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Singular patterns in Keller–Segel-type models

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The aim of this paper is to elucidate the existence of patterns for Keller–Segel-type models that are solutions of the traveling pulse form. The idea is to search for transport mechanisms that describe this type of waves with compact support, which we find in the so-called nonlinear diffusion through saturated flux mechanisms for the movement cell. At the same time, we analyze various transport operators for the chemoattractant. The techniques used combine the analysis of the phase diagram in dynamic systems together with its counterpart in the system of partial differential equations through the concept of entropic solution and the admissible jump conditions of the Rankine–Hugoniot type. We found traveling pulse waves of two types that correspond to those found experimentally.

Keywords: Flux-saturated; Keller–Segel; traveling waves; patterns; block solution; cross-diffusion; soliton.

AMS Subject Classification: 35K57, 35A15, 35C07

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1. Introduction

The collective behavior of species and how dynamic patterns emerge (defense, invasion, resilience, ...) is one of the most important topics in this research that requires a multidisciplinary approach to be addressed. In addition to the intrinsic value of studying the dynamics of a population of birds, fish, ants or sheep, these models could provide foundations for understanding other, more microscopic problems such as morphogen-cell interaction or the evolution of tumors. However, the impressive evolution in microscopy and in antibody concentration morphogenesis cell signaling allowed the study of collective behavior at the subcellular and cellular level to be analyzed and modeled directly.[1] This provides a new impetus in which the models initially developed by Keller and Segel (KS)[39, 40, 45] for chemotaxis processes (the movement of biological entities in response to chemical gradients) take on a new dimension. In addition, in recent years, various applications have been developed in exotic contexts[11] beyond cell signaling mechanisms that have provided more flexible and diverse variants of KS-type models.

The classical KS model consists in a reaction–diffusion system of two coupled parabolic equations

$$\begin{aligned} \partial_t U &= \operatorname{div}_x(D_U \nabla_x U - \chi U \nabla_x Q) + H(U,Q), \quad x \in \mathbb{R}^N, \ \ t > 0, \\ \tau \partial_t Q &= D_Q \Delta Q + K(U,Q), \quad x \in \mathbb{R}^N, \ \ t > 0. \end{aligned} \tag{1.1}$$

In the biological context of cell dynamics, U represents the cell-density and Q the chemoattractant concentration. The positive definite terms D_Q and D_U are, respectively, the diffusivity of the chemoattractant and of the cells, $\chi \geq 0$ is the chemotactic sensitivity and the functions $H(U,Q)$ and $K(U,Q)$ in (1.1) model the interaction (production and degradation) between the cell density and the chemical substance. In most simplified models and in the original KS system, these terms are modeled as $K(U,Q) = U - Q$ and $H(U,Q) = 0$. The hyperbolic limit (high-field limit) and some special parabolic limit (low-field limit) have been derived from kinetic equations describing the run and tumble process for bacterial motion.[7–11, 28, 29, 38, 44] The parameter $\tau \geq 0$ is introduced to distinguish if there is an adjustment of the chemo-attacker during the evolution of the process, it is standard that it only takes values only 0 or 1 giving rise to different models that may not be dynamically equivalents.

In this paper, we will consider two variants of the flux-saturated Keller–Segel (FSKS) model:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - aU\frac{\partial Q}{\partial x}\right), \\ \mathcal{T}(Q) &= U, \end{aligned} \tag{1.2}$$

where $\mathcal{T} = \mathcal{T}(Q)$ is one of the following linear differential operators:

$$\mathcal{T}(Q) = \frac{\partial^2 Q}{\partial t \partial x} - \nu \frac{\partial^2 Q}{\partial x^2}, \tag{1.3}$$

or

$$\mathcal{T}(Q) = \tau\frac{\partial Q}{\partial t} + \alpha\frac{\partial Q}{\partial x} - \nu\frac{\partial^2 Q}{\partial x^2}. \tag{1.4}$$

The parameters $\alpha \geq 0$ and $\nu \geq 0$ stand for the transport and diffusion coefficients, respectively. The flux function $\Phi = \Phi(s)$ is a bounded, regular, increasing and odd function. The value $c > 0$ is defined as

$$c = \lim_{s\to\infty} \Phi(s),$$

and it is finite. Also $\Phi \in C^1(\mathbb{R})$ in order to have uniqueness of the initial value problems. The value $\mu = \Phi'(0)$ is the kinematic viscosity for small velocities and near $u_x = 0$ the flow means

$$U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) \sim \mu U^{m-1}U_x,$$

being $m \geq 1$ a parameter that measures the porosity of the medium. In some sense, we have a flux-saturated combined with a porous media operator.[16] Different proposals to ours to use flux-saturated operators as an alternative to linear diffusion for the KS model use hyperbolic, fractional diffusion or porous medium type approximations, see, for example, Refs. 15, 27 and 30 and the references therein.

One of the main objectives of our study is to consider the so-called relativistic heat case

$$\Phi(s) = \mu\frac{s}{\sqrt{1 + \frac{\mu^2}{c^2}s^2}}$$

that leads to

$$\frac{\partial U}{\partial t} = \mu\frac{\partial}{\partial x}\left(\frac{U^m \frac{\partial U}{\partial x}}{\sqrt{U^2 + \frac{\mu^2}{c^2}\left|\frac{\partial U}{\partial x}\right|^2}} - aU\frac{\partial Q}{\partial x}\right).$$

Other examples of great interest are $\Phi(s) = \mu\frac{s}{1+\frac{\mu}{c}|s|}$ usually referred as Wilson operator,[43] the Larson operator[16] $\Phi(s) = \mu\frac{s}{\sqrt[p]{1+\frac{\mu^p}{c^p}s^p}}$ that include the relativistic case, and $\Phi(s) = c\tanh(\frac{\mu s}{c})$ usually referred as the hyperbolic tangent operator, see, for instance, Ref. 42.

If $m = 1$, which is the case of the relativistic heat equation, then c is the speed at which the solution support moves.[3] In the general choice of Φ, c represents the maximum speed at which the solution support can move.[16] Therefore, c is a parameter that can be taken from the biological experimental data.[47] Note that for any flux-saturated Φ, if c tends to infinity the heat equation or the porous media equation are recovered for the different values of $m \geq 1$.[26]

The aim of this paper is to find, in the context of KS models, soliton-type patterns with compact support, which represent collective models of cell invasion, propagation or behavior in which the interface with the medium is singular. This type of patterns usually appears with diverse geometry in the experimental data,

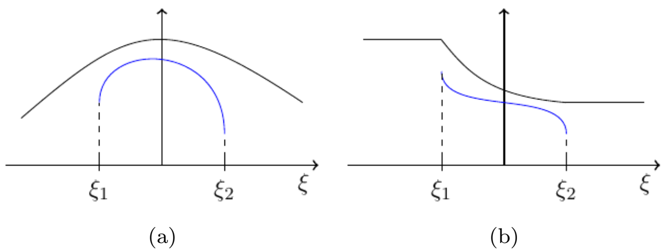


Fig. 1. (Color online) Pattern prototypes with compact support associated with flux-saturated operators. Pattern prototypes with compact support associated with saturated-flux operators. In blue we represent the cell concentration and in black the profile of the chemoattractant.

and cannot be captured with linear diffusion terms in the classical KS model. In Fig. 1, we provide various configurations of some of the results that we obtain here, coming from the analysis of the nonlinear variants of the KS models that we will study throughout the paper.

Experimental data show that the movement of cells affected by a chemoattractant occurs through a pulse or soliton-type solution.[33, 47] Moreover, from the point of view of modeling, the KS model combines a system of partial differential equations that represents the evolution of the cell density and the chemoattractant concentration. However, the classical KS system, although it admits regular traveling waves with a birth term of either a Fisher–KPP term-type, does not seem to admit soliton-type solutions. The modification of the transport terms, especially preventing free diffusion, allows to build solutions that better reflect the experimental results. This fact was rigorously proved in the case of a flux-saturated as an alternative to linear diffusion in cell density in Ref. 6. A great effort has been devoted in recent years to study the properties of the evolution by flux-saturated mechanisms, in particular the existence of traveling waves, see, for instance, Refs. 3, 4, 12, 13, 17–25, 31, 32, 36, 41 and 46 and the references therein.

