A note on stability criteria in the periodic Lotka–Volterra predator-prey model

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ABSTRACT

We present a stability result for $T$-periodic solutions of the periodic predator–prey Lotka–Volterra model. In 2021, R. Ortega gave a stability criteria in terms of the $L^1$ norm of the coefficients of a planar linear system associated to the model. Previously, in 1994, Z. Amine and R. Ortega proved another stability criteria formulated in terms of the $L^\infty$ norm. The present work gives a $L^p$ criterion, building a bridge between the two previous results.

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1. Introduction

In this work we consider the periodic predator–prey Lotka–Volterra model:

\[
\begin{align*}
\dot{u} &= u(a(t) - b(t)u - c(t)v), \\
\dot{v} &= v(d(t) + e(t)u - f(t)v)
\end{align*}
\]  

(1)

with $u \geq 0$, $v \geq 0$. All the coefficients are $T$-periodic, $a, d \in L^p(T_T)$, $p \in [1, \infty]$, and $b, c, e$ and $f$ are positive functions in $C(T_T)$, where we denote the quotient set $\mathbb{R}/T\mathbb{Z}$ as $T_T$. This model is a classical non-autonomous model for predator–prey interaction studied by many authors (see [1] and the references therein). In [1] the authors study the existence of coexistence states and in particular prove that if one among the trivial and semi-trivial states is linear stable then it attracts all the solutions with positive initial conditions. As
an immediate consequence, for the existence of a coexistence state it is necessary that the trivial and the possible semi-trivial states are linearly unstable. In the same paper the authors prove that this is also a sufficient condition.

Assuming the existence of a coexistence state, to know if it is stable or not is an important problem. In [2] and in [3] the authors addressed this question and gave conditions for the existence of one stable coexistence state.

The stability of the coexistence state was obtained using a homotopy from a non-autonomous linear system to an autonomous one. In both results plays an important role to prove the non-existence of $2T$-periodic solutions for a linear system of the form

$$
\begin{align*}
\dot{x}_1 &= -a_{11}(t)x_1 - a_{12}(t)x_2, \\
\dot{x}_2 &= a_{21}(t)x_1 - a_{22}(t)x_2,
\end{align*}
$$

where the $a_{ij}$ are non negative. In order to guarantee this nonexistence, in [2] a condition which implies

$$
T^{1/2}||a_{12}||_{L^\infty(\mathbb{T}_T)}^{1/2} ||a_{21}||_{L^\infty(\mathbb{T}_T)}^{1/2} + \frac{1}{2}||a_{11} - a_{22}||_{L^1(\mathbb{T}_T)} \leq \pi,
$$

is given while in [3] the analogous expression

$$
||a_{12}||_{L^1(\mathbb{T}_T)}^{1/2} ||a_{21}||_{L^1(\mathbb{T}_T)}^{1/2} + \frac{1}{2}||a_{11} - a_{22}||_{L^1(\mathbb{T}_T)} \leq 2,
$$

but concerning the $L^1(0,T)$ norms, was obtained.

The main purpose of our paper is to extend these results allowing to use other $L^p$ norms, see Proposition 2.1 below. With this result we connect the results in [2,3]. We also give an example of a case in which the results in [2] and in [3] do not apply but ours does with $p = 2$. Finally, as in the previous papers, we give conditions for the local asymptotic stability of a coexistence state of (1).

2. Planar linear system: $L^p$ stability result

Our aim is to give a $L^p$ stability condition for the system

$$
\begin{align*}
\dot{x}_1 &= -a_{11}(t)x_1 - a_{12}(t)x_2, \\
\dot{x}_2 &= a_{21}(t)x_1 - a_{22}(t)x_2,
\end{align*}
$$

where the coefficients $a_{ij}$ belong to $L^p(\mathbb{T}_T)$ with $p \in [1, \infty]$ and satisfy

$$
\pi_{11} \geq 0, \pi_{22} \geq 0 \quad \text{and} \quad a_{12}(t) \geq \delta, a_{21}(t) \geq \delta \quad \text{a.e} \quad t \in \mathbb{R},
$$

for some $\delta > 0$ where $\pi_{ij} = \frac{1}{T} \int_0^T a_{ij}(t)dt$.

