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Common Best Proximity Points and Completeness of $\mathcal{F}-$Metric Spaces

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Abstract: In this paper, we introduce three classes of proximal contractions that are called the proximally $\lambda-\psi-$dominated contractions, generalized $\eta_{\beta}^{\gamma}-$proximal contractions and Berinde-type weak proximal contractions, and obtain common best proximity points for these proximal contractions in the setting of $\mathcal{F}-$metric spaces. Further, we obtain the best proximity point result for generalized $\alpha-\varphi-$proximal contractions in $\mathcal{F}-$metric spaces. As an application, fixed point and coincidence point results for these contractions are obtained. Some examples are provided to support the validity of our main results. Moreover, we obtain a completeness characterization of the $\mathcal{F}-$metric spaces via best proximity points.

Keywords: common best proximity point; proximally $\lambda-\psi-$dominated contraction; generalized $\eta_{\beta}^{\gamma}-$proximal contraction; Berinde-type weak proximal contraction; generalized $\alpha-\varphi-$proximal contraction; $\mathcal{F}-$metric spaces

MSC: 47H10; 54H25

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1. Introduction and Preliminaries

In 1922, Banach [1] introduced his well-known contraction principle, which states that any single-valued self-mapping T defined on a complete metric space satisfying the contractivity condition admits a unique fixed point. Afterward, fixed point theory appeared as a fundamental and broad subject in nonlinear analysis, which is still developing at a rapid pace. It has useful applications in mathematics as well as various scientific disciplines, such as physics, chemistry, computer science, and others. In fact, many practical and research problems in science and engineering can be reduced to fixed point problems. As a result, this theory has offered a remarkable scope of research. Most of the research on this theory can be roughly divided into two directions: generalizing the contraction conditions, the underlying metric spaces and extending the applications. We refer readers to [2–12] and the references therein. Very recently, a new generalization of the notion of metric space, called an $\mathcal{F}-$metric space, was given in [6], for which the authors used a certain class of auxiliary functions to establish the idea of such abstract spaces. It is obvious that any metric space is an $\mathcal{F}-$metric space, but the converse is not valid in general. A comparison of the $\mathcal{F}-$metric with the existing generalizations of metric in [6] illustrates that any $s-$relaxed $p-$metric is an $\mathcal{F}-$metric, but the converse is not true, which confirms that the class of $\mathcal{F}-$metric spaces is larger than the class of $s-$relaxed $p-$metric spaces. Moreover, some examples in [6] also figured out that $\mathcal{F}-$metric and $b-$metric are two distinct notions. For more details about other generalizations of metric, we refer readers to the book [13] written by Kirk.

Metric fixed point theory gives sufficient conditions that ensure the existence of solutions to the equation $T(x) = x$, where T is a self mapping defined on a metric space (X, d). On the other hand, for a non-self mapping $T : A \to B$, if $T(A) \cap A = \emptyset$, the mapping T has no fixed point. In this case, it is vital to find an element u_0 from the domain spaces whose distance from its image is a minimum. This type of problem is often stated as follows: *"Does there exist a point u_0 in the metric space (X, d) such that $d(u_0, Tu_0) = d(A, B)$, where A, B are two nonempty subsets of X, $T : A \to B$ is a non-self mapping and $d(A, B) = \inf\{d(a, b), (a, b) \in A \times B\}$."* Recall that the point u_0 is called the *best proximity point*, which was introduced by Basha and Veeramani [14]. By the definition of the best proximity point, it is clear that every best proximity point is a fixed point of the mapping T when $A \cap B \neq \emptyset$. This new setting is richer and more general than the metric fixed point theory. A noteworthy best approximation theorem (see [15]) states that if A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space X and $f : A \to X$ is a continuous single-valued function, then either f has a fixed point in A, or there exists an element $x_0 \in A$ and a continuous semi-norm p on X such that $0 < p(x_0 - fx_0) = \min_{x \in A} p(x - f(x_0))$. In best proximity point theory, many authors attempted to find minimum conditions on the non-self mapping T to ensure the existence and uniqueness of the best proximity point. For more details, we refer to [16–20] and the references therein.

The aim of this paper is to present some common best proximity point results in the setting of $\mathcal{F}-$metric spaces. Based on the notion of commuting mappings introduced by Jungck [5], a common fixed point theorem, due to Das and Naik [4], is a special case of our common best proximity point theorem for commuting self-mappings. In the following discussion, the concepts of proximally $\lambda - \psi-$dominated contraction, $\eta_\beta^\gamma-$proximal contraction and Berinde-type weak proximal contraction are introduced. Further, some common best proximity point results are proven, which generalize the main results of [21,22] in the setting of $\mathcal{F}-$metric spaces. Moreover, we will introduce the notion of generalized $\alpha - \varphi-$proximal contractions and prove the best proximity point result for such contractions. As an application, coincidence point and fixed point theorems corresponding to the above proximal contractions are also presented. Some examples are also presented to support our main results. Moreover, a completeness characterization of an $\mathcal{F}-$metric space will be studied in connection with the best proximity points.

