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A note on stability criteria in the periodic Lotka–Volterra predator-prey model

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ABSTRACT

We present a stability result for T-periodic solutions of the periodic predator–prey Lotka–Volterra model. In 2021, R. Ortega gave a stability criteria in terms of the L^1 norm of the coefficients of a planar linear system associated to the model. Previously, in 1994, Z. Amine and R. Ortega proved another stability criteria formulated in terms of the L^∞ norm. The present work gives a L^p criterion, building a bridge between the two previous results.

1. Introduction

In this work we consider the periodic predator–prey Lotka–Volterra model:

$$\begin{cases} \dot{u} = u(a(t) - b(t)u - c(t)v), \\ \dot{v} = v(d(t) + e(t)u - f(t)v) \end{cases} \tag{1}$$

with $u \geq 0$, $v \geq 0$. All the coefficients are T-periodic, $a, d \in L^p(\mathbb{T}_T)$, $p \in [1, \infty]$, and b, c, e and f are positive functions in $C(\mathbb{T}_T)$, where we denote the quotient set $\mathbb{R}/T\mathbb{Z}$ as $\mathbb{T}_T$. This model is a classical non-autonomous model for predator–prey interaction studied by many authors (see [1] and the references therein). In [1] the authors study the existence of coexistence states and in particular prove that if one among the trivial and semi-trivial states is linear stable then it attracts all the solutions with positive initial conditions. As

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an immediate consequence, for the existence of a coexistence state it is necessary that the trivial and the possible semi-trivial states are linearly unstable. In the same paper the authors prove that this is also a sufficient condition.

Assuming the existence of a coexistence state, to know if it is stable or not is an important problem. In [2] and in [3] the authors addressed this question and gave conditions for the existence of one stable coexistence state.

The stability of the coexistence state was obtained using a homotopy from a non-autonomous linear system to an autonomous one. In both results plays an important role to prove the non-existence of $2T$-periodic solutions for a linear system of the form

$$\begin{cases} \dot{x}_1 = -a_{11}(t)x_1 - a_{12}(t)x_2, \\ \dot{x}_2 = a_{21}(t)x_1 - a_{22}(t)x_2, \end{cases} \tag{2}$$

where the a_{ij} are non negative. In order to guarantee this nonexistence, in [2] a condition which implies

$$T\|a_{12}\|^{1/2}_{L^\infty(\mathbb{T}_T)}\ \|a_{21}\|^{1/2}_{L^\infty(\mathbb{T}_T)} + \frac{1}{2}\|a_{11} - a_{22}\|_{L^1(\mathbb{T}_T)} \leq \pi, \tag{3}$$

is given while in [3] the analogous expression

$$\|a_{12}\|^{1/2}_{L^1(\mathbb{T}_T)}\ \|a_{21}\|^{1/2}_{L^1(\mathbb{T}_T)} + \frac{1}{2}\|a_{11} - a_{22}\|_{L^1(\mathbb{T}_T)} \leq 2, \tag{4}$$

but concerning the $L^1(0,T)$ norms, was obtained.

The main purpose of our paper is to extend these results allowing to use other L^p norms, see Proposition 2.1 below. With this result we connect the results in [2,3]. We also give an example of a case in which the results in [2] and in [3] do not apply but ours does with $p = 2$. Finally, as in the previous papers, we give conditions for the local asymptotic stability of a coexistence state of (1).

2. Planar linear system: L^p stability result

Our aim is to give a L^p stability condition for the system

$$\begin{cases} \dot{x}_1 = -a_{11}(t)x_1 - a_{12}(t)x_2, \\ \dot{x}_2 = a_{21}(t)x_1 - a_{22}(t)x_2, \end{cases} \tag{5}$$

where the coefficients a_{ij} belong to $L^p(\mathbb{T}_T)$ with $p \in [1,\infty]$ and satisfy

$$\overline{a}_{11} \geq 0,\ \overline{a}_{22} \geq 0 \quad \text{and} \quad a_{12}(t) \geq \delta,\ a_{21}(t) \geq \delta \quad \text{a.e} \quad t \in \mathbb{R}, \tag{6}$$

for some $\delta > 0$ where $\overline{a}_{ij} = \frac{1}{T}\int_0^T a_{ij}(t)\mathrm{d}t$.

