PÉNDULOS INVERTIDOS Y SUS ECUACIONES

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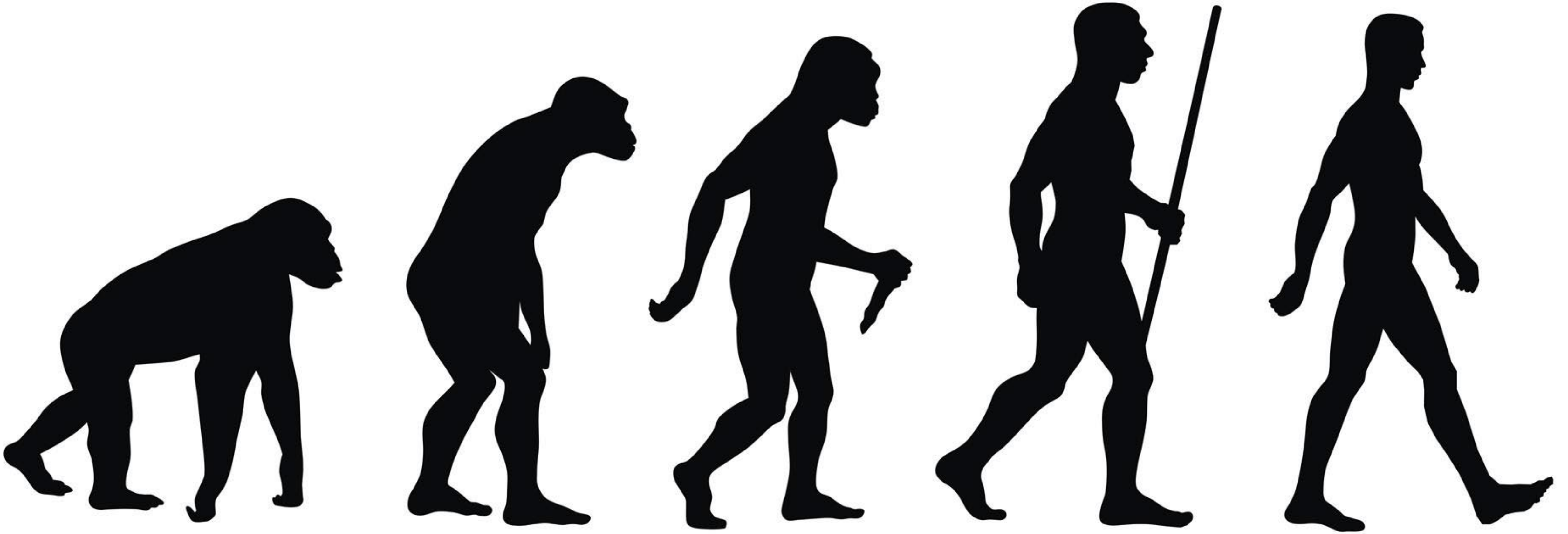
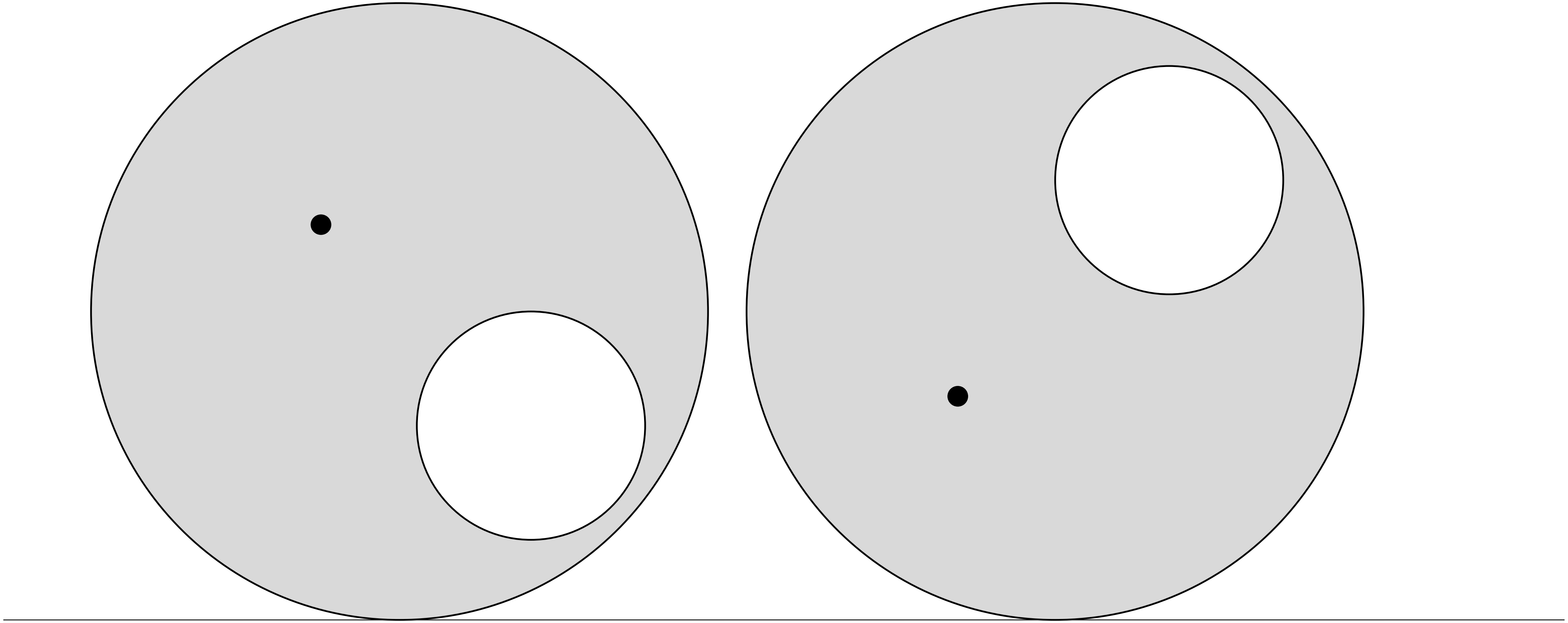


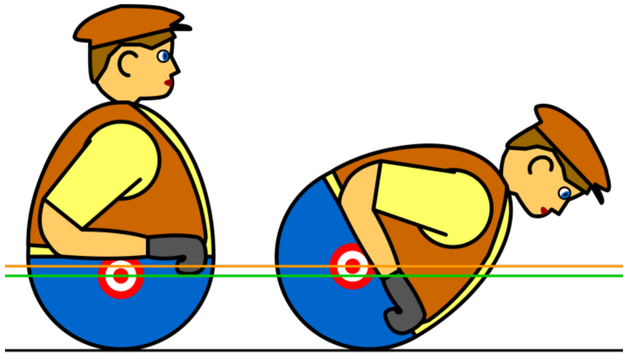
EL VUELCO



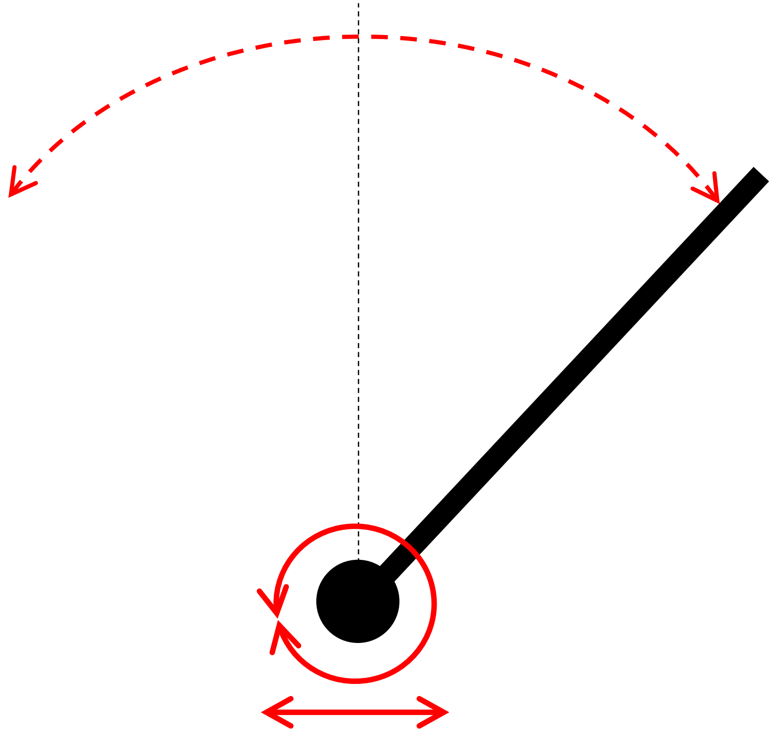
INESTABLE ante el vuelco

ESTABLE ante el vuelco

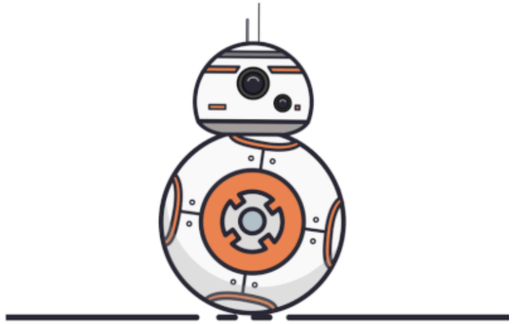
SISTEMAS ANTIVUELCO/AUTONIVELADO/AUTOBALANCEO



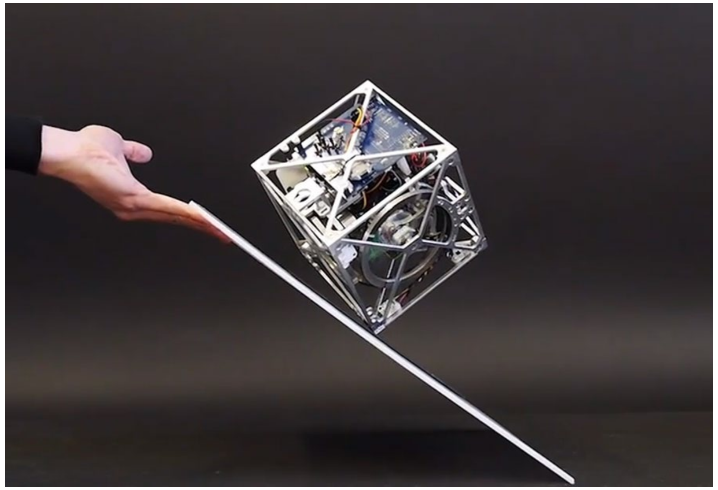
Centro de masas "cerca" del punto pivote/giro: estable espontáneamente (guiado por el peso)



Centro de masas "lejos" del punto pivote/giro: requiere actuación de contrabalanceo (sobre el pivote o por un contragiro)



BB8



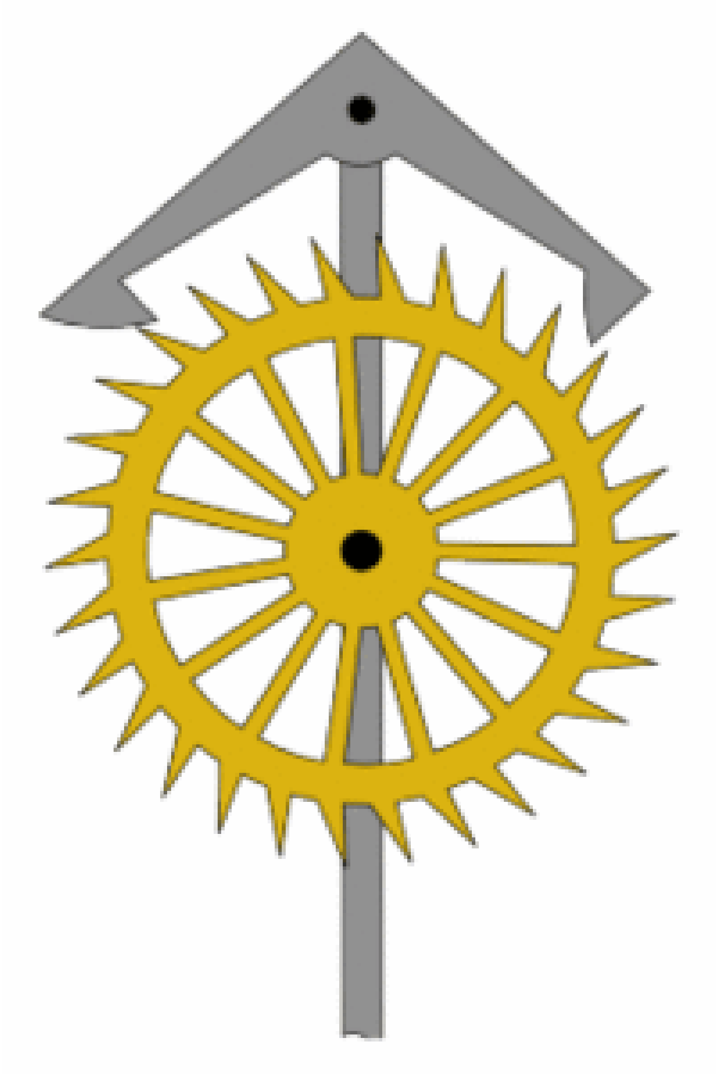
Gyro-cube

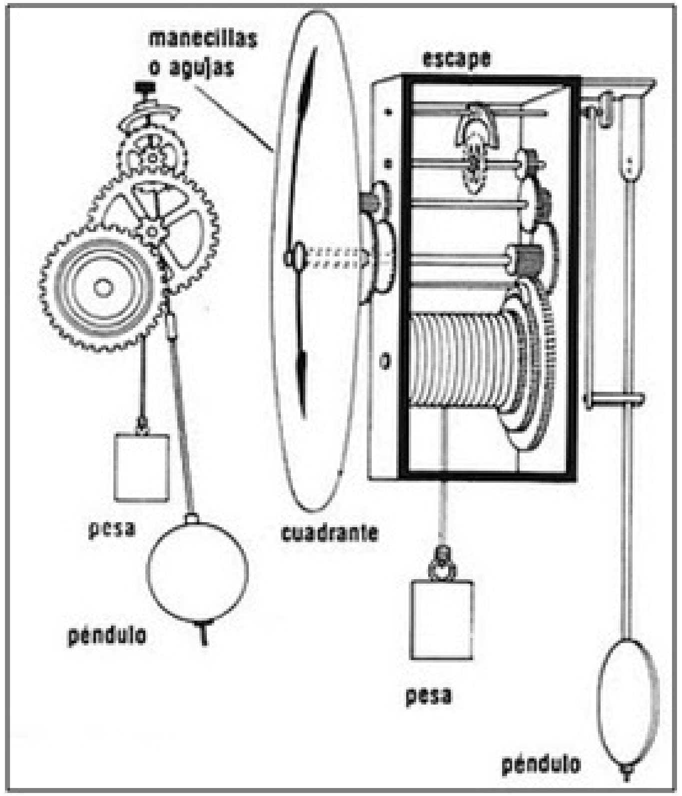
SISTEMAS ANTIVUELCO/AUTONIVELADO

GÖMBÖC: cuerpo convexo tridimensional homogéneo, que cuando descansa sobre una superficie plana, tiene un solo punto de equilibrio estable y otro inestable.