In order to do that we are going to give conditions which guarantee the nonexistence of $2T$-periodic solutions for this linear system in the next Proposition.

**Proposition 2.1.** The system (5) has no $2T$-periodic solutions except $x \equiv 0$ if the periodic coefficients $a_{ij}$ satisfy (with $\frac{1}{p} + \frac{1}{q} = 1$):

$$
T^{1/q}||a_{12}||_{L^p(\mathbb{T}_T)}^{1/2} ||a_{21}||_{L^p(\mathbb{T}_T)}^{1/2} + \frac{1}{2}||a_{11} - a_{22}||_{L^1(\mathbb{T}_T)} \leq \frac{\mathcal{J}(q)}{2^{2-1/q}},
$$

where

$$
\mathcal{J}(q) = \int_0^{2\pi} \frac{d\theta}{(|\cos \theta|^{2q} + |\sin \theta|^{2q})^{1/q}}, \quad q \in [1, \infty] \quad \text{and} \quad \mathcal{J}(\infty) := \lim_{q \to + \infty} \mathcal{J}(q).
$$
Proof. We make a change of variables to the elliptic-polar coordinates with a weight \( \mu > 0 \) that will be determined later:

\[
x_1 = \sqrt{\mu r} \cos \theta, \quad x_2 = \frac{1}{\sqrt{\mu}} r \sin \theta.
\]

Then the equation of motion for the variable \( \theta \) is,

\[
\dot{\theta} = \mu a_{21}(t) \cos^2 \theta + \frac{1}{\mu} a_{12}(t) \sin^2 \theta + (a_{11}(t) - a_{22}(t)) \cos \theta \sin \theta.
\] (8)

From (8), we can write

\[
\dot{\theta} \leq \left( \left( \mu a_{21}(t) + \left( \frac{1}{\mu} a_{12}(t) \right)^p \right)^{1/p} + |a_{11}(t) - a_{22}(t)| \left( \frac{2}{2^p} \right)^{1/p} \right) (|\cos \theta|^{2q} + |\sin \theta|^{2q})^{1/q}.
\] (9)

First, let us consider \( p \in ]1, \infty[ \). By the Hölder inequality in \( \mathbb{R}^2 \), we have

\[
\dot{\theta} \leq \left( \left( \mu a_{21}(t) + \left( \frac{1}{\mu} a_{12}(t) \right)^p \right)^{1/p} + |a_{11}(t) - a_{22}(t)| \left( \frac{2}{2^p} \right)^{1/p} \right) (|\cos \theta|^{2q} + |\sin \theta|^{2q})^{1/q}.
\] (10)

Let us integrate on an interval \( I \in \mathbb{R} \) where a solution of (5) is well defined,

\[
\int_{\theta(I)} \frac{d\theta}{(|\cos \theta|^{2q} + |\sin \theta|^{2q})^{1/q}} \leq \int_I \left( \left( \mu a_{21}(t) + \left( \frac{1}{\mu} a_{12}(t) \right)^p \right)^{1/p} + |a_{11}(t) - a_{22}(t)| \left( \frac{2}{2^p} \right)^{1/p} \right) (|\cos \theta|^{2q} + |\sin \theta|^{2q})^{1/q} dt.
\]

Now, by the Hölder inequality in \( L \)-spaces norms, we have

\[
\int_{\theta(I)} \frac{d\theta}{(|\cos \theta|^{2q} + |\sin \theta|^{2q})^{1/q}} \leq |I|^{1/q} \left( \int_I \left( \mu a_{21}(t) + \left( \frac{1}{\mu} a_{12}(t) \right)^p \right) dt + \int_I |a_{11}(t) - a_{22}(t)| dt \right)^{1/p} + 2^{1/p} \int_I |a_{11}(t) - a_{22}(t)| dt.
\]

