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Approximation via gradients on the ball. The Zernike case ☆

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ABSTRACT

In this work, we deal in a d dimensional unit ball equipped with an inner product constructed by adding a mass point at zero to the classical ball inner product applied to the gradients of the functions. Apart from determining an explicit orthogonal polynomial basis, we study approximation properties of Fourier expansions in terms of this basis. In particular, we deduce relations between the partial Fourier sums in terms of the new orthogonal polynomials and the partial Fourier sums in terms of the classical ball polynomials. We also give an estimate of the approximation error by polynomials of degree at most n in the corresponding Sobolev space, proving that we can approximate a function by using its gradient. Numerical examples are given to illustrate the approximation behavior of the Sobolev basis.

1. Introduction

The natural field of application of orthogonal polynomials is that of the approximation of functions, which can be found in multiple technological applications. The reconstruction and representation of surfaces is a basic tool of graphical computing, medical imaging and other branches. For example, in ophthalmological practice, the Hartmann–Shack sensor (or wavefront sensor) is used to determine the refractive errors of the human optical system, measuring slopes or normals of the wavefront at different points starting from the displacement of some luminous points in a target. A systematic method of classifying forms of aberration is to express the corresponding function on an appropriate basis. The so-called Zernike polynomials, originally described by Frits Zernike in 1934 [1] to describe the diffraction of the wavefront in the phase contrast image microscope, are recognized as the standard basis of wavefront developments by the Optical Society of America, (OSA). In addition, they are implemented in the standard measuring devices used in optics.

From our point of view, Zernike polynomials are polynomials in two variables which are orthogonal on the unit disk with respect to the Lebesgue measure. They are represented in polar coordinates as a product of a radial polynomial part times a trigonometric function. The even polynomials are multiples of the cosine, and the odd polynomials are multiples

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of the sine. Any sufficiently regular phase function defined on the unit disk can be represented by its Fourier expansion in terms of the Zernike polynomials with certain coefficients. The alteration of these coefficients allows detection of the possible aberrations of the studied optical system. The Zernike polynomials show many applications in the manufacture of precision optical devices, because they allow the characterization of higher order errors observed in the interferometric analysis to achieve the desired performance of the system. They are also used to describe the aberrations of the cornea or lens from an ideal spherical shape in optometry and ophthalmology. Finally, they can be effectively used in adaptive optics to cancel atmospheric distortion, allowing images to be improved in IR or visual astronomy and satellite images.

However, in practice, Zernike polynomials present convergence problems when working on the edges of the disk, producing distortions that could be eliminated by dealing with a modification of the associated inner product ([2,3], among others). This modification could be, for instance, of Sobolev type, that is, including into the original inner product one or several terms on one or several points on the interior or on the boundary of the unit disk, using function values and/or derivative operators, since it is known that measuring devices are also capable of describing local gradients. Sobolev orthogonal polynomials in several variables have already been applied in the analysis of polishing tools in the manufacture of optical surfaces [4]. In the case of applications of orthogonal polynomials to the clinical problems related to human vision, we consider interesting, for example, the study of the efficiency of bivariate Zernike–Sobolev orthogonal polynomials within this context.

In this paper we start dealing in a d dimensional unit ball equipped with a inner product constructed by adding a mass point at zero to the classical ball inner product applied to the gradients of the functions in the form

$$\langle f, g\rangle_{\nabla,\mu} = f(0)g(0) + \lambda \int_{\mathbf{B}^d} \nabla f(x)\cdot\nabla g(x)(1-\|x\|^2)^{\mu}\,dx,$$

for $\lambda > 0$ and $\mu > -1$. While these conditions are necessary for the positive definiteness of the inner product, our contribution involves only the case when $\mu \geqslant 0$. Our paper is divided in three parts. In the first part, we work in the general frame, then we study the particular case when $\mu = 0$, that we have called *the Zernike case*, noting that it cannot be deduced from the general case, and finally, we have designed and implemented numerical experiments to contrast the improvements offered by these new approaches. We show that if we only know the gradients of the functions, we can compute approximants by using the Sobolev Fourier orthogonal expansions, and the approximation is similar or even better than the classical one.

Since the above inner product involves a derivation operator, it is usually called a *Sobolev* inner product. Despite the fact that there are many works about univariate orthogonal polynomials with respect to this kind of inner product, the multivariate case has been considered only in a few classical cases (see, for example, [5,6] and the references therein). In the introduction of [7], Li and Xu explain why and how the approximation based on the Sobolev orthogonal expansions could be better than the classical orthogonal expansions.

The particular inner product involving gradients that we will study in our paper was introduced and studied in the particular case $\mu = 0$ by Xu in [4], where the author finds a complete system of orthonormal polynomials with respect to these inner products and explores their properties.

Our objective in this paper, apart from the extension and completion of the results obtained by Xu in [4] to the more general case $\mu \geqslant 0$, is to study the analytic properties of approximation by means of the corresponding Fourier sums. We remark that, using our results, we can compute the coefficients of the Sobolev–Fourier sums without using derivatives.

The paper is organized as follows. Section 2 presents background on the orthogonal structure and approximation on the unit ball. The Sobolev inner product with a mass point at the origin and associated bases of orthogonal polynomials are discussed in Section 3. Section 4 is devoted to the study of Sobolev Fourier orthogonal expansions and their approximation behavior. Finally, in Section 6, we illustrate our results with numerical examples.

2. Preliminaries

In this section, we collect the basic background on the classical orthogonal polynomials on the unit ball and orthogonal expansions that will be used throughout this paper.

For $x \in \mathbb{R}^d$, we denote by $\|x\|$ the usual Euclidean norm of x. The unit ball and the unit sphere in $\mathbb{R}^d$ are denoted by $\mathbf{B}^d = \left\{x \in \mathbb{R}^d : \|x\| \leqslant 1\right\}$ and $\mathbf{S}^{d-1} = \left\{\xi \in \mathbb{R}^d : \|\xi\| = 1\right\}$, respectively.

For $\mu > -1$, let W_μ be the weight function on the unit ball defined as

$$W_\mu(x) = \left(1-\|x\|^2\right)^{\mu}, \quad x \in \mathbf{B}^d.$$

This weight function can be used to define the inner product

$$\langle f, g\rangle_\mu = b_\mu \int_{\mathbf{B}^d} f(x)g(x)\left(1-\|x\|^2\right)^{\mu} dx, \quad f, g \in L^2(W_\mu; \mathbf{B}^d), \tag{2.1}$$

where b_μ is the normalization constant such that $\langle 1, 1\rangle_\mu = 1$ given by

$$b_\mu = \left(\int_{\mathbf{B}^d} W_\mu(x)\,dx\right)^{-1} = \frac{\Gamma\left(\mu+1+\frac{d}{2}\right)}{\Gamma(\mu+1)\pi^{d/2}}.$$

In turn, this inner product induces the norm $\|\cdot\|_\mu$ defined by

$$\|f\|_\mu = \left(b_\mu \int_{\mathbf{B}^d} f(x)^2\, W_\mu(x) dx\right)^{1/2}, \quad f \in L^2(W_\mu;\mathbf{B}^d). \tag{2.2}$$

For $n \geqslant 0$, let us denote by Π_n^d the linear space of real polynomials in d variables of total degree less than or equal to n, and by $\Pi^d = \bigcup_{n\geqslant 0} \Pi_n^d$ the linear space of all real polynomials in d variables.

A polynomial $P \in \Pi_n^d$ is said to be an orthogonal polynomial of (total) degree n if $\langle P, Q\rangle_\mu = 0$ for all $Q \in \Pi_{n-1}^d$. For $n \geqslant 0$, let $\mathcal{V}_n^d(W_\mu)$ denote the space of orthogonal polynomials of total degree n. Then $\dim \mathcal{V}_n^d(W_\mu) = \binom{n+d-1}{n} := r_n^d$.

For $n \geqslant 0$, let $\{P_\nu^n(x) : 0 \leqslant \nu \leqslant r_n^d\}$ be a basis of $\mathcal{V}_n^d(W_\mu)$. Notice that every element of $\mathcal{V}_n^d(W_\mu)$ is orthogonal to polynomials of lower degree. If the elements of the basis are also orthogonal to each other, that is, $\langle P_\nu^n, P_\eta^n\rangle = 0$ whenever $\nu \neq \eta$, we call the basis *mutually orthogonal*. If, in addition, $\langle P_\nu^n, P_\nu^n\rangle = 1$, we say that the basis is *orthonormal*.

2.1. Spherical harmonics

Let $\mathcal{H}_n^d$ denote the space of harmonic polynomials in d variables of degree n, that is, homogeneous polynomials of degree n satisfying the Laplace equation $\Delta Y = 0$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ is the usual Laplace operator. It is well known that

$$a_n^d := \dim\ \mathcal{H}_n^d = \binom{n+d-1}{n} - \binom{n+d-3}{n}.$$

Spherical harmonics are the restriction of harmonic polynomials to the unit sphere. If $Y \in \mathcal{H}_n^d$, then in spherical-polar coordinates $x = r\xi$ where $r > 0$ and $\xi \in \mathbf{S}^{d-1}$, we get

$$Y(x) = r^n\, Y(\xi),$$

so that Y is uniquely determined by its restriction to the sphere. We will also use $\mathcal{H}_n^d$ to denote the space of spherical harmonics of degree n.

If we define the operator

$$x \cdot \nabla = \sum_{i=1}^d x_i \frac{\partial}{\partial x_i},$$

then, by Euler's equation for homogeneous polynomials, we deduce

$$x \cdot \nabla Y(x) = n\, Y(x), \quad \forall Y \in \mathcal{H}_n^d.$$

The differential operators Δ and $x\cdot\nabla$ can be expressed in spherical-polar coordinates as [8]

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0, \tag{2.3}$$

$$x\cdot\nabla = r\frac{\partial}{\partial r}, \tag{2.4}$$

where Δ_0 is the spherical part of the Laplacian and is called the Laplace–Beltrami operator. The operator Δ_0 has spherical harmonics as eigenfunctions. More precisely, it holds that [8]

$$\Delta_0\, Y(\xi) = -n\,(n+d-2)\, Y(\xi), \quad \forall Y \in \mathcal{H}_n^d, \quad \xi \in \mathbf{S}^{d-1}.$$

We will also need the following family of differential operators, $D_{i,j}$, defined by

$$D_{i,j} = x_i\,\partial_j - x_j\,\partial_i, \quad 1 \leqslant i < j \leqslant d.$$

These are angular derivatives since $D_{i,j} = \partial_{\theta_{i,j}}$ in the polar coordinates of the x_i, x_j–plane, $(x_i, x_j) = r_{i,j}(\cos\theta_{i,j}, \sin\theta_{i,j})$. Furthermore, the angular derivatives $D_{i,j}$ and the Laplace–Beltrami operator Δ_0 are related by

$$\Delta_0 = \sum_{1\leqslant i<j\leqslant d} D_{i,j}^2.$$

Spherical harmonics are orthogonal polynomials on $\mathbf{S}^{d-1}$ with respect to the inner product

$$\langle f, g\rangle_{\mathbf{S}^{d-1}} = \frac{1}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} f(\xi)\, g(\xi)\, d\sigma(\xi),$$

where $d\sigma$ denotes the surface measure and σ_{d-1} denotes the surface area,

$$\sigma_{d-1} = \int_{\mathbf{S}^{d-1}} d\sigma(\xi) = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}.$$

2.2. Mutually orthogonal polynomials on the unit ball

A mutually orthogonal basis of $\mathcal{V}_n^d(W_\mu)$ can be given in terms of Jacobi polynomials and spherical harmonics.

For $\alpha, \beta > -1$, the Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ of degree n is defined as

$$P_n^{(\alpha,\beta)}(t) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(k+\alpha+1)_{n-k}(k+\alpha+\beta+1)_k\left(\frac{t-1}{2}\right)^k,$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. They are orthogonal with respect to the Jacobi weight function $w_{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ on $[-1, 1]$.

Proposition 2.1 ([9]). *For $n \geqslant 0$ and $0 \leqslant j \leqslant \frac{n}{2}$, let $\{Y_\nu^{n-2j}(x) : 1 \leqslant \nu \leqslant a_{n-2j}^d\}$ denote an orthonormal basis of $\mathcal{H}_{n-2j}^d$. For $\mu > -1$, define the polynomials*

$$P_{j,\nu}^{n,\mu}(x) = P_j^{\left(\mu,n-2j+\frac{d-2}{2}\right)}\left(2\|x\|^2-1\right)Y_\nu^{n-2j}(x). \tag{2.5}$$

Then the set $\{P_{j,\nu}^{n,\mu} : 0 \leqslant j \leqslant \frac{n}{2},\ 1 \leqslant \nu \leqslant a_{n-2j}^d\}$ constitutes a mutually orthogonal basis of $\mathcal{V}_n^d(W_\mu)$.

