FINITE ENERGY TRAVELING WAVES FOR THE GROSS-PITAEVSKII EQUATION IN THE SUBSONIC REGIME

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Abstract. In this paper we study the existence of finite energy traveling waves for the Gross-Pitaevskii equation. This problem has deserved a lot of attention in the literature, but the existence of solutions in the whole subsonic range was a standing open problem till the work of Mariş in 2013. However, such result is valid only in dimension 3 and higher. In this paper we first prove the existence of finite energy traveling waves for almost every value of the speed in the subsonic range. Our argument works identically well in dimensions 2 and 3.

With this result in hand, a compactness argument could fill the range of admissible speeds. We are able to do so in dimension 3, recovering the aforementioned result by Mariş. The planar case turns out to be more intricate and the compactness argument works only under an additional assumption on the vortex set of the approximating solutions.

1. Introduction

In this paper we are concerned with the Gross-Pitaevskii equation

$$i\partial_t \Psi = \Delta\Psi + \Psi\left(1 - |\Psi|^2\right) \text{ on } \mathbb{R}^d \times \mathbb{R} \tag{1.1}$$

when $d = 2$ or $d = 3$. Observe that this is no more than a Nonlinear Schrödinger Equation with a Ginzburg-Landau potential. The Gross-Pitaevskii equation was proposed in 1961 ([31, 47]) to model a quantum system of bosons in a Bose-Einstein condensate, via a Hartree-Fock approximation (see also [3, 6, 36, 37]). It appears also in other contexts such as the study of dark solitons in nonlinear optics ([39, 40]).

From the point of view of the dynamics, the Cauchy problem for the Gross-Pitaevskii equation was first studied in one space dimension by Zhidkov [51] and in dimension $d = 2, 3$ by Béthuel and Saut [13]. At least formally, equation (1.1) presents two invariants, namely:

- *Energy:*

$$\mathcal{E} = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\Psi|^2 + \frac{1}{4}\left(1 - |\Psi|^2\right)^2,$$

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- *Momentum:*
$$\mathbf{P} = \frac{1}{2}\int_{\mathbb{R}^d} \langle i\nabla\Psi, \Psi\rangle,$$
where $\langle f, g\rangle = Re(f)Re(g) + Im(f)Im(g)$.

See also [18, 24, 25, 38] and the references therein for more information on the dynamics of the Gross-Pitaevskii equation.

This paper is focused on the existence of traveling wave solutions to (1.1), that is, solutions in the form

$$\Psi(x,t) = \psi(x_1 - ct, \tilde{x}), \quad \tilde{x} = (x_2 \ldots x_d) \in \mathbb{R}^{d-1}, \tag{1.2}$$

where the parameter $c \in \mathbb{R}$ characterizes the speed of the traveling wave. Without any lack of generality we will consider $c > 0$ throughout the paper. By the ansatz (1.2) the equation for the profile ψ is given by

$$ic\,\partial_{x_1}\psi + \Delta\psi + \left(1 - |\psi|^2\right)\psi = 0. \tag{1.3}$$

The study of finite energy traveling waves for (1.1) has also implications in the dinamics of the equation. In particular, their pressence is an obstruction to scattering of solutions. Scattering of small energy solutions has been proved in [32, 33] for $d = 3$, and such result is not true in dimension $d = 2$. This latter fact may seem surprising for a defocusing Schrödinger equation; the reason is that finite energy solutions of (1.1) do not vanish at infinity.

Nontrivial finite energy traveling waves in dimension $d = 1$ are explicitly known, and they are uniquely given (up to rotation or translation) by the expression

$$\psi_c(x) = \sqrt{\frac{2-c^2}{2}}\tanh\left(\frac{\sqrt{2-c^2}}{2}x\right) + i\frac{c}{\sqrt{2}},$$

if $c < \sqrt{2}$. In the literature the function ψ_0 is called black soliton whereas ψ_c $(c \neq 0)$ receives the name of dark soliton. Their orbital and asymptotic stability has been studied, see [10, 11].

The problem of finding solutions to (1.3) in dimension $d \geq 2$ has a long story. In the pioneer work of Jones, Putterman and Roberts ([36, 37]), formal calculations and numerical analysis gave rise to a set of conjectures regarding existence, asymptotic behavior and stability of finite energy travelling waves: the so-called the Jones-Putterman-Roberts program. In particular, the existence of finite energy traveling waves is expected if and only if $c \in (0, \sqrt{2})$ (the sub-sonic case). The threshold value $c = \sqrt{2}$ comes from the linearization of the problem around the constant solutions of modulus 1. In a certain sense, those solutions correspond to local minima if $c < \sqrt{2}$.

In the last years much progress has been made to give rigorous proofs of those conjectures. Nontrivial finite energy traveling waves for supersonic speed $c > \sqrt{2}$ do not exist, see [26]. In dimension $d = 2$ this nonexistence result holds also for $c = \sqrt{2}$, see [30]. For general nonlinearities analogous results have been proved in [46].

Concerning the asymptotics of finite energy solutions, for any $d \geq 2$, finite energy solutions of (1.3) converge at infinity to a fixed complex number of modulus 1. By the phase invariance of the problem, we can assume that

$$\psi(x) \to 1 \text{ as } |x| \to +\infty. \tag{1.4}$$

A more precise asymptotic description of ψ is indeed available, see [27, 28, 29].

A very active field of research is the study of the location and dynamics of vortices, namely, the zeroes of the wave function ψ. The existence of multi-vortices traveling waves with small speed has been proved in dimension $d = 2$, see [15, 16, 43]. The existence of solutions with speeds close to $\sqrt{2}$ has been recently addressed in [44]. In dimension 3 there are traveling vortex rings ([2, 42]) as well as leapfrogging vortex rings, see [35].

At least formally, the Lagrangian associated to (1.3) is defined as:

$$I^c(\psi) = \mathcal{E}(\psi) - c\mathcal{P}(\psi) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\psi|^2 - c\mathcal{P}(\psi) + \frac{1}{4}\int_{\mathbb{R}^d} \left(1 - |\psi|^2\right)^2, \tag{1.5}$$

where $\mathcal{P}$ is the first component of the momentum $\mathbf{P}$ that, under suitable integrability conditions (and taken into account (1.4)) can be written as:

$$\mathcal{P}(\psi) := -\int_{\mathbb{R}^d} \partial_{x_1}(Im\Psi)(Re\Psi - 1). \tag{1.6}$$

A classical approach to prove existence of traveling waves (starting from [36, 37]) is a minimization procedure of the energy functional $\mathcal{E}$ under the constraint $\mathcal{P}(\psi) = p$ in a suitable functional space. This approach has been pursued in a number of papers, see for instance [8, 12] for the Gross-Pitaevskii equation and [17] for more general nonlinearities. A major difficulty in this strategy is to find a natural definition of the momentum for functions with finite energy, since the integrand in 1.6 might be non-integrable (see [12]). This approach has the advantage of providing orbital stability of the solutions found (more precisely, of the set of minimizers). As a drawback, the speed c appears as a Lagrange multiplier and is not under control. In particular the possibility of gaps in the subsonic range of velocity cannot be excluded with the constrained minimization approach (see [8]).

We shall also quote existence results for small values of c, see [13] in dimension 2 and [14] in dimension 3, but a complete existence result in the sub-sonic case remained for many years as a standing open problem. Finally, Mariş proved in [45] the existence result for any $c \in (0, \sqrt{2})$ in dimension $d \geq 3$. His approach is, summing up, to minimize $I^c(\psi)$ under a Pohozaev-type constraint. Once this is accomplished, Mariş proves that the corresponding Lagrange multiplier is 0, concluding the proof. This approach works also for more general nonlinearities with nonvanishing conditions at infinity, such as the cubic-quintic nonlinearity. As commented in [45], this minimization approach breaks down in dimension 2 because of different scaling properties: the infimum is 0 and is never attained.

One important tool in Mariş' argument is the use of the fiber $t \mapsto u_t$, where $u_t(x_1, \tilde{x}) = u(x_1, t\tilde{x})$. For instance, in dimension $d \geq 4$ all solutions correspond to a maximizer of I^c with respect to that fiber. In dimension 3, $I^c(u_t)$ is independent of t for any solution: the argument needs to be adapted, but still the use of the fiber is essential. Those cases have an analogy in the study of the Nonlinear Schödinger equation, see [5], [4], respectively. However, in dimension 2 this approach breaks down, and the fiber u_t seems of no use; $I^c(u_t)$ attains a minimum at $t = 1$ for any solution u.

One of the main motivations of this paper is to deal with the physically relevant $2D$ model where the existence of finite energy traveling waves in the full subsonic range is still an open problem. Our main result is the following:

Theorem 1.1. *There exists a subset $E \subset (0, \sqrt{2})$ of plein measure such that, for any $c \in E$, there exists a nontrivial finite energy solution of* (1.3) *ψ_c such that:*

(1) *For any $c_0 \in (0, \sqrt{2})$ there exists $\chi = \chi(c_0) > 0$ such that*
$$0 < I^c(\psi_c) \leq \chi \quad \textit{for all } c \in E, \ c \geq c_0;$$
(2) *$ind(\psi_c) \leq 1$, where $ind(\psi_c)$ stands for the Morse index of ψ_c, that is,*
$$\sup\{\textit{dim}\, Y : \ Y \subset C_0^\infty(\mathbb{R}^d) \ \textit{vector space}, \ (I^c)''(\psi_c)(\phi, \phi) < 0 \ \forall \, \phi \in Y\} \leq 1.$$

The proof deals directly with the Lagrangian I^c and is focused on searching critical points by using min-max arguments. Our proofs use several ingredients:

- Several regularization (or relaxation) techniques have been used in the literature to deal with the Gross-Pitaevskii equation ([13, 45]). Alternatively, some authors have proposed an approach by approximating domains, like flat tori, see [8, 12]. In this paper we choose the second approach, but we use as approximating domains the slabs:
$$\Omega_N = \Big\{(x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}, \quad -N < x_1 < N\Big\}, \ N \in \mathbb{N}. \tag{1.7}$$
In other words, we first use a mountain-pass argument to address the question of existence of solutions to the problem:
$$\begin{aligned} ic\partial_{x_1}\psi + \Delta\psi + \left(1 - |\psi|^2\right)\psi &= 0 \quad \text{on } \Omega_N, \\ \psi &= 1 \text{ on } \partial\Omega_N. \end{aligned} \tag{1.8}$$
The boundary condition is motivated by (1.4). This approach has several advantages. First, as Ω_N is bounded in the x_1 direction, Poincaré inequality holds and we can work on the space $1 + H_0^1(\Omega_N)$. As a consequence the momentum given by formula (1.6) is well defined. Secondly, as Ω_N is invariant along the variable $\tilde{x}$, a Pohozaev type inequality is satisfied without boundary terms (see Lemma 2.3). This allows us to avoid the problem of unfolding choices of tori, as in [8, 12].
- A second fundamental tool is an energy bound argument via monotonicity in order to control the energy of (PS) sequences for almost all values of c. This idea has been used many times in literature starting from [49]. The main point here is that we are able to obtain a *uniform bound on the energy for a subsequence of enlarging slabs* $\Omega_{k(N)}$. This is based in a key analytic argument, and it is fundamental in what follows. To the best of our knowledge, this abstract argument is completely new and could be of use in other frameworks where a monotonicity argument is used together with a relaxation procedure.

- The next step is to pass to the limit, and for that we need to deal with the problem of vanishing. Here we rely on arguments of [8], and we use in an essential way that ψ_N are solutions of (1.8). We can also exclude the concentration of solutions near the boundary of Ω_N, since the problem posed in the half-space

$$\begin{aligned} ic\partial_{x_1}\psi + \Delta\psi + \left(1-|\psi|^2\right)\psi &= 0 \quad \text{on } \mathbb{R}^d_+, \\ \psi &= 1 \quad \text{on } \partial\mathbb{R}^d_+, \end{aligned}$$

  does not admit nontrivial solutions.
- Finally, we use the arguments of [20] to obtain a Morse index bound of the solutions obtained. Roughly speaking, since our solutions come from a mountain pass argument, their Morse index is at most 1. This will be used in an essential way in the proof of Theorem 1.3.

With Theorem 1.1 in hand, one could ask whether we can pass to the limit and obtain a nontrivial solution for all values of $c \in (0, \sqrt{2})$. This is relatively easy, see Proposition 6.1. The problem here is to show that the limit solution has finite energy. Let us point out that the boundedness of the energy cannot be deduced only by using Pohozaev-type identities, and more delicate arguments are needed. We give two results on this aspect.

The only requirement of the next theorem is $d = 3$:

Theorem 1.2. *Assume that $d = 3$. Let $c \in (0, \sqrt{2})$, $c_n \in E$, $c_n \to c$, where E is the set given by Theorem 1.1. Let ψ_n be the finite energy solutions with speed c_n given by that theorem. Then there exists $\xi_n \in \mathbb{R}^d$ such that:*

$$\psi_n(\cdot - \xi_n) \to \psi \quad \textit{in } C^k_{loc}(\mathbb{R}^d),$$

where ψ is a nontrivial finite energy solution of (1.3) *with speed c.*

Observe that Theorems 1.1 and 1.2 give an alternative proof of the result of Mariş [45] for the Gross-Pitaevskii equation.

Under minor changes, Theorems 1.1 and 1.2 can be adapted to $d \geq 4$: the problem there is the fact that the term $(1-|\psi|^2)^2$ becomes critical or supercritical with respect to the Sobolev embedding. However, since this term has a positive sign in the functional, this issue could be fixed by changing suitably the functional setting, or, alternatively, by using a convenient truncation argument. For the sake of brevity we will not do so and restrict ourselves to the relevant spatial dimensions $d = 2$ or $d = 3$.

Regarding compactness of solutions, the case $d = 2$ is, again, more involved. It presents analytical difficulties and also topological obstructions, see Remarks 6.4, 6.8. In dimension 2 we are able to conclude only under some assumptions on the vortex set of the solutions:

Theorem 1.3. *Take $c \in (0, \sqrt{2})$, $c_n \in E$ with $c_n \to c$ and ψ_n the finite energy solutions with speed c_n given by Theorem 1.1. Assume that*

1. *either ψ_n are vortexless, that is, $\psi_n(x) \neq 0$ for all $x \in \mathbb{R}^d$,*
2. *or there exists $R > 0$, $\delta > 0$ such that:*

(1.9) $$\{x \in \mathbb{R}^d : \ \psi_n(x) = 0\} \subset B(0,R) \ \textit{and} \ |\psi_n(x)| \geq \delta \ \forall\, x \in \partial B(0,R).$$

Then there exists $\xi_n \in \mathbb{R}^d$ such that:

$$\psi_n(\cdot - \xi_n) \to \psi \quad \textit{in } C^k_{loc}(\mathbb{R}^d),$$

where ψ is a nontrivial finite energy solution of (1.3) with speed c.