Let us assume that exists a non-trivial 2T-periodic solution of (5) \( (x_1(t), x_2(t)) \). We claim that every non-trivial solution of (8) crosses the axes in a counter-clockwise sense. Intuitively, if the coefficients \( a_{12}(t) \) and \( a_{21}(t) \) are continuous and we consider small neighborhoods of the axes, the angular evolution is always positive since the sign in the right hand side of Eq. (8) is positive. In [3] the author gives a proof for coefficients in \( L^3(\mathbb{T}^2) \). Therefore we have in the angular variable \( \theta(t + 2T) = \theta(t) + 2\pi k \), being \( k \) a non-negative integer. Take \( k = 0 \) and let us consider that the solution \( (x_1(t), x_2(t)) \) cross an axis. Due to the periodicity it cannot cross the axis in a clockwise sense to came back. We conclude that the solution must lie in an open quadrant, and we have two possibilities, either \( x_1(t) \cdot x_2(t) > 0 \) or \( x_1(t) \cdot x_2(t) < 0 \) for \( t \in \mathbb{R} \).

In the first case we divide the first equation in (5) by \( x_1 \) and integrate over a 2T-period to get \( \overline{a}_{11} < 0 \), in contradiction with (6). The second case has an analog treatment considering the second equation. See [3] for more details.

For \( k \geq 1 \), integrating from 0 to 2T,

\[
kJ(q) < (2T)^{1/q} \left( \int_0^{2T} (\mu a_{21}(t))^p dt + \int_0^{2T} \left( \frac{1}{\mu} a_{12}(t) \right)^p dt \right)^{1/p} + 2^{1/p} \int_0^{2T} |a_{11}(t) - a_{22}(t)| dt.
\]

Additionally, we have dropped the equal sign since the equality in (9) only occurs when \( \theta = \pi/4 \pm \pi k \). Now, defining \( \mu_p := \left( \int_0^{2T} a_{12}(t)^p dt / \int_0^{2T} a_{21}(t)^p dt \right)^{1/2} \), we arrive to a contradiction with the hypothesis (7) in the more restrictive case associated with \( k = 1 \).

Concerning the limiting cases, we can found a proof for the case \( p = 1 \) in [3] and for \( p = \infty \) in [2].
In the next proposition we give a result which, together to the fact that $J(1) = 2\pi$, allows us to conclude that (7) connects the results in [2,3].

**Proposition 2.2.** We have that $\lim_{q \to +\infty} J(q) = 8$.

**Proof.** Let us study the limit of the integrand of $J(q)$:

$$f(\theta; q) = \left( |\cos \theta|^{2q} + |\sin \theta|^{2q} \right)^{-1/q}.$$  

(11)

It is useful to write the function in two different ways

$$f(\theta; q) = \cos^{-2} \theta \left( 1 + |\tan \theta|^{2q} \right)^{-1/q} = \sin^{-2} \theta \left( 1 + |\cot \theta|^{2q} \right)^{-1/q}.$$  

For $\theta \in [-\pi/4, \pi/4] \cup [3\pi/4, 5\pi/4]$ we have that

$$\lim_{q \to \infty} f(\theta; q) = \lim_{q \to \infty} \cos^{-2} \theta \left( 1 + |\tan \theta|^{2q} \right)^{-1/q} = \cos^{-2} \theta.$$  

(12)