In order to do that we are going to give conditions which guarantee the nonexistence of $2T$-periodic solutions for this linear system in the next Proposition.

Proposition 2.1. *The system (5) has no $2T$-periodic solutions except $x \equiv 0$ if the periodic coefficients a_{ij} satisfy (with $1/p + 1/q = 1$):*

$$T^{1/q}\|a_{12}\|^{1/2}_{L^p(\mathbb{T}_T)}\ \|a_{21}\|^{1/2}_{L^p(\mathbb{T}_T)} + \frac{1}{2}\|a_{11} - a_{22}\|_{L^1(\mathbb{T}_T)} \leq \frac{\mathcal{J}(q)}{2^{2-1/q}}, \tag{7}$$

where

$$\mathcal{J}(q) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{\left(|\cos\theta|^{2q} + |\sin\theta|^{2q}\right)^{1/q}}, \quad q \in [1,\infty[\quad \textit{and} \quad \mathcal{J}(\infty) := \lim_{q\to+\infty} \mathcal{J}(q).$$

Proof. We make a change of variables to the elliptic-polar coordinates with a weight $\mu > 0$ that will be determined later:

$$x_1 = \sqrt{\mu} r \cos\theta, \quad x_2 = \frac{1}{\sqrt{\mu}} r \sin\theta.$$

Then the equation of motion for the variable θ is,

$$\dot\theta = \mu a_{21}(t)\cos^2\theta + \frac{1}{\mu}a_{12}(t)\sin^2\theta + (a_{11}(t) - a_{22}(t))\cos\theta\,\sin\theta. \tag{8}$$

From (8), we can write

$$\dot\theta \le \left\langle \left(\mu a_{21}(t), \frac{1}{\mu}a_{12}(t)\right), (\cos^2\theta, \sin^2\theta)\right\rangle + |a_{11}(t) - a_{22}(t)| \left\langle \left(\frac{1}{2}, \frac{1}{2}\right), (\cos^2\theta, \sin^2\theta)\right\rangle. \tag{9}$$

First, let us consider $p \in]1, \infty[$. By the Hölder inequality in $\mathbb{R}^2$, we have

$$\dot\theta \le \left(\left((\mu a_{21}(t))^p + \left(\frac{1}{\mu}a_{12}(t)\right)^p\right)^{1/p} + |a_{11}(t) - a_{22}(t)|\left(\frac{2}{2^p}\right)^{1/p}\right)\left(|\cos\theta|^{2q} + |\sin\theta|^{2q}\right)^{1/q}. \tag{10}$$

Let us integrate on an interval $I \in \mathbb{R}$ where a solution of (5) is well defined,

$$\int_{\theta(I)} \frac{\mathrm{d}\theta}{\left(|\cos\theta|^{2q} + |\sin\theta|^{2q}\right)^{1/q}} \le \int_I \left((\mu a_{21}(t))^p + \left(\frac{1}{\mu}a_{12}(t)\right)^p\right)^{1/p} \mathrm{d}t \,+\, 2^{\frac{1-p}{p}} \int_I |a_{11}(t) - a_{22}(t)|\,\mathrm{d}t.$$

Now, by the Hölder inequality in L-spaces norms , we have

$$\int_{\theta(I)} \frac{\mathrm{d}\theta}{\left(|\cos\theta|^{2q} + |\sin\theta|^{2q}\right)^{1/q}} \le |I|^{1/q}\left(\int_I (\mu a_{21}(t))^p\,\mathrm{d}t + \int_I \left(\frac{1}{\mu}a_{12}(t)\right)^p \mathrm{d}t\right)^{1/p} \\ +\, 2^{\frac{1-p}{p}} \int_I |a_{11}(t) - a_{22}(t)|\,\mathrm{d}t.$$