The proofs of both Theorem 1.2 and Theorem 1.3 follow similar ideas, which include the following:

- A fundamental tool is the use of a lifting, that is, the existence of real functions $\rho_n(x)$, $\theta_n(x)$ such that $\psi_n = \rho_n e^{i\theta_n}$. This is always possible if the solutions are vortexless. If the solutions present vortices, one needs some information on the location of the vortex set. In Theorem 1.2 one can show that the vortices are included in a set of disjoint balls, and that the number of balls and their radius is bounded. Generally speaking, a nonvanishing function ψ admits a lifting if its domain is simply connected. Since the complement of a disjoint union of closed balls is simply connected if $d = 3$, we can find a lifting outside those balls. In dimension 2 this is no longer true, though, and we can use a lifting only in the complement of one ball, since the total degree of a finite energy solution is 0 (see [27]).
- We reason by contradiction assuming that $\mathcal{E}(\psi_n) \to +\infty$. A Pohozaev-type identity implies that
$$\sum_{k=2}^{d} \int_{\mathbb{R}^d} |\partial_{x_k} \psi_n|^2 = (d-1) I^{c_n}(\psi_n),$$
and $I^{c_n}(\psi_n)$ is bounded by Theorem 1.1. In our arguments we can pass to a limit (locally) which is a 1-D solution of the Gross-Pitaevskii equation (with finite or infinite energy). The knowledge of those 1-D solutions is essential at this point. For instance, in the proof of Theorem 1.3 we are able to obtain in the limit a circular solution $\psi(x_1) = \rho_0 e^{i\omega_0 x_1}$, with $\rho_0^2 < \frac{2}{3}(1 + c^2/4)$. But it turns out that such solution has infinite Morse index, and we reach a contradicion.

Under minor changes, it is possible to adapt the results of this paper to an equation with more general nonlinearities, namely:

$$ic\partial_{x_1}\psi + \Delta\psi + F(|\psi|)\psi = 0 \quad \text{on } \mathbb{R}^d.$$

Several assumptions on the nonlinearity F would be in order. However, for the sake of brevity and clarity, we have preferred to focus on the prototype model of the Gross-Pitaevskii equation in this paper.

The rest of the paper is organized as follows. Section 2 is devoted to the setting of the notation and some preliminary results. In Section 3 we begin the proof of Theorem 1.1 by considering problem (1.8) from a variational point of view. A main issue here is that we are not able to show that (PS) sequences have bounded energy. This problem is solved for almost all values of c via the monotonicity trick of Struwe in Section 4.

We are able to find sequences of slabs $\Omega_{k(N)}$ for which those solutions have uniformly bounded energy. In Section 5 we pass to the limit avoiding vanishing or concentration on the boundary, concluding the proof of Theorem 1.1. Sections 6 and 7 are devoted to the proofs of Theorems 1.2, 1.3, respectively. The appendix deals with the Morse index computation of the 1-D circular solutions of the Gross-Pitaevskii equation, which is needed in the conclusion of Theorem 1.3.

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2. Preliminaries

In this section we collect some well-known properties of solutions of the Gross-Pitaevskii equation. We begin by stablishing the notation that we will use throughout the paper.

Notation: We denote by $\langle z_1, z_2 \rangle$ the real scalar product of two elements in $\mathbb{C}$, that is, $\langle z_1, z_2 \rangle = Re(z_1 \overline{z_2})$. We denote instead by $\xi_1 \cdot \xi_2$ the real scalar product in $\mathbb{R}^d$, to avoid confusion.

We shall use the letter ψ for complex valued functions, and we will denote its real and imaginary part by u and v, respectively, so that $\psi = u + iv$. Moreover, we will write ρ to denote its modulus, that is, $\rho^2 = u^2 + v^2 = \langle \psi, \overline{\psi} \rangle$.

We denote the partial derivatives by $\partial_{x_1}\psi$, but sometimes we will use ψ_{x_1} for convenience.

In next lemma we are concerned with the regularity of solutions and the uniform boundedness of their derivatives.

Lemma 2.1. *Any solution ψ of (1.3) or (1.8) is of class C^∞ and, for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that $|D^k\psi(x)| \leq C_k$ for any $x \in \mathbb{R}^d$.*

The above result is well-known. The starting point is the L^∞ estimate:

$$\|\psi\|_{L^\infty} \leq \sqrt{1 + c^2/4}.$$

This was proved in [22] for all entire solutions of (1.3) (not only those with finite energy). The argument works equally well for problem (1.8) since the boundary condition is compatible with the L^∞ bound. From this, one can obtain the result via local elliptic regularity estimates.

Indeed the solutions are analytic, see [8][Theorem 2.1] for more details.

Next lemma gives a Pohozaev identity:

Lemma 2.2. *Let ψ be a finite energy solution of (1.3). Then:*

$$\frac{d-2}{2} \int_{\mathbb{R}^d} |\nabla \psi|^2 - (d-1)c\mathcal{P}(\psi) + \frac{d}{4} \int_{\mathbb{R}^d} \left(1 - |\psi|^2\right)^2 = 0.$$

Proof. See for instance [26], or [8, Lemma 2.5 and following]. □

Next identity is also of Pohozaev-type, but only uses the invariance of the domain by dilations in the $\tilde{x}$ variable:

Lemma 2.3. *Let ψ be a finite energy solution of either* (1.3) *or* (1.8). *Then the following identity holds:*

$$(d-3)A(\psi) + (d-1)B(\psi) = 0,$$

where

$$A(\psi) = \frac{1}{2}\sum_{j=2}^{d}\int |\partial_{x_j}\psi|^2$$

and

$$B(\psi) = \frac{1}{2}\int |\partial_{x_1}\psi|^2 + \frac{1}{4}\int \left(1-|\psi|^2\right)^2 - c\mathcal{P}(\psi).$$

Moreover, by the definition of the Lagrangian (1.5), *we conclude that*

$$I^c(\psi) = \frac{2}{d-1}A(\psi) \geq 0. \tag{2.1}$$

Finally, $I^c(\psi) = 0$ if and only if ψ is a constant function of modulus 1.

Proof. The case of (1.3) has actually been proved in [26][Proposition 5], taking into account [8][Lemma 2.5] (see also [46, Proposition 4.1] or Section 4 in [45]). The case of the domain Ω_N is completely analogous and is based on the fact that the dilations $(x_1, \tilde{x}) \mapsto (x_1, \lambda\tilde{x})$ leave the domain Ω_N invariant. □

The following decay estimate has been proved in [27]:

Lemma 2.4. *Let ψ be a finite energy solution of* (1.3) *satisfying* (1.4). *Then the following asymptotics hold:*

$$|v(x)| \leq \frac{K}{1+|x|^{d-1}}, \quad |u(x)-1| \leq \frac{K}{1+|x|^d},$$

$$|\nabla v(x)| \leq \frac{K}{1+|x|^d}, \quad |\nabla u(x)| \leq \frac{K}{1+|x|^{d+1}}.$$

Outside a ball $B(0,R)$ containing all vortices, ψ can be lifted as $\psi = \rho e^{i\theta}$. Then the above decay estimates can be written as:

$$|\theta(x)| \leq \frac{K}{1+|x|^{d-1}}, \quad |\rho(x)-1| \leq \frac{K}{1+|x|^d},$$

$$|\nabla\theta(x)| \leq \frac{K}{1+|x|^d}, \quad |\nabla\rho(x)| \leq \frac{K}{1+|x|^{d+1}}.$$

In particular, the definition (1.6) *of the momentum is well defined for any finite energy solution of* (1.3).

We now define the Morse index of a solution of (1.3):

Definition 2.5. *Let ψ be a solution of* (1.3) *(either with finite or infinite energy). We define its Morse index $ind(\psi)$ as:*

$$\sup\{dim\, Y : \; Y \subset C_0^\infty(\mathbb{R}^d) \text{ vector space, } Q(\phi) < 0 \;\forall\, \phi \in Y\},$$

where

$$Q(\phi) = \int_{\mathbb{R}^d} |\nabla \phi|^2 - c\langle \phi, i\partial_{x_1}\phi\rangle - (1-|\psi|^2)|\phi|^2 + 2\langle \phi, \psi\rangle^2. \tag{2.2}$$

If that set is not bounded from above, we will say that its Morse index is $+\infty$.

Observe that, at least formally, $Q(\phi) = (I^c)''(\psi)[\phi, \phi]$, and hence the Morse index is nothing but the maximal dimension for which $(I^c)''(\psi)$ is negative definite.

Remark 2.6. *An useful property of the so-defined Morse index is that it is decreasing under convergence in compact sets. Being more specific, assume that ψ_n is a sequence of solutions of* (1.3) *or* (1.8)*. Assume also that $ind(\psi_n) \leq m$ and ψ_n converges to ψ_0 in C^1_{loc} sense. Then $ind(\psi_0) \leq m$. This property will be essential, in particular, in the proof of Theorem 1.3.*

Indeed, assume that $ind(\psi_0) > m$; this implies that there exists $E \subset C_0^\infty(\mathbb{R}^d)$ with $dim\; E = m + 1$ and such that ψ_0 is negative definite in E. If ψ_n converges to ψ_0 in C^1_{loc} sense, by compactness we obtain that ψ_n is also negative definite in E for large n, and hence $ind(\psi_n) > m$.

3. The variational approach of Problem (1.8)

We first recall the definition of Ω_N (1.7) and observe that in the Sobolev Space $H_0^1(\Omega_N)$ the Poincaré inequality holds:

$$\int_{\Omega_N} |\phi|^2 \leq C_N \int_{\Omega_N} |\nabla \phi|^2 \quad \forall\; \phi \in H_0^1(\Omega_N). \tag{3.1}$$

If we combine this with the Sobolev inequality we obtain that

$$\|\phi\|_{L^p} \leq C_N \|\nabla \phi\|_{L^2}, \quad \begin{cases} p \in [2,6] & \text{if } d = 3, \\ p \geq 2 & \text{if } d = 2. \end{cases} \tag{3.2}$$

Let us define the action functional I^c_N as the Lagrangian I^c defined in (1.5) restricted to the affine space $1 + H_0^1(\Omega_N)$, that is,

$$I^c_N(\psi) := \mathcal{E}(\psi) - c\mathcal{P}(\psi) = \frac{1}{2}\int_{\Omega_N} |\nabla \psi|^2 - c\mathcal{P}(\psi) + \frac{1}{4}\int_{\Omega_N} \left(1 - |\psi|^2\right)^2,$$

with

$$\mathcal{P}(u + iv) = -\int_{\Omega_N} (u(x) - 1)\partial_{x_1} v(x).$$

Observe that, integrating by parts, we obtain:

$$(I^c_N)'(\psi)(\phi) = \int_{\Omega_N} \langle \nabla \psi, \, \nabla \phi\rangle - c\langle i\partial_{x_1}\psi, \, \phi\rangle - (1 - |\psi|^2)\langle \psi, \, \phi\rangle,$$

for any $\phi \in H_0^1(\Omega_N)$. In particular the action functional I_N^c is C^2 (actually, it is C^∞) in $H_0^1(\Omega_N)$.

Our aim is to prove the existence of critical points of the action functional where the velocity parameter if fixed; these critical points correspond to solution to (1.8). Let us point out that $H_0^1(\Omega_N)$ is included in $H_0^1(\Omega_{N'})$ if $N' > N$ (up to extension by 0).

Our strategy is to prove that I_N^c has a mountain pass geometry on $1 + H_0^1(\Omega_N)$. More precisely we aim to prove that

$$\gamma_N(c) := \inf_{g\in\Gamma} \max_{t\in[0,1]} I_N^c(g(t)) > 0, \tag{3.3}$$

where

$$\Gamma(N) = \{g \in C([0,1], (1 + H_0^1(\Omega_N)) : \ g(0) = 1, \ g(1) = \psi_0\}, \tag{3.4}$$

where ψ_0 is chosen so that $I_N^c(\psi_0) < 0$.

Proposition 3.1. *Given any $c_0 \in (0, \sqrt{2})$, there exist $N_0 > 0$, $\psi_0 \in 1 + H_0^1(\Omega_{N_0})$ and $\chi(c_0) > 0$ such that $\forall N \geq N_0$, $c \in [c_0, \sqrt{2})$:*

a) $I_N^c(\psi_0) < 0$.
b) $0 < \gamma_N(c) \leq \chi(c_0)$.

Proof. We can write the action functional $I_N^c(\psi)$, as:

$$I_N^c(\psi) = \int_{\Omega_N} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2 - c(1-u)\partial_{x_1} v + \frac{1}{4}(2(u-1) + (u-1)^2 + v^2)^2.$$

Moreover we have the elementary inequality $cxy \leq \frac{c^2}{4}x^2 + y^2$, so that

$$\begin{aligned} I_N^c(\psi) &\geq \int_{\Omega_N} \frac{1}{2}|\nabla u|^2 + \left(\frac{1}{2} - \frac{c^2}{4}\right)|\nabla v|^2 - (u-1)^2 + \frac{(2(u-1) + (u-1)^2 + v^2)^2}{4} \\ &\geq \int_{\Omega_N} \frac{1}{2}|\nabla u|^2 + \left(\frac{1}{2} - \frac{c^2}{4}\right)|\nabla v|^2 - |u-1|^3 - |u-1|v^2. \end{aligned}$$

By using Holder inequality and (3.2), we obtain:

$$I_N^c(\psi) \geq \left(\frac{1}{2} - \frac{c^2}{4}\right)||\psi - 1||^2_{H_0^1(\Omega_N)} - K||\psi - 1||^3_{H_0^1(\Omega_N)},$$

and hence $\psi = 1$ is a local minimum of the action functional whenever $c^2 < 2$.

In [45], Lemma 4.4, a compactly supported function ϕ_0 is found so that $I^{c_0}(1+\phi_0) < 0$. So it suffices to take sufficiently large N_0 such that $\Omega_N \supset supp\, \psi_0$, to obtain a).

Finally, define $\gamma_0(t) = 1 + t\phi_0$, which obviously belongs to $\Gamma(N)$ for all $N \geq N_0$. Observe that:

$$I_N^c(\gamma_0(t)) = \mathcal{E}(\gamma_0(t)) - c\, t^2 \mathcal{P}(\psi_0).$$

As commented above $I_N^c(\psi_0) < 0$, which implies that $\mathcal{P}(\psi_0) > 0$. Hence, for all $c \geq c_0$,

$$I_N^c(\gamma_0(t)) \leq I_N^{c_0}(\gamma_0(t)) \leq \max_{t\in[0,1]} I_{N_0}^{c_0} \circ \gamma_0(t) = \chi(c_0),$$

by definition. As a consequence, $\gamma_N(c) \leq \chi(c_0)$ for all $N \geq N_0$, $c \geq c_0$. □

It is standard (see for instance [1, 50]) that the mountain pass geometry induces the existence of a Palais-Smale sequence at the level γ_N. Namely, a sequence ψ_n such that

$$I_N^c(\psi_n) = \gamma_N(c) + o(1), \quad ||(I_N^c)'(\psi_n)||_{H_0^{-1}(\Omega_N)} = o(1).$$

It is not clear if such Palais-Smale sequences are bounded or not; this is one of the main difficulties. The question of the existence of Palais-Smale sequences with bounded energy for almost all values of c will be addressed in next section. In what follows we show that, if bounded, such sequences give rise to critical points of I_N^c.

