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Approximation of 3D trapezoidal fuzzy data using radial basis functions

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Abstract

We present a new methodology to approximate a trapezoidal fuzzy numbers set by using smoothing radial basis functions ($RBFs$). The methodology uses different error and similarity indices to determine and compare the accuracy of the approximation of the given trapezoidal fuzzy data. For the proposed approximation method a fuzzy radial basis functions type are defined, called fuzzy smoothing radial basis functions under tension. The computation of one of these approximation functions from a given trapezoidal fuzzy data set is described and some convergence results are proved. Finally, some examples in two-dimensions are given to compare the behavior of the presented method by using the proposed error and similarity indices for different configurations of the fuzzy smoothing radial basis functions under tension.

Keywords: Fuzzy data; Fuzzy functions; Approximation methods; Radial basis functions; Error and similarity measures

1. Introduction

One of the most interesting and important problem in various scientific fields is approximation. In this paper, we present a new methodology to approximate a trapezoidal fuzzy numbers set by using smoothing radial basis functions ($RBFs$). The methodology uses different error and similarity indices to determine and compare the accuracy of the approximation of the given trapezoidal fuzzy data. For the proposed approximation method a fuzzy radial basis functions type are defined, called fuzzy smoothing radial basis functions under tension. The computation of one of these approximating functions from a given trapezoidal fuzzy data set is described and some convergence results are proved. Finally, some examples of two-dimensions are given to compare the behavior of the presented method by using the proposed error and similarity indices for different configurations of the fuzzy smoothing radial basis functions under tension.

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Multivariate interpolation of fuzzy data has been reviewed in several papers (see [3–6], [10–14], among others), and it is sufficient here to mention how it works. The scattered data approximation of fully fuzzy data by a quasi-interpolation method is studied in [22] and some fuzzy Taylor approximation methods are introduced in [1]. In [24,25] the authors define some new similarity indices to determine the accuracy of approximation of fuzzy data by fuzzy cubic spline functions.

In this paper we describe a novel approach based on $(RBFs)$ for trapezoidal fuzzy data approximation of two variables. Radial basis functions constitute a widely used and researched tool for (nonlinear) function approximation (see [8], [7] and [19–21], for example), which is a central theme in pattern analysis and recognition [18]; see also [15] for a recent and comprehensive overview and further references. Often, the (RBF)-method is seen as a neural network [12].

The rest of this paper is organized as follows: In the next Section we present some preliminaries which are necessary for this study, the fuzzy numbers concept, particularly the trapezoidal fuzzy numbers and some similarity indices to compare the similarity between two trapezoidal fuzzy numbers. Section 3 presents a radial basis functional space where we construct the approximating function, concretely the called fuzzy radial basis functions under tension space. For this, we introduce both the smoothing radial basis function approximating a data set of $\mathbb{R}^4$ and the fuzzy radial basis function under tension approximating a trapezoidal fuzzy data set, we describe the computational method of these approximates, and we give some convergence results. The followed methodology is presented in Section 5. Several simulations for different parameter values are showed in Section 6. Finally, in Section 7 the conclusions are presented.

2. Preliminaries

This section aims to introduce both the notations and some of the preliminary results necessary for the development of the theories which are studied in the remainder.

2.1. Fuzzy numbers

Definition 1. A fuzzy number is a mapping $u : \mathbb{R} \longrightarrow [0, 1]$ with the following properties.

i) u is an upper semi-continuous function on $\mathbb{R}$.
ii) $u(x) = 0$ outside some interval $[a_1, a_4] \subset \mathbb{R}$.
iii) There exist real numbers a_2 and a_3 such that $a_1 \leq a_2 \leq a_3 \leq a_4$ with
   a) $u(x)$ is a monotonic increasing function on $[a_1, a_2]$,
   b) $u(x)$ is a monotonic decreasing function on $[a_3, a_4]$,
   c) $u(x) = 1$, for all $x \in [a_2, a_3]$.

2.2. Trapezoidal fuzzy numbers

A popular type of fuzzy number is the set of trapezoidal fuzzy numbers, $\mathbb{TFN}$, that can be defined as $a = (a_1, a_2, a_3, a_4)$, and their membership function is defined by

$$\mu(a) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \leq x \leq a_4, \\ 0, & otherwise. \end{cases}$$

Definition 2. Let $u = (u_1, u_2, u_3, u_4) \in \mathbb{TFN}$ and $0 < \alpha \leq 1$, then it is called α-cut of u the set

$$[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}.$$

It is defined the 0-cut of u as its support, i.e.,

$$[u]^0 = \bigcup_{0<\alpha\leq 1} [u]^\alpha = [u_1, u_4]$$

An equivalent definition of a trapezoidal fuzzy number $u = (u_1, u_2, u_3, u_4)$ is a function $u : [0, 1] \longrightarrow I$ given by

$$u(\alpha) = [\underline{u}(\alpha), \bar{u}(\alpha)],$$

with

$$\begin{aligned} \underline{u}(\alpha) &= u_1 + (u_2 - u_1)\alpha, \\ \bar{u}(\alpha) &= u_4 + (u_3 - u_4)\alpha, \end{aligned} \tag{2.1}$$

where I is the set of all real closed intervals.