Given two nonempty subsets, A and B, of a metric space $(X, \mathfrak{D})$, the following notations and notions will be used.

$$\mathfrak{D}(A, B) = \inf\{\mathfrak{D}(x, y) : x \in A, y \in B\}.$$
$$A_0 = \{x \in A : \mathfrak{D}(x, y) = \mathfrak{D}(A, B), \text{ for some } y \in B\}.$$
$$B_0 = \{y \in B : \mathfrak{D}(x, y) = \mathfrak{D}(A, B), \text{ for some } x \in A\}.$$

If $A \cap B \neq \emptyset$, then A_0 and B_0 are nonempty. Further, it is interesting to notice that A_0 and B_0 are included in the boundaries of A and B, respectively, provided that A and B are closed subsets of a normed linear space such that $d(A, B) > 0$ (see [14]).

In 2018, the concept of $\mathcal{F}-$metric space was presented by Jleli and Samet in [6] as a generalization of the notion of metric space. More precisely, let $\mathcal{F}$ be the set of functions $f : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:

$(\mathcal{F}_1)$ f is non-decreasing, i.e., $0 < s < t \Rightarrow f(s) \leq f(t)$.

$(\mathcal{F}_2)$ For every sequence $\{t_n\} \subset (0, +\infty)$, we have $\lim_{n \to +\infty} t_n = 0 \Leftrightarrow \lim_{n \to +\infty} f(t_n) = -\infty$.

Here are some examples of f belonging to $\mathcal{F}$.

(i) $f_1(t) = -\frac{1}{t}, t \in (0, +\infty)$;

(ii) $f_2(t) = \ln t, t \in (0, +\infty)$;

(iii) $f_3(t) = -e^{\frac{1}{t}}, t \in (0, +\infty)$.

Using such functions, Jleli and Samet [6] generalized the concept of ordinary metric space and introduced the notion of $\mathcal{F}-$metric space as follows:

Definition 1 ([6])**.** *Let X be a nonempty set, and $\mathfrak{D} : X \times X \to [0, +\infty)$ be a given mapping. Assume that there exists $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that*

$(\mathfrak{D}_1)$ $(x, y) \in X \times X, \mathfrak{D}(x, y) = 0 \Leftrightarrow x = y$;

$(\mathfrak{D}_2)$ $\mathfrak{D}(x, y) = \mathfrak{D}(y, x), \forall (x, y) \in X \times X$;

$(\mathfrak{D}_3)$ *for every* $(x, y) \in X \times X, N \in \mathbb{N}$ *with* $N \geq 2$ *and* $(u_i)_{i=1}^{N} \subset X$ *with* $(u_1, u_N) = (x, y)$, *we have*

$$\mathfrak{D}(x, y) > 0 \Rightarrow f(\mathfrak{D}(x, y)) \leq f\left(\sum_{i=1}^{N-1} \mathfrak{D}(u_i, u_{i+1})\right) + \alpha.$$

Then $\mathfrak{D}$ is said to be an $\mathcal{F}-$metric on X, and the pair $(X, \mathfrak{D})$ is said to be an $\mathcal{F}-$metric space.

Example 1 ([6])**.** *Let $X = \mathbb{N}$ and let $\mathfrak{D} : X \times X \to [0, +\infty)$ be a mapping defined for all $(x, y) \in X \times X$,*

$$\mathfrak{D}(x, y) = \begin{cases} \exp(|x - y|), & \text{if} \quad x \neq y \\ 0, \quad \text{if} \quad x = y \end{cases}.$$

For $f(t) = \frac{-1}{t}, t > 0$ and $\alpha = 1$, $\mathfrak{D}$ is an $\mathcal{F}-$ metric.

Definition 2 ([6])**.** *Let $(X, \mathfrak{D})$ be an $\mathcal{F}-$metric space. A subset $\mathcal{O}$ of X is said to be $\mathcal{F}-$open if for $\forall x \in \mathcal{O}, \exists r > 0$ such that $B(x, r) \subset \mathcal{O}$, where $B(x, r) = \{y \in X : \mathfrak{D}(x, y) < r\}$.*

We say that a subset $\mathcal{C}$ of X is $\mathcal{F}-$closed if $X \backslash \mathcal{C}$ is $\mathcal{F}-$open. We denote by $\tau_{\mathcal{F}}$ the family of all $\mathcal{F}-$open subsets of X.

