essential to control the singularity of V at $q = 0$.

Lemma 3.2. *Assume that W is bounded and V satisfies the following hypothesis:* (V_0) *There exists $c_0 > 0$ such that*

$$\limsup_{q \to 0} \ V(t, q)|q| \le -c_0 < 0.$$

If a sequence $\{q_n\}$ in $\mathcal{K}_\Lambda$ converges uniformly to some q which vanishes at some point of $[0, T]$, then

$$\lim_{n \to \infty} \int_0^T V(t, q_n)\, dt = -\infty \quad \text{and} \quad \lim_{n \to \infty} \mathcal{I}(q_n) = +\infty.$$

Remark 3.3. A sufficient condition for (V_0) is that there exist $\alpha > 1$ and $c_1 > 0$ such that $\limsup_{q\to 0} V(t,q)|q|^\alpha \leq -c_1 < 0$.

Proof. Since $\{q_n\} \subset \mathcal{K}_\Lambda$ is bounded in $C([0,T],\mathbb{R})$ and W is bounded, the first and second integrals of

$$\mathcal{I}(q_n) = \int_0^T [1 - \sqrt{1-|q_n'|^2}]\mathrm{d}t + \int_0^T W(t,q_n)\cdot \widetilde{q}_n'\mathrm{d}t - \int_0^T V(t,q_n)\mathrm{d}t$$

are bounded. Hence to prove the lemma, it suffices to show that $\int_0^T V(t,q_n)\mathrm{d}t$ converges to $-\infty$.

By the hypothesis (V_0), there exists $\varepsilon_0 > 0$ such that

$$V(t,q) \leq -\frac{c_0}{|q|} \text{ for } 0 < |q| \leq \varepsilon_0.$$

Let $t_0 \in [0,T]$ be such that $q(t_0) = 0$. Two cases can occur: either $q \equiv 0$, or (up to a change of the zero t_0 by other zero) we can assume that there exists $t_1 \in (t_0, T]$ such that $q(t_0) = 0 < |q(t)| \leq \varepsilon_0$, for every $t \in (t_0, t_1]$.

In the first case, $q \equiv 0$, we have $|q_n(t)| \leq \varepsilon_0$ for every $t \in [0,T]$ provided that n is large enough. Thus, the above hypothesis implies

$$\int_0^T V(t,q_n)\,\mathrm{d}t \leq -\int_0^T \frac{c_0}{|q_n|}\,\mathrm{d}t, \text{ for } n \text{ large enough.}$$

By Fatou lemma, we deduce

$$\limsup_{n\to\infty}\int_0^T V(t,q_n)\,\mathrm{d}t \leq -\liminf_{n\to\infty}\int_0^T \frac{c_0}{|q_n|}\,\mathrm{d}t \leq -\int_0^T \liminf_{n\to\infty}\frac{c_0}{|q_n|}\,\mathrm{d}t = -\infty,$$

which shows that

$$\lim_{n\to\infty}\int_0^T V(t,q_n)\,\mathrm{d}t = -\infty$$

and the lemma is proved in this case.

In the second case, observe that

$$\int_0^T V(t,q_n)\,\mathrm{d}t = \left\{\int_{|q_n|\leq\varepsilon_0} + \int_{|q_n|>\varepsilon_0}\right\} V(t,q_n)\,\mathrm{d}t \leq -\int_{|q_n|\leq\varepsilon_0}\frac{c_0}{|q_n|}\,\mathrm{d}t + \int_{|q_n|>\varepsilon_0} V(t,q_n)\,\mathrm{d}t.$$

Using $V(t,\xi)$ is bounded from above for $t \in [0,T]$ and $\varepsilon_0 < |\xi| \leq \sup_n \|q_n\|_\infty < \infty$, we have

$$\int_{|q_n|>\varepsilon_0} V(t,q_n)\,\mathrm{d}t$$

is also bounded from above for all n. Therefore, to conclude the proof it suffices to show that

$$\lim_{n\to\infty}\int_{|q_n|\leq\varepsilon_0}\frac{1}{|q_n|}\,\mathrm{d}t = \infty.$$

In order to prove it, since $\|q_n'\|_\infty \le 1$, we note that

$$\int\limits_{|q_n|\le\varepsilon_0} \frac{1}{|q_n|}\,\mathrm{d}t \ge \int\limits_{t_0}^{t_1} \frac{1}{|q_n|}\,\mathrm{d}t \ge \int\limits_{t_0}^{t_1} \frac{q_n q_n'}{|q_n|^2}\,\mathrm{d}t = \log|q_n(t_1)| - \log|q_n(t_0)|.$$

Consequently, using that $q_n(t_1)$ converges to $q(t_1) \neq 0$ and $q_n(t_0)$ to $q(t_0) = 0$, we deduce that

$$\lim_{n\to\infty} \int\limits_{|q_n|\le\varepsilon_0} \frac{1}{|q_n|}\,\mathrm{d}t \ge \lim_{n\to\infty} \log|q_n(t_1)| - \log|q_n(t_0)| = \infty,$$

and the proof is concluded. □

Remark 3.4. Observe that, as a consequence of the above lemma, the functional $\mathcal{I}$ satisfies the condition (2.1) of Theorem 2.1. Indeed, let $\{q_n\} \subset E$ be a sequence satisfying $\lim_{n\to\infty} \mathrm{dist}\,(q_n, E\backslash\Lambda) = 0$ and assume by contradiction that condition (2.1) does not hold true. Then, up to a subsequence, we can suppose that $\{\mathcal{I}(q_n)\}$ is bounded from above. In particular, $q_n \in K_\Lambda$. By using this and choosing $p_n \in E\backslash\Lambda$ such that $\|q_n - p_n\| = \mathrm{dist}\,(q_n, E\backslash\Lambda)$ we get the uniform boundedness of p_n' in $[0, T]$, which together to the fact that each p_n vanishes at some point in $[0, T]$, implies that the sequence $\{p_n\}$ is bounded in E. By the compact embedding of E into the continuous functions in $[0, T]$ we can extract a subsequence $\{p_{n_k}\}$ which uniformly converges in $[0, T]$ to some function q which vanishes at some point of $[0, T]$. The convergence to zero of $\|q_n - p_n\|$ gives also the uniform convergence of $\{q_{n_k}\}$ to q in $[0, T]$. By the previous lemma we obtain the convergence of $\{\mathcal{I}(q_{n_k})\}$ to infinity contradicting the boundedness from above of the sequence $\{\mathcal{I}(q_n)\}$.