Definition 3. For any $u, v \in \mathbb{TFN}$ is defined the Hausdorff distance between u and v as the quantity

$$d(u, v) = \sup_{\alpha\in[0,1]} \max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\}.$$

From this definition we have that if $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$ then

$$d(u, v) = \max_{i=1,2,3,4} |u_i - v_i| \tag{2.2}$$

Definition 4. A fuzzy function defined on $\Omega \subset \mathbb{R}^2$ into the trapezoidal fuzzy number set $\mathbb{TFN}$ is an application $f : \Omega \to \mathbb{TFN}$ such that $f = (f_1, f_2, f_3, f_4)$ where f_i is a real function defined on Ω, $i = 1, 2, 3, 4$, and $f(x, y) \in \mathbb{TFN}$, for any $(x, y) \in \Omega$.

2.3. Similarity indices

The concept of similarity or dissimilarity between two data sets is fundamental on almost every scientific field. The analysis of similarity or dissimilarity measures between fuzzy sets has gained importance due to the widespread of applications in diverse fields, including fuzzy risk analysis problem [23], decision making [13] and fuzzy function approximation [9]. Now, we introduce some existing similarity measures of fuzzy numbers.

If $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$, then the degree of similarity $S(A, B)$ between the trapezoidal fuzzy numbers A and B is defined

i) by Chen as follows:

$$S_{CHEN}(A, B) = 1 - \frac{\sum_{i=1}^4 \mid a_i - b_i \mid}{4} \in [0, 1], \tag{2.3}$$

where $\mid a \mid$ is the absolute value of the real number a.

ii) Hsieh et al. proposed a similarity measure using the *graded mean integration-representation distance* where the degree of similarity $S(A, B)$ between the fuzzy numbers A and B is calculated as follows:

$$S_{HSIEH}(A, B) = \frac{1}{1 + d(A, B)}, \tag{2.4}$$

where $d(A, B) =\mid P(A) - P(B) \mid$, and $P(A),\ P(B)$ are the graded mean integration representations of A and B, respectively. If A and B are trapezoidal fuzzy numbers, with $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$, then the graded mean integration of these fuzzy numbers is defined as:

$$P(A) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6},$$
$$P(B) = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6}.$$

iii) Chen & Chen presented another similarity measure between generalized trapezoidal fuzzy numbers. They presented the (simple center of gravity method) denoted as SCGM to calculate the center of gravity points (x_A^*, y_A^*) and (x_B^*, y_B^*) of the generalized trapezoidal fuzzy number A and B respectively. $A = (a_1, a_2, a_3, a_4)$, $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$, and $B = (b_1, b_2, b_3, b_4)$, $0 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq 1$. Then the degree of similarity $S(A, B)$ between the trapezoidal fuzzy numbers A and B, using the SCGM methodology, is calculated as follows:

$$\begin{aligned} S_{SCGM}(A,B) = {} & 1 - \frac{\sum_{i=1}^{4} | a_i - b_i |}{4} \times \\ & \left(1 - | x_A^* - x_B^* |\right)^{B(S_A,S_B)} \times \frac{min(y_A^*, y_B^*)}{max(y_A^*, y_B^*)}, \end{aligned} \tag{2.5}$$

where $S(A, B) \in [0, 1]$, and

$$x_A^* = \frac{y_A^*(a_3 + a_2) + (a_4 + a_1)(1 - y_A^*)}{2},$$

$$y_A^* = \begin{cases} \frac{1}{2}, & if \quad a_1 = a_4, \\ \frac{1}{6}(\frac{a_3 - a_2}{a_4 - a_1} + 2), & if \quad a_1 \neq a_4, \end{cases}$$

and

$$B(S_A, S_B) = \begin{cases} 1, & if \quad S_A + S_B > 0, \\ 0, & if \quad S_A + S_B = 0, \end{cases}$$

where S_A and S_B are the lengths of the bases of trapezoidal fuzzy numbers A and B, respectively, and are defined by:

$$S_A = a_4 - a_1,$$
$$S_B = b_4 - b_1.$$

In all cases it is verified that $S(A, B)$ tends to 1 as A tends to be equal B, and $S(A, B)$ tends to 0 as the common support of A and B tends to $\emptyset$.

3. A radial basis functions space

Let $\langle \cdot, \cdot \rangle_k$ and $\langle \cdot \rangle_k$ be the Euclidean inner product and norm in $\mathbb{R}^k$, respectively.

Let Ω be an open bounded connected nonempty subset of $\mathbb{R}^2$.

Let $m > 1$ and $k \geq 1$ be two positive integers and let $H^m(\Omega, \mathbb{R}^k)$ ($H^m(\Omega)$ if $k = 1$) be the usual Sobolev space of order m equipped with the inner products

$$(u, v)_\ell = \sum_{|\beta| \leq \ell} \int_\Omega \langle D^\beta u(p), D^\beta v(p) \rangle_k dp, \ \forall 0 \leq \ell \leq m,$$

being $|\beta| = \beta_1 + \beta_2$, for any $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$. Let be the following semi-norms in $H^m(\Omega, \mathbb{R}^k)$

$$|u|_\ell = (u, u)_\ell^{\frac{1}{2}}, \ \forall 0 \leq \ell \leq m,$$

and the norm $\|u\|_m = \left(\sum_{\ell=0}^{m} |u|_\ell^2 \right)^{\frac{1}{2}}$.