Indeed, for $\theta \in [-\pi/4, \pi/4] \cup [3\pi/4, 5\pi/4]$,

$$2^{-1/q} \leq \left( 1 + \tan^2 \theta \right)^{-1/q} \leq \left( 1 + |\tan \theta|^{2q} \right)^{-1/q} \leq 1,$$

where we have used that $\tan^2 \theta \leq 1$ and $q \geq 1$ and the result follows. Analogously, for $\theta \in [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]$, we have that

$$\lim_{q \to \infty} f(\theta; q) = \lim_{q \to \infty} \sin^{-2} \theta \left( 1 + |\cot \theta|^{2q} \right)^{-1/q} = \sin^{-2} \theta.$$  

(13)

Now, by Lebesgue's dominated convergence theorem,

$$\lim_{q \to \infty} \int_0^{2\pi} f(\theta; q) \, d\theta = \int_{-\pi/4}^{\pi/4} \cos^{-2} \theta \, d\theta + \cdots + \int_{5\pi/4}^{7\pi/4} \sin^{-2} \theta \, d\theta = 2 + \cdots + 2 = 8. \quad \Box$$

From this convergence of $J(q)$ we recover the condition in [3] for $p = 1$,

$$\|a_{12}\|_{L^1(\mathbb{T}_T)}^{1/2} \|a_{21}\|_{L^1(\mathbb{T}_T)}^{1/2} + \frac{1}{2} \|a_{11} - a_{22}\|_{L^1(\mathbb{T}_T)} \leq 2.$$  

(14)

Additionally, we can check numerically that $J(q) \in [2\pi, 8]$ for $q \in [1, \infty]$.

**Remark 2.1.** In the Hill’s equation

$$\ddot{x} + \alpha(t) x = 0, \quad \alpha(t) > 0 \quad \text{a.e.} \quad t \in \mathbb{R},$$  

(15)

our criterion becomes

$$\|\alpha\|_{L^p(\mathbb{T}_T)} \leq \left( \frac{J(q)}{2^{1+1/q}} \right)^{2} \frac{1}{T^{1+1/q}} := B_T(q).$$  

(16)

The classical stability criterion due to Lyapunov (see [4] or [3]) follows for $q = \infty$. Additionally, if $q = 1$, we recover the condition in Lemma 4.4 of [2]. In [5], Zhang and Li extended the Lyapunov stability criterion using $L^p$ norms, as follows

$$\|\alpha\|_{L^p(\mathbb{T}_T)} < K_T(2q) \text{ if } 1 < p \leq \infty, \quad \text{and} \quad \|\alpha\|_{L^1(\mathbb{T}_T)} < K_T(\infty) = \frac{4}{T}, \text{ if } p = 1.$$  

(17)
Here, $K_T(q)$ is the Sobolev constant defined as the optimal constant for the following inequality

$$K_T(q)\|u\|_{L^q(T)}^2 \leq \|\dot{u}\|_{L^2(T)}^2,$$

where $u$ is any function in the Sobolev space $H^1_0[0,T]$. This Sobolev constant is given by

$$K_T(q) = \begin{cases} \frac{2\pi}{q} \left( \frac{2}{2+q} \right)^{1-2/q} \left( \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{q}{2+q}\right)} \right)^2 \frac{1}{T^{1+2/q}}, & \text{if } 1 \leq q < \infty, \\ \frac{4}{T}, & \text{if } q = \infty. \end{cases}$$

(18)

where $\Gamma$ is the usual Gamma function (see [6] for the proof). The upper bounds in (17) are best possible in the sense that for any $\varepsilon > 0$, there is some $\alpha$ such that

$$\|\alpha\|_{L^p(T)} < K_T(2q) + \varepsilon,$$

while (15) is unstable. In Fig. 1 we compare numerically $B(q) := T^{1+2/q}B_T(q)$ with $K(q) := T^{1+2/q}K_T(q)$. It seems that our bound is not the best possible for the Hill's equation since $B(q) < K(2q)$ for $1 < q < \infty$. In the cases $q = 1$ and $q = \infty$ both are equal.