Lemma 3.2. *Let $d = 2, 3$ and $\{Q_j\}$ be the set of disjoint unitary cubes that covers Ω_N. Assume that $\psi_n = u_n + iv_n$ is a bounded vanishing sequence in $1 + H_0^1(\Omega_n)$, i.e. such that*

$$\sup_j \int_{Q_j} |u_n - 1|^p + |v_n|^p \to 0,$$

for some $2 \leq p < \infty$ if $d = 2$, $2 \leq p < 6$ if $d = 3$. Then

$$\int_{\Omega_N} |u_n - 1|^r + |v_n|^r \to 0,$$

for any $2 < r < \infty$ if $d = 2$, $2 < r < 6$ if $d = 3$.

Proof. The proof is standard, see e.g [41, Lemma I.1]. □

Proposition 3.3 (Splitting property)**.** *Given $0 < c < \sqrt{2}$ and ψ_n a bounded Palais-Smale sequence at the energy level $\gamma_N(c)$. Then there exist k sequences of points $\left\{y_n^j\right\} \subset \{0\} \times \mathbb{R}^{d-1}$, $1 \leq j \leq k$, with $|y_n^j - y_n^k| \to +\infty$ if $j \neq k$, such that, up to subsequence,*

$$\psi_n - 1 = w_n + \sum_{j=1}^{k} (\psi^j(\cdot - y_n^j) - 1) \text{ with } w_n \to 0 \text{ in } H_0^1(\Omega_N), \tag{3.5}$$

$$||\psi_n - 1||^2_{H_0^1(\Omega_N)} \to \sum_{j=1}^{k} ||\psi^j - 1||^2_{H_0^1(\Omega_N)}, \tag{3.6}$$

$$I_N^c(\psi_n) \to \sum_{j=1}^{k} I_N^c(\psi^j), \tag{3.7}$$

where ψ^j are nontrivial finite energy solutions to (1.8)*.*

In particular $I_N^c(\psi^j) \leq \gamma_N(c) \leq \chi(c_0)$, $\mathcal{E}(\psi^j) \leq \limsup_{n \to +\infty} \mathcal{E}(\psi_n)$ for all $j = 1, \dots k$.

Proof. In this proof, for the sake of clarity, we drop the dependence on c.

We first claim that ψ_n is not vanishing. Reasoning by contradiction, by means of Lemma 3.2 we have

$$\int_{\Omega_N} |u_n - 1|^r + |v_n|^r \to 0, \tag{3.8}$$

for any $2 < r < \infty$ if $d = 2$, $2 < r < 6$ if $d = 3$.
Then,

$$\int_{\Omega_N} (1 - u_n^2 - v_n^2)^2 = \int_{\Omega_N} 4(u_n - 1)^2 + (u_n - 1)^4 + v_n^4 + \\ + \int_{\Omega_N} 4(u_n - 1)^3 + 4(u_n - 1)v_n^2 + 2(u_n - 1)^2 v_n^2,$$

and it follows by Hölder and (3.8) that

$$\int_{\Omega_N} (1 - u_n^2 - v_n^2)^2 = \int_{\Omega_N} 4(u_n - 1)^2 + o(1). \tag{3.9}$$

Therefore

$$I_N(u_n, v_n) = \frac{1}{2}\int_{\Omega_N} |\nabla u_n|^2 + \frac{1}{2}\int_{\Omega_N} |\nabla v_n|^2 - c\int_{\Omega_N} (1 - u_n(x))\partial_{x_1} v_n(x) + \\ + \int_{\Omega_N} (u_n - 1)^2 + o(1). \tag{3.10}$$

On the other hand direct computation gives

$$o(1) = I_N'[\psi_n](1 - \psi_n) = \int_{\Omega_N} |\nabla u_n|^2 + |\nabla v_n|^2 - 2c\int_{\Omega_N} (1 - u_n)\partial_{x_1} v_n(x) - \\ + \int_{\Omega_N} (1 - u_n^2 - v_n^2)(u_n(1 - u_n) - v_n^2).$$

Arguing as before we notice that

$$\int_{\Omega_N} (1 - u_n^2 - v_n^2)v_n^2 = o(1),$$

and, thanks to Hölder inequality and (3.9)

$$\int_{\Omega_N} (1 - u_n^2 - v_n^2)(u_n(1 - u_n)) = \int_{\Omega_N} (1 - u_n^2 - v_n^2)(u_n - 1 + 1)(1 - u_n)) \\ = \int_{\Omega_N} (1 - u_n^2 - v_n^2)(1 - u_n) + o(1) = \int_{\Omega_N} (1 - u_n^2)(1 - u_n) + o(1) = \\ \int_{\Omega_N} (2(1 - u_n) - (1 - u_n)^2)(1 - u_n) + o(1) = 2||u_n - 1||^2_{L^2(\Omega_N)} + o(1).$$

We get hence that

$$I_N'[\psi_n](1-\psi_n) = \int_{\Omega_N} |\nabla \psi_n|^2 - 2c\int_{\Omega_N} (1-u_n)\partial_{x_1} v_n(x) + 2||u_n - 1||^2_{L^2(\Omega_N)} + o(1). \tag{3.11}$$

Taken into account (3.10) and(3.11) we conclude

$$\gamma(N) + o(1) = I_N(\psi_n) - \frac{1}{2}I_N'[\psi_n](1 - \psi_n) = o(1),$$

a contradiction.

Once vanishing is excluded, there exists a sequence $y_n^1 \in \{0\} \times \mathbb{R}^{d-1}$ and $\psi^1 \in 1 + H_0^1(\Omega_N)$, $\psi^1 \neq 1$, such that

$$\psi_n(\cdot + y_n^1) - \psi^1 \rightharpoonup 0 \text{ in } H_0^1(\Omega_N),$$

up to a subsequence. By translation invariance we can assume that $y_n^1 = 0$, and define $z_n^1 = \psi_n - \psi^1$.

From the definition of weak convergence we obtain that

$$\frac{1}{2}\int_{\Omega_N} |\nabla \psi_n|^2 - c\mathcal{P}(\psi_n) = \frac{1}{2}\int_{\Omega_N} |\nabla \psi^1|^2 - c\mathcal{P}(\psi^1) +$$

$$+\frac{1}{2}\int_{\Omega_N} |\nabla z_n^1|^2 - c\mathcal{P}(1 + z_n^1) + o(1).$$

Now we notice that the nonlinear term fulfills the following splitting property

$$\int_{\Omega_N} \left(1 - |\psi_n|^2\right)^2 = \int_{\Omega_N} \left(1 - |\psi^1|^2\right)^2 + \int_{\Omega_N} \left(1 - |1 + z_n^1|^2\right)^2 + o(1).$$

As a consequence the action and the norm split as

$$\|\psi_n - 1\|^2 = \|\psi^1 - 1\|^2 + \|z_n^1\|^2 + o(1), \tag{3.12}$$

$$I_N(\psi_n) = I_N(\psi^1) + I_N(1 + z_n^1) + o(1). \tag{3.13}$$

Clearly ψ^1 is a weak solution of (1.8). Now if $z_n^1 \to 0$ in $H_0^1(\Omega_N)$ the proposition is proved. Let us assume the contrary, i.e. that $z_n^1 \rightharpoonup 0$ but $z_n^1 \nrightarrow 0$ in $H_0^1(\Omega_N)$.

We aim to prove that there exists a sequence of points $y_n^2 \in \{0\} \times \mathbb{R}^{d-1}$, $|y_n^2| \to +\infty$, and $\psi^2 \in H_0^1(\Omega_N)$, $\psi^2 \neq 1$, such that $z_n^1(\cdot + y_n^2) + 1 - \psi^2 \rightharpoonup 0$. Let us argue again by contradiction assuming that the sequence z_n^1 vanishes which means by Lemma 3.2 that

$$\int_{\Omega_N} |u_n - u^1|^r + |v_n - v^1|^r \to 0,$$

for any $2 < r < \infty$ if $d = 2$, $2 < r < 6$ if $d = 3$, where $\psi^1 = u^1 + iv^1$. We have

$$I'_N[\psi_n](1 - \psi_n) = \int_{\Omega_N} |\nabla u_n|^2 + |\nabla v_n|^2 - 2c\int_{\Omega_N} (1 - u_n)\partial_{x_1} v_n(x) +$$

$$+\int_{\Omega_N} (1 - u_n^2 - v_n^2)(u_n(1 - u_n) - v_n^2) = o(1),$$

and

$$I'_N[\psi^1](1 - \psi^1) = \int_{\Omega_N} |\nabla u^1|^2 + |\nabla v^1|^2 - 2c\int_{\Omega_N} (1 - u^1)\partial_{x_1} v^1(x) -$$

$$+\int_{\Omega_N} (1 - (u^1)^2 - (v^1)^2)(u^1(1 - u^1) - (v^1)^2) = 0.$$

Using the splitting property (3.13) and using $I'_N[\psi_n](1 - \psi_n) - I'_N[\psi^1](1 - \psi^1) = o(1)$, we get

$$\int_{\Omega_N} |\nabla z_n^1|^2 - 2c\int_{\Omega_N} Re(z_n^1)\partial_{x_1} Im(z_n^1(x)) + o(1) =$$

$$\underbrace{\int_{\Omega_N} (1-u_n^2-v_n^2)(u_n(u_n-1)+v_n^2) - \int_{\Omega_N} (1-(u^1)^2-(v^1)^2)(u^1((u^1)-1)+(v^1)^2)}_{=I_4}.$$

Now, using the elementary inequality $cxy \leq \frac{c^2}{4}x^2+y^2$ we have

$$\int_{\Omega_N} |\nabla Re(z_n^1)|^2 + (1-\frac{c^2}{2}) \int_{\Omega_N} |\nabla Im(z_n^1)|^2 \leq 2\int_{\Omega_N} |Re(z_n^1)|^2 + I_4 + o(1). \tag{3.14}$$

Notice that

$$(1-u_n^2-v_n^2)(u_n(1-u_n)-v_n^2) = (1-u_n^2-v_n^2)^2 + (1-u_n^2-v_n^2)(u_n-1),$$

and moreover

$$(1-u_n^2-v_n^2)^2 = 4(u_n-1)^2+(u_n-1)^4+v_n^4+4(u_n-1)^3+4(u_n-1)v_n^2+2(u_n-1)^2v_n^2.$$

On the other hand

$$(1-u_n^2-v_n^2)(u_n-1) = -2(1-u_n)^2+(1-u_n)^3+v_n^2(1-u_n).$$

Therefore, assuming that $z_n^1 = \psi_n - \psi^1 \rightharpoonup 0$ we get

$$2\int_{\Omega_N} |Re(z_n^1)|^2 + I_4 = o(1),$$

and hence we get a contradiction with (3.14).

We have hence proved the existence of a sequence $y_n^2 \in \{0\} \times \mathbb{R}^{d-1}$ and $\psi^2 \in 1 + H_0^1(\Omega_N)$, $\psi^2 \neq 1$, such that

$$z_n^1(\cdot + y_n^2) + 1 - \psi^2 \rightharpoonup 0.$$

Clearly, $|y_n^2| \to +\infty$, and ψ^2 is also a (weak) solution of (1.8). We now define

$$z_n^2 = z_n^1(\cdot) + 1 - \psi^2(\cdot - y_n^2).$$

Now we can iterate the splitting argument to obtain:

$$\|\psi_n - 1\|^2 = \|\psi^1 - 1\|^2 + \|\psi^2 - 1\|^2 + \|z_n^2\|^2 + o(1), \tag{3.15}$$

$$I_N(\psi_n) = I_N(\psi^1) + I_N(\psi^2) + I_N(1+z_n^2) + o(1). \tag{3.16}$$

Again, if $z_n^2 \to 0$, we are done. Instead, if $z_n^2 \not\to 0$, we can repeat the previous procedure to find $y_n^3 \in \{0\} \times \mathbb{R}^{d-1}$ ($|y_n^3| \to +\infty$, $|y_n^3 - y_n^2| \to +\infty$), z_n^3 and ψ^3 in $H_0^1(\Omega_N)$ as before.

We aim to show that we can have only a finite number of iterative steps. On this purpose, we claim that

$$\inf_{\psi \in \mathcal{N}} ||1-\psi||_{H_0^1(\Omega_N)} > 0,$$

where

$$\mathcal{N} := \left\{ \psi \in 1 + H_0^1(\Omega_N), \psi \neq 1,\ I_N'[\psi](1-\psi) = 0 \right\}.$$

In order to prove the claim we notice the identity

$$\int_{\Omega_N}(1-u^2-v^2)(u(1-u)-v^2)=\int_{\Omega_N} 2(u-1)^2+3(u-1)^3+$$

$$+\int_{\Omega_N}\left(3v^2(u-1)+2(u-1)^2v^2+3(u-1)^3+(u-1)^4+v^4\right), \tag{3.17}$$

such that, thanks to the inequality

$$-2c\int_{\Omega_N}(1-u)\partial_{x_1}v(x)\geq -\frac{c^2}{2}\int_{\Omega_N}|\nabla v|^2-2\int_{\Omega_N}(u(x)-1)^2,$$

we obtain

$$0=I_N'[\psi](1-\psi)\geq\int_{\Omega_N}|\nabla u|^2+(1-\frac{c^2}{2})\int_{\Omega_N}|\nabla v|^2+ \tag{3.18}$$

$$\int_{\Omega_N}\left(3(u-1)^3+3v^2(u-1)+2(u-1)^2v^2+3(u-1)^3+(u-1)^4+v^4\right).$$

From (3.18) we get

$$\alpha||\psi-1||^3_{H^1_0(\Omega_N)}+\beta||\psi-1||^4_{H^1_0(\Omega_N)}\geq(1-\frac{c^2}{2})||\psi-1||^2_{H^1_0(\Omega_N)},$$

and the claim is proved.

Observe now that, as in (3.15), at the k-th step we would have:

$$\|\psi_n-1\|^2=\sum_{j=1}^{k}\|\psi^j-1\|^2+\|z_n^k\|^2+o(1). \tag{3.19}$$

Taking into account that $\psi^j\in\mathcal{N}$, the claim implies that the iterative procedure finishes in a finite number of steps.

Finally, recall that by (2.1), $I_n(\psi^j)>0$. Now, taking into account (3.16),

$$\gamma_N+o(1)=I_N(\psi_n)=\sum_{j=1}^{k}I_N(\psi^j)+o(1)\geq I_N(\psi^j)+o(1),$$

and hence $I_N(\psi^j)\leq\gamma_N$. □

4. Uniformly bounded energy solutions in approximating domains

In this section we prove shall prove the following result:

Proposition 4.1. *There exists a subset $E\subset(0,\sqrt{2})$ of plein measure satisfying that, for any $c\in E$, there exists a subsequence $k:\mathbb{N}\to\mathbb{N}$ strictly increasing such that:*

(1) *There exists a nontrivial finite energy solution ψ_N of the problem:*

$$\begin{aligned} ic\partial_{x_1}\psi_N+\Delta\psi+\left(1-|\psi_N|^2\right)\psi_N &= 0 \quad on\ \Omega_{k(N)},\\ \psi_N &= 1\ on\ \partial\Omega_{k(N)}.\end{aligned}$$

(2) *$\mathcal{E}(\psi_N)\leq M$ for some positive constant $M=M(c)$ independent of $N\in\mathbb{N}$.*

(3) $I^c_{k(N)}(\psi_N) \leq \gamma_{k(N)}(c)$.
(4) $ind(\psi_N) \leq 1$.