Finally, let $\Pi_{m-1}(\mathbb{R}^2)$ denote the space of polynomials on $\mathbb{R}^2$ of degree at most $m - 1$ whose dimension is denoted $d(m)$, where $d(m) = \dfrac{m(m+1)}{2}$ and $\{q_1, \cdots, q_{d(m)}\}$ be the standard basis of $\Pi_{m-1}(\mathbb{R}^2)$.

Let us give an arbitrary finite set $B = \{b_1, \cdots, b_M\} \subset \mathbb{R}^2$ of M distinct approximation points, $b_i = (x_i, y_i) \in \Omega$, $i = 1, \ldots, M$, and a set of fuzzy numbers $U = \{u_1, \cdots, u_M\}$ such as $u_i = (u_{i1}, u_{i2}, u_{i3}, u_{i4})$ is a trapezoidal fuzzy number, with $i = 1, \cdots, M$, i.e., $U \subset \mathbb{TFN}$, where $\mathbb{TFN}$ is the set of the trapezoidal fuzzy numbers.

Suppose that

$$B \text{ contains a } \Pi_{m-1} - \text{unisolvent subset.} \tag{3.1}$$

We denote

$$h = \sup_{p\in\Omega} \min_{b\in B} \langle p - b\rangle_2.$$

We need a center points set $A = \{a_1, \cdots, a_N\} \subset \mathbb{R}^2$ and for each $i = 1, \ldots, N$, a radial function $\Phi(\cdot - a_i)$.

We denote

$$d = \sup_{p\in\Omega} \min_{a\in A} \langle p - a\rangle_2.$$

The aim of this work is to approximate the data set $\{(b_1, u_1), \cdots, (b_M, u_M)\} \subset \mathbb{R}^2 \times \mathbb{TFN}$.

To conclude this section we define a radial basis functions space type.

For this, we consider the following primitive function [7]:

$$\phi_\varepsilon(t) = -\frac{1}{2\varepsilon^3}\left(e^{-\varepsilon\sqrt{t}} + \varepsilon\sqrt{t}\right), \quad \varepsilon \in \mathbb{R}^+, \ t \geq 0 \tag{3.2}$$

and the following radial function

$$\Phi_\varepsilon(x) = \phi_\varepsilon(< x >_2^2) = -\frac{1}{2\varepsilon^3}\left(e^{-\varepsilon < x >_2} + \varepsilon < x >_2\right), \quad \varepsilon \in \mathbb{R}^+, \ x \in \mathbb{R}^2.$$

Consider $m = 3$ and let H be the real functional space generated by the restrictions on Ω of the functions

$$\{q_1, \cdots, q_6, \Phi_\epsilon(\cdot - a_1), \cdots, \Phi_\epsilon(\cdot - a_N)\}.$$

In [7] the authors present an interpolation method of a real function f in H which has a unique solution called the interpolation radial basis function under tension of f associated with A and ε.

Proposition 1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and connected set having the cone property and a Lipschitz continuous boundary and $f \in H^3(\Omega)$. Let $s \in H$ be the interpolation radial basis spline under tension of f associated with A and ε. There exist a constant $d_0 > 0$ and a constant $C > 0$, independent of d, such that, for any $d \leq d_0$,*

$$|f - s|_\ell \leq C d^{3-\ell}, \quad \forall \ell = 0, 1, 2.$$

Proof. The result is obtained reasoning as [16, Theorem 4.7] for the partial derivatives of order less or equal 2, taking into account [17, Theorem 4.4]. □

4. Smoothing fuzzy radial basis functions under tension

4.1. Defining the problem

Let $m = 3$ and let U be a set of fuzzy numbers $U = \{u_i : i = 1, \cdots, M\}$ such that every $u_i = (u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4})$, $i = 1, \cdots, M$, is a trapezoidal fuzzy number, i.e., $U \subset \mathbb{TFN}$.

Let $H^{\mathbb{T}}$ the set of fuzzy functions $s : \Omega \to \mathbb{TFN}$ such that

$$s(x) = \sum_{l=1}^{N+6} \alpha_l w_l(x), \quad x \in \Omega,$$

where $\alpha_1, \cdots, \alpha_{N+6} \in \mathbb{TFN}$ and

$$w_l = \left\{ \begin{array}{ll} \Phi_\varepsilon(\cdot - a_l), & l = 1, \cdots, N \\ q_{l-N}, & l = N+1, \cdots, N+6, \end{array} \right\}.$$

The fuzzy function set $H^{\mathbb{T}}$ is called the set of the fuzzy radial basis functions under tension defined on Ω.