Let us assume that exists a non-trivial $2T$-periodic solution of (5) $(x_1(t), x_2(t))$. We claim that every non-trivial solution of (8) crosses the axes in a counter-clockwise sense. Intuitively, if the coefficients $a_{12}(t)$ and $a_{21}(t)$ are continuous and we consider small neighborhoods of the axes, the angular evolution is always positive since the sign in the right hand side of Eq. (8) is positive. In [3] the author gives a proof for coefficients in $L^1(\mathbb{T}_T)$. Therefore we have in the angular variable $\theta(t + 2T) = \theta(t) + 2\pi k$, being k a non-negative integer. Take $k = 0$ and let us consider that the solution $(x_1(t), x_2(t))$ cross an axis. Due to the periodicity it cannot cross the axis in a clockwise sense to came back. We conclude that the solution must lie in an open quadrant, and we have two possibilities, either $x_1(t) \cdot x_2(t) > 0$ or $x_1(t) \cdot x_2(t) < 0$ for $t \in \mathbb{R}$. In the first case we divide the first equation in (5) by x_1 and integrate over a $2T$-period to get $\overline{a}_{11} < 0$, in contradiction with (6). The second case has an analog treatment considering the second equation. See [3] for more details.

For $k \ge 1$, integrating from 0 to $2T$,

$$k\mathcal{J}(q) < (2T)^{1/q}\left(\int_0^{2T} (\mu a_{21}(t))^p\,\mathrm{d}t + \int_0^{2T}\left(\frac{1}{\mu}a_{12}(t)\right)^p \mathrm{d}t\right)^{1/p} + 2^{\frac{1-p}{p}}\int_0^{2T} |a_{11}(t) - a_{22}(t)|\,\mathrm{d}t.$$

Additionally, we have dropped the equal sign since the equality in (9) only occurs when $\theta = \pi/4 \pm \pi k$. Now, defining $\mu^p := \left(\int_0^T a_{12}(t)^p \mathrm{d}t / \int_0^T a_{21}(t)^p \mathrm{d}t\right)^{1/2}$, we arrive to a contradiction with the hypothesis (7) in the more restrictive case associated with $k = 1$.

Concerning the limiting cases, we can found a proof for the case $p = 1$ in [3] and for $p = \infty$ in [2]. □

In the next proposition we give a result which, together to the fact that $\mathcal{J}(1) = 2\pi$, allows us to conclude that (7) connects the results in [2,3].

Proposition 2.2. *We have that* $\lim_{q\to+\infty} \mathcal{J}(q) = 8$.

Proof. Let us study the limit of the integrand of $\mathcal{J}(q)$:

$$f(\theta; q) = \left(|\cos\theta|^{2q} + |\sin\theta|^{2q}\right)^{-1/q}. \tag{11}$$

It is useful to write the function in two different ways

$$f(\theta; q) = \cos^{-2}\theta \left(1 + |\tan\theta|^{2q}\right)^{-1/q} = \sin^{-2}\theta \left(1 + |\cot\theta|^{2q}\right)^{-1/q}.$$

For $\theta \in [-\pi/4, \pi/4] \cup [3\pi/4, 5\pi/4]$ we have that

$$\lim_{q\to\infty} f(\theta; q) = \lim_{q\to\infty} \cos^{-2}\theta \left(1 + |\tan\theta|^{2q}\right)^{-1/q} = \cos^{-2}\theta. \tag{12}$$

Indeed, for $\theta \in [-\pi/4, \pi/4] \cup [3\pi/4, 5\pi/4]$,

$$2^{-1/q} \leq \left(1 + \tan^2\theta\right)^{-1/q} \leq \left(1 + |\tan\theta|^{2q}\right)^{-1/q} \leq 1,$$

where we have used that $\tan^2\theta \leq 1$ and $q \geq 1$ and the result follows. Analogously, for $\theta \in [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]$, we have that

$$\lim_{q\to\infty} f(\theta; q) = \lim_{q\to\infty} \sin^{-2}\theta \left(1 + |\cot\theta|^{2q}\right)^{-1/q} = \sin^{-2}\theta. \tag{13}$$

Now, by Lebesgue's dominated convergence theorem,

$$\lim_{q\to\infty} \int_0^{2\pi} f(\theta; q)\mathrm{d}\theta = \int_{-\pi/4}^{\pi/4} \cos^{-2}\theta\,\mathrm{d}\theta + \cdots + \int_{5\pi/4}^{7\pi/4} \sin^{-2}\theta\,\mathrm{d}\theta = 2 + \cdots + 2 = 8. \quad \square$$

From this convergence of $\mathcal{J}(q)$ we recover the condition in [3] for $p = 1$,

$$\|a_{12}\|^{1/2}_{L^1(\mathbb{T}_T)}\, \|a_{21}\|^{1/2}_{L^1(\mathbb{T}_T)} + \frac{1}{2}\|a_{11} - a_{22}\|_{L^1(\mathbb{T}_T)} \leq 2. \tag{14}$$

Additionally, we can check numerically that $\mathcal{J}(q) \in [2\pi, 8]$ for $q \in [1, \infty]$.