In the case where the time evolution of the chemoattractant is negligible, the resulting model produces a self-generated potential in terms of cell density. This has been the most studied approach in the context of the KS models.[14, 34, 35, 37, 38]

The main reason to modify the linear diffusion by a nonlinear one is that it reproduces more faithfully the experimental data. In this context, the FSKS is a macroscopic model describing cell motion by chemotaxis, in which saturation of the velocity is taken into account. The FSKS model also has the advantage that traveling pulse or soliton-type solutions with compact support emerging as a prototype of pattern under this system.[6] The existence of this type of solutions is relevant for biological applications since, from a modeling perspective, the compactly supported property is well suited.

The paper is structured as follows. Section 2 is devoted to defining the solution concept and the soliton-type geometric structure of the solutions we seek. In this sense, we define the block-type solution, which will be the object of study in this paper. Section 3 deals with the case in which the chemoattractant gradient is transported without diffusion, proving that any maximal solution of the associated dynamic system is a block-type solution. Section 4 deals with the case where transport and diffusion terms are combined in the chemoattractant. Conditions are given for the existence and non-existence of block-type solutions. In the case of diffusion without transport of the chemoattractant, a complete analysis of all types of traveling waves is carried out (particularly those cases in which there are block-type solutions), classifying all types of solutions according to the system parameters.

2. Block Solitons Moving at a Constant Speed

Let us consider the general system of cell evolution U together with the chemoattractant Q

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - aU\frac{\partial Q}{\partial x}\right) \\ \tau_1\frac{\partial Q}{\partial t} &+ \tau_2\frac{\partial}{\partial t}\left(\frac{\partial Q}{\partial x}\right) + \alpha\frac{\partial Q}{\partial x} - \nu\frac{\partial^2 Q}{\partial x^2} = U. \end{aligned} \tag{2.1}$$

In this model, the coefficient $\nu \geq 0$ is the viscosity of the chemoattractant. Additionally, we assume that $a > 0$, $\alpha > 0$ and $m \geq 1$. The parameters τ_1 and τ_2 are taken greater than or equal to zero, in fact in the models considered τ_1 and τ_2 take values $\{0, 1\}$.

We are going to study the existence of biological blocks that move at a constant speed. Mathematically, the concept of block solutions is associated to that of traveling waves type solutions of the previous problem (2.1) for cell dynamics whose mass is concentrated in a bounded region. If $\sigma > 0$ denotes a speed of propagation, we look for solutions of the kind $U(t, x) = u(x - \sigma t)$, $Q(t, x) = q(x - \sigma t)$, where $u, q : \mathbb{R} \to \mathbb{R}$ are scalar functions and u has the mass concentrated in a compact interval. Formally, the resulting system for u, q verifies

$$\begin{aligned} -\sigma u'(\xi) &= \left(u^m(\xi)\Phi\left(\frac{u'(\xi)}{u(z)}\right) - au(\xi)q'(\xi)\right)', \\ (\alpha - \sigma\tau_1)q'(\xi) &- (\sigma\tau_2 + \nu)q''(\xi) = u(\xi), \end{aligned} \tag{2.2}$$

where $\xi := x - \sigma t$.

A first question to consider is the concept of solution for (2.1). The appropriate framework for our analysis is that of solutions of bounded variation. However, the theory of existence in the context of bounded variation solutions for KS-type systems is not sufficiently fully established, and this is not the aim of our paper. To avoid entering the theory of bounded variation functions of several variables, we are going to focus on our study on Eq. (2.2) directly.

The cell structure that gives rise to the u component is going to be assumed to be much larger and heavier than the molecular structure of the chemoattractant given by the q component, therefore a singularization of the component u can be expected. This appreciation is supported by the presence of a flux-saturated as the basis of the movement of u in the first equation. On the other hand, we expect a milder behavior of the chemoattractant q. The formation of discontinuities in q is not expected if

$$\sigma\tau_2 + \nu > 0. \tag{2.3}$$

Even in the degenerate case $\tau_2 = 0 = \nu$ the existence of fronts is not apparent because in q, even in that case, we have a linear transport equation. These reasons make us assume that q is of class 1 in $\mathbb{R}$ while u is only going to be a bounded variation function. We are in a position to consider distributional solutions imposing that

$$\begin{aligned} &\int_{\mathbb{R}} \left(u^m(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right) - au(\xi)q'(\xi) + \sigma u(\xi) \right) \psi'(\xi)d\xi = 0, \\ &(\alpha - \sigma\tau_1) \int_{\mathbb{R}} q(\xi)\psi'(\xi)d\xi - (\sigma\tau_2 + \nu) \int_{\mathbb{R}} q'(\xi)\psi'(\xi)d\xi = -\int_{\mathbb{R}} u(\xi)\psi(\xi)d\xi, \end{aligned} \tag{2.4}$$

holds, for each $\psi \in \mathcal{D}(\mathbb{R})$. In this expression, the role of $u'(\xi)$ has to be clarified. When a function $u \in BV(\mathbb{R})$ its derivative in the sense of the distributions Du decomposes as an absolutely continuous part $u'(\xi)$ and a singular part $D_s u$ that is orthogonal to the Lebesgue measure. The singular part of the measure is not easy to absorb, see Ref. 2. The set $\mathcal{S} = \operatorname{supp}\{D_s u\}$ is called the set of singularities of u. It is common in this type of operators that $\mathcal{S}$ is a finite set and also $u \in C^1(\mathbb{R}\backslash\mathcal{S})$.

A block structure is going to be requested on u. This concept of block solution materializes in the existence of a compact interval $[\xi_1, \xi_2]$, not reduced to a point, such that $u(\xi) > 0$, for a.e. $\xi \in [\xi_1, \xi_2]$, and $u(\xi) = 0$, otherwise. For q no restrictions will be imposed on its support. One last assumption is that the singularities of u have been formed by the saturation of the cell flux. If $\bar{\xi} \in \mathcal{S}$ is a singular point, then the lateral limit values are always defined. The point $\bar{\xi}$ is said of saturation to the left if

$$\lim_{\xi \to \bar{\xi}} u'(\xi) = -\infty, \quad \lim_{\xi \to \bar{\xi}^-} u(\xi) \geq \lim_{\xi \to \bar{\xi}^+} u(\xi),$$

while it will be saturation to the right if

$$\lim_{\xi \to \bar{\xi}} u'(\xi) = +\infty, \quad \lim_{\xi \to \bar{\xi}^-} u(\xi) \leq \lim_{\xi \to \bar{\xi}^+} u(\xi).$$

If $\bar{\xi}$ is a boundary point of the support, then the saturation condition is only one-sided, that is, if $\bar{\xi} = \xi_1$, then $u(\xi) = 0$, for $\xi < \xi_1$, which means

$$\lim_{\xi \to \xi_1^+} u'(\xi) = \infty,$$

and a symmetric condition on ξ_2.

Lemma 2.1. *Assume that* (2.3) *holds, then there are no saturation points inside the support.*

Proof. It follows from (2.4) that the function $u^m(\xi)\Phi\big(\frac{u'(\xi)}{u(\xi)}\big) - au(\xi)q'(\xi) + \sigma u(\xi)$ has zero weak derivative. By Stampacchia's Lemma $u^m(\xi)\Phi\big(\frac{u'(\xi)}{u(\xi)}\big) - au(\xi)q'(\xi) + \sigma u(\xi) = K$, for some constant K. We can assume that this constant is going to be zero when considering ξ outside the support of u. Therefore, we have

$$u^{m-1}(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right) - aq'(\xi) + \sigma = 0, \quad \text{a.e. } \xi \in [\xi_1, \xi_2]. \tag{2.5}$$

Whence, both values

$$\lim_{\xi \to \bar{\xi}^{\pm}} u(\xi) = u_{\pm},$$

verify

$$u_{\pm}^{m-1} c - aq'(\bar{\xi}) + \sigma = 0,$$

where it has been used that we have a saturation on the left. Then, both one-sided limits are equal. That implies the continuity of u in $\bar{\xi}$. Using the regularity and the (2.3) condition in the second equation of (2.4) we get that q'' is defined in $\bar{\xi}$. In particular, the function

$$\xi \to cu^{m-1}(\xi) - q'(\xi) + \sigma,$$

has infinite derivative at $\bar{\xi}$, and it is an increasing function in a neighborhood of that point. This would give us ξ values such that $\xi \to cu^{m-1}(\xi) - q'(\xi) + \sigma < 0$, which is contradictory to (2.5) since $\Phi(s) < c$, for all $s \in \mathbb{R}$. □

In conclusion, assuming (2.3) we can define a block-type solution as follows.