In the incoming results we will use this direct sum decomposition

$$E = \overline{E} \oplus \widetilde{E}, \quad q = \overline{q} + \widetilde{q}, \quad \overline{q} = \frac{1}{T}\int\limits_0^T q\,\mathrm{d}t, \quad \widetilde{q} = q - \overline{q}.$$

Lemma 3.5. (Palais–Smale condition) *Assume in addition to* (V_0) *that* V *and* W *satisfy also [*(V_∞)*]* $\lim\limits_{|q|\to\infty} |V(t,q)| + |\nabla_q V(t,q)| = 0$ *uniformly in* $t \in [0,T]$, (W_∞) $\lim\limits_{|q|\to\infty} |W(t,q)| + |D_{q_1}W(t,q)| + |D_{q_2}W(t,q)| + |D_{q_3}W(t,q)| = 0$ *uniformly in* $t \in [0,T]$*. If* $\{\varepsilon_n\}$ *is a sequence of positive numbers converging to zero and* $\{q_n\}$ *is a sequence in* $\mathcal{K}_\Lambda$ *satisfying that*

$$\lim_{n\to\infty} \mathcal{I}(q_n) = c \in \mathbb{R}\backslash\{0\} \tag{3.1}$$

and

$$\int\limits_0^T [\sqrt{1-|q_n'|^2} - \sqrt{1-|\varphi'|^2}]\,dt + \int\limits_0^T [\mathcal{E}(t,q_n,q_n') - \nabla_q V(t,q_n)]\cdot(\varphi - q_n)\,dt$$
$$+ \int\limits_0^T W(t,q_n)\cdot(\varphi' - q_n')\,dt \ge -\varepsilon_n\|\varphi - q_n\|, \quad \forall \varphi \in \mathcal{K},\backslash\{0\}, \tag{3.2}$$

then there exists a subsequence $\{q_{n_k}\}$ *of* $\{q_n\}$ *converging in* $C([0,T],\mathbb{R}^3)$ *to a critical point* $q \in \mathcal{K}_\Lambda$ *of* $\mathcal{I}$ *with level* $\mathcal{I}(q) = c$*.*

Proof. Let $\{q_n\}$ be a sequence satisfying (3.1) and (3.2). Since $q_n = \overline{q}_n + \widetilde{q}_n$ with $\|\widetilde{q}_n\| = \|\widetilde{q}_n\|_\infty + \|\widetilde{q}_n'\|_\infty = \|\widetilde{q}_n\|_\infty + \|q_n'\|_\infty \le T+1$, to deduce the boundedness of $\{q_n\}$ in E it suffices to show that $\{\overline{q}_n\}$ is bounded.

Suppose by contradiction that, up to a subsequence, $|\overline{q}_n|$ converges to infinity. Choosing $\varphi = \overline{q}_n$ in (3.2) we obtain

$$\int_0^T [\sqrt{1-|q_n'|^2} - 1]\mathrm{d}t - \int_0^T [\mathcal{E}(t, q_n, q_n') - \nabla_q V(t, q_n)] \cdot \widetilde{q}_n \mathrm{d}t$$
$$- \int_0^T W(t, q_n) \cdot \widetilde{q}_n' \mathrm{d}t \geq -\varepsilon_n \| \, \widetilde{q}_n \| \geq -\varepsilon_n (T+1).$$

Since $|q_n(t)| = |\overline{q}_n + \widetilde{q}_n(t)|$ converges to infinity,

$$\begin{aligned} |[\mathcal{E}(t, q_n, q_n') - \nabla_q V(t, q_n)] \cdot \widetilde{q}_n| &\leq T|\mathcal{E}(t, q_n, q_n')| + |\nabla_q V(t, q_n)| \\ &\leq T\left[|D_{q_1} W(t, q_n)| + |D_{q_2} W(t, q_n)| + |D_{q_3} W(t, q_n)|\right] + |\nabla_q V(t, q_n)|, \end{aligned}$$

and

$$|W(t, q_n) \cdot \widetilde{q}_n'| \leq |W(t, q_n)|,$$

by hypotheses (V_∞) and (W_∞), we have

$$\lim_{n\to\infty} \int_0^T [\mathcal{E}(t, q_n, q_n') - \nabla_q V(t, q_n)] \cdot \widetilde{q}_n \mathrm{d}t + \int_0^T W(t, q_n) \cdot \widetilde{q}_n' \mathrm{d}t = 0.$$

Thus,

$$\limsup_{n\to\infty} \int_0^T [1 - \sqrt{1-|q_n'|^2}]\mathrm{d}t \leq \limsup_{n\to\infty} \varepsilon_n (T+1) = 0;$$

which means (by the positiveness of the function $1 - \sqrt{1-|s|^2}$) that

$$\lim_{n\to\infty} \int_0^T [1 - \sqrt{1-|q_n'|^2}]\mathrm{d}t = 0.$$

Therefore, again by (V_∞) and (W_∞) we would obtain from (3.1) that

$$c = \lim_{n\to\infty} \mathcal{I}(q_n) = \lim_{n\to\infty} \int_0^T [1 - \sqrt{1-|q_n'|^2}]\mathrm{d}t + \int_0^T W(t, q_n) \cdot \widetilde{q}_n' \mathrm{d}t - \int_0^T V(t, q_n)\mathrm{d}t = 0,$$

a contradiction proving that the sequence $\{\overline{q}_n\}$ is necessarily bounded.

By the compact embedding of E into $C([0,T], \mathbb{R})$ we can assume, up to subsequences, that $q_n(t) \to q(t)$ uniformly in $[0,T]$. Since each q_n is Lipschitz with Lipschitz constant equal $\|q_n\|_\infty \leq 1$, we deduce that q is also Lipschitz with Lipschitz constant smaller or equal to one; i.e., $q \in \mathcal{K}$. By Lemma 3.2 and (3.1), we have

$$q(t) \neq 0, \quad \forall t \in [0,T],$$

concluding the proof. □

Theorem 3.6. *Assume the hypotheses* (V_0), (V_∞) *and* (W_∞). *If there exists* $c_0 > 0$ *such that*

$$\frac{\pi^2}{2T^2}|q|^2 - |W(t,q)| - V(t,q) \geq c_0, \quad \forall q \in \mathbb{R}^3, /\{0\} \tag{3.3}$$

then there exists a T*-periodic solution of the Lorentz force equation* (1.1).