Remark 2.1. In the Hill's equation

$$\ddot{x} + \alpha(t)\,x = 0, \qquad \alpha(t) > 0 \quad \text{a.e} \quad t \in \mathbb{R}, \tag{15}$$

our criterion becomes

$$\|\alpha\|_{L^p(\mathbb{T}_T)} \leq \left(\frac{\mathcal{J}(q)}{2^{2-1/q}}\right)^2 \frac{1}{T^{1+1/q}} := B_T(q). \tag{16}$$

The classical stability criterion due to Lyapunov (see [4] or [3]) follows for $q = \infty$. Additionally, if $q = 1$, we recover the condition in Lemma 4.4 of [2]. In [5], Zhang and Li extended the Lyapunov stability criterion using L^p norms, as follows

$$\|\alpha\|_{L^p(\mathbb{T}_T)} < K_T(2q) \text{ if } 1 < p \leq \infty, \quad \text{and} \quad \|\alpha\|_{L^1(\mathbb{T}_T)} < K_T(\infty) = \frac{4}{T}, \text{ if } p = 1. \tag{17}$$

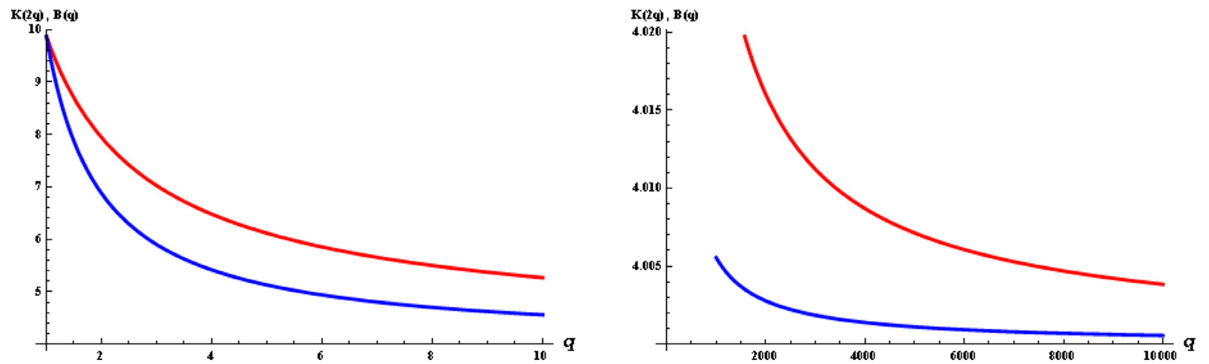

Fig. 1. Comparison of the Sobolev constant $K(2q)$ (in red) with the upper bound $B(q)$ (in blue). On the left, for $q \in [1, 10]$; on the right, asymptotic behavior for large q, note that $K(q)$ and $B(q)$ tend to 4 according to the classical Lyapunov criterion.

Here, $K_T(q)$ is the Sobolev constant defined as the optimal constant for the following inequality

$$K_T(q)\|u\|^2_{L^q(\mathbb{T}_T)} \leq \|\dot{u}\|^2_{L^2(\mathbb{T}_T)},$$

where u is any function in the Sobolev space $H^1_0[0,T]$. This Sobolev constant is given by

$$K_T(q) = \begin{cases} \frac{2\pi}{q}\left(\frac{2}{2+q}\right)^{1-2/q}\left(\frac{\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{q}\right)}\right)^2 \frac{1}{T^{1+2/q}}, & \text{if} \quad 1 \leq q < \infty, \\ \frac{4}{T}, \quad \text{if} \quad q = \infty. \end{cases} \tag{18}$$

Γ is the usual Gamma function (see [6] for the proof). The upper bounds in (17) are best possible in the sense that for any $\varepsilon > 0$, there is some α such that

$$\|\alpha\|_{L^p(\mathbb{T}_T)} < K_T(2q) + \varepsilon,$$

while (15) is unstable. In Fig. 1 we compare numerically $B(q) := T^{1+2/q}B_T(q)$ with $K(q) := T^{1+2/q}K_T(q)$. It seems that our bound is not the best possible for the Hill's equation since $B(q) < K(2q)$ for $1 < q < \infty$. In the cases $q = 1$ and $q = \infty$ both are equal.