One of the key points here is that in (2) the energy is bounded uniformly in N. This will be essential later when passing to the limit as $N \to +\infty$.

In a first subsection we will give an abstract result, which is basically well-known but maybe not in this specific form. Later we will apply that result to prove Proposition 4.1.

4.1. **Energy and Morse index bounds.** Energy bounds on Palais-Smale sequences via monotonicity (also called monotonicity trick argument), is a tool first devised in [49] that has been used many times since then, applied to a wide variety of problems. Here we need to adapt this argument to obtain uniform bounds in N, for a subsequence $k(N)$. Moreover, we will also use Morse index bounds for Palais-Smale sequences, in the spirit of [20, 21]. For the sake of completeness, we state and give a proof of a general result in this subsection.

Proposition 4.2. *Let X be a Banach space and A, $B : X \to \mathbb{R}$ two C^1 functionals. Assume that either $A(\psi) \geq 0$ or $B(\psi) \geq 0$ for all $\psi \in X$. For any $c \in J \subset \mathbb{R}^+_0$, we define $I^c : X \to \mathbb{R}$,*

$$I^c(\psi) = A(\psi) - cB(\psi).$$

We assume that there are two points ψ_0, ψ_1 in X, such that setting

$$\Gamma = \{g \in C([0,1], X),\ g(0) = \psi_0,\ g(1) = \psi_1\},$$

the following strict inequality holds for all $c \in J$:

$$\gamma(c) = \inf_{g \in \Gamma} \max_{t \in [0,1]} I^c(g(t)) > \max\{I^c(\psi_0), I^c(\psi_1)\}.$$

Then the following assertions hold true:

(1) *If $B \geq 0$, γ is decreasing. If instead $A \geq 0$, then the map $\sigma(c) = \frac{\gamma(c)}{c}$ is decreasing. As a consequence, both the maps γ, σ are almost everywhere differentiable.*
(2) *Let $c \in J$, $c > 0$, be a point of differentiability of γ. Then, there exists a sequence $\{\psi_n\}$ such that*
    (a) *$I^c(\psi_n) \to \gamma(c)$,*
    (b) *$(I^c)'(\psi_n) \to 0$ in X^{-1}, and*
    (c) *$dist(\psi_n, G_n) \to 0$, where*

$$G_n = \{\psi \in X :\ B(\psi) \leq\ -\gamma'(c) + 1/n,\ A(\psi) \leq \gamma(c) - \gamma'(c)c + \frac{1}{n}\}.$$

(3) *Let us define, for any $\delta > 0$, the sets*

$$F_\delta = \{\psi \in X :\ |I^c(\psi) - \gamma(c)| < 2\delta\},$$

$$G_\delta = \{\psi \in X :\ B(\psi) < -\gamma'(c) + \delta,\ A(\psi) < -c^2\sigma'(c) + \delta\}, \tag{4.1}$$

$$H_\delta = \{\psi \in F_\delta :\ dist(\psi, G_\delta) < 2\delta\}.$$

Let us assume that A and B are uniformly $C^{2,\alpha}$ functionals in H_δ for some $\delta > 0$. Then in (2) we can choose ψ_n satisfying also that:

d) *There exists a sequence $\delta_n < 0$, $\delta_n \to 0$, such that*

$$\sup\{dim\, Y : \ Y \subset X \ linear\ subspace : (I^c)''(\psi_n)(\phi,\phi) \le \delta_n \|\phi\|^2 \ \forall \ \phi \in Y\} \le 1.$$

Remark 4.3. *Observe that, in general, there exist (PS) sequences for I^c for any $c \in J$; see for instance* [1, 50]. *The above proposition shows that, for almost all values $c \in J$, there exist (PS) sequences for I^c that satisfy also condition c). This extra condition c) can be useful in order to show convergence of the (PS) sequence. For instance, if either A or B is coercive, Proposition 4.2 implies the existence of bounded (PS) sequences, which is an important information in order to derive convergence. This is the result of* [34].

Assertion (3) comes from [20, 21] *and gives also a Morse index bound of the (PS) sequence. The only novelty is that we have assumed uniform $C^{2,\alpha}$ regularity on the set H_δ. If A or B is coercive, it suffices to have uniform $C^{2,\alpha}$ estimates on bounded sets.*

Proof. The proof of (1) is inmediate. Indeed, if $B \ge 0$, $I^c(u)$ is decreasing in c. Since the family Γ is independent of c, we have that γ is decreasing. Instead, if $A \ge 0$, then the expression $\frac{I^c(u)}{c}$ is decreasing in c, and we conclude.

In any of the two cases, the maps γ, σ are differentiable in a set $E \subset J$ of plein measure.

In order to prove (2), we are largely inspired by [34]. We first state and prove the following lemma:

Lemma 4.4. *Let $c \in E$, $c > 0$, then there exists $g_n \in \Gamma$ such that*

(1) $\max_{t\in[0,1]} I^c(g_n(t)) \to \gamma^c$.
(2) *There exists $\rho_n > 0$, $\rho_n \to 0$ such that for all $t \in [0,1]$ with $I^c(g_n(t)) \ge \gamma(c) - \frac{1}{n}$, we have:*

$$B(g_n(t)) \le -\gamma'(c) + \rho_n, \quad A(g_n(t)) \le -c^2\sigma'(c) + \rho_n.$$

Proof of the lemma. Take $c_n \in J$ an increasing sequence converging to c. For any $n \in \mathbb{N}$, there exists $g_n \in \Gamma$ such that $\max_{t\in[0,1]} I^{c_n}(g_n(t)) \le \gamma(c_n) + |c_n - c|^2$.

If $B \ge 0$ we have that:

$$\max_{t\in[0,1]} I^c(g_n(t)) \le \max_{t\in[0,1]} I^{c_n}(g_n(t)) \le \gamma(c_n) + |c_n - c|^2 \to \gamma(c).$$

Instead, if $A \ge 0$,

$$\max_{t\in[0,1]} I^c(g_n(t)) \le \frac{c}{c_n} \max_{t\in[0,1]} I^{c_n}(g_n(t)) \le \frac{c}{c_n}(\gamma(c_n) + |c_n - c|^2) \to \gamma(c).$$

We now take $t \in [0,1]$ such that $I^c(g_n(t)) \ge \gamma(c) - |c - c_n|^2$. Then:

$$\begin{aligned} B(g_n(t)) &= \frac{I^{c_n}(g_n(t)) - I^c(g_n(t))}{c - c_n} \\ &\le \frac{\gamma(c_n) + |c_n - c|^2 - \gamma(c) + |c_n - c|^2}{c - c_n} \to -\gamma'(c). \end{aligned}$$

Moreover,

$$\limsup_{n\to+\infty} A(g_n(t)) = \limsup_{n\to+\infty} I^c(g_n(t)) + cB(g_n(t)) \le \gamma(c) - c\gamma'(c).$$

It suffices then to take $c_n = c - \frac{1}{\sqrt{n}}$.

□

Recall now the definitions of F_δ, G_δ and H_δ given in (4.1). By the previous lemma the set $F_\delta \cap G_\delta$ is not empty: indeed, the curves g_n pass through $F_\delta \cap G_\delta$ for sufficiently large n . Proposition 4.2, (2) is proved if we show that for any $\delta > 0$,

$$\inf\{\|(I^c)'(\psi)\| : \ \psi \in H_\delta\} = 0.$$

We argue by contradiction, and assume that there exists $\delta > 0$ such that $\inf\{\|(I^c)'(\psi)\| : \psi \in H_\delta\} \ge \delta > 0$. A classical deformation argument shows that there exists $\varepsilon > 0$, $\eta \in C([0,1] \times X : \ X)$ such that:

i) $\eta(s,\psi) = \psi$ if $s = 0$, $|I^c(\psi) - \gamma(c)| > 2\varepsilon$ or $dist(\psi, G_\delta) > 2\delta$.
ii) $I^c(\eta(1,\psi)) \le \gamma(c) - \varepsilon$ for all $\psi \in G_\delta$ with $I^c(\psi) \le \gamma(c) + \varepsilon$.
iii) $\eta(s,\cdot)$ is a homeomorphism of X.
iv) $\|\eta(s,\psi) - \psi\| < \delta$,
v) $I^c(\eta(s,\psi)) \le I^c(\psi)$ for all $\psi \in X$.

The existence of the above deformation can be found in [50, Lemma 2.3], for instance. Actually our notation is compatible with that reference, setting $S = G_\delta$, and taking $\varepsilon = \delta^2/8$, for instance.

We now take n large enough and the curve g_n given by the lemma. If $I^c(g_n(t)) < \gamma(c) - \frac{1}{n}$, by b), we have that $I^c(\eta(1, g_n(t))) < \gamma(c) - \frac{1}{n}$. In on the contrary, $I^c(g_n(t)) \ge \gamma(c) - \frac{1}{n}$, we can combine the lemma with ii) to conclude that $I^c(\eta(1, g_n(t))) \le \gamma(c) - \varepsilon$. As a consequence,

$$\max_t I^c(\eta \circ g_n(t)) < \gamma(c),$$

a contradiction.

For the proof of (3) of Proposition 4.2, we use Theorem 1 of [20] to our sequence of paths g_n. It is important to point out that, in our setting, the uniform $C^{2,\alpha}$ regularity assumption in [20] is needed only in the set H_δ defined above. Indeed, the proof of Theorem 1.bis of [20] (pages 93-94) only needs a uniform $C^{2,\alpha}$ bound, independent of n, in a certain ball $B(g_n(t), 2\bar{\delta})$, where $|I^c(g_n(t)) - \gamma(c)| \le \bar{\varepsilon}$, for some $\bar{\delta} > 0$, $\bar{\varepsilon} > 0$. Observe that by Lemma 4.4, such ball is contained in H_δ, with a suitable choice of the constants.

Finally, since by [20] $d(\psi_n, g_n[0,1]) \to 0$ and $I^c(\psi_n) \to \gamma(c)$, again Lemma 4.4 implies that $\psi_n \in H_\delta$ for large n. By a diagonal argument, we can take ψ_n such that $d(\psi_n, G_n) \to 0$.

□

4.2. **Proof of Proposition 4.1.** A direct application of the above results to our setting, combined with Proposition 3.3, yields the existence of finite energy solutions in any domain Ω_N, for almost all values of c. The problem here is that the energy of those solutions could diverge if we make $N \to +\infty$. In order to obtain uniform bounds independent of the parameter N, we need a more subtle application of Proposition 4.2.

Define:

(1) $X = 1 + H_0^1(\Omega_N)$, which is an affine Banach space, for which Proposition 4.2 also holds;
(2) $A(\psi) = \mathcal{E}(\psi)$, which is positive and coercive;
(3) $B(\psi) = \mathcal{P}(\psi)$, the momentum;
(4) $J = (c_0, \sqrt{2})$ for a fixed value $c_0 > 0$.

For $N \geq N_0$ the functional I_N^c has a min-max geometry (see Proposition 3.1); recall that $\gamma_N(c) > 0$ is the function that associates to a speed $c \in J$ the min-max value of I_N^c. Clearly, $\sigma_N(c) = \frac{\gamma_N(c)}{c}$ is decreasing in c as Proposition 4.2 shows. By Proposition 4.2, there exists a bounded (PS) sequence in $H_0^1(\Omega_N)$ at level $\gamma_N(c)$. Proposition 3.3 yields then the existence of a solution ψ_N with:

$$I_N^c(\psi_N) \leq \gamma_N(c), \ \mathcal{E}(\psi_N) = A(\psi_N) \leq -c^2 \sigma_N'(c).$$

Since A is coercive, here the set H_δ is uniformly bounded, and I is clearly uniformly $C^{2,\alpha}$ in bounded sets. By Proposition 4.2, 3), we have that:

$$\sup\{dim\, Y : \ Y \subset H_0^1(\Omega_N) : (I_N^c)''(\psi_N)(\phi, \phi) < 0 \ \forall \ \phi \in Y\} \leq 1.$$

We are now concerned with passing to the limit as $N \to +\infty$. In order to control the energy of the solutions ψ_N, we reason as follows.

Recall Proposition 3.1, b), and that $\sigma_N(c)$ is decreasing in c; then, for $N \geq N_0$,

$$\frac{\chi(c_0)}{c_0} \geq \frac{\gamma_N(c_0)}{c_0} \geq \frac{\gamma_N(c_0)}{c_0} - \frac{\gamma_N(c)}{c} \geq \int_{c_0}^{c} |\sigma_N'(s)| \, ds. \tag{4.2}$$

Let us now define the sets

$$D_{N,M} = \{c \in (c_0, \sqrt{2}) : \sigma_N \text{ is not differentiable or } |\sigma_N'(c)| > M\},$$

for all N, $M \in \mathbb{N}$, $N \geq N_0$. Clearly the sets $D_{N,M}$ also depend on c_0, but we avoid to make that dependence explicit in the notation for the sake of clarity.

By (4.2), we have that

$$|D_{N,M}| \leq \frac{\chi(c_0)}{c_0 M}.$$

The following claim is the key to be able to pass to the limit for enlarging slabs preserving bounded energy.

Claim: The set $D(c_0)$ defined as:

$$D(c_0) = \cap_{M \in \mathbb{N}} \cup_{N \geq N_0} \cap_{k \geq N} D_{k,M}$$

has 0 measure.

Indeed, the sets $\cap_{k\geq N} D_{k,M}$ are increasing in N, and all of them satisfy that have measure smaller than $\frac{\chi(c_0)}{c_0 M}$. Hence the same estimate works also for the union in N. Now, $D(c_0)$ is a set given by an intersection of sets of measure $\frac{\chi(c_0)}{c_0 M}$, $M \in \mathbb{N}$, so that $D(c_0)$ has 0 measure.

Finally, we can set

$$D = \cup_{n=1}^{+\infty} D(1/n),$$

which has also 0 measure.

Let us define $E = (0, \sqrt{2}) \setminus D$, and take $c \in E$. We can fix $n \in N$ such that $c_0 = 1/n < c$, and $c \notin D(c_0)$. Then, there exists $M(c)$ and a subsequence $k(N)$ such that $|\sigma'_{k(N)}(c)| \leq M(c)$. By Proposition 4.2, for any of these slabs $\Omega_{k(N)}$ there exists a Palais-Smale sequence with bounded energy. According to Proposition 3.3, this gives rise to a solution $\psi_{k(N)} \in 1 + H_0^1(\Omega_{k(N)})$ such that:

$$I^c_{k(N)}(\psi_{k(N)}) \leq \gamma_{k(N)}(c),\ \mathcal{E}(\psi_{k(N)}) \leq M(c)c^2.$$

This concludes the proof of Proposition 4.1.

5. Proof of Theorem 1.1

In view of Proposition 4.1, we aim to conclude the proof of Theorem 1.1 by passing to the limit. This is indeed possible thanks to Lemma 2.1. However, we need to face two difficulties: vanishing of solutions (that is, the limit solution is trivial) and concentration near the boundary (that is, the limit solution is defined in a half-space). The purpose of this section is to exclude both scenarios.