Proof. Recalling that $\frac{\pi^2}{T^2}$ is the second eigenvalue of the periodic problem associated to the operator $-q''(t)$, we deduce by its variational characterization that

$$\int_0^T |\widetilde{q}'|^2 \, \mathrm{d}t \geq \frac{\pi^2}{T^2} \int_0^T |\widetilde{q}|^2 \, \mathrm{d}t, \quad \forall \widetilde{q} \in \widetilde{E}.$$

Thus, using this and the inequality $1-\sqrt{1-|p|^2} \geq \frac{1}{2}|p|^2$ for every $|p| \leq 1$, we obtain for every $\widetilde{q} \in \widetilde{E} \cap \mathcal{K}_\Lambda$ that

$$\begin{aligned} \mathcal{I}(\widetilde{q}) &\geq \frac{1}{2} \int_0^T |\widetilde{q}'|^2 \, \mathrm{d}t + \int_0^T [\widetilde{q}' \cdot W(t,\widetilde{q}) - V(t,\widetilde{q})] \mathrm{d}t \\ &\geq \frac{\pi^2}{2T^2} \int_0^T |\widetilde{q}|^2 \, \mathrm{d}t - \int_0^T |W(t,\widetilde{q})| \, \mathrm{d}t - \int_0^T V(t,\widetilde{q}) \, \mathrm{d}t. \end{aligned}$$

The condition (3.3) implies then that

$$\inf_{\widetilde{E}} \mathcal{I} \geq c_0 T > 0.$$

On the other hand, by (V_∞), when $\overline{q} \in \overline{E}$ converges to infinity we have

$$\lim_{|\overline{q}| \to \infty} \mathcal{I}(\overline{q}) = -\lim_{|\overline{q}| \to \infty} \int_0^T V(t,\overline{q}) \, \mathrm{d}t = 0,$$

and we can choose $\rho > 0$ such that the boundary ∂B_ρ of the ball B_ρ in $\overline{E}$ of center zero and radius ρ satisfies that

$$\sup_{\partial B_\rho} \mathcal{I} < c_0 T \leq \inf_{\widetilde{E}} \mathcal{I},$$

that is, $\mathcal{I}$ verifies the geometry of the Rabinowitz's saddle point theorem [12].

Therefore, since $\dim \overline{E} < \infty$, we can apply Theorem 2.1 with K the closed ball B_ρ in $\overline{E}$ of center zero and radius ρ, K_0 the boundary in $\overline{E}$ of this ball and γ_0 the identity function in K_0 to deduce the existence of a sequence $\{q_n\} \subset E$ such that

$$\lim_{n \to \infty} \mathcal{I}(q_n) = c := \inf_{\gamma \in \Gamma} \sup_{x \in \overline{B}_\rho} \mathcal{I}(\gamma(x)),$$

and there exists $0 < \varepsilon_n \to 0$ such that

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\varepsilon_n \|\varphi - q_n\|$$

for all positive integer n and for all $\varphi \in \operatorname{Dom} \Psi$.

By Lemma 3.5, there exists a subsequence $\{q_{n_k}\}$ of $\{q_n\}$ converging in $C([0,T],\mathbb{R})$ to a critical point $q \in \mathcal{K}_\Lambda$ of $\mathcal{I}$ with critical level $\mathcal{I}(q) = c$ □

Observe that the periodic solution given by Theorem 3.6 can be trivial. Indeed, if for instance, V and W does not depend on t, then $q = \xi$ is a constant solution if and only if $\nabla V(\xi) = 0$. In this section we show a sufficient condition in order to obtain a second solution of (1.1).

Theorem 3.7. *Assume that V and W only depends on the variable q, satisfies conditions (V_0), (V_∞) and (W_∞) and $V \in C^2(\mathbb{R}^3 \setminus \{0\})$. If there exists $\xi \in \mathbb{R}^3 \setminus \{0\}$ such that $V(\xi) < 0$, $\nabla V(\xi) = 0$ and the matrix*

$$\left(2\frac{\partial W_j}{\partial q_i}(\xi) - \frac{\partial^2 V}{\partial q_i \partial q_j}(\xi) \right)_{i,j=1,2,3} \tag{3.4}$$

is positive definite, then the Lorentz force equation (1.1) *has a periodic solution which is different from the constant solution* ξ.

Proof. As it has been mentioned, since V and W only depends on the variable q, the hypothesis $\nabla V(\xi) = 0$ means that $q = \xi$ is a constant (thus periodic) solution of (1.1). Moreover, the condition about the positive definiteness of the matrix given by (3.4) implies that the functional $\mathcal{F}$ presents a strict local minimum at $q = \xi$. Since any constant is trivially a local minimum of the function Ψ, we deduce that $\mathcal{I} = \Psi + \mathcal{F}$ has a strict local minimum at $q = \xi$. In particular, there exists $r_0 > 0$ such that

$$\inf_{\|q-\xi\|=r_0} \mathcal{I}(q) > \mathcal{I}(\xi) = -V(\xi)T > 0.$$

On the other hand, by (V_∞) when the constant $\eta \in \mathbb{R}^3$ converges to infinity we have

$$\lim_{|\eta|\to\infty} \mathcal{I}(\eta) = -\lim_{|\eta|\to\infty} V(\eta)T = 0,$$

and consequently it is possible to choose $\eta \in \mathbb{R}^3$ with $\|\eta - \xi\| > r_0$ such that $\mathcal{I}(\eta) < \mathcal{I}(\xi)$.

Therefore, the functional $\mathcal{I}$ satisfies the geometry of the Mountain Pass Theorem [2] and applying Theorem 2.1 with $K = [0, 1]$, $K_0 = \{0, 1\}$, $\gamma_0(0) = 0$, $\gamma(1) = \eta$ and $\Gamma = \{\gamma : [0, 1] \to E : \gamma \text{ is continuous and } \gamma(0) = 0,\ \gamma(1) = \eta\}$ we deduce the existence of a sequence $\{q_n\} \subset E$ such that

$$\lim_{n\to\infty} \mathcal{I}(q_n) = c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathcal{I}(\gamma(t)),$$

and there exists $0 < \varepsilon_n \to 0$ such that

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\varepsilon_n\|\varphi - q_n\|$$

for all positive integer n and for all $\varphi \in \operatorname{Dom}\Psi$.

By Lemma 3.5, there exists a subsequence $\{q_{n_k}\}$ of $\{q_n\}$ converging in $C([0, T], \mathbb{R}^3)$ to a critical point $q \in \mathcal{K}_\Lambda$ of $\mathcal{I}$ with critical level $\mathcal{I}(q) = c$. Since $\mathcal{I}(q) = c \geq \inf_{\|q-\xi\|=r_0} \mathcal{I}(q) > \mathcal{I}(\xi)$, we obtain $q \neq \xi$ and the proof is concluded in this case. □

4. Periodic oscillations of a relativistic type-pendulum

This section is devoted to the study of the scalar equation (1.3). Our main existence result is achieved by means of both variational and topological arguments. We divide the section in two subsections.