Definition 2.1. Given an interval $[\xi_1, \xi_2]$, we will say that a pair of functions

$$u \in C^0[\xi_1, \xi_2] \cap C^1(\xi_1, \xi_2) \quad \text{and} \quad q \in C^0[\xi_1, \xi_2] \cap C^2(\xi_1, \xi_2)$$

constitute a block-type solution as long as

- $u(\xi) > 0$, for each $\xi \in [\xi_1, \xi_2]$ and both verify

$$\begin{aligned} &u^{m-1}(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right) - aq'(\xi) + \sigma = 0 \\ &(\alpha - \sigma\tau_1)q'(\xi) - (\sigma\tau_2 + \nu)q''(\xi) = u(\xi). \end{aligned} \tag{2.6}$$

- The singular points are $\mathcal{S} = \{\xi_1, \xi_2\}$, and both are lateral saturation points for u, that is

$$\lim_{\xi \to \xi_1^+} u'(\xi) = \infty, \qquad \lim_{\xi \to \xi_2^-} u'(\xi) = -\infty.$$

Under these conditions, we will discuss throughout the paper the existence of a cell block moving at speed σ, and this will be obtained by extending by zero the cell density u outside the interval $[\xi_1, \xi_2]$, q extends to a function from class 1 to $\mathbb{R}$ through two straight lines.

Taking $g = \Phi^{-1}$, in the sense of the composition of applications, then $g : (-c, c) \to \mathbb{R}$ is a C^1 function defined as

$$g(y) = s \leftrightarrow y = \Phi(s).$$

If, in addition, we define $r(\xi) = g'(\xi)$, we get

$$u' = ug\left(\frac{ar - \sigma}{u^{m-1}}\right).$$

Finally, we can rebuild from (2.6) the following system:

$$\begin{aligned} u' &= ug\left(\frac{ar - \sigma}{u^{m-1}}\right), \\ r' &= \frac{(\alpha - \sigma\delta_1)r - u}{\sigma\delta_2 + \nu}. \end{aligned} \tag{2.7}$$

Note that under the hypothesis on g in the previous section, this system is defined only for values of (u, r) belonging to the domain Γ defined by

$$\Gamma := \{(u, r) : u > 0, |(ar - \sigma)u^{1-m}| < c\}. \tag{2.8}$$

such that $\Gamma = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$, where

$$\Gamma_0 = \left\{(u, r) \in \Gamma : u > 0, r = \frac{\sigma}{a}\right\}, \quad \Gamma_- = \left\{(u, r) \in \Gamma : r > \frac{\sigma}{a}\right\}$$

and

$$\Gamma_+ = \left\{(u, r) \in \Gamma : r < \frac{\sigma}{a}\right\}.$$

We denote by $\gamma = \partial\Gamma = \gamma_+ \cup \gamma_- \cup \left\{\left(0, \frac{\sigma}{a}\right)\right\}$, where

$$\gamma_\pm := \left\{(u, r) \in (0, \infty) \times \mathbb{R} : \frac{ar - \sigma}{u^{m-1}} = \mp c\right\}. \tag{2.9}$$

A block-type solution corresponds to a maximal solution of (2.7) such that

$$\begin{aligned} \lim_{\xi \to \bar{\xi}^\pm} u(\xi) &= u_\pm, \\ \lim_{\xi \to \bar{\xi}^\pm} r(\xi) &= r_\pm, \end{aligned} \tag{2.10}$$

where (ξ_-, ξ_+) is the maximum interval of definition and $(u_\pm, r_\pm) \in \gamma_\pm$, see Fig. 2.

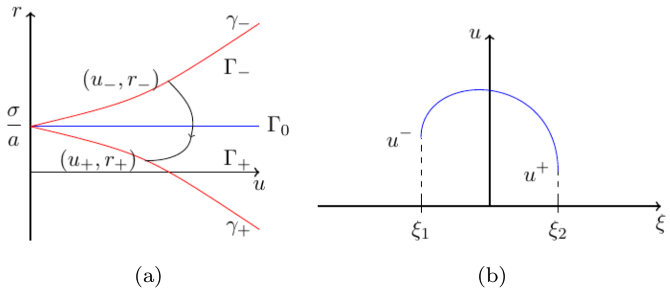


Fig. 2. (a) Representation of the field of tangent vectors of (2.7) and (b) representation of a block-type solution.

Proposition 2.1. *Under these conditions, the spatial support of the solution is necessarily bounded.*

Proof. Let us prove it by reductio ad absurdum. Assume that $\xi^+ = +\infty$. Then, there exists a sequence $\xi_n \to +\infty$ such that

$$u'(\xi_n) \to 0.$$

This implies

$$\frac{ar(\xi_n) - \sigma}{u^{m-1}(\xi_n)} \to 0.$$

Since u^+ is bounded, this implies that $r(\xi^n) \to \frac{\sigma}{a}$, but this is not possible since $r^+ \in \gamma^+$. Therefore $\xi^+ < +\infty$. The reasoning for ξ^- is analogous. □

3. Transport in the Gradient of Q

This section is devoted to analyze the case where the gradient of the chemoattractant $(\partial_x Q)$ is solution of the non-homogeneous linear transport equation:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - aU\frac{\partial Q}{\partial x}\right), \\ \frac{\partial}{\partial t}\left(\frac{\partial Q}{\partial x}\right) &- \nu \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial x}\right) = U. \end{aligned} \tag{3.1}$$

In this model, we assume that $\nu > 0$ so we are in a non-degenerate situation (remember that $a > 0$). The case $m = 1$ can be analyzed following the guidelines of the case $m > 1$, therefore we also assume that $m > 1$. As we will see, this model can be seen as a particular case of the one analyzed in the following section where the values would have another expression, but we have considered studying this case first for clarity in the exposition.

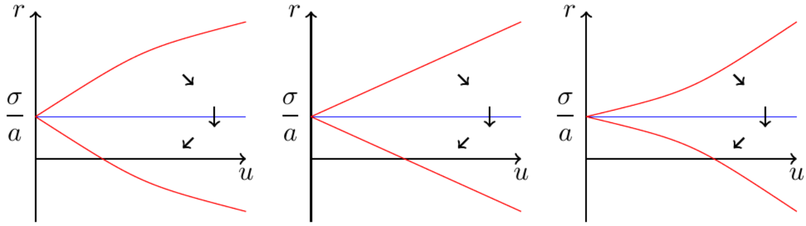
Hence, we will focus on study the existence of block solutions, defined in the previous section, which correspond with the search of orbits of the differential equation

$$\begin{aligned} u' &= ug\left(\frac{ar - \sigma}{u^{m-1}}\right), \\ r' &= -\frac{u}{\sigma + \nu}. \end{aligned} \tag{3.2}$$

This orbits connect γ_- with γ_+, where $\gamma_\pm$ were defined in (2.9). Therefore, for any initial condition in Γ defined in (2.8), we will be able to find a block solution, see Fig. 3.

Theorem 3.1. *Every maximal solution of* (3.2) *is a block solution.*

A key ingredient to prove this theorem are the following results, in which we will show that there exist an orbit connecting γ^- with γ^+, for every initial condition in the line $s = \frac{\sigma}{a}$.



(a) $1 < m < 2$ (b) $m = 2$ (c) $m \geq 2$

Fig. 3. Representation of the tangent vector field associated to (3.2).