Next result deals with the question of vanishing and is actually a version of Proposition 2.4 of [8] adapted to problem (1.8).

Proposition 5.1. *Let ψ be a nontrivial finite energy solution of* (1.8) *with $0 < c < \sqrt{2}$, then*

$$\|1 - |\psi|\|_{L^\infty(\Omega_N)} \geq \frac{2}{5}(1 - \frac{c}{\sqrt{2}}).$$

The proof is actually the same as in [8], with one difference: when integrating by parts, the authors use the decay estimates of the solutions to avoid contributions from infinity, and those estimates are available only for the Euclidean space case. Instead, here we use integrability bounds.

In our argument we will use liftings of the solutions, that is, we write $\psi = \rho e^{i\theta}$. The existence of liftings is always guaranteed, for instance, if $|\psi(x)| \neq 0$ for all x.

5.1. Liftings for solutions in Ω_N without vortices.

We consider here solutions without vortices, i.e. that do not vanish. The energy density is given by the following formula

$$e(\rho, \theta) = \frac{1}{2}\left(|\nabla\rho|^2 + |\nabla\theta|^2\rho^2\right) + \frac{1}{4}\left(1 - |\rho|^2\right)^2$$

and the associated energy is

$$\mathcal{E}(\rho,\theta) := \int_{\Omega_N} e(\rho,\theta)$$

By using the fact that $\psi = \rho e^{i\theta}$ is a solution of (1.3), ρ, θ fulfill the following system of equations

$$\begin{cases} \frac{c}{2}\partial_{x_1}\rho^2 + \nabla\cdot(\rho^2\nabla\theta) = 0, \\ c\rho\partial_{x_1}\theta - \Delta\rho - \rho(1-\rho^2) + \rho|\nabla\theta|^2 = 0. \end{cases} \tag{5.1}$$

The following pointwise inequality (Lemma 2.3 in BGS)

$$\left|(\rho^2-1)\partial_{x_1}\theta\right| \le \frac{\sqrt{2}}{\rho} e(\rho,\theta) \tag{5.2}$$

that holds for arbitary C^1 scalar function that can be written as $\psi = \rho e^{i\theta}$ (not necessary being a solution) are crucial in the sequel.

Lemma 5.2. *Let ψ be a vortexless finite energy solution in Ω_N, then $1-\rho$ and θ belong to $H^1_0(\Omega_N)$.*

Proof. Let us notice that for a vortexless finite energy solution

$$\mathcal{E}(\psi) = \frac{1}{2}\int_{\Omega_N} |\nabla\rho|^2 + \rho^2|\nabla\theta|^2 < +\infty \tag{5.3}$$

which implies, by means of Poincaré inequality, that $\rho - 1 \in H^1_0(\Omega_N)$. Since $\nabla\rho$ bounded in L^∞ (by Lemma 2.1), one concludes that $\rho(x) \to 1$ uniformly as $|x| \to +\infty$. Hence $\rho(x) \ge \rho_0 > 0$ for all $x \in \Omega_N$. Again by (5.3), $\nabla\theta \in L^2(\Omega_N)$. To conclude the proof we shall prove that $\theta = 0$ on $\partial\Omega_N$ which will allows to use Poincaré inequality.

Let us assume that $\theta = 0$ if $x_1 = -N$ and that $\theta = 2\pi$ if $x_1 = N$. For any $\tilde{x} \in \mathbb{R}^{d-1}$ there exists $y \in (-N, N)$ such that $u(y,\tilde{x}) = 0$. We get

$$1 = |u(y,\tilde{x}) - u(-N,\tilde{x})|^2 = \left|\int_{-N}^{y} \partial_{x_1}u(s,\tilde{x})ds\right|^2 \le 2N\int_{-N}^{N} |\partial_{x_1}u(s,\tilde{x})|^2\, ds.$$

By Fubini we get

$$\int_{\Omega_N} |\partial_{x_1}u|^2 = \int_{\mathbb{R}^{d-1}} \left(\int_{-N}^{N} |\partial_{x_1}u(s,\tilde{x})|^2\, ds\right) d\tilde{x} = +\infty,$$

which implies that the energy is infinity.

□

From (5.1) we derive three useful identities that are important in the sequel. These identities have been stablished in [8, Lemmas 2.8, 2.10] for solutions in the whole euclidean space: here we adapt these arguments to the problem in the domain Ω_N.

Lemma 5.3. *Let ψ be a vortexless finite energy solution of* (1.8)*. Then:*

$$\mathcal{P}(\psi) = \frac{1}{2}\int_{\Omega_N} (1-\rho^2)\partial_{x_1}\theta. \tag{5.4}$$

$$c\mathcal{P} = \int_{\Omega_N} \rho^2|\nabla\theta|^2. \tag{5.5}$$

$$\int_{\Omega_N} \left(2\rho|\nabla\rho|^2 + \rho(1-\rho^2)^2\right) = c\int_{\Omega_N} \rho(1-\rho^2)\partial_{x_1}\theta + \int_{\Omega_N} \rho(1-\rho^2)|\nabla\theta|^2. \tag{5.6}$$

Proof. Straightforward computation gives

$$\mathcal{P}(\psi) = \frac{1}{2}\int_{\Omega_N} \partial_{x_1}(\rho\sin\theta) - \rho^2\partial_{x_1}\theta = \frac{1}{2}\int_{\Omega_N} \partial_{x_1}(\rho\sin\theta - \theta) + (1-\rho^2)\partial_{x_1}\theta.$$

We will prove that $\int_{\Omega_N} \partial_{x_1}(\rho\sin\theta-\theta) = 0$. Thanks to Lemma 5.2 $\psi_1 = (1-\rho^2)\partial_{x_1}\theta$ is integrable in Ω_N and hence we derive that $\psi_2 = \partial_{x_1}(\rho\sin\theta-\theta)$ is integrable as well. By integration by parts together with Lemma 5.2 we get

$$\int_{\Omega_N} \psi_2 = \int_{\partial\Omega_N} (\rho\sin\theta - \theta)\,\eta_1 = 0.$$

To get (5.5) we multiply the first equation of (5.1) by θ and we integrate in $\Omega_{N,M}$, defined as

$$\Omega_{N,M} = \left\{x \in \mathbb{R}^d, \quad -N < x_1 < N, \ |x_j| < M, \quad 2 \le j \le d\right\} \subset \Omega_N.$$

By integrating by parts we obtain:

$$\frac{c}{2}\int_{\Omega_{N,M}} (1-\rho^2)\partial_{x_1}\theta - \int_{\Omega_{N,M}} \rho^2|\nabla\theta|^2$$
$$= \int_{\partial\Omega_{N,M}} \theta\left(\frac{c}{2}(1-\rho^2)\eta_1 - \rho^2\nabla\theta\cdot\eta\right).$$

Observe that by Lemma 5.2 all functions involved in the expression above belong to $L^1(\Omega_N)$, and recall that $\theta = 0$ on $\partial\Omega_N$. Then, there exists a sequence M_n such that

$$\lim_{n\to\infty}\int_{\partial\Omega_{N,M_n}} \theta\rho^2\nabla\theta\cdot\eta = 0.$$

This proves (5.5).

By multiplying the second equation of (5.1) by ρ^2-1 and integrating over $\Omega_{N,M}$ by parts we obtain

$$\int_{\Omega_{N,M}} \left(2\rho|\nabla\rho|^2 + \rho(1-\rho^2)^2\right) + \int_{\partial\Omega_{N,M}} (1-\rho^2)\nabla\rho\cdot\eta =$$
$$= c\int_{\Omega_{N,M}} \rho(1-\rho^2)\partial_{x_1}\theta + \int_{\Omega_{N,M}} \rho(1-\rho^2)|\nabla\theta|^2.$$

Again by Lemma 5.2, all functions involved in the above expression belong to $L^1(\Omega_N)$. Hence we can finde a sequence M_n such that

$$\lim_{n\to\infty}\int_{\partial\Omega_{N,M_n}}(1-\rho^2)\nabla\rho\cdot\eta=0.$$

This proves (5.6) passing to the limit. □

5.2. **Proof of Proposition 5.1.** Let us call $\delta=||1-|\psi|||_{L^\infty(\Omega_N)}$. If $\delta>\frac{1}{2}>\frac{2}{5}(1-\frac{c}{\sqrt{2}})$ there is nothing to prove. Let us suppose hence that $\delta<\frac{1}{2}$ which implies that $\rho(x)\geq 1-\delta>\frac{1}{2}$ for any $x\in\Omega_N$. In particular ψ admits a lifting $\psi=\rho e^{i\theta}$. We notice that

$$4(1-\delta)\left(\int_{\Omega_N}\frac{1}{2}|\nabla\rho|^2+\frac{1}{4}\left(1-|\rho|^2\right)^2\right)\leq\int_{\Omega_N}2\rho|\nabla\rho|^2+\rho\left(1-|\rho|^2\right)^2$$

and thanks to (5.6) we get

$$\int_{\Omega_N}e(\rho,\theta)\leq\frac{1}{4(1-\delta)}\int_{\Omega_N}\rho(1-\rho^2)\left(c\partial_{x_1}\theta+|\nabla\theta|^2\right)+\frac{1}{2}\int_{\Omega_N}\rho^2|\nabla\theta|^2 \tag{5.7}$$

The strategy is to estimate r.h.s of (5.7) using the pointwise bound given by (5.2). We have, thanks to (5.5) and (5.2)

$$\frac{c}{4(1-\delta)}\int_{\Omega_N}\rho(1-\rho^2)\partial_{x_1}\theta+\frac{1}{2}\int_{\Omega_N}\rho^2|\nabla\theta|^2\leq\left(\frac{\sqrt{2}c}{4(1-\delta)}+\frac{\sqrt{2}c}{4}\right)\int_{\Omega_N}e(\rho,\theta)$$

and hence

$$\frac{c}{4(1-\delta)}\int_{\Omega_N}\rho(1-\rho^2)\partial_{x_1}\theta+\frac{1}{2}\int_{\Omega_N}\rho^2|\nabla\theta|^2\leq\frac{c}{\sqrt{2}(1-\delta)}\int_{\Omega_N}e(\rho,\theta).$$

Now we claim that

$$\left|\int_{\Omega_N}\rho(1-\rho^2)|\nabla\theta|^2\right|\leq 6\delta\int_{\Omega_N}e(\rho,\theta) \tag{5.8}$$

such that we obtain

$$\int_{\Omega_N}e(\rho,\theta)\leq(\frac{c}{\sqrt{2}(1-\delta)}+\frac{3\delta}{2(1-\delta)})\int_{\Omega_N}e(\rho,\theta).$$

The fact that $e(\rho,\theta)\geq 0$ and that $1-(\frac{c}{\sqrt{2}(1-\delta)}+\frac{3\delta}{2(1-\delta)})\leq 0$ if $\delta\geq\frac{2}{5}(1-\frac{c}{\sqrt{2}})$ concludes the proof. Now we prove claim (5.8). Notice that

$$\left|\int_{\Omega_N}\rho(1-\rho^2)|\nabla\theta|^2\right|\leq\delta\int_{\Omega_N}\rho(1+\rho)|\nabla\theta|^2.$$

Now, $\rho(1+\rho)\leq 3\rho^2$ if $\rho\geq\frac{1}{2}$, such that thanks to (5.5)

$$\left|\int_{\Omega_N}\rho(1-\rho^2)|\nabla\theta|^2\right|\leq 3\delta\int_{\Omega_N}\rho^2|\nabla\theta|^2\leq\frac{3\delta c}{2}\int_{\Omega_N}(1-\rho^2)\partial_{x_1}\theta\leq 3\sqrt{2}\delta c\int_{\Omega_N}e(\rho,\theta).$$

The proof of the claim ends noticing that $0<c<\sqrt{2}$.

5.3. **Conclusion of the proof of Theorem 1.1.** Take $c \in E$, $c_0 \in (0, c)$ and N_0 given by Proposition 3.1.

By Proposition 5.1, there exists ξ_N such that $|\psi_{k(N)}(\xi_N) - 1| \nrightarrow 0$, where $\psi_{k(N)}$ are the solutions given by Proposition 4.1. We consider now a space translation defining $\tilde{\psi}_{k(N)}(x) = \psi_{k(N)}(x - \xi_N)$. Observe that the uniform bounds of Lemma 2.1 allow us to use Ascoli-Arzelà Theorem for the sequence $\tilde{\psi}_{k(N)}$, which converges locally to a certain function ψ_c.

If $d(\xi_N, \partial\Omega_{k(N)}) \to +\infty$, the limit function ψ_c is a nontrivial solution of (1.3) in $\mathbb{R}^d$. Moreover, by Fatou Lemma, $\mathcal{E}(\psi_c)$ is finite. Finally, again by Fatou lemma and (2.1),

$$\begin{aligned} I(\psi_c) = \frac{1}{d-1} \sum_{j=2}^{d} \int |\partial_{x_j} \psi_c|^2 &\le \frac{1}{d-1} \liminf_{N \to +\infty} \sum_{j=2}^{d} \int |\partial_{x_j} \tilde{\psi}_{k(N)}|^2 \\ &= \liminf_{N \to +\infty} I_{k(N)}(\psi_{k(N)}) \le \liminf_{N \to +\infty} \gamma_{k(N)}(c) \le \chi(c_0). \end{aligned}$$

This shows the validity of Theorem 1.1 in this case.

Assume now that $d(\xi_N, \partial\Omega_{k(N)})$ is bounded. Up to a subsequence, the limit function ψ_c is a nontrivial solution of (1.3) defined in a half-space $\{x \in \mathbb{R}^d : \ x_1 > -m\}$ or $\{x \in \mathbb{R}^d : \ x_1 < m\}$, for some $m > 0$, and with boundary condition $\psi_c = 1$. Again by Fatou Lemma, $\mathcal{E}(\psi_c)$ is finite. In next proposition we rule out this possibility, and this concludes the proof of Theorem 1.1.

Proposition 5.4. *Let ψ be a finite energy solution of the problem:*

$$\begin{array}{rcll} ic\partial_{x_1}\psi + \Delta\psi + \left(1 - |\psi|^2\right)\psi &=& 0 & \text{on } \mathbb{R}^d_+, \\ \psi &=& 1 & \text{on } \partial\mathbb{R}^d_+, \end{array} \tag{5.9}$$

where $c \in \mathbb{R}$ and $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : \ x_1 > 0\}$. Then $\psi = 1$.

