Calc. Var. (2023) 62:185
https://doi.org/10.1007/s00526-023-02528-0

Calculus of Variations

Compact anisotropic stable hypersurfaces with free boundary in convex solid cones

César Rosales[1]

Received: 17 February 2023 / Accepted: 18 June 2023

Abstract
We consider a convex solid cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ with vertex at the origin and boundary $\partial\mathcal{C}$ smooth away from 0. Our main result shows that a compact two-sided hypersurface Σ immersed in $\mathcal{C}$ with free boundary in $\partial\mathcal{C} \setminus \{0\}$ and minimizing, up to second order, an anisotropic area functional under a volume constraint is contained in a Wulff-shape. The technique of proof also works for a non-smooth convex cone $\mathcal{C}$ provided the boundary of Σ is away from the singular set of $\partial\mathcal{C}$.

Mathematics Subject Classification 49Q20 · 53C42

1 Introduction

In this paper we study a variational problem for Euclidean hypersurfaces associated to an energy functional of *anisotropic character*. This means that the energy is computed by integrating an elliptic parametric function (or surface tension) which depends on the normal direction along the hypersurface. As it is explained in the introduction of Taylor [38] these functionals provide a mathematical model to study solid crystals. In the particular case of a constant surface tension we obtain the isotropic case, where the energy is proportional to the Euclidean area.

The surface tension that we consider in this work is the asymmetric norm given by the support function h_K of a smooth strictly convex body $K \subset \mathbb{R}^{n+1}$ containing the origin in its interior. The corresponding *anisotropic area* of a two-sided hypersurface Σ with unit normal N has the expression $A_K(\Sigma) := \int_\Sigma h_K(N)\, d\Sigma$, where $d\Sigma$ is the Euclidean area element. By using the metric projection onto K it is possible to define also anisotropic counterparts

Communicated by Andrea Mondino.

The author was supported by the research grant PID2020-118180GB-I00 funded by MCIN/AEI/10.13039/501100011033 and the Junta de Andalucía grant PY20-00164.

✉ César Rosales
crosales@ugr.es

1 Departamento de Geometría y Topología and Excellence Research Unit, “Modeling Nature” (MNat) , Universidad de Granada, 18071 Granada, Spain

Published online: 03 July 2023

to some of the classical notions in the extrinsic geometry of hypersurfaces, see Reilly [32] and Sect. 2 for a precise description. When K is the round unit ball about the origin we have $h_K(N) = 1$, so that we recover the isotropic situation. We point out that the anisotropic geometry of hypersurfaces is typically introduced by means of a function $F : \mathbb{S}^n \to \mathbb{R}^+$ instead of a convex body K, see Remark 2.3 for more details.

It is well known that the round spheres uniquely solve the isoperimetric problem in $\mathbb{R}^{n+1}$. In our anisotropic setting, the strictly convex hypersurface ∂K uniquely minimizes, up to translations and dilations centered at the origin, the anisotropic area computed with respect to the outer unit normal among hypersurfaces enclosing the same Euclidean volume. There are different proofs of this statement relying on analytical and geometric techniques, see Taylor [37, 38], Fonseca and Müller [13], Brothers and Morgan [3], Milman and Rotem [28], and Cabré et al. [4]. The minimizer ∂K is usually referred to as the *Wulff shape* in honor of the crystallographer Georg Wulff, who first constructed the optimal crystal for a specified integrand [43].

As in the isotropic case, it is interesting to analyze the critical points (stationary hypersurfaces) and the second order minima (stable hypersurfaces) of the anisotropic area under a volume constraint. From the first variational formulas, a hypersurface Σ is *anisotropic stationary* if and only if its *anisotropic mean curvature* H_K defined in (2.6) is constant, see for instance Palmer [31] or Clarenz [7]. This motivated the generalization to the anisotropic context of some classical results for constant mean curvature hypersurfaces in $\mathbb{R}^{n+1}$. In this direction, an Alexandrov type theorem (resp. a Hopf type theorem) characterizing the Wulff shape among compact embedded hypersurfaces (resp. immersed spheres) with H_K constant was obtained by Morgan [29] for $n = 1$, and by He et al. [15] in higher dimension (resp. by He and Li [17], and Koiso and Palmer [25]). On the other hand, Palmer [31] and Winklmann [42] employed the second variation of the anisotropic area to show that, up to translation and homothety, the Wulff shape ∂K is the unique compact, two-sided, *anisotropic stable hypersurface* immersed in $\mathbb{R}^{n+1}$. This is an anisotropic extension of a celebrated theorem of Barbosa and do Carmo [2] also discussed by Wente [41].

When we consider a smooth proper domain Ω instead of the whole space $\mathbb{R}^{n+1}$, we are naturally led to study the *anisotropic partitioning problem*. Here, we seek minimizers of the functional A_K among hypersurfaces inside Ω, possibly with non-empty boundary contained in $\partial\Omega$, and separating a fixed Euclidean volume. A critical point Σ for this problem is called an anisotropic stationary hypersurface *with free boundary* in $\partial\Omega$. From the calculus of $A'_K(0)$, and reasoning as Koiso and Palmer [22], this is equivalent to that H_K is constant on Σ and the anisotropic normal N_K defined in (2.2) is tangent to $\partial\Omega$ along $\partial\Sigma$. The more general case of *anisotropic capillary hypersurfaces* arises when both, H_K and the angle between N_K and the inner unit normal to $\partial\Omega$, are constant. These are critical points of an energy functional which involves not only $A_K(\Sigma)$ but also the wetting area of the set bounded by $\partial\Sigma$ in $\partial\Omega$. We remark that isotropic capillary hypersurfaces in convex domains of $\mathbb{R}^{n+1}$ have been extensively investigated, see Ros and Souam [34], Wang and Xia [39], and the references therein.

An anisotropic stationary hypersurface with free boundary in $\partial\Omega$ is *stable* if $A''_K(0)$ is nonnegative under variations preserving the volume separated by Σ and the boundary of Ω. The computation of $A''_K(0)$ and the subsequent analysis of anisotropic stable hypersurfaces have been treated in several previous works. In [22–24], Koiso and Palmer considered stable capillary surfaces when Ω is a slab of $\mathbb{R}^3$ and the Wulff shape is rotationally symmetric about a line orthogonal to $\partial\Omega$. In [19, 21], Koiso classified compact anisotropic stable capillary hypersurfaces disjoint from the edges in wedge-shaped domains of $\mathbb{R}^{n+1}$. Inside a smooth domain of revolution $\Omega \subset \mathbb{R}^3$, and for certain rotationally symmetric surface tensions, Barbosa and Silva [1] established that the totally geodesic disks orthogonal to the revolution axis are the unique compact stationary surfaces with free boundary in $\partial\Omega$ and meeting $\partial\Omega$ orthogonally. By assuming convexity of Ω they also discussed the stability of these surfaces. More recently, Jia et al. [18] have employed a Heintze-Karcher inequality and a Minkowski formula to prove that any compact, embedded, anisotropic capillary hypersurface inside an open half-space of $\mathbb{R}^{n+1}$ is part of a Wulff shape. In [14], besides the computation of a very general second variation formula, Guo and Xia have generalized the argument of Winklmann [42] after Barbosa and do Carmo [2] to show that a compact, two-sided, anisotropic stable capillary hypersurface immersed in a Euclidean half-space of $\mathbb{R}^{n+1}$ is a truncated Wulff shape.

In this paper we are interested in the stability question when the ambient domain is an open solid cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ with vertex at the origin and boundary $\partial\mathcal{C}$ smooth away from 0. We do not require $\mathcal{C}$ to have any kind of symmetry. In the isotropic case it is known that round balls about the vertex provide the unique solutions to the partitioning problem in a (possibly non-smooth) convex cone $\mathcal{C}$, up to translations leaving $\mathcal{C}$ invariant, see Lions and Pacella [26], and Figalli and Indrei [12]. When $\partial\mathcal{C} \setminus \{0\}$ is smooth, Ritoré and the author [33] solved the problem by combining existence of minimizers with the classification of compact stable hypersurfaces with free boundary in $\partial\mathcal{C}$. A similar characterization result for homogenous weights in convex cones is due to Cañete and the author [5]. We must also mention the papers of Choe and Park [6], and Pacella and Tralli [30], where an Alexandrov type theorem for compact, embedded, stationary hypersurfaces in convex cones is proven.

In the anisotropic case it is easy to check that the truncated Wulff shape $\partial K \cap \overline{\mathcal{C}}$ is an anisotropic stationary hypersurface with free boundary in $\partial\mathcal{C}$, see Example 4.2. In [40], Weng derived an anisotropic version of the aforementioned result of Pacella and Tralli [30, Thm. 1.2]. By assuming an integral condition along $\partial\Sigma$ that involves the conormal vector it is shown that a compact embedded hypersurface Σ in a smooth convex cone $\mathcal{C}$ with constant anisotropic mean curvature and boundary $\partial\Sigma \subset \partial\mathcal{C} \setminus \{0\}$ must be the truncated Wulff shape, up to translation and homothety. Also when the cone $\mathcal{C}$ is convex, results of Milman and Rotem [28], and Cabré et al. [4] for anisotropic area functionals with homogeneous weights entail that $\partial K \cap \overline{\mathcal{C}}$ minimizes the anisotropic area (computed with respect to the outer unit normal) for fixed volume. We remark that the techniques in [28] and [4] still hold under weaker regularity assumptions on the cone $\mathcal{C}$ and the surface tension. By extending arguments of Figalli and Indrei [12], the uniqueness of $\partial K \cap \overline{\mathcal{C}}$ as a solution to the partitioning problem in $\mathcal{C}$ was recently discussed by Dipierro et al. [11, Thm. 4.2]. The fact that $\partial K \cap \overline{\mathcal{C}}$ is a minimizer implies, in particular, the weaker property that it is an anisotropic stable hypersurface with free boundary in $\partial\mathcal{C}$. Our main result, Theorem 4.8, is the following uniqueness statement:

Up to a translation and a dilation centered at the vertex, any compact, connected, two-sided, anisotropic stable hypersurface immersed in a convex solid cone $\mathcal{C}$ with free boundary in $\partial\mathcal{C} \setminus \{0\}$ is part of the Wulff shape.

We must observe that, in general, we cannot expect $\partial K \cap \overline{\mathcal{C}}$ to be the only anisotropic stable hypersurface, up to translation and homothety, see Example 4.7. This motivates us to find additional conditions on the cone in order to deduce stronger rigidity consequences. With this idea in mind, in Corollary 4.11 we prove that a compact anisotropic stable hypersurface in a cone $\mathcal{C}$ over a smooth strictly convex domain of the sphere $\mathbb{S}^n$ must be a dilation of $\partial K \cap \overline{\mathcal{C}}$. On the other hand, for the case of a Euclidean half-space $\mathcal{H}$, recently treated by Guo and Xia [14], we can conclude that $\partial K \cap \overline{\mathcal{H}}$ is the only compact anisotropic stable hypersurface, up to translations along the boundary hyperplane.