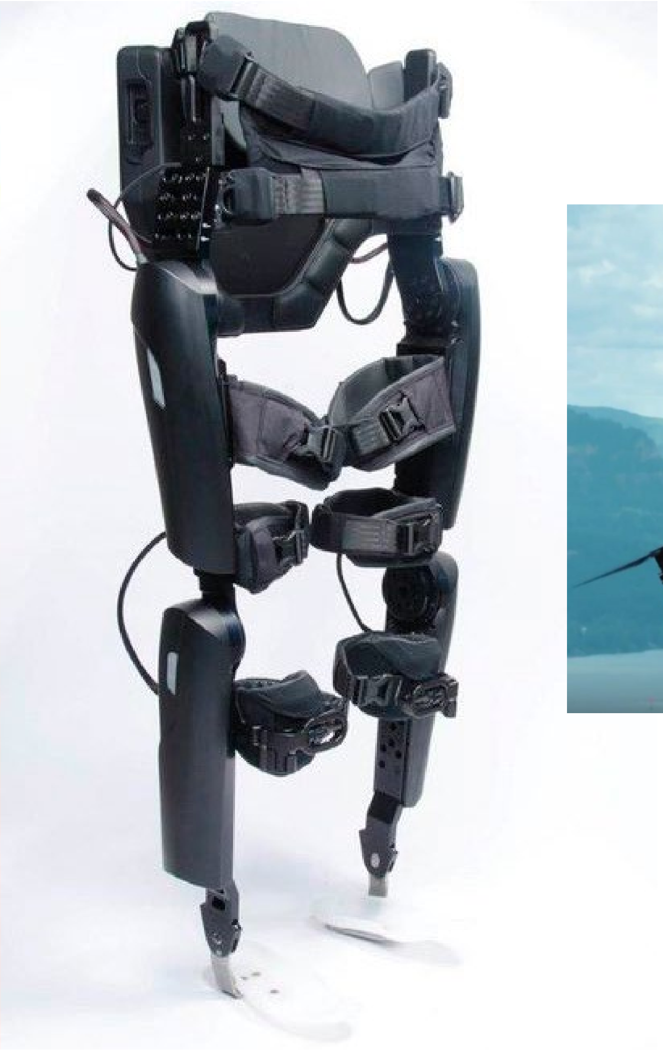
MECANISMO DE ESCAPE DE UN RELOJ DE PÉNDULO

Tic-tac





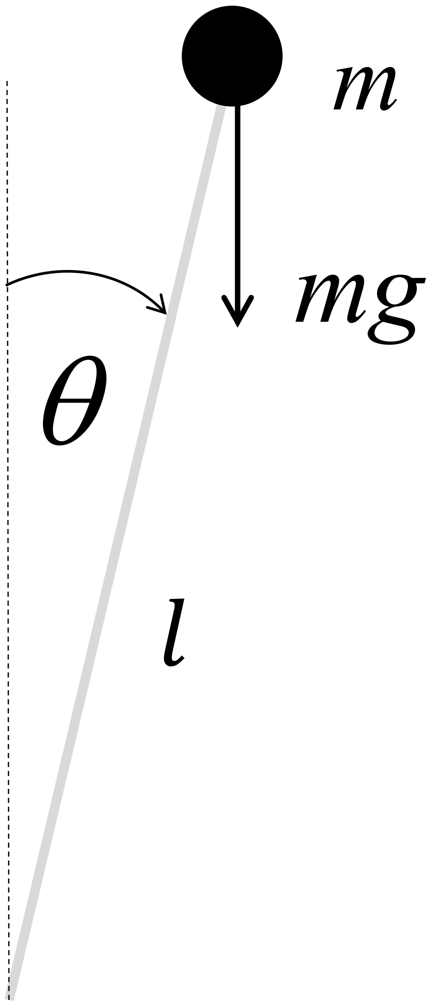
BIOMECÁNICA



Balancing with Vibration: A Prelude for "Drift and Act" Balance Control, PLoS One. 2009; 4(10): e7427.

EL PÉNDULO INVERTIDO LIBRE MÁS SIMPLE

$$\dot{\theta} \equiv \frac{d\theta}{dt}; \ddot{\theta} \equiv \frac{d^2\theta}{dt^2}$$



$$t = mgl \operatorname{sen} \theta$$

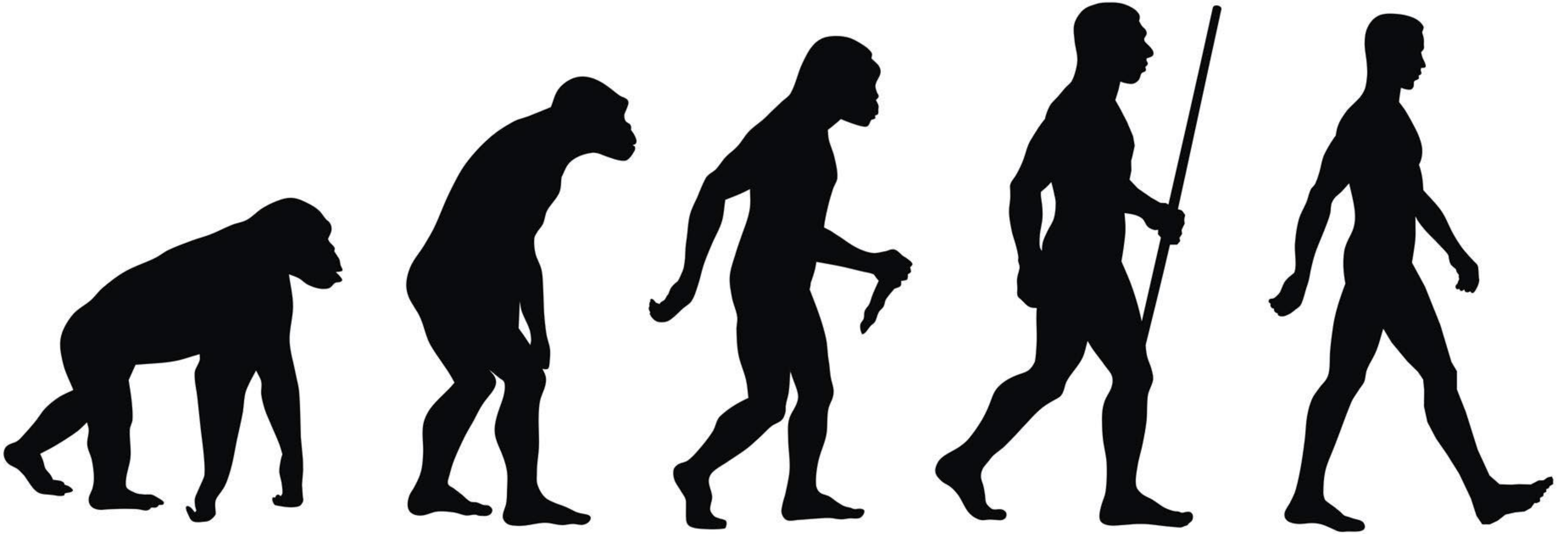
$$I = ml^2$$

$$\tau = I\ddot{\theta}$$

$$\ddot{\theta} = \frac{g}{l} \operatorname{sen} \theta$$

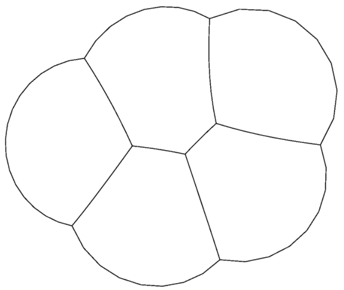
PROBLEMA DE ESTABILIDAD (ENERGÉTICA)

LAS MATEMÁTICAS QUE NECESITAMOS



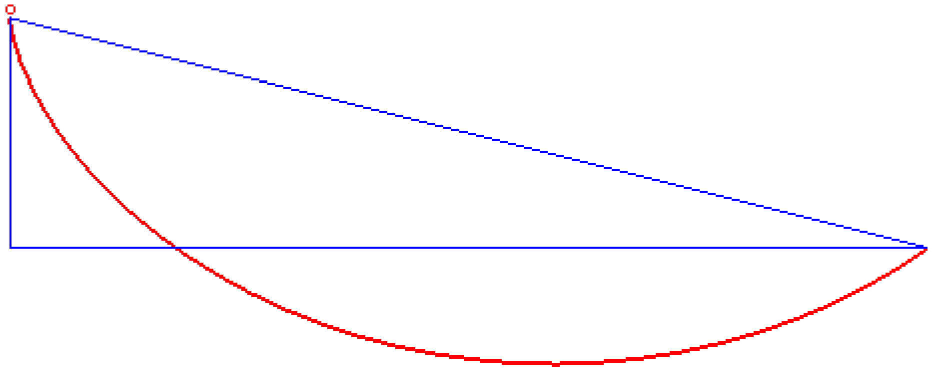
CÁLCULO VARIACIONAL

Problemas isovolúmicos/isoperimétricos: ¿Cuerpo regular de menor área para un volumen/perímetro dado? (i.e. compacidad máxima y angulosidad mínima)

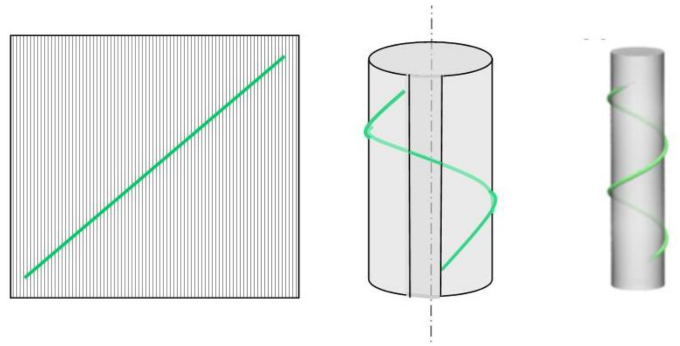


La catenaria (mínima energía potencial gravitatoria)

La braquistócrona (mínimo tiempo)



Línea Geodésica (mínima distancia entre dos puntos)



$$\text{extremo}\left\{\int_{x_1}^{x_2} \mathbb{F}\left[y(x), y'(x); x\right] dx\right\} \Leftrightarrow \frac{d}{dx}\left(\frac{\partial \mathbb{F}}{\partial y'}\right) - \frac{\partial \mathbb{F}}{\partial y} = 0$$ Ec. EULER

CÁLCULO VARIACIONAL EN FÍSICA

De todas las trayectorias posibles (compatibles con las ligaduras), que puede seguir un sistema dinámico para desplazarse de un punto a otro en un intervalo de tiempo determinado, la trayectoria verdaderamente seguida es aquella que hace mínima la integral temporal de la diferencia entre las energías cinética y potencial.

Energía cinética: energía por estar en movimiento
Energía potencial: energía por estar en una posición o configuración determinada

$$\mathbb{L} \equiv T[\dot{q};t] - U[q;t] \Rightarrow \min \left\{ \int_{t_1}^{t_2} \mathbb{L}[\dot{q},q;t]\,dt \right\} \Leftrightarrow \frac{d}{dt}\left(\frac{\partial \mathbb{L}}{\partial \dot{q}}\right) - \frac{\partial \mathbb{L}}{\partial q} = 0$$

LAGRANGIANO

Ec. EULER-LAGRANGE

Transformada de Legendre

$$\mathbb{H} \equiv \dot{q}\frac{\partial \mathbb{L}}{\partial \dot{q}} - \mathbb{L}$$

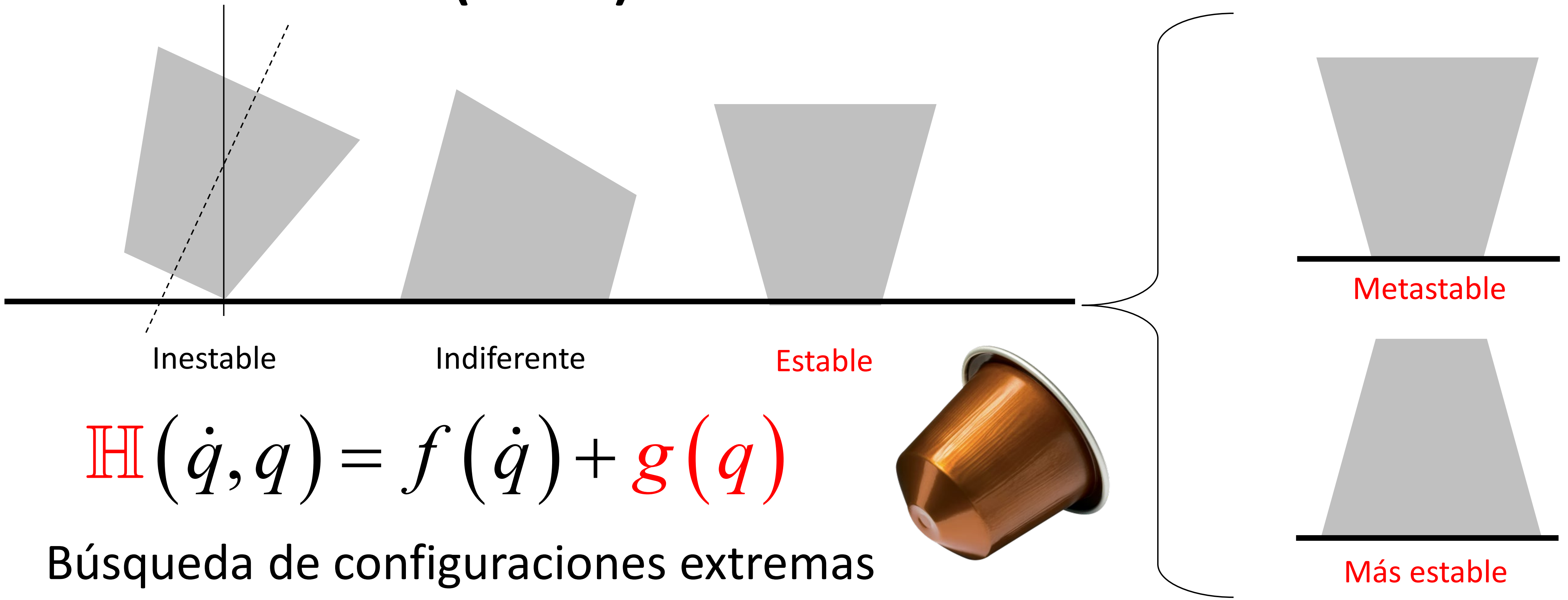
HAMILTONIANO=verdadero sentido energético

Minimización condicionada

$$f(q) = 0$$

$$\tilde{\mathbb{L}} \equiv \mathbb{L} - \lambda(t) f(q)$$

CONFIGURACIONES (META)ESTABLES E INESTABLES



$$\mathbb{H}(\dot{q},q) = f(\dot{q}) + g(q)$$

Búsqueda de configuraciones extremas

$$\left.\frac{\partial g(q)}{\partial q}\right|_{q_{eq}} = 0; \left(\left.\frac{\partial^2 g(q)}{\partial q^2}\right|_{q_{eq}}\right) \begin{matrix} < 0 \\ > 0 \end{matrix}$$