4.1. Existence via a variational approach

Following [1], in this subsection we study the existence of T-periodic solutions of (1.3) where $h \in L^1(0, T)$ and the singular function $G : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ satisfies the hypothesis

(G_0) there exists $c_0 > 0$ such that

$$\liminf_{s\to 0} G(s)|s| \geq c_0 > 0.$$

Remark 4.1. Observe that a sufficient condition for the hypothesis (G_0) is that there exists a constant $c_1 > 0$ such that

$$\liminf_{s\to 0} G'(s)s|s| \leq -c_1.$$

Remark 4.2. Another sufficient condition for (G_0) is that there exist $\alpha > 1$ and $c_1 > 0$ such that

$$\liminf_{s\to 0} G(s)|s|^\alpha \geq c_1.$$

Even if, contrary to the system equation (1.1), the Eq. (1.3) is scalar, we keep the notation q for the unknown of the Eq. (1.3). Thus, in this case, q belongs to the space $E = W_T^{1,\infty}$ of all T-periodic Lipschitz functions in $\mathbb{R}$ equipped with the norm

$$\|q\| = \|q\|_\infty + \|q'\|_\infty, \qquad \forall q \in E.$$

Since the function G has a singularity at $q = 0$, we work in the subset

$$\Lambda = \{q \in E : q(t) \neq 0, \ \forall t \in [0, T]\}.$$

Taking

$$\mathcal{K} = \{q \in E : \|q'\|_\infty \leq 1\}, \quad \mathcal{K}_\Lambda = \mathcal{K} \cap \Lambda = \{q \in \Lambda : \|q'\|_\infty \leq 1\},$$

the functional $\mathcal{I}$ related to problem (1.3) is given for every $q \in \Lambda$ by

$$\mathcal{I}(q) = \begin{cases} \int_0^T [1 - \sqrt{1-(q')^2}]\mathrm{d}t - \frac{1}{2}\int_0^T q^2 \mathrm{d}t + \int_0^T G(q)\mathrm{d}t + \int_0^T hq\mathrm{d}t, & \text{if } q \in \mathcal{K}_\Lambda; \\ +\infty, & \text{if } q \in \Lambda \backslash \mathcal{K}_\Lambda. \end{cases} \tag{4.1}$$

A critical point q of $\mathcal{I}$ is a point $q \in \mathcal{K}_\Lambda$ such that

$$\int_0^T [1 - \sqrt{1-(w')^2}]\mathrm{d}t - \int_0^T [1 - \sqrt{1-(q')^2}]\mathrm{d}t - \int_0^T q(w-q)\mathrm{d}t$$
$$+ \int_0^T G'(q)(w-q)\mathrm{d}t + \int_0^T h(w-q)\mathrm{d}t \geq 0, \ \forall w \in \mathcal{K}_\Lambda.$$

Similarly to the previous section, the critical points q of $\mathcal{I}$ in E are just the T-periodic solutions of the Eq. (1.3); i.e., satisfying $|q'(t)| < 1$ for all t, $q(0) = q(T)$ and (1.3) are satisfied pointwise.

We state the similar results to the Lemmas 3.2 and 3.5.

Lemma 4.3. *Assume that $h \in L^1(0, T)$ and that G satisfies the condition (G_0). If a sequence $\{q_n\}$ in $\mathcal{K}_\Lambda$ converges uniformly in $[0, T]$ to some q which vanishes at some point of $[0, T]$, then*

$$\lim_{n\to\infty} \int_0^T G(q_n)\, dt = +\infty \quad \text{ and } \quad \lim_{n\to\infty} \mathcal{I}(q_n) = +\infty.$$

Proof. Since $\{q_n\}$ is bounded in $C([0, T])\mathbb{R})$, the first, second and fourth integrals of $\mathcal{I}(q_n)$ ($\mathcal{I}$ is given by (4.1)) are bounded. Hence to prove the lemma, it suffices to show that $\int_0^T G(q_n)\, dt$ converges to infinity. By the hypothesis (G_0), there exists $\varepsilon_0 > 0$ such that

$$G(s) \geq \frac{c_0}{|s|}, \qquad \forall |s| \leq \varepsilon_0, \ s \neq 0.$$

Let $t_0 \in [0, T]$ be such that $q(t_0) = 0$. Two cases can occur: either $q \equiv 0$, or (up to a change of the zero t_0 by other zero) we can assume that there exists $t_1 \in (t_0, T]$ such that $q(t_0) = 0 < |q(t)| \leq \varepsilon_0$, for every $t \in (t_0, t_1]$. In the first case, $q \equiv 0$, we have $|q_n(t)| \leq \varepsilon_0$ for every $t \in [0, T]$ provided that n is large enough. Thus, the above hypothesis implies

$$\int_0^T G(q_n)\, \mathrm{d}t \geq \int_0^T \frac{c_0}{|q_n|}\, \mathrm{d}t, \text{ for } n \text{ large enough.}$$

By Fatou lemma, we deduce

$$\liminf_{n\to\infty}\int_0^T G(q_n)\,\mathrm{d}t \geq \liminf_{n\to\infty}\int_0^T \frac{c_0}{|q_n|}\,\mathrm{d}t \geq \int_0^T \liminf_{n\to\infty}\frac{c_0}{|q_n|}\,\mathrm{d}t = +\infty,$$

which shows that

$$\lim_{n\to\infty}\int_0^T G(q_n)\,\mathrm{d}t = +\infty$$

and the lemma is proved in this case.

In the second case, we have

$$\int_0^T G(q_n)\,\mathrm{d}t = \left\{\int_{|q_n|\leq\varepsilon_0} + \int_{|q_n|>\varepsilon_0}\right\} G(q_n)\,\mathrm{d}t \geq \int_{|q_n|\leq\varepsilon_0}\frac{c_0}{|q_n|}\,\mathrm{d}t + \int_{|q_n|>\varepsilon_0} G(q_n)\,\mathrm{d}t.$$

Using that $G(s)$ is bounded from below for $0\leq t\leq T$ and $\varepsilon_0<|s|\leq \sup_n \|q_n\|_\infty<\infty$, we have

$$\int_{|q_n|>\varepsilon_0} G(q_n)\,\mathrm{d}t$$

is also bounded from below for all n. Therefore, to conclude the proof it suffices to show that

$$\lim_{n\to\infty}\int_{|q_n|\leq\varepsilon_0}\frac{1}{|q_n|}\,\mathrm{d}t = \infty.$$

In order to prove it, since $\|q_n'\|_\infty \leq 1$, we note that

$$\int_{|q_n|\leq\varepsilon_0}\frac{1}{|q_n|}\,\mathrm{d}t \geq \int_{t_0}^{t_1}\frac{1}{|q_n|}\,\mathrm{d}t \geq \int_{t_0}^{t_1}\frac{q_n q_n'}{q_n^2}\,\mathrm{d}t = \log|q_n(t_1)| - \log|q_n(t_0)|.$$

Consequently, using that, as n tends to infinity, $q_n(t_1)$ converges to $q(t_1)\neq 0$ and $q_n(t_0)$ to $q(t_0)=0$, we deduce that

$$\lim_{n\to\infty}\int_{|q_n|\leq\varepsilon_0}\frac{1}{|q_n|}\,\mathrm{d}t \geq \lim_{n\to\infty}\log|q_n(t_1)| - \log|q_n(t_0)| = \infty,$$

and the proof is concluded. □

Remark 4.4. As in the Remark 3.4, the above lemma implies that the functional $\mathcal{I}$ given by (4.1) satisfies the condition (2.1) required in Theorem 2.1.