Consider the following problem: Given the approximation data set

$$\{(b_i, u_i) : i = 0, \cdots, M\} \in \mathbb{R}^2 \times \mathbb{TFN},$$

we want to obtain a fuzzy function $s_h \in H^{\mathbb{T}}$ such that

$$s_h(b_i) \approx u_i, \;\; i = 1, \cdots, M.$$

4.2. Smoothing radial function under tension

Consider the following minimization problem: Given $\tau \in (0, \infty)$, find $\sigma_h \in H^4$ such that:

$$J(\sigma_h) \le J(\upsilon), \quad \forall \upsilon \in H^4, \tag{4.1}$$

where

$$J(\upsilon) = \sum_{i=1}^{M} < u_i - \upsilon(b_i) >_4^2 + \tau |\upsilon|_2^2,$$

considering $u_i = (u_{i,1}, \ldots, u_{i,4}) \in \mathbb{R}^4$, for any $i = 1, \ldots, M$, i.e. $U \subset \mathbb{R}^4$.

Theorem 1. *The minimization problem (4.1) has a unique solution that is the unique solution of the following variational problem: Find $\sigma_h \in H^4$ such that for all $\upsilon \in H^4$:*

$$\sum_{i=1}^{M} < \sigma_h(b_i), \upsilon(b_i) >_4 + \tau(\sigma_h, \upsilon)_2 = \sum_{i=1}^{M} < \upsilon(b_i), u_i >_4 \tag{4.2}$$

Proof. Consider the application $a : H^4 \times H^4 \to \mathbb{R}$, given by

$$a(u, v) = \sum_{i=1}^{M} \langle u(b_i), v(b_i) \rangle_4 + \tau(u, v)_2.$$

Obviously, the form $a(\cdot, \cdot)$ is bilinear, symmetric and continuous on H^4.

Moreover, from (3.1), $a(u, u)$ defines on H^4 a norm equivalent to the usual Sobolev norm $\| \cdot \|_2$. Thus we have that a is H^4-coercive.

Let $\varphi : H^4 \to \mathbb{R}$ defined on H^4 by $\varphi(v) = \sum_{i=1}^{M} \langle v(b_i), u_i \rangle_4$, which clearly is a linear and continuous application.

So, by applying Lax-Milgramm Lemma there exists a unique $\sigma_h \in H^4$ such that $a(\sigma_h, v) = \varphi(v)$, for any $v \in H^4$, and (4.2) holds.

Furthermore, σ_h is the minimum in H^4 of the functional $\psi(v) = \frac{1}{2}a(v, v) - \varphi(v)$, which is the minimum of J since

$$J(v) = 2\psi(v) + \sum_{i=1}^{M} < u_i >_4^2 .$$

Hence we conclude the result. □

Definition 5. Given the center points set $A = \{a_1, \ldots, a_N\} \subset \mathbb{R}^2$, the approximation data set $B \times U = \{(b_i, u_i), \; i = 1, \ldots M\} \subset \mathbb{R}^2 \times \mathbb{R}^4$ and $\tau > 0$, the unique solution of problem (4.2), $\sigma_h \in H^4$, is called the smoothing radial basis function under tension associated with A, B, U, τ and ε.

4.3. Computing the solution

Let $\sigma_h \in H^4$ be the solution of Problem (4.1), then $\sigma_h = \sum_{1}^{N+6} \alpha_i w_i$, where $\alpha_1, \ldots, \alpha_{N+6} \in \mathbb{R}^4$ are the unknowns of the problem.

By linearity, applying Theorem 1, we can reduce this problem to the following linear system:

$$\left(\mathcal{A}\mathcal{A}^T + \tau\mathcal{R}\right)\alpha = \mathcal{A}^T U, \tag{4.3}$$

where

$$\mathcal{A} = \left(w_i(b_j)\right)_{\substack{i=1,\ldots,N+6, \\ j=1,\ldots,M}},$$
$$\mathcal{R} = \left((w_i, w_j)_2\right)_{i,j=1,\ldots,N+6},$$
$$\alpha = (\alpha_1, \ldots, \alpha_{N+6})^T,$$
$$U = (u_i)_{i=1,\ldots,M}.$$

Let be $\sigma_h(x) = \sum_{i=1}^{N+6} \alpha_i \omega_i(x)$, for any $x \in \Omega$, the unique solution of (4.2), where $\alpha = (\alpha_1, \ldots, \alpha_{N+6})^T$ is the unique solution of the linear system (4.3).

For all $i = 1, \ldots, N+6$, let $\overline{\alpha_i} \in \mathbb{R}^4$ be such that its components are the same as those of α_i ordered from lowest to highest, i.e., if $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \alpha_{i4})$ and $\overline{\alpha_i} = (\overline{\alpha_{i1}}, \overline{\alpha_{i2}}, \overline{\alpha_{i3}}, \overline{\alpha_{i4}})$ then $\overline{\alpha_{ij}} = \alpha_{i\gamma(j)}$, $j = 1, 2, 3, 4$, being $\gamma : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ the permutation such that

$$\overline{\alpha_{i1}} \leq \overline{\alpha_{i2}} \leq \overline{\alpha_{i3}} \leq \overline{\alpha_{i4}}.$$

Definition 6. The function

$$s_h(x) = \sum_{i=1}^{N+6} \overline{\alpha_i}\omega_i(x), \ x \in \Omega,$$

verifies that $s_h \in H^{\mathbb{TFN}}$ and it is called the smoothing fuzzy radial basis function under tension associated with A, B, U, τ and ε.