Our proof of Theorem 4.8 is different from the ones in [5, 33] and [14], which are mainly inspired by Barbosa and do Carmo [2, Thm. (1.3)]. Instead, we adapt to our setting the idea employed by Wente [41] to characterize the round spheres as the only compact, two-sided, stable hypersurfaces immersed in $\mathbb{R}^{n+1}$. Wente applied the stability inequality with the special volume-preserving deformation obtained by parallel hypersurfaces dilated to keep the volume constant. After computing the second variation formulas for this particular variation, he observed that the area decreases unless the hypersurface is a round sphere. By using dilations of *anisotropic parallel hypersurfaces*, Palmer [31] proved the uniqueness of the Wulff shape as a compact, two-sided, anisotropic stable hypersurface immersed in $\mathbb{R}^{n+1}$. The variation of a hypersurface Σ by anisotropic parallels is defined as $\psi_t(p) := p+t\,N_K(p)$, where $p \in \Sigma$, $t \in \mathbb{R}$ and N_K is the anisotropic normal introduced in (2.2). With a similar argument, Koiso [19, Thm. 4], [21, Thm. 1] was able to classify compact anisotropic stable hypersurfaces disjoint from the edges in a wedge-shape domain Ω of $\mathbb{R}^{n+1}$. Moreover, after a suitable translation, the same variation allows to treat the capillary case. We remark that such a deformation is possible because $\partial\Omega$ consists of finitely many pieces of hyperplanes.

In a convex cone $\mathcal{C}$ different from a half-space, Palmer's deformation does not work in general. Indeed, even though N_K is tangent to $\partial\mathcal{C}$ along $\partial\Sigma$, we cannot ensure that $\partial\mathcal{C}$ contains a small anisotropic normal segment $p + t\,N_K(p)$ centered at any point $p \in \partial\Sigma$. To solve this difficulty we replace the anisotropic parallels deformation of Σ with a variation $\Sigma_t := \psi_t(\Sigma)$ associated to the one-parameter group $\{\psi_t\}_{t\in\mathbb{R}}$ of a vector field X on $\mathbb{R}^{n+1}$ such that $X(0) = 0$, X is tangent on $\partial\mathcal{C} \setminus \{0\}$ and $X_{|\Sigma} = N_K$. After this, as $\partial\mathcal{C}$ is invariant under dilations centered at 0, we can apply to Σ_t a suitable dilation in order to construct a deformation that preserves both, $\partial\mathcal{C}$ and the volume of Σ. To finish the proof we compute the second derivative of the anisotropic area for this variation, and we discover that it is strictly negative unless Σ is contained in the Wulff shape (up to translation and homothety). In the isotropic case this is a new demonstration of the stability result by Ritoré and the author [33] when we consider immersed hypersurfaces without singularities.

We emphasize that, to prove Theorem 4.8, we only need to calculate $A_K''(0)$ for deformations associated to a smooth vector field X on $\mathbb{R}^{n+1}$ which is tangent on $\partial\mathcal{C} \setminus \{0\}$ and satisfies $X_{|\Sigma} = N_K$. This is done in Proposition 3.3 for arbitrary compact hypersurfaces with non-empty boundary in $\mathbb{R}^{n+1}$. As a difference with respect to the anisotropic parallels deformation, a boundary term involving the acceleration vector field Z appears. For anisotropic stationary hypersurfaces in $\mathcal{C}$ we see in Proposition 4.5 that this term is related to the extrinsic Euclidean geometry of $\partial\mathcal{C}$. As the formula for $A_K''(0)$ in Proposition 4.5 is valid for any smooth domain Ω, this shows that the stability condition is more restrictive when Ω is convex. It is worth mentioning that, in most of the aforementioned works about anisotropic capillary hypersurfaces in a Euclidean domain Ω, this boundary term had no relevance since $\partial\Omega$ was contained in the union of finitely many hyperplanes.

Finally, we remark that the proof of Theorem 4.8 is still valid in non-smooth convex cones, provided the boundary $\partial\Sigma$ of an anisotropic stable hypersurface Σ is inside a smooth open part of $\partial\mathcal{C}$.

The paper is organized into three sections besides this introduction. In Sect. 2 we review some basic facts about the anisotropic geometry of hypersurfaces. In Sect. 3 we compute the second derivative of the anisotropic area for certain deformations of a compact hypersurface with non-empty boundary. Finally, in Sect. 4 we prove our classification result of anisotropic stable hypersurfaces in convex solid cones.

2 Preliminaries

In this section we gather some definitions and results that will be needed throughout this work. Starting with a Euclidean convex body we will use its support function and metric projection to introduce the anisotropic area and notions of anisotropic extrinsic geometry for two-sided hypersurfaces. The relation between this point of view and previous approaches is shown in Remark 2.3.

By a *convex body* (about the origin) we mean a compact convex set $K \subset \mathbb{R}^{n+1}$ containing 0 in its interior. The associated *support function* $h_K : \mathbb{R}^{n+1} \to \mathbb{R}$ is given by

$$h_K(w) := \max\{\langle u, w\rangle;\ u \in K\},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^{n+1}$. This defines an *asymmetric norm* in $\mathbb{R}^{n+1}$ (which is a norm when K is centrally symmetric about 0), see [35, Sect. 1.7.1]. For any $w \neq 0$, the *supporting hyperplane* of K with exterior normal w is the set

$$\Pi_w := \{p \in \mathbb{R}^{n+1};\ \langle p, w\rangle = h_K(w)\}.$$

The corresponding *support set* is $\Pi_w \cap K$. This set is not empty because K is compact. If $\Pi_w \cap K$ is a single point p, then h_K is differentiable at w, and its gradient satisfies $(\nabla h_K)(w) = p$, see [35, Cor. 1.7.3].

Henceforth, we suppose that K is a strictly convex body with smooth boundary ∂K. Thus, for any $w \neq 0$, there is a point $\pi_K(w) \in \partial K$ for which $\Pi_w \cap K = \{\pi_K(w)\}$. This provides a map $\pi_K : \mathbb{R}^{n+1}_* \to \partial K$, which is called the *K-projection*, and verifies these identities

$$\begin{aligned} \langle \pi_K(w), w\rangle &= h_K(w), \quad \text{for any } w \neq 0, \\ (\nabla h_K)(w) &= \pi_K(w), \quad \text{for any } w \neq 0. \end{aligned} \tag{2.1}$$

On the other hand, if η_K is the outer unit normal on ∂K, then $(\eta_K \circ \pi_K)(w) = w$ for any $w \in \mathbb{S}^n$. This equality and the fact that $\pi_K(\lambda\, w) = \pi_K(w)$ for any $w \neq 0$ and $\lambda > 0$ entail that $\pi_K(w) = \eta_K^{-1}(w/|w|)$ for any $w \neq 0$ (here $|\cdot|$ stands for the Euclidean norm). As $\eta_K : \partial K \to \mathbb{S}^n$ is a diffeomorphism, the support function h_K is C^∞ on $\mathbb{R}^{n+1}_*$. Observe that, when K is the round unit ball about 0, then $h_K(w) = |w|$ and $\pi_K(w) = w/|w|$ for any $w \neq 0$, whereas $\eta_K(p) = p$ for any $p \in \partial K = \mathbb{S}^n$.

Let $\varphi_0 : \Sigma \to \mathbb{R}^{n+1}$ be a smooth two-sided immersed hypersurface, possibly with smooth boundary $\partial\Sigma$. Most of the time we will omit the map φ_0. We will also identify any set $S \subseteq \Sigma$ with $\varphi_0(S)$, and the tangent space $T_p\Sigma$ at a point $p \in \Sigma$ with $(d\varphi_0)_p(T_p\Sigma)$. For a fixed smooth unit normal vector field N on Σ, the associated *shape operator* at p is the endomorphism $B_p : T_p\Sigma \to T_p\Sigma$ introduced by $B_p(w) := -D_w N$. Here D denotes the Levi-Civita connection for the Euclidean metric. It is well known that B_p is self-adjoint with respect to the metric induced by the scalar product in $\mathbb{R}^{n+1}$.

The *anisotropic Gauss map* or *anisotropic normal* in Σ is the map $N_K : \Sigma \to \partial K$ given by

$$N_K := \pi_K \circ N. \tag{2.2}$$

By setting $\varphi_K := h_K(N)$ we infer from (2.1) that

$$\langle N_K, N\rangle = \varphi_K \quad \text{on } \Sigma. \tag{2.3}$$

When $\partial\Sigma \neq \emptyset$ we introduce the *anisotropic conormal* by setting

$$\nu_K := \varphi_K\, \nu - \langle N_K, \nu\rangle\, N, \tag{2.4}$$

where ν is the *inner conormal* along $\partial\Sigma$. Note that ν_K is a normal vector to $\partial\Sigma$ with $\langle N_K, \nu_K\rangle = 0$.

For any $p \in \Sigma$ we have $T_{N_K(p)}(\partial K) = [\eta_K(N_K(p))]^\perp = N(p)^\perp = T_p\Sigma$, so that the differential $(dN_K)_p$ is an endomorphism of $T_p\Sigma$. The *anisotropic shape operator* at p is defined by

$$(B_K)_p := -(dN_K)_p = (d\pi_K)_{N(p)} \circ B_p. \tag{2.5}$$

The *anisotropic mean curvature* at p is the number

$$H_K(p) := \frac{\operatorname{tr}((B_K)_p)}{n} = -\frac{(\operatorname{div}_\Sigma N_K)(p)}{n}, \tag{2.6}$$

where $\operatorname{div}_\Sigma$ is the divergence relative to Σ and $\operatorname{tr}(f)$ is the trace of an endomorphism $f : T_p\Sigma \to T_p\Sigma$. It is known, see for instance Palmer [31, p. 3666], that

$$\operatorname{tr}\big((B_K)_p^2\big) \geqslant n H_K(p)^2, \quad \text{for any } p \in \Sigma, \tag{2.7}$$

and equality holds if and only p is an *anisotropic umbilical point*, i.e., $(B_K)_p$ is a multiple of the identity map in $T_p\Sigma$. This fact has a short proof that we reproduce here for the sake of completeness. Since $\nabla h_K = \pi_K$ in $\mathbb{R}^{n+1}_*$ and K is a strictly convex body, then $(d\pi_K)_{N(p)}$ can be represented as a positive definite symmetric matrix of order n. Thus, there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_p\Sigma$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}^+$ such that $(d\pi_K)_{N(p)}(e_j) = \lambda_j\, e_j$ for any $j = 1, \ldots, n$. From (2.5) we get $(B_K)_p(e_i) = \sum_{j=1}^n \sigma_{ij}\,\lambda_j\, e_j$, where $\sigma_{ij} := \langle B_p(e_i), e_j\rangle$. By using the Cauchy-Schwarz inequality in $\mathbb{R}^n$ with the vectors $(\sigma_{11}\lambda_1, \ldots, \sigma_{nn}\lambda_n)$ and $(1, \ldots, 1)$, we obtain

$$\begin{aligned}\operatorname{tr}(B_K^2) - nH_K^2 &= \sum_{i,j=1}^n \sigma_{ij}^2\,\lambda_i\,\lambda_j - \frac{1}{n}\left(\sum_{i=1}^n \sigma_{ii}\,\lambda_i\right)^2 \\ &\geqslant \sum_{i,j=1}^n \sigma_{ij}^2\,\lambda_i\,\lambda_j - \sum_{i=1}^n \sigma_{ii}^2\,\lambda_i^2 = \sum_{i\neq j} \sigma_{ij}^2\,\lambda_i\,\lambda_j \geqslant 0.\end{aligned}$$

Equality holds if and only if $\sigma_{ij} = 0$ for any $i \neq j$ and $\sigma_{ii}\,\lambda_i = \sigma_{jj}\,\lambda_j$ for any $i, j = 1, \ldots, n$. By denoting $\alpha := \sigma_{ii}\,\lambda_i$, this is equivalent to $(B_K)_p(w) = \alpha\, w$ for any $w \in T_p\Sigma$, as desired.