< 0 Máximo local/global inestabilidad

> 0 Mínimo local/global metaestabilidad

ECUACIONES DIFERENCIALES DE SEGUNDO ORDEN HOMOGÉNEAS NO LINEALES

$$\frac{d^2y}{dx^2} - \omega^2 \sin(y) = 0$$ Solución no oscilatoria

$$\frac{d^2y}{dx^2} + \omega^2 \sin(y) = 0$$ Solución oscilatoria

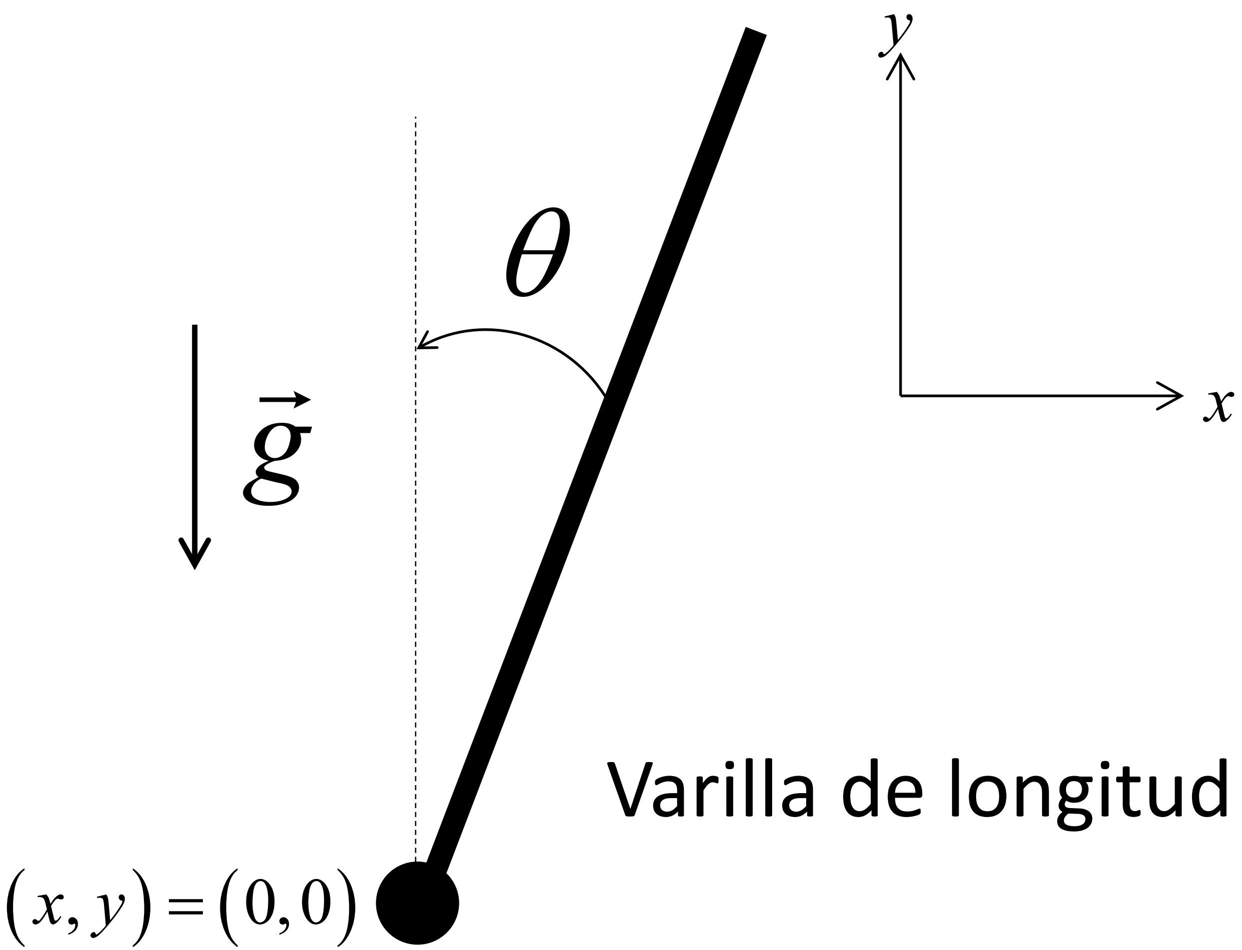
$$\frac{d^2y}{dx^2} \pm \omega^2(y) \sin(y) = 0$$ Solución dependiente del signo

$$\frac{d^2y}{dx^2} \pm \omega^2(x) \sin(y) = 0$$ Solución más compleja dependiente del signo

Modelado del caminar natural

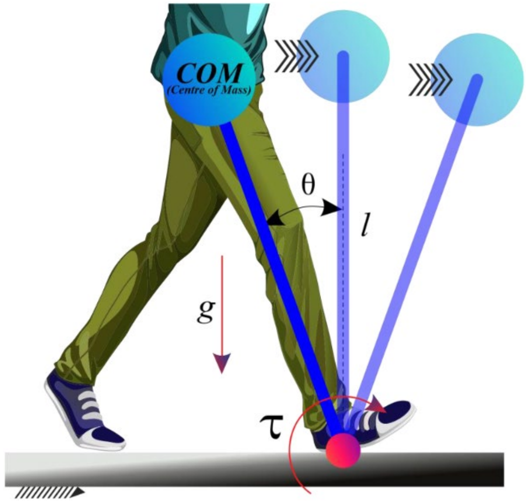
Modelo: Pierna rígida (sin rodilla) libre= varilla pivotada en el suelo

Un paso= medio ciclo



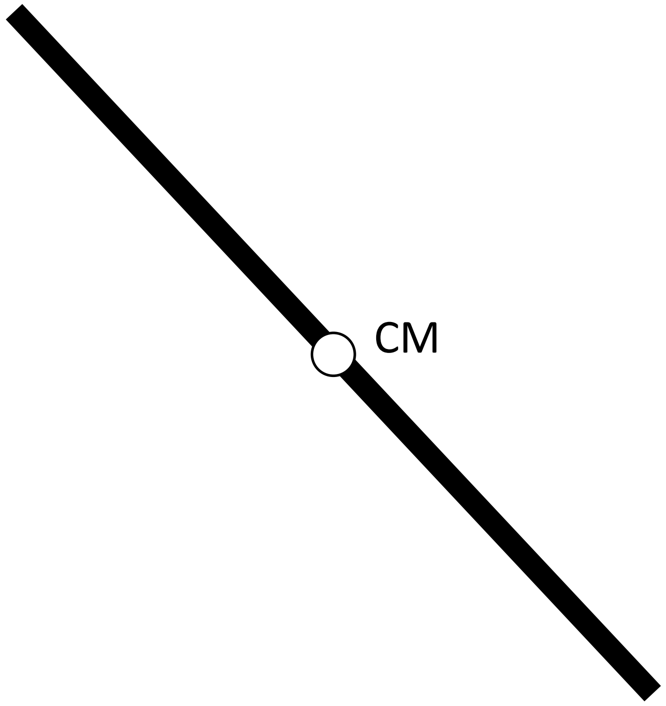
1 Grado de libertad

Varilla de longitud *l* y masa *m*



Energía cinética de un sólido rígido

$$T = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2 = \frac{1}{2} m \left(\left(\dot{x}_{cm} \right)^2 + \left(\dot{y}_{cm} \right)^2 \right) + \frac{1}{2} I_{cm} \dot{\theta}^2$$



$$\begin{cases} x_{cm} = \frac{l}{2} \sin\theta \\ y_{cm} = -\frac{l}{2} \cos\theta \\ I_{cm} = \frac{1}{12} m l^2 \end{cases} \Rightarrow \begin{cases} \dot{x}_{cm} = \frac{l}{2} \dot{\theta} \cos\theta \\ \dot{y}_{cm} = \frac{l}{2} \dot{\theta} \sin\theta \end{cases}$$

$$T = \frac{1}{2} m \left(\frac{l}{2} \dot{\theta} \right)^2 + \frac{1}{24} m l^2 \dot{\theta}^2$$

$$U = mgy_{cm} - mg\frac{l}{2} = mg\frac{l}{2}(\cos\theta - 1)$$

Energía potencial gravitatoria

$$\mathbb{L} = \frac{1}{6}m\left(l\dot{\theta}\right)^2 - \frac{1}{2}mgl\left(\cos\theta - 1\right)$$

Lagrangiano

$$\frac{\partial\mathbb{L}}{\partial\theta} = \frac{1}{2}mgl\sin\theta; \frac{\partial\mathbb{L}}{\partial\dot{\theta}} = \frac{1}{3}ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial\mathbb{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathbb{L}}{\partial\theta} = 0; \frac{1}{3}ml^2\ddot{\theta} - \frac{1}{2}mgl\sin\theta = 0$$

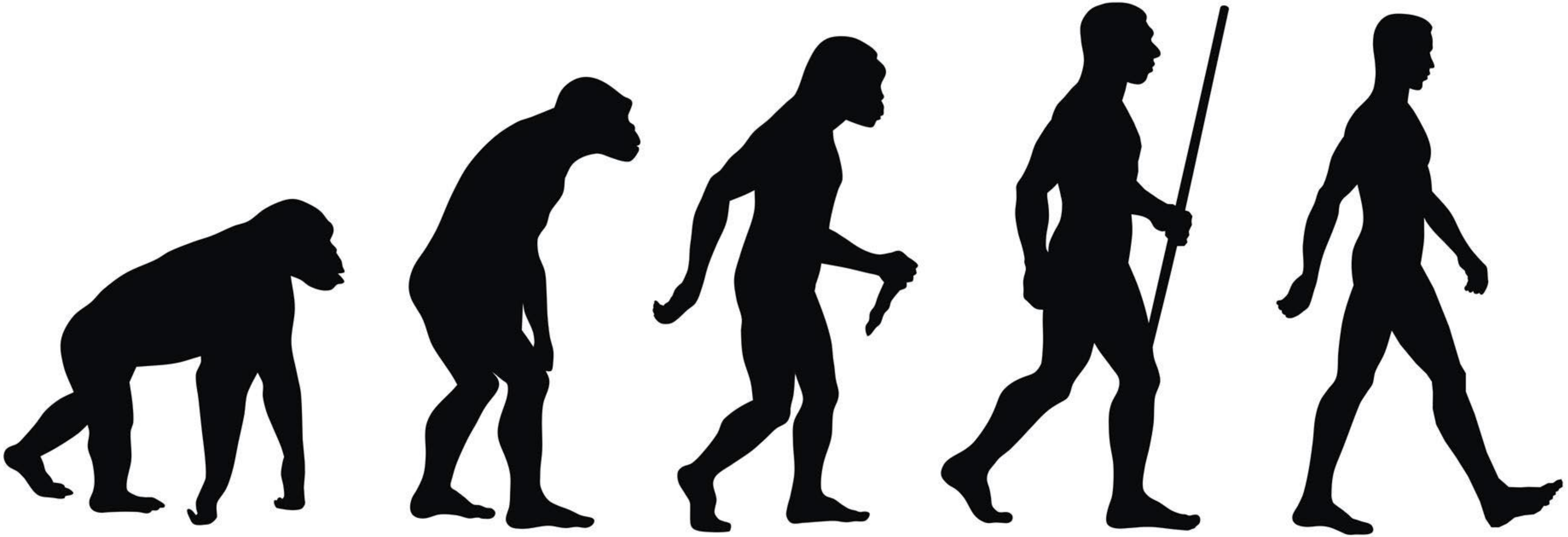
$$\ddot{\theta} - \frac{3g}{2l}\sin\theta = 0$$

$$\theta \ll 1rad \Rightarrow \ddot{\theta} - \frac{3g}{2l}\theta = 0$$

Solución no oscilatoria

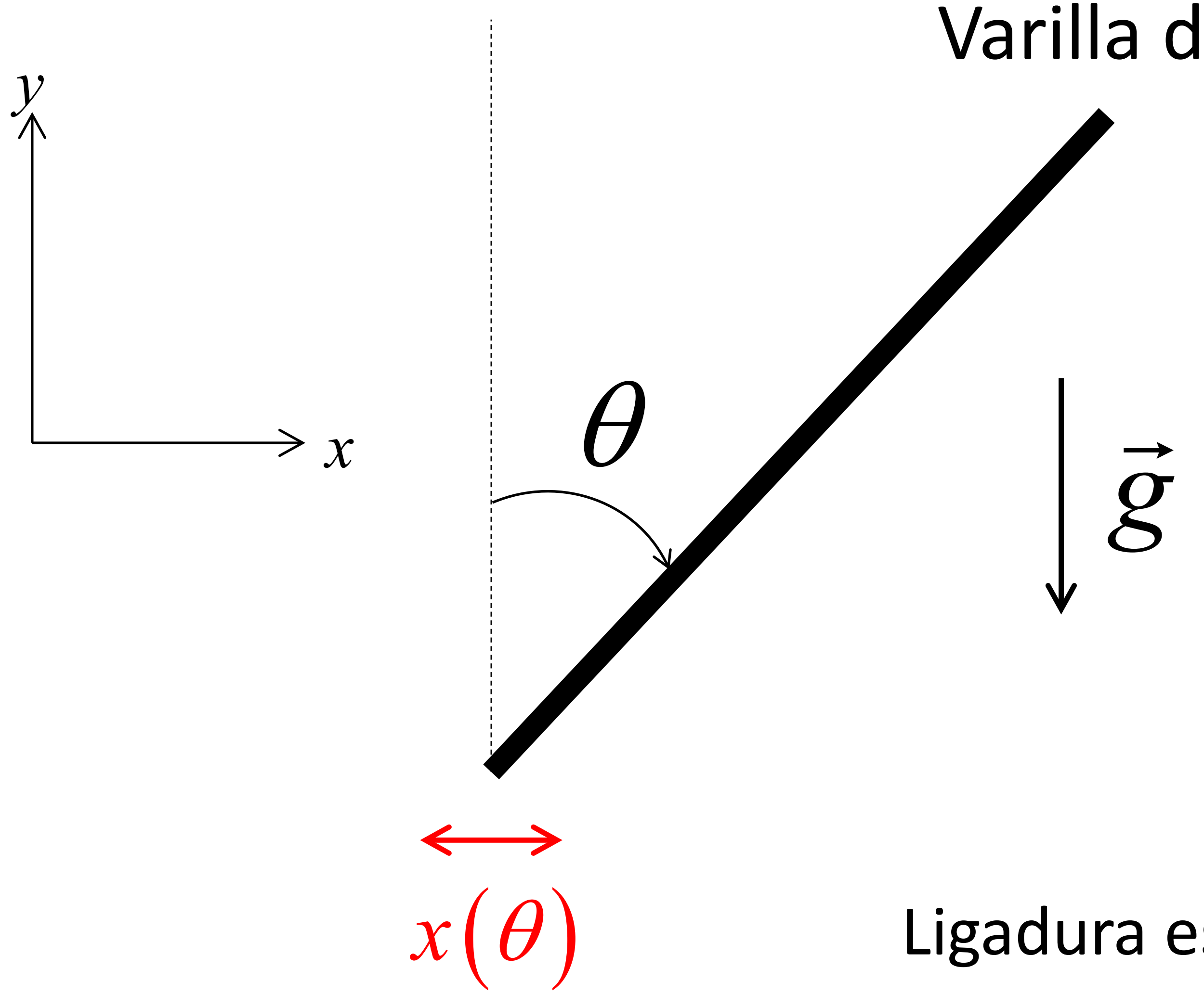
$$\sqrt{\frac{3g}{2l}} \approx 1.25\,pasos/s$$

ACTUACIÓN SOBRE PÉNDULO INVERTIDO



Péndulo invertido y forzado horizontalmente con una señal dependiente de la inclinación