We are willing to get our stability result as in [3], that is, using a homotopy argument. We consider the family of continuous matrices $\{A_\lambda\}$ as

$$A_\lambda(t) = (1-\lambda)A(t) + \lambda\overline{A}$$

where $A(t)$ is the coefficient matrix of the system (5) whose elements satisfy (6), the elements of the matrix $\overline{A}$ are the average of a_{ij} and the matrix $A_\lambda(t)$ is the coefficient matrix of the associated system in the plane

$$\dot{x} = A_\lambda(t)x. \tag{19}$$

We have that $\forall\lambda \in [0,1]$,

$$\overline{a}_{11}(\cdot,\lambda) \geq 0,\ \overline{a}_{22}(\cdot,\lambda) \geq 0 \quad \text{and} \quad a_{12}(t,\lambda) \geq \delta,\ a_{21}(t,\lambda) \geq \delta \quad \text{a.e} \quad t \in \mathbb{R}. \tag{20}$$

The continuous matrices A_0, A_1 will be called homotopic if the family of systems (19) is continuous on $\lambda \in [0,1]$ and has no $2T$-periodic solutions excepting $x \equiv 0$. The continuity of $\{A_\lambda\}$ means that for each element $a_{ij}(t,\lambda)$ of A_λ and $\lambda \in [0,1]$, $\lim_{h\to 0}\|a_{ij}(t,\lambda+h) - a_{ij}(t,\lambda)\|_{L^1(\mathbb{T}_T)} = 0$. It is important to note that requirement of non-existence of $2T$-periodic solutions is crucial to guarantee the stability properties of the family of systems (19) along the whole homotopy.

The following lemma establishes that $A(t)$ and $\overline{A}$ can be connected by the homotopy $A_\lambda(t)$.

Lemma 2.1. *Assume (7), then $A_0 = A(t)$ and $A_1 = \overline{A}$ are homotopic.*

Proof. The family of systems (19) satisfies the criterion (7) of the previous section. The case $\lambda = 0$ is immediate. The case $\lambda = 1$, easily follows when $p = \infty$ and for $p \in [1, \infty[$ is a consequence of the Hölder inequality as

$$\|\overline{a}_{ij}\|_{L^p(\mathbb{T}_T)} \le T^{1/p-1}\|a_{ij}\|_{L^1(\mathbb{T}_T)} \le T^{1/p-1}T^{1/q}\|a_{ij}\|_{L^p(\mathbb{T}_T)} = \|a_{ij}\|_{L^p(\mathbb{T}_T)}.$$

The remaining details of the proof are easy to check. □

Finally, following the lines of the proof of Lemma 2.1 in [3] we obtain our main theorem, taking into account that the system of constant coefficients $\dot{x} = \overline{A}\,x$ is stable as we assume (6). The key idea of this proof is the relation between the eigenvalues of the monodromy matrix $X(T)$ associated to the periodic system (5) and the non-existence of $2T$-periodic solutions. It can be shown that the system (5) is asymptotically stable if and only if the trace of $X(T)$ satisfies

$$|\mathrm{tr}\, X(T)| < 1 + e^{-(\overline{a}_{11}+\overline{a}_{22})T}.$$

If $|\mathrm{tr}\, X(T)| = 1 + e^{-(\overline{a}_{11}+\overline{a}_{22})T}$, the system (5) has a non-trivial $2T$-periodic solution since in that case there is a real characteristic multiplier with absolute value equal to one. The homotopy between $A_0 = A(t)$ and $A_1 = \overline{A}$ applied to $\Delta(\lambda) := \mathrm{tr}\, X(T, \lambda)$, $\lambda \in [0, 1]$ concludes the proof. The example 1.2 in [7] also helps to understand this relation between the stability properties of a periodic system and the trace of the associated monodromy matrix.