As in the previous section, we will use the direct sum decomposition

$$E = \overline{E}\oplus\widetilde{E},\quad q=\overline{q}+\widetilde{q},\quad \overline{q}=\frac{1}{T}\int_0^T q\,\mathrm{d}t,\quad \widetilde{q}=q-\overline{q}. \tag{4.2}$$

Lemma 4.5. *Assume that $h\in L^1(0,T)$ and that G satisfies the conditions (G_0) and (G'_∞) $\limsup_{|s|\to\infty}|G'(s)|<\infty$.*

If $\{\varepsilon_n\}_n$ is a sequence of positive numbers converging to zero and $\{q_n\}$ is a sequence in $\mathcal{K}_\Lambda$ satisfying that

$$\lim_{n\to\infty}\mathcal{I}(q_n) = c\in\mathbb{R} \tag{4.3}$$

and

$$\int_0^T [1-\sqrt{1-(w')^2}]\mathrm{d}t - \int_0^T [1-\sqrt{1-(q_n')^2}]\mathrm{d}t - \int_0^T q_n(w-q_n)\mathrm{d}t$$
$$+\int_0^T G'(q_n)(w-q_n)\mathrm{d}t + \int_0^T h(w-q_n)\mathrm{d}t \geq -\varepsilon_n\|w-q_n\|, \ \forall w\in\mathcal{K}_\Lambda, \tag{4.4}$$

then there exists a subsequence $\{q_{n_k}\}$ of $\{q_n\}$ converging in $C([0,T],\mathbb{R})$ to a critical point $q\in\mathcal{K}_\Lambda$ of $\mathcal{I}$ with level $\mathcal{I}(q)=c$.

Proof. Let $\{q_n\}\subset E$ be a sequence satisfying (4.3) and (4.4). Observe that for every $q\in\mathcal{K}_\Lambda$ we have $\|\widetilde{q}'\|_\infty = \|q'\|_\infty \leq 1$, which together to the existence of a zero of $\widetilde{q}$ in $[0,T]$ (consequence of the zero mean value of $\widetilde{q}$) implies that

$$\|\widetilde{q}\|_\infty \leq T, \quad \|\widetilde{q}\| \leq 1+T, \quad \forall q\in\mathcal{K}_\Lambda. \tag{4.5}$$

Hence, in order to prove that $\{q_n\}$ is bounded in E, it suffices to show that $\{\overline{q}_n\}$ is bounded. To this aim, choosing $w=q_n+\overline{q}_n$ in (4.4), it follows that

$$\overline{q}_n^2 T \leq \int_0^T G'(q_n)\overline{q}_n\mathrm{d}t + |\overline{q}_n|\|h\|_{L^1} + \varepsilon_n|\overline{q}_n|. \tag{4.6}$$

This implies that the sequence $\{\overline{q}_n\}$ is bounded. Indeed, otherwise we can assume, up to subsequences, that $|\overline{q}_n|$ converges to ∞. Thus, by (4.5), $q_n=\overline{q}_n+\widetilde{q}_n$ is away from zero for large n; that is, $s_0, n_0 \gg 0$ exist such that $|q_n|\geq s_0$ for every $n\geq n_0$. By assumption (G'_∞), there exists $\eta>0$ such that

$$|G'(q_n)| = |G'(\overline{q}_n+\widetilde{q}_n)| \leq \eta, \quad \forall n\geq n_0.$$

By (4.6) we infer for every $n\geq n_0$ that

$$\overline{q}_n^2 T \leq \eta T|\overline{q}_n| + |\overline{q}_n|\|h\|_{L^1} + \varepsilon_n|\overline{q}_n|$$

i.e. $|\overline{q}_n| \leq \eta + T^{-1}\|h\|_{L^1} + \varepsilon_n T^{-1}$, (for every $n\geq n_0$) contradicting the convergence of $|\overline{q}_n|$ to ∞ and proving that the sequence $\{\overline{q}_n\}$, and thus the sequence $\{q_n\}$, is bounded in E.

By the compact embedding of E into $C([0,T])\mathbb{R}$ we can assume, up to subsequences, that $q_n(t)\to q(t)$ uniformly in $[0,T]$.

Since each q_n is Lipschitz with Lipschitz constant equal $\|q_n\|_\infty \leq 1$, we deduce that q is also Lipschitz with Lipschitz constant smaller or equal to one; i.e., $q\in\mathcal{K}$. By Lemma 4.3 and (4.3), we have

$$q(t)\neq 0, \quad \forall t\in[0,T],$$

concluding the proof. □

Theorem 4.6. *If the conditions (G_0), (G'_∞) and*

(G_∞) $\limsup_{|s|\to\infty} \frac{G(s)}{s} < \frac{1}{2}$,

hold true, then the Eq. (1.3) has at least a T-periodic solution.

Remark 4.7. A sufficient condition for (G'_∞) and (G_∞) is that

$$\limsup_{|s|\to\infty} |G'(s)| < \frac{1}{2}.$$

Proof. Taking into account the decomposition given by (4.2) and using (4.5) we have

$$\int_0^T \widetilde{q}^2 \mathrm{d}t \le T^3 \text{ and } \int_0^T h(t)\widetilde{q}\mathrm{d}t \le \|h\|_{L^1}\|\widetilde{q}\|_\infty \le T\|h\|_{L^1}, \quad \forall \widetilde{q} \in \widetilde{E} \cap \mathcal{K}_\Lambda.$$

By this and since $1 - \sqrt{1-s^2} \ge \frac{1}{2}s^2 \ge 0$ we deduce that the functional $\mathcal{I}$ defined by (4.1) satisfies

$$\mathcal{I}(\widetilde{q}) \ge -\frac{T^3}{2} + \int_0^T G(\widetilde{q})\mathrm{d}t - T\|h\|_{L^1}, \quad \forall \widetilde{q} \in \widetilde{E} \cap \mathcal{K}_\Lambda. \tag{4.7}$$

We claim that this inequality implies that $\mathcal{I}$ is bounded from below over $\widetilde{E}$. Indeed, if we take a minimizing sequence $\{\widetilde{q}_n\}$ in $\widetilde{E} \cap \mathcal{K}_\Lambda$ of $\mathcal{I}$; i.e., such that

$$\lim_{n\to\infty} \mathcal{I}(\widetilde{q}_n) = \inf_{\widetilde{E}} \mathcal{I} \in \mathbb{R} \cup \{-\infty\},$$

by Lemma 4.3 the accumulation points in the $C([0,T])$-topology of the sequence $\{\widetilde{q}_n\}$ can not vanish at any point of $[0,T]$. This means that there exists $\varepsilon > 0$ such that $\|\widetilde{q}_n\|_\infty \ge \varepsilon$, for every n. Therefore, by using again that $\|\widetilde{q}_n\|_\infty \le T$, we obtain from (4.7) that

$$\mathcal{I}(\widetilde{q}_n) \ge -\frac{T^3}{2} - T \sup_{\varepsilon \le |s| \le T} |G(s)| - T\|h\|_{L^1}.$$

This implies that $\inf_{\widetilde{E}} \mathcal{I} \in \mathbb{R}$ and the claim is proved.