Varilla de longitud *l* y masa *m*



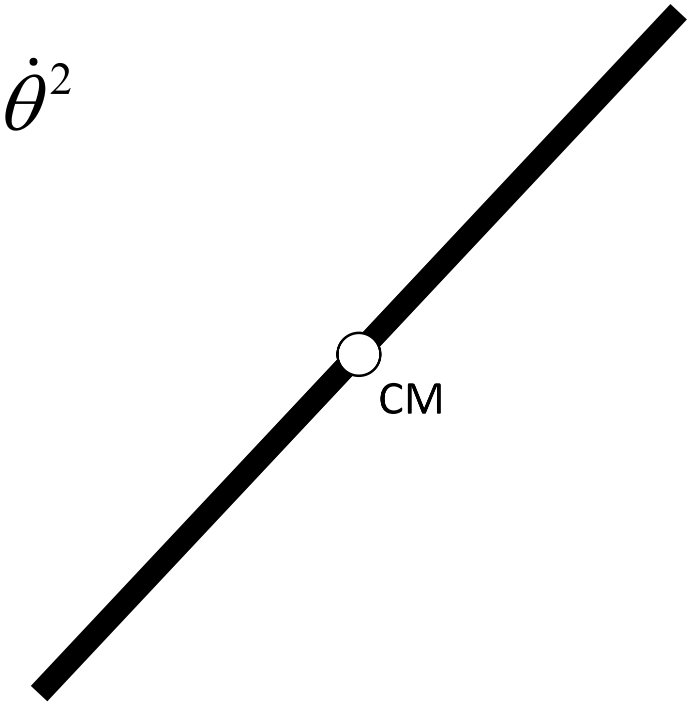
1 Grado de libertad

Ligadura esclerónoma holónoma

Energía cinética de un sólido rígido

$$T=\frac{1}{2}mv_{cm}^2+\frac{1}{2}I_{cm}\omega^2=\frac{1}{2}m\left(\left(\dot{x}_{cm}\right)^2+\left(\dot{y}_{cm}\right)^2\right)+\frac{1}{2}I_{cm}\dot{\theta}^2$$

$$\begin{cases} x_{cm}=\dfrac{l}{2}\sin\theta+x(\theta) \\ y_{cm}=\dfrac{l}{2}\cos\theta \\ I_{cm}=\dfrac{1}{12}ml^2 \end{cases} \Rightarrow \begin{cases} \dot{x}_{cm}=\dfrac{l}{2}\dot{\theta}\cos\theta+\dot{x}(\theta) \\ \dot{y}_{cm}=-\dfrac{l}{2}\dot{\theta}\sin\theta \end{cases}$$



$$T=\frac{1}{2}m\left(\left(\frac{l}{2}\dot{\theta}\right)^2+l\dot{\theta}\cos\theta\dot{x}(\theta)+\left(\dot{x}(\theta)\right)^2\right)+\frac{1}{24}ml^2\dot{\theta}^2$$

$$U = mgy_{cm} - mg\frac{l}{2} = mg\frac{l}{2}(\cos\theta - 1)$$

Energía potencial gravitatoria

$$\mathbb{L} = \frac{1}{2}m\left(\frac{1}{3}(l\dot{\theta})^2 + l\dot{\theta}\cos\theta\dot{x}(\theta) + (\dot{x}(\theta))^2\right) - mg\frac{l}{2}(\cos\theta - 1)$$

Lagrangiano

$$\frac{\partial\mathbb{L}}{\partial\theta} = \frac{1}{2}m\left(-l\dot{\theta}\sin\theta\dot{x}(\theta) + l\cos\theta\ddot{x}(\theta) + 2\dot{x}(\theta)\frac{d\dot{x}(\theta)}{d\theta}\right) + mg\left(\frac{l}{2}\sin\theta\right)$$

$$\frac{\partial\mathbb{L}}{\partial\dot{\theta}} = \frac{1}{2}m\left(\frac{2}{3}l^2\dot{\theta} + l\cos\theta\dot{x}(\theta)\right)$$

$$\frac{d}{dt}\left(\frac{\partial\mathbb{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathbb{L}}{\partial\theta} = 0$$

$$\frac{1}{2}m\left(\frac{2}{3}l^2\ddot{\theta} - l\dot{\theta}\sin\theta\dot{x}(\theta) + l\cos\theta\ddot{x}(\theta)\right) - \frac{1}{2}m\left(-l\dot{\theta}\sin\theta\dot{x}(\theta) + l\cos\theta\ddot{x}(\theta) + 2\dot{x}(\theta)\frac{d\dot{x}(\theta)}{d\theta}\right) - mg\left(\frac{l}{2}\sin\theta\right) = 0$$

$$\ddot{\theta} + \frac{3g}{2l}\left(-1 - \frac{2\dot{x}(\theta)}{gl\sin\theta}\frac{d\dot{x}(\theta)}{d\theta}\right)\sin\theta = 0$$

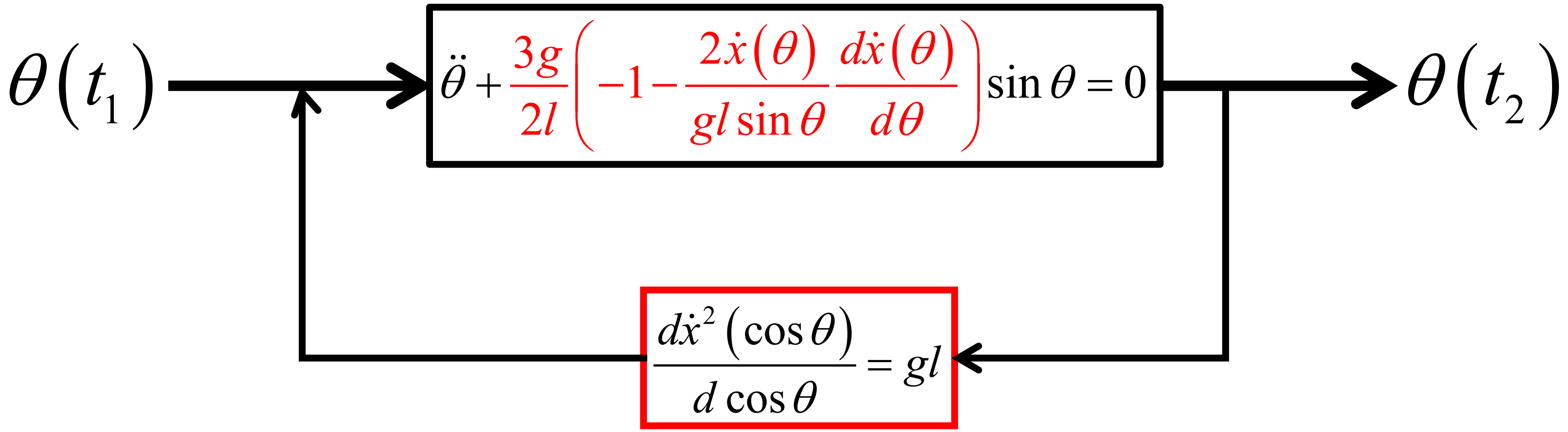
$$\frac{3g}{2l}\left(\frac{1}{gl}\frac{d\dot{x}^2(\cos\theta)}{d\cos\theta} - 1\right) = \omega^2$$

Solución oscilatoria: Péndulo

$$\frac{d\dot{x}^2(\cos\theta)}{d\cos\theta} = gl$$

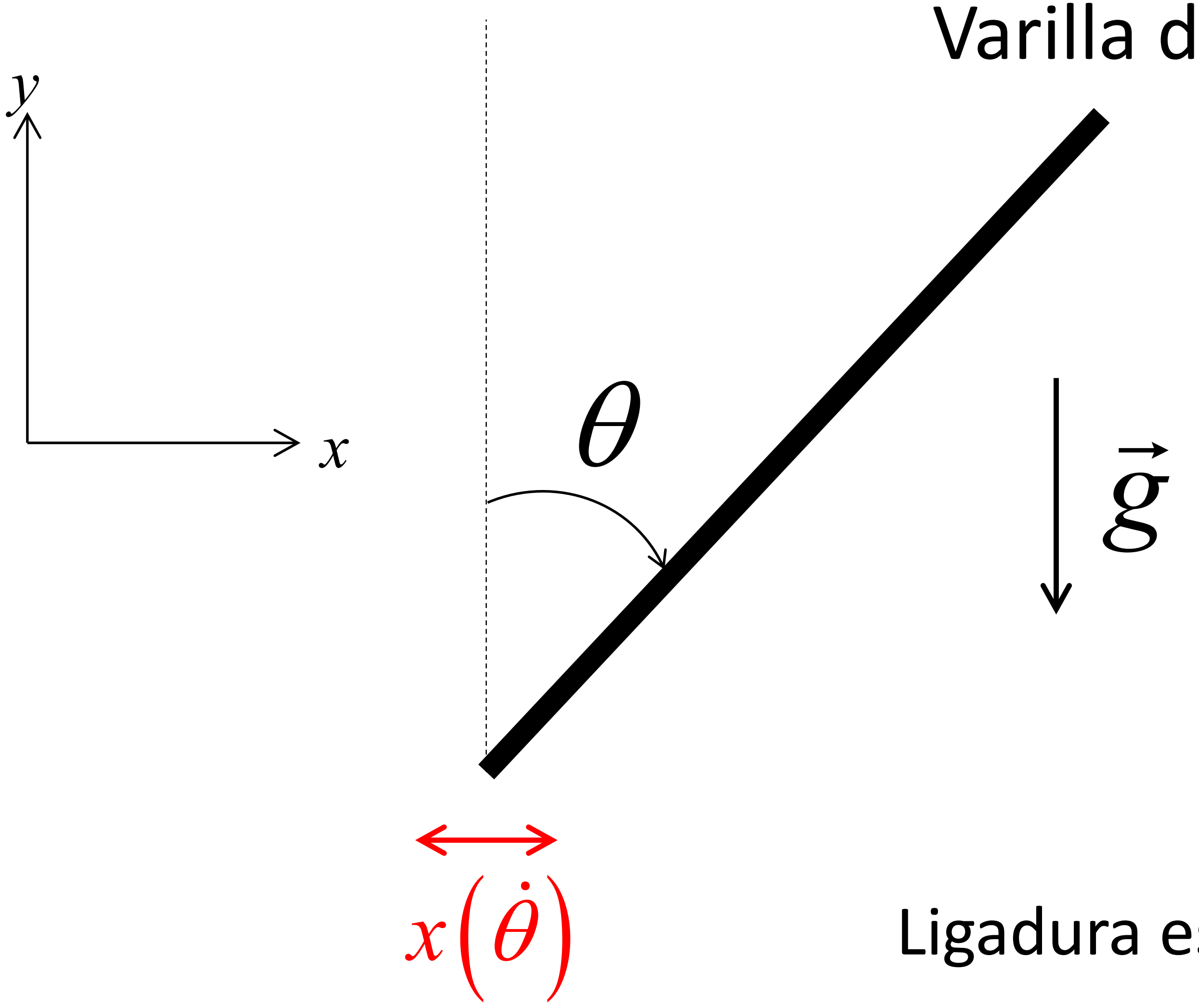
Solución nula: Equilibrio

Lazo de control: cómo lo resuelve un ingeniero



$$\dot{x}(\theta \neq 0) = \operatorname{sgn}(\theta)\sqrt{2gl}\cos\frac{\theta}{2}$$

Péndulo invertido y forzado horizontalmente con una señal dependiente del cambio de la inclinación



Varilla de longitud *l* y masa *m*

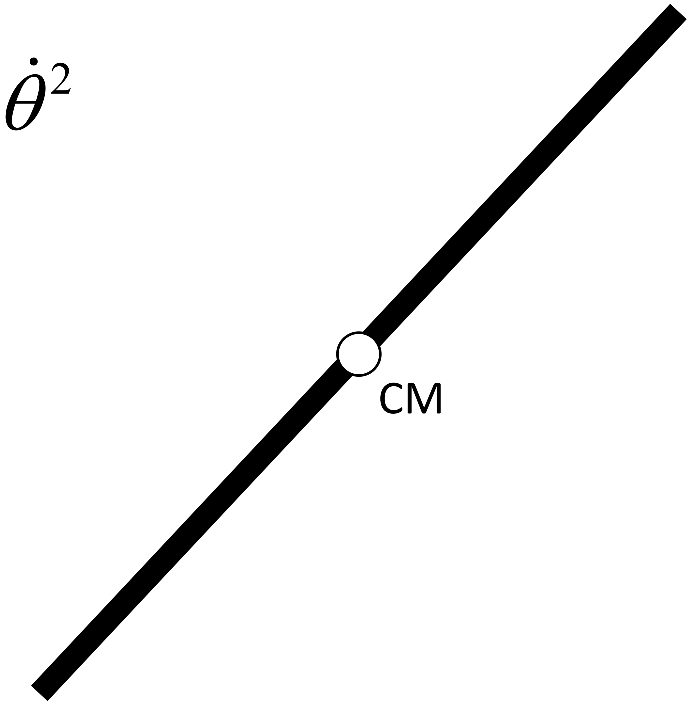
1 Grado de libertad

Ligadura esclerónoma holónoma

Energía cinética de un sólido rígido

$$T = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm}\omega^2 = \frac{1}{2} m\left(\left(\dot{x}_{cm}\right)^2 + \left(\dot{y}_{cm}\right)^2\right) + \frac{1}{2} I_{cm}\dot{\theta}^2$$

$$\begin{cases} x_{cm} = \frac{l}{2}\sin\theta + x\left(\dot{\theta}\right) \\ y_{cm} = \frac{l}{2}\cos\theta \\ I_{cm} = \frac{1}{12} m l^2 \end{cases} \Rightarrow \begin{cases} \dot{x}_{cm} = \frac{l}{2}\dot{\theta}\cos\theta + \dot{x}\left(\dot{\theta}\right) \\ \dot{y}_{cm} = -\frac{l}{2}\dot{\theta}\sin\theta \end{cases}$$



$$T = \frac{1}{2} m\left(\left(\frac{l}{2}\dot{\theta}\right)^2 + l\dot{\theta}\cos\theta\,\dot{x}\left(\dot{\theta}\right) + \left(\dot{x}\left(\dot{\theta}\right)\right)^2\right) + \frac{1}{24} m l^2\dot{\theta}^2$$

$$U = mgy_{cm} - mg\frac{l}{2} = mg\frac{l}{2}(\cos\theta - 1)$$

Energía potencial gravitatoria

$$\mathbb{L} = \frac{1}{2}m\left(\frac{1}{3}(l\dot{\theta})^2 + l\dot{\theta}\cos\theta\dot{x}(\dot{\theta}) + (\dot{x}(\dot{\theta}))^2\right) - mg\frac{l}{2}(\cos\theta - 1)$$

Lagrangiano

$$\frac{\partial\mathbb{L}}{\partial\theta} = \frac{1}{2}m(-l\dot{\theta}\sin\theta\dot{x}) + mg\left(\frac{l}{2}\sin\theta\right)$$

$$\frac{\partial\mathbb{L}}{\partial\dot{\theta}} = \frac{1}{2}m\left(\frac{2}{3}l^2\dot{\theta} + l\cos\theta\dot{x} + (l\dot{\theta}\cos\theta + 2\dot{x})\frac{d\dot{x}}{d\dot{\theta}}\right)$$

$$\frac{d}{dt}\left(\frac{\partial\mathbb{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathbb{L}}{\partial\theta} = 0$$