Examples 2.1 (i). If K is the round unit ball about 0, then $N_K = N$, $B_K = B$ and H_K equals the Euclidean mean curvature of Σ.

(ii). For a hyperplane $\Sigma \subset \mathbb{R}^{n+1}$ oriented with a unit normal N, the map N_K is constant, so that $B_K = 0$ and $H_K = 0$ in Σ.

(iii). Consider the hypersurface $\Sigma = \partial K$ with outer unit normal $N = \eta_K$. Then $N_K(p) = p$, $(B_K)_p(w) = -w$ and $H_K(p) = -1$, for any $p \in \Sigma$ and $w \in T_p\Sigma$. Observe that, if K is not centrally symmetric about 0, and we take the inner normal $N = -\eta_K$ on Σ, then similar equalities need not hold and H_K may be nonconstant. So, the role of the chosen unit normal N is very important for the computations.

The previous examples show that ∂K and Euclidean hyperplanes are *anisotropic umbilical hypersurfaces*, i.e., all of its points are anisotropic umbilical. Next, we provide a converse statement after Palmer [31, p. 3666] and Clarenz [8, p. 358].

Proposition 2.2 *Let Σ be a two-sided, connected, anisotropic umbilical hypersurface immersed in $\mathbb{R}^{n+1}$, and such that H_K is constant. If $H_K = 0$ then Σ is contained inside a hyperplane. If $H_K \neq 0$ then, up to a translation and a dilation centered at* 0*, we have $\Sigma \subset \partial K$.*

Proof For any $p \in \Sigma$, there is $\alpha(p) \in \mathbb{R}$ such that $(B_K)_p(w) = \alpha(p)\, w$ for any $w \in T_p\Sigma$. By definition (2.6) it is clear that $\alpha(p) = H_K$ for any $p \in \Sigma$. By Eq. (2.5) we deduce that the differential of $N_K + H_K\,\mathrm{Id}$ vanishes on Σ. In case $H_K = 0$ this implies that N_K is a constant vector n, and so Σ is within a hyperplane with unit normal $\eta_K(n)$. When $H_K \neq 0$ we can find $c \in \mathbb{R}^{n+1}$ satisfying

$$p = \frac{c}{H_K} - \frac{1}{H_K}\, N_K(p), \quad \text{for any } p \in \Sigma.$$

From here we infer that $\Sigma \subseteq (c/H_K) - (1/H_K)\,(\partial K)$, as we claimed. □

We finish this section with the notions of *anisotropic area* and *algebraic volume* for a two-sided hypersurface Σ. For a fixed smooth unit normal N on Σ, the anisotropic area is given by

$$A_K(\Sigma) := \int_\Sigma \varphi_K \, d\Sigma = \int_\Sigma h_K(N)\, d\Sigma,$$

where $d\Sigma$ denotes the area element of Σ. This coincides with the Euclidean area of Σ when K is the round unit ball about 0. When K is not centrally symmetric about 0 the value of $A_K(\Sigma)$ may depend on the normal N that we consider on Σ. On the other hand, for a compact hypersurface Σ, we follow Barbosa and do Carmo [2, Eq. (2.2)] to define

$$V(\Sigma) := \frac{1}{n+1} \int_\Sigma \big\langle p, N(p)\big\rangle\, d\Sigma. \tag{2.8}$$

When Σ is embedded and $\Sigma' \subset \Sigma$ is a sufficiently small open set, we can apply the divergence theorem to the position vector field $X(p) = p$ in the cone $\mathcal{C}$ over Σ' with vertex at 0 to conclude that $|V(\Sigma')|$ equals the Lebesgue measure of $\mathcal{C}$. If Σ is immersed then it is possible that $V(\Sigma) = 0$.

Consider a dilation $\delta_\lambda(p) := \lambda\, p$ with $\lambda > 0$ and $p \in \mathbb{R}^{n+1}$. A unit normal N_λ on the hypersurface $\delta_\lambda(\Sigma)$ is determined by $N_\lambda(\delta_\lambda(p)) := N(p)$ for any $p \in \Sigma$. Therefore, the change of variables formula and the fact that the Jacobian of the diffeomorphism $\delta_{\lambda|\Sigma} : \Sigma \to \delta_\lambda(\Sigma)$ equals λ^n, entail that

$$\begin{aligned} A_K(\delta_\lambda(\Sigma)) &= \lambda^n\, A_K(\Sigma), \\ V(\delta_\lambda(\Sigma)) &= \lambda^{n+1}\, V(\Sigma). \end{aligned} \tag{2.9}$$

These identities will play a relevant role in the proof of our main result in Sect. 4.

Remark 2.3 It is usual to introduce the anisotropic area A_F associated to an arbitrary function $F : \mathbb{S}^n \to \mathbb{R}^+$. This has the expression $A_F(\Sigma) := \int_\Sigma F(N)\, d\Sigma$. Convexity assumptions on F are necessary to deduce fine properties for A_F and for the F-Laplacian, see for instance Maggi [27, Ch. 20] or Palmer [31]. Other ellipticity conditions for more general integrands, including equivalences between them and first variation formulas for the corresponding energies are found in De Rosa and Kolasiński [10], and De Lellis et al. [9]. By extending F as a 1-homogeneous function we get an asymmetric norm Φ in $\mathbb{R}^{n+1}$. The *Wulff shape* for Φ is the convex body $\mathcal{W}_\Phi$ supported by Φ, see [27, Eq. (20.8)]. As we remembered in the Introduction, $\mathcal{W}_\Phi$ minimizes the anisotropic perimeter among sets of the same Euclidean volume. In our context we have $F = h_{K|\mathbb{S}^n}$, $\Phi = h_K$ and $\mathcal{W}_\Phi = K$. So, the initial convex body K is the optimal shape for the anisotropic area A_K defined from its support function h_K. In Palmer [31, Sect. 1] the *Wulff hypersurface* of F is the strictly convex hypersurface defined as $\phi(\mathbb{S}^n)$, where $\phi(w) := F(w)\, w + (\nabla_{\mathbb{S}^n} F)(w)$. When $F = h_{K|\mathbb{S}^n}$ it follows from Eq. (2.1) that $\phi = \pi_K$ on $\mathbb{S}^n$, so that the corresponding Wulff hypersurface $\phi(\mathbb{S}^n)$ equals ∂K.

3 A second variation formula for the anisotropic area

In this section we compute the second derivative of the anisotropic area for certain deformations of an immersed hypersurface with non-empty boundary. The resulting formula will be employed in the proof of our main result in Theorem 4.8. We begin with some preliminary definitions.

A *flow* (*of diffeomorphisms*) in $\mathbb{R}^{n+1}$ is a smooth map $\phi:\mathbb{R}^{n+1}\times\mathbb{R}\to\mathbb{R}^{n+1}$ such that $\phi(p,0)=p$ for any $p\in\mathbb{R}^{n+1}$, and the map $\phi_t:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ defined by $\phi_t(p):=\phi(p,t)$ is a diffeomorphism for any $t\in\mathbb{R}$. Usually we will denote a flow by $\{\phi_t\}_{t\in\mathbb{R}}$. The associated *velocity vector field* is given by

$$X(p):=\frac{d}{dt}\bigg|_{t=0}\phi_t(p),\quad \text{for any } p\in\mathbb{R}^{n+1}.$$

When Y is a smooth complete vector field on $\mathbb{R}^{n+1}$, the corresponding one-parameter group of diffeomorphisms $\{\phi_t\}_{t\in\mathbb{R}}$ is a flow with velocity Y.

For a hypersurface Σ immersed in $\mathbb{R}^{n+1}$ the *variation of* Σ induced by a flow $\{\phi_t\}_{t\in\mathbb{R}}$ is the family $\{\Sigma_t\}_{t\in\mathbb{R}}$, where $\Sigma_t:=\phi_t(\Sigma)$ for any $t\in\mathbb{R}$. The flow is *compactly supported on* Σ if there is a compact set $C\subseteq\Sigma$ such that $\phi_t(p)=p$ for any $p\in\Sigma\setminus C$ and $t\in\mathbb{R}$. If Σ is a two-sided hypersurface with unit normal N then, along the variation $\{\Sigma_t\}_{t\in\mathbb{R}}$, we can find a smooth vector field $\overline{N}$ whose restriction to Σ_t provides a unit normal N_t with $N_0=N$. For a fixed smooth strictly convex body $K\subset\mathbb{R}^{n+1}$ with support function h_K, the *anisotropic area functional* for the variation $\{\Sigma_t\}_{t\in\mathbb{R}}$ is the function

$$A_K(t):=A_K(\Sigma_t)=\int_{\Sigma_t}h_K(N_t)\,d\Sigma_t=\int_\Sigma h_K(\overline{N}\circ\phi_t)\,\mathrm{Jac}\,\phi_t\,d\Sigma, \tag{3.1}$$

where $\mathrm{Jac}\,\phi_t$ is the Jacobian of the diffeomorphism $\phi_{t|\Sigma}:\Sigma\to\Sigma_t$.

Our objective is to provide a useful expression of $A''_K(0)$ for some variations. For that we must compute $A'_K(0)$ first. We need some preliminary calculations that we gather below.

Lemma 3.1 *Let $K\subset\mathbb{R}^{n+1}$ be a smooth strictly convex body with support function h_K. For a two-sided hypersurface Σ immersed in $\mathbb{R}^{n+1}$ with unit normal N, the function $\varphi_K:=h_K(N)$ satisfies*

$$\nabla_\Sigma\varphi_K=-B(N_K^\top), \tag{3.2}$$

where ∇_Σ is the gradient relative to Σ and $N_K^\top$ is the tangent projection of the anisotropic normal N_K. Moreover, for any smooth vector field U with compact support on Σ and normal component $u:=\langle U,N\rangle$, we have

$$\int_\Sigma \mathrm{div}_\Sigma(\varphi_K\,U)\,d\Sigma=-\int_\Sigma nH_K\,u\,d\Sigma+\int_\Sigma\langle N_K,\nabla_\Sigma u\rangle\,d\Sigma-\int_{\partial\Sigma}\langle U,\nu_K\rangle\,d(\partial\Sigma), \tag{3.3}$$

where H_K is the anisotropic mean curvature and ν_K is the anisotropic conormal.