Moreover,

$$\langle P_{j,\nu}^{n,\mu}, P_{k,\eta}^{m,\mu}\rangle_\mu = H_{j,n}^\mu\,\delta_{n,m}\,\delta_{j,k}\,\delta_{\nu,\eta},$$

where

$$H_{j,n}^\mu = \frac{(\mu+1)_j\ (d/2)_{n-j}\ (n-j+\mu+d/2)}{j!\ (\mu+d/2+1)_{n-j}\ (n+\mu+d/2)}. \tag{2.6}$$

The square of the norm of the Jacobi polynomial $P_j^{(\alpha,\beta)}(x)$, denoted by $h_j^{(\alpha,\beta)}$, is related with $H_{j,n}^\mu$ as follows:

$$H_{j,n}^\mu = \frac{\gamma_{\mu,d}}{2^{n-2j}}\,h_j^{(\mu,n-2j+\frac{d-2}{2})}, \tag{2.7}$$

where $\gamma_{\mu,d} = \dfrac{b_\mu\,\sigma_{d-1}}{2^{\mu+\frac{d}{2}+1}}$.

It is known that the elements of the basis of $\mathcal{V}_n^d(W_\mu)$ defined in (2.5) are eigenfunctions of a second order linear partial differential operator $\mathcal{L}_\mu$. More precisely, we have [10,11]

$$\mathcal{L}_\mu[P_{j,\nu}^{n,\mu}(x)] = \lambda_{n,j}^\mu\,P_{j,\nu}^{n,\mu}(x), \tag{2.8}$$

where

$$\mathcal{L}_\mu = (1-\|x\|^2)\,\Delta - (2\,\mu+1)(x\cdot\nabla), \tag{2.9}$$

and

$$\lambda_{n,j}^\mu = -[4j(n-j+\mu+d/2)+2(n-2j)(\mu+1)]. \tag{2.10}$$

We will need the following proposition in the sequel, which states that the basis defined in (2.5) satisfies another orthogonality on the unit ball.

Proposition 2.2 ([11]). *Let $\mu > -1$ and let $P_{j,\nu}^{n,\mu}(x)$ be the mutually orthogonal polynomials in $\mathcal{V}_n^d(W_\mu)$ defined in (2.5). Then,*

$$b_\mu\int_{\mathbf{B}^d}\nabla P_{j,\nu}^{n,\mu}(x)\cdot\nabla P_{k,\eta}^{m,\mu}(x)\,W_{\mu+1}(x)\,dx = H_{j,n}^\mu(\nabla)\,\delta_{n,m}\,\delta_{j,k}\,\delta_{\nu,\eta},$$

where

$$H_{j,n}^\mu(\nabla) = [4j(n-j+\mu+d/2)+2(n-2j)(\mu+1)]\,H_{j,n}^\mu.$$

The following lemma will be useful in the sequel. For convenience we define $P_{j,\nu}^{n,\mu}(x) = 0$ if $j < 0$.

Lemma 2.3 ([11]). *Let $\mu > -1$. Then*

$$\Delta\,P_{j,\nu}^{n,\mu}(x) = \kappa_{n-j}^\mu\,P_{j-1,\nu}^{n-2,\mu+2}(x) \quad \text{and} \quad \Delta_0\,P_{j,\nu}^{n,\mu}(x) = \varrho_{n-2j}\,P_{j,\nu}^{n,\mu}(x),$$

where

$$\kappa_n^\mu = 4\left(n+\mu+\frac{d}{2}\right)\left(n+\frac{d-2}{2}\right) \quad \text{and} \quad \varrho_n = -n(n+d-2).$$

Motivated by Proposition 2.2, we define the Sobolev space: For $\mathbf{m} \in \mathbb{N}_0^d$, let $|\mathbf{m}| = m_1 + \cdots + m_d$ and $\partial^{\mathbf{m}} = \partial_1^{m_1} \cdots \partial_d^{m_d}$. For $\mu > -1$ and $s \geqslant 1$, we denote by $\mathcal{W}_2^s(W_\mu; \mathbf{B}^d)$ the Sobolev space

$$\mathcal{W}_2^s(W_\mu; \mathbf{B}^d) = \left\{ f \in L^2(W_\mu; \mathbf{B}^d);\ \partial^{\mathbf{m}} f \in L^2(W_{\mu+|\mathbf{m}|}; \mathbf{B}^d),\ |\mathbf{m}| \leqslant s,\ \mathbf{m} \in \mathbb{N}_0^d \right\}.$$

2.3. Fourier orthogonal expansion and approximation

With respect to the basis (2.5), the Fourier orthogonal expansion of $f \in L^2(W_\mu; \mathbf{B}^d)$ is defined by

$$f(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}^d} \widehat{f}_{j,\nu}^{\,n,\mu} P_{j,\nu}^{n,\mu}(x) \quad \text{with} \quad \widehat{f}_{j,\nu}^{\,n,\mu} := \frac{1}{H_{j,n}^{\mu}} \left\langle f, P_{j,\nu}^{n,\mu} \right\rangle_\mu . \tag{2.11}$$

Since $\|f\|_\mu$ is finite, the Parseval identity holds: for $\mu > -1$,

$$\|f\|_\mu^2 = \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}^d} \left| \widehat{f}_{j,\nu}^{\,n,\mu} \right|^2 H_{j,n}^{\mu}. \tag{2.12}$$

Let $\operatorname{proj}_n^\mu : L^2(W_\mu; \mathbf{B}^d) \to \mathcal{V}_n^d(W_\mu)$ and $S_n^\mu : L^2(W_\mu; \mathbf{B}^d) \to \Pi_n^d$ denote the projection operator and the partial sum operator, respectively. Then,

$$\operatorname{proj}_m^\mu f(x) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu=1}^{a_{m-2j}^d} \widehat{f}_{j,\nu}^{\,m,\mu} P_{j,\nu}^{m,\mu}(x) \quad \text{and} \quad S_n^\mu f(x) = \sum_{m=0}^{n} \operatorname{proj}_m^\mu f(x).$$

By definition, $S_n^\mu f = f$ if $f \in \Pi_n^d$ and $\langle f - S_n^\mu f, Q \rangle_\mu = 0$ for all $Q \in \Pi_n^d$.

It turns out that the partial derivatives commute with the partial sum operator.

Proposition 2.4 ([11]). *Let $\mu > -1$. Then,*

$$\partial_i S_n^\mu f \ = \ S_{n-1}^{\mu+1}(\partial_i f), \quad 1 \leqslant i \leqslant d,$$

and

$$D_{i,j} S_n^\mu f \ = \ S_n^\mu(D_{i,j} f), \quad 1 \leqslant i < j \leqslant d.$$

The relations in the above proposition pass down to the Fourier coefficients.

Proposition 2.5. *Let $f \in \mathcal{W}_2^2(W_\mu; \mathbf{B}^d)$, $\mu > -1$. Then,*

$$\widehat{\Delta f}_{j,\nu}^{\,n-2,\mu+2} \ = \ \kappa_{n-j-1}^{\mu} \widehat{f}_{j+1,\nu}^{\,n,\mu}, \quad 0 \leqslant j \leqslant \frac{n-2}{2},$$

and

$$\widehat{\Delta_0 f}_{j,\nu}^{\,n,\mu} \ = \ \varrho_{n-2j} \widehat{f}_{j,\nu}^{\,n,\mu}, \quad 0 \leqslant j \leqslant \frac{n}{2}.$$

We consider the error, $\mathcal{E}_n(f)_\mu$, of best approximation by polynomials in Π_n^d in the space $L^2(W_\mu; \mathbf{B}^d)$, defined by

$$\mathcal{E}_n(f)_\mu = \inf_{p_n \in \Pi_n^d} \|f - p_n\|_\mu,$$

and notice that the infimum is achieved by $S_n^\mu f$. The following estimate was proved in [11]: for $n \geqslant 2s$ and $f \in \mathcal{W}_2^{2s}(W_\mu; \mathbf{B}^d)$,

$$\mathcal{E}_n(f)_\mu \leqslant \frac{c}{n^{2s}} \left[\mathcal{E}_{n-2s}(\Delta^s f)_{\mu+2s} + \mathcal{E}_n(\Delta_0^s f)_\mu \right], \tag{2.13}$$

and for $n \geqslant 2s+1$ and $f \in \mathcal{W}_2^{2s+1}(W_\mu; \mathbf{B}^d)$,

$$\mathcal{E}_n(f)_\mu \leqslant \frac{c}{n^{2s+1}} \left[\sum_{i=1}^{d} \mathcal{E}_{n-2s-1}(\partial_i \Delta^s f)_{\mu+2s+1} + \sum_{1 \leqslant i < j \leqslant d} \mathcal{E}_n(D_{i,j} \Delta_0^s f)_\mu \right]. \tag{2.14}$$

Here and in the sequel, c is a generic constant independent of n and f but may depend on μ and d, and its value may be different from one instance to the next. As pointed out in [11], each term involving Δ and Δ_0 on the right hand side of the above inequalities is necessary since the first term deals with the radial component of f and the second one deals with the harmonic component of f defined on the ball.

3. Sobolev orthogonal polynomials with a mass point at zero

This section is devoted to the study of the orthogonal structure on the unit ball with respect to the *Sobolev inner product*

$$\langle f, g\rangle_{\nabla,\mu} = f(0)g(0) + \lambda \int_{\mathbf{B}^d} \nabla f(x)\cdot\nabla g(x) W_\mu(x)dx, \quad \lambda > 0. \tag{3.1}$$

Orthogonal polynomials with respect to inner products involving derivatives are called Sobolev orthogonal polynomials. Let us denote by $\mathcal{V}_n^d(\nabla, W_\mu)$ the space of Sobolev orthogonal polynomials of degree n with respect (3.1).

Let $\mathcal{U}(W_\mu;\mathbf{B}^d)$ denote the Sobolev space

$$\mathcal{U}(W_\mu;\mathbf{B}^d) = \{f \in \mathcal{W}_2^1(W_\mu;\mathbf{B}^d) : |f(0)| < +\infty\},$$

and let $\|\cdot\|_{\nabla,\mu}$ denote the norm of $\mathcal{U}(W_\mu;\mathbf{B}^d)$ defined by

$$\|f\|_{\nabla,\mu} = \left(f(0)^2 + \frac{\lambda}{b_\mu}\sum_{i=1}^d \|\partial_i f\|_\mu^2\right)^{1/2}. \tag{3.2}$$

In the following proposition, we construct a mutually orthogonal polynomial basis with respect to the inner product (3.1).

Proposition 3.1. *For $\mu > 0$, define the polynomials*

$$\begin{aligned} Q_{j,\nu}^{n,\mu}(x) &:= P_{j,\nu}^{n,\mu-1}(x) - P_{j,\nu}^{n,\mu-1}(0), \quad n \geqslant 1, \\ Q_{0,0}^{0,\mu}(x) &:= 1. \end{aligned} \tag{3.3}$$

Then $\{Q_{j,\nu}^{n,\mu} : 0 \leqslant j \leqslant \frac{n}{2},\ 1 \leqslant \nu \leqslant a_{n-2j}^d\}$ constitutes a mutually orthogonal basis of $\mathcal{V}_n^d(\nabla, W_\mu)$. Moreover,

$$\left\langle Q_{j,\nu}^{n,\mu}, Q_{k,\eta}^{m,\mu}\right\rangle_{\nabla,\mu} = H_{j,n}^{\nabla,\mu}\,\delta_{n,m}\,\delta_{j,k}\,\delta_{\nu,\eta},$$

where

$$H_{j,n}^{\nabla,\mu} = \frac{\lambda}{b_{\mu-1}}[4j(n-j+\mu+d/2-1) + 2(n-2j)\mu]\,H_{j,n}^{\mu-1}. \tag{3.4}$$

Proof. We note that since $Y_\nu^{n-2j} \in \mathcal{H}_{n-2j}^d$ are homogeneous polynomials, $Y_\nu^{n-2j}(0) = 0$ when $n-2j \geqslant 1$, and, therefore, $P_{j,\nu}^{n,\mu-1}(0) = 0$ for $n-2j \geqslant 1$. Moreover, $Q_{j,\nu}^{n,\mu}(0) = 0$ for $n \geqslant 1$.