Proposition 3.1. *Consider the initial conditions* $r(0) = \frac{\sigma}{a}$ *and* $u(0) = u_0 > 0$ *associated to* (3.2). *Then, there exists a block solution* (u, r) *corresponding to these initial data.*

Proof. Let us prove that the maximal solution of the previous problem gives rise to a block solution, by seeing that it connect a point in γ^- with a point in γ^+, according to the definition of the previous section.

We can check that $u(\xi)$ has uni-modal shape with a unique maximum at $\xi = 0$, and $r(\xi)$ is a strictly decreasing function. Therefore, we can prove the existence of the finite limits

$$\lim_{\xi \to \xi_\pm} (u(\xi), r(\xi)) =: (u_\pm, r_\pm). \tag{3.3}$$

Due to the decrease of r we obtain $r_+ < \frac{\sigma}{a} < r_-$, and taking limits in the inequality

$$\left| \frac{ar - \sigma}{u^{m-1}} \right| < c,$$

we deduce that $u_\pm$ are necessarily strictly positive.

Let us see that

$$r_+ = \frac{\sigma - cu_+^{m-1}}{a}.$$

If $r_+ \neq \frac{\sigma - cu_+^{m-1}}{a}$, then (u^+, r^+) is not in the boundary of Γ, and by a prolongation argument we get $\xi^+ = +\infty$. Therefore (u^+, r^+) will be a critical point. However, there are no critical points in the problem, so $\xi^+ < +\infty$. Using again a prolongability argument we obtain that (u^+, r^+) is in the boundary of Γ.

Also, we can prove that

$$r_- = \frac{\sigma + cu_-^{m-1}}{a},$$

by using similar arguments. □

Once demonstrated the existence of solutions for initial conditions in the vertical isocline, we will proceed to prove that for every initial condition in Γ, the associated solutions always reach the vertical isocline.

Lemma 3.1. *Every maximal solution of* (3.2) *intersects the curve* Γ_0.

Proof. Let $u, r : (\xi^-, \xi^+) \to \mathbb{R}$ a solution of (3.2), and assume that for some value $\xi_0 \in (\xi^-, \xi^+)$ we have $(u_0, r_0) := (u(\xi_0), r(\xi_0)) \in \Gamma$. Let us prove that if $r_0 > \frac{\sigma}{a}$, then there exits a $\xi_1 \in (\xi^-, \xi^+)$, such that $r(\xi_1) = \frac{\sigma}{a}$. Similarly, if $r_0 < \frac{\sigma}{a}$, we can find a value ξ_2 such that $r(\xi_2) = \frac{\sigma}{a}$, which will conclude the proof.

Assume that $r_0 > \frac{\sigma}{a}$ and $r(\xi) > \frac{\sigma}{a}$, for all $\xi \in (\xi^-, \xi^+)$ and hence, $u'(\xi) > 0$ and there exist the $\lim_{\xi \to \xi^+} u(\xi)$. If this limit is finite, then $\xi^+ = +\infty$ and we will have a critical point, but this is not possible. Therefore, we have

$$\lim_{\xi \to \xi^+} u(\xi) = +\infty. \tag{3.4}$$

Since $u(\xi) \geq u_0$ and $r(\xi) \leq r_0$, then we obtain

$$g\left(\frac{ar(\xi) - \sigma}{u^{m-1}(\xi)}\right) \leq g\left(\frac{ar_0 - \sigma}{u_0^{m-1}}\right),$$

for $\xi \in (\xi_0, \xi^+)$. Using that $r \to \frac{ar-\sigma}{u^{m-1}}$ is increasing in r, then we deduce that $u \to \frac{ar-\sigma}{u^{m-1}}$ is decreasing in u and $y \to g(y)$ is decreasing. Defining M as the value obtained above, we have that

$$u'(\xi) \leq Mu(\xi).$$

Then, combining the Gronwall Lemma together with (3.4) we deduce $\xi_+ = +\infty$. However, this cannot be possible, because $r'(\xi) = -\frac{1}{\sigma+\mu} u(\xi)$ and $u(\xi) \geq u_0$, for $\xi \in (\xi_0, \xi^+)$, and we obtain

$$r'(\xi) \leq -\frac{1}{\sigma + \mu} u_0 < 0, \quad \forall \xi \in (\xi_0, \xi^+),$$

which is a contradiction. □

Therefore, thanks to these two results and the fact that the solutions are invariant under time translation, the proof of Proposition 3.1 follows.

4. Transport and Diffusion in the Chemoattractant

In this section, we consider that the chemoattractant concentration is solution of a linear transport-diffusion equation.

$$\begin{aligned} &\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1} \frac{\partial U}{\partial x}\right) - aU \frac{\partial Q}{\partial x}\right), \\ &\frac{\partial Q}{\partial t} + \alpha \frac{\partial Q}{\partial x} - \nu \frac{\partial^2 Q}{\partial x^2} = U, \end{aligned} \tag{4.1}$$

where α is the transport speed coefficient of the chemoattractant density and ν stands for its diffusion coefficient. Our goal is to study the existence of orbits that connect γ^- with γ^+, defined in (2.9), of the differential equation:

$$\begin{aligned} u' &= ug\left(\frac{ar-\sigma}{u^{m-1}}\right), \\ r' &= \frac{1}{\nu}((\alpha-\sigma)r-u). \end{aligned} \tag{4.2}$$

The analysis of the existence of such orbits will be studied in terms of the parameter m and the relation between α and σ.

Remark 4.1. If $\sigma = \alpha$, the differential equation (4.2) is similar to (3.2). Therefore, we will have existence of block solutions for $\sigma > 0$, thanks to Theorem 3.1.

The main result describing the existence of block solution in this context is the following one.

Theorem 4.1. *Block solutions exist if one of the following conditions holds true*:

- *If* $1 < m < 2$ *and* $\sigma > 0$.
- *If* $m \geq 2$, $\sigma > 0$ *and* $\alpha \leq \alpha^*$, *for* $\alpha^* > 0$, *where*

$$\alpha^* = \left(\frac{a}{m-1}\right)^{\frac{m-1}{2m-3}} \frac{1}{c^{3-2m}}(m-2)^{\frac{m-2}{2m-3}}. \tag{4.3}$$

- *If* $m \geq 2$, $\sigma > \sigma^*(\alpha) > 0$ *and* $\alpha > \alpha^* > 0$, *where*

$$\sigma^*(\alpha) = \alpha - \frac{a}{m-1}c^{\frac{1}{1-m}}\left(\frac{m-2}{\alpha}\right)^{\frac{m-2}{m-1}}. \tag{4.4}$$

Remark 4.2. In the case $m = 2$, we have that $\sigma^*(\alpha) = \alpha - \frac{a}{c}$, which is the limit when $m \to 2$ of (4.4), and therefore solving $\sigma^*(\alpha) = 0$, we obtain the value $\alpha^* = \frac{a}{c}$. In Figure 4 we can see a qualitative representation, for $m \geq 2$, of the region of σ values for which a solution exists as a function of the parameter α.

To carry out the proof of Theorem 4.1, we are going to introduce a series of previous results.

Let us denote by η the horizontal isocline, whose equation is $r = \frac{1}{\alpha-\sigma}u$ and represents the points $(u, r) \in \Gamma$, where $r' = 0$. First, we will analyze the case in which η has positive slope, and we will focus on finding some initial values $(\bar{u}, \bar{r})$ from which we can construct the solution.

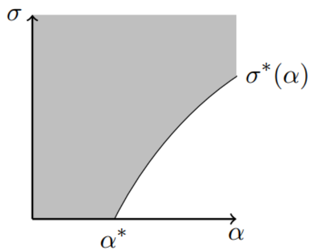


Fig. 4. Representation of the region of existence obtained in Theorem 4.1 for $m \geq 2$.

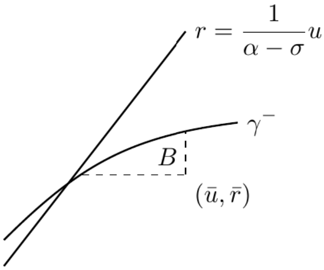


Fig. 5. Scheme of the proof of Lemma 4.1.