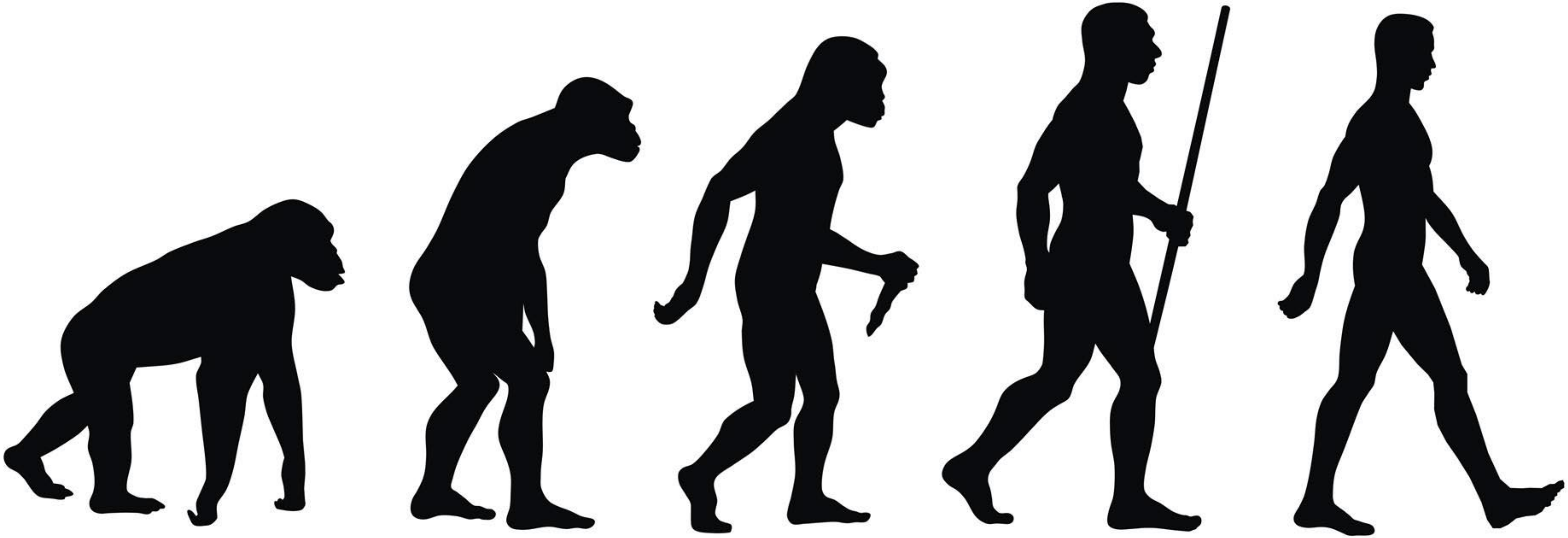
$$\frac{1}{2}m\left(\frac{2}{3}l^2\ddot{\theta} + l\cos\theta\ddot{x} + \frac{d}{dt}\left((l\dot{\theta}\cos\theta + 2\dot{x})\frac{d\dot{x}}{d\dot{\theta}}\right)\right) - mg\left(\frac{l}{2}\sin\theta\right) = 0$$

$$\frac{1}{2}m\left(\frac{2}{3}l^2\ddot{\theta} + l\cos\theta\ddot{x} + (l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta + 2\ddot{x})\frac{d\dot{x}}{d\dot{\theta}} + (l\dot{\theta}\cos\theta + 2\dot{x})\frac{d\ddot{x}}{d\dot{\theta}}\right) - mg\left(\frac{l}{2}\sin\theta\right) = 0$$

$(\ldots)$

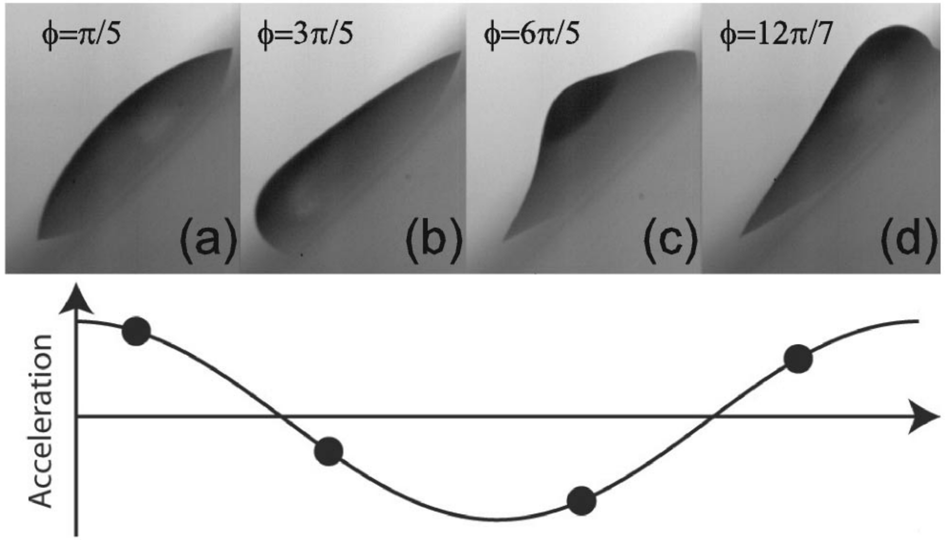
SEÑAL PARECIDA, SOLUCIÓN MUY DIFERENTE

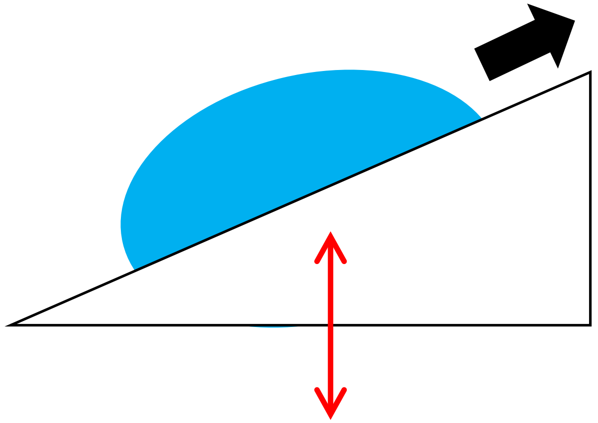
VIBRACIÓN VERTICAL



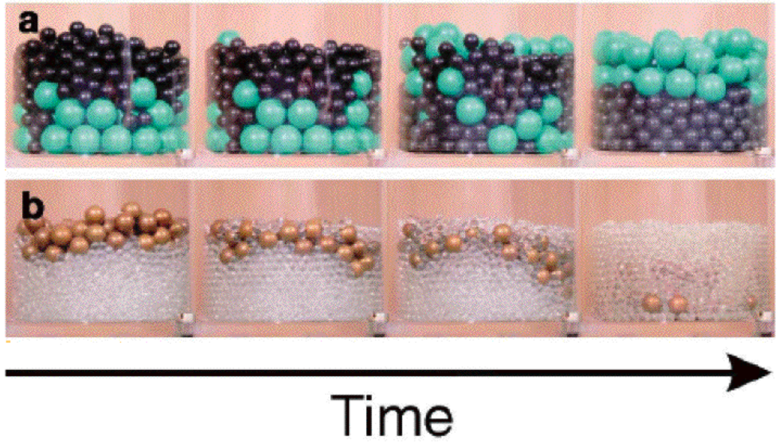
Consecuencias exóticas de vibraciones (verticales)

Vibration-Induced Climbing of Drops   PRL **99,** 144501 (2007)



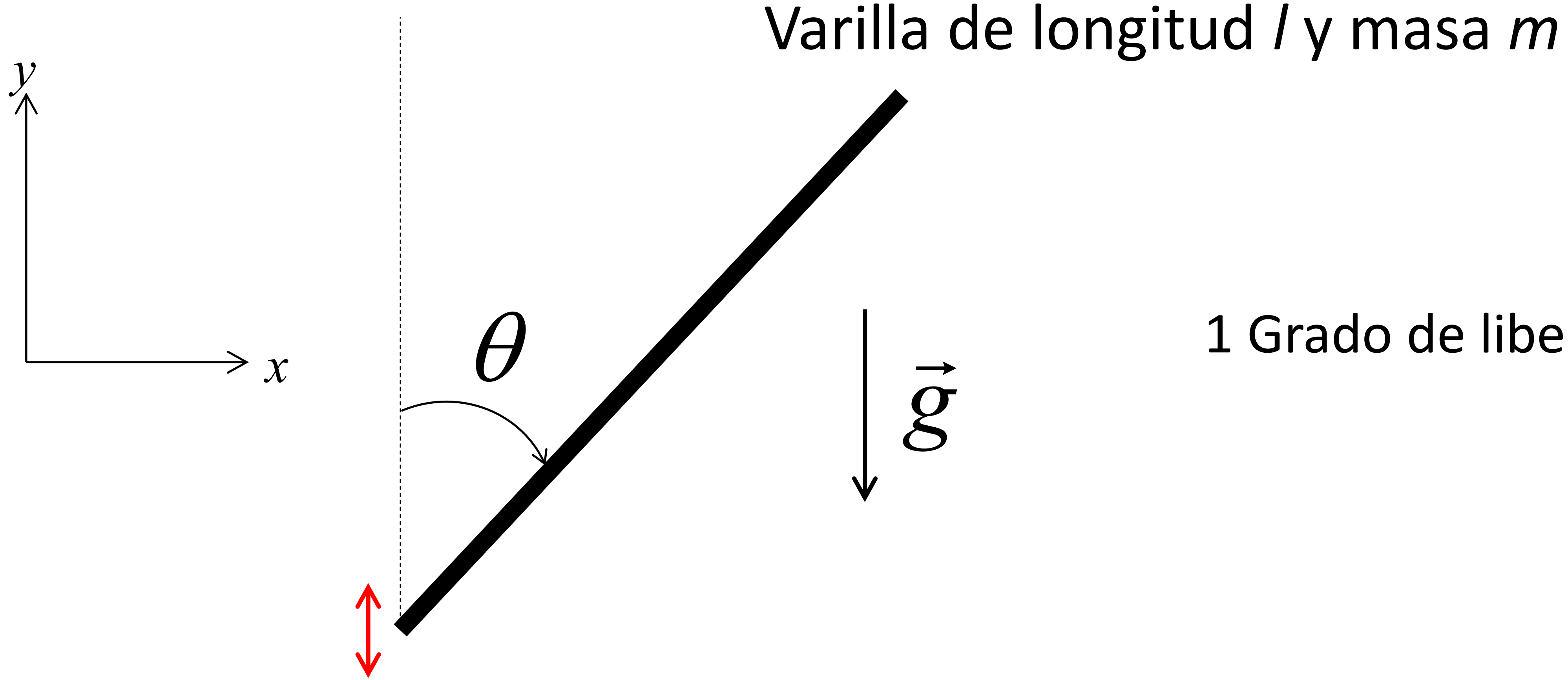


Brazil nut effect: segregation/stratification induced by vertical vibrations



Spontaneous stratification in granular mixtures,
Nature 386, 379–382 (1997)

Péndulo invertido y excitado verticalmente con una señal senoidal



1 Grado de libertad

$$y(t) = A\sin(\omega_0 t)$$

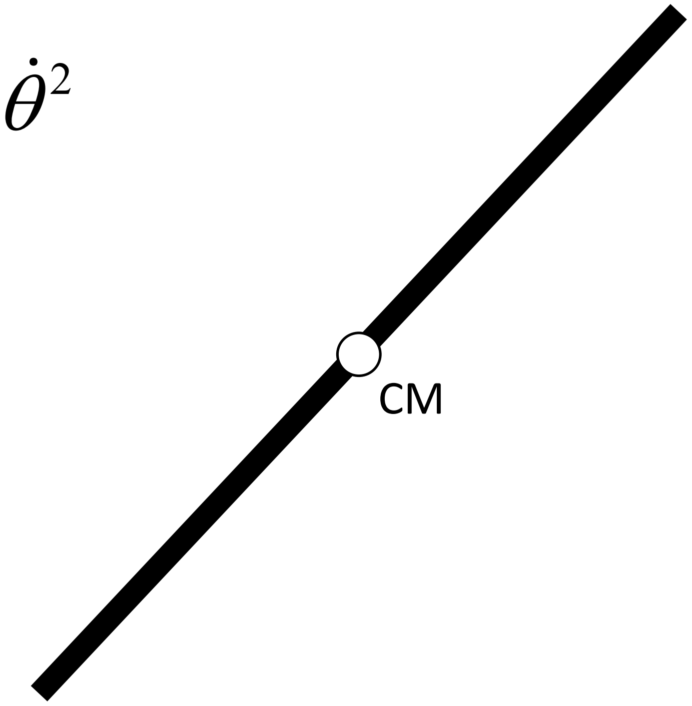
Ligadura reónoma

Stabilization by vertical vibrations, Math. Meth. Appl. Sci.2009;32:1118–1128

Energía cinética de un sólido rígido

$$T=\frac{1}{2}mv_{cm}^2+\frac{1}{2}I_{cm}\omega^2=\frac{1}{2}m\left(\left(\dot{x}_{cm}\right)^2+\left(\dot{y}_{cm}\right)^2\right)+\frac{1}{2}I_{cm}\dot{\theta}^2$$

$$\begin{cases} x_{cm}=\frac{l}{2}\sin\theta \\ y_{cm}=\frac{l}{2}\cos\theta+A\sin\left(\omega_0 t\right) \\ I_{cm}=\frac{1}{12}ml^2 \end{cases}\Rightarrow\begin{cases} \dot{x}_{cm}=\frac{l}{2}\dot{\theta}\cos\theta \\ \dot{y}_{cm}=-\frac{l}{2}\dot{\theta}\sin\theta+\omega_0 A\cos\left(\omega_0 t\right) \end{cases}$$



$$T=\frac{1}{2}m\left(\left(\frac{l}{2}\dot{\theta}\right)^2-l\dot{\theta}\omega_0 A\sin\theta\cos\left(\omega_0 t\right)+\left(\omega_0 A\right)^2\cos^2\left(\omega_0 t\right)\right)+\frac{1}{24}ml^2\dot{\theta}^2$$

$$U = mgy_{cm} - mg\frac{l}{2} = mg\left(\frac{l}{2}(\cos\theta - 1) + A\sin(\omega_0 t)\right)$$

Energía potencial gravitatoria

Lagrangiano

$$\mathbb{L} = \frac{1}{2}m\left(\frac{1}{3}(l\dot{\theta})^2 - l\dot{\theta}\omega_0 A\sin\theta\cos(\omega_0 t) + (\omega_0 A)^2\cos^2(\omega_0 t)\right) - \frac{1}{2}m\left(gl(\cos\theta - 1) + 2gA\sin(\omega_0 t)\right)$$

$$\frac{\partial\mathbb{L}}{\partial\theta} = \frac{1}{2}m\left(-l\dot{\theta}\omega_0 A\cos\theta\cos(\omega_0 t)\right) + mg\left(\frac{l}{2}\sin\theta\right)$$

$$\frac{\partial\mathbb{L}}{\partial\dot{\theta}} = \frac{1}{2}m\left(\frac{2}{3}l^2\dot{\theta} - l\omega_0 A\sin\theta\cos(\omega_0 t)\right)$$

$$\frac{d}{dt}\left(\frac{\partial\mathbb{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathbb{L}}{\partial\theta} = 0$$