On one hand, it is clear that

$$\left\langle Q_{0,0}^{0,\mu}, Q_{0,0}^{0,\mu}\right\rangle_{\nabla,\mu} = 1 \quad \text{and} \quad \left\langle Q_{0,0}^{0,\mu}, Q_{k,\eta}^{m,\mu}\right\rangle_{\nabla,\mu} = 0, \quad m \geqslant 1.$$

On the other hand, we compute

$$\left\langle Q_{j,\nu}^{n,\mu}, Q_{k,\eta}^{m,\mu}\right\rangle_{\nabla,\mu} = \lambda\int_{\mathbf{B}^d} \nabla P_{j,\nu}^{n,\mu-1}(x)\cdot\nabla P_{k,\eta}^{m,\mu-1}(x)\,W_\mu(x)\,dx.$$

From Proposition 2.2 we get

$$\begin{aligned}&\left\langle Q_{j,\nu}^{n,\mu}, Q_{k,\eta}^{m,\mu}\right\rangle_{\nabla,\mu} \\ &\quad= \frac{\lambda}{b_{\mu-1}}[4j(n-j+\mu+d/2-1) + 2(n-2j)\mu]\,H_{j,n}^{\mu-1}\,\delta_{n,m}\,\delta_{j,k}\,\delta_{\nu,\eta}. \quad \square\end{aligned}$$

Observe that we can write the basis (3.3) as follows:

$$\begin{aligned} Q_{0,\nu}^{n,\mu}(x) &:= Y_\nu^n(x), \\ Q_{j,\nu}^{n,\mu}(x) &:= P_j^{\left(\mu-1,n-2j+\frac{d-2}{2}\right)}\left(2\|x\|^2-1\right)\, Y_\nu^{n-2j}(x), \quad 1 \leqslant j \leqslant \lfloor\frac{n-1}{2}\rfloor, \\ Q_{\frac{n}{2},1}^{n,\mu}(x) &:= P_{\frac{n}{2}}^{(\mu-1,\frac{d-2}{2})}(2\|x\|^2-1) - (-1)^{\frac{n}{2}}\frac{(d/2)_{\frac{n}{2}}}{(n/2)!}, \end{aligned}$$

where $Q_{\frac{n}{2},1}^{n,\mu}(x)$ holds only when n is even. Here, we have used the fact that $P_k^{(\alpha,\beta)}(-1) = (-1)^k(\beta+1)_k/k!$

The case when $\mu = 0$ was previously studied in [4]. Here, we recall the explicit expression for the basis in this case. Let us denote by $q_k(x)$ the univariate polynomial defined by

$$q_0(x) = 1, \quad q_k(x) = \int_{-1}^{x} P_{k-1}^{(0,\frac{d}{2})}(t)\,dt, \quad k \geqslant 1.$$

The Jacobi polynomials $P_k^{(-1,\frac{d-2}{2})}(x)$ are well defined for $k \geqslant 1$, satisfying [12, (4.5.5), p.72]

$$\frac{d}{dx} P_k^{(-1,\frac{d-2}{2})}(x) = \frac{1}{2}\left(k + \frac{d-2}{2}\right) P_{k-1}^{(0,\frac{d}{2})}(x). \tag{3.5}$$

Hence, we have

$$q_k(x) = \frac{4}{2k+d-2}\left(P_k^{(-1,\frac{d-2}{2})}(x) - (-1)^k \frac{(d/2)_k}{k!}\right).$$

Proposition 3.2. *A mutually orthogonal basis for $\mathcal{V}_n^d(\nabla, W_0)$ is given by*

$$\begin{aligned} Q_{0,\nu}^{n,0}(x) &= Y_\nu^n(x), \\ Q_{j,\nu}^{n,0}(x) &= (1-\|x\|^2)\,P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(2\|x\|^2-1)\,Y_\nu^{n-2j}(x), \quad 1 \leqslant j \leqslant \lfloor \tfrac{n-1}{2} \rfloor, \\ Q_{\frac{n}{2},1}^{n,0}(x) &= \frac{4}{n+d-2}\left(P_{\frac{n}{2}}^{(-1,\frac{d-2}{2})}(2\|x\|^2-1) - (-1)^{\frac{n}{2}} \frac{(d/2)_{\frac{n}{2}}}{(n/2)!}\right), \end{aligned} \tag{3.6}$$

where $Q_{\frac{n}{2},1}^{n,0}(x)$ holds only when n is even. Furthermore,

$$\begin{aligned} \left\langle Q_{0,\nu}^{n,0}, Q_{0,\nu}^{n,0}\right\rangle_{\nabla,0} &= n\,\lambda\,\sigma_{d-1}, \\ \left\langle Q_{j,\nu}^{n,0}, Q_{j,\nu}^{n,0}\right\rangle_{\nabla,0} &= \frac{2j^2}{n+\frac{d-2}{2}}\lambda\,\sigma_{d-1}, \quad 1 \leqslant j \leqslant \lfloor \tfrac{n-1}{2} \rfloor, \\ \left\langle Q_{\frac{n}{2},1}^{n,0}, Q_{\frac{n}{2},1}^{n,0}\right\rangle_{\nabla,0} &= \frac{8}{n+\frac{d-2}{2}}\,\lambda\,\sigma_{d-1}. \end{aligned} \tag{3.7}$$

4. Sobolev Fourier orthogonal expansions and approximation

For $\mu \geqslant 0$ and $f \in \mathcal{U}(W_\mu; \mathbf{B}^d)$, let us denote by $\widehat{f}_{j,\nu}^{\,n,\mu}(\nabla)$ the Fourier coefficients with respect to the basis of $\mathcal{V}_n^d(\nabla, W_\mu)$ defined in (3.3), that is,

$$\widehat{f}_{j,\nu}^{\,n,\mu}(\nabla) = \frac{1}{H_{j,n}^{\nabla,\mu}} \left\langle f, Q_{j,\nu}^{n,\mu}\right\rangle_{\nabla,\mu}.$$

Let $\mathrm{proj}_m^{\nabla,\mu} : \mathcal{U}(W_\mu; \mathbf{B}^d) \to \mathcal{V}_n^d(\nabla, W_\mu)$ and $S_n^{\nabla,\mu} : \mathcal{U}(W_\mu; \mathbf{B}^d) \to \Pi_n^d$ denote the projection operator and partial sum operators

$$\mathrm{proj}_m^{\nabla,\mu} f(x) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu=1}^{a_{m-2j}^d} \widehat{f}_{j,\nu}^{\,m,\mu}(\nabla)\, Q_{j,\nu}^{m,\mu}(x) \quad \text{and} \quad S_n^{\nabla,\mu} f(x) = \sum_{m=0}^{n} \mathrm{proj}_m^{\nabla,\mu} f(x).$$

4.1. The case $\mu > 0$

The Fourier coefficients with respect to the basis (2.5) of $\mathcal{V}_n^d(W_\mu)$ are related to the Fourier coefficients $\widehat{f}_{j,\nu}^{\,n,\mu}(\nabla)$.

Proposition 4.1. *Let $\mu > 0$. Then, for $f \in \mathcal{U}(W_\mu; \mathbf{B}^d)$,*

$$\begin{aligned} &\widehat{f}_{j,\nu}^{\,n,\mu}(\nabla) = \widehat{f}_{j,\nu}^{\,n,\mu-1}, \quad n \geqslant 1, \\ &\widehat{f}_{0,1}^{\,0,\mu}(\nabla) = f(0). \end{aligned}$$

Proof. Since $Q_{0,0}^{0,\mu}(x) = 1$ and $H_{0,0}^{\nabla,\mu} = 1$ for $\mu \geqslant 0$, $\widehat{f}_{0,1}^{\,0,\mu}(\nabla) = \langle f, Q_{0,1}^{0,\mu}\rangle_{\nabla,\mu} = f(0)$.

Let $\mu > 0$. For $n \geqslant 1$,

$$\begin{aligned} \left\langle f, Q_{j,\nu}^{n,\mu}\right\rangle_{\nabla,\mu} &= f(0)\,Q_{j,\nu}^{n,\mu}(0) + \lambda \int_{\mathbf{B}^d} \nabla f(x) \cdot \nabla Q_{j,\nu}^{n,\mu}(x)\, W_\mu(x)\,dx \\ &= \lambda \int_{\mathbf{B}^d} \nabla f(x) \cdot \nabla P_{j,\nu}^{n,\mu-1}(x)\, W_\mu(x)\,dx. \end{aligned}$$

Using Green's formula and (2.8), we get

$$\begin{aligned}\left\langle f, Q^{n,\mu}_{j,\nu}\right\rangle_{\nabla,\mu} &= \lambda\,[4j(n-j+\mu+d/2-1)+2(n-2j)\,\mu]\int_{\mathbf{B}^d} f(x)\,P^{n,\mu-1}_{j,\nu}(x)\,W_{\mu-1}(x)\,dx\\ &= \frac{\lambda}{b_{\mu-1}}[4j(n-j+\mu+d/2-1)+2(n-2j)\,\mu]\left\langle f, P^{n,\mu-1}_{j,\nu}\right\rangle_{\mu-1}.\end{aligned}$$

From (3.4), we get

$$\widehat{f}^{\,n,\mu}_{j,\nu}(\nabla) = \frac{1}{H^{\nabla,\mu}_{j,n}}\left\langle f, Q^{n,\mu}_{j,\nu}\right\rangle_{\nabla,\mu} = \frac{1}{H^{\mu-1}_{j,n}}\left\langle f, P^{n,\mu-1}_{j,\nu}\right\rangle_{\mu-1} = \widehat{f}^{\,n,\mu-1}_{j,\nu}. \quad \square$$

It is important to note that the Fourier coefficients can be computed without involving the derivatives of f.

For $\mu > 0$, we can deduce the relationship between the projection operators with respect to the classical and Sobolev bases.

Proposition 4.2. *For $\mu > 0$,*

$$\begin{aligned}&\mathrm{proj}^{\nabla,\mu}_0 f(x) = f(0),\\ &\mathrm{proj}^{\nabla,\mu}_m f(x) = \mathrm{proj}^{\mu-1}_m f(x) - \mathrm{proj}^{\mu-1}_m f(0), \quad m \geqslant 1,\end{aligned}$$

and

$$\begin{aligned}&S^{\nabla,\mu}_0 f(x) = f(0),\\ &S^{\nabla,\mu}_n f(x) = f(0) + S^{\mu-1}_n f(x) - S^{\mu-1}_n f(0), \quad n \geqslant 0.\end{aligned}$$

Proof. Clearly, $\mathrm{proj}^{\nabla,\mu}_0 f(x) = f(0)$. For $m \geqslant 1$,

$$\begin{aligned}\mathrm{proj}^{\nabla,\mu}_m f(x) &= \sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\sum_{\nu=1}^{a^d_{m-2j}} \widehat{f}^{\,m,\mu}_{j,\nu}(\nabla)\,Q^{m,\mu}_{j,\nu}(x)\\ &= \sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\sum_{\nu=1}^{a^d_{m-2j}} \left[\widehat{f}^{\,m,\mu-1}_{j,\nu} P^{m,\mu-1}_{j,\nu}(x) - \widehat{f}^{\,m,\mu-1}_{j,\nu} P^{m,\mu-1}_{j,\nu}(0)\right].\end{aligned}$$

Then,

$$\mathrm{proj}^{\nabla,\mu}_m f(x) = \mathrm{proj}^{\mu-1}_m f(x) - \mathrm{proj}^{\mu-1}_m f(0), \quad m \geqslant 1,$$

where we have used Proposition 4.1 to write $\widehat{f}^{\,n,\mu}_{j,\nu}(\nabla) = \widehat{f}^{\,n,\mu-1}_{j,\nu}$ for $n \geqslant 1$.

Moreover, since $\mathrm{proj}^{\mu-1}_0 f(x) - \mathrm{proj}^{\mu-1}_0 f(0) = 0$

$$\begin{aligned}S^{\nabla,\mu}_n f(x) &= \mathrm{proj}^{\nabla,\mu}_0 f(x) + \sum_{m=1}^{n}\mathrm{proj}^{\nabla,\mu}_m f(x)\\ &= f(0) + \sum_{m=0}^{n}\left[\mathrm{proj}^{\mu-1}_m f(x) - \mathrm{proj}^{\mu-1}_m f(0)\right].\end{aligned}$$

Therefore,

$$S^{\nabla,\mu}_n f(x) = S^{\mu-1}_n f(x) + f(0) - S^{\mu-1}_n f(0), \quad \mu > 0, \tag{4.1}$$

and, consequently, $S^{\nabla,\mu}_n f(0) = f(0)$. $\square$

We have the following proposition stating the interaction between differentiation and the partial sum operator $S^{\nabla,\mu}_n$ for $\mu > 0$.

Proposition 4.3. *Let $\mu > 0$ and $n \geqslant 1$. Then,*

$$\partial_i S^{\nabla,\mu}_n f(x) = S^{\mu}_{n-1}(\partial_i f)(x), \quad 1 \leqslant i \leqslant d,$$

or, equivalently,

$$\partial_i S^{\nabla,\mu}_n f(x) = S^{\nabla,\mu+1}_{n-1}(\partial_i f)(x) + S^{\mu}_{n-1}(\partial_i f)(0) - (\partial_i f)(0).$$

Proof. Differentiating (4.1) and using Proposition 2.4, we obtain

$$\partial_i S_n^{\nabla,\mu} f(x) = \partial_i S_n^{\mu-1} f(x) = S_{n-1}^{\mu}(\partial_i f)(x).$$

Using (4.1) again, we get

$$S_{n-1}^{\mu}(\partial_i f)(x) = S_{n-1}^{\nabla,\mu+1}(\partial_i f)(x) + S_{n-1}^{\mu}(\partial_i f)(0) - (\partial_i f)(0),$$

and the result follows. □

We have the following expression for the Parseval identity.