Proof Take a point $p\in\Sigma$ and a vector $w\in T_p\Sigma$. By using (2.1), (2.2) and that the shape operator B_p is a self-adjoint endomorphism of $T_p\Sigma$, we obtain

$$\begin{aligned}\langle(\nabla_\Sigma\varphi_K)(p),w\rangle&=\langle(\nabla h_K)(N(p)),D_wN\rangle=-\langle\pi_K(N(p)),B_p(w)\rangle\\&=-\langle N_K(p),B_p(w)\rangle=-\langle B_p(N_K^\top(p)),w\rangle.\end{aligned} \tag{3.4}$$

This implies (3.2). Let us prove (3.3). For any $w\in\mathbb{R}^{n+1}$ we denote by $w^\top$ and $w^\perp$ the projections of w with respect to $T_p\Sigma$ and $(T_p\Sigma)^\perp$, respectively. For a smooth vector field U on Σ, note that

$$\varphi_K\, U = \varphi_K\, U^\top + \varphi_K\, u\, N = \varphi_K\, U^\top + u\, N_K^\perp = \varphi_K\, U^\top + u\, N_K - u\, N_K^\top,$$

where in the second equality we have employed (2.3). By taking divergences relative to Σ and having in mind (2.6), we get

$$\mathrm{div}_\Sigma(\varphi_K\, U) = \mathrm{div}_\Sigma(\varphi_K\, U^\top) - nH_K\, u + \big\langle N_K, \nabla_\Sigma u\big\rangle - \mathrm{div}_\Sigma(u\, N_K^\top).$$

From here the desired formula follows by using (2.4) and the divergence theorem on Σ. □

Next, we compute the first variation of A_K. This was previously derived by many authors, see for instance Clarenz [7, Sect. 1], or Koiso and Palmer [22, Proof of Prop. 3.1]. As in Koiso [20, Lem. 9.1], our formula holds for arbitrary deformations of a Euclidean hypersurface with non-empty boundary. We include a short proof that will be helpful in the subsequent calculus of $A''_K(0)$.

Proposition 3.2 *Let $K \subset \mathbb{R}^{n+1}$ be a smooth strictly convex body, Σ a two-sided immersed hypersurface with boundary, and $\{\phi_t\}_{t\in\mathbb{R}}$ a flow in $\mathbb{R}^{n+1}$ with compact support on Σ. Then, we have*

$$A'_K(0) = -\int_\Sigma nH_K\, u\, d\Sigma - \int_{\partial\Sigma} \big\langle X, \nu_K\big\rangle\, d(\partial\Sigma),$$

where H_K is the anisotropic mean curvature, $u := \big\langle X, N\big\rangle$ is the normal component of the velocity vector field X, and ν_K is the anisotropic conormal along $\partial\Sigma$.

Proof Recall that we denote $\varphi_K := h_K(N)$. For a fixed point $p \in \Sigma$, we define

$$h_p(t) := h_K(N_t \circ \phi_t)(p), \quad j_p(t) := (\mathrm{Jac}\,\phi_t)(p), \quad \text{for any } t \in \mathbb{R}. \tag{3.5}$$

By differentiating under the integral sign in (3.1) and taking into account that $\phi_0 = \mathrm{Id}$, we obtain

$$A'_K(0) = \int_\Sigma \big(h'_p(0) + \varphi_K(p)\, j'_p(0)\big)\, d\Sigma. \tag{3.6}$$

On the one hand it is well known, see Simon [36, §9], that

$$j'_p(0) = \frac{d}{dt}\bigg|_{t=0} (\mathrm{Jac}\,\phi_t)(p) = (\mathrm{div}_\Sigma\, X)(p). \tag{3.7}$$

On the other hand, from Eqs. (2.1) and (2.2), it follows that

$$h'_p(0) = \big\langle(\nabla h_K)(N(p)), D_{X(p)}\overline{N}\big\rangle = \big\langle N_K(p), D_{X(p)}\overline{N}\big\rangle.$$

The computation of $D_{X(p)}\overline{N}$ is found in [34, Lem. 4.1(1)]. We get

$$D_{X(p)}\overline{N} = \frac{d}{dt}\bigg|_{t=0} (N_t \circ \phi_t)(p) = -(\nabla_\Sigma u)(p) - B_p(X^\top(p)), \tag{3.8}$$

where $X^\top(p)$ is the projection of $X(p)$ onto $T_p\Sigma$. By substituting this information into the previous expression for $h'_p(0)$ and having in mind the third equality in (3.4), we arrive at

$$h'_p(0) = -\big\langle N_K, \nabla_\Sigma u\big\rangle(p) + \big\langle \nabla_\Sigma \varphi_K, X^\top\big\rangle(p). \tag{3.9}$$

Thus, the integrand in (3.6) is the evaluation at p of the function

$$-\big\langle N_K, \nabla_\Sigma u\big\rangle + \big\langle \nabla_\Sigma\varphi_K, X^\top\big\rangle + \varphi_K\, \mathrm{div}_\Sigma\, X = -\big\langle N_K, \nabla_\Sigma u\big\rangle + \mathrm{div}_\Sigma(\varphi_K\, X).$$

The proof finishes by applying the formula in (3.3) with $U = X$. □

Second variation formulas for the anisotropic area under different hypotheses on Σ and the deformation can be found in Koiso and Palmer [22, Prop. 3.3], [23, Prop. 3.3], Barbosa and Silva [1, Prop. 3], and Guo and Xia [14, Prop. 3.5]. In this work we only need to compute $A''_K(0)$ when we move a hypersurface Σ with non-empty boundary by means of some special flows.

Proposition 3.3 *Let $K \subset \mathbb{R}^{n+1}$ be a smooth strictly convex body, Σ a compact two-sided immersed hypersurface with boundary, and X a smooth complete vector field on $\mathbb{R}^{n+1}$ such that $X_{|\Sigma} = N_K$. Then, for the one-parameter group of diffeomorphisms $\{\phi_t\}_{t\in\mathbb{R}}$ associated to X, we have*

$$A''_K(0) = \int_\Sigma \big(n^2 H_K^2 - \mathrm{tr}(B_K^2)\big)\,\varphi_K\,d\Sigma - \int_\Sigma nH_K\,v\,d\Sigma - \int_{\partial\Sigma} \big\langle Z, \nu_K\big\rangle\,d(\partial\Sigma),$$

where H_K is the anisotropic mean curvature, B_K is the anisotropic shape operator, ν_K is the anisotropic conormal and $v := \big\langle Z, N\big\rangle$ is the normal component of the vector field $Z := D_X X$.

Proof For any $w \in \mathbb{R}^{n+1}$ the notations $w^\top$ and $w^\perp$ will stand for the projections of w onto $T\Sigma$ and $(T\Sigma)^\perp$, respectively. For a given point $p \in \Sigma$ we define $h_p(t)$ and $j_p(t)$ as in (3.5). By differentiating under the integral sign twice in (3.1) and taking into account that $\phi_0 = \mathrm{Id}$, we get

$$A''_K(0) = \int_\Sigma \big(h''_p(0) + 2h'_p(0)\,j'_p(0) + \varphi_K(p)\,j''_p(0)\big)\,d\Sigma. \tag{3.10}$$

Let us compute all the derivatives in the integrand above.

For the calculus of $j'_p(0)$ and $j''_p(0)$ we refer the reader to Simon [36, §9]. From (3.7), (2.6), and the fact that $X_{|\Sigma} = N_K$, we obtain

$$j'_p(0) = (\mathrm{div}_\Sigma\,X)(p) = (\mathrm{div}_\Sigma\,N_K)(p) = -nH_K(p). \tag{3.11}$$

On the other hand, we have

$$j''_p(0) = (\mathrm{div}_\Sigma\,Z)(p) + (\mathrm{div}_\Sigma\,X)^2(p) + \sum_{i=1}^n |(D_{e_i}X)^\perp|^2 - \sum_{i,j=1}^n \big\langle D_{e_i}X, e_j\big\rangle\big\langle D_{e_j}X, e_i\big\rangle,$$

where $\{e_1,\ldots,e_n\}$ is an orthonormal basis of $T_p\Sigma$. Equation (2.5) and the fact that $X_{|\Sigma} = N_K$ yield

$$D_{e_i}X = D_{e_i}N_K = -(B_K)_p(e_i),$$

so that $(D_{e_i}X)^\perp = 0$ for any $i = 1,\ldots,n$. It is also clear that

$$\sum_{i,j=1}^n \big\langle D_{e_i}X, e_j\big\rangle\big\langle D_{e_j}X, e_i\big\rangle = \mathrm{tr}\big((B_K)_p^2\big).$$

All this together with (3.11) shows that

$$j''_p(0) = \big(\mathrm{div}_\Sigma\,Z + n^2H_K^2 - \mathrm{tr}(B_K^2)\big)(p). \tag{3.12}$$

Next, we compute $h'_p(0)$ and $h''_p(0)$. Note that $\big\langle X, N\big\rangle = \big\langle N_K, N\big\rangle = \varphi_K$ by (2.3). Hence, Eq. (3.9) implies that

$$h'_p(0) = -\big\langle N_K, \nabla_\Sigma\varphi_K\big\rangle(p) + \big\langle \nabla_\Sigma\varphi_K, N_K^\top\big\rangle(p) = 0. \tag{3.13}$$

In order to calculate $h''_p(0)$ we need an expression for $h'_p(t)$. From the definition in (3.5) we deduce

$$h'_p(t) = \big\langle (\nabla h_K)((N_t \circ \phi_t)(p)), (D_X \overline{N})(\phi_t(p)) \big\rangle = \big\langle \pi_K((N_t \circ \phi_t)(p)), (D_X \overline{N})(\phi_t(p)) \big\rangle$$

because of Eq. (2.1). This entails that

$$h''_p(0) = \big\langle (d\pi_K)_{N(p)}(D_{X(p)} \overline{N}), D_{X(p)} \overline{N} \big\rangle + \big\langle N_K(p), D_{X(p)} D_X \overline{N} \big\rangle.$$

As $\langle X, N \rangle = \varphi_K$ on Σ, Eqs. (3.8) and (3.2) lead to

$$D_{X(p)} \overline{N} = -(\nabla_\Sigma \varphi_K)(p) - B_p(N_K^\top(p)) = 0. \tag{3.14}$$

On the other hand, by Lemma 3.4 below we know that

$$D_{X(p)} D_X \overline{N} = -(\nabla_\Sigma v)(p) - B_p(Z^\top(p)),$$

and so

$$\begin{aligned} h''_p(0) &= -\big\langle N_K, \nabla_\Sigma v \big\rangle(p) - \big\langle N_K(p), B_p(Z^\top(p)) \big\rangle \\ &= -\big\langle N_K, \nabla_\Sigma v \big\rangle(p) + \big\langle \nabla_\Sigma \varphi_K, Z^\top \big\rangle(p), \end{aligned} \tag{3.15}$$

where we have employed the third equality in Eq. (3.4).