On the other hand, for every $\overline{q} \in \overline{E}$ we have

$$\mathcal{I}(\overline{q}) = -\frac{T}{2}\overline{q}^2 + TG(\overline{q}) + T\overline{q}\|h\|_{L^1},$$

and thus, by (G_∞), we have

$$\lim_{\|\overline{q}\|\to\infty} \mathcal{I}(\overline{q}) = -\infty.$$

In consequence, we can choose $\rho > 0$ such that the boundary ∂B_ρ of the ball B_ρ in $\overline{E}$ of center zero and radius ρ satisfies that

$$\sup_{\partial B_\rho} \mathcal{I} < \inf_{\widetilde{E}} \mathcal{I},$$

that is, $\mathcal{I}$ verifies the geometry of the Rabinowitz's saddle point theorem [12]. By Lemma 4.5 (instead of Lemma 3.5) we can repeat the argument in the proof of Theorem 3.6 to deduce the existence a subsequence (q_{n_k}) of (q_n) converging in $C([0,T],\mathbb{R})$ to a critical point $q \in \mathcal{K}_\Lambda$ of $\mathcal{I}$ with critical level $\mathcal{I}(q) = c$. $\square$

Now we look for T-periodic solutions $q \in E$ of the Eq. (1.3) with $h \equiv 0$, i.e.,

$$\left(\frac{q'}{\sqrt{1-(q')^2}}\right)' + q = G'(q) \tag{4.8}$$

Firstly, observe that every constant $\xi \in \mathbb{R}\backslash\{0\}$ verifying $\xi = G'(\xi)$ is a trivial (constant) T-periodic solution of (4.8). In this case, in order to find another solution we apply the Mountain Pass Theorem to the functional $\mathcal{I}$ given by (4.1) with $h \equiv 0$; that is,

$$\mathcal{I}(q) = \begin{cases} \int_0^T [1-\sqrt{1-(q')^2}]\mathrm{d}t - \frac{1}{2}\int_0^T q^2\mathrm{d}t + \int_0^T G(q)\mathrm{d}t, & \text{if } q \in \mathcal{K}_\Lambda, \\ +\infty, & \text{if } q \in \Lambda\backslash\mathcal{K}_\Lambda. \end{cases}$$

Theorem 4.8. *Assume that hypotheses* (G_0), (G_∞) *and* (G'_∞) *hold true with the function* G *of class* C^2 *in* $\mathbb{R}\backslash\{0\}$*. If* $\xi \in \mathbb{R}\backslash\{0\}$ *verifies* $\xi = G'(\xi)$ *and* $G''(\xi) > 1$*, then the Eq.* (4.8) *has at least one* T*-periodic solution different from the trivial one* ξ*.*

Proof. Consider the functional $\mathcal{F}$ defined for $q \in E$ by

$$\mathcal{F}(q) = -\frac{1}{2}\int_0^T q^2 \mathrm{d}t + \int_0^T G(q)\mathrm{d}t.$$

Observe that it is of class C^2 (because $G \in C^2$) with the first and second derivatives given for every $q, w_1, w_2 \in E$ by

$$\mathcal{F}'(q)[w_1] = -\int_0^T q w_1 \mathrm{d}t + \int_0^T G'(q) w_1 \mathrm{d}t$$

and

$$\mathcal{F}''(q)[w_1, w_2] = -\int_0^T w_1 w_2 \mathrm{d}t + \int_0^T G''(q) w_1 w_2 \mathrm{d}t.$$

In particular, the condition $\xi = G'(\xi)$ implies $q = \xi$ is a critical point of $\mathcal{F}$ and the hypothesis $G''(\xi) > 1$ means that the second derivative of $\mathcal{F}$ at $q = \xi$ is positive definite. Thus, $\mathcal{F}$ has a strict local minimum at $q = \xi$. Taking into account that

$$\mathcal{I}(q) = \Psi(q) + \mathcal{F}(q), \quad \forall q \in \mathcal{K}_\Lambda$$

with

$$\Psi(q) := \int_0^T [1 - \sqrt{1-(q')^2}]\mathrm{d}t \geq 0\,(= \Psi(\xi)),\ \forall q \in \mathcal{K},$$

we deduce that $q = \xi$ is also a strict local minimum of $\mathcal{I}$. Hence, there exist $\delta, r > 0$ such that

$$\mathcal{I}(q) \geq \mathcal{I}(\xi) + \delta, \text{ when } \|q - \xi\| = r.$$

As in the proof of Theorem 4.6, by assumption (G_∞), for every $\overline{q} \in \overline{E}$ we have

$$\lim_{\|\overline{q}\|\to\infty} \mathcal{I}(\overline{q}) = \lim_{\|\overline{q}\|\to\infty} -\frac{T}{2}\overline{q}^2 + TG(\overline{q}) = -\infty$$

and we can choose a constant $\eta \in \overline{E}$ such that $\mathcal{I}(\eta) < \mathcal{I}(\xi)$.

In conclusion, the functional $\mathcal{I}$ satisfies the geometry of the Mountain Pass Theorem [2] and applying Theorem 2.1 with $K = [0,1]$, $K_0 = \{0,1\}$, $\gamma_0(0) = \xi$, $\gamma(1) = \eta$ and $\Gamma = \{\gamma : [0,1] \to E : \gamma \text{ is continuous and } \gamma(0) = \xi,\ \gamma(1) = \eta\}$ we deduce the existence of a sequence $(q_n) \subset E$ such that

$$\lim_{n\to\infty} \mathcal{I}(q_n) = c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathcal{I}(\gamma(t)) > \mathcal{I}(\xi) > \mathcal{I}(\eta),$$

and a sequence $0 < \varepsilon_n \to 0$ such that

$$\Psi(\varphi) - \Psi(q_n) + \mathcal{F}'(q_n)[\varphi - q_n] \geq -\varepsilon_n \|\varphi - q_n\|$$

for all positive integer n and for all $\varphi \in \mathrm{Dom}\,\Psi$.

By Lemma 4.5, there exists a subsequence $\{q_{n_k}\}$ of $\{q_n\}$ converging in $C([0,T],\mathbb{R})$ to a critical point $q \in \mathcal{K}_\Lambda$ of $\mathcal{I}$ with critical level $\mathcal{I}(q) = c$. Since $\mathcal{I}(q) = c \geq \inf_{\|q-\xi\|=r} \mathcal{I}(q) > \mathcal{I}(\xi)$, we obtain $q \neq \xi$ and the proof is finished. □

4.2. Existence by global continuation theorem

In contrast with the variational techniques used so far, here we will derive the existence of at least one T-periodic solution of (1.3) by using the global continuation theorem to the homotopic system

$$\left(\frac{q'}{\sqrt{1-(q')^2}}\right)' + q = g_\lambda(q) + h_\lambda(t), \quad \lambda \in [0,1], \tag{4.9}$$

with $g_\lambda(q) = \lambda G'(q) + (1-\lambda)\dfrac{c_0}{q|q|}$ and $h_\lambda(t) = \lambda h(t)$. Observe that (4.9) turns into (1.3) when $\lambda = 1$.