Corollary 4.4. *For $\mu > 0$ and $f \in \mathcal{U}(W_\mu; \mathbf{B}^d)$,*

$$\|f\|_{\nabla,\mu}^2 = f(0)^2 + \frac{\lambda}{b_{\mu-1}} \sum_{n=1}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}} \left|\lambda_{n,j}^{\mu-1}\right| \left|\widehat{f}_{j,n}^{\,n,\mu-1}\right|^2 H_{j,n}^{\mu-1},$$

where $\lambda_{n,j}^{\mu-1}$ are defined in (2.10).

Consequently,

$$b_{\mu-1} \int_{B^d} \|\nabla f(x)\|^2 \, W_\mu(x)\, dx = \sum_{n=1}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}} \left|\lambda_{n,j}^{\mu-1}\right| \left|\widehat{f}_{j,n}^{\,n,\mu-1}\right|^2 H_{j,n}^{\mu-1}.$$

Proof. Parseval's identity for $f \in \mathcal{U}(W_\mu; \mathbf{B}^d)$ with respect to the orthogonal basis defined in (3.3) is written as

$$\|f\|_{\nabla,\mu}^2 = f(0)^2 + \sum_{n=1}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}} \left|\widehat{f}_{j,n}^{\,n,\mu}(\nabla)\right|^2 H_{j,n}^{\nabla,\mu}.$$

The result follows from Proposition 4.1 and (3.4).

The last equation follows from

$$\int_{B^d} \nabla f(x) \cdot \nabla f(x)\, W_\mu(x)\, dx = \lim_{\lambda \to +\infty} \frac{\|f\|_{\nabla,\mu}^2}{\lambda}. \quad \square$$

Let $\mathcal{E}_n(f)_{\nabla,\mu}$ denote the error of best approximation by polynomials in Π_n^d in the space $\mathcal{U}(W_\mu; \mathbf{B}^d)$:

$$\mathcal{E}_n(f)_{\nabla,\mu} = \|f - S_n^{\nabla,\mu} f\|_{\nabla,\mu}.$$

We have the following estimate.

Theorem 4.5. *Let $\mu > 0$. Then, for $n \geqslant 2s+1$ and $f \in \mathcal{U}(W_\mu; \mathbf{B}^d) \cap \mathcal{W}_2^{2s+1}(W_\mu; \mathbf{B}^d)$,*

$$\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s}} \sum_{i=1}^{d} \left[\mathcal{E}_{n-2s-1}(\Delta^s \partial_i f)_{\mu+2s} + \mathcal{E}_{n-1}(\Delta_0^s \partial_i f)_\mu \right],$$

and for $n \geqslant 2s+2$ and $f \in \mathcal{U}(W_\mu; \mathbf{B}^d) \cap \mathcal{W}_2^{2s+2}(W_\mu; \mathbf{B}^d)$,

$$\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s+1}} \sum_{i=1}^{d} \left[\sum_{k=1}^{d} \mathcal{E}_{n-2s-2}(\partial_k \Delta^s \partial_i f)_{\mu+2s+1} + \sum_{1 \leqslant k < \ell \leqslant d} \mathcal{E}_{n-1}(D_{k,\ell} \Delta_0^s \partial_i f)_\mu \right].$$

Proof. For $n \geqslant 1$, we have

$$\mathcal{E}_n(f)_{\nabla,\mu}^2 = \|f - S_n^{\nabla,\mu} f\|_{\nabla,\mu}^2 = \frac{\lambda}{b_\mu} \sum_{i=1}^{d} \|\partial_i f - \partial_i S_n^{\nabla,\mu} f\|_\mu^2.$$

Using Proposition 4.3, we get

$$\mathcal{E}_n(f)_{\nabla,\mu}^2 = \frac{\lambda}{b_\mu} \sum_{i=1}^{d} \left\|\partial_i f - S_{n-1}^{\mu}(\partial_i f)\right\|_\mu^2 = \frac{\lambda}{b_\mu} \sum_{i=1}^{d} \mathcal{E}_{n-1}(\partial_i f)_\mu^2,$$

and the result follows from (2.13) and (2.14). □

Lemma 4.6. *Let $\mu \geqslant 0$. For $f \in \mathcal{U}(W_{\mu+1}; \mathbf{B}^d)$,*

$$\mathcal{E}_n(f)_\mu \leqslant \frac{c}{\sqrt{n}} \mathcal{E}_n(f)_{\nabla,\mu+1}.$$

Proof. From Corollary 4.4 and the Parseval identity (2.12), we get

$$\begin{aligned}\|f - S_n^\mu f\|_\mu^2 &= \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu=1}^{a_{m-2j}} \left|\widehat{f}_{j,\nu}^{\,m,\mu}\right|^2 H_{j,m}^\mu \\ &= \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{\left|\lambda_{m,j}^\mu\right|} \sum_{\nu=1}^{a_{m-2j}} \left|\lambda_{m,j}^\mu\right| \left|\widehat{f}_{j,\nu}^{\,n,\mu}\right|^2 H_{j,n}^\mu \\ &\leqslant \frac{c}{2(\mu+1)n} \mathcal{E}_n(f)_{\nabla,\mu+1}^2,\end{aligned}$$

where c is a constant. The inequality follows from $\left|\lambda_{n,j}^\mu\right|^{-1} \leqslant \frac{1}{2(\mu+1)n}$ for $n \geqslant 1$ and $0 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor$. □

Moreover, the following proposition shows that the rate of convergence of $S_n^{\nabla,\mu} f$ towards f is faster than the rate stated in Theorem 4.5.

Theorem 4.7. *Let $\mu > 0$. Then, for $n \geqslant 2s+1$ and $f \in \mathcal{U}(W_{\mu+2s+1}; \mathbf{B}^d) \cap \mathcal{W}_2^{2s+1}(W_\mu; \mathbf{B}^d)$,*

$$\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s+1/2}} \sum_{i=1}^{d} \left[\mathcal{E}_{n-2s-1}(\Delta^s \partial_i f)_{\nabla,\mu+2s+1} + \mathcal{E}_{n-1}(\Delta_0^s \partial_i f)_{\nabla,\mu+1} \right],$$

and for $n \geqslant 2s+2$ and $f \in \mathcal{U}(W_{\mu+2s+2}; \mathbf{B}^d) \cap \mathcal{W}_2^{2s+2}(W_\mu; \mathbf{B}^d)$,

$$\begin{aligned}&\mathcal{E}_n(f)_{\nabla,\mu} \\ &\leqslant \frac{c}{(n-1)^{2s+3/2}} \sum_{i=1}^{d} \left[\sum_{k=1}^{d} \mathcal{E}_{n-2s-2}(\partial_k \Delta^s \partial_i f)_{\nabla,\mu+2s+2} + \sum_{1\leqslant k<\ell\leqslant d} \mathcal{E}_{n-1}(D_{k,\ell} \Delta_0^s \partial_i f)_{\nabla,\mu+1} \right].\end{aligned}$$

Consequently,

$$\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s+1/2}} \sum_{i=1}^{d} \left[\|\Delta^s \partial_i f\|_{\nabla,\mu+2s+1} + \|\Delta_0^s \partial_i f\|_{\nabla,\mu+1} \right],$$

and

$$\begin{aligned}&\mathcal{E}_n(f)_{\nabla,\mu} \\ &\leqslant \frac{c}{(n-1)^{2s+3/2}} \sum_{i=1}^{d} \left[\sum_{k=1}^{d} \|\partial_k \Delta^s \partial_i f\|_{\nabla,\mu+2s+2} + \sum_{1\leqslant k<\ell\leqslant d} \|D_{k,\ell} \Delta_0^s \partial_i f\|_{\nabla,\mu+1} \right],\end{aligned}$$

respectively.

Proof. For $n \geqslant 2s+1$ and $f \in \mathcal{U}(W_{\mu+2s+1}; \mathbf{B}^d) \cap \mathcal{W}_2^{2s+1}(W_\mu; \mathbf{B}^d)$, Theorem 4.5, together with Lemma 4.6, implies

$$\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s+1/2}} \sum_{i=1}^{d} \left[\mathcal{E}_{n-2s-1}(\Delta^s \partial_i f)_{\nabla,\mu+2s+1} + \mathcal{E}_{n-1}(\Delta_0^s \partial_i f)_{\nabla,\mu+1} \right].$$

Moreover, from the fact that, for $i = 1, \ldots, d$,

$$\mathcal{E}_{n-2s-1}(\Delta^s \partial_i f)_{\nabla,\mu+2s+1} \leqslant c \, \|\Delta^s \partial_i f\|_{\nabla,\mu+2s+1},$$

and

$$\mathcal{E}_{n-1}(\Delta_0^s \partial_i f)_{\nabla,\mu+1} \leqslant c \, \|\Delta_0^s \partial_i f\|_{\nabla,\mu+1},$$

we get

$$\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s+1/2}} \sum_{i=1}^{d} \left[\|\Delta^s \partial_i f\|_{\nabla,\mu+2s+1} + \|\Delta_0^s \partial_i f\|_{\nabla,\mu+1} \right].$$

The remaining part of the theorem follows similarly. □

5. The Zernike case $\mu = 0$

This case is more complicated than the previous one. Hence, we consider it separately here.

Proposition 5.1. *For $f \in \mathcal{U}(W_0; \mathbf{B}^d)$,*

$$\begin{aligned}
\widehat{f}^{\,n,0}_{0,\nu}(\nabla) &= \langle f, Y^n_\nu\rangle_{\mathbf{S}^{d-1}}, \\
\widehat{f}^{\,n,0}_{j,\nu}(\nabla) &= \frac{n+\frac{d-2}{2}}{j}\left[\frac{2\left(n-j+\frac{d-2}{2}\right)}{\sigma_{d-1}}\int_{\mathbf{B}^d} f(x)\,P^{n-2,1}_{j-1,\nu}(x)\,dx \right.\\
&\qquad\left. -\langle f, Y^{n-2j}_\nu\rangle_{\mathbf{S}^{d-1}}\right], \quad 1 \leqslant j \leqslant \lfloor\frac{n-1}{2}\rfloor, \\
\widehat{f}^{\,n,0}_{\frac{n}{2},1}(\nabla) &= \frac{n+\frac{d-2}{2}}{2}\left[\langle f, 1\rangle_{\mathbf{S}^{d-1}} - \frac{(n+d-2)}{\sigma_{d-1}}\int_{\mathbf{B}^d} f(x)\,P^{n-2,1}_{\frac{n}{2}-1,1}(x)\,dx\right], \\
\widehat{f}^{\,0,0}_{0,1}(\nabla) &= f(0),
\end{aligned}$$

where $\widehat{f}^{\,n,0}_{\frac{n}{2},1}(\nabla)$ holds when n is even.

Proof. Here, we use spherical-polar coordinates $x = r\,\xi$ where $r > 0$ and $\xi \in \mathbf{S}^{d-1}$.

Let $n \geqslant 1$. Using Green's identity and the fact that $Q^{n,0}_{0,\nu}(0) = Y^n_\nu(0) = 0$, we get

$$\left\langle f, Q^{n,0}_{0,\nu}\right\rangle_{\nabla,0} = \lambda \int_{\mathbf{S}^{d-1}} f(\xi)\,\frac{\partial\, Q^{n,0}_{0,\nu}}{\partial r}(\xi)\,d\sigma(\xi) - \lambda\int_{\mathbf{B}^d} f(x)\,\Delta Q^{n,0}_{0,\nu}(x)\,dx.$$

Observe that $\Delta\, Q^{n,0}_{0,\nu}(x) = \Delta\, Y^n_\nu(x) = 0$. Moreover, by Euler's equation for homogeneous polynomials, we have that

$$\left.\frac{\partial}{\partial r} Y^n_\nu(x)\right|_{r=1} = n\,Y^n_\nu(\xi).$$

Then, from (3.7), we get

$$\widehat{f}^{\,n,0}_{0,\nu}(\nabla) = \frac{\left\langle f, Q^{n,0}_{0,\nu}\right\rangle_{\nabla,0}}{\left\langle Q^{n,0}_{0,\nu}, Q^{n,0}_{0,\nu}\right\rangle_{\nabla,0}} = \langle f, Y^n_\nu\rangle_{\mathbf{S}^{d-1}}.$$

Similarly, for $j \geqslant 1$, we have

$$\left\langle f, Q^{n,0}_{j,\nu}\right\rangle = \lambda \int_{\mathbf{S}^{d-1}} f(\xi)\frac{\partial\, Q^{n,0}_{j,\nu}}{\partial r}(\xi)\,d\sigma(\xi) - \lambda\int_{\mathbf{B}^d} f(x)\,\Delta\, Q^{n,0}_{j,\nu}(x)\,dx.$$