We are willing to get our stability result as in [3], that is, using a homotopy argument. We consider the family of continuous matrices $\{A_\lambda\}$ as

$$A_\lambda(t) = (1 - \lambda)A(t) + \lambda \overline{A}$$

where $A(t)$ is the coefficient matrix of the system (5) whose elements satisfy (6), the elements of the matrix $\overline{A}$ are the average of $a_{ij}$ and the matrix $A_\lambda(t)$ is the coefficient matrix of the associated system in the plane

$$\dot{x} = A_\lambda(t)x. \quad (19)$$

We have that $\forall \lambda \in [0,1],

$$\varpi_{11}(\cdot,\lambda) \geq 0, \varpi_{22}(\cdot,\lambda) \geq 0 \quad \text{and} \quad a_{12}(t,\lambda) \geq \delta, a_{21}(t,\lambda) \geq \delta \quad \text{a.e. } t \in \mathbb{R}. \quad (20)$$

The continuous matrices $A_0, A_1$ will be called homotopic if the family of systems (19) is continuous on $\lambda \in [0,1]$ and has no $2T$-periodic solutions excepting $x \equiv 0$. The continuity of $\{A_\lambda\}$ means that for each element $a_{ij}(t,\lambda)$ of $A_\lambda$ and $\lambda \in [0,1]$, $\lim_{h \to 0} \|a_{ij}(t,\lambda + h) - a_{ij}(t,\lambda)\|_{L^1(T)} = 0$. It is important to note that requirement of non-existence of $2T$-periodic solutions is crucial to guarantee the stability properties of the family of systems (19) along the whole homotopy.

The following lemma establishes that $A(t)$ and $\overline{A}$ can be connected by the homotopy $A_\lambda(t)$.
Lemma 2.1. Assume (7), then $A_0 = A(t)$ and $A_1 = \overline{A}$ are homotopic.

Proof. The family of systems (19) satisfies the criterion (7) of the previous section. The case $\lambda = 0$ is immediate. The case $\lambda = 1$, easily follows when $p = \infty$ and for $p \in [1, \infty]$ is a consequence of the Hölder inequality as

$$\|\sigma_{ij}\|_{L^p(TT)} \leq T^{1/p-1}\|a_{ij}\|_{L^1(TT)} \leq T^{1/p-1}T^{1/q}\|a_{ij}\|_{L^p(TT)} = \|a_{ij}\|_{L^p(TT)}.$$ 

The remaining details of the proof are easy to check. □

Finally, following the lines of the proof of Lemma 2.1 in [3] we obtain our main theorem, taking into account that the system of constant coefficients $\dot{x} = \overline{A}x$ is stable as we assume (6). The key idea of this proof is the relation between the eigenvalues of the monodromy matrix $X(T)$ associated to the periodic system (5) and the non-existence of $2T$-periodic solutions. It can be shown that the system (5) is asymptotically stable if and only if the trace of $X(T)$ satisfies

$$|\text{tr} X(T)| < 1 + e^{-(\overline{\pi}_{11}+\overline{\pi}_{22})T}.$$ 

If $|\text{tr} X(T)| = 1 + e^{-(\overline{\pi}_{11}+\overline{\pi}_{22})T}$, the system (5) has a non-trivial $2T$-periodic solution since in that case there is a real characteristic multiplier with absolute value equal to one. The homotopy between $A_0 = A(t)$ and $A_1 = \overline{A}$ applied to $\Delta(\lambda) := \text{tr} X(T, \lambda)$, $\lambda \in [0, 1]$ concludes the proof. The example 1.2 in [7] also helps to understand this relation between the stability properties of a periodic system and the trace of the associated monodromy matrix.

Theorem 2.1. Assume (7), then the system (5) is stable. Moreover, it is asymptotically stable if $\overline{\pi}_{11}+\overline{\pi}_{22} > 0$.

2.1. An example

In the following example we ask if there are coefficients $a_{ij}(t)$ such that for some $p \in [1, \infty]$ and some $T > 0$ condition (7) is satisfied but the conditions (3) or (4) are not fulfilled.