Now, by having in mind (3.15), (3.13) and (3.12), we conclude that the integrand in (3.10) is the evaluation at p of the function

$$-\big\langle N_K, \nabla_\Sigma v \big\rangle + \mathrm{div}_\Sigma(\varphi_K\, Z) + \big(n^2 H_K^2 - \mathrm{tr}(B_K^2)\big)\, \varphi_K.$$

From here the proof finishes by applying the formula in (3.3) with $U = Z$. $\square$

Lemma 3.4 *In the conditions of Proposition 3.3, for any $p \in \Sigma$, we have*

$$D_{X(p)} D_X \overline{N} = -(\nabla_\Sigma v)(p) - B_p(Z^\top(p)).$$

Proof Take an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_p\Sigma$. We use the flow $\{\phi_t\}_{t \in \mathbb{R}}$ associated to X to construct, for any $i = 1, \ldots, n$, a smooth vector field E_i around p which is tangent on any $\Sigma_t := \phi_t(\Sigma)$ while satisfying $E_i(p) = e_i$ and $[X, E_i] = 0$ (here $[\cdot, \cdot]$ stands for the Lie bracket of vector fields in $\mathbb{R}^{n+1}$). It is clear that

$$D_{X(p)} D_X \overline{N} = \sum_{i=1}^n \big\langle D_{X(p)} D_X \overline{N}, e_i \big\rangle e_i + \big\langle D_{X(p)} D_X \overline{N}, N(p) \big\rangle N(p). \tag{3.16}$$

We will compute the different terms in the previous equation.

Since $|\overline{N}|^2 = 1$ we get $\langle D_X \overline{N}, \overline{N} \rangle = 0$. By differentiating and applying Eq. (3.14), we deduce

$$0 = \big\langle D_{X(p)} D_X \overline{N}, N(p) \big\rangle + |D_{X(p)} \overline{N}|^2 = \big\langle D_{X(p)} D_X \overline{N}, N(p) \big\rangle. \tag{3.17}$$

On the other hand, we differentiate twice with respect to X in equality $\langle \overline{N}, E_i \rangle = 0$. By taking into account that $D_{X(p)} \overline{N} = 0$ (see Eq. (3.14)) and $[X, E_i] = 0$, we obtain

$$\big\langle D_{X(p)} D_X \overline{N}, e_i \big\rangle = -\big\langle N(p), D_{X(p)} D_X E_i \big\rangle = -\big\langle N(p), D_{X(p)} D_{E_i} X \big\rangle.$$

Since the Riemann curvature tensor vanishes for the standard metric in $\mathbb{R}^{n+1}$, we infer

$$0 = D_X D_{E_i} X - D_{E_i} D_X X - D_{[X, E_i]} X = D_X D_{E_i} X - D_{E_i} Z$$

because $D_X X = Z$. This shows that $D_{X(p)} D_{E_i} X = D_{e_i} Z$. As a consequence

$$\begin{aligned}\big\langle D_{X(p)} D_X \overline{N}, e_i\big\rangle &= -\big\langle N(p), D_{e_i} Z\big\rangle = -e_i\big(\langle Z, N\rangle\big) + \big\langle Z(p), D_{e_i} N\big\rangle \\ &= -\big\langle(\nabla_\Sigma v)(p), e_i\big\rangle - \big\langle Z^\top(p), B_p(e_i)\big\rangle \\ &= -\big\langle(\nabla_\Sigma v)(p), e_i\big\rangle - \big\langle B_p(Z^\top(p)), e_i\big\rangle. \end{aligned} \tag{3.18}$$

By substituting (3.17) and (3.18) into (3.16) the proof follows. □

Remarks 3.5 (i). If $K \subset \mathbb{R}^{n+1}$ is the round unit ball about 0, then the anisotropic area coincides with the Euclidean area and the obtained formulas are well known.

(ii). It is interesting to observe that, unless K is centrally symmetric about 0, the formulas for $A'_K(0)$ and $A''_K(0)$ may depend on the unit normal vector N fixed on Σ.

(iii). In Proposition 4.5 we will see that, under some extra conditions, the boundary integrand in the expression of $A''_K(0)$ has a geometric interpretation.

4 Anisotropic stable hypersurfaces in solid cones

In this section we consider a Euclidean solid cone (not necessarily convex) and study compact hypersurfaces immersed in the cone and minimizing the anisotropic area up to second order for deformations preserving the volume of the hypersurface and the boundary of the cone. In this situation we will show that the variational formulas computed in Sect. 3 can be slightly simplified. After that we will prove our main theorem, where we characterize these second order minima when the cone is convex. It is worth mentioning that, excluding this classification statement and the Minkowski-type formula in Eq. (4.5), all the results in this section still hold when we replace the cone with any smooth Euclidean open set.

We begin by introducing some notation and definitions. For a domain $\mathcal{D} \subset \mathbb{S}^n$ with smooth boundary, the *solid cone* over $\mathcal{D}$ is the set

$$\mathcal{C} := \{\lambda\, p;\ \lambda > 0,\ p \in \mathcal{D}\}.$$

This is a domain of $\mathbb{R}^{n+1}$ with boundary $\partial\mathcal{C}$ smooth away from 0. We call ξ the inner unit normal along $\partial\mathcal{C} \setminus \{0\}$. Note that $\mathcal{C}$ and $\partial\mathcal{C}$ are invariant under the dilations δ_λ centered at 0. It is also clear that $\mathcal{C}$ coincides with an open half-space when $\mathcal{D}$ is an open hemisphere.

Let Σ be a smooth, compact, two-sided hypersurface immersed in $\overline{\mathcal{C}}$ with smooth boundary $\partial\Sigma$ in $\partial\mathcal{C} \setminus \{0\}$. We suppose that $\Sigma \cap \partial\mathcal{C} = \partial\Sigma$, so that $\Sigma \setminus \partial\Sigma \subseteq \mathcal{C}$. We fix a smooth unit normal vector field N on Σ. The inner conormal vector of $\partial\Sigma$ in Σ is represented by ν. In the planar distribution $T(\partial\Sigma)^\perp$ we choose the orientation induced by $\{\nu, N\}$. Thus, for any $p \in \partial\Sigma$, there is a unique $\mu(p) \in T_p(\partial\Sigma)^\perp$ such that $\{\xi(p), \mu(p)\}$ is a positively oriented orthonormal basis. Observe that μ is tangent to $\partial\mathcal{C}$ and normal to $\partial\Sigma$. It is easy to check that these equalities hold along $\partial\Sigma$

$$\begin{aligned} \nu &= (\cos\theta)\,\xi - (\sin\theta)\,\mu, \qquad & \mu &= -(\sin\theta)\,\nu + (\cos\theta)\,N, \\ N &= (\sin\theta)\,\xi + (\cos\theta)\,\mu, \qquad & \xi &= (\cos\theta)\,\nu + (\sin\theta)\,N, \end{aligned} \tag{4.1}$$

where θ is the oriented angle function between ν and ξ in $T(\partial\Sigma)^\perp$. As a consequence, for a fixed smooth strictly convex body $K \subset \mathbb{R}^{n+1}$, the anisotropic normal N_K on Σ and the anisotropic conormal ν_K along $\partial\Sigma$ given in (2.2) and (2.4) verify

$$\big\langle N_K, \mu\big\rangle = \big\langle \nu_K, \xi\big\rangle, \qquad \big\langle N_K, \xi\big\rangle = -\big\langle \nu_K, \mu\big\rangle. \tag{4.2}$$

A flow of diffeomorphisms $\{\phi_t\}_{t\in\mathbb{R}}$ in $\mathbb{R}^{n+1}$ is *admissible* for $\mathcal{C}$ if $\phi_t(\partial\mathcal{C}) = \partial\mathcal{C}$ and $\phi_t(0) = 0$, for any $t \in \mathbb{R}$. The induced variation $\{\Sigma_t\}_{t\in\mathbb{R}}$ of Σ satisfies $\partial\Sigma_t \subset \partial\mathcal{C} \setminus \{0\}$ and $\Sigma_t \cap \partial\mathcal{C} = \partial\Sigma_t$, for any $t \in \mathbb{R}$. Moreover, the velocity vector field X is tangent on $\partial\mathcal{C} \setminus \{0\}$ and vanishes at 0. We consider the *anisotropic area functional* $A_K(t)$ introduced in (3.1) and the *volume functional* $V(t)$, which assigns to any $t \in \mathbb{R}$ the *algebraic volume enclosed* by Σ_t as defined in (2.8). We computed $A'_K(0)$ in Proposition 3.2. On the other hand, it is well known ([2, Eq. (2.3)]) that

$$V'(0) = \int_\Sigma u\, d\Sigma, \tag{4.3}$$

where $u := \langle X, N\rangle$. We say that the flow preserves the volume of Σ if $V(t)$ is constant for any t small enough. This implies that $\int_\Sigma u\, d\Sigma = 0$. Conversely, for any smooth function $u : \Sigma \to \mathbb{R}$ with $\int_\Sigma u\, d\Sigma = 0$, there is a flow $\{\phi_t\}_{t\in\mathbb{R}}$ admissible for $\mathcal{C}$, preserving the volume of Σ, and such that $\langle X, N\rangle = u$ on Σ, see Barbosa and do Carmo [2, Lem. (2.4)].

A hypersurface Σ in the previous conditions is *anisotropic stationary* (with respect to the fixed unit normal N) if $A'_K(0) = 0$ for any admissible flow for $\mathcal{C}$ preserving the volume of Σ. We emphasize that this property may depend on the unit normal N when K is not centrally symmetric. Thus, Σ could be anisotropic stationary with respect to N but not with respect to $-N$.

The first variational formulas in Proposition 3.2 and Eq. (4.3), together with the aforementioned construction of volume-preserving flows, lead to a characterization of anisotropic stationary hypersurfaces. Indeed, we can reason as Koiso and Palmer [22, Prop. 3.1] to deduce the next result, which generalizes the isotropic situation.

Proposition 4.1 *A two-sided hypersurface Σ immersed in $\overline{\mathcal{C}}$ with boundary $\partial\Sigma$ in $\partial\mathcal{C} \setminus \{0\}$ is anisotropic stationary if and only if the anisotropic mean curvature H_K is constant on Σ and $\langle N_K, \xi\rangle = 0$ along $\partial\Sigma$.*

Example 4.2 Take the hypersurface $\Sigma = \partial K \cap \overline{\mathcal{C}}$ with unit normal $N = \eta_K$ (the one pointing outside K). Note that ∂K meets $\partial\mathcal{C}$ transversally because K is convex and contains 0 in its interior. Hence, Σ is a compact hypersurface with boundary $\partial\Sigma$ such that $\Sigma \cap \partial\mathcal{C} = \partial\Sigma$. In Examples 2.1 (iii) we saw that $H_K(p) = -1$ and $N_K(p) = p$, for any $p \in \Sigma$. As $\partial\mathcal{C}$ is invariant under dilations centered at 0 then $N_K(p) \in T_p(\partial\mathcal{C})$ for any $p \in \partial\Sigma$ and so, $\langle N_K, \xi\rangle = 0$. From the previous proposition we conclude that Σ is anisotropic stationary.