Proposition 1 ([6])**.** *Let $(X, \mathfrak{D})$ be an $\mathcal{F}-$metric space. Then $\tau_{\mathcal{F}}$ is a topology on X.*

Proposition 2 ([6])**.** *Let $(X, \mathfrak{D})$ be an $\mathcal{F}-$metric space. Then for any nonempty subset A of X, the following statements are equivalent:*

(i) *A is $\mathcal{F}-$closed.*
(ii) *For any sequence $\{x_n\} \subset A$, we have*

$$\lim_{n \to +\infty} \mathfrak{D}(x_n, x) = 0, x \in X \Rightarrow x \in A.$$

Definition 3 ([6])**.** *Let $(X, \mathfrak{D})$ be an $\mathcal{F}-$metric space and $\{x_n\}$ be a sequence in X. Then*

(i) *$\{x_n\}$ is $\mathcal{F}-$convergent to $x \in X$, if $\{x_n\}$ is convergent to x with respect to the topology $\tau_{\mathcal{F}}$, i.e., for every $\mathcal{F}-$open subset $\mathcal{O}_x$ of X containing x, there exists some $N \in \mathbb{N}$ such that $x_n \in \mathcal{O}_x$ for $\forall n \geq N$. In this case, we say that x is the limit of $\{x_n\}$.*
(ii) *$\{x_n\}$ is $\mathcal{F}-$Cauchy, if $\lim_{n,m \to +\infty} \mathfrak{D}(x_n, x_m) = 0$.*
(iii) *$(X, \mathfrak{D})$ is $\mathcal{F}-$complete, if every $\mathcal{F}-$Cauchy sequence in X is $\mathcal{F}-$convergent to a certain element in X.*

Proposition 3 ([6])**.** *Let $(X, \mathfrak{D})$ be an $\mathcal{F}-$metric space and $\{x_n\}$ be a sequence in X. Then we have*

(1) *a sequence $\{x_n\}$ that is $\mathcal{F}-$convergent to $x \Rightarrow \lim_{n \to +\infty} \mathfrak{D}(x_n, x) = 0$.*
(2) *the limit of an $\mathcal{F}-$convergent sequence is unique, i.e., $(x, y) \in X \times X$, $\lim_{n \to +\infty} \mathfrak{D}(x_n, x) = \lim_{n \to +\infty} \mathfrak{D}(x_n, y) = 0 \Rightarrow x = y$.*
(3) *if a sequence is $\{x_n\}$ is $\mathcal{F}-$convergent, then it is $\mathcal{F}-$Cauchy.*

Definition 4 ([6]). *Let X be a nonempty set and $\mathfrak{D} : X \times X \to [0, \infty)$ be a given mapping satisfying $(\mathfrak{D}_1)$ and $(\mathfrak{D}_2)$. Then the pair $(X, \mathfrak{D})$ is $\mathcal{F}-$metric bounded with respect to $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ if there exists a metric d on X such that*

$$(x, y) \in X \times X, \mathfrak{D}(x, y) > 0 \Rightarrow f(d(x, y)) \leq f(\mathfrak{D}(x, y)) \leq f(d(x, y)) + \alpha.$$

Definition 5 ([23]). *Let $\alpha : X \times X \to \mathbb{R}$ be a given function. We say that $T : X \to X$ is $\alpha-$admissible if*

$$(x, y) \in X \times X, \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Based on the existing definitions introduced in normal metric spaces, we will introduce the analogous definitions in the setting of an $\mathcal{F}-$metric space as follows.

Definition 6 ([21]). *Let A and B be two nonempty subsets of an $\mathcal{F}-$metric space $(X, \mathfrak{D})$. A mapping $T : A \to B$ is said to be a proximal contraction if there exists a non-negative real number $\lambda < 1$ such that*

$$\mathfrak{D}(u_1, Tx_1) = \mathfrak{D}(A, B) = \mathfrak{D}(u_2, Tx_2) \Rightarrow \mathfrak{D}(u_1, u_2) \leq \lambda \mathfrak{D}(x_1, x_2), \quad \forall u_1, u_2, x_1, x_2 \in A.$$

Definition 7 ([21]). *Let A and B be two nonempty subsets of an $\mathcal{F}-$metric space $(X, \mathfrak{D})$. Two mappings $T, S : A \to B$ are said to proximally commute if the following holds true*

$$\mathfrak{D}(u, Sx) = \mathfrak{D}(v, Tx) = \mathfrak{D}(A, B) \Rightarrow Sv = Tu, \quad \forall u, v, x \in A.$$