4.4. Convergence results

Theorem 2. *Let $f \in H^3(\Omega, \mathbb{R}^4)$ and let $\sigma_h \in H^4$ be the smoothing radial basis function under tension associated with A, B, $f(B)$, τ and ε. Suppose that (3.1) and the hypothesis*

$$\tau = o(1), \quad h \to 0, \tag{4.4}$$

and

$$\frac{Md^2}{\tau} = o(1), \quad h \to 0 \tag{4.5}$$

hold. Then

$$\lim_{h\to 0} \|f - \sigma_h\|_2 = 0. \tag{4.6}$$

Proof. Let $f = (f_1, f_2, f_3, f_4)$, with $f_i \in H^3(\Omega)$, for $i = 1, \ldots, 4$. Let s_i be the interpolation radial basis spline under tension associated with A and ε. Then, from Proposition 1 there exist some constants $d_i > 0$ and $C_i > 0$ such that

$$\|f_i - s_i\|_2 \leq C_i d, \ \forall d \leq d_i.$$

Hence there exist some constants $C > 0$ and $d_0 > 0$ such that

$$\|f - s\|_2 \leq Cd, \ \forall d \leq d_0, \tag{4.7}$$

being $s = (s_1, s_2, s_2, s_4) \in H^4$.

On the other hand, from (4.4) and (4.5), it is also verified that

$$d = o(1), \quad h \to 0.$$

Thus, from (4.7) one has

$$|s|_2 = |f|_2 + o(1), \ h \to 0. \tag{4.8}$$

Let $\sigma_h \in H^4$ be the unique solution of Problem (4.1), then $J(\sigma_h) \leq J(s)$ and thus

$$\sum_{i=1}^{M} \langle \sigma_h(b_i) - f(b_i) \rangle_4^2 + \tau |\sigma_h|_2^2 \leq \sum_{i=1}^{M} \langle s(b_i) - f(b_i) \rangle_4^2 + \tau |s|_2^2,$$

which implies that

$$\sum_{i=1}^{M} \langle \sigma_h(b_i) - f(b_i) \rangle_4^2 \leq \sum_{i=1}^{M} \langle s(b_i) - f(b_i) \rangle_4^2 + \tau |s|_2^2 \tag{4.9}$$

and

$$|\sigma_h|_2 \leq \frac{1}{\tau} \sum_{i=1}^{M} \langle s(b_i) - f(b_i) \rangle_4^2 + |s|_2^2. \tag{4.10}$$

From (4.7) it follows that there exist some constants $C > 0$ and $d_0 > 0$ such that

$$\sum_{i=1}^{M} \langle s(b_i) - f(b_i) \rangle_4^2 \leq MCd^2, \ d \leq d_0$$

and, from Proposition 1,

$$|s|_2^2 \leq Cd^2 + |f|_2^2, \ d \leq d_0.$$

Consequently, from (4.10)

$$|\sigma_h|_2^2 \leq C\frac{Md^2}{\tau} + Cd^2 + |f|_2^2, \ d \leq d_0$$

and thus, from (4.9),

$$\sum_{i=1}^{M} \langle \sigma_h(b_i) - f(b_i) \rangle_4^2 \leq CMd^2 + C\tau d^2 + \tau |f|_2^2, \ d \leq d_0.$$

Hence

$$\sum_{i=1}^{M} \langle \sigma_h(b_i) - f(b_i) \rangle_4^2 = o(1), \ h \to 0$$

and

$$|\sigma_h - f|_2^2 = O(1), \ h \to 0.$$

Taking into account (3.1) we have that

$$\lfloor\lfloor v \rfloor\rfloor = \left(\sum_{i=1}^{M} \langle v(b_i) \rangle_4^2 + |v|_2^2 \right)^2$$

is a norm in $H^2(\Omega, \mathbb{R}^4)$ equivalent to the usual norm $\|v\|_2$.

And we obtain that

$$\|\sigma_h - f\|_2 = O(1),\ h \to 0.$$

Thus there exists $h_1 > 0$ such that the family $(\sigma_h)_{h \le h_1}$ is bounded in $H^2(\Omega, \mathbb{R}^4)$ and hence there exists a subsequence $(\sigma_{h_\ell})_{\ell \in \mathbb{N}}$ and a element $f^* \in H^2(\Omega, \mathbb{R}^4)$ such that

$$(\sigma_{h_\ell})_{\ell \in \mathbb{N}} \text{ converges weakly to } f^* \text{ in } H^2(\Omega, \mathbb{R}^4).$$

From here, by reasoning as in the points 3), 4) and 5) of the proof of [2, VI-Theorem 3.2] we conclude the result. □

Theorem 3. *Consider a fuzzy function $f : \Omega \to \mathbb{TFN}$, $f = (f_1, f_2, f_3, f_4)$, with $f_i \in H^3(\Omega)$, $i = 1, \ldots, 4$, and let $s_h = (s_{h,1}, s_{h,2}, s_{h,3}, s_{h,4}) \in H^{\mathbb{T}}$ the smoothing fuzzy radial basis function under tension associated with A, B, $f(B)$, τ and ε. Suppose the hypotheses (3.1) and (4.4) hold. Then, one has*

$$\lim_{h \to 0} S(f(p), s_h(p)) = 1, \quad \forall\, p \in \Omega, \tag{4.11}$$

where S is the Chen index (S_{CHEN}), the Hsieh index (S_{HSIEH}) or the Chen and Chen index (S_{SCGM}) defined in Subsection 2.3.