Theorem 2.1. *Assume (7), then the system (5) is stable. Moreover, it is asymptotically stable if $\overline{a}_{11}+\overline{a}_{22} > 0$.*

2.1. An example

In the following example we ask if there are coefficients $a_{ij}(t)$ such that for some $p \in]1, \infty[$ and some $T > 0$ condition (7) is satisfied but the conditions (3) or (4) are not fulfilled.

Example 2.1. Let us consider $a_{12}(t) = 1+\delta+\sin\left(\frac{2\pi t}{T}\right)$, $a_{21}(t) = 1+\delta+\cos\left(\frac{2\pi t}{T}\right)$ and $p = 2$. Also we assume that $a_{11}(t) = a_{11}(\frac{2\pi t}{T})$ and $a_{22}(t) = a_{22}(\frac{2\pi t}{T})$ and both are in $L^1(\mathbb{T}_T)$. Then $D(T) := \frac{1}{2}\|a_{11} - a_{22}\|_{L^1(\mathbb{T}_T)}$ is a linear function, $D(T) = \alpha(a_{11}, a_{22})T$ with $\alpha \ge 0$. The conditions to fulfill become

$$\begin{cases} f(T) := \left[(1+\delta)^2 + \frac{1}{2}\right]^{1/2} T + \alpha(a_{11}, a_{22})\, T - \frac{\mathcal{J}(2)}{2^{3/2}} \le 0, & \left(\frac{\mathcal{J}(2)}{2^{3/2}} \approx 2,622\right) \\ g(T) := (1+\delta)\, T + \alpha(a_{11}, a_{22})\, T - 2 > 0, \\ h(T) := (2+\delta)\, T + \alpha(a_{11}, a_{22})\, T - \pi > 0. \end{cases} \tag{21}$$

Let us observe that the slope K of these linear functions is always positive and $K_h > K_f > K_g$. This guarantees that the functions intersect each other. Analyzing the ordering of the zeros and the cross-points of the previous functions with respect to the parameters δ and α, we can prove that there exists non-empty sets of possible values of T where the three previous conditions hold simultaneously. The zeros are given by the equations:

$$T_f = \frac{\frac{\mathcal{J}(2)}{2^{3/2}}}{\left[(1+\delta)^2 + \frac{1}{2}\right]^{1/2} + \alpha}, \quad T_g = \frac{2}{1+\delta+\alpha}, \quad T_h = \frac{\pi}{2+\delta+\alpha},$$

and we want that $T_f > T_g$ and $T_f > T_h$, simultaneously. We can check that

(i) Since $\delta > 0$ and $\alpha \geq 0$, there are no restrictions to satisfy $T_f > T_g$.
(ii) $T_f > T_h$ holds if $\psi(\delta) := \frac{\mathcal{J}(2)}{2^{3/2}}(2+\delta) - \pi\left[(1+\delta)^2 + \frac{1}{2}\right]^{1/2} > \left(\pi - \frac{\mathcal{J}(2)}{2^{3/2}}\right)\alpha \geq 0$. It is easy to find a set of the form $\mathcal{D} = \{(\alpha,\delta) : \alpha \in [0,\alpha^*[, \delta \in]\delta_1(\alpha), \delta_2(\alpha)[\subset]0, \delta_+[\}$, in which this inequality holds.
Here, $\alpha^* = \frac{\psi(\delta_{max})}{\pi - \frac{\mathcal{J}(2)}{2^{3/2}}} \approx 2.692$, $\delta_{max} \approx 0,07145$ and $\delta_+ \approx 3,729$ is the positive root of $\psi(\delta)$.
(iii) On the other hand, $T_h > T_g \Rightarrow \alpha > \frac{4-\pi}{\pi-2} \approx 0.752$ and $T_g > T_h \Rightarrow \alpha < \frac{4-\pi}{\pi-2}$, $\delta \in]0, \delta_3(\alpha)[\subset]0, \frac{4-\pi}{\pi-2}[$.