Definition 8 ([21]). *Let A and B be two nonempty subsets of an $\mathcal{F}-$metric space $(X, \mathfrak{D})$. A mapping $T : A \to B$ is said to proximally dominate a mapping $S : A \to B$ if there exists a non-negative real number $\xi < 1$ such that*

$$\begin{aligned} &\mathfrak{D}(u_1, Sx_1) = \mathfrak{D}(v_1, Tx_1) = \mathfrak{D}(A, B) = \mathfrak{D}(u_2, Sx_2) = \mathfrak{D}(v_2, Tx_2) \\ &\quad \Rightarrow \mathfrak{D}(u_1, u_2) \leq \xi \mathfrak{D}(v_1, v_2), \quad \forall u_1, u_2, x_1, x_2, v_1, v_2 \in A. \end{aligned}$$

Definition 9 ([16]). *Let $(X, \mathfrak{D})$ be an $\mathcal{F}-$metric space and $T : A \to B$ and $\alpha : A \times A \to [0, +\infty)$ be two mappings. We say that T is $\alpha-$proximal admissible if*

$$\alpha(x_1, x_2) \geq 1, \mathfrak{D}(u_1, Tx_1) = \mathfrak{D}(A, B) = \mathfrak{D}(u_2, Tx_2) \Rightarrow \alpha(u_1, u_2) \geq 1, \forall x_1, x_2, u_1, u_2 \in A.$$

Definition 10 ([14]). *Let A and B be nonempty subsets of a metric space $\mathcal{F}-$metric space $(X, \mathfrak{D})$ and $T : A \to B$ be a given mapping. A point $x_0 \in A$ is said to be the best proximity point of T if $d(x_0, Tx_0) = \mathfrak{D}(A, B)$.*

Definition 11 ([24]). *Let A and B be two nonempty subsets of an $\mathcal{F}-$metric space $(X, \mathfrak{D})$. An element $x \in A$ is said to be a common best proximity point of the pair (T, S) if x satisfies*

$$\mathfrak{D}(x, Sx) = \mathfrak{D}(x, Tx) = \mathfrak{D}(A, B),$$

where $T, S : A \to B$ are two non-self mappings.

Definition 12 ([17]). *Let A and B be two nonempty subsets of an $\mathcal{F}-$metric space $(X, \mathfrak{D})$ with $A_0 \neq \emptyset$. Then the pair (A, B) is said to satisfy the $P-$property $\Leftrightarrow$ for all $x_1, x_2 \in A_0, y_1, y_2 \in B_0$,*

$$\mathfrak{D}(x_1, y_1) = \mathfrak{D}(A, B) = \mathfrak{D}(x_2, y_2) \Rightarrow \mathfrak{D}(x_1, x_2) = \mathfrak{D}(y_1, y_2).$$

Definition 13 ([18])**.** *Let A and B be two nonempty subsets of an $\mathcal{F}-$metric space $(X, \mathfrak{D})$ with $A_0 \neq \emptyset$. Then the pair (A, B) is said to satisfy the weak $P-$property $\Leftrightarrow$ for all $x_1, x_2 \in A_0, y_1, y_2 \in B_0$,*

$$\mathfrak{D}(x_1, y_1) = \mathfrak{D}(A, B) = \mathfrak{D}(x_2, y_2) \Rightarrow \mathfrak{D}(x_1, x_2) \leq \mathfrak{D}(y_1, y_2).$$

Here is an example to show that (A, B) satisfies the weak $P-$property but not the $P-$property.

Example 2. *Consider an $\mathcal{F}-$metric space illustrated in Example 4 in [25] as follows.*

Define $g : \mathbb{R}^2 \to [0, +\infty)$ by

$$g(x, y) = \begin{cases} 2|x| & y = 0, \\ |x| + |y| & y \neq 0. \end{cases}$$

Let $\mathfrak{D}((x_1, y_1), (x_2, y_2)) = g(x_1 - x_2, y_1 - y_2)$, where $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Then, it can be easily checked that $(\mathbb{R}^2, \mathfrak{D})$ is an $\mathcal{F}-$metric space with $f(t) = \ln t \in \mathcal{F}$ and $\alpha = \ln 2$.

Set $A = \{(0, 0)\}$ and $B = \{(x, y) : y = 1 - |x|\}$. Obviously, $A_0 = \{(0, 0)\}$, $B_0 = \{(x, y) : y = 1 - |x|, |x| \leq 1\}$ and