Moreover

$$\lim_{h \to 0} d(f(p), s_h(p)) = 0, \quad \forall\, p \in \Omega. \tag{4.12}$$

Proof. Let $\sigma_h = (\sigma_{h,1}, \sigma_{h,2}, \sigma_{h,3}, \sigma_{h,4})$ the smoothing radial basis function under tension associated with A, B, $f(B)$, τ and ε, considering $f(b_i)$ as an element of $\mathbb{R}^4$, for $i = 1, \ldots, M$.

From Theorem 2 we can deduce that, for all $i = 1, 2, 3, 4$ and all $p \in \Omega$, we obtain

$$|f_i(p) - \sigma_{h,i}(p)| = o(1), \quad h \to 0.$$

Thus, for $i = 1, 2, 3$ and for any $p \in \Omega$ we have

$$\sigma_{h,i+1}(p) - \sigma_{h,i}(p) = \sigma_{h,i+1}(p) - f_{i+1}(p) + f_{i+1}(p) - f_i(p) + f_i(p) - \sigma_{h,i}(p)$$

and thus

$$\sigma_{h,i+1}(p) - \sigma_{h,i}(p) \ge o(1) \quad h \to 0.$$

Then,

$$\sigma_{h,1}(p) \le \sigma_{h,2}(p) \le \sigma_{h,3}(p) \le \sigma_{h,4}(p), \quad h \to 0$$

and thus $\sigma_h(p) \in \mathbb{TFN}$, as $h \to 0$ for any $p \in \Omega$.

Hence $s_h = \sigma_h$ as $h \to 0$.

Consequently, from Theorem 2, we have

$$\lim_{h \to 0} |f_i(p) - s_{h,i}(p)| = 0, \quad \forall\, p \in \Omega,\ i = 1, 2, 3, 4,$$

and we can confirm that (4.11) and (4.12) hold. □

5. Methodology

In order to verify the ability of approximation of the smoothing fuzzy radial basis function under tension, we will illustrate the different phases carried out for a given trapezoidal fuzzy data set. The steps to be undertaken in the simulation process are as follows.

i) Let TDS be a test data set $B^{Test} = \{x_1^T, ..., x_{ntest}^T\} \in \Omega$ to verify the ability of the presented fuzzy approximation method and its corresponding output fuzzy numbers, $U^{Test} = \{f(x_i^T) \mid i = 1, \dots, ntest\}$, i.e., $TDS = \{X^{Test}; U^{Test}\}$.

ii) With the data of TDS, it is possible to obtain the output fuzzy data approximation using the presented smoothing fuzzy radial basis function under tension s_h. This output set is termed as $\hat{U}^{Test} = \{s_h(x_i^T) \mid i = 1, \dots, ntest\}$.

iii) Measure the error and similarity between the trapezoidal fuzzy numbers $f(x_i^T)$ and $s_h(x_i^T)$, for any $i = 1, ..., ntest$ (the error measure is defined as $d(f(x_i^T),\ s_h(x_i^T))$ and the similarity measure is defined as $S(f(x_i^T), s_h(x_i^T))$, where S is the Chen index (S_{CHEN}), the Hsieh index (S_{HSIEH}) or the Chen & Chen index (S_{SCGM})).

iv) In order to analyze the fuzzy data approximation using smoothing fuzzy radial basis functions under tension we consider the following error and similarity average indices:

$$\begin{aligned} \overline{d} &= \frac{1}{ntest} \sum_{i=0}^{ntest} d(f(x_i^T), s_h(x_i^T)), \\ \overline{S} &= \frac{1}{ntest} \sum_{i=0}^{ntest} S(f(x_i^T), s_h(x_i^T)), \end{aligned} \tag{5.1}$$

6. Numerical examples

6.1. Parameters of the problem

To analyze the behavior of the approximation of fuzzy numbers, performed by smoothing fuzzy radial basis functions under tension, various simulations have been carried out, in which five important factors can be modified:

1) The number of center points to build the radial basis functions, that will be denoted as $nctrs$.
2) The number of approximation points used, denoted by np.
3) The parameter ε of the definition of the basis functions. This variable ε is called the shape parameter.
4) The parameter τ, that reflects the relative importance which we give to the two conflicting objectives: accuracy in approximation (remaining close to the data), and obtaining a smooth fuzzy function s. It is a tradeoff between precision and smoothness.
5) The number of random test points for measure the similarity average index, denoted by $ntest$

6.2. Original fuzzy function

We consider the fuzzy function $f : [0, 1] \times [0, 1] \longrightarrow \mathbb{TFN}$, given by

$$\begin{aligned} f(x, y) = &\ (a_1(x, y), a_2(x, y), a_3(x, y), a_4(x, y)) = \\ &(-0.8 + 0.75e^{-((5x-2)^2+(5y-2)^2)}, 0.75e^{-((5x-1)^2+(5y-1)^2)}, \\ &0.75 + 0.5e^{-((6x-2)^2+(6y-3)^2)}, 1.5 + 0.9e^{-((6x-4)^2+(5y-2)^2)}). \end{aligned}$$

6.3. Numerical results

Fig. 1 shows the original fuzzy function and its approximation for $nctrs = 81$, $np = 100$, $\tau = 1e - 9$ and $\varepsilon = 1$, from left to right.