Example 2.1. Let us consider $a_{12}(t) = 1+\delta+\sin\left(\frac{2\pi t}{T}\right)$, $a_{21}(t) = 1+\delta+\cos\left(\frac{2\pi t}{T}\right)$ and $p = 2$. Also we assume that $a_{11}(t) = a_{11}\left(\frac{2\pi t}{T}\right)$ and $a_{22}(t) = a_{22}\left(\frac{2\pi t}{T}\right)$ and both are in $L^1(TT)$. Then $D(T) := \frac{1}{2}\|a_{11} - a_{22}\|_{L^1(TT)}$ is a linear function, $D(T) = \alpha(a_{11}, a_{22})T$ with $\alpha \geq 0$. The conditions to fulfill become

$$\begin{cases} f(T) := \left(1 + \delta\right)^2 + \frac{1}{2}1/2 + \alpha(a_{11}, a_{22})T - \frac{\mathcal{J}(2)}{2^{3/2}} \leq 0, \quad \left(\frac{\mathcal{J}(2)}{2^{3/2}} \approx 2, 622\right) \\ g(T) := (1 + \delta)T + \alpha(a_{11}, a_{22})T - 2 > 0, \\ h(T) := (2 + \delta)T + \alpha(a_{11}, a_{22})T - \pi > 0. \end{cases}$$

(21)

Let us observe that the slope $K$ of these linear functions is always positive and $K_h > K_f > K_g$. This guarantees that the functions intersect each other. Analyzing the ordering of the zeros and the cross-points of the previous functions with respect to the parameters $\delta$ and $\alpha$, we can prove that there exists non-empty sets of possible values of $T$ where the three previous conditions hold simultaneously. The zeros are given by the equations:

$$T_f = \frac{\mathcal{J}(2)}{2^{3/2}\left(1 + \delta\right)^2 + \frac{1}{2}} + \alpha, \quad T_g = \frac{2}{1 + \delta + \alpha}, \quad T_h = \frac{\pi}{2 + \delta + \alpha},$$

and we want that $T_f > T_g$ and $T_f > T_h$, simultaneously. We can check that
(i) Since $\delta > 0$ and $\alpha \geq 0$, there are no restrictions to satisfy $T_J > T_g$.

(ii) $T_J > T_h$ holds if $\psi(\delta) := \frac{J(2)}{2\sqrt{2}} (2 + \delta - \pi \left[ (1 + \delta)^2 + \frac{2}{3} \right])^{1/2} > \left( \pi - \frac{J(2)}{2\sqrt{2}} \right) \alpha \geq 0$. It is easy to find a set of the form $D = \{ (\alpha, \delta) : \alpha \in [0, \alpha^*], \delta \in [\delta_1(\alpha), \delta_2(\alpha)] \} \subset [0, \delta_+ [ ]$, in which this inequality holds.

Here, $\alpha^* = \frac{\psi(\delta_{\max})}{\pi - J(2) / 2\sqrt{2}} \approx 2.692$, $\delta_{\max} \approx 0.07145$ and $\delta_+ \approx 3.729$ is the positive root of $\psi(\delta)$.

(iii) On the other hand, $T_h > T_g \Rightarrow \alpha > \frac{4 - \pi}{\pi - 2} \approx 0.752$ and $T_g > T_h \Rightarrow \alpha < \frac{4 - \pi}{\pi - 2}, \delta \in [0, \delta_3(\alpha)] \subset [0, \frac{4 - \pi}{\pi - 2} [ ]$.