Proof. The proof follows well-known ideas that date back to [19]. If ψ is a finite energy solution, then $\nabla\psi$ and $(1 - |\psi|^2)$ are functions in $L^2(\mathbb{R}^d_+)$. Since ψ is in $L^\infty(\mathbb{R}^d_+)$ and is a strong solution, standard regularity results allow us to conclude that $D^2\psi$ belongs to $L^2(\mathbb{R}^d_+)$. Hence we can multiply equation (5.9) by $\partial_{x_1}\psi$ and integrate by parts, obtaining:

$$c \int_{\mathbb{R}^d_+} \langle i\partial_{x_1}\psi, \partial_{x_1}\psi \rangle = 0;$$

$$\begin{aligned} \int_{\mathbb{R}^d_+} \langle \Delta\psi, \partial_{x_1}\psi \rangle &= \int_{\partial\mathbb{R}^d_+} \langle (\nabla\psi \cdot \nu), \partial_{x_1}\psi \rangle - \int_{\mathbb{R}^d_+} \frac{1}{2} \partial_{x_1} \left(|\nabla\psi|^2\right) \\ &= - \int_{\partial\mathbb{R}^d_+} |\partial_{x_1}\psi|^2 + \frac{1}{2} \int_{\partial\mathbb{R}^d_+} |\partial_{x_1}\psi|^2 = -\frac{1}{2} \int_{\partial\mathbb{R}^d_+} |\partial_{x_1}\psi|^2; \end{aligned}$$

$$\int_{\mathbb{R}^d_+} \left(1 - |\psi|^2\right) \langle \psi, \partial_{x_1}\psi \rangle = -\frac{1}{4} \int_{\mathbb{R}^d_+} \partial_{x_1} \left((1 - |\psi|^2)^2\right) = 0.$$

These computations imply that:

$$\int_{\partial \mathbb{R}^d_+} |\partial_{x_1} \psi|^2 = 0.$$

In other words, $\partial_{x_1}\psi = 0$ in $\partial \mathbb{R}^d_+$. By unique continuation, we conclude that $\psi = 1$.

□

6. Proof of Theorem 1.2

In this section we prove the compactness criterion given in 1.2. We start by the following result, which is independent of the dimension:

Proposition 6.1. *Let $d = 2$ or 3, $c_n \to c$, $c_n \in E$ where the set E is given by Theorem 1.1. Let ψ_n be the sequence of solutions provided by that theorem. Then there exists $\xi_n \in \mathbb{R}^d$ such that $\psi_n(\cdot - \xi_n)$ converges locally in C^k (up to a subsequence) to a nontrivial solution ψ_0 of* (1.3).

Proof. Let ψ be a nontrivial finite energy solution of (1.3) with $0 < c < \sqrt{2}$. Then there exists $\varepsilon = \varepsilon(c) > 0$ such that

$$\| 1 - |\psi| \|_{L^\infty(\mathbb{R}^d)} \geq \varepsilon. \tag{6.1}$$

Statement (6.1) is just Proposition 2.4 of [8]. Compare it with Proposition 5.1, which is nothing but its version for problem (1.8) (with a slight change of the constants).

Then, there exists ξ_n such that $|1 - |\psi_n(\xi_n)|| > \varepsilon$ for some fixed $\varepsilon > 0$. By Lemma 2.1 we can use Ascoli-Arzelà Theorem to obtain that $\psi_n(\cdot - \xi_n)$ converges locally in C^k to a nontrivial solution ψ_0 of (1.3).

□

The main problem to conclude the proof of Theorem 1.2 or 1.3 is to assure that ψ_0 has finite energy. Let us point out that the boundedness of the energy cannot be deduced only by using the Pohozaev identities given in Lemmas 2.2, 2.3.

Observe that since $I_{c_n}(\psi_n)$ is bounded, Lemma 2.3 implies that

$$\sum_{j=2}^{3} \int_{\mathbb{R}^3} |\partial_{x_j} \psi_n|^2 = O(1).$$

The idea of the proof is to try to relate the behavior of ψ_n with that of the 1-D solutions of the Gross-Pitaevskii equation. Next proposition is a first step in this line (see also Remark 6.3).

Proposition 6.2. *Let ψ_n be solutions of* (1.3) *for c_n, $c_n \to c$, such that $I_{c_n}(\psi_n) \leq C$. Then,*

$$\int_{\mathbb{R}^3} |\nabla g_n|^2 + \int_{\mathbb{R}^3} |\nabla h_n|^2 = O(1),$$

where

$$g_n = (\partial_{x_1} u_n) v_n - (\partial_{x_1} v_n) u_n - \frac{c_n}{2}(\rho_n^2 - 1), \tag{6.2}$$

$$h_n = \frac{1}{2}|\partial_{x_1}\psi_n|^2 - \frac{1}{4}(1-\rho_n^2)^2. \tag{6.3}$$

Remark 6.3. *The two quantities defined above correspond to the invariants of the 1-D Gross-Pitaevskii equation. Indeed h represents its hamiltonian, whereas g is another invariant given by the fact that the problem, after a change of variables, is radially symmetric (see equations* (8.1), (8.2)*). On this aspect, see for instance* [9]*, pages 3-4.*

Proof. For the sake of clarity we drop the subscript n in the proof of this proposition.

We first consider the function g, which is an L^2 function, but with L^2 norm out of control. Observe that equation (1.3) implies that $\nabla \cdot G = 0$, where

$$G = (g, u_{x_2}v - v_{x_2}u, u_{x_3}v - v_{x_3}u). \tag{6.4}$$

Straightforward computations give:

$$curl\, G = \begin{pmatrix} 2u_{x_3}v_{x_2} - 2v_{x_3}u_{x_2} \\ 2v_{x_3}u_{x_1} - 2u_{x_3}v_{x_1} - \frac{c}{2}(\rho^2-1)_{x_3} \\ -2v_{x_2}u_{x_1} + 2u_{x_2}v_{x_1} + \frac{c}{2}(\rho^2-1)_{x_2} \end{pmatrix}.$$

Observe that by (2.1), the derivatives with respect to x_2, x_3 are uniformly bounded (with respect to n) in L^2. Moreover, all factors involved are bounded in L^∞ by Lemma 2.1. As a consequence, $curlG$ is uniformly bounded in L^2. Observe now that:

$$G = curl(-\Delta^{-1} curl\, G),$$

where Δ^{-1} is given by convolution with the Coulomb potential $\frac{1}{4\pi|x|}$, see for instance [7, Subsection 2.4.1].

By using the Fourier Transform and Plancherel, all partial derivatives of G are uniformly bounded in L^2, independently of n. This concludes the proof for g.

For h, the proof follows the same ideas. Let us define the vector field:

$$H = \left(h, u_{x_1}u_{x_2} + v_{x_1}v_{x_2}, u_{x_1}u_{x_3} + v_{x_1}v_{x_3}\right).$$

Let us recall here that $|\psi_{x_1}|^2 = u_{x_1}^2 + v_{x_1}^2$. Observe first that H is an L^2 vector field, even if its L^2 norm could be unbounded as $n \to +\infty$. Taking into account (1.3), straightforward computations give:

$$\nabla \cdot H = u_{x_1x_2}u_{x_2} + u_{x_1x_3}u_{x_3} + v_{x_1x_2}v_{x_2} + v_{x_1x_3}v_{x_3}$$

which is uniformly bounded in L^2 norm, again, by (2.1) and Lemma 2.1. Moreover, we can compute:

$$curl\, H = \begin{pmatrix} u_{x_1x_2}u_{x_3} + v_{x_1x_2}v_{x_3} - u_{x_1x_3}u_{x_2} - v_{x_1x_3}u_{x_2} \\ -u_{x_1x_1}u_{x_3} - v_{x_1x_1}v_{x_3} - (1-\rho^2)(uu_{x_3} + vv_{x_3}) \\ u_{x_1x_1}u_{x_2} + v_{x_1x_1}v_{x_2} + (1-\rho^2)(uu_{x_2} + vv_{x_2}) \end{pmatrix}.$$

which is also uniformly bounded in L^2 norm. We now recall that:

$$H = \nabla(\Delta^{-1}(\nabla \cdot H)) - curl(\Delta^{-1}\, curl H),$$

see again [7, Subsection 2.4.1]. By using the Fourier Transform and Plancherel, all partial derivatives of H are uniformly bounded in L^2, finishing the proof.

□

Remark 6.4. *Let us point out that the above result can be easily extended to any dimension. However, in dimension 3 it implies, by Sobolev inequality, that:*

$$\int_{\mathbb{R}^3} |g_n|^6 + \int_{\mathbb{R}^3} |h_n|^6 = O(1). \tag{6.5}$$

In dimension $d > 3$ the Sobolev exponent is $\frac{2d}{d-2}$. However we cannot deduce a similar expression in dimension 2. The lack of a Sobolev inequality in dimension 2 is one of the obstacles for this approach to work also in the planar case.

Definition 6.5. *We define the set $S_n^r = \{x \in \mathbb{R}^3 : \rho_n(x) < r\}$. The behavior of these sets will be important in our arguments.*

Next lemma is a key ingredient in our proof.

Lemma 6.6. *Under the assumptions of Theorem 1.2, assume that for some $r \in (0,1)$, $|S_n^r| \to +\infty$. Then, there exists $\xi_n \in S_n^r$ and $R_n \to +\infty$ such that:*

$$\int_{B(\xi_n, R_n)} |g_n|^6 + |h_n|^6 + \sum_{i=2}^{3} |\partial_{x_i} \psi_n|^2 \to 0.$$

Proof. Take $x_1^n \in S_n^r$, and define $R_n = |S_n^r|^{\frac{1}{6}}$. Observe that

$$|B(x_1^n, 2R_n)| = c_0 |S_n^r|^{\frac{1}{2}}, \quad c_0 = \frac{32}{3}\pi.$$

As a consequence, there exists $x_2^n \in S_n^r \setminus B(x_1^n, 2R_n)$. Clearly,

$$B(x_1^n, R_n) \cap B(x_2^n, R_n) = \emptyset \text{ and } |B(x_1^n, 2R_n) \cup B(x_2^n, 2R_n)| \le 2c_0 |S_n^r|^{\frac{1}{2}}.$$

We can choose then $x_3^n \in S_n^r \setminus (B(x_1^n, 2R_n) \cup B(x_2^n, 2R_n))$, with

$$B(x_1^n, R_n) \cap B(x_2^n, R_n) \cap B(x_3^n, 2R_n) = \emptyset$$

and

$$|B(x_1^n, 2R_n) \cup B(x_2^n, 2R_n) \cup B(x_3^n, R_n)| \le 3c_0 |S_n^r|^{\frac{1}{2}}.$$

In this way we find $x_n^1 \dots x_n^{j_n} \in S_n^r$ with:

$$B(x_j^n, R_n) \cap B(x_k^n, R_n) = \emptyset, \quad j,\ k \in \{1, \dots j_n\},\ j \ne k,$$

where $j_n = [\frac{|S_n|^{1/2}}{c_0}]$ (here $[a]$ denotes the largest integer smaller than or equal to a). Hence we can choose $\xi_n = x_n^k$ such that, taking into account (6.5) and (2.1):

$$\int_{B(\xi_n,R_n)} |g_n|^6 + |h_n|^6 + \sum_{i=2}^{d} |\partial_{x_i}\psi_n|^2 \leq \frac{C}{j_n} \to 0.$$

□

The above result will be the key to prove next proposition, which allows us to have some control on the set of vortices of the solutions.

Proposition 6.7. *Let us fix $r \in (0, c/\sqrt{2})$. Then, there exists $N \in \mathbb{N}$ and N sequences of disjoint closed balls $\overline{B_k^n} = \overline{B}(\xi_k^n, R_k)$ $(k = 1 \dots n)$ with $R_k \in (1, N)$ such that*

$$S_n^r \subset \cup_{k=1}^N B_k^n.$$

Proof. The proof is divided into several steps:

Step 1: $|S_n^r|$ remains bounded.

Assume by contradiction that $|S_n^r| \to +\infty$ as $n \to +\infty$. Take $\xi_n \in \mathbb{R}^3$ given by Lemma 6.6, and define $\tilde{\psi}_n = \psi_n(\cdot - \xi_n)$. By Lemma 2.1 we can use Ascoli-Arzelà theorem to conclude that, up to a subsequence, $\tilde{\psi}_n$ converges C^k locally to a solution ψ of (1.3). By the choice of ξ_n, this solution satisfies that $\rho(0) \leq r$. Moreover, by Fatou lemma we have that:

$$\partial_{x_2}\psi = 0, \ \partial_{x_3}\psi = 0, \ g = 0, \ h = 0,$$

where g and h are the analogous of (6.2), (6.3), namely:

$$g = u_{x_1}v - v_{x_1}u - \frac{c}{2}(\rho^2 - 1),$$

$$h = \frac{1}{2}|\psi_{x_1}|^2 - \frac{1}{4}(1 - \rho^2)^2.$$

As a consequence, ψ is a 1-D solution to the Gross-Pitaevskii equation with $g = 0$, $h = 0$. But those are precisely the finite energy 1-D travelling waves (see [9, pages 3, 4]); hence, after a rotation, ψ has the explicit expression:

$$\psi(x_1) = \sqrt{\frac{2 - c^2}{2}} \tanh\left(\frac{\sqrt{2 - c^2}}{2}(x_1 + t)\right) + i\frac{c}{\sqrt{2}}, \ t \in \mathbb{R}.$$

But this is in contradiction with $|\psi(0)| \leq r < c/\sqrt{2}$, concluding the proof.

Step 2: There exists $N \in \mathbb{N}$, $\xi_k^n \in \mathbb{R}^3$ such that:

$$S_n^r \subset \cup_{k=1}^N B(\xi_k^n, 1).$$

Fix $s \in (r, \frac{c}{\sqrt{2}})$, and define ξ_1^n as any point in S_n^r. Since $\nabla\rho_n$ is uniformly bounded (Lemma 2.1), there exists $\delta > 0$ such that $B(\xi_1^n, \delta) \subset S_n^s$. It suffices to take

$$\delta \leq \frac{s - r}{sup_n \|\nabla\rho_n\|_{L^\infty}}.$$

Without loss of generality we can assume that $\delta < 1/2$.

Take now ξ_2^n any point in $S_n^r \setminus B(\xi_1^n, 1)$; again, $B(\xi_2^n, \delta) \subset S_n^s$, and observe that $B(\xi_1^n, \delta) \cap B(\xi_2^n, \delta) = \emptyset$.

We follow by taking ξ_3^n any point in $S_n^r \setminus (B(\xi_1^n, 1) \cup B(\xi_2^n, 1))$, if there exists one. Since $|S_n^s|$ is bounded by the step 1, this procedure has to finish at a certain point, yielding the thesis of the proposition. Indeed we cannot find more than N such points, where

$$N = \left[\frac{\sup_n |S_n^s|}{\frac{4}{3}\pi\delta^3} \right].$$

Recall that $[a]$ stands for the largest integer smaller than or equal to a.

Step 3: Conclusion

By step 2, we have already S_n^r contained in N balls of radius 1. The problem is that they might not be disjoint. We now make a procedure of aggregation of balls which is gererally described as follows:

Take a closed ball $\overline{B}(x, R_x)$. If it intersects a closed ball $\overline{B}(y, R_y)$, we replace both balls by $\overline{B}(x, R_x + R_y)$. We now repeat the procedure to the new set of balls.

We apply this procedure iteratively to the balls given in Step 2, and in this way we conclude.