Tables 1, 2 and 3 illustrate the performance of the approximation error and similarity average indices, when analyzing some different values of $nctrs$, np and τ, for $ntest = 5000$ and $\varepsilon = 0.1$, $\varepsilon = 1$ and $\varepsilon = 10$, respectively.

7. Conclusions

1) In this work a new approximation method of 3D trapezoidal fuzzy data by smoothing fuzzy radial basis functions under tension has been studied.

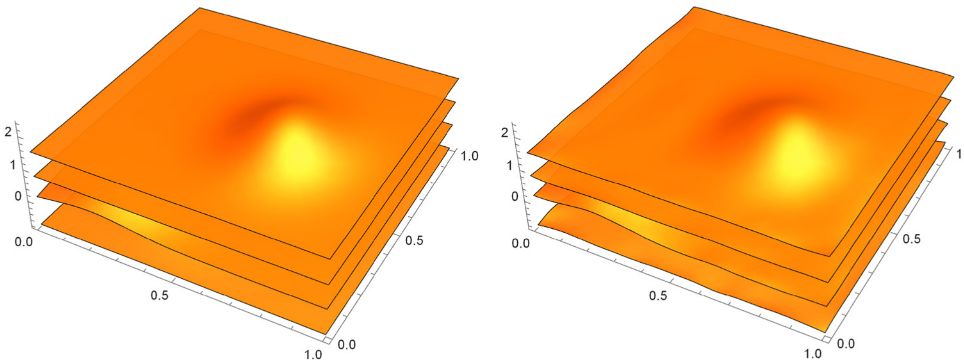

Fig. 1. From left to right, the original fuzzy function and its approximation for $nctrs = 81$, $np = 100$, $\tau = 1e-9$ and $\varepsilon = 1$.

Table 1
Function $f(x, y)$. Similarity average indices estimates for different values of the approximation method parameters.

ε	$nctrs$	np	τ	$\overline{S}_{CHEN}$	$\overline{S}_{HSIEH}$	$\overline{S}_{SCGM}$	$\overline{d}$
0.1	25	50	1e-5	0.988907	0.993159	0.989209	2.2823e-2
			1e-7	0.997818	0.998400	0.997858	4.6650e-3
			1e-9	0.997747	0.998642	0.997779	4.7419e-3
		100	1e-5	0.998483	0.999056	0.998499	3.3843e-3
			1e-7	0.998290	0.999139	0.998312	3.6414e-3
			1e-9	0.998503	0.999133	0.998516	3.3634e-3
		500	1e-5	0.996072	0.997154	0.996112	9.1060e-3
			1e-7	0.992329	0.994216	0.992428	1.9417e-2
			1e-9	0.997381	0.998794	0.997397	6.0180e-3
	81	50	1e-7	0.983675	0.996465	0.993789	1.0590e-2
			1e-9	0.997996	0.999392	0.998018	4.6631e-3
			1e-11	0.997968	0.998882	0.997991	4.4555e-3
		100	1e-7	0.971020	0.987234	0.974210	5.2693e-2
			1e-9	0.990503	0.999151	0.991206	1.9918e-2
			1e-11	0.998514	0.999201	0.998523	3.2482e-3
		500	1e-7	0.995777	0.996849	0.995845	8.4172e-3
			1e-9	0.999045	0.999730	0.999049	2.3118e-3
			1e-11	0.998777	0.999752	0.998786	2.7374e-3
	225	50	1e-7	0.994403	0.999105	0.994493	1.2200e-2
			1e-9	0.996426	0.998983	0.996587	7.8953e-3
			1e-11	0.995828	0.997608	0.995902	9.5876e-3
		100	1e-7	0.998293	0.999216	0.998304	3.7257e-3
			1e-9	0.995974	0.998551	0.996155	1.2654e-2
			1e-11	0.996415	0.99862	0.996544	6.7600e-3
		500	1e-7	0.996818	0.997337	0.996857	7.2918e-3
			1e-9	0.998274	0.999098	0.998289	2.9491e-3
			1e-11	0.995912	0.99831	0.996352	9.3615e-3

2) For this we have presented the definition, computation and some convergence results of the smoothing fuzzy radial basis function under tension from the input 3D trapezoidal fuzzy data set.
3) According to the presented tables, for fixed values of the remainder of the parameters, we can conjecture the existence of an optimal value of parameter τ, which can be approximated by the cross validation method.
4) Also, according to the presented tables, for fixed values of the remainder of the parameters, we can conjecture the existence of an optimal value of parameter ε.
4) Finally we can observe that the error average index $\overline{d}$ tends to 0 and the similarity average indices $\overline{S}$ tend to the unity (for the different similarity measures used S_{CHEN}, S_{HSIEH} and S_{SCGM}) under adequate hypotheses for the problem parameters, as $nctrs$ and np tend to $+\infty$, as $h \to 0$, which confirms the given convergence results.