In the isotropic case the orthogonality condition for a stationary hypersurface Σ entails that $\nu = \xi$ along $\partial\Sigma$. In the next lemma we establish a similar equality for the anisotropic conormal ν_K that will be useful in future results.

Lemma 4.3 *Let Σ be a two-sided hypersurface immersed in $\overline{\mathcal{C}}$ with boundary $\partial\Sigma$ in $\partial\mathcal{C} \setminus \{0\}$. If $\langle N_K, \xi\rangle = 0$ in $\partial\Sigma$, then $\cos\theta$ never vanishes and*

$$\nu_K = \frac{\varphi_K}{\cos\theta}\,\xi \quad \text{along } \partial\Sigma,$$

where $\varphi_K := h_K(N)$ and θ is the oriented angle function between ν and ξ in $T(\partial\Sigma)^\perp$.

Proof Recall that $\nu_K := \varphi_K\,\nu - \langle N_K, \nu\rangle N$, which is normal to $\partial\Sigma$. By Eq. (4.2) we have $\langle \nu_K, \mu\rangle = -\langle N_K, \xi\rangle = 0$. This shows that ν_K is normal to $\partial\mathcal{C}$ and so, $\nu_K = \langle \nu_K, \xi\rangle\xi$ along $\partial\Sigma$.

Let us compute $\langle \nu_K, \xi \rangle$. Consider the projection N_K^* of N_K onto $T(\partial\Sigma)^\perp$. By using (4.2), equality $\langle N_K, \xi \rangle = 0$, and that $\{\nu, N\}$ and $\{\xi, \mu\}$ are orthonormal basis of $T(\partial\Sigma)^\perp$, we obtain

$$\langle \nu_K, \xi \rangle^2 = \langle N_K, \mu \rangle^2 = |N_K^*|^2 = \langle N_K, \nu \rangle^2 + \varphi_K^2,$$

which is a positive number. Hence, $\langle \nu_K, \xi \rangle$ never vanishes along $\partial\Sigma$. On the other hand, by substituting the expression for ξ in (4.1) into equality $\langle N_K, \xi \rangle = 0$, we get

$$\sin\theta = -\frac{\langle N_K, \nu \rangle \cos\theta}{\varphi_K} \quad \text{along } \partial\Sigma.$$

The definition of ν_K and the two previous relations lead to

$$\langle \nu_K, \xi \rangle = \varphi_K \cos\theta - \langle N_K, \nu \rangle \sin\theta = \frac{\varphi_K^2 + \langle N_K, \nu \rangle^2}{\varphi_K} \cos\theta = \frac{\langle \nu_K, \xi \rangle^2}{\varphi_K} \cos\theta,$$

and so

$$\frac{\langle \nu_K, \xi \rangle}{\varphi_K} \cos\theta = 1 \quad \text{along } \partial\Sigma.$$

This implies that $\cos\theta$ never vanishes and allows to deduce the announced expression for ν_K. $\square$

Now, we can provide new expressions for the derivatives of the anisotropic area when we consider an anisotropic stationary hypersurface Σ in $\mathcal{C}$. On the one hand, the boundary term in the formula for $A_K'(0)$ obtained in Proposition 3.2 vanishes for any flow $\{\phi_t\}_{t\in\mathbb{R}}$ admissible for $\mathcal{C}$. This comes from Proposition 4.1 and Lemma 4.3 since the velocity vector field X is tangent to $\partial\mathcal{C} \setminus \{0\}$. Hence

$$A_K'(0) = -nH_K \int_\Sigma u \, d\Sigma, \tag{4.4}$$

where $u := \langle X, N \rangle$ on Σ. Note that, for the flow $\phi_t(p) := e^t p$, the first equation in (2.9) implies that $A_K'(0) = nA_K(\Sigma)$. Thus, by having in mind (4.4) and Eq. (2.8), it follows that

$$A_K(\Sigma) = -(n+1)\, H_K\, V(\Sigma). \tag{4.5}$$

This identity is a *Minkowski-type formula* for compact anisotropic stationary hypersurfaces in $\mathcal{C}$.

Remark 4.4 We can use (2.8) to write (4.5) as

$$\int_\Sigma \big(\varphi_K + H_K\langle p, N(p)\rangle\big)\, d\Sigma = 0.$$

The proof of (4.5) implies that the previous identity is true for any compact hypersurface Σ immersed in $\mathcal{C}$ with boundary $\partial\Sigma$ in $\partial\mathcal{C} \setminus \{0\}$ and satisfying $\langle N_K, \xi \rangle = 0$ along $\partial\Sigma$. Similar formulas were previously obtained by He and Li [16, Thm. 1.1], and Jia et al. [18, Thm. 1.3].

On the other hand, the boundary integrand appearing in $A_K''(0)$, see Proposition 3.3, can be written in terms of the Euclidean extrinsic geometry of $\partial\mathcal{C}$. We prove this in the next proposition, where we also compute the second derivative of the volume for certain flows.

Proposition 4.5 *Let Σ be a two-sided anisotropic stationary hypersurface immersed in $\overline{\mathcal{C}}$ with boundary $\partial\Sigma$ in $\partial\mathcal{C}\setminus\{0\}$. Consider a smooth complete vector field X on $\mathbb{R}^{n+1}$ such that $X_{|\Sigma}=N_K$, $X(0)=0$ and X is tangent to $\partial\mathcal{C}\setminus\{0\}$. Then, for the admissible flow $\{\phi_t\}_{t\in\mathbb{R}}$ defined by the one-parameter group of diffeomorphisms associated to X, we have*

$$\begin{aligned} A_K''(0) &= \int_\Sigma \big(n^2H_K^2 - \mathrm{tr}(B_K^2)\big)\,\varphi_K\,d\Sigma \\ &\quad - \int_\Sigma nH_K\,v\,d\Sigma \\ &\quad - \int_{\partial\Sigma} \frac{\mathrm{II}(N_K,N_K)}{\cos\theta}\,\varphi_K\,d(\partial\Sigma), \end{aligned} \tag{4.6}$$

$$V''(0) = \int_\Sigma (-nH_K\,\varphi_K + v)\,d\Sigma, \tag{4.7}$$

where B_K is the anisotropic shape operator of Σ, II is the second fundamental form of $\partial\mathcal{C}\setminus\{0\}$ with respect to the inner normal, θ is the oriented angle function between ν and ξ in $T(\partial\Sigma)^\perp$, and $v:=\langle Z,N\rangle$ is the normal component of $Z:=D_XX$.

Remark 4.6 Observe that N_K is tangent to $\partial\mathcal{C}$ along $\partial\Sigma$ because Σ is anisotropic stationary. This guarantees the existence of a vector field X in the conditions of the statement and that the term $\mathrm{II}(N_K,N_K)$ is well defined. Note also that $\cos\theta$ never vanishes by Lemma 4.3.

Proof of Proposition 4.5 We first check that (4.6) holds. From Proposition 3.3 it suffices to see that

$$\langle Z,\nu_K\rangle = \frac{\mathrm{II}(N_K,N_K)}{\cos\theta}\,\varphi_K \quad \text{along } \partial\Sigma.$$

By Lemma 4.3 we know that

$$\langle Z,\nu_K\rangle = \frac{\varphi_K}{\cos\theta}\langle Z,\xi\rangle \quad \text{along } \partial\Sigma.$$

It is clear that $\langle X,\xi\rangle=0$ along $\partial\mathcal{C}$ because X is tangent to $\partial\mathcal{C}\setminus\{0\}$ and $X(0)=0$. By differentiating with respect to X, we obtain

$$0=\langle D_XX,\xi\rangle+\langle X,D_X\xi\rangle=\langle Z,\xi\rangle-\mathrm{II}(X,X),$$

so that $\langle Z,\xi\rangle=\mathrm{II}(X,X)=\mathrm{II}(N_K,N_K)$ along $\partial\Sigma$.

Now we compute $V''(0)$. As the family $\{\phi_t\}_{t\in\mathbb{R}}$ is a one-parameter group of diffeomorphisms, we can employ Eq. (4.3) to get

$$V'(t)=\int_{\Sigma_t}\langle X,N_t\rangle\,d\Sigma_t=\int_\Sigma\big(\langle X,N_t\rangle\circ\phi_t\big)\,\mathrm{Jac}\,\phi_t\,d\Sigma.$$

By differentiating and having in mind (3.8), (3.7), (3.14), (2.6) and (2.3), it follows that

$$V''(0)=\int_\Sigma\big(\langle D_XX,N\rangle+\langle X,D_X\overline{N}\rangle+\langle X,N\rangle\,\mathrm{div}_\Sigma X\big)\,d\Sigma=\int_\Sigma(v-nH_K\,\varphi_K)\,d\Sigma.$$

This completes the proof. □

We now turn to the main result of the paper. This is a classification of anisotropic stable hypersurfaces in a Euclidean solid cone.

Let Σ be a two-sided hypersurface immersed in $\overline{\mathcal{C}}$ with boundary $\partial\Sigma$ in $\partial\mathcal{C}\setminus\{0\}$. We fix a smooth unit normal vector field N on Σ. We say that Σ is *anisotropic stable* (with respect to

N) if $A'_K(0) = 0$ and $A''_K(0) \geqslant 0$ for any flow admissible for $\mathcal{C}$ and preserving the volume of Σ. When $\mathcal{C}$ is convex, the hypersurface $\partial K \cap \overline{\mathcal{C}}$ is anisotropic stable with respect to the outer unit normal since it minimizes the anisotropic area among compact hypersurfaces in $\mathcal{C}$ separating the same volume, see [28, Cor. 1.2] and [4, Thm. 1.3]. So, it is natural to ask if this property characterizes $\partial K \cap \overline{\mathcal{C}}$ up to translations and dilations centered at 0. The next example shows that, in general, the answer is negative because other anisotropic stable hypersurfaces may appear.

Example 4.7 Let $\mathcal{C}$ be a convex cone different from a Euclidean half-space and such that $\partial\mathcal{C}$ contains a half-hyperplane P (any solid cone in $\mathbb{R}^2$ satisfies this property). Consider the open half-space $\mathcal{H} \subset \mathbb{R}^{n+1}$ with $P \subset \partial\mathcal{H}$ and $\mathcal{C} \subset \mathcal{H}$. Since $\mathcal{H}$ is a convex cone, we can apply the isoperimetric result in [28, Cor. 1.2] and [4, Thm. 1.3] to deduce that the truncated Wulff shape $\partial K \cap \overline{\mathcal{H}}$ minimizes the anisotropic area in $\mathcal{H}$ for fixed volume. Next, we apply to $\partial K \cap \overline{\mathcal{H}}$ a translation along P so that the resulting hypersurface Σ is contained in $\overline{\mathcal{C}}$. Note that Σ is anisotropic stable in $\mathcal{C}$ because $\partial K \cap \overline{\mathcal{H}}$ is anisotropic stable in $\mathcal{H}$. Finally, as $\mathcal{C} \neq \mathcal{H}$, then $\Sigma \neq p_0 + \lambda\,(\partial K \cap \overline{\mathcal{C}})$ for any $p_0 \in \mathbb{R}^{n+1}$ and $\lambda > 0$.