□

The above proposition is the first milestone in our proof: it allows us to control the vortices of the solutions, as they are always contained in a fixed number of disjoint balls of bounded radii. Since $\mathbb{R}^3 \setminus \cup_{k=1}^N \overline{B}_k^n$ is a simply connected open set, we can guarantee the existence of a lifting of ψ_n outside these balls. Being more specific, taking $\frac{c}{2}$ as the value r (for instance), we can write:

$$\psi_n(x) = \rho_n(x) e^{i\theta_n(x)} \ \forall\, x \in \mathbb{R}^d \setminus \cup_{k=1}^N B_k^n$$

where the balls B_k^n are given by Proposition 6.7. Since ψ_n is a solution of (1.3), we have that ρ_n, θ_n satisfy equations (5.1).

Remark 6.8. *This is a second crucial point in which the requirement $d \geq 3$ is crucial. If $d = 2$ we can have liftings of finite energy solutions outside one ball (see* [27, Lemma 15]*), but this is not possible in the complement of two or more disjoint balls.*

Next lemma is inspired in [8][Lemmas 2.8, 2.10], which are concerned with the case without vortices. Compare it with the identities (5.4), (5.5) for the vortexless case.

Lemma 6.9. *Take $c \in (0, \sqrt{2})$, $c_n \in E$ with $c_n \to c$ and ψ_n the solutions given by Theorem 1.1. Then*

$$\mathcal{P}(\psi_n) = \frac{1}{2} \int_{\mathbb{R}^3 \setminus \cup_{k=1}^N B_k^n} (1 - \rho_n^2) \partial_{x_1} \theta_n + O(1), \tag{6.6}$$

$$c\mathcal{P}(\psi_n) = \int_{\mathbb{R}^3 \setminus \cup_{k=1}^N B_k^n} \rho_n^2 |\nabla \theta_n|^2 + O(1), \tag{6.7}$$

$$\int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} |\nabla\rho_n|^2 = O(1). \tag{6.8}$$

Proof. First of all, observe that

$$\nabla\theta = \frac{u\nabla v - v\nabla u}{\rho^2},$$

so that

$$|\nabla\theta_n| = O(1) \text{ in } \partial B_k^n \Rightarrow |\theta_n(p) - \theta_n(q)| \leq C \ \forall\ p,\ q \in \partial B_k^n. \tag{6.9}$$

This is useful in what follows; observe that we do not know whether $\|\theta_n\|_{L^\infty}$ is bounded or not.

Direct computation gives

$$\mathcal{P}(\psi_n) = \frac{1}{2}\int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} \partial_{x_1}(\rho_n\sin\theta_n) - \rho_n^2\partial_{x_1}\theta + \frac{1}{2}\int_{\cup_{k=1}^N B_k^n} \langle i\partial_{x_1}\psi_n, \psi_n - 1\rangle,$$

which implies that

$$\mathcal{P}(\psi_n) = \frac{1}{2}\int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} \partial_{x_1}(\rho_n\sin\theta_n - \theta_n) + (1-\rho_n^2)\partial_{x_1}\theta_n + O(1). \tag{6.10}$$

In order to get (6.6) it suffices hence to prove that

$$\int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} \partial_{x_1}(\rho_n\sin\theta_n - \theta_n) = O(1). \tag{6.11}$$

We recall that $\partial_{x_1}(\rho_n\sin\theta_n - \theta_n)$ is integrable thanks to (5.2) and (6.10). By integration by parts, using the the decay estimates at the infinity, we get

$$\int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} \partial_{x_1}(\rho_n\sin\theta_n - \theta_n) = \sum_{k=1}^N \int_{\partial B_k^n} (\rho_n\sin\theta_n - \theta_n)\,\eta_1,$$

where η_1 is the first component of the inward unit normal vector to the spheres B_k^n. Relation (6.11) follows now from (6.9) together with the fact that the outward unit surface normal η_1 has zero average on the sphere, which implies that

$$\int_{\partial B_k^n} \theta_n\eta_1 = \int_{\partial B_k^n} (\theta_n - \theta_n(p_0))\,\eta_1 = O(1),$$

where p_0 is an arbitrary point on the sphere B_k^n.

In order to prove (6.7) we argue as in Lemma 5.3, i.e. multiplying first equation of (5.1) by θ_n and then integrating on $\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n$. By integration by parts we get

$$\frac{c}{2}\int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} (1-\rho_n^2)\partial_{x_1}\theta_n - \int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} \rho_n^2|\nabla\theta_n|^2,$$

$$= \sum_{k=1}^N \int_{\partial B_k^n} \theta_n\left(\frac{c}{2}(1-\rho_n^2)\eta_1 - \rho_n^2\nabla\theta\cdot\eta\right).$$

Observe that $G_n(x) = (\frac{c}{2}(1-\rho_n^2), 0, 0) - \rho_n^2 \nabla\theta_n$, where G_n is defined in (6.4). In particular it is defined in the whole euclidean space and $\nabla \cdot G_n = 0$. By integrating by parts in B_k^n, we obtain that

$$\int_{\partial B_k^n} \frac{c}{2}(1-\rho_n^2)\eta_1 - \rho_n^2 \nabla\theta \cdot \eta = 0.$$

As a consequence, we can use (6.9) to obtain:

$$\int_{\partial B_k^n} \theta_n \left(\frac{c}{2}(1-\rho_n^2)\eta_1 - \rho_n^2 \nabla\theta \cdot \eta\right)$$
$$= \int_{\partial B_k^n} (\theta_n - \theta_n(p_0)) \left(\frac{c}{2}(1-\rho_n^2)\eta_1 - \rho_n^2 \nabla\theta \cdot \eta\right) = O(1).$$

Now we prove (6.8). From Lemma 2.2 we get

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla\psi_n|^2 - 2c\mathcal{P}(\psi_n) + \frac{3}{4}\int_{\mathbb{R}^3} \left(1-|\psi_n|^2\right)^2 = 0,$$

which implies, thanks to (6.7)

$$\frac{1}{2}\int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} |\nabla\rho_n|^2 + \frac{3}{2}\int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} \rho^2|\nabla\theta_n|^2 + \frac{3}{4}\int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} \left(1-|\rho_n|^2\right)^2$$
$$= 3\int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} \rho^2|\nabla\theta_n|^2 + O(1) = 3c\mathcal{P}(\psi_n) + O(1).$$

As a consequence we get

$$3\left(\mathcal{E}(\psi_n) - c\mathcal{P}(\psi_n)\right) = \int_{\mathbb{R}^3\setminus\cup_{k=1}^N B_k^n} |\nabla\rho_n|^2 + O(1),$$

and hence (6.8) follows by the fact that $\mathcal{E}(\psi_n) - c\mathcal{P}(\psi_n) = I(\psi_n) = O(1)$. □

For next proposition it is useful to recall the definition 6.5.

Proposition 6.10. *Take $c \in (0, \sqrt{2})$, $c_n \in E$ with $c_n \to c$ and ψ_n the solutions given by Theorem 1.1. Assume that:*

$$\mathcal{E}(\psi_n) \to +\infty. \tag{6.12}$$

Then, for any $r \in (\frac{c}{\sqrt{2}}, 1)$, $|S_n^r| \to +\infty$.

Proof. We recall the form of the energy for functions ψ given by a lifting $\psi = \rho e^{i\theta}$:

$$e(\rho,\theta) = \frac{1}{2}\left(|\nabla\rho|^2 + |\nabla\theta|^2\rho^2\right) + \frac{1}{4}\left(1-|\rho|^2\right)^2.$$

The following function represents the lagrangian in the vortexless case, and is an approximation of the real lagrangian in view of (6.6):

$$l(\rho,\theta) = e(\rho,\theta) - \frac{c_n}{2}(1-\rho^2)\partial_{x_1}\theta.$$

Assume by contradiction that $|S_n^r|$ is bounded for some $r > \frac{c}{\sqrt{2}}$. Observe that:

$$I^{c_n}(\psi_n) = \int_{\mathbb{R}^3 \setminus \cup_{k=1}^N B_k^n} l(\rho_n, \theta_n) + O(1) = \int_{\{\rho_n \geq r\}} l(\rho_n, \theta_n) + O(1).$$

We now use the inequality $|\frac{c}{2}(1-\rho_n^2)\partial_{x_1}\theta_n| \leq \frac{(1-\rho^2)^2}{4(1+\varepsilon)} + \frac{c^2}{4}(\partial_{x_1}\theta_n)^2(1+\varepsilon)$ with suitable $\varepsilon > 0$ to obtain:

$$\int_{\{\rho_n \geq r\}} l(\rho_n, \theta_n) \geq \int_{\{\rho_n \geq r\}} \frac{1}{2}|\nabla \rho_n|^2 + \left[\frac{1}{2} - \frac{c^2(1+\varepsilon)}{4\rho_n^2}\right]|\nabla\theta_n|^2\rho_n^2 + \frac{\varepsilon}{1+\varepsilon}\frac{(1-\rho_n^2)^2}{4}$$
$$\geq \varepsilon_0 \int_{\{\rho_n \geq r\}} e(\rho_n, \theta_n) = \varepsilon_0\, \mathcal{E}(\psi_n) + O(1),$$

for suitable $\varepsilon_0 > 0$. Then,

$$O(1) = I^{c_n}(\psi_n) \geq \varepsilon_0\, \mathcal{E}(\psi_n) + O(1),$$

and this allows us to conclude.

□

6.1. **Proof of Theorem 1.2.** With all the results above we can inmediately conclude the proof of Theorem 1.2. Indeed, by (6.8) and Sobolev inequality, we have that

$$\int_{\mathbb{R}^3} (1-\rho_n)^6 = O(1).$$

If $\mathcal{E}(\psi_n) \to +\infty$, Proposition 6.10 implies $|S_n^r|$ is unbounded for $r > \frac{c}{\sqrt{2}}$, and this is a contradiction with the above estimate. Hence $\mathcal{E}(\psi_n)$ is bounded. By Fatou Lemma, the solution ψ_0 given in Proposition 6.1 has finite energy, concluding the proof.

7. Proof of Theorem 1.3

In this section we prove the compactness criterion given in Theorem 1.3. The proof follows some of the ideas of the previous section, but with important differences. As previously, we will be done if we show that $\mathcal{E}(\psi_n)$ is bounded.

By (1.9), we have that $\psi_n \neq 0$ outside $B(0,R)$; as a consequence, ψ_n admit a lifting $\psi_n(x) = \rho_n(x)e^{i\theta_n(x)}$ for all $x \in \mathbb{R}^2 \setminus B(0,R)$. This is a consequence of the fact that ψ_n have finite energy, see [27][Lemma 15]. In the vortexless case, this lifting holds in the whole euclidean space.

Next lemma is a version of Lemma 6.9:

Lemma 7.1. *Take $c \in (0, \sqrt{2})$, $c_n \in E$ with $c_n \to c$ and ψ_n the solutions given by Theorem 1.1. Assume also that there exists $R > 0$ and $\delta > 0$ such that* (1.9) *is satisfied. Then*

$$\mathcal{P}(\psi_n) = \frac{1}{2}\int_{B(0,R)^c} (1-\rho_n^2)\partial_{x_1}\theta_n + O(1), \tag{7.1}$$

$$c\mathcal{P}(\psi_n) = \int_{B(0,R)^c} \rho_n^2|\nabla\theta_n|^2 + O(1), \tag{7.2}$$

$$\int_{B(0,R)^c} |\nabla \rho_n|^2 = O(1). \tag{7.3}$$

Proof. The proof is completely analogue to that of Lemma 6.9. Observe that in the vortexless case we have exact identities in (7.1), (7.2), (7.3). □

With Lemma 6.9 in hand, we can adapt the proof of Proposition 6.10 to our setting, obtaining the following result:

Proposition 7.2. *Take $c \in (0, \sqrt{2})$, $c_n \in E$ with $c_n \to c$ and ψ_n the solutions given by Theorem 1.1. Assume that:*

$$\mathcal{E}(\psi_n) \to +\infty.$$

Then, for any $r \in (\frac{c}{\sqrt{2}}, 1)$, $|S_n^r| \to +\infty$.

Next result is analogue to Lemma 6.6. The only difference is that now we do not know that (6.5) holds, but instead we have (7.3).

Lemma 7.3. *Under the assumptions of Theorem 1.3, assume that for some $r \in (0,1)$, $|S_n^r| \to +\infty$. Then, there exists $\xi_n \in S_n^r$ and $R_n \to +\infty$ such that:*

$$\int_{B(\xi_n, R_n)} |\nabla \rho_n|^2 + |\partial_{x_2} \psi_n|^2 \to 0.$$

Proof. The proof is analogue to that of Lemma 6.6. □

7.1. **Proof of Theorem 1.3.** Assume by contradiction that $\mathcal{E}(\psi_n) \to +\infty$. By Proposition 7.2, we can apply Lemma 7.3 to a value r satisfying that:

$$\frac{c}{\sqrt{2}} < r < \sqrt{\frac{2}{3}(1 + c^2/4)} < 1.$$

Notice that this is possible if $c < \sqrt{2}$.

Let $\xi_n \in \mathbb{R}^d$ given by Lemma 7.3 and define $\tilde{\psi}_n(x) = \psi_n(x - \xi_n)$. Up to a subsequence we have that:

$$\tilde{\psi}_n \to \psi_0 \text{ in } C^k_{loc}(\mathbb{R}^d).$$

Taking into account Remark 2.6, $ind(\psi_0) \leq 1$. By Lemma 6.6, ψ_0 depends only of the x_1 variable. Moreover $\nabla \rho_0 = 0$ where $\rho_0 = |\psi_0| \leq r$. That is, $\psi_0(x_1)$ is a $1D$ circular solution,

$$\psi_0(x_1) = \rho_0 e^{i\omega(x_1 - t)},$$

where $\omega^2 + c\omega + \rho_0^2 = 1$. By the choice of r, we have that $\rho_0^2 < \frac{2}{3}(1 + c^2/4)$. But those solutions have infinite Morse index, as shown in Proposition 8.1 (see Appendix). This contradiction shows that $\mathcal{E}(\psi_n)$ is bounded.

By Fatou Lemma, the solution ψ_0 given in Proposition 6.1 has finite energy, concluding the proof.