By considering the position q and the momentum $p = q'/\sqrt{1-(q')^2}$ as new coordinates (and taking into account that the inverse of $\phi(s) = s/\sqrt{1-s^2}$ is given by $\phi^{-1}(p) = p/\sqrt{1+p^2}$), Eq. (4.9) turns into the first order system of differential equations

$$\begin{cases} q' = \dfrac{p}{\sqrt{1+p^2}} \\ p' = -q + g_\lambda(q) + h_\lambda(t), \end{cases}$$

that is,

$$(q,p)' = f_\lambda(t,(q,p)) := \left(\frac{p}{\sqrt{1+p^2}}, -q + g_\lambda(q) + h_\lambda(t)\right).$$

Setting $x := (q,p)$ and denoting by $\mathcal{N}_{f_\lambda}$ the Nemitskii operator associated to the function $f_\lambda(t,x)$, the previous problem can be written as the first order ordinary differential equation

$$x' = \mathcal{N}_{f_\lambda}(x)$$

in the Banach space $X = \{x \in C([0,T],\mathbb{R}^2) : x(0) = x(T)\}$.

If $P : X \to X$ is the projection given by

$$Px = \overline{x} := \frac{1}{T}\int_0^T x(t)\mathrm{d}t \qquad \forall x \in X,$$

then

$$x = \overline{x} + \widetilde{x}, \quad \overline{x} = Px, \quad \widetilde{x} := (I-P)(x), \qquad \forall x \in X.$$

Note that, if $\widetilde{X} := \operatorname{Ker} P = \left\{x \in X : \frac{1}{T}\int_0^T x(t)\mathrm{d}t = 0\right\}$, we can consider the operator $K : L^1([0,T],\mathbb{R}^2) \to X$ defining for each $g \in L^1([0,T],\mathbb{R}^2)$, the function Kg as the unique solution $\widetilde{x} \in \widetilde{X}$ of the equation

$$\widetilde{x}'(t) = g(t).$$

Thus, using the previous notations, (4.9) turns into

$$x = Px + K\big[\mathcal{N}_{f_\lambda}(x)\big] =: T_\lambda x, \tag{4.10}$$

where

- P has a finite range,
- $\mathcal{N}_{f_\lambda}$ is continuous with $\mathcal{N}_{f_\lambda}(\overline{\Omega})$ bounded in X,
- and $K|_X : X \to C^1([0,T],\mathbb{R}^2)$ is linear and continuous.

Thus, by the compact embedding of $C^1([0,T])$ into $C([0,T],\mathbb{R}^2)$ (due to the Ascoli-Arzelà theorem) we have that $T_\lambda : X \to X$ is compact and we can employ [7, Theorem 2] to address problem (4.10). For the sake of completeness, we recall it here in our particular case.

Theorem 4.9. *Let $\Omega \subset X = \{x \in C([0,T],\mathbb{R}^2) : x(0) = x(T)\}$ be an open bounded subset and assume that the operators $T_\lambda : X \to X$ are compact on $\overline{\Omega}$. If the following conditions hold true* *(i)* *there is no solution* $x \in \partial\Omega$ *of the homotopic problem* (4.10) *for every* $\lambda \in [0, 1]$;

(ii) $\deg_B(f_0, \Omega \cap \mathbb{R}^2, 0) \neq 0$, *where* $\Omega \cap \mathbb{R}^2$ *denotes the subset of the constant functions in* Ω *(observe that* $T_0|_{\Omega\cap\mathbb{R}^2} = f_0$*) and* $\deg_B$ *denotes the Brouwer degree;*

then the problem (4.10) *has at least a solution* $x \in \Omega$. □

Remark 4.10. Observe that if condition (i) in the previous theorem were not satisfied for $\lambda = 1$, then a (trivial) solution of (4.10) would exist on $\partial\Omega$.

In order to properly define the open bounded subset Ω required to apply the previous theorem, we derive some a priori bounds for the position q and the momentum $p = q'/\sqrt{1-(q')^2}$ of every T–periodic solution $q(t)$ of (4.9).

Lemma 4.11. *If condition* (G_∞) *and* $(\widetilde{G}_0)$

$$\liminf_{s\to 0} G'(s)s|s| > 0$$

hold true, then there exist positive constants C, c *and* M *such that for every* T*-periodic solution* $q(t)$ *of* (4.9) *we have*

(i) $\|q\|_\infty \leq C$,

(ii) $|q(t)| \geq c$, *for every* $t \in [0, T]$,

(iii) $\left|\dfrac{q'}{\sqrt{1-(q')^2}}\right| \leq M$, *for every* $t \in [0, T]$.

Remark 4.12. Similarly to Remark 4.1, assumption $(\widetilde{G}_0)$ implies that

$$\limsup_{s\to 0} G(s)|s| < 0.$$

Proof. (i) By hypothesis (G_∞), there exists a positive constant η (independent of $\lambda \in [0, 1]$) such that

$$g_\lambda(s) \leq G'(s) + \frac{c_0}{s|s|} \leq \eta, \quad \forall |s| \geq 1.$$

Fixing $R \geq \max\{\eta + \|h\|_{L^1}/T, 1\}$ we claim that the above identity implies that for every solution q of (4.9) there exists at least one $t_0 \in [0, T]$ such that $q(t_0) \leq R$. Indeed, supposing by contradiction that a solution q of (4.9) satisfies $q(t) > R(\geq 1)$ for all t, we would have

$$\int_0^T g_\lambda(q(t))\mathrm{d}t < T\eta \leq TR - \|h\|_{L^1} \leq \left|\left|\int_0^T q(t)\mathrm{d}t\right| - \left|\int_0^T h_\lambda(t)\mathrm{d}t\right|\right|$$
$$\leq \left|\int_0^T q(t)\mathrm{d}t - \int_0^T h_\lambda(t)\mathrm{d}t\right|.$$

This inequality contradicts that choosing $v = 1$ as test function in (4.9), we have

$$\int_0^T g_\lambda(q(t))\mathrm{d}t = \int_0^T q(t)\mathrm{d}t - \int_0^T h_\lambda(t)\mathrm{d}t.$$

Therefore, for every T-periodic solution q of (4.9) there is at least one $t_0 \in [0, T]$ such that $q(t_0) \leq R$. Thus

$$|q(t)| = \left|q(t_0) + \int_{t_0}^t q'(s)ds\right| < R + T,$$

i.e. every solution q of the problem (4.9) satisfies $\|q\|_\infty \leq R + T$ and it suffices to choose $C = R + T$ to conclude the proof of (i).