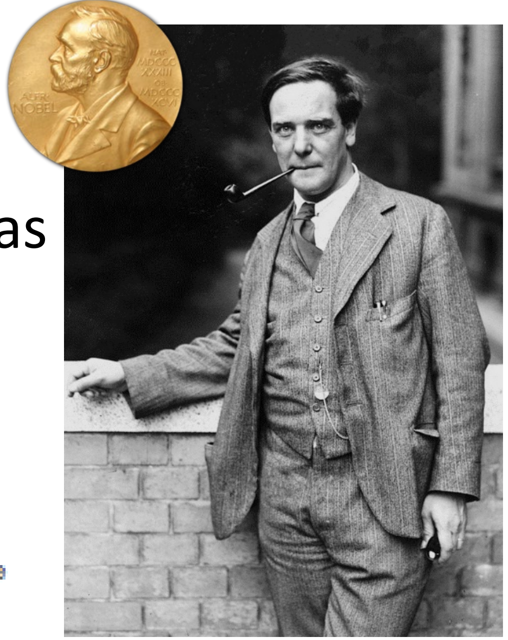
$$\frac{1}{2}m\left(\frac{2}{3}l^2\ddot{\theta} - l\omega_0 A\dot{\theta}\cos\theta\cos(\omega_0 t) + l(\omega_0)^2 A\sin\theta\sin(\omega_0 t)\right) - \frac{1}{2}m\left(-l\dot{\theta}\omega_0 A\cos\theta\cos(\omega_0 t)\right) - mg\left(\frac{l}{2}\sin\theta\right) = 0$$

$$\ddot{\theta} + \frac{3g}{2l}\left(-1 + \frac{A(\omega_0)^2}{g}\sin(\omega_0 t)\right)\sin\theta = 0$$

$$\boxed{\frac{A(\omega_0)^2}{g}\sin(\omega_0 t) > 1}$$

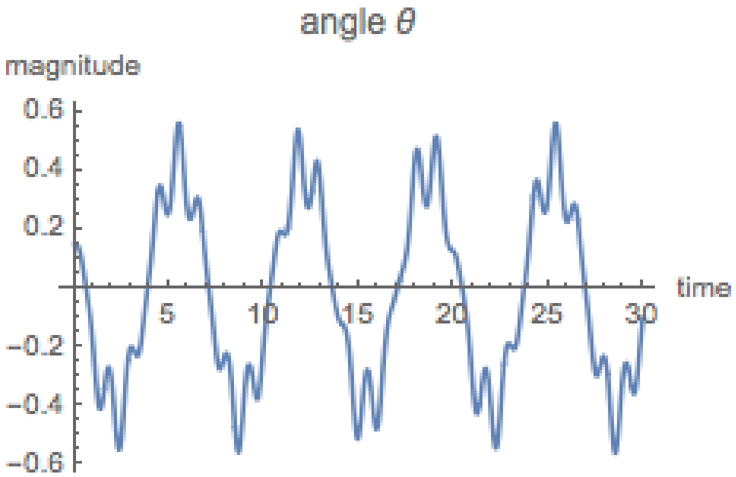
Péndulo (paramétrico)

Kapitza averaging



Perturbación de ángulo promedio (lento) con vibraciones pequeñas y rápidas

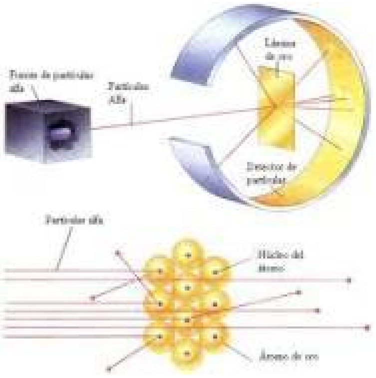
$$\theta = \phi + \xi(t) \qquad \phi = cte\ durante\ \ \Delta t = \frac{2\pi}{\omega_0}$$



Kapitza P. L. (1951). "Dynamic stability of a pendulum when its point of suspension vibrates". Soviet Phys. JETP. ***21****: 588–597.; Kapitza P. L. (1951). "Pendulum with a vibrating suspension". Usp. Fiz. Nauk.* ***44****: 7–15*

The effect of Kapitza pendulum and price equilibrium, Physica A 324 (2003) 388 – 395

- En 1908, A. Stephenson encontró que en la posición vertical, con el péndulo arriba, puede ser estable si la frecuencia de excitación es los suficientemente alta
- En 1951, el científico ruso Pyotr Kapitza analizó exitosamente este inusual y contraintuitivo fenómeno, dividiendo el movimiento en uno rápido y otro lento, e introduciendo un potencial efectivo



Hamiltoniano

$$\mathbb{H} = \dot{\theta}\frac{\partial \mathbb{L}}{\partial \dot{\theta}} - \mathbb{L} = \frac{1}{2}m\left(\frac{1}{3}l^2\left(\dot{\theta}\right)^2 - \left(\omega_0 A\right)^2\cos^2\left(\omega_0 t\right)\right) + \frac{1}{2}m\left(gl\left(\cos\theta - 1\right) + 2gA\sin\left(\omega_0 t\right)\right)$$

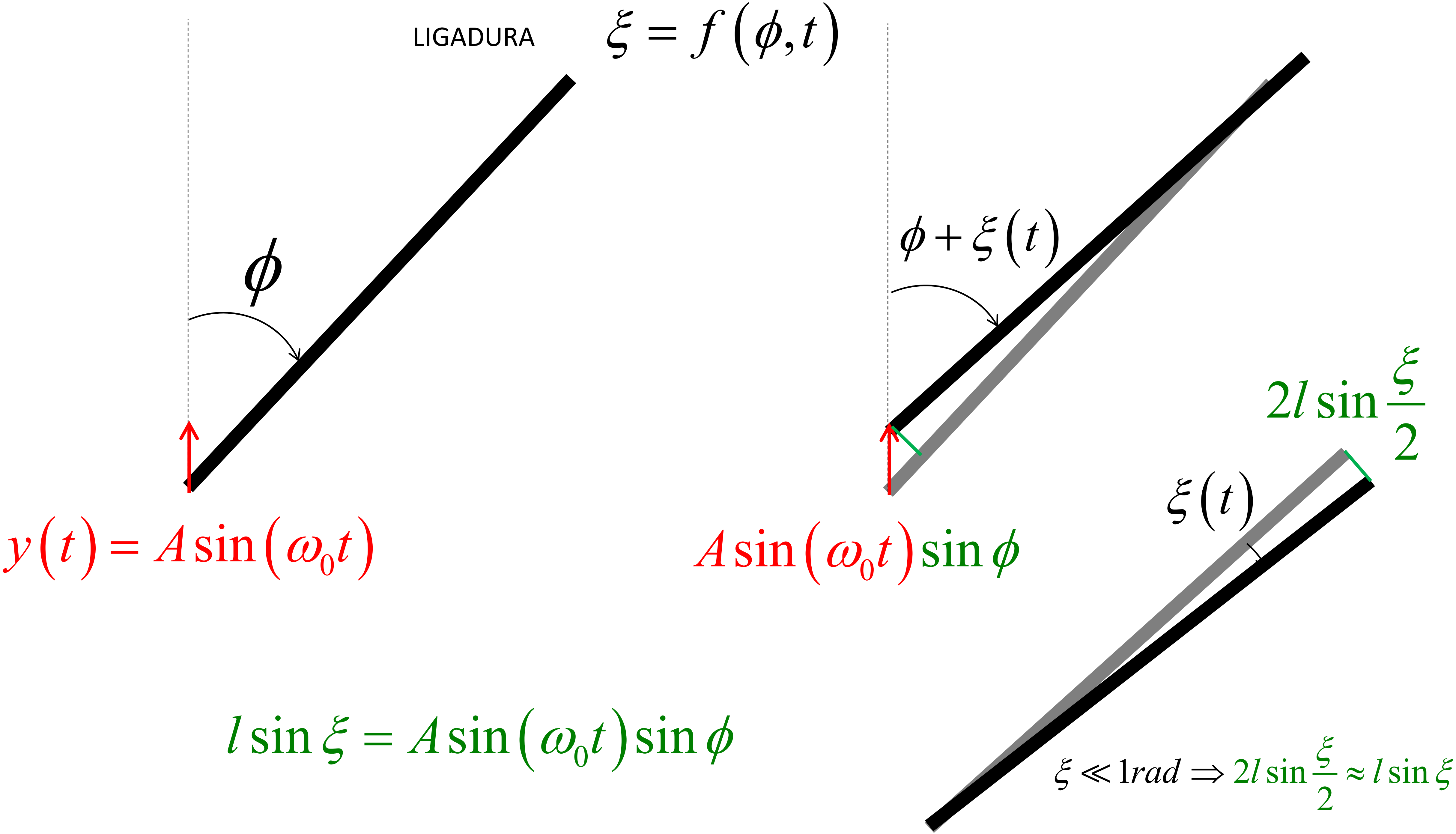
$$= \frac{1}{6}ml^2\left(\dot{\theta}\right)^2 + \frac{1}{2}m\left(-\left(\omega_0 A\right)^2\cos^2\left(\omega_0 t\right) + gl\left(\cos\theta - 1\right) + 2gA\sin\left(\omega_0 t\right)\right)$$

$$\mathbb{H}\left(\dot{\theta},\theta,t\right) = \frac{1}{6}ml^2\left(\dot{\theta}\right)^2 + V_{eff}\left(\theta,t\right)$$

Tratamiento de ángulo promedio (lento)

$$\theta = \phi + \xi\left(t\right) \qquad A \ll l, \left(\omega_0\right)^2 \gg \frac{3g}{2l}; \frac{A\left(\omega_0\right)^2}{g} = cte$$

$$\left\langle \mathbb{H}\left(\dot{\theta},\theta,t\right)\right\rangle = \left\langle \frac{1}{6}ml^2\left(\dot{\theta}\right)^2 \right\rangle + \left\langle V_{eff}\left(\theta,t\right)\right\rangle = T\left(\dot{\phi}\right) + \tilde{V}_{eff}\left(\phi\right)$$


LIGADURA
$\xi = f(\phi, t)$
ϕ
$y(t) = A\sin(\omega_0 t)$
$\phi + \xi(t)$
$2l\sin\frac{\xi}{2}$
$\xi(t)$
$A\sin(\omega_0 t)\sin\phi$
$l\sin\xi = A\sin(\omega_0 t)\sin\phi$
$\xi \ll 1rad \Rightarrow 2l\sin\frac{\xi}{2} \approx l\sin\xi$

$$\xi \ll 1rad \Rightarrow l\sin\xi \approx \xi l \Rightarrow \xi = \frac{A}{l}\sin(\omega_0 t)\sin\phi$$

$$\dot{\xi} = \frac{A}{l}\omega_0 \cos(\omega_0 t)\sin\phi$$

$$\langle\dot{\xi}\rangle = 0; \langle\dot{\xi}^2(t)\rangle = \frac{1}{2}\left(\frac{A\omega_0}{l}\right)^2 \sin^2\phi$$

$$\langle \mathbb{H}(\dot{\theta},\theta,t)\rangle = \underline{\left\langle \frac{1}{6}ml^2(\dot{\theta})^2 \right\rangle} + \langle V_{eff}(\theta,t)\rangle$$

$$\frac{1}{6}ml^2(\dot{\theta})^2 = \frac{1}{6}ml^2\left(\dot{\phi}^2 + \dot{\xi}^2 + 2\dot{\phi}\dot{\xi}\right)$$

$$\left\langle \frac{1}{6}ml^2(\dot{\theta})^2 \right\rangle = \frac{1}{6}ml^2\left(\dot{\phi}^2 + \langle\dot{\xi}^2\rangle\right) = \frac{1}{6}ml^2\left(\dot{\phi}^2 + \frac{1}{2}\left(\frac{A\omega_0}{l}\right)^2 \sin^2\phi\right)$$

$$\xi \ll 1rad \Rightarrow 2l\sin\frac{\xi}{2} \approx \xi l \Rightarrow \xi = \frac{A}{l}\sin(\omega_0 t)\sin\phi$$

$$\langle\xi\rangle = 0; \langle\xi^2(t)\rangle = \frac{1}{2}\left(\frac{A}{l}\right)^2 \sin^2\phi$$

$$\langle\mathbb{H}(\dot{\theta},\theta,t)\rangle = \left\langle\frac{1}{6}ml^2(\dot{\theta})^2\right\rangle + \underline{\langle V_{eff}(\theta,t)\rangle}$$

$$\langle V_{eff}\rangle = \frac{1}{2}m\left(-(\omega_0 A)^2\langle\cos^2(\omega_0 t)\rangle + gl\left(\langle\cos(\phi+\xi(t))\rangle - 1\right) + 2gA\langle\sin(\omega_0 t)\rangle\right)$$

$$\langle\cos(\phi+\xi(t))\rangle \approx \left\langle\cos\phi - \sin\phi\,\xi(t) - \frac{1}{2}\cos\phi\,\xi^2(t)\right\rangle; \xi^3 \sim 0$$

$$\langle V_{eff}\rangle = \frac{1}{2}m\left(-\frac{1}{2}(\omega_0 A)^2 - gl + gl\cos\phi\left(1 - \left(\frac{A}{2l}\right)^2\sin^2\phi\right)\right)$$

$$\left\langle \mathbb{H}\left(\dot{\theta},\theta,t\right)\right\rangle = \frac{1}{6}ml^2\left(\dot{\phi}^2 + \frac{1}{2}\left(\frac{A\omega_0}{l}\right)^2 \sin^2\phi\right) + \frac{1}{2}m\left(-\frac{1}{2}(\omega_0 A)^2 - gl + gl\cos\phi\left(1 - \left(\frac{A}{2l}\right)^2 \sin^2\phi\right)\right)$$

$$= \frac{1}{6}ml^2\left(\dot{\phi}^2\right) + \frac{1}{2}m\left(\left(-gl\cos\phi + \frac{2}{3}(\omega_0 l)^2\right)\left(\frac{A}{2l}\right)^2 \sin^2\phi + gl\cos\phi - \frac{1}{2}(\omega_0 A)^2 - gl\right)$$