Remark 7.4. *Let us point out that Theorem 1.2 does not need the information on the Morse index of the solutions. The main tool there is that $I^{c_n}(\psi_n) = O(1)$. Instead, Theorem 1.3 requires in a essential way that the Morse index of the solutions obtained is bounded.*

8. Appendix (by Rafael Ortega)

In this appendix we prove the following result:

Proposition 8.1. *Given $t \in \mathbb{R}$, $\omega_0 \in \mathbb{R}$, $\rho_0 > 0$ satisfying that $\omega_0^2 + c\omega_0 + \rho_0^2 = 1$, the function $\psi_0(x) = \rho_0 e^{i\omega_0(x-t)}$ is a (infinite energy) solution of (1.3). Assume also that $\rho_0^2 < \frac{2}{3}(1 + c^2/4)$. Then its Morse index, as defined in Definition 2.5, is infinity.*

Proof. The problem is autonomous so that we can assume $t = 0$. The proof is based on the study of the $1D$ problem:

$$\psi'' + ic\psi' + \left(1 - |\psi|^2\right)\psi = 0 \quad \text{on } \mathbb{R}. \tag{8.1}$$

By the change of variables $\phi = e^{ixc/2}\psi$ we pass to a problem:

$$\phi'' + \left(1 + c^2/4 - |\phi|^2\right)\phi = 0 \quad \text{on } \mathbb{R}. \tag{8.2}$$

The Morse index of this problem depends on the existence of conjugate points to some solutions of the linearized equation, see for instance [23, Chapter 5]. The function $\phi(x) = \rho_0 e^{i\omega_1 x}$ is a solution of (8.2), where , $\omega_1 = \omega_0 + c/2$. Observe that

$$\omega_1^2 + \rho_0^2 = 1 + c^2/4 \tag{8.3}$$

The linearized equation to (8.2) around the solution ϕ is:

$$\zeta'' + (1 + c^2/4)\zeta - 2\overline{\phi(s)}\phi(s)\zeta - \phi(s)^2\overline{\zeta} = 0.$$

We will follow the lines of [48, Section 21] to analyze the oscillatory properties of this equation.

$$\zeta'' + (1 + c^2/4)\zeta - 2\rho_0^2\zeta - \rho_0^2 e^{2i\omega_1 s}\overline{\zeta} = 0.$$

We now make the change of variable $\zeta = e^{i\omega_1 s}\eta$, to obtain a constant coefficient linear system:

$$\eta'' + 2i\omega_1\eta' - \rho_0^2\eta - \rho_0^2\overline{\eta} = 0. \tag{8.4}$$

If $\rho_0^2 < 2\omega_1^2$ (which, by (8.3), reduces to $\rho_0^2 < \frac{2}{3}(1 + c^2/4)$) we can find the explicit solution to (8.4):

$$\eta(s) = \frac{\sin\left(s\sqrt{4\omega_1^2 - 2\rho_0^2}\right)}{\sqrt{4\omega_1^2 - 2\rho_0^2}} + i\omega_1 \frac{\cos\left(s\sqrt{4\omega_1^2 - 2\rho_0^2}\right) - 1}{2\omega_1^2 - \rho_0^2}.$$

Clearly, $\zeta(s) = e^{i\omega_1 s}\eta(s)$ has infinitely many conjugate points $\frac{2\pi n}{\sqrt{4\omega_1^2 - 2\rho_0^2}}$, $n \in \mathbb{N}$.

Given any interval I, the quadratic functional $\tilde{Q}_{1,I} : H_0^1(I, \mathbb{C}) \to \mathbb{R}$,

$$\tilde{Q}_{1,I}(\sigma_k) = \int_I |\sigma_k'|^2 - (1 + c^2/4 - |\phi|^2)|\sigma_k|^2 + 2(\langle \sigma_k, \phi \rangle)^2 < 0$$

is in the conditions of Section 29.2 of [23]. We can apply [23, Theorem 3' in page 122] to deduce that $\tilde{Q}_{1,I}$ takes negative values as soon as the length of the interval I is greater than $\frac{2\pi}{\sqrt{4\omega_1^2 - 2\rho_0^2}}$. Then we can find infinitely many functions $\sigma_k \in C_0^\infty(\mathbb{R})$ *with disjoint support* such that

$$\tilde{Q}_1(\sigma_k) = \int_{-\infty}^{\infty} |\sigma_k'|^2 - (1 + c^2/4 - |\phi|^2)|\sigma_k|^2 + 2(\langle \sigma_k, \phi \rangle)^2 < 0.$$

We now want to pass to the original problem (8.1) and estimate its Morse index. In order to do so, define $\tau_k(s)$ by $\sigma_k(s) = e^{ics/2}\tau_k(s)$. Simple computations give:

$$\sigma_k'(s) = (i\frac{c}{2}\tau_k(s) + \tau_k'(s))e^{ics/2},$$

$$|\sigma_k'(s)|^2 = |\tau_k'(s)|^2 + \frac{c^2}{4}|\tau_k(s)|^2 + c\langle i\tau_k(s), \tau_k'(s)\rangle = |\tau_k'(s)|^2 + \frac{c^2}{4}|\tau_k(s)|^2 - c\langle \tau_k(s), i\tau_k'(s)\rangle.$$

Moreover,

$$\langle \sigma_k(s), \phi(s) \rangle = \langle \tau_k(s), \psi(s) \rangle.$$

As a consequence $\tilde{Q}_1(\sigma_k) = Q_1(\tau_k) < 0$, where

$$Q_1(\tau_k) = \int_{-\infty}^{\infty} |\tau_k'|^2 - c\langle \tau_k, i\tau_k' \rangle - (1 - |\psi|^2)|\tau_k|^2 + 2(\langle \tau_k, \psi \rangle)^2.$$

Observe that this is the quadratic form associated to (8.1).

Take now a C_0^∞ function $\chi_k : \mathbb{R}^{d-1} \to \mathbb{R}^+$, and let us estimate Q on the function $\iota_k(x) = \chi_k(\tilde{x})\tau_k(x_1)$, where Q is defined in (2.2):

$$Q(\iota_k) = Q_1(\tau_k) \int_{\mathbb{R}^{d-1}} \chi_k(\tilde{x})^2 \, d\tilde{x} + \left(\int_{-\infty}^{+\infty} |\tau_k(x_1)|^2 \, dx_1 \right) \left(\int_{\mathbb{R}^{d-1}} |\nabla \chi_k(\tilde{x})|^2 \, d\tilde{x} \right).$$

It suffices to take now χ_k such that $\int_{\mathbb{R}^{d-1}} \chi_k^2 = 1$ and $\int_{\mathbb{R}^{d-1}} |\nabla \chi_k|^2$ is sufficiently small, to conclude that $Q(\iota_k) < 0$.

Observe also that $supp\ \iota_k \cap supp\ \iota_{k'} = \emptyset$ if $k \neq k'$, since an analogue property holds for σ_k and τ_k. Hence, Q is negative definite on the vector space generated by the linearly independent functions $\{\iota_1, \ \ldots, \iota_k\}$ for any $k \in \mathbb{N}$, concluding the proof.

□

References

[1] A. Ambrosetti, A. Malchiodi, Nonlinear analysis and semilinear elliptic problems, Cambridge Studies in Advanced Mathematics, 104. Cambridge University Press, Cambridge, 2007.

[2] WW. Ao, Y. Huang, Y. Li and J. Wei, Generalized Adler-Moser polynomials and multiple vortex rings for the Gross-Pitaevskii equation, SIAM J. Math. Anal. 53 (2021), no. 6, 6959-6992..

[3] I. V. Barashenkov and V. G. Makhan'kov, Soliton-like bubbles in a system of interacting bosons, Phys. Lett. A 128 (1988), 52-56.

[4] H. Berestycki, T. Gallouet and O. Kavian, Équations de champs scalaires euclidiens non linéaires dans le plan, C. R. Acad. Sci. Paris Sr. I Math. 297 (1983) 307-310.

[5] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations, I, Arch. Ration. Mech. Anal. 82 (1983) 313-346.

[6] N. G. Berloff, Quantised vortices, travelling coherent structures and superfluid turbulence, in Stationary and Time Dependent Gross-Pitaevskii Equations, Contemp. Math. 473, Amer. Math. Soc., Providence, RI, 2008, pp. 27-54.

[7] A. L. Bertozzi and A. Majda, Vorticity and incompressible flow, Cambridge Texts in Applied Mathematics, Cambridge University Press 2002.

[8] F. Béthuel, P. Gravejat and J.-C. Saut, Traveling Waves for the Gross-Pitaevskii Equation II, Comm. Math. Physics 285, 567-651 (2009).

[9] F. Béthuel, P. Gravejat and J.-C. Saut, Existence and properties of travelling waves for the Gross-Pitaevskii equation, Alberto Farina and Jean-Claude Saut. Stationary and time dependent Gross-Pitaevskii equations, 473, American Mathematical Society, pp.55-104, 2008, Contemporary Mathematics, 978-0-8218-4357-4. 10.1090/conm/473. http/www.ams.org/books/conm/473/. hal-00363329.

[10] F. Béthuel, P. Gravejat, J.-C. Saut, and D. Smets, Orbital stability of the black soliton for the Gross–Pitaevskii equation, Indiana Univ. Math. J, 57(6):26112642, 2008.

[11] F. Béthuel, P. Gravejat and Didier Smets, Asymptotic stability in the energy space for dark solitons of the Gross–Pitaevskii equation, Annales Scientifiques de l'École Normale Supérieure, (2015), 48 (6), pp.1327-1381

[12] F. Béthuel, G., Orlandi and D. Smets, Vortex rings for the Gross-Pitaevskii equation, J. Eur. Math. Soc. 6(1),17-94 (2004).

[13] F. Béthuel and J. C. Saut, Travelling waves for the Gross-Pitaevskii equation I, Ann. Inst. Henri Poincaré, Physique Théorique. 70(2), 147-238 (1999).

[14] D. Chiron, Travelling waves for the Gross-Pitaevskii equation in dimension larger than two, Nonlinear Anal., 58(1-2):175204, 2004.

[15] D. Chiron, E. Pacherie, Coercivity for travelling waves in the Gross-Pitaevskii equation in $\mathbb{R}^2$ for small speed, preprint arXiv 1911.03944.

[16] D. Chiron, E. Pacherie, Smooth branch of travelling waves for the Gross-Pitaevskii equation in $\mathbb{R}^2$ for small speed, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) Vol. XXII (2021), 1937-2038.

[17] D. Chiron and M. Mariş, Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity, Arch. Ration. Mech. Anal. 226, no. 1, 143-242, (2017)

[18] A. Enciso and D. Peralta-Salas, Approximation theorems for the Schrödinger equation and quantum vortex reconnection, Comm. Math. Phys. 387 (2021), no. 2, 1111-1149.

[19] M.J. Esteban and P. L. Lions, Existence and non-existence results for semilinear elliptic problems in unbounded domains, Proc. Royal Soc. Edinburgh 93 A (1982), 1-14.

[20] G. Fang, N. Ghoussoub, Second-order information on Palais-Smale sequences in the mountain pass theorem. Manuscripta Math. 75 (1992), no. 1, 81-95.

[21] G. Fang, N. Ghoussoub, Morse-type information on Palais-Smale sequences obtained by min-max principles, Comm. Pure Appl. Math. 47 (1994), 1595-1653.

[22] A. Farina, From Ginzburg-Landau to Gross-Pitaevskii, Monatsh. Math. 139, 265-269 (2003).

[23] I. M. Gelfand and S. V. Fomin, Calculus of variations. Revised English edition translated and edited by Richard A. Silverman Prentice-Hall, Inc., Englewood Cliffs, N.J. 1963.

[24] P. Gérard, The Cauchy problem for the Gross-Pitaevskii equation, Ann. Inst. H. Poincaré ANL 23(5) (2006), 765779.

[25] P. Gérard, The Gross-Pitaevskii equation in the energy space. Stationary and time dependent Gross-Pitaevskii equations, Contemp. Math. 473, Amer. Math. Soc., Providence, RI, (2008), 129148.

[26] P. Gravejat, A Non-Existence Result for Supersonic traveling Waves in the Gross-Pitaevskii Equation, Commun. Math. Phys. 243, 93-103 (2003).

[27] P. Gravejat, Decay for traveling waves in the Gross-Pitaevskii equation, Ann. I. H. Poincaré ANL 21 (2004) 591-637.

[28] P. Gravejat, Asymptotics for the travelling waves in the Gross-Pitaevskii equation,Asymptot. Anal., 45(3-4):227299, 2005

[29] P. Gravejat, First order asymptotics for the travelling waves in the Gross-Pitaevskii equation, Adv. Differential Equations, 11(3):259280, 2006

[30] P. Gravejat, Limit at infinity and nonexistence results for sonic travelling waves in the Gross-Pitaevskii equation, Differential Integral Equations 17, no. 11-12, 12131232 (2004)

[31] E.P. Gross, Hydrodynamics of a superfluid condensate, J. Math. Phys., 4(2):195-207, 1963.

[32] S. Gustafson, K. Nakanishi and T.P. Tsai, Scattering theory for the Gross-Pitaevskii equation in three dimensions, Commun. Contemp. Math. 11(4) (2009), 657-707.

[33] S. Gustafson, K. Nakanishi and T.P. Tsai, Scattering for the GrossPitaevskii equation, Math. Res. Letters 13(2) (2006), 273285

[34] L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbb{R}^N$, Proceedings of the Royal, Society of Edinburgh, 129A, 787-809, 1999.

[35] R.L. Jerrard and D. Smets, Leapfrogging vortex rings for the three-dimensional Gross–Pitaevskii equation, Ann. PDE 4 (2018), no. 1, Art. 4, 48 pp

[36] C. A. Jones, S. J. Putterman and P. H. Roberts, Motions in a Bose condensate V. Stability of solitary wave solutions of nonlinear Schrödinger equations in two and three dimensions, J. Phys. A, Math. Gen. 19, 2991-3011 (1986).

[37] C. A. Jones and P. H. Roberts, Motions in a Bose condensate IV, Axisymmetric Solitary Waves. J. Phys. A, Math. Gen. 5, 2599-2619 (1982).

[38] R. Killip, T. Oh, O. Pocovnicu, and M. Vişan, Global well-posedness of the Gross-Pitaevskii and cubic-quintic nonlinear Schrödinger equations with nonvanishing boundary conditions, Math. Res. Lett. 19 (2012), 969-986.

[39] Y.S. Kivshar and B. Luther-Davies, Dark optical solitons: physics and applications, Phys. Rep., 298:81-197, 1998.

[40] Y.S. Kivshar, D.E. Pelinovsky, and Y.A. Stepanyants, Self-focusing of plane dark solitons in nonlinear defocusing media, Phys. Rev. E, 51(5):5016-5026, 1995.

[41] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case II, Ann. Inst. H. Poincaré Anal. Non Linéaire 1, (1984), 223-283

[42] T. Lin, J. Wei and J. Yang, Vortex rings for the Gross-Pitaevskii equation in $\mathbb{R}^3$, J. Math. Pures Appl. 100 (2013) 69-112

[43] Y. Liu and J. Wei Multi-vortex traveling waves for the Gross-Pitaevskii equation and the Adler-Moser polynomials, SIAM Journal of Mathematical Analysis 52 (2020), no. 4, 3546-3579.

[44] Y. Liu, Z. Wang, J. Wei and W. Yang, From KP-I lump solution to travelling waves of Gross-Pitaevskii equation, preprint arXiv 2110.15472.

[45] M. Mariş, Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity, Ann. of Math. (2) 178 (2013), no. 1, 107-182.

[46] M. Mariş, Nonexistence of supersonic traveling waves for nonlinear Schrodinger equations with nonzero conditions at infinity, SIAM J. Math. Anal. 40 (2008), 1076-1103.

[47] L.P. Pitaevskii, Vortex lines in an imperfect Bose gas. Sov. Phys. JETP, 13(2): 451-454, 1961.

[48] C. L. Siegel and J. K. Moser, Celestial Mechanics, Springer, 1971.

[49] M. Struwe, The existence of surfaces of constant mean curvature with free boundaries, Acta Math. 160 (1988), no. 1-2, 19-64.

[50] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications, 24. Birkhuser Boston, Inc., Boston, MA, 1996.

[51] P.E. Zhidkov, The Cauchy problem for the nonlinear Schrödinger equation, Joint Inst. Nucl. Res., Dubna (1987), 15 pp.

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