Using the following facts [4]:

$$\begin{aligned}
\left.\frac{\partial}{\partial r} Q^{n,0}_{j,\nu}(x)\right|_{r=1} &= -2\,P^{(1,n-2j+\frac{d-2}{2})}_{j-1}(1)\,Y^{n-2j}_\nu(\xi), \\
P^{(1,n-2j+\frac{d-2}{2})}_{j-1}(1) &= j, \\
\Delta\, Q^{n,0}_{j,\nu}(x) &= -4j\left(n-j+\frac{d-2}{2}\right)P^{n-2,1}_{j-1,\nu}(x),
\end{aligned}$$

we get

$$\begin{aligned}
\left\langle f, Q^{n,0}_{j,\nu}\right\rangle &= -2j\,\lambda\int_{\mathbf{S}^{d-1}} f(\xi)\,Y^{n-2j}_\nu(\xi)\,d\sigma(\xi) \\
&\qquad + 4j\left(n-j+\frac{d-2}{2}\right)\lambda\int_{\mathbf{B}^d} f(x)\,P^{n-2,1}_{j-1,\nu}(x)\,dx,
\end{aligned}$$

and, therefore, from (3.7), we obtain

$$\begin{aligned}
\widehat{f}^{\,n,0}_{j,\nu}(\nabla) &= -\frac{n+\frac{d-2}{2}}{j}\langle f, Y^{n-2j}_\nu\rangle_{\mathbf{S}^{d-1}} \\
&\qquad + \frac{2\left(n+\frac{d-2}{2}\right)\left(n-j+\frac{d-2}{2}\right)}{j\,\sigma_{d-1}}\int_{\mathbf{B}^d} f(x)\,P^{n-2,1}_{j-1\,\nu}(x)\,dx.
\end{aligned}$$

Similarly, $\widehat{f}^{n,0}_{\frac{n}{2},1}(\nabla)$ is deduced by using the fact that $P^{(-1,\frac{d-2}{2})}_{\frac{n}{2}}(2\|x\|^2-1)$ is a radial function and that

$$\frac{\partial}{\partial r}P^{(-1,\frac{d-2}{2})}_{\frac{n}{2}}(2r^2-1)\Big|_{r=1}=(n+d-2),$$

$$\Delta P^{(-1,\frac{d-2}{2})}_{\frac{n}{2}}(2\|x\|^2-1)=(n+d-2)^2\,P^{(1,\frac{d-2}{2})}_{\frac{n}{2}-1}(2\|x\|^2-1), \tag{5.1}$$

where we have used (2.3) and the identities ([10,12], respectively)

$$\frac{d}{dx}P^{(\alpha,\beta)}_n(x)=\frac{1}{2}(n+\alpha+\beta+1)\,P^{(\alpha+1,\beta+1)}_{n-1}(x),$$

$$\beta\,P^{(\alpha,\beta)}_n(t)+(1+t)\,\frac{d}{dt}P^{(\alpha,\beta)}_n(t)=(\beta+n)\,P^{(\alpha+1,\beta-1)}_n(t),$$

to compute $\Delta P^{(-1,\frac{d-2}{2})}_{\frac{n}{2}}(2\|x\|^2-1)$. □

For the case when $\mu=0$, the Parseval identity reads

$$\|f\|^2_{\nabla,0}=f(0)^2+\lambda\,\sigma_{d-1}\sum_{n=1}^{\infty}\Bigg[\sum_{\nu=1}^{a_n^d} n\,\big|\langle f, Y_\nu^n\rangle_{\mathbf{S}^{d-1}}\big|^2$$
$$+\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\nu=1}^{a^d_{n-2j}}(2n+d-2)$$
$$\times\left|\frac{2\left(n-j+\frac{d-2}{2}\right)}{\sigma_{d-1}}\int_{\mathbf{B}^d}f(x)\,P^{n-2,1}_{j-1\,\nu}(x)\,dx-\langle f, Y_\nu^{n-2j}\rangle_{\mathbf{S}^{d-1}}\right|^2\Bigg]$$
$$+\lambda\,\sigma_{d-1}\sum_{k=1}^{\infty}(4k+d-2)\left|\langle f,1\rangle_{\mathbf{S}^{d-1}}-\frac{(2k+d-1)}{\sigma_{d-1}}\int_{\mathbf{B}^d}f(x)\,P^{2k-2,1}_{k-1,1}(x)\,dx\right|^2.$$

Since

$$\int_{\mathbf{B}^d}\nabla f(x)\cdot\nabla f(x)\,dx=\lim_{\lambda\to+\infty}\frac{\|f\|^2_{\nabla,0}}{\lambda},$$

we have the following corollary.

Corollary 5.2. *For $f\in\mathcal{U}(W_0;\mathbf{B}^d)$,*

$$\frac{1}{\sigma_{d-1}}\int_{\mathbf{B}^d}\|\nabla f(x)\|^2\,dx=\sum_{n=1}^{\infty}\Bigg[\sum_{\nu=1}^{a_n} n\,\big|\langle f, Y_\nu^n\rangle_{\mathbf{S}^{d-1}}\big|^2$$
$$+\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\nu=1}^{a_{n-2j}}(2n+d-2)$$
$$\times\left|\frac{2\left(n-j+\frac{d-2}{2}\right)}{\sigma_{d-1}}\int_{B^d}f(x)\,P^{n-2,1}_{j-1\,\nu}(x)\,dx-\langle f, Y_\nu^{n-2j}\rangle_{\mathbf{S}^{d-1}}\right|^2\Bigg]$$
$$+\sum_{k=1}^{\infty}(4k+d-2)\left|\langle f,1\rangle_{\mathbf{S}^{d-1}}-\frac{(2k+d-1)}{\sigma_{d-1}}\int_{\mathbf{B}^d}f(x)\,P^{2k-2,1}_{k-1,1}(x)\,dx\right|^2.$$

Therefore, if $f(x)=(1-\|x\|^2)\,g(x)\in\mathcal{U}(W_0;\mathbf{B}^d)$,

$$\sigma_{d-1}\,b_1^2\int_{\mathbf{B}^d}\|\nabla f(x)\|^2\,dx$$
$$=4\sum_{n=1}^{\infty}\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\nu=1}^{a_{n-2j}}(2n+d-2)\left(n-j+\frac{d-2}{2}\right)^2\left|H^1_{j-1,n-2}\,\widehat{g}^{\,n-2,1}_{\,j-1\,\nu}\right|^2$$
$$+\sum_{k=1}^{\infty}(4k+d-2)\,(2k+d-1)^2\left|H^1_{k-1,2k-2}\,\widehat{g}^{\,2k-2,1}_{\,k-1,1}\right|^2.$$

We will denote the projection operator on $\mathcal{H}_n^d$ by $\operatorname{proj}_{\mathcal{H}_n^d}$. It is well known that

$$\operatorname{proj}_{\mathcal{H}_n^d} f(x) = \|x\|^2 \frac{n+\frac{d-2}{2}}{\frac{d-2}{2}} \frac{1}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} f(y)\, C_n^{(\frac{d-2}{2})}(x'\cdot y)\, d\sigma(y),$$

for $x \in \mathbf{B}^d$ and $x' = x/\|x\| \in \mathbf{S}^{d-1}$, where $C_n^{(\lambda)}(t)$ denotes the Gegenbauer polynomial of degree n and $x\cdot y$ is the usual dot product in $\mathbb{R}^d$. Moreover, we will denote by $\mathbb{P}_n^\mu(\cdot,\cdot)$ the reproducing kernel on $\mathcal{V}_n^d(W_\mu)$ given by

$$\mathbb{P}_n^\mu(x,y) = \sum_{j=0}^{\lfloor \frac{n}{2}\rfloor} \sum_{\nu=1}^{a_{n-2j}^d} \frac{P_{j,\nu}^{n,\mu}(x)\, P_{j,\nu}^{n,\mu}(y)}{H_{j,n}^\mu}.$$

Proposition 5.3. *For $f \in \mathcal{U}(W_0; \mathbf{B}^d)$ and $n \geqslant 1$,*

$$\begin{aligned}\operatorname{proj}_n^{\nabla,0} f(x) &= \operatorname{proj}_{\mathcal{H}_n^d} f(x)\\ &\quad + (1-\|x\|^2)\left[\frac{d(d/2+1)}{\sigma_{d-1}}\int_{\mathbf{B}^d} f(y)\mathbb{P}_{n-2}^1(x,y)\,dy\right.\\ &\quad \left. - \left(n+\frac{d-2}{2}\right)\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{1}{j} P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(2\|x\|^2-1)\operatorname{proj}_{\mathcal{H}_{n-2j}^d} f(x)\right]\\ &\quad + \widehat{f}_{\frac{n}{2},1}^{n,0}(\nabla)\, Q_{\frac{n}{2},1}^{n,0}(x),\end{aligned}$$

where the last term holds when n is even.

Consequently, if $f(x) = (1-\|x\|^2)g(x) \in \mathcal{U}(W_0; \mathbf{B}^d)$, then

$$\begin{aligned}\operatorname{proj}_n^{\nabla,0} f(x) &= (1-\|x\|^2)\operatorname{proj}_{n-2}^1 g(x)\\ &\quad + \frac{(n+d-2)\left(n+\frac{d-2}{2}\right)(n+d)}{8\,\sigma_{d-1}} \frac{H_{\frac{n}{2},n}^1}{b_1}\, \widehat{g}_{\frac{n}{2},1}^{n,1}\, Q_{\frac{n}{2},1}^{n,0}(x).\end{aligned}$$

Proof. From Proposition 4.1, we have $\widehat{f}_{0,\nu}^{n,0}(\nabla) = \langle f, Y_\nu^n\rangle_{\mathbf{S}^{d-1}}$. Then,

$$\sum_{\nu=1}^{a_n^d} \widehat{f}_{0,\nu}^{n,0}(\nabla)\, Y_\nu^n(x) = \sum_{\nu=1}^{a_n^d} \langle f, Y_\nu^n\rangle_{\mathbf{S}^{d-1}}\; Y_\nu^n(x) = \operatorname{proj}_{\mathcal{H}_n^d} f(x).$$

Again, from Proposition 4.1, we have that for $1 \leqslant j \leqslant \lfloor\frac{n-1}{2}\rfloor$,

$$\begin{aligned}\widehat{f}_{j,\nu}^{n,0}(\nabla) &= -\frac{n+\frac{d-2}{2}}{j}\langle f, Y_\nu^{n-2j}\rangle_{\mathbf{S}^{d-1}}\\ &\quad + \frac{2\left(n+\frac{d-2}{2}\right)\left(n-j+\frac{d-2}{2}\right)}{j\,\sigma_{d-1}} \int_{\mathbf{B}^d} f(x)\, P_{j-1,\nu}^{n-2,1}(x)\,dx.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\nu=1}^{a_{n-2j}^d} \widehat{f}_{j,\nu}^{n,0}(\nabla)\, Q_{j,\nu}^{n,0}(x)\\ &= -\left(n+\frac{d-2}{2}\right)(1-\|x\|^2)\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{1}{j}P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(2\|x\|^2-1)\operatorname{proj}_{\mathcal{H}_{n-2j}^d} f(x)\\ &+ 2\left(n+\frac{d-2}{2}\right)(1-\|x\|^2)\sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\sum_{\nu=1}^{a_{n-2-2j}^d}\frac{n-j-1+\frac{d-2}{2}}{j+1}\\ &\qquad\times\frac{1}{\sigma_{d-1}}\int_{\mathbf{B}^d} f(y)P_{j,\nu}^{n-2,1}(y)\,P_{j,\nu}^{n-2,1}(x)\,dy,\end{aligned}$$

where we have made the change $j-1 \to j$ in the last line. Using

$$H_{j,n-2}^1 = \frac{(j+1)(d/2)(d/2+1)}{(n+\frac{d-2}{2})(n-1-j+\frac{d-2}{2})},$$

we obtain

$$\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\nu=1}^{a^d_{n-2j}}\widehat{f}^{\,n,0}_{j,\nu}(\nabla)\,Q^{n,0}_{j,\nu}(x)$$

$$= -\left(n+\frac{d-2}{2}\right)(1-\|x\|^2)\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{1}{j}P^{(1,n-2j+\frac{d-2}{2})}_{j-1}(2\|x\|^2-1)\,\mathrm{proj}_{\mathcal{H}^d_{n-2j}}f(x)$$

$$+\ \frac{d\,(d/2+1)}{\sigma_{d-1}}(1-\|x\|^2)\int_{\mathbf{B}^d}f(y)\mathbb{P}^1_{n-2}(x,y)\,dy.$$

Note that

$$b_1=\frac{d\,(d/2+1)}{\sigma_{d-1}},$$

is the normalization constant for $W_1(x)$. □

The study of the interaction between differentiation and the partial sums $S^{\nabla,0}_n$ depends on the following proposition. First, we recall integration by parts in higher dimensions. From the Divergence Theorem and the product rule, if u is a real valued function and $\mathbf{V}$ is a vector field, then

$$\int_{\mathbf{S}^{d-1}}u\,\mathbf{V}\cdot\xi\,d\sigma(\xi)=\int_{\mathbf{B}^d}\nabla\cdot(u\,\mathbf{V})\,dx=\int_{\mathbf{B}^d}u\,\nabla\cdot\mathbf{V}\,dx+\int_{\mathbf{B}^d}\nabla u\cdot\mathbf{V}\,dx.$$

Therefore, the integration by parts formula on the unit ball is

$$\int_{\mathbf{B}^d}u\,\nabla\cdot\mathbf{V}\,dx=\int_{\mathbf{S}^{d-1}}u\,\mathbf{V}\cdot\xi\,d\sigma(\xi)-\int_{\mathbf{B}^d}\nabla u\cdot\mathbf{V}\,dx.$$

Proposition 5.4. *For $f\in\mathcal{U}(W_0;\mathbf{B}^d)$ and $m\geqslant 1$, we have*

$$\partial_i\,\mathrm{proj}^{\nabla,0}_m f(x)\in\mathcal{V}^d_{m-1}(W_0),\quad 1\leqslant i\leqslant d,$$

and

$$D_{i,j}\,\mathrm{proj}^{\nabla,0}_m f(x)\in\mathcal{V}^d_m(\nabla,W_0),\quad 1\leqslant i<j\leqslant d.$$

Proof. By the definition of $\mathrm{proj}^{\nabla,0}_m f(x)$, it is sufficient to show that $\partial_i\,Q^{m,0}_{j,\nu}(x)\in\mathcal{V}^d_{m-1}(W_0)$ for $0\leqslant j\leqslant\lfloor\frac{m}{2}\rfloor$, $1\leqslant\nu\leqslant a^d_{m-2j}$, and $1\leqslant i\leqslant d$.