Consequently, it exists non-empty intervals of possible periods where the conditions (21) are satisfied simultaneously:

$$T \in ]T_h, T_J[ \text{ if } \alpha \in \left[ \frac{4 - \pi}{\pi - 2}, \alpha^* \right], \delta \in [\delta_1(\alpha), \delta_2(\alpha)] \subset [0, \delta_+ [ ; \quad T \in ]T_g, T_J[ \text{ if } \alpha \in \left[ 0, \frac{4 - \pi}{\pi - 2} \right], \delta \in [0, \delta_3(\alpha)] \subset [0, \frac{4 - \pi}{\pi - 2} [ .$$

3. Stability result in the predator–prey Lotka–Volterra model

Let us consider once again (1) and assume that $(u(t), v(t))$ is a coexistence state. In [1] the authors give sufficient and necessary conditions for the existence of such solutions of the system (1). The following theorem gives an additional condition in $L^p$ spaces to guarantee the uniqueness and the asymptotic stability of the $T$-periodic solution (coexistence state). Additionally, we give some estimates on the $L^p$ norm of this solution $(u(t), v(t))$ in terms of the coefficients of system (1).

**Theorem 3.1.** Assume that all possible coexistence states satisfy

$$T^{1/p} \sqrt{\| eu \|_{L^p(T_T)}} \| cv \|_{L^p(T_T)} + \frac{T}{2} \| bu - f v \|_{L^1(T_T)} \leq \frac{J(q)}{2^{q-1} q^q},$$

(22)

where $p$ and $q$ are conjugate indices and $p, q \in [1, \infty]$. Then the $T$-periodic solution $(u(t), v(t))$ is unique and asymptotically stable. In addition, for $p \in [1, \infty]$ we have the following upper bounds of this solution:

$$\| u \|_{L^p(T_T)} < \frac{\| a \|_{L^p(T_T)}}{b_L}, \quad \| v \|_{L^p(T_T)} < \frac{\| d \|_{L^p(T_T)} + e_M \| a \|_{L^p(T_T)}}{f_L b_L}.$$  

(23)

where $\varphi_L := \min_{t \in [0, T]} \varphi(t)$ and $\varphi_M := \max_{t \in [0, T]} \varphi(t)$ for a $T$-periodic function, $\varphi(t)$.

**Proof.** The result of existence and uniqueness follows as [3] using the theorem of the previous section.

Concerning the estimates, let us multiply the first equation of system (1) by $v^{p-2}$ and the second one by $u^{p-2}$. After integrating both equations over a period $T$ and using the $T$-periodicity of $(u(t), v(t))$ together with the following inequality for $f, g \in L^p(T_T), p \in [1, \infty[, \| f^{p-1} g \|_{L^1(T_T)} \leq \| f \|_{L^p(T_T)}^{p-1} \| g \|_{L^p(T_T)}$, we get:

$$b_L \| u \|_{L^p(T_T)}^{p} + \| e u^{p-1} v \|_{L^1(T_T)} \leq \| a \|_{L^p(T_T)} \| u \|_{L^p(T_T)}^{p-1} \Rightarrow \| u \|_{L^p(T_T)} < \frac{\| a \|_{L^p(T_T)}}{b_L},$$

for the first equation and

$$f_L \| v \|_{L^p(T_T)}^{p} \leq \| (d + e u) v^{p-1} \|_{L^1(T_T)} \leq \| (d + e u) \|_{L^p(T_T)} \| v \|_{L^p(T_T)}^{p-1} \Rightarrow \| v \|_{L^p(T_T)} < \frac{\| d \|_{L^p(T_T)} + e_M \| a \|_{L^p(T_T)}}{f_L b_L},$$

for the second one. \[ \square \]

**Remark 3.1.** By the previous proof, we conclude that the previous theorem holds if (22) is verified for all $u$ and $v$ such that

$$\| u \|_{L^p(T_T)} < \frac{\| a \|_{L^p(T_T)}}{b_L}, \quad \| v \|_{L^p(T_T)} < \frac{\| d \|_{L^p(T_T)} + e_M \| a \|_{L^p(T_T)}}{f_L b_L}. $$

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Data availability

No data was used for the research described in the article.

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References