Lemma 4.1. *Let* $1 < m \leq 2$. *If* η *has positive slope and cuts* γ^-, *then there exist* $(\bar{u}, \bar{r}) \in \Gamma_-$ *such as*

$$r' < 0, \quad u' > 0, \quad \forall (u, r) \in B,$$

where $B := B_{(\bar{u},\bar{r})} = \{(u, r) \in \Gamma_- : u \leq \bar{u}, r \geq \bar{r}\}$.

Proof. Since $1 < m \leq 2$, then γ^- is the graph of a concave function or a straight line. It is easy to see that an increasing line below $u = 0$ will intersect γ^- at a single point. u^*, see Fig. 5. Therefore, it allows us to find a curved triangular region, denoted by B, just build taking as vertex of B any points $(\bar{u}, \bar{r}) \in \Gamma^-$ such that $\bar{u} > u^*$ and $\bar{r} > \gamma_-(u^*)$. □

The points $(\bar{u}, \bar{r})$, defined in Lemma 4.1, will allow us to construct the desired orbits of (4.2) when these are taken as initial data.

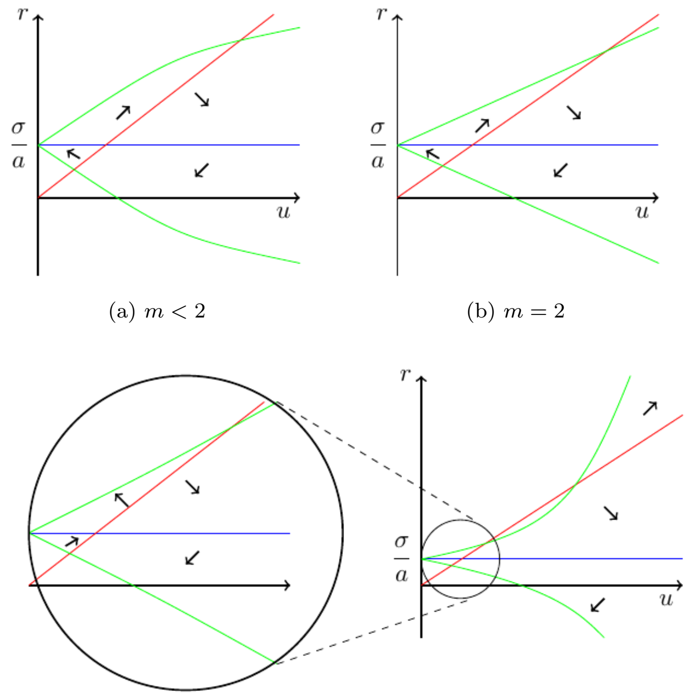
Proposition 4.1. *If* η *has positive slope and cuts* γ^-, *then there exists a block solution of* (4.2).

Proof. In the case $1 < m \leq 2$ (see Fig. 6), let us take the points $(\bar{u}, \bar{r})$, previously defined in Lemma 4.1, as the initial condition of the problem (4.2). The solution of the initial value problem $(\bar{u}, \bar{r})$ remains in B as long as it is defined. Bendixson's theorem assures us that this solution has to touch γ^-, as $\xi \to \xi^-$, since there is no equilibrium points in the set B. Moreover, this solution will always intersect the line $r = \frac{\sigma}{a}$, for some $\xi_1 \in (\xi^-, \xi^+)$. This is because $r' < 0$ and $u' > 0$, for $r > \frac{\sigma}{a}$. An argument similar to that used in Lemma 3.1 allows us to prove the existence of a value ξ_1 at which the solution intersects the straight line $r = \frac{\sigma}{a}$.

We cannot know how the solutions (u_1, r_1), connecting γ^- with $r = \frac{\sigma}{a}$, will behave once they go through the line $r = \frac{\sigma}{a}$.

Let us define $u^* = u_1(\xi_1)$. Observe that the solution of (4.2), with initial condition $u(0) = \tilde{u}$, $r(0) = \frac{\sigma}{a}$, will touch γ^- when $\xi \to \xi^-$, for any value $\tilde{u} \geq u^*$. This is due to the fact that the orbits of the autonomous systems cannot intersect.

Therefore, to finish the proof it remains to find a value $\tilde{u}$ such that the solution of the initial value problem $(\tilde{u}, \frac{\sigma}{a})$ reaches γ^+.



(a) $m < 2$ (b) $m = 2$

(c) $m > 2$

Fig. 6. Representation of the phase diagram as a function of m, when η intersects γ^+.

Let $(\hat{u}, \hat{r})$ be the intersection of the straight line η with the curve γ^+. Then, the solution of the initial value problem $(\hat{r}, u^*)$ touches γ^+ as $\xi \to \xi^+$, due to the fact that $r' < 0$ for $r \le \hat{r}$. In addition, this solution will intersect the straight line $r = \frac{\sigma}{a}$ at a point $(\tilde{u}, \frac{\sigma}{a})$, by a symmetric argument to the one made in Lemma 3.1.

In the case $m > 2$, the proof is carried out in a similar way. Indeed, if η intersects γ^-, this intersection can be made at two points or at one tangent point (see Fig. 6). In both cases, we can define

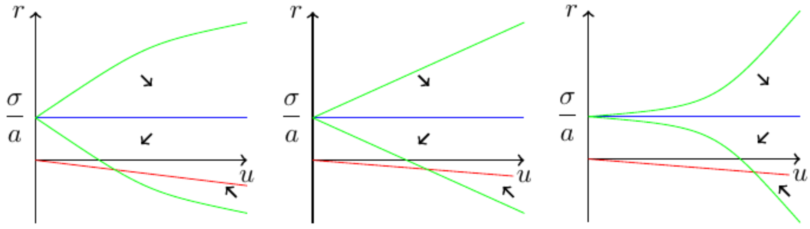
$$\underline{u} = \max\left\{u \in (0, +\infty) : \frac{1}{\alpha - \sigma} u = \gamma^-(u)\right\}.$$

Therefore, the set

$$A = \{(u, r) \in \Gamma_- : u > \underline{u}, r \ge \eta(u)\}$$

is negatively invariant, since $u' > 0$ and $r' \ge 0$, for all $(u, r) \in A$. Therefore, following the same ideas as in the previous case, we can show that the solution connects γ^- to $r = \frac{\sigma}{a}$, reaching the line $r = \frac{\sigma}{a}$, for some $\xi_1 \in (\xi^-, \xi^+)$. So, as in the previous case, we are able to find an initial condition $(\tilde{u}, \frac{\sigma}{a})$ whose solution connects γ^- with γ^+. □

Let us now study the case in which η has negative slope.



(a) $1 < m < 2$ (b) $m = 2$ (c) $m > 2$

Fig. 7. Representation of the phase diagram as a function of m, when η is decreasing.

Proposition 4.2. *If η has negative slope, then there exists a block solution of* (4.2).

Proof. The directions of the vector field, represented in Fig. 7, show that the solution with initial data $(\hat{u}, \frac{\sigma}{a})$ will touch γ^- and γ^+, for any point $(\hat{u}, \frac{\sigma}{a})$ in the straight line $r = \frac{\sigma}{a}$ which is to the left of the intersection point of the straight line η with γ^+. The proof argument is similar to the one made in the proof of Proposition 2.1. □

These results inform us under which conditions we can find block solutions. To finish the proof of Theorem 4.1, it is necessary to see what relationships between the parameters allow us to obtain these solutions.

Proof of Theorem 4.1. The equation of the line η is $\frac{1}{\alpha-\sigma}u$. If $\sigma > \alpha$, η has negative slope, and by Proposition 4.2 we have existence of a block solution. If $\sigma = \alpha$, we have also existence of solution by arguing as in Remark 4.1.

On the other hand, if $\sigma < \alpha$, we have to study the relative position between γ^- and η. Proposition 4.1 establishes the existence of solution when η has positive slope. Therefore, we have to analyze the possibilities of intersection between η and γ^-.

If $1 < m < 2$, η will always intersect γ^-, when $0 < \sigma < \alpha$.