Fix $i\in\{1,2,\ldots,d\}$. For $j=0$, we have

$$\partial_i Q^{m,0}_{0,\nu}(x)=\partial_i\,Y^m_\nu(x).$$

We compute

$$\left\langle\partial_i\,Y^m_\nu,P^{k,0}_{\ell,\eta}\right\rangle_0=b_0\int_{\mathbf{B}^d}\partial_i\,Y^m_\nu(x)\,P^{k,0}_{\ell,\eta}(x)\,dx.$$

Applying the integration by parts formula to $P^{k,0}_{\ell,\eta}(x)$ and the vector field $Y^m_\nu(x)\,\mathbf{e}_i$, where $\mathbf{e}_i$ is the ith canonical vector in $\mathbb{R}^n$, we obtain

$$\int_{\mathbf{B}^d}\partial_i\,Y^m_\nu(x)\,P^{k,0}_{\ell,\eta}(x)\,dx=P^{(0,k-2\ell+\frac{d-2}{2})}_\ell(1)\int_{\mathbf{S}^{d-1}}\xi_i\,Y^{k-2\ell}_\eta(\xi)\,Y^m_\nu(\xi)\,d\sigma(\xi)$$

$$-\int_{\mathbf{B}^d}\partial_i P^{k,0}_{\ell,\eta}(x)\,Y^m_\nu(x)\,dx.$$

The integral over $\mathbf{S}^{d-1}$ vanishes for $k\leqslant m-2$. Moreover, since $Y^m_\eta(x)=P^{m,0}_{0,\eta}(x)$, then the second integral on the right also vanishes for $k\leqslant m-2$. Therefore,

$$\left\langle\partial_i\,Y^m_\nu,P^{k,0}_{\ell,\eta}\right\rangle_0=0,\quad k\leqslant m-2.$$

Consequently, $\partial_i Q^{m,0}_{0,\nu}(x)\in\mathcal{V}^d_{m-1}(W_0)$.

For $1\leqslant j\leqslant\lfloor\frac{m-1}{2}\rfloor$, we have

$$\partial_i Q^{m,0}_{j,\nu}(x)=\partial_i\,(1-\|x\|^2)\,P^{m-2,1}_{j-1,\nu}(x).$$

From the integration by parts formula and the fact that $1-\|x\|^2$ vanishes on the sphere, we get

$$\int_{\mathbf{B}^d} \partial_i (1-\|x\|^2) P^{m-2,1}_{j-1,\nu}(x) P^{k,0}_{\ell,\eta}(x)\, dx = -\int_{\mathbf{B}^d} \partial_i P^{k,0}_{\ell,\eta}(x) P^{m-2,1}_{j-1,\nu}(x) W_1(x)\, dx.$$

Hence,

$$\left\langle \partial_i (1-\|x\|^2) P^{m-2,1}_{j-1,\nu}(x), P^{k,0}_{\ell,\eta}(x)\right\rangle_0 = 0, \quad k \leqslant m-2,$$

and, thus, $\partial_i Q^{m,0}_{j,\nu}(x) \in \mathcal{V}^d_{m-1}(W_0)$.

Finally, from (3.5), we have

$$\begin{aligned}\int_{\mathbf{B}^d} \partial_i Q^{m,0}_{\frac{m}{2},1}(x) P^{k,0}_{\ell,\eta}(x)\, dx &= 4\int_{\mathbf{B}^d} x_i P^{(0,\frac{d}{2})}_{\frac{m}{2}-1}(2\|x\|^2-1) P^{k,0}_{\ell,\eta}(x)\, dx \\ &= 4\int_{\mathbf{B}^d} P^{m-1,0}_{\frac{m}{2}-1,i}(x) P^{k,0}_{\ell,\eta}(x)\, dx,\end{aligned}$$

where we have used the fact that $Y^1_i(x) = x_i$. Therefore,

$$\left\langle \partial_i Q^{m,0}_{\frac{m}{2},1}, P^{k,0}_{\ell,\eta}\right\rangle_0 = 0, \quad k \leqslant m-2.$$

Hence, we conclude that $\partial_i Q^{m,0}_{\frac{m}{2},1}(x) \in \mathcal{V}^d_{m-1}(W_0)$.

Now, $D_{i,j}$ maps $\mathcal{H}^d_n$ to itself, and

$$D_{i,j}\, Q^{m,0}_{\frac{m}{2},1}(x) = 0, \quad 1 \leqslant i < j \leqslant d,$$

since $Q^{m,0}_{\frac{m}{2},1}(x)$ is a radial function. This implies that $D_{i,j}\,\mathrm{proj}^{\nabla,0}_m f(x) \in \mathcal{V}^d_{m-1}(\nabla, W_0)$. □

We use the previous result to show that differentiation commutes with the partial Fourier sum $S^{\nabla,0}_n$.

Proposition 5.5. *Let $\mu = 0$. Then,*

$$\partial_i S^{\nabla,0}_n f \;=\; S^0_{n-1}(\partial_i f), \quad 1 \leqslant i \leqslant d,$$

and

$$D_{i,j} S^{\nabla,0}_n f \;=\; S^{\nabla,0}_n(D_{i,j} f), \quad 1 \leqslant i < j \leqslant d.$$

Proof. By its definition, $f - S^{\nabla,0}_n f = \sum_{m=n+1}^{+\infty} \mathrm{proj}^{\nabla,0}_m f$. From Proposition 5.4 we get that $\langle \partial_i (f - S^{\nabla,0}_n f), P\rangle_0 = 0$ for all $P \in \Pi^d_{n-1}$. Consequently, $S^0_{n-1}(\partial_i f - \partial_i S^{\nabla,0}_n f) = 0$. Since S^0_{n-1} reproduces polynomials of degree at most $n-1$, then $S^0_{n-1}(\partial_i S^{\nabla,0}_n f) = \partial_i S^{\nabla,0}_n f$, which implies that

$$0 = S^0_{n-1}(\partial_i f - \partial_i S^{\nabla,0}_n f) = S^0_{n-1}(\partial_i f) - \partial_i S^{\nabla,0}_n f,$$

and the first commutation relation is proved. The second relation can be established in a similar way. □

The relation in the proposition above passes down to the Fourier coefficients.

Proposition 5.6. *Let $f \in \mathcal{U}(W_0; \mathbf{B}^d) \cap \mathcal{W}^2_2(W_1; \mathbf{B}^d)$. Then,*

$$\begin{aligned}\widehat{\Delta f}^{\,n-2,1}_{j,\nu} &= -4(j+1)\left(n-j-1+\frac{d-2}{2}\right)\widehat{f}^{\,n,0}_{j+1,\nu}(\nabla), \quad 0 \leqslant j \leqslant \lfloor \frac{n-3}{2} \rfloor, \\ \widehat{\Delta f}^{\,n-2,1}_{\frac{n-2}{2},1} &= 4(n+d)(n+d-1)\widehat{f}^{\,n,0}_{\frac{n}{2},1}(\nabla),\end{aligned}$$

where the last relation holds only when n is even. Moreover,

$$\begin{aligned}\widehat{\Delta_0 f}^{\,n,0}_{j,\nu}(\nabla) &= -(n-2j)(n-2j+d-2)\widehat{f}^{\,n,0}_{j,\nu}(\nabla), \quad 0 \leqslant j \leqslant \lfloor \frac{n-1}{2} \rfloor, \\ \widehat{\Delta_0 f}^{\,n,0}_{\frac{n}{2},1}(\nabla) &= 0.\end{aligned}$$

Proof. From $\mathrm{proj}^{\nabla,0}_n f = S^{\nabla,0}_n f - S^{\nabla,0}_{n-1} f$, Proposition 5.5, and Proposition 2.4, we obtain $\Delta\, \mathrm{proj}^{\nabla,0}_m f = \mathrm{proj}^1_{m-2}(\Delta f)$.

On the other hand, we have

$$\Delta\, \mathrm{proj}^{\nabla,0}_n = \sum_{\nu=1}^{a^d_{n-2j}} \widehat{f}^{\,n,0}_{0,\nu}(\nabla)\, \Delta Y^n_\nu(x)$$

$$+ \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\nu} \widehat{f}^{\,n,0}_{j,\nu}(\nabla)\, \Delta (1-\|x\|^2)\, P^{n-2,1}_{j-1,\nu}(x)$$

$$+ \frac{4}{n+d-2} \widehat{f}^{\,n,0}_{\frac{n}{2},1}(\nabla)\, \Delta\, P^{(-1,\frac{d-2}{2})}_{\frac{n}{2}}(2\|x\|^2-1).$$

Using $\Delta Y^n_\nu(x) = 0$, together with (5.1) and

$$\Delta(1-\|x\|^2)\, P^{n-2,1}_{j-1,\nu}(x) = -4j\left(n-j+\frac{d-2}{2}\right) P^{n-2,1}_{j-1,\nu}(x), \quad [4]$$

we obtain

$$\Delta\, \mathrm{proj}^{\nabla,0}_n = -\,4 \sum_{j=0}^{\lfloor \frac{n-3}{2} \rfloor} \sum_{\nu} (j+1)\left(n-j-1-\frac{d-2}{2}\right) \widehat{f}^{\,n,0}_{j+1,\nu}(\nabla)\, P^{n-2,1}_{j,\nu}(x)$$

$$+\, 4(n+d)(n+d-1)\widehat{f}^{\,n,0}_{\frac{n}{2},1}(\nabla)\, P^{n-2,1}_{\frac{n}{2}-1,1}(x).$$

Hence, by $\Delta\, \mathrm{proj}^{\nabla,0}_m f = \mathrm{proj}^1_{m-2}(\Delta f)$, the first result follows.

Similarly, using $D_{i,j}\, S^{\nabla,0}_n f = S^{\nabla,0}_n(D_{i,j} f)$, we get $\Delta_0 \mathrm{proj}^{\nabla,0}_n f = \mathrm{proj}^{\nabla,0}_n(\Delta_0 f)$. Then, using

$$\Delta_0\, Y(\xi) = -n(n+d-2)\, Y(\xi), \quad \forall Y \in \mathcal{H}^d_n, \quad \xi \in \mathbf{S}^{d-1}.$$

we get the second result. □

The main results of this section are the following theorems.

