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A non-existence result for periodic solutions of the relativistic pendulum with friction

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ABSTRACT

It is proved that if the damped periodically forced Newtonian pendulum does not have periodic solutions, the same happens for the relativistic version of the problem for high values of the parameter c representing the speed of light in the vacuum.

1. Introduction and main result

In this paper, we present for the first time in the literature a non-existence result for periodic solutions for the forced pendulum equation with relativistic acceleration

$$\left(\frac{x'}{\sqrt{1-\frac{x'^2}{c^2}}}\right)' + kx' + a\sin x = p(t) \tag{1.1}$$

where $c > 0$ is the speed of light in the vacuum, $k \geq 0$ is a possible viscous friction coefficient, $a > 0$ and p is a continuous and T-periodic forcing term with mean value $\overline{p} = \frac{1}{T}\int_0^T p(t)dt = 0$.

The classical forced pendulum equation

$$x'' + kx' + a\sin x = p(t), \tag{1.2}$$

has been a fundamental source of inspiration for researchers working on Dynamical Systems for many years [1,2]. Concerning the existence of periodic solutions, the history comes back one century ago when

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Hamel [3] in 1922 proved that there exists at least one T-periodic solution when $k = 0$ and $p(t) = b \sin t$. The proof is of variational nature. The general proof for any forcing term with zero mean value was given independently by Dancer [4] and Willem [5]. A conjecture by J. Mawhin [6], asking if a topological approach may be useful to prove the existence of periodic solutions in the presence of friction, generated considerable interest in the community. By the time, many specialists expected a positive answer to Mawhin's conjecture for a good reason: by Massera's theorem, if there are no periodic solutions then all the solutions must be unbounded, but then the addition of a friction term would turn a periodic motion into unbounded, which seems counter-intuitive. In spite of this consideration, Ortega [7] devised a remarkable example of Eq. (1.2) without periodic solutions. A second example was constructed in [8] by using a different idea. Finally, [9] provided the most general result for non-existence in the Newtonian case. From now on, let us denote by C_T the Banach space of the continuous and T-periodic functions and by $\tilde{C}_T$ the space of the functions of C_T with zero mean value.

Theorem 1 *([9]). Given positive constants a, k and T, there exists $p \in \tilde{C}_T$ such that Eq. (1.2) has no T-periodic solutions.*

In the relativistic case, this result is not true: it was proved in [10] that if $2cT \leq 1$, Eq. (1.2) has at least one T -periodic solution for any values a, k and for any $p \in \tilde{C}_T$. Other sufficient conditions for existence can be found in [11–16], but non-existence results are not available in the literature up to the date. Our main result partially fills this gap.

Theorem 2. *Let $p \in \tilde{C}_T$ be such that (1.2) has no T-periodic solutions. Then, there exists $c_* > 0$ such that (1.1) (with the same choice of k, a, p) has no T-periodic solutions for any $c > c_*$.*

The proof relies on a priori bounds of solutions not depending on c and a pass to the limit via Ascolí–Arzela Theorem.

2. Proof of the main result

Let $\|.\|_\infty$ be the usual norm of the supremum. Any periodic solution of (1.1) has a natural bound $\|x'\|_\infty < c$. A key point of the proof is a different priori bound for the derivative given in the next lemma.

Lemma 1. *Any T-periodic solution of (1.1) satisfies the bound*

$$\|x'\|_\infty < \|p\|_2 \sqrt{T} + aT. \tag{2.3}$$

Proof. Suppose that $x(t)$ is a given T-periodic solution. Eq. (1.1) can be written as

$$\frac{x''}{\left(1 - \frac{x'^2}{c^2}\right)^{3/2}} + kx' + a \sin x = p(t). \tag{2.4}$$

Multiplying by x'' and integrating, a basic application of Cauchy–Schwarz inequality gives

$$\|x''\|_2^2 < \int_0^T \frac{(x'')^2}{\left(1 - \frac{x'^2}{c^2}\right)^{2/3}} dt = \int_0^T (p(t) - a \sin x) x'' dt \leq (\|p\|_2 + a\sqrt{T}) \|x''\|_2 \,,$$

from where

$$\|x''\|_2 < \|p\|_2 + a\sqrt{T}.$$

Now, taking $t_0 \in [0,T)$ such that $x'(t_0) = 0$,

$$|x'(t)| = \left| \int_{t_0}^{t} x''(s)ds \right| \le \|x''\|_1 \le \sqrt{T}\,\|x''\|_2 < \|p\|_2\,\sqrt{T} + aT$$

for any $t \in [0,T]$. □

Let us prove now the main result. By a contrapositive argument, let us assume that there exists a sequence $c_n \to +\infty$ and corresponding T-periodic solutions $x_n(t)$ of (1.1) with $c = c_n$. By the 2π-periodic character of the nonlinearity, it is not restrictive to assume that $x_n(0) \in [-\pi, \pi]$. Then, we can derive the uniform bound

$$|x_n(t) - x_n(0)| = \left| \int_0^t x_n'(s)ds \right| \le T\,\|x'\|_\infty \le \|p\|_2\,T\sqrt{T} + aT^2.$$

Hence, the sequence of $x_n(t)$ and its derivatives are uniformly bounded, so Ascoli-Arzela Theorem implies that a subsequence of $x_n(t)$ (not necessarily $x_n(t)$ itself, but we keep the notation for convenience) is uniformly convergent in C_T. Besides, the sequence of x_n'' are also uniformly bounded because the nonlinearity and the forcing term are bounded, therefore x_n is uniformly convergent to a certain $x_\infty(t)$ in C_T^1, the space of T-periodic functions with continuous derivatives. Note that all the derived bounds are independent of c_n.

The final step is to write the equation as an integral equation and a pass to the limit. Starting from (2.4), we write

$$x'' - x = \left(1 - \frac{x'^2}{c^2}\right)^{3/2} (p(t) - kx' - a\sin x) - x, \tag{2.5}$$

then to find a T-periodic solution of (1.1) is equivalent that to find a T-periodic solution of the integral equation

$$x = \int_0^T G(t,s) \left[\left(1 - \frac{x'^2(s)}{c^2}\right)^{3/2} (p(s) - kx'(s) - a\sin x(s)) - x(s) \right] ds, \tag{2.6}$$

where $G(t,s)$ is the Green function of the linear operator $x'' - x$ with periodic conditions. Consequently, x_n verifies

$$x_n = \int_0^T G(t,s) \left[\left(1 - \frac{x_n'^2(s)}{c_n^2}\right)^{3/2} (p(s) - kx_n'(s) - a\sin x_n(s)) - x_n(s) \right] ds. \tag{2.7}$$

The Green function $G(t,s)$ has an explicit expression, but for our purposes is it enough to know that it is uniformly bounded on the square $[0,T] \times [0,T]$. Taking limits when $n \to +\infty$, the uniform boundedness of $G(t,s)$ implies that the limit can pass inside the integral and we get

$$x_\infty = \int_0^T G(t,s) \left[(p(s) - kx_\infty'(s) - a\sin x_\infty(s)) - x_\infty(s) \right] ds,$$

or equivalently, $x_\infty(t)$ is a T-periodic solution of the Newtonian Eq. (1.2). This concludes the proof.

Data availability

No data was used for the research described in the article.

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