This example illustrates that the optimal conclusion to be deduced for an anisotropic stable hypersurface Σ in $\mathcal{C}$ is that $\Sigma \subset \partial K$, up to translation and homothety. We prove this fact in the next uniqueness statement under our regularity conditions for Σ, $\mathcal{C}$ and K.

Theorem 4.8 *Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a solid convex cone over a smooth domain of $\mathbb{S}^n$. Consider a compact, connected, two-sided hypersurface Σ immersed in $\overline{\mathcal{C}}$ with smooth boundary $\partial\Sigma$ in $\partial\mathcal{C} \setminus \{0\}$ and such that $\Sigma \cap \partial\mathcal{C} = \partial\Sigma$. If Σ is anisotropic stable for the area A_K defined by a smooth strictly convex body $K \subset \mathbb{R}^{n+1}$ then, there is $p_0 \in \mathbb{R}^{n+1}$ and $\lambda > 0$ such that $\Sigma \subset p_0 + \lambda\,(\partial K)$.*

Proof We will follow the idea explained in the Introduction. As Σ is anisotropic stationary, Proposition 4.1 implies that H_K is constant on Σ and $\langle N_K, \xi\rangle = 0$ along $\partial\Sigma$. So, we can find a smooth complete vector field X on $\mathbb{R}^{n+1}$ such that $X_{|\Sigma} = N_K$, $X(0) = 0$ and X is tangent to $\partial\mathcal{C} \setminus \{0\}$. By Eq. (2.3) it is clear that $u = \varphi_K$ on Σ, where $u := \langle X, N\rangle$ and $\varphi_K := h_K(N)$. Let $\{\psi_t\}_{t\in\mathbb{R}}$ be the one-parameter group of diffeomorphisms associated to X. We take the functionals $A_K(t) := A_K(\Sigma_t)$ and $V(t) := V(\Sigma_t)$ associated to the variation $\Sigma_t := \psi_t(\Sigma)$. From the Minkowski formula in Eq. (4.5) we have $H_K \neq 0$ and $V(\Sigma) \neq 0$. Hence, there is $\varepsilon > 0$ such that $V(t)$ has the same sign as $V(\Sigma)$ for any $t \in (-\varepsilon, \varepsilon)$. Next, for any $t \in (\varepsilon, \varepsilon)$, we apply to Σ_t a dilation $\delta_{\lambda(t)}(p) := \lambda(t)\,p$ with $\lambda(t) > 0$, so that the volume of $\delta_{\lambda(t)}(\Sigma_t)$ equals the volume of Σ. Since $V(\delta_{\lambda(t)}(\Sigma_t)) = \lambda(t)^{n+1}\,V(t)$, see (2.9), we get

$$\lambda(t) := \left(\frac{V(\Sigma)}{V(t)}\right)^{\frac{1}{n+1}}, \quad \text{for any } t \in (-\varepsilon, \varepsilon). \tag{4.8}$$

In particular, $\lambda(0) = 1$. We consider a smooth positive function on $\mathbb{R}$ extending $\lambda(t)$ for small values of t. We also denote this function by $\lambda(t)$. If we define

$$\phi_t := \delta_{\lambda(t)} \circ \psi_t, \quad \text{for any } t \in \mathbb{R},$$

then we produce an admissible flow for $\mathcal{C}$ that preserves the volume of Σ. Hence, the anisotropic stability of Σ entails that the functional $a_K(t) := A_K(\phi_t(\Sigma))$ satisfies $a''_K(0) \geqslant 0$. To prove the theorem we need to compute $a''_K(0)$. By Eq. (2.9) we know that

$$a_K(t) = A_K(\delta_{\lambda(t)}(\Sigma_t)) = \lambda(t)^n\,A_K(t). \tag{4.9}$$

Thus, the calculus of $a''_K(0)$ relies on the values of $\lambda'(0)$, $\lambda''(0)$, $A'_K(0)$ and $A''_K(0)$.

By using (4.3), (4.4), the fact that $u = \varphi_K$ on Σ, and that $A_K(\Sigma) = \int_\Sigma \varphi_K \, d\Sigma$, we obtain

$$V'(0) = A_K(\Sigma), \tag{4.10}$$

$$A'_K(0) = -nH_K A_K(\Sigma). \tag{4.11}$$

On the other hand, from the expression of $\lambda(t)$ in (4.8), we have

$$\lambda'(t) = -\frac{1}{n+1} \, V(\Sigma)^{\frac{1}{n+1}} \, V(t)^{\frac{-n-2}{n+1}} \, V'(t). \tag{4.12}$$

Hence, from Eqs. (4.10) and (4.5), it follows that

$$\lambda'(0) = -\frac{1}{n+1} \, \frac{A_K(\Sigma)}{V(\Sigma)} = H_K. \tag{4.13}$$

Now we compute $A''_K(0)$ and $\lambda''(0)$. From (4.7) and (4.6), since H_K is constant, we get

$$V''(0) = -nH_K A_K(\Sigma) + \alpha, \tag{4.14}$$

$$\begin{aligned} A''_K(0) = n^2 H_K^2 \, A_K(\Sigma) - \int_\Sigma \operatorname{tr}(B_K^2) \, \varphi_K \, d\Sigma - nH_K \, \alpha \\ - \int_{\partial\Sigma} \frac{\mathrm{II}(N_K, N_K)}{\cos\theta} \, \varphi_K \, d(\partial\Sigma), \end{aligned} \tag{4.15}$$

where $\alpha := \int_\Sigma v \, d\Sigma$ and $v := \langle D_X X, N \rangle$. Recall that II is the second fundamental form of $\partial\mathcal{C} \setminus \{0\}$ with respect to the inner normal ξ. On the other hand, from (4.12) we deduce

$$\lambda''(t) = \frac{-1}{n+1} \, V(\Sigma)^{\frac{1}{n+1}} \left(-\frac{n+2}{n+1} \, V(t)^{\frac{-2n-3}{n+1}} \, V'(t)^2 + V(t)^{\frac{-n-2}{n+1}} \, V''(t) \right).$$

By evaluating at $t = 0$ and simplifying, Eqs. (4.10) and (4.14) give us

$$\lambda''(0) = \frac{n+2}{(n+1)^2} \, \frac{A_K(\Sigma)^2}{V(\Sigma)^2} + \frac{n}{n+1} \, \frac{H_K A_K(\Sigma)}{V(\Sigma)} - \frac{1}{n+1} \, \frac{\alpha}{V(\Sigma)}.$$

When we employ the identity (4.5) in the three summands above, we arrive at

$$\lambda''(0) = 2H_K^2 + \frac{H_K}{A_K(\Sigma)} \, \alpha. \tag{4.16}$$

Finally, we differentiate into Eq. (4.9) to infer

$$a'_K(t) = n \, \lambda(t)^{n-1} \, \lambda'(t) \, A_K(t) + \lambda(t)^n \, A'_K(t).$$

As a consequence

$$\begin{aligned} a''_K(t) = n \left\{ (n-1) \, \lambda(t)^{n-2} \, \lambda'(t)^2 \, A_K(t) + \lambda(t)^{n-1} \, \lambda''(t) \, A_K(t) + 2\lambda(t)^{n-1} \, \lambda'(t) \, A'_K(t) \right\} \\ + \lambda(t)^n \, A''_K(t). \end{aligned}$$

By substituting above the expressions in (4.13), (4.16), (4.11), (4.15), and simplifying, we obtain

$$a''_K(0) = -\int_\Sigma \left(\operatorname{tr}(B_K^2) - nH_K^2 \right) \varphi_K \, d\Sigma - \int_{\partial\Sigma} \frac{\mathrm{II}(N_K, N_K)}{\cos\theta} \, \varphi_K \, d(\partial\Sigma). \tag{4.17}$$

The first integrand in the previous formula is nonnegative by (2.7). The convexity of $\mathcal{C}$ implies that $\mathrm{II}(N_K, N_K) \geqslant 0$ along $\partial\Sigma$. Let us see that $\cos\theta > 0$ along $\partial\Sigma$. From the first equation in (4.1) we know that $\cos\theta = \langle \nu, \xi \rangle$. Take a point $p \in \partial\Sigma$. Since $\nu(p)$ is the inner conormal of $\partial\Sigma$ in Σ, there is a smooth curve $\gamma : [0, \tau) \to \Sigma$ such that $\gamma(0) = p$ and $\gamma'(0) = \nu(p)$. As $\Sigma \subset \overline{\mathcal{C}}$ and $\mathcal{C}$ is convex, the function $f(t) := \langle \gamma(t) - p, \xi(p) \rangle$ satisfies $f(t) \geqslant 0$ for any $t \in [0, \tau)$ and $f(0) = 0$. This gives us $\langle \nu(p), \xi(p) \rangle = f'(0) \geqslant 0$. Indeed we have $\langle \nu(p), \xi(p) \rangle > 0$ because $\cos\theta$ never vanishes, see Lemma 4.3. All this together with the stability inequality $a_K''(0) \geqslant 0$ entails that $\mathrm{tr}(B_K^2) = nH_K^2$ on Σ (this means that Σ is anisotropic umbilical) and

$$\mathrm{II}(N_K, N_K) = 0 \quad \text{along } \partial\Sigma. \tag{4.18}$$

Since $H_K \neq 0$ (as a consequence of Eq. (4.5)) the proof finishes by invoking Proposition 2.2. □

Remark 4.9 (Non-smooth cones) The proof of Theorem 4.8 is still valid when $\mathcal{C}$ is an arbitrary open convex cone (the base domain $\mathcal{D} \subset \mathbb{S}^n$ need not be smooth), provided the boundary $\partial\Sigma$ is contained in a smooth open portion of $\partial\mathcal{C}$. For instance, the conclusion holds for a compact anisotropic stable hypersurface Σ disjoint from the edge of a domain $\mathcal{C}$ bounded by two transversal hyperplanes. In this direction, Koiso [19, Thm. 4], [21, Thm. 1] has characterized compact anisotropic stable capillary hypersurfaces in wedge-shaped domains of $\mathbb{R}^{n+1}$.

Remark 4.10 (Planar cones) For a solid cone $\mathcal{C} \subset \mathbb{R}^2$, the boundary $\partial\mathcal{C}$ is the union of two-closed half-lines leaving from 0. Thus, we have $\mathrm{II} = 0$ along $\partial\mathcal{C} \setminus \{0\}$ and the boundary term in (4.17) disappears. Hence, the conclusion of Theorem 4.8 remains valid even if the cone is not convex.

The situation described in Example 4.7 leads us to seek additional conditions on the cone in order to deduce stronger uniqueness conclusions. In this direction we can prove the next statement.