Theorem 5.7. *Let $s \geqslant 1$ be an integer and $f \in \mathcal{U}(W_0;\mathbf{B}^d) \cap \mathcal{W}^{2s}_2(W_1,\mathbf{B}^d)$. Then, for $n \geqslant 2s+2$,*

$$\mathcal{E}_n(f)_{\nabla,0} \leqslant \frac{c}{n^{2s-1}}\Big[\mathcal{E}_{n-2s-2}(\Delta^s f)_{2s+1} + \mathcal{E}_n(\Delta^s_0 f)_{\nabla,0}\Big],$$

and, consequently,

$$\mathcal{E}_n(f)_{\nabla,0} \leqslant \frac{c}{n^{2s-1}}\Big[\|\Delta^s f\|_{2s+1} + \|\Delta^s_0 f\|_{\nabla,0}\Big].$$

Proof. The Parseval identity reads,

$$\mathcal{E}_n(f)^2_{\nabla,0} = \|f - S^{\nabla,0}_n f\|^2_{\nabla,0} = \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu} \left|\widehat{f}^{\,m,0}_{j,\nu}(\nabla)\right|^2 \widetilde{H}^0_{j,m} = \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where we split the sum as

$$\Sigma_1 = \sum_{m=n+1}^{\infty} \sum_{j=\lfloor \frac{m}{4} \rfloor}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{\nu} \left|\widehat{f}^{\,m,0}_{j,\nu}(\nabla)\right|^2 \widetilde{H}^0_{j,m},$$

$$\Sigma_2 = \sum_{m=\lfloor \frac{n+2}{2} \rfloor}^{\infty} \left|\widehat{f}^{\,m,0}_{\frac{m}{2},1}(\nabla)\right|^2 \widetilde{H}^0_{\frac{m}{2},m},$$

$$\Sigma_3 = \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{4} \rfloor -1} \sum_{\nu} \left|\widehat{f}^{\,m,0}_{j,\nu}(\nabla)\right|^2 \widetilde{H}^0_{j,m}.$$

We estimate Σ_1 first. Using Proposition 5.6, we get

$$\left|\widehat{f}^{\,m,0}_{j,\nu}(\nabla)\right|^2 = \frac{1}{16 j^2 (m-j+\frac{d-2}{2})^2} \left|\widehat{\Delta f}^{\,m-2,1}_{j-1,\nu}\right|^2,$$

and iterating the first identity in Proposition 2.5, we obtain

$$\left|\widehat{f}^{\,m,0}_{j,\nu}(\nabla)\right|^2 = \frac{1}{16 j^2 (m-j+\frac{d-2}{2})^2} \prod_{i=1}^{s-1} \left(\kappa^{2i-1}_{m-j-1}\right)^{-2} \left|\widehat{\Delta^s f}^{\,m-2s-2,2s+1}_{j-s-1,\nu}\right|^2.$$

For $\lfloor \frac{m}{4} \rfloor \leqslant j \leqslant \lfloor \frac{m}{2} \rfloor$, we have $j \sim m$, and, thus

$$\left|\widehat{f}^{\,m,0}_{j,\nu}(\nabla)\right|^2 \sim \frac{1}{m^{4s}} \left|\widehat{\Delta^s f}^{\,m-2s-2,2s+1}_{j-s-1,\nu}\right|^2.$$

Furthermore,

$$\frac{\widetilde{H}^0_{j,m}}{H^{2s+1}_{j-s-1,m-2s-2}} = \frac{\widetilde{H}^0_{j,m}}{H^0_{j,m}}\,\frac{H^0_{j,m}}{H^1_{j-1,m-2}}\,\frac{H^1_{j-1,m-2}}{H^{2s+1}_{j-s-1,m-2s-2}}.$$

From (2.6) and (3.7), we have

$$\begin{aligned}\frac{\widetilde{H}^0_{j,m}}{H^0_{j,m}} &= \frac{4\,\lambda\,\sigma_{d-1}\,(m+\frac{d}{2})(m-j+\frac{d}{2})j^2}{d\,(m+\frac{d-2}{2})(m-j+\frac{d}{2})},\\ \frac{H^0_{j,m}}{H^1_{j-1,m-2}} &= \frac{(m-2+\frac{d}{2})(m-j+\frac{d-2}{2})}{(\frac{d}{2}+1)(m+\frac{d}{2})j},\\ \frac{H^1_{j-1,m-2}}{H^{2s+1}_{j-s-1,m-2s-2}} &= \frac{(2)_{2s}\,(m-j-s+\frac{d-2}{2})_s\,(m-j+\frac{d+2}{2})_s}{(1+\frac{d+2}{2})_{2s}\,(j-s)_s\,(j+1)_s}.\end{aligned}$$

It is easy to verify that when $j \sim m$,

$$\frac{\widetilde{H}^0_{j,m}}{H^{2s+1}_{j-s-1,m-2s-2}} \sim m^2.$$

Consequently, it follows that

$$\Sigma_1 \leqslant c \sum_{m=n+1}^{\infty} \sum_{j=\lfloor\frac{m}{4}\rfloor}^{\lfloor\frac{m-1}{2}\rfloor} \sum_{\nu} m^{-4s+2} \left|\widehat{\Delta^s f}^{\,m-2s-2,2s+1}_{j-s-1,\nu}\right|^2 H^{2s+1}_{j-s-1,m-2s-2}.$$

Similarly, we obtain

$$\Sigma_2 \leqslant c \sum_{m=\lfloor\frac{n+2}{2}\rfloor}^{\infty} m^{-4s+2} \left|\widehat{\Delta^s f}^{\,m-2s-2,2s+1}_{\frac{m}{2}-s-1,1}\right|^2 H^{2s+1}_{\frac{m}{2}-s-1,m-2s-2}.$$

Next, we estimate Σ_3. Iterating the identities involving Δ_0 in Proposition 5.6, we obtain

$$\left|\widehat{f}^{\,m,0}_{j,\nu}(\nabla)\right|^2 = \frac{1}{(m-2j)^{2s}\,(m-2j+d-2)^{2s}} \left|\widehat{\Delta_0^s f}^{\,m,0}_{j,\nu}(\nabla)\right|^2 \sim \frac{1}{m^{4s}} \left|\widehat{\Delta_0^s f}^{\,m,0}_{j,\nu}(\nabla)\right|^2,$$

for $0 \leqslant j \leqslant \lfloor\frac{m}{4}\rfloor$. Consequently, it follows that

$$\begin{aligned}\Sigma_3 &\leqslant c \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor\frac{m}{4}\rfloor-1} \sum_{\nu} m^{-4s} \left|\widehat{\Delta_0^s f}^{\,m,0}_{j,\nu}(\nabla)\right|^2 \widetilde{H}^0_{j,m}\\ &\leqslant \frac{c}{n^{4s}}\,\mathcal{E}_n(\Delta_0^s f)^2_{\nabla,0}.\end{aligned}$$

Putting these estimates together completes the proof of the theorem. □

5.1. Approximation behavior in terms of the fractional Laplace–Beltrami operator

In the proof of Theorem 5.7, we do not need to specify the basis of spherical harmonics in the definition of $Q^{n,0}_{j,\nu}$. It is far more complicated to give a bound for the error $\mathcal{E}_n(f)_{\nabla,0}$ in terms of derivatives of odd order involving Δ and Δ_0, for which we do need to specify the basis as in [11]. Thus, here we shall choose a more convenient distributional differential operator in order to avoid having to specify a basis.

Recall that the space $\mathcal{H}^d_n$ of spherical harmonics can be characterized as the eigenfunction space of the Laplace–Beltrami operator Δ_0 on $\mathbf{S}^{d-1}$:

$$\mathcal{H}^d_n = \left\{f \in C^2(\mathbf{S}^{d-1}) :\ -\Delta_0 f = n(n+d-2)f\right\}.$$

Therefore, we can define the fractional powers of $-\Delta_0$.

Definition 5.8. For $\alpha \in \mathbb{R}$, we define

$$(-\Delta_0)^{\alpha/2} f = \sum_{n=0}^{\infty} (n(n+d-2))^{\alpha/2}\,\mathrm{proj}_{\mathcal{H}^d_n} f.$$

It is shown in [13] that

$$\left\|(-\Delta_0)^{1/2} f\right\|_{\mathbf{S}^{d-1}} = \|\nabla_0 f\|_{\mathbf{S}^{d-1}},$$

where $\|\cdot\|_{\mathbf{S}^{d-1}}$ is the norm induced by $\langle\cdot,\cdot\rangle_{\mathbf{S}^{d-1}}$ and ∇_0 denotes the tangential gradient defined as

$$\nabla_0 f = \nabla F|_{\mathbf{S}^{d-1}} \quad \text{with} \quad F(x) = f\left(\frac{x}{\|x\|}\right), \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Theorem 5.9. *Let $s \geqslant 1$ be an integer and $f \in \mathcal{U}(W_0; \mathbf{B}^d) \cap \mathcal{W}_2^{2s}(W_1; \mathbf{B}^d)$. Then, for $n \geqslant 2s+3$,*

$$\mathcal{E}_n(f)_{\nabla,0} \leqslant \frac{c}{n^{2s-1}}\left[\sum_{i=1}^{d} \mathcal{E}_{n-2s-3}(\partial_i \Delta^s f)_{2s+2} + \mathcal{E}_n\left((-\Delta_0)^{1/2}\Delta_0^s f\right)_{\nabla,0}\right].$$

Consequently,

$$\mathcal{E}_n(f)_{\nabla,0} \leqslant \frac{c}{n^{2s-1}}\left[\sum_{i=1}^{d} \|\partial_i \Delta^s f\|_{2s+2} + \|(-\Delta_0)^{1/2}\Delta_0^s f\|_{\nabla,0}\right].$$

Proof. On one hand, from Lemma 4.6, we have

$$\mathcal{E}_{n-2s-2}(\Delta^s f)^2_{2s+1} \leqslant \frac{c}{n}\mathcal{E}_{n-2s-2}(\Delta^s f)^2_{\nabla,2s+2} \leqslant \frac{c}{n}\left[\sum_{i=1}^{d} \mathcal{E}_{n-2s-3}(\partial_i \Delta^s f)^2_{2s+2}\right].$$

On the other hand, we have

$$\begin{aligned}
\mathcal{E}_n\left(\Delta_0^s f\right)^2_{\nabla,0} &= \sum_{m=n+1}^{\infty}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\sum_{\nu}\left|\widehat{\Delta_0^s f}^{m,0}_{j,\nu}(\nabla)\right|^2 \widetilde{H}^0_{j,m}(\nabla)\\
&\leqslant \sum_{m=n+1}^{\infty}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\sum_{\nu}(m-2j)(m-2j+d-2)\left|\widehat{\Delta_0^s f}^{m,0}_{j,\nu}(\nabla)\right|^2 \widetilde{H}^0_{j,m}(\nabla)\\
&= \mathcal{E}_n\left((-\Delta_0)^{1/2}\Delta_0^s f\right)^2_{\nabla,0}.
\end{aligned}$$

The result follows from combining the inequalities above and Theorem 5.7. □

6. Numerical experiments

In this section we present numerical experiments to compare the approximation behavior of Fourier orthogonal expansions with respect to classical and Sobolev ball polynomials with $d = 2$ variables. To this end, we consider different functions defined on $\mathbf{B}^2$. For each function $f(x, y)$, we compute $S_n^{\mu} f$ and $S_n^{\nabla,0} f$ for different values of μ and n. The two approximations were compared by computing their respective root mean square error (RMSE) as follows. We generate a circular mesh consisting of 1441 points

$$\left\{(r_i\cos(\theta_j), r_i\sin(\theta_j)) : \ r_i = i/20,\ \theta_j = j\pi/36,\ 0 \leqslant i \leqslant 20,\ 0 \leqslant j \leqslant 71\right\}.$$

We set $z_{i,j} = f(r_i\cos(\theta_j), r_i\sin(\theta_j))$, and $\hat{z}_{i,j}$ equal to the value of the approximation (classical or Sobolev) at the same point, and compute the RMSE as:

$$\mathrm{RMSE}(S) = \left(\frac{(z_{0,0}-\hat{z}_{0,0})^2 + \sum_{i=1}^{20}\sum_{j=0}^{71}(z_{i,j}-\hat{z}_{i,j})^2}{1441}\right)^{1/2}.$$

where S denotes either $S_n^{\mu} f(x, y)$ or $S_n^{\nabla,0} f(x, y)$.

We consider three different continuous functions and provide figures showing their approximation overlapped with their graph. We also provide tables with the approximation error of S_n^{μ} and $S_n^{\nabla,0}$ for different values of μ and n. The figures and errors were obtained using Wolfram Mathematica. We point out that the approximation error in the Sobolev case seems to be smaller than the classical approximation error as the value of n gets large.

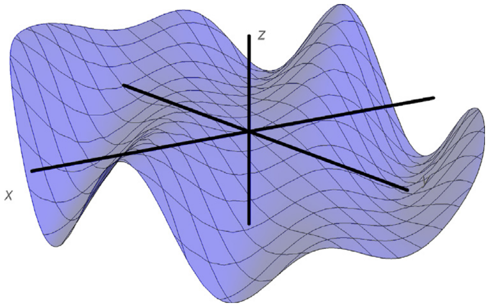

Fig. 1. Graph of $f(x,y)=x\sin(5x-6y)+y$.

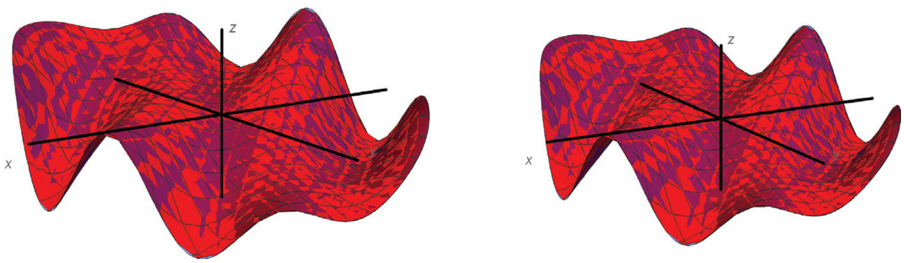

Fig. 2. Approximations overlapped with the graph of $f(x,y)$. Left: $S_{20}^{1}f(x,y)$. Right: $S_{20}^{\nabla,1}f(x,y)$.