If $m = 2$, the slope of η must be greater than the slope of γ^-, which is a line in this case. This is fulfilled when $\frac{c}{a} < \frac{1}{\alpha-\sigma}$, i.e.

$$\sigma > \alpha - \frac{a}{c}. \tag{4.5}$$

Finally, for $m > 2$, the intersection points are given by the roots of the following equation:

$$\frac{cu^{m-1}}{a} + \frac{\sigma}{a} - \frac{u}{\alpha - \sigma} = 0.$$

The existence of roots of this equation is equivalent to prove that the minimum takes negative values, this holds true for

$$\sigma > \alpha - \frac{a}{m-1} c^{\frac{1}{1-m}} \left(\frac{m-2}{\alpha} \right)^{\frac{m-2}{m-1}} = f(\alpha).$$

Note that if $m = 2$ this equation coincides with (4.5), as $m \to 2$.

It can be seen that $f(\alpha) \leq 0$, if $\alpha \leq \alpha^*$, where

$$\alpha^* = \left(\frac{a}{m-1}\right)^{\frac{m-1}{2m-3}} \frac{1}{c^{3-2m}}(m-2)^{\frac{m-2}{2m-3}}.$$

If $\alpha > \alpha^*$, then the expression $\sigma > f(\alpha)$ is equivalent to $\sigma > \sigma^*(\alpha)$ which is given by (4.4). □

4.1. *Non-existence of block solution*

Once we have analyzed the existence of solution in the previous section, let us see under what conditions we can prove the non-existence of block solutions.

For this purpose, we will consider the function $\theta : (-c, c) \to \mathbb{R}$ defined as

$$\theta(y) = \frac{(\alpha - \sigma)y}{\nu} - (m-1)yg(y).$$

Observe that the function satisfies that $\theta(-c) = \theta(c) = -\infty$, therefore there exists the value $\theta_0 = \max_{y \in (-c,c)} \theta(y)$.

On the other hand, let us consider the function $\omega : (0, +\infty) \to \mathbb{R}$ defined as

$$\omega(u) = \frac{au - \sigma(\alpha - \sigma)}{\nu u^{m-1}}.$$

Arguing as before, let us consider the value $\omega_0 = \max_{u \in (0,+\infty)} \omega(u)$, whose expression is

$$\omega_0 = \frac{1}{v}\left(\frac{\sigma(\alpha-\sigma)}{m-2}\right)^{2-m}\left(\frac{a}{(m-1)}\right)^{m-1}.$$

With these two constants we obtain the following non-existence result.

Theorem 4.2. *If $\theta_0 > \omega_0$, then there is no block-type solution.*

Proof. Let us take $y(\xi) = \frac{ar(\xi)-\sigma}{u^{m-1}(\xi)}$, which satisfies the differential equation

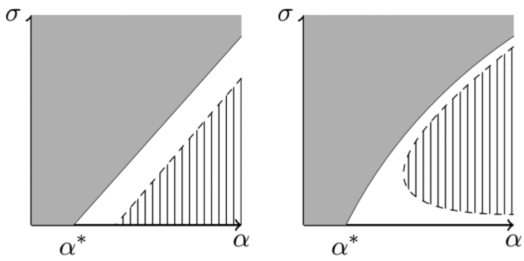
$$u' = ug(y), \quad y' = \frac{\frac{a}{v}[(\alpha-\sigma)[\frac{u^{m-1}y+\sigma}{a}] - u] - (m-1)u^{m-1}yg(y)}{u^{m-1}}. \tag{4.6}$$

Observe that a block-type solution is now a connection between $y = c$ and $y = -c$. Taking $\bar{y}$ such that $\theta(\bar{y}) > \max_{u \in (0,+\infty)} \omega(u)$ then the expression of the second equation of (4.6) provides $y' > 0$, for all $u \in (0, +\infty)$. Therefore, it would not be possible to connect $y = c$ with $y = -c$. □

Remark 4.3. For example, in the Wilson operator $g(u)$ is defined by $g(u) = \frac{1}{\mu}\frac{u}{1-\frac{|u|}{c}}$. Then, we can calculate explicitly the values of θ_0 and ω_0. Those values are

$$\theta_0 = c\left(\sqrt{\left(\frac{\alpha-\sigma}{\nu} + (m-1)\frac{c}{\mu}\right)} - \sqrt{(m-1)\frac{c}{\mu}}\right)^2,$$

$$\omega_0 = \frac{1}{\nu}\left(\frac{\sigma(\alpha-\sigma)}{m-2}\right)^{2-m}\left(\frac{a}{m-1}\right)^{m-1}.$$



(a) $m = 2$ (b) $m = 3$

Fig. 8. Representation of the region of existence (gray region) and non-existence (pattern region) of solution in Wilson's model, for $c = \mu = a = \nu = 1$.

Combining and approximating them we obtain the following inequality:

$$\sigma^{\frac{m}{2}-1}(\alpha - \sigma)^{\frac{m}{2}} \geq \Theta\left(\sqrt{\left(\frac{\alpha}{\nu} + (m-1)\frac{c}{\mu}\right)} + \sqrt{(m-1)\frac{c}{\mu}}\right), \tag{4.7}$$

where

$$\Theta = \sqrt{\frac{\nu}{c}}\left(\frac{a}{m-1}\right)^{\frac{m-1}{2}} \frac{1}{(m-2)^{\frac{2-m}{2}}}.$$

Using Theorems 4.1 and 4.2, we can establish the region of existence and non-existence of solution for a given parametric configuration, (see Fig. 8).

Observe that the left-hand side of the inequality (4.7) has uni-modal shape and, therefore, we obtain an interval of σ-values for which there is no solution, with all parameters fixed. We can see this behavior in Fig. 8. Note that in the limit case $m = 2$, the region of non-existence is bounded by a straight line.

Moreover, it has been numerically observed, it is possible to find a block-type solution under certain parameter settings for $m > 2$ in the region between the non-existence zone and $\sigma = 0$.

4.2. *Case* $m = 1$

In the previous sections, we have always considered the case $m > 1$, for which we have shown the existence and non-existence of solution under certain configurations of the parameters. But what happens in the case $m = 1$?

Taking $m = 1$, the system can be written as follows:

$$\begin{aligned} u' &= ug(ar - \sigma), \\ r' &= \frac{1}{\nu}((\alpha - \sigma)r - u). \end{aligned} \tag{4.8}$$

Here, γ^+ and γ^- are horizontal straight lines and it is possible that the point $(u, r) = (0, 0)$ belongs to Γ. If we remove the $\sigma = \alpha$ and $\sigma = c$ cases, an isocline analysis reveals the situations given by Fig. 9. In this case, due to the shape of

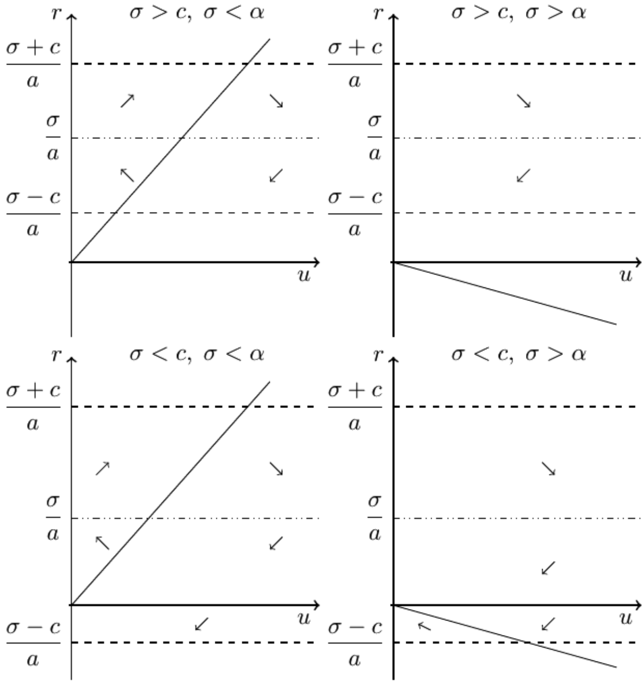


Fig. 9. Representation of the phase diagram of (4.8), as a function of σ, m and α.

the curves γ^+ and γ^-, we have more difficulties in finding block solutions of the equation, i.e. solutions that touch these curves. This fact is due to the behavior of the function $g(u)$ as $u \to \pm c$. In fact, we can show that we may not find a block solution, for a certain behavior of the function g, as the following result shows.