Table 2
Function $f(x, y)$. Similarity average indices estimates for different values of the approximation method parameters.

ε	$nctrs$	np	τ	$\overline{S}_{CHEN}$	$\overline{S}_{HSIEH}$	$\overline{S}_{SCGM}$	$\overline{d}$
1	25	50	1e-5	0.998817	0.999453	0.998841	2.8786e-3
			1e-7	0.999039	0.999468	0.999054	2.0472e-3
			1e-9	0.999027	0.999483	0.999042	2.0280e-3
		100	1e-5	0.999526	0.999725	0.999530	1.0622e-3
			1e-7	0.999547	0.999737	0.999550	9.7867e-4
			1e-9	0.999546	0.999737	0.999549	9.7863e-4
		500	1e-5	0.999639	0.999809	0.999640	8.0633e-4
			1e-7	0.999644	0.999809	0.999645	7.8797e-4
			1e-9	0.999645	0.999809	0.999646	7.8238e-4
	81	50	1e-7	0.998238	0.999630	0.998311	4.3516e-3
			1e-9	0.988555	0.998336	0.995505	2.1941e-2
			1e-11	0.988555	0.998336	0.995505	2.1941e-2
		100	1e-7	0.999789	0.999910	0.999790	4.9729e-4
			1e-9	0.998793	0.999686	0.998797	3.1962e-3
			1e-11	0.999780	0.999900	0.999781	5.7181e-4
		500	1e-7	0.999902	0.999956	0.999902	2.4780e-4
			1e-9	0.999898	0.999953	0.999898	2.5945e-4
			1e-11	0.999778	0.999867	0.999778	6.0303e-4
	225	50	1e-7	0.995997	0.998450	0.996096	7.5452e-3
			1e-9	0.932759	0.972674	0.94248	4.1420e-2
			1e-11	0.998909	0.999484	0.998914	2.3016e-3
		100	1e-7	0.999733	0.999484	0.999737	6.4181e-4
			1e-9	0.99959	0.9998	0.999597	1.3297e-3
			1e-11	0.99954	0.999845	0.999543	1.3506e-3
		500	1e-7	0.999925	0.999959	0.999925	1.9554e-4
			1e-9	0.999954	0.999977	0.999954	1.2862e-4
			1e-11	0.999965	0.999986	0.999965	9.0140e-5

Table 3
Function $f(x, y)$. Similarity average indices estimates for different values of the approximation method parameters.

ε	$nctrs$	np	τ	$\overline{S}_{CHEN}$	$\overline{S}_{HSIEH}$	$\overline{S}_{SCGM}$	$\overline{d}$
10	25	50	1e-5	0.998425	0.999407	0.998459	3.4761e-3
			1e-7	0.996894	0.998415	0.997103	7.0671e-3
			1e-9	0.996481	0.998126	0.997779	7.9208e-3
		100	1e-5	0.998594	0.999487	0.998630	3.1538e-3
			1e-7	0.998260	0.999805	0.998349	3.6527e-3
			1e-9	0.998251	0.999402	0.998342	3.6672e-3
		500	1e-5	0.998883	0.999578	0.998894	2.4538e-3
			1e-7	0.998885	0.999581	0.998893	2.4527e-3
			1e-9	0.998886	0.999582	0.998895	2.4526e-3
	81	100	1e-5	0.999520	0.999824	0.999527	1.1695e-3
			1e-7	0.999900	0.999823	0.999467	1.4075e-3
			1e-9	0.999241	0.999755	0.999258	1.5961e-3
		500	1e-5	0.999478	0.999861	0.999480	1.1190e-3
			1e-7	0.999829	0.999912	0.999829	3.9295e-4
			1e-9	0.999638	0.999797	0.999638	8.2272e-4
		1500	1e-5	0.999677	0.999779	0.999678	7.1248e-4
			1e-7	0.999672	0.999842	0.999673	7.1546e-4
			1e-9	0.999602	0.999870	0.999602	9.3121e-4
	225	100	1e-5	0.999364	0.999758	0.999376	1.5926e-3
			1e-7	0.999685	0.999848	0.999690	8.2711e-4
			1e-9	0.991246	0.999271	0.994475	2.6304e-3
		500	1e-5	0.999936	0.999969	0.999937	1.7874e-4
			1e-7	0.999945	0.999978	0.999946	1.4029e-4
			1e-9	0.999953	0.999979	0.999955	1.2489e-4
		1500	1e-9	0.999932	0.999957	0.999952	1.4492e-4
			1e-11	0.999959	0.999988	0.999960	9.6331e-5
			1e-13	0.999971	0.999989	0.999973	7.6271e-5

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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