Table 1
Approximation errors for $f(x,y)$.

μ	n	RMSE($S_n^{\mu}f(x,y)$)	RMSE($S_n^{\nabla,\mu}f(x,y)$)
0	5	0.29919	0.29001
	10	0.01235	0.01704
	15	1.26607×10^{-4}	1.74117×10^{-4}
	20	8.9264×10^{-7}	5.35549×10^{-7}
1	5	0.30721	0.29003
	10	0.01913	0.01196
	15	2.65266×10^{-4}	1.23246×10^{-4}
	20	3.54923×10^{-6}	9.9071×10^{-7}
1.5	5	0.32634	0.29171
	10	0.02730	0.01365
	15	4.22629×10^{-4}	1.63851×10^{-4}
	20	9.80412×10^{-6}	4.10402×10^{-6}
2	5	0.35935	0.30226
	10	0.03816	0.01882
	15	6.42325×10^{-4}	2.62762×10^{-4}
	20	4.32089×10^{-6}	6.58669×10^{-6}
3.5	5	0.53568	0.41311
	10	0.08591	0.05138
	15	1.73557×10^{-3}	9.31567×10^{-4}
	20	2.1150×10^{-4}	1.10783×10^{-4}

6.1. Example 1

First, we consider the function

$$f(x,y)=x\sin(5x-6y)+y.$$

The graph of $f(x,y)$ is shown in Fig. 1, and the approximations $S_{20}^{1}f(x,y)$ and $S_{20}^{\nabla,1}f(x,y)$ are shown in Fig. 2. We list the RMSE of both approximations for different values of n and μ in Table 1. For $\mu>0$, a seemingly faster rate of convergence can be observed for the Sobolev approximation. Nevertheless, for $\mu=0$, the rate of convergence of the Sobolev expansion does not seem to be much faster than the classical one. This is consistent with the theoretical rates of convergence that appear in Section 4.

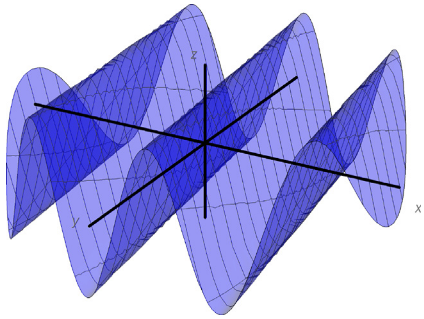

Fig. 3. Graph of $g(x, y) = \sin(10x + y)$.

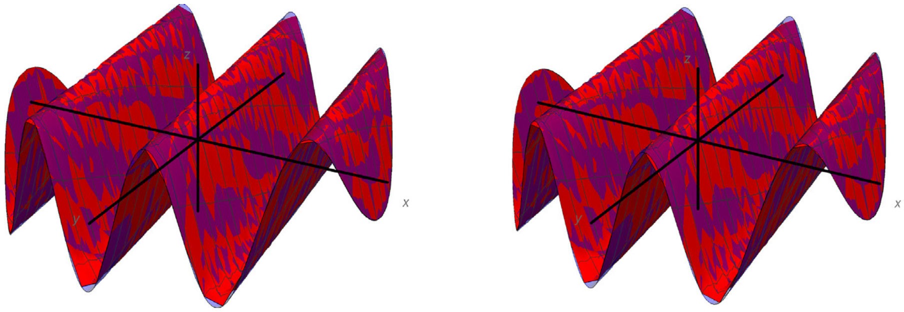

Fig. 4. Approximations overlapped with the graph of $g(x, y)$. Left: $S^{1}_{20}g(x, y)$. Right: $S^{\nabla,1}_{20}g(x, y)$.

Table 2
Errors of the expansions of $g(x, y)$.

μ	n	RMSE($S^{\mu}_{n}g(x, y)$)	RMSE($S^{\nabla,\mu}_{n}g(x, y)$)
0	5	0.69811	0.75427
	10	0.17290	0.20945
	15	7.21219×10^{-4}	8.09666×10^{-4}
	20	4.35377×10^{-6}	4.70028×10^{-6}
1	5	0.80258	0.69811
	10	0.32401	0.17290
	15	0.00210	0.00071
	20	1.59163×10^{-5}	4.35377×10^{-6}
1.5	5	1.01491	0.70814
	10	0.48232	0.21956
	15	0.00356	0.00117
	20	2.87009×10^{-5}	1.08525×10^{-5}
2	5	1.34532	0.80258
	10	0.69211	0.32401
	15	0.00568	0.00210
	20	5.24189×10^{-5}	1.59163×10^{-5}
3.5	5	2.89127	1.77811
	10	1.62760	0.95300
	15	0.01719	0.00856
	20	3.98356×10^{-4}	1.1693×10^{-4}

6.2. Example 2

Now, we consider the continuous sinusoidal function

$$g(x, y) = \sin(10x + y).$$

Its graph is shown in Fig. 3, and the approximations $S^{\mu}_{20}f(x, y)$ and $S^{\nabla,\mu}_{20}f(x, y)$ are shown in Fig. 4. We note that the approximation error in the classical and Sobolev case seems to be larger at the maximum and minimum values of the function. Table 2 shows the errors corresponding to the approximations of $g(x, y)$. Again, the rates of convergence seem to corresponding to the theoretical rates.

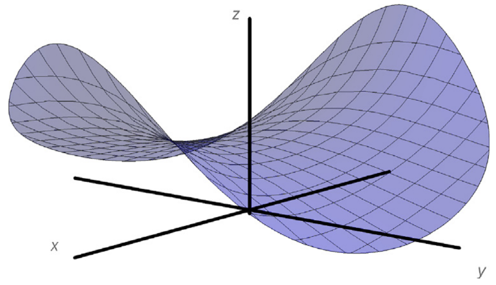

Fig. 5. Graph of $h(x,y) = e^{x^2-y^2} - xy$.

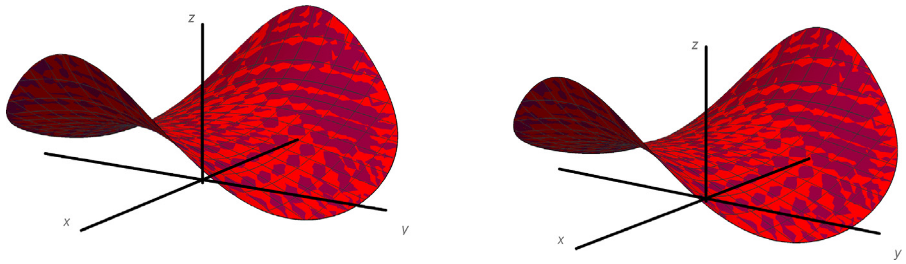

Fig. 6. Approximations overlapped with the graph of $h(x,y)$. Left: $S_{20}^{1} h(x,y)$. Right: $S_{20}^{\nabla,1} h(x,y)$.

Table 3
Errors of the expansions of $h(x,y)$.

μ	n	RMSE($S_n^{\mu} h(x,y)$)	RMSE($S_n^{\nabla,\mu} h(x,y)$)
0	5	0.01063	0.01141
	10	8.52813×10^{-6}	1.80036×10^{-5}
	15	8.41258×10^{-7}	1.50115×10^{-5}
	20	9.02632×10^{-7}	1.5485×10^{-5}
1	5	0.01097	0.01061
	10	8.7175×10^{-6}	8.5254×10^{-6}
	15	1.80786×10^{-7}	6.95056×10^{-7}
	20	1.42097×10^{-6}	6.95056×10^{-7}
1.5	5	0.01182	0.01098
	10	8.98758×10^{-6}	1.57057×10^{-5}
	15	2.99213×10^{-6}	1.3121×10^{-5}
	20	1.29664×10^{-5}	1.75648×10^{-5}
2	5	0.01218	0.01126
	10	9.2924×10^{-6}	8.7124×10^{-6}
	15	1.74896×10^{-6}	1.57222×10^{-6}
	20	2.61056×10^{-6}	1.57222×10^{-6}
3.5	5	0.01342	0.01235
	10	1.12976×10^{-5}	9.79411×10^{-6}
	15	1.27364×10^{-5}	9.79411×10^{-6}
	20	1.9156×10^{-5}	2.72889×10^{-5}

6.3. Example 3

Here, we consider the continuous function

$$h(x,y) = e^{x^2-y^2} - xy,$$

whose graph is shown in Fig. 7. Both approximations are shown in Fig. 8, and their respective RSME are listed in Table 4. Observe that, in this case, the RSME for both approximations are significantly smaller than in the previous examples. Note that the errors apparently do not change for a large n but this may be due to rounding errors (see Figs. 5 and 6 and Table 3).

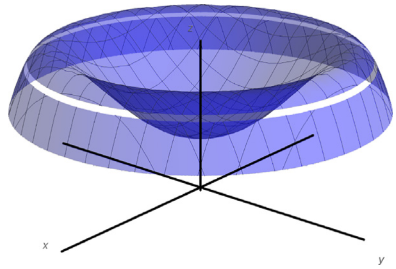

Fig. 7. Graph of $h(x, y) = q(x^2 + y^2)$.

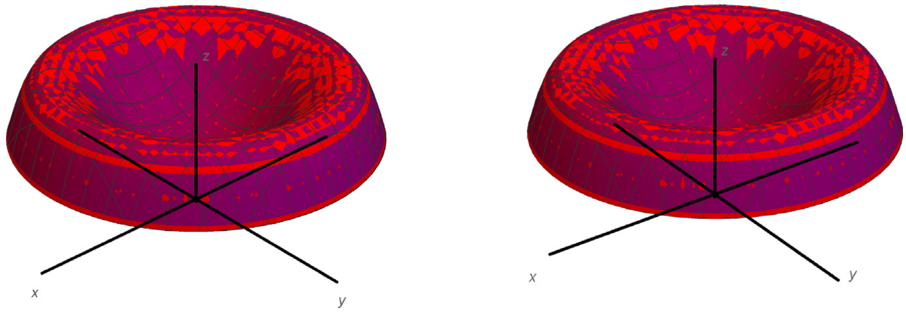

Fig. 8. Approximations overlapped with the graph of $h(x, y)$. Left: $S^1_{20}h(x, y)$. Right: $S^{\nabla,1}_{20}h(x, y)$.

Table 4
Approximation errors for $h(x, y)$.

μ	n	$\mathrm{RMSE}(S^{\mu}_n f(x, y))$	$\mathrm{RMSE}(S^{\nabla,\mu}_n f(x, y))$
0	5	0.032115	0.105139
	10	1.69904×10^{-3}	7.90909×10^{-3}
	15	2.81159×10^{-4}	8.211484×10^{-4}
	20	7.60244×10^{-5}	1.19557×10^{-4}
1	5	0.03692	0.06976
	7	9.17275×10^{-3}	7.2697×10^{-3}
	10	3.88087×10^{-3}	3.06606×10^{-3}
	15	2.59168×10^{-4}	4.45460×10^{-4}
	20	2.53382×10^{-4}	2.18760×10^{-4}
1.5	5	0.047003	0.060414
	10	5.42702×10^{-3}	2.71346×10^{-3}
	15	2.34768×10^{-4}	5.35595×10^{-4}
	20	6.85998×10^{-3}	1.81918×10^{-4}
2	5	0.059031	0.058599
	10	6.93792×10^{-3}	3.79857×10^{-3}
	15	9.35486×10^{-4}	4.16972×10^{-4}
	20	1.35988×10^{-3}	2.93696×10^{-4}
3.5	5	0.09692	0.08112
	10	0.010308	8.25693×10^{-3}
	15	4.58444×10^{-3}	1.9499×10^{-3}
	20	3.93952×10^{-3}	2.22019×10^{-3}

6.4. Example 4

Here, we consider the following univariate $\mathcal{C}^2$ spline defined on [0, 1] by:

$$q(t) = \begin{cases} -1.50391(t+0.8)^3 + 3.96995(t+0.8)^2 - 2.22067(t+0.8) + 0.35, & 0 \leqslant t \leqslant 0.8, \\ 5.41466(t-0.8)^3 - 3.2488(t-0.8)^2 - 1.06683(t-0.8) + 0.8, & 0.8 \leqslant t \leqslant 1. \end{cases}$$

Then, we construct the radially symmetric function $h(x, y) = q(x^2 + y^2)$ defined on $\mathbf{B}^2$ whose graph is shown in Fig. 7. The classical and Sobolev approximations are shown in Fig. 8, and their respective RSME are listed in Table 4. We remark that, contrary to the previous examples, $h(x, y)$ is not an analytic function. In spite of this, the approximation errors appear to behave similarly than in the previous examples.

Data availability

No data was used for the research described in the article.

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