(ii) Choosing $v = q$ as test function in (4.9) we deduce that

$$-\int_0^T \frac{(q')^2}{\sqrt{1-(q')^2}}\mathrm{d}t + \int_0^T q^2(t)\mathrm{d}t = \int_0^T g_\lambda(q(t))q(t)\mathrm{d}t + \int_0^T h_\lambda(t)q(t)\mathrm{d}t,$$

for every T-periodic solution of (4.9). Now, since

$$\int_0^T \frac{(q'(t))^2}{\sqrt{1-(q'(t))^2}}\mathrm{d}t \geq 0,$$

we infer that

$$\int_0^T q^2(t)\mathrm{d}t \geq \int_0^T g_\lambda(q(t))q(t)\mathrm{d}t + \int_0^T h_\lambda(t)q(t)\mathrm{d}t.$$

Since assumption $(\widetilde{G}_0)$ implies that there exists $c_0, \varepsilon_0 > 0$ such that

$$G'(s)s \geq \frac{c_0}{|s|}, \quad \forall 0 < |s| \leq \varepsilon_o,$$

we have

$$g_\lambda(s)s = \lambda G'(s)s + (1-\lambda)\frac{c_0}{|s|} \geq \frac{c_0}{|s|}, \quad \forall\, 0 < |s| \leq \varepsilon_0, \quad \forall\, \lambda \in [0,1].$$

Then, it follows that

$$\begin{aligned}\int_{|q(t)|<\varepsilon_0} \frac{c_0}{|q(s)|}\mathrm{d}s &= \int_{|q(t)|<\varepsilon_0} g_\lambda(q(s))q(s)\mathrm{d}s \\ &\leq \int_0^T q^2(s)\mathrm{d}s - \int_{|q(t)|>\varepsilon_0} g_\lambda(q(s))q(s)\mathrm{d}s - \int_0^T h_\lambda(s)q(s)\mathrm{d}s \\ &\leq \int_0^T q^2(s)\mathrm{d}s + \left|\int_{|q(t)|>\varepsilon_0} g_\lambda(q(s))q(s)\mathrm{d}s\right| + \left|\int_0^T h_\lambda(s)q(s)\mathrm{d}s\right|.\end{aligned}$$

By (i) we obtain the inequality

$$\int_{|q(t)|<\varepsilon_0} \frac{c_0}{|q(s)|}\mathrm{d}s \leq C^2T + T \max_{\varepsilon_0<|s|<C}\left|G'(s)s + \frac{c_0}{|s|}\right| + C\|h\|_{L^1}.$$

that allows us to prove the item (ii). Indeed, observe first that if the T-periodic solution $q(t)$ of (4.9) is always out the ball $B_{L^\infty}(0,\varepsilon_0)$ the result is proved. On the other hand, if there exists a closed interval $[t_1,t_2] \subset [0,T]$ such that

$$|q(t_1)| = \varepsilon_0 \quad \text{and} \quad |q(t)| < \varepsilon_0 \quad \forall\, t \in (t_1,t_2],$$

then, using that $|q'| \le 1$, we deduce for $t \in (t_1, t_2)$ that

$$\big|\log|q(t)|\big| = \left|\log|q(t_1)| + \int_{t_1}^{t} \frac{q(s)q'(s)}{q(s)^2}\mathrm{d}s\right| \le |\log\varepsilon_0| + \int_{t_1}^{t_2} \frac{1}{|q(s)|}\mathrm{d}s$$
$$\le |\log\varepsilon_0| + \int_{|q(t)|<\varepsilon_0} \frac{1}{|q(s)|}\mathrm{d}s$$
$$\le |\log\varepsilon_0| + \frac{C^2T}{c_0} + \frac{T}{c_0}\max_{\varepsilon_0<|s|<C}\left|G'(s)s + \frac{c_0}{|s|}\right| + \frac{C}{c_0}\|h\|_{L^1} =: \bar{c},$$

that is,

$$|q(t)| > e^{-\bar{c}},$$

and (ii) is also proved in this case.

(iii) Let $q(t)$ be a T–periodic solution of (4.9). Choosing $t_0 \in [0,T]$ such that $q'(t_0) = 0$ and using the mean value theorem, we have

$$\frac{q'(t)}{\sqrt{1-(q'(t))^2}} = \int_{t_0}^{t} \Big(\frac{q'(s)}{\sqrt{1-(q'(s))^2}}\Big)' ds$$

Taking again $v = 1$ as test function in (4.9) (as in item (i)) we deduce that

$$\frac{q'(t)}{\sqrt{1-(q'(t))^2}} = \int_{t_0}^{t} \left(-q(s) + g_\lambda(q(s)) + h_\lambda(s)\right) ds,$$

and item (iii) holds true by (i) and (ii) with $M = T\max_{c<|s|<C}(|s| + |G'(s)| + \frac{c_0}{|s|^2}) + \|h\|_{L^1}$. □

Remark 4.13. Observe that, unlike in [10], in our setting we have no conditions on $\overline{h}$ to get the upper bound of case (i) in the previous lemma.

We prove our main result.

Theorem 4.14. *If $h \in L^1(0,T)$ and the function G defined in $\mathbb{R}\backslash\{0\}$ satisfies assumptions (G_∞) and $(\widetilde{G}_0)$, then problem (1.3) admits at least one T-periodic solution.*

Proof. We define the open bounded subset Ω in X as

$$\Omega := \{x = (q,p) \in X : c < |q(t)| < C,\ |p(t)| < M,\ \forall t \in [0,T]\},$$

which, by Lemma 4.11, contains each T–periodic solution of system (4.10). The proof will be concluded by applying Theorem 4.9 if we show that

$$\deg_B(f_0, \Omega\cap\mathbb{R}^2, 0) \neq 0,$$

where $\Omega\cap\mathbb{R}^2$ stands for the identification with constants functions and $f_0 \in C^\infty(\Omega\cap\mathbb{R}^2)$ is given by

$$f_0(x) = f_0(t,x) = f_0(t,(q,p)) = \left(\frac{p}{\sqrt{1+p^2}}, -q + \frac{c_0}{q|q|}\right).$$

To compute this degree we observe that

$$\det \mathrm{Jac} f_0(x) = (1+p^2)^{-3/2}\left(-1 - \frac{2c_0}{|q|^3}\right).$$

and that the only zeroes of $f_0(x)$ are $(\pm\sqrt[3]{c_0}, 0)$. We apply the additivity property of the degree to infer that

$$\deg_B(f_0, \Omega \cap \mathbb{R}^2, 0) = \det \mathrm{Jac} f_0(\sqrt[3]{c_0}, 0) + \det \mathrm{Jac} f_0(-\sqrt[3]{c_0}, 0) = -6 \neq 0,$$

and the proof is concluded. □

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