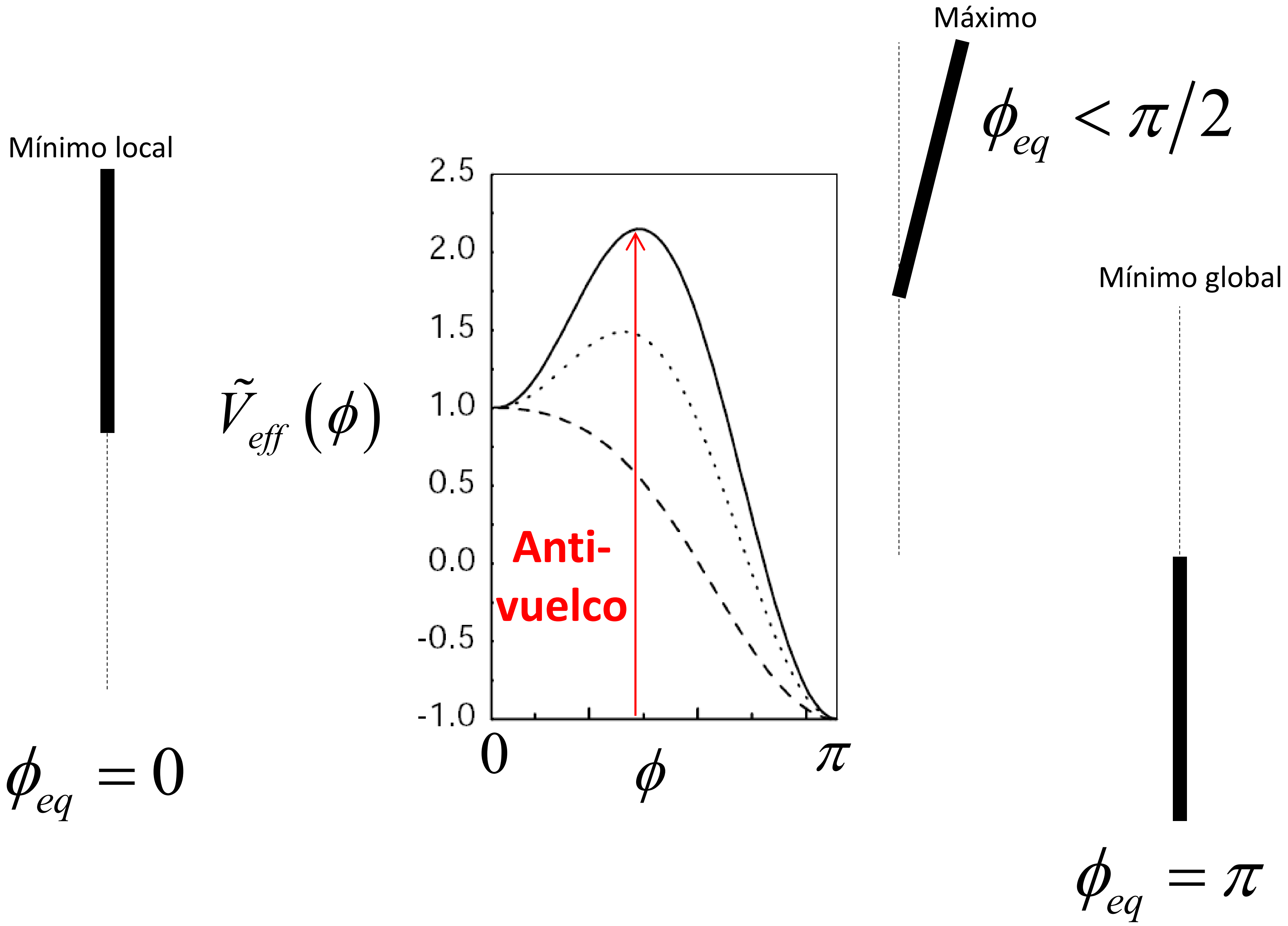
BÚSQUEDA DE CONFIGURACIONES DE EQUILIBRIO

$$\tilde{V}_{eff}(\phi) = \frac{1}{2}m\left(\left(-gl\cos\phi + \frac{2}{3}(\omega_0 l)^2\right)\left(\frac{A}{2l}\right)^2 \sin^2\phi + gl\cos\phi - \frac{1}{2}(\omega_0 A)^2 - gl\right)$$

$$\left.\frac{\partial\tilde{V}_{eff}}{\partial\phi}\right|_{\phi_{eq}} = \frac{1}{2}m\left(\left(gl\sin\phi_{eq}\right)\left(\frac{A}{2l}\right)^2 \sin^2\phi_{eq} + 2\left(-gl\cos\phi_{eq} + \frac{2}{3}(\omega_0 l)^2\right)\left(\frac{A}{2l}\right)^2 \sin\phi_{eq}\cos\phi_{eq} - gl\sin\phi_{eq}\right) = 0$$

$$\boxed{\frac{1}{2}mgl\sin\phi_{eq}\left(\left(\frac{A}{2l}\right)^2 - 3\left(\frac{A}{2l}\right)^2 \cos^2\phi_{eq} + \frac{4}{3gl}(\omega_0 l)^2\left(\frac{A}{2l}\right)^2 \cos\phi_{eq} - 1\right) = 0}$$

Ec. algebraica (tres raíces reales)


Mínimo local
$\phi_{eq} = 0$
$\tilde{V}_{eff}(\phi)$
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
0
ϕ
π
Anti-vuelco
Máximo
$\phi_{eq} < \pi/2$
Mínimo global
$\phi_{eq} = \pi$

$$\left(\frac{A}{2l}\right)^2 \sin\phi_{eq}\left(-3\cos^2\phi_{eq}+\frac{4}{3gl}(\omega_0 l)^2\cos\phi_{eq}+1-\left(\frac{2l}{A}\right)^2\right)=0$$

$$\begin{cases}\phi_{eq}=0,\pi \\ \cos\phi_{eq}\approx\frac{2(\omega_0)^2 l}{9g}\left(1\mp\sqrt{1-3\left(\frac{3g}{A(\omega_0)^2}\right)^2}\right)\end{cases} > 0 \Leftrightarrow 3\sqrt{3}<\frac{A(\omega_0)^2}{g};\left(\frac{2l}{A}\right)^2<\frac{4(\omega_0)^2 l}{3g}-3$$

Aseguradas para frecuencias muy altas

Configuraciones estables
(Mínimos energéticos)

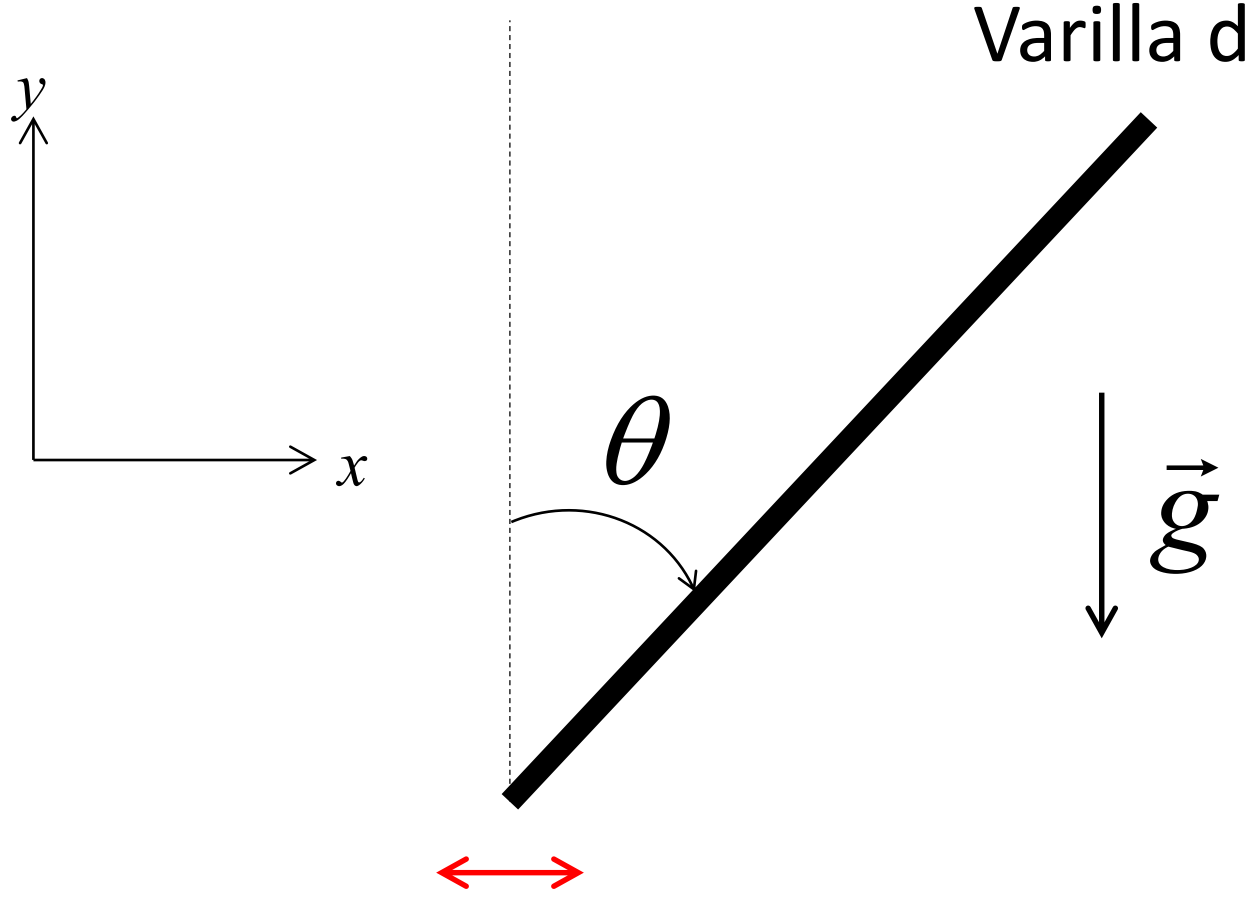
$$\xi=0$$

$$\langle\xi^2(t)\rangle=\langle\dot{\xi}^2(t)\rangle=0$$

$$\begin{cases}\tilde{V}_{eff}(0)=-\frac{1}{4}m(\omega_0 A)^2 \\ \tilde{V}_{eff}(\pi)=-\frac{1}{4}m(\omega_0 A)^2-mgl<\tilde{V}_{eff}(0)\end{cases}$$

Péndulo invertido y excitado horizontalmente con una señal senoidal

Varilla de longitud *l* y masa *m*



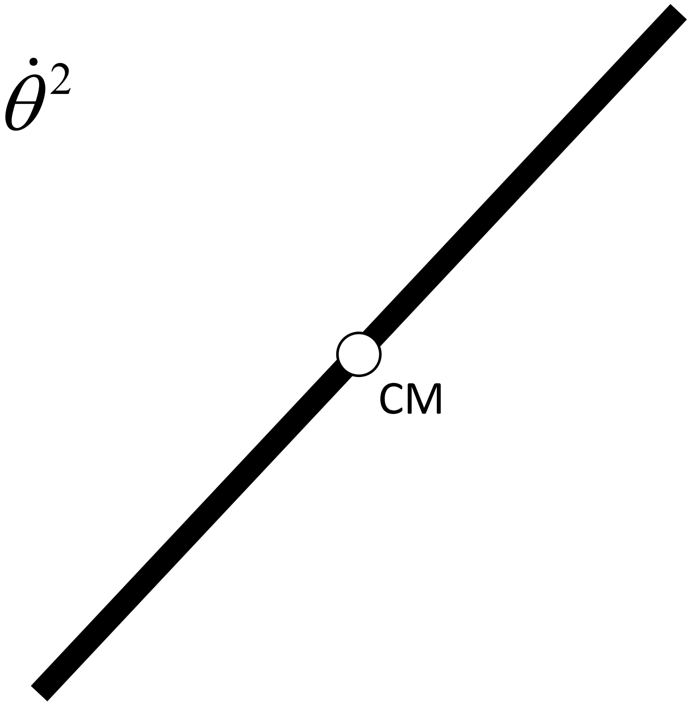
1 Grado de libertad

$$x(t) = A\sin(\omega_0 t)$$

Ligadura reónoma

Energía cinética de un sólido rígido

$$T=\frac{1}{2}mv_{cm}^2+\frac{1}{2}I_{cm}\omega^2=\frac{1}{2}m\left(\left(\dot{x}_{cm}\right)^2+\left(\dot{y}_{cm}\right)^2\right)+\frac{1}{2}I_{cm}\dot{\theta}^2$$



$$\begin{cases} x_{cm}=\frac{l}{2}\sin\theta+A\sin\left(\omega_0 t\right) \\ y_{cm}=\frac{l}{2}\cos\theta \\ I_{cm}=\frac{1}{12}ml^2 \end{cases} \Rightarrow \begin{cases} \dot{x}_{cm}=\frac{l}{2}\dot{\theta}\cos\theta+\omega_0 A\cos\left(\omega_0 t\right) \\ \dot{y}_{cm}=-\frac{l}{2}\dot{\theta}\sin\theta \end{cases}$$

$$T=\frac{1}{2}m\left(\left(\frac{l}{2}\dot{\theta}\right)^2-l\dot{\theta}\omega_0 A\cos\theta\cos\left(\omega_0 t\right)+\left(\omega_0 A\right)^2\cos^2\left(\omega_0 t\right)\right)+\frac{1}{24}ml^2\dot{\theta}^2$$

$$U = mgy_{cm} - mg\frac{l}{2} = mg\frac{l}{2}(\cos\theta - 1)$$

Energía potencial gravitatoria

$$\mathbb{L} = \frac{1}{2}m\left(\frac{1}{3}(l\dot{\theta})^2 - l\dot{\theta}\omega_0 A\cos\theta\cos(\omega_0 t) + (\omega_0 A)^2\cos^2(\omega_0 t)\right) - mg\frac{l}{2}(\cos\theta - 1)$$

Lagrangiano

$$\frac{\partial\mathbb{L}}{\partial\theta} = \frac{1}{2}m\left(l\dot{\theta}\omega_0 A\sin\theta\cos(\omega_0 t)\right) + mg\left(\frac{l}{2}\sin\theta\right)$$

$$\frac{\partial\mathbb{L}}{\partial\dot{\theta}} = \frac{1}{2}m\left(\frac{2}{3}l^2\dot{\theta} - l\omega_0 A\cos\theta\cos(\omega_0 t)\right)$$

$$\frac{d}{dt}\left(\frac{\partial\mathbb{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathbb{L}}{\partial\theta} = 0$$

$$\frac{1}{2}m\left(\frac{2}{3}l^2\ddot{\theta} + l\omega_0 A\dot{\theta}\sin\theta\cos(\omega_0 t) + l(\omega_0)^2 A\cos\theta\sin(\omega_0 t)\right) - \frac{1}{2}m\left(l\dot{\theta}\omega_0 A\sin\theta\cos(\omega_0 t)\right) - mg\left(\frac{l}{2}\sin\theta\right) = 0$$

$$\ddot{\theta} + \frac{3g}{2l}\left(-1 + \frac{A(\omega_0)^2}{g}\cot\theta\sin(\omega_0 t)\right)\sin\theta = 0$$

Hamiltoniano

$$\mathbb{H} = \dot{\theta}\frac{\partial \mathbb{L}}{\partial \dot{\theta}} - \mathbb{L} = \frac{1}{2}m\left(\frac{1}{3}l^2\dot{\theta}^2 - (\omega_0 A)^2\cos^2(\omega_0 t)\right) + \frac{1}{2}mgl(\cos\theta - 1)$$