Corollary 4.11 *Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a solid cone over a smooth strictly convex domain of $\mathbb{S}^n$. Consider a compact, connected, two-sided hypersurface Σ immersed in $\overline{\mathcal{C}}$ with smooth boundary $\partial\Sigma$ in $\partial\mathcal{C} \setminus \{0\}$ and such that $\Sigma \cap \partial\mathcal{C} = \partial\Sigma$. If Σ is anisotropic stable for the area A_K defined by a smooth strictly convex body $K \subset \mathbb{R}^{n+1}$, then $\Sigma = \lambda\,(\partial K \cap \overline{\mathcal{C}})$ for some $\lambda > 0$.*

Proof From Theorem 4.8 we know that $\Sigma \subset p_0 + \lambda\,(\partial K)$ for some $p_0 \in \mathbb{R}^{n+1}$ and $\lambda > 0$. If we define on Σ the unit normal vector $N(p) := \eta_K\big((p - p_0)/\lambda\big)$ then, by using (2.2) and equality $(\pi_K \circ \eta_K)(w) = w$ for any $w \in \partial K$, we get $N_K(p) = (p - p_0)/\lambda$ for any $p \in \Sigma$.

The fact that the base set $\mathcal{D} \subset \mathbb{S}^n$ of the cone $\mathcal{C}$ is a strictly convex domain means that the second fundamental form of $\partial\mathcal{D}$ as a hypersurface of $\mathbb{S}^n$ is always positive definite with respect to the inner unit normal. This implies that $\mathrm{II}_p(w, w) > 0$ for any $p \in \partial\Sigma$ and any vector $w \in T_p(\partial\mathcal{C})$ non-proportional to p. From the identity in (4.18) it follows that $(p - p_0)/\lambda$ is proportional to p, for any $p \in \partial\Sigma$. From here we get $p_0 = 0$, and this completes the proof. □

A special example of convex cone is a half-space $\mathcal{H} \subset \mathbb{R}^{n+1}$ with $0 \in \partial\mathcal{H}$. Since $\partial\mathcal{H}$ is smooth we do not need to assume $\partial\Sigma \subset \partial\mathcal{H} \setminus \{0\}$. Note that Corollary 4.11 cannot be applied in this situation because $\mathrm{II} = 0$ on $\partial\mathcal{C}$. However, in the next statement we deduce that $\partial K \cap \overline{\mathcal{H}}$ is the unique compact anisotropic stable hypersurface in $\mathcal{H}$, up to dilations about 0 and translations along $\partial\mathcal{H}$. A similar result for anisotropic stable capillary hypersurfaces in $\mathcal{H}$ has been given by Guo and Xia [14, Thm. 1.1] with a different proof.

Corollary 4.12 *Let Σ be a compact, connected, two-sided hypersurface immersed in an open half-space $\mathcal{H} \subset \mathbb{R}^{n+1}$ with smooth boundary $\partial\Sigma \subset \partial\mathcal{H}$ such that $\Sigma \cap \partial\mathcal{H} = \partial\Sigma$. If Σ is anisotropic stable, then $\Sigma = p_0 + \lambda\,(\partial K \cap \overline{\mathcal{H}})$ for some $p_0 \in \partial\mathcal{H}$ and $\lambda > 0$.*

Proof After applying Theorem 4.8 it remains to see that $p_0 \in \partial\mathcal{H}$. By reasoning as in the proof of Corollary 4.11 we obtain $N_K(p) = (p - p_0)/\lambda$, for any $p \in \Sigma$. The orthogonality condition $\langle N_K, \xi \rangle = 0$ along $\partial\Sigma$ entails that $p - p_0 \in T_p(\partial\mathcal{H})$ for any $p \in \partial\Sigma$. As $\partial\mathcal{H}$ is a Euclidean hyperplane, the straight line starting from a point $p \in \partial\Sigma$ with $p \neq p_0$ and generated by the vector $p - p_0$ is entirely contained in $\partial\mathcal{H}$. This shows that $p_0 \in \partial\mathcal{H}$, as we claimed. □

Funding Funding for open access publishing: Universidad de Granada/CBUA

References

1. Barbosa, E., Silva, L.C.: Surfaces of constant anisotropic mean curvature with free boundary in revolution surfaces. Manuscr. Math. **169**(3–4), 439–459 (2022)
2. Barbosa, J.L., do Carmo, M.P.: Stability of hypersurfaces with constant mean curvature. Math. Z. **185**(3), 339–353 (1984)
3. Brothers, J.E., Morgan, F.: The isoperimetric theorem for general integrands. Mich. Math. J. **41**(3), 419–431 (1994)
4. Cabré, X., Ros-Oton, X., Serra, J.: Sharp isoperimetric inequalities via the ABP method. J. Eur. Math. Soc. (JEMS) **18**(12), 2971–2998 (2016)
5. Cañete, A., Rosales, C.: Compact stable hypersurfaces with free boundary in convex solid cones with homogeneous densities. Calc. Var. Partial Differ. Equ. **51**(3–4), 887–913 (2014)
6. Choe, J., Park, S.-H.: Capillary surfaces in a convex cone. Math. Z. **267**(3–4), 875–886 (2011)
7. Clarenz, U.: Enclosure theorems for extremals of elliptic parametric functionals. Calc. Var. Partial Differ. Equ. **15**(3), 313–324 (2002)
8. Clarenz, U.: The Wulff shape minimizes an anisotropic Willmore functional. Interfaces Free Bound. **6**(3), 351–359 (2004)
9. De Lellis, C., De Philippis, G., Kirchheim, B., Tione, R.: Geometric measure theory and differential inclusions. Ann. Fac. Sci. Toulouse Math. **30**(4), 899–960 (2021)
10. De Rosa, A., Kolasiński, S.: Equivalence of the ellipticity conditions for geometric variational problems. Commun. Pure Appl. Math. **73**(11), 2473–2515 (2020)
11. Dipierro, S., Poggesi, G., Valdinoci, E.: Radial symmetry of solutions to anisotropic and weighted diffusion equations with discontinuous nonlinearities. Calc. Var. Partial Differ. Equ. **61** , no. 2, Paper No. 72, 31 (2022)
12. Figalli, A., Indrei, E.: A sharp stability result for the relative isoperimetric inequality inside convex cones. J. Geom. Anal. **23**(2), 938–969 (2013)
13. Fonseca, I., Müller, S.: A uniqueness proof for the Wulff theorem. Proc. Roy. Soc. Edinb. Sect. A **119**(1–2), 125–136 (1991)
14. Guo, J., Xia, C.: Stable anisotropic capillary hypersurfaces in the half-space. arXiv:2301.03020 (2023)
15. He, Y., Li, H., Ma, H., Ge, J.: Compact embedded hypersurfaces with constant higher order anisotropic mean curvatures. Indiana Univ. Math. J. **58**(2), 853–868 (2009)
16. He, Y.J., Li, H.Z.: Integral formula of Minkowski type and new characterization of the Wulff shape. Acta Math. Sin. (Engl. Ser.) **24**(4), 697–704 (2008)

17. He, Y.J., Li, H.Z.: Anisotropic version of a theorem of H. Hopf, Ann. Global Anal. Geom. **35**(3), 243–247 (2009)
18. Jia, X., Wang, G., Xia, C., Zhang, X.: Alexandrov's theorem for anisotropic capillary hypersurfaces in the half-space. Arch. Ration. Mech. Anal. **247**(2), Paper No. 25, 19 (2023)
19. Koiso, M.: Uniqueness of closed equilibrium hypersurfaces for anisotropic surface energy and application to a capillary problem, Math. Comput. Appl. **24**(4), Paper No. 88, 15 (2019)
20. Koiso, M.: Uniqueness of stable closed non-smooth hypersurfaces with constant anisotropic curvature. arXiv:1903.03951 (2019)
21. Koiso, M.: Stable anisotropic capillary hypersurfaces in a wedge. Math. Eng. **5**(2) Paper No. 029, 22 (2023)
22. Koiso, M., Palmer, B.: Stability of anisotropic capillary surfaces between two parallel planes. Calc. Var. Partial Differ. Equ. **25**(3), 275–298 (2006)
23. Koiso, M., Palmer, B.: Anisotropic capillary surfaces with wetting energy. Calc. Var. Partial Differ. Equ. **29**(3), 295–345 (2007)
24. Koiso, M., Palmer, B.: Uniqueness theorems for stable anisotropic capillary surfaces. SIAM J. Math. Anal. **39**(3), 721–741 (2007)
25. Koiso, M., Palmer, B.: Anisotropic umbilic points and Hopf's theorem for surfaces with constant anisotropic mean curvature. Indiana Univ. Math. J. **59**(1), 79–90 (2010)
26. Lions, P.-L., Pacella, F.: Isoperimetric inequalities for convex cones. Proc. Am. Math. Soc. **109** (2), 477–485 (90i:52021) (1990)
27. Maggi, F.: Sets of Finite Perimeter and Geometric Variational Problems, Cambridge Studies in Advanced Mathematics, vol. 135. Cambridge University Press, Cambridge (2012)
28. Milman, E., Rotem, L.: Complemented Brunn–Minkowski inequalities and isoperimetry for homogeneous and non-homogeneous measures. Adv. Math. **262**, 867–908 (2014)
29. Morgan, F.: Planar Wulff shape is unique equilibrium. Proc. Am. Math. Soc. **133**(3), 809–813 (2005)
30. Pacella, F., Tralli, G.: Overdetermined problems and constant mean curvature surfaces in cones. Rev. Mat. Iberoam. **36**(3), 841–867 (2020)
31. Palmer, B.: Stability of the Wulff shape. Proc. Am. Math. Soc. **126**(12), 3661–3667 (1998)
32. Reilly, R.C.: The relative differential geometry of nonparametric hypersurfaces. Duke Math. J. **43**(4), 705–721 (1976)
33. Ritoré, M., Rosales, C.: Existence and characterization of regions minimizing perimeter under a volume constraint inside Euclidean cones. Trans. Am. Math. Soc. **356**(11), 4601–4622 (2004)
34. Ros, A., Souam, R.: On stability of capillary surfaces in a ball. Pac. J. Math. **178**(2), 345–361 (1997)
35. Schneider, R.: Convex Bodies: The Brunn–Minkowski Theory, Expanded ed., Encyclopedia of Mathematics and Its Applications, vol. 151, Cambridge University Press, Cambridge (2014)
36. Simon, L.: Lectures on geometric measure theory. In: Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University Centre for Mathematical Analysis, Canberra (1983)
37. Taylor, J.E.: Unique structure of solutions to a class of nonelliptic variational problems, Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 1, Stanford Univ., Stanford, Calif., 1973) American Mathematical Society, Providence, RI (1975), pp. 419–427
38. Taylor, J.E.: Crystalline variational problems. Bull. Am. Math. Soc. **84**(4), 568–588 (1978)
39. Wang, G., Xia, C.: Uniqueness of stable capillary hypersurfaces in a ball. Math. Ann. **374**(3–4), 1845–1882 (2019)
40. Weng, L.: An overdetermined problem of anisotropic equations in convex cones. J. Differ. Equ. **268**(7), 3646–3664 (2020)
41. Wente, H.C.: A note on the stability theorem of J. L. Barbosa and M. Do Carmo for closed surfaces of constant mean curvature. Pac. J. Math. **147**(2) pp. 375–379 (1991)
42. Winklmann, S.: A note on the stability of the Wulff shape. Arch. Math. (Basel) **87**(3), 272–279 (2006)
43. Wulff, G.: Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Kristallflächen. Z. Kryst. Miner. **34**, 449–530 (1901)