Proposition 4.3. *Assume that*

$$\frac{1}{g(u)} = \mathcal{O}(c-u), \quad \text{as } u \to c,$$

then there is no block-type solution.

Remark 4.4. A type of flux-saturated function that satisfies this condition is the Wilson operator.

Proof. We are going to use a reductio ad absurdum argument to prove the lemma. Suppose that there is a block solution, then there would be a connection between γ^+ and γ^-. This means we can find a solution branch $(\bar{u}, \bar{r})$ in the interval $(\xi^-, \xi^- + \epsilon]$ that starts over points of γ^-.

On the other hand, we can consider the problem

$$r'(u) = \frac{1}{\nu} \frac{(\alpha - \sigma)r - u}{ug(ar - \sigma)}, \quad r(u_0) = r_0, \tag{4.9}$$

where $(u_0, r_0) \in \gamma^-$. The Picard–Lindelof theorem can be applied to (4.9) extended by zero, prove uniqueness of solution that takes the form $r(u) = \frac{\sigma+c}{a}$.

However, we had assumed that there was a solution branch $(\bar{u}, \bar{r})$ that in the form $\bar{r}(\bar{u})$ would be the solution of the problem (4.9), which is not possible due to the previous uniqueness argument. □

On the other hand, we can also establish conditions for the existence of solution.

Proposition 4.4. *Assume that*

$$\frac{1}{g(u)} = \mathcal{O}((c-u)^{1/p}), \quad \text{as } u \to c,$$

then there exists a block-type solution.

Remark 4.5. We can find some flux-saturated functions satisfying these conditions on g, such as the hyperbolic tangent operator, or the Larson operator to which it is associated $g(u) = \frac{1}{\mu}\frac{u}{\sqrt[p]{1-(\frac{|u|}{c})^p}}$.

The non-uniqueness of the problem (4.9), under the hypothesis of Proposition 4.4, will allow us to prove the existence of a solution for this model. To do this, we will consider the following result.

Lemma 4.2. *Consider the equation*

$$x' = x^{\frac{1}{p}} a(t,x), \quad x > 0, \ \ p > 1, \tag{4.10}$$

where the function a admits a continuous extension in a neighborhood of $(0,0)$ and $a(0,0) > 0$. Then, the initial value problem (4.10) with $x(0) = 0$ has a solution $x(t) > 0$ in a neighborhood of the right-hand side of $t = 0$.

Proof. Let us make the change of variable $y = x^{\frac{p-1}{p}}$. We have the following initial value problem:

$$y' = \frac{p-1}{p} a\left(t, y^{\frac{p}{p-1}}\right), \quad y(0) = 0.$$

By using the Picard theorem, the problem has a solution $y(t)$ with $y'(0) = \frac{p-1}{p}a(0,0) > 0$. Therefore, we have that $y(t) > 0$ on positive values in the neighborhood of $t = 0$. □

Remark 4.6. If $a(0,0) < 0$, then the neighborhood is on the left-hand side of $t = 0$.

We can now proceed to prove Proposition 4.4.

Proof. We will use the same constructive scheme developed in the proof of Proposition 4.1.

First, we will consider $\tilde{u}$ sufficiently large, such that $r' < 0$ and u' has an unimodal shape, under the conditions $u > \tilde{u}$ and $r \in \left(\frac{\sigma+c}{a}, \frac{\sigma-c}{a}\right)$.

Using Lemma 4.2 and system (4.9), we are be able to launch solutions from the curves γ^+ and γ^-. Taking the initial conditions such that $u_0 > \tilde{u}$, these solutions will always touch the curve $r = \frac{\sigma}{a}$ (see the proof of Lemma 3.1).

Let us denote by (u^-, r^-) one of the solution launched from γ^- and (u^+, r^+) one of the solution launched from γ^+. Those solutions touch the curve $r = \frac{\sigma}{a}$ at some value u_1^- and u_1^+, respectively. Assume $u_1^- < u_1^+$, the opposite case can be treated similarly. Then, we have that the solution (u^+, r^+), once it crosses the line $r = \frac{\sigma}{a}$, it will always touch the curve γ', since $u' > 0$, and it cannot touch the orbit of the solution (u^-, r^-). Therefore, the solution (u^+, r^+) connects γ^- with γ^+. □

4.3. *Diffusion without transport*

Formally, letting $\alpha = 0$ in the model (4.1) leads to the following system:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - aU\frac{\partial Q}{\partial x}\right), \\ \frac{\partial Q}{\partial t} - \nu\frac{\partial^2 Q}{\partial x^2} &= U, \end{aligned} \tag{4.11}$$

corresponding to a process where diffusion of the chemoattractant dominates the dynamics. As we discussed in the previous section, using the jump condition we can derive a differential system for the description of entropy solutions. In this case, the equations can be obtained by letting $\alpha = 0$ in Eq. (4.2)

$$\begin{aligned} u' &= ug\left(\frac{ar - \sigma}{u^{m-1}}\right), \\ r' &= -\frac{1}{\nu}(u + \sigma r). \end{aligned} \tag{4.12}$$

Therefore, we can analyze the existence of block-type solutions of this problem as a particular case of (4.2). From Theorem 4.1, we have the following result.

Corollary 4.1. *There is a block-type solution of* (4.12), *for all* $\sigma > 0$.

4.4. *Transport without diffusion in Q*

In this last section, we consider that the chemoattractant concentration is solution of a linear transport-diffusion equation, which corresponds to the case $\tau_1 = 1$ and $\tau_2 = 0 = \nu$ in Eq. (2.1), namely

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - aU\frac{\partial Q}{\partial x}\right), \\ \frac{\partial Q}{\partial t} + \alpha\frac{\partial Q}{\partial x} &= U. \end{aligned} \tag{4.13}$$

With a similar argument to the one in Sec. 2, we would obtain (2.4). However, in this case, the expression obtained for q' does not need to be regular since (2.3) is not satisfied. However, if $\sigma \neq \alpha$ we can expect that $q \in H^1_{\text{loc}}(\mathbb{R})$ and

$$(\alpha - \sigma)q'(\xi) = u(\xi), \quad \text{a.e. } \xi \in \mathbb{R}.$$

Using this in the first expression of (2.4), as in Sec. 2, we obtain the existence of a value K such as

$$u^m(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right) - \frac{a}{\alpha - \sigma}u^2(\xi) + \sigma u(\xi) = K, \quad \text{a.e. } \xi \in \mathbb{R},$$

where u' is the Radon–Nikodym derivative. Since $u^m(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right)$ is assumed to be 0 if $u(\xi) = 0$, it follows that if u has compact support, then outside this support $u = 0$, and thus $K = 0$. Therefore, solutions of (4.13) will satisfy the equation

$$\begin{aligned} u' &= ug\left(\frac{\frac{a}{\alpha-\sigma}u - \sigma}{u^{m-1}}\right), \\ q' &= \frac{u}{\alpha - \sigma}, \end{aligned} \tag{4.14}$$

at the points of its support.

Remark 4.7. Since condition (2.3) is not verified, there is no clear description of the block-type solutions. Therefore, we will describe the maximal branches solutions of (4.16) in order to describe a possible connection between them.

Because of the large casuistry, the description of the chemoattractant will not be discussed here. In this section, it will be assumed that $\sigma \neq \alpha$, since the solution $u = cte$ is obtained for $\sigma = \alpha$.

Setting

$$H(u) = \frac{\frac{a}{\alpha-\sigma}u - \sigma}{u^{m-1}} \tag{4.15}$$

the first equation of (4.14) can be written as

$$u' = ug(H(u)), \tag{4.16}$$

and the expression of the chemoattractant follows after integrate the second equation of (4.14).

Since the function g is only defined in $(-c, c)$, it is important to know the values of u for which $H(u) \in (-c, c)$. We will distinguish several cases based on the values of α, σ, m.

Case 1. If $\sigma > \alpha$, H is always negative because $H(0) = -\infty$ and there is no root of H.

Case 1.1. If $m > 2$, H is increasing and $H(\infty) = 0$.