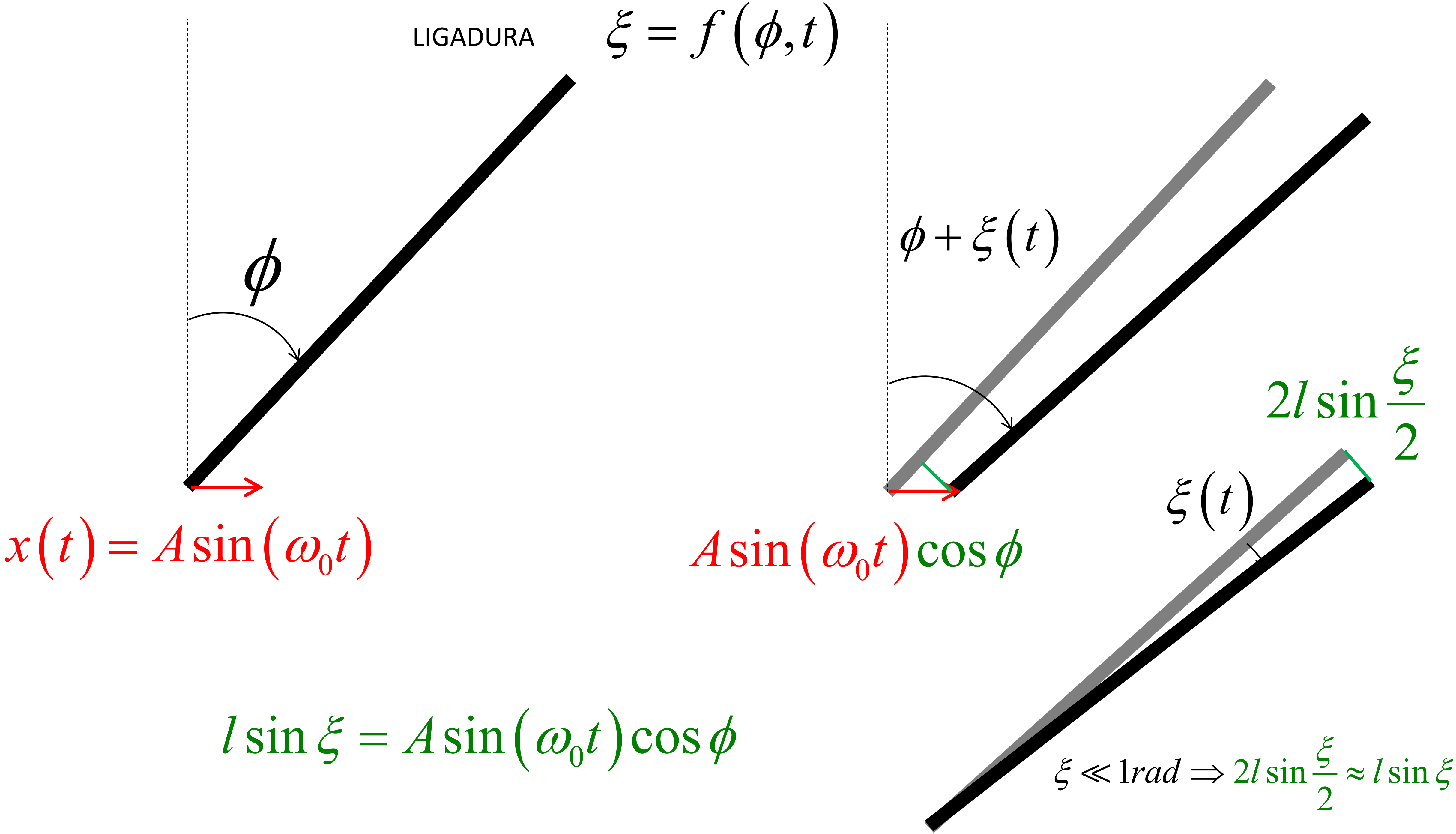
$$= \frac{1}{6}ml^2(\dot{\theta})^2 + \frac{1}{2}m\left(-(\omega_0 A)^2\cos^2(\omega_0 t) + gl(\cos\theta - 1)\right)$$

$$\mathbb{H}(\dot{\theta},\theta,t) = \frac{1}{6}ml^2(\dot{\theta})^2 + V_{eff}(\theta,t)$$

Tratamiento de ángulo promedio (lento)

$$\theta = \phi + \xi(t) \qquad A \ll l, (\omega_0)^2 \gg \frac{3g}{2l}$$

$$\left\langle \mathbb{H}(\dot{\theta},\theta,t)\right\rangle = \left\langle \frac{1}{6}ml^2(\dot{\theta})^2 \right\rangle + \left\langle V_{eff}(\theta,t)\right\rangle = T(\dot{\phi}) + \tilde{V}_{eff}(\phi)$$


LIGADURA
$\xi = f(\phi, t)$
ϕ
$x(t) = A\sin(\omega_0 t)$
$\phi + \xi(t)$
$2l\sin\frac{\xi}{2}$
$\xi(t)$
$A\sin(\omega_0 t)\cos\phi$
$l\sin\xi = A\sin(\omega_0 t)\cos\phi$
$\xi \ll 1rad \Rightarrow 2l\sin\frac{\xi}{2} \approx l\sin\xi$

$$\xi \ll 1rad \Rightarrow l\sin\xi \approx \xi l \Rightarrow \xi = \frac{A}{l}\sin(\omega_0 t)\cos\phi$$

$$\dot{\xi} = \frac{A}{l}\omega_0 \cos(\omega_0 t)\cos\phi$$

$$\langle\dot{\xi}\rangle = 0; \langle\dot{\xi}^2(t)\rangle = \frac{1}{2}\left(\frac{A\omega_0}{l}\right)^2 \cos^2\phi$$

$$\langle\mathbb{H}(\dot{\theta},\theta,t)\rangle = \underline{\left\langle\frac{1}{6}ml^2(\dot{\theta})^2\right\rangle} + \langle V_{eff}(\theta,t)\rangle$$

$$\frac{1}{6}ml^2(\dot{\theta})^2 = \frac{1}{6}ml^2(\dot{\phi}^2 + \dot{\xi}^2 + 2\dot{\phi}\dot{\xi})$$

$$\left\langle\frac{1}{6}ml^2(\dot{\theta})^2\right\rangle = \frac{1}{6}ml^2(\dot{\phi}^2 + \langle\dot{\xi}^2\rangle) = \frac{1}{6}ml^2\left(\dot{\phi}^2 + \frac{1}{2}\left(\frac{A\omega_0}{l}\right)^2\cos^2\phi\right)$$

$$\xi \ll 1rad \Rightarrow 2l\sin\frac{\xi}{2} \approx \xi l \Rightarrow \xi = \frac{A}{l}\sin\left(\omega_0 t\right)\cos\phi$$

$$\left\langle \xi \right\rangle = 0; \left\langle \xi^2\left(t\right) \right\rangle = \frac{1}{2}\left(\frac{A}{l}\right)^2 \cos^2\phi$$

$$\left\langle \mathbb{H}\left(\dot{\theta},\theta,t\right) \right\rangle = \left\langle \frac{1}{6} m l^2 \left(\dot{\theta}\right)^2 \right\rangle + \underline{\left\langle V_{eff}\left(\theta,t\right) \right\rangle}$$

$$\left\langle V_{eff} \right\rangle = \frac{1}{2} m\left(-\left(\omega_0 A\right)^2 \left\langle \cos^2\left(\omega_0 t\right) \right\rangle + gl\left(\left\langle \cos\left(\phi + \xi\left(t\right)\right) \right\rangle - 1\right)\right)$$

$$\left\langle \cos\left(\phi + \xi\left(t\right)\right) \right\rangle \approx \left\langle \cos\phi - \sin\phi\,\xi\left(t\right) - \frac{1}{2}\cos\phi\,\xi^2\left(t\right) \right\rangle; \xi^3 \sim 0$$

$$\left\langle V_{eff} \right\rangle = \frac{1}{2} m\left(-\frac{1}{2}\left(\omega_0 A\right)^2 - gl + gl\cos\phi\left(1 - \left(\frac{A}{2l}\right)^2 \cos^2\phi\right)\right)$$

$$\left\langle \mathbb{H}\left(\dot{\theta},\theta,t\right)\right\rangle = \frac{1}{6}ml^2\left(\dot{\phi}^2 + \frac{1}{2}\left(\frac{A\omega_0}{l}\right)^2 \cos^2\phi\right) + \frac{1}{2}m\left(-\frac{1}{2}(\omega_0 A)^2 - gl + gl\cos\phi\left(1 - \left(\frac{A}{2l}\right)^2 \cos^2\phi\right)\right)$$

$$= \frac{1}{6}ml^2\left(\dot{\phi}^2\right) + \frac{1}{2}m\left(\left(-gl\cos\phi + \frac{2}{3}(\omega_0 l)^2\right)\left(\frac{A}{2l}\right)^2 \cos^2\phi + gl\cos\phi - \frac{1}{2}(\omega_0 A)^2 - gl\right)$$

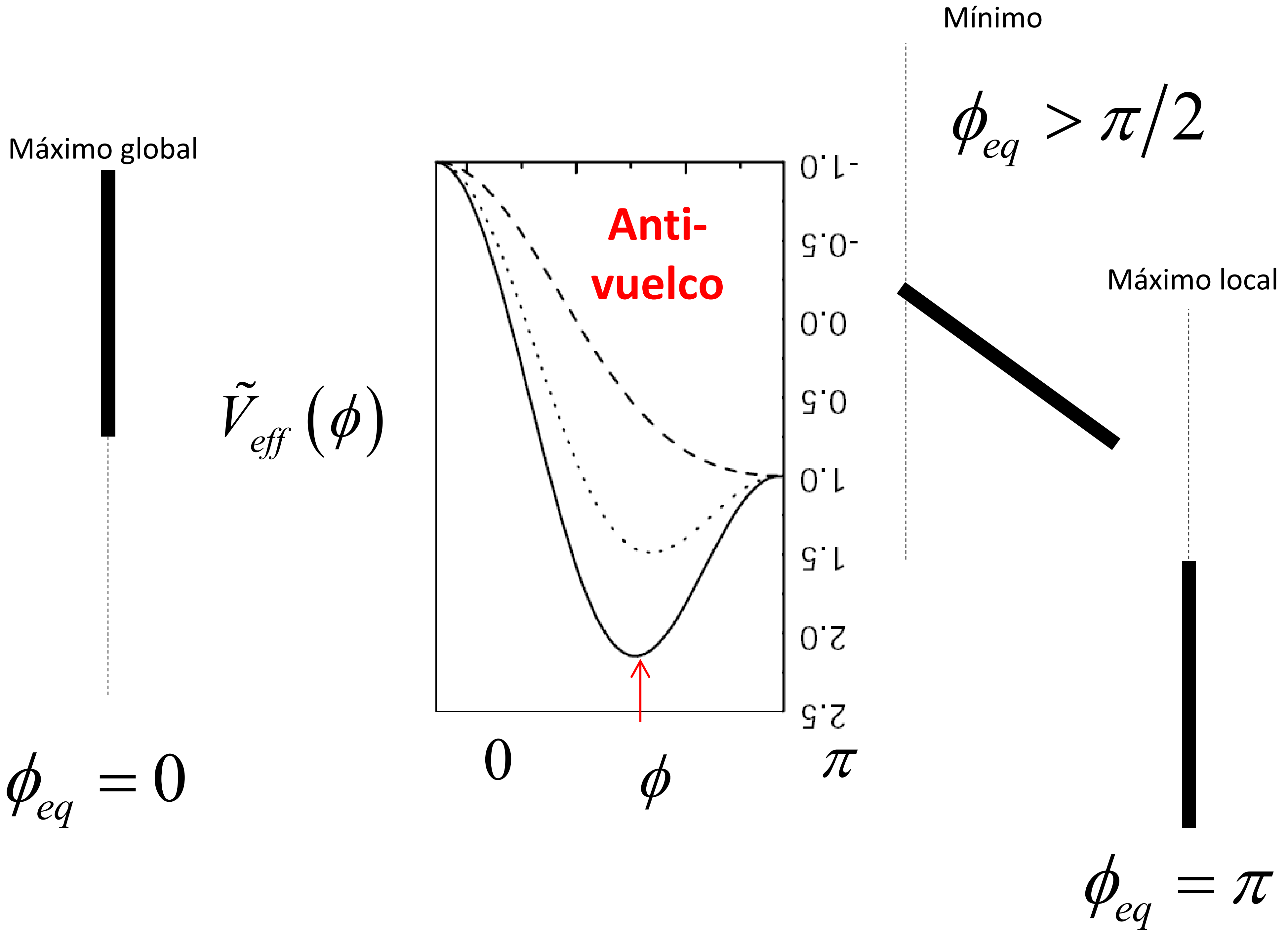
BÚSQUEDA DE CONFIGURACIONES DE EQUILIBRIO

$$\tilde{V}_{eff}(\phi) = \frac{1}{2}m\left(\left(-gl\cos\phi + \frac{2}{3}(\omega_0 l)^2\right)\left(\frac{A}{2l}\right)^2 \cos^2\phi + gl\cos\phi - \frac{1}{2}(\omega_0 A)^2 - gl\right)$$

$$\left.\frac{\partial \tilde{V}_{eff}}{\partial \phi}\right|_{\phi_{eq}} = \frac{1}{2}m\left(\left(gl\sin\phi_{eq}\right)\left(\frac{A}{2l}\right)^2 \cos^2\phi_{eq} - 2\left(-gl\cos\phi_{eq} + \frac{2}{3}(\omega_0 l)^2\right)\left(\frac{A}{2l}\right)^2 \cos\phi_{eq}\sin\phi_{eq} - gl\sin\phi_{eq}\right) = 0$$

$$\boxed{\frac{1}{2}mgl\sin\phi_{eq}\left(3\left(\frac{A}{2l}\right)^2 \cos^2\phi_{eq} - \frac{4}{3gl}(\omega_0 l)^2\left(\frac{A}{2l}\right)^2 \cos\phi_{eq} - 1\right) = 0}$$

Ec. algebraica (tres raíces reales)



Máximo global
$\phi_{eq} = 0$
$\tilde{V}_{eff}(\phi)$
Anti-
vuelco
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
2.5
0
ϕ
π
Mínimo
$\phi_{eq} > \pi/2$
Máximo local
$\phi_{eq} = \pi$

$$\left(\frac{A}{2l}\right)^2 \sin\phi_{eq}\left(3\cos^2\phi_{eq} - \frac{4}{3gl}(\omega_0 l)^2\cos\phi_{eq} - \left(\frac{2l}{A}\right)^2\right) = 0$$

$$\begin{cases} \phi_{eq} = 0, \pi \\ \cos\phi_{eq} = \dfrac{2l}{9g}(\omega_0)^2\left(1 - \sqrt{1 + 3\left(\dfrac{3g}{A(\omega_0)^2}\right)^2}\right) < 0 \Leftrightarrow \left(\dfrac{2l}{A}\right)^2 < 3 + \dfrac{4l(\omega_0)^2}{3g} \end{cases}$$

Menos restrictivo

Configuraciones inestables
(Máximos energéticos)

$$\xi = \pm\frac{A}{l}\sin(\omega_0 t)$$

$$\left\langle \xi^2(t)\right\rangle \neq \left\langle \dot{\xi}^2(t)\right\rangle \neq 0$$

$$\begin{cases} \tilde{V}_{eff}(0) = \dfrac{1}{8}mA^2\left(-\dfrac{g}{l} - \dfrac{4}{3}(\omega_0)^2\right) \\ \tilde{V}_{eff}(\pi) = \dfrac{1}{8}mA^2\left(\dfrac{g}{l}\left(1 - \dfrac{8l^2}{A^2}\right) - \dfrac{2}{3}(\omega_0)^2\right) < \tilde{V}_{eff}(0) \end{cases}$$