Consequently, it exists non-empty intervals of possible periods where the conditions (21) are satisfied simultaneously:

$$T \in]T_h, T_f[\text{ if } \alpha \in]\frac{4-\pi}{\pi-2}, \alpha^*[, \delta \in]\delta_1(\alpha), \delta_2(\alpha)[\subset]0, \delta_+[; \qquad T \in]T_g, T_f[\text{ if } \alpha \in [0, \frac{4-\pi}{\pi-2}[, \delta \in]0, \delta_3(\alpha)[\subset]0, \frac{4-\pi}{\pi-2}[.$$

3. Stability result in the predator–prey Lotka–Volterra model

Let us consider once again (1) and assume that $(u(t), v(t))$ is a coexistence state. In [1] the authors give sufficient and necessary conditions for the existence of such solutions of the system (1). The following theorem gives an additional condition in L^p spaces to guarantee the uniqueness and the asymptotic stability of the T-periodic solution (coexistence state). Additionally, we give some estimates on the L^p norm of this solution $(u(t), v(t))$ in terms of the coefficients of system (1).

Theorem 3.1. *Assume that all possible coexistence states satisfy*

$$T^{1/q}\sqrt{\|eu\|_{L^p(\mathbb{T}_T)}\|cv\|_{L^p(\mathbb{T}_T)}} + \frac{T}{2}\|bu - fv\|_{L^1(\mathbb{T}_T)} \leq \frac{\mathcal{J}(q)}{2^{2-1/q}}, \tag{22}$$

where p and q are conjugate indices and $p, q \in [1, \infty]$. Then the T-periodic solution $(u(t), v(t))$ is unique and asymptotically stable. In addition, for $p \in [1, \infty[$ we have the following upper bounds of this solution:

$$\|u\|_{L^p(\mathbb{T}_T)} < \frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L}, \quad \|v\|_{L^p(\mathbb{T}_T)} < \frac{\|d\|_{L^p(\mathbb{T}_T)} + e_M\frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L}}{f_L}. \tag{23}$$

where $\varphi_L := \min_{t\in[0,T]}\varphi(t)$ and $\varphi_M := \max_{t\in[0,T]}\varphi(t)$ for a T-periodic function, $\varphi(t)$.

Proof. The result of existence and uniqueness follows as [3] using the theorem of the previous section.

Concerning the estimates, let us multiply the first equation of system (1) by u^{p-2} and the second one by v^{p-2}. After integrating both equations over a period T and using the T-periodicity of $(u(t), v(t))$ together with the following inequality for $f, g \in L^p(\mathbb{T}_T)$, $p \in [1, \infty[$, $\|f^{p-1}g\|_{L^1(\mathbb{T}_T)} \leq \|f\|^{p-1}_{L^p(\mathbb{T}_T)}\|g\|_{L^p(\mathbb{T}_T)}$, we get:

$$b_L\,\|u\|^p_{L^p(\mathbb{T}_T)} + \|c\,u^{p-1}v\|_{L^1(\mathbb{T}_T)} \leq \|a\|_{L^p(\mathbb{T}_T)}\|u\|^{p-1}_{L^p(\mathbb{T}_T)} \Longrightarrow \|u\|_{L^p(\mathbb{T}_T)} < \frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L},$$

for the first equation and

$$f_L\,\|v\|^p_{L^p(\mathbb{T}_T)} \leq \|(d + e\,u)\,v^{p-1}\|_{L^1(\mathbb{T}_T)} \leq \|(d+e\,u)\|_{L^p(\mathbb{T}_T)}\,\|v\|^{p-1}_{L^p(\mathbb{T}_T)} \Longrightarrow \|v\|_{L^p(\mathbb{T}_T)} < \frac{\|d\|_{L^p(\mathbb{T}_T)} + e_M\frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L}}{f_L}$$

for the second one. □

Remark 3.1. By the previous proof, we conclude that the previous theorem holds if (22) is verified for all u and v such that

$$\|u\|_{L^p(\mathbb{T}_T)} < \frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L}, \quad \|v\|_{L^p(\mathbb{T}_T)} < \frac{\|d\|_{L^p(\mathbb{T}_T)} + e_M\frac{\|a\|_{L^p(\mathbb{T}_T)}}{b_L}}{f_L}.$$

Data availability

No data was used for the research described in the article.

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