Solutions live in the interval (u^+, ∞) and are decreasing. It can be extended to $-\infty$, and $u(-\infty) = +\infty$. For a finite value $u(\xi^+) = u^+$ and $u'(\xi^+) = -\infty$. This can be observed in Fig. 10.

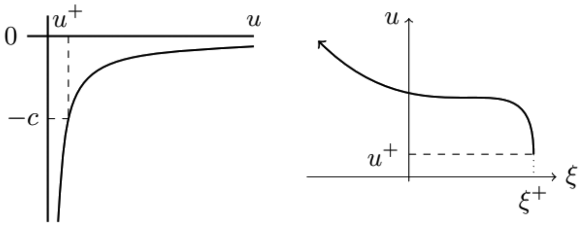


Fig. 10. Behavior of H in Case 1.1 and the profile of the solution.

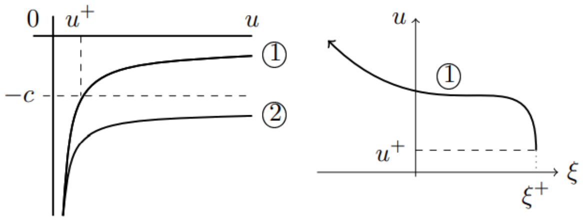


Fig. 11. Behavior of H in Case 1.2 and the profile of the solution. Note that for ② there is no solution because the differential equation is not defined.

Case 1.2. When $m = 2$, H is again increasing, negative but $H(+\infty) = -\frac{a}{\sigma-\alpha} < 0$. This case opens two possibilities see Fig. 11.

1.2.① The first alternative is $\sigma > \alpha + \frac{a}{c}$, which is similar to Case 1.1, see Fig. 11.
1.2.② The second case corresponds with $\sigma \leq \alpha + \frac{a}{c}$. In this situation there are no solutions because there are no points in which the differential equation is defined.

Case 1.3. In the case $1 < m < 2$ the function H satisfies $H(\infty) = -\infty$ and there is only a critical point at u^* with a maximum value H^*, which are given by

$$H^* := H(u^*) < 0, \quad u^* = \frac{\sigma(\alpha - \sigma)(1 - m)}{a(2 - m)}. \tag{4.17}$$

According to the relative position of H^* with respect to c, we can distinguish the following cases, see Fig. 12:

1.3.① If $H^* \leq -c$, there are no solutions since there are no points for which the differential equation is defined.
1.3.② If $-c < H^*$, the differential equation is only defined for $u \in (u^+, u^-)$. Therefore, the solutions are defined in a bounded interval (ξ_-, ξ_+), where $u(\xi^-) = u^-$, $u(\xi^+) = u^+$, and $u'(\xi^-) = u'(\xi^+) = -\infty$.

Case 2. If $0 < \sigma < \alpha$, then $H(0) = -\infty$, but H changes sign in $\hat{u}$, which is given by

$$\hat{u} = \frac{\sigma(\alpha - \sigma)}{a}. \tag{4.18}$$

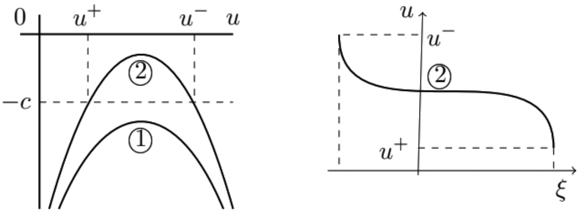


Fig. 12. Behavior of H in Case 1.3 and the profile of the solution. Note that for ① there is no solution because the differential equation is not defined.

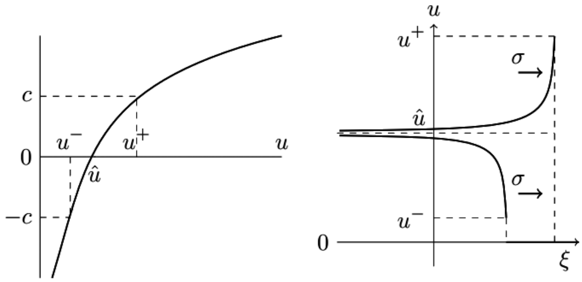


Fig. 13. Behavior of H in the case $1 < m < 2$ and $0 < \sigma < \alpha$. Note that both traveling waves are continued by zero.

Case 2.1. In the case $1 < m < 2$, H has no critical points and $H(+\infty) = +\infty$. Therefore, there are two types of traveling waves, one increasing and one decreasing that are represented in Fig. 13.

Case 2.2. If $m = 2$ and $0 < \alpha < \sigma$, H has no critical points, but there is a finite asymptotic value $H(+\infty) = \dfrac{a}{\alpha - \sigma} > 0$. The position of this asymptotic value with respect to c gives us three different situations:

2.2.① If $\frac{a}{\alpha-\sigma} > c$. This is a scenario similar to Case 2.1, see the graph in Fig. 14.

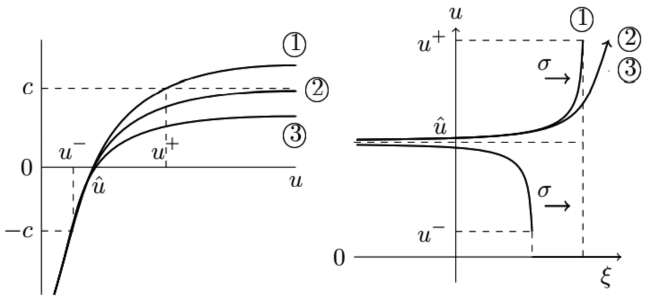


Fig. 14. Behavior of H in the case $m = 2$ and $0 < \alpha < \sigma$. In this case, the traveling waves that have a finite height are continued by zero. The traveling wave of type ② and ③ are not bounded, and, therefore, is conditioned by a more general theory of the initial value problem.

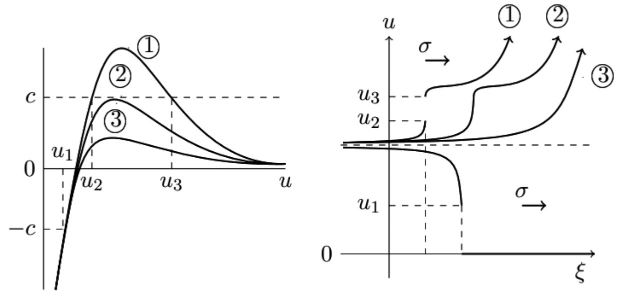


Fig. 15. Behavior of H in the case $m > 2$ and $0 < \alpha < \sigma$.

2.2.② If $\frac{a}{\alpha-\sigma} = c$. The modification in this case is the non-existence of a point r_+ that cuts the graph of H to the c level (see Fig. 14), and now the solutions end at infinity. They can reach infinity in finite or infinite time depending on the properties of Φ.

2.2.③ If $0 < \frac{a}{\alpha-\sigma} < c$. This is a situation similar to the previous one, but infinite is reached in infinite time, see Fig. 14.

Case 2.3. In the case $m > 2$, $H(+\infty) = 0$, and the function H reaches a maximum level $H^* > 0$. Then, it is necessary to compare this number with c and we can define three different scenarios and the values of σ for which the different traveling waves are defined, see Fig. 15.

4.4.1. *Conclusion and summary*

The idea of this last section is to determine under which conditions we can find block-type solutions. To do this we have analyzed all the different solution profiles satisfying Eq. (4.14).

We have basically found two types of profiles that we can denominate, according to their character, as increasing or decreasing profiles. Both are separated by the level u^*, defined in (4.18), which defines the point of possible sign change of the H function.

However, the only type of compact support solution we have found has a decreasing profile, see Fig. 16. This compact support solution exists for $\sigma > \alpha$,

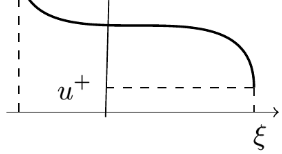


Fig. 16. Representation of the only block solution of (4.13).

$1 < m < 2$ and $H^* > -c$. The last condition can be expressed, after several standard calculations, as

$$\frac{\sigma^{(2-m)}}{(\sigma-\alpha)^{m-1}} > \frac{c}{2-m}\left(\frac{m-1}{a(2-m)}\right)^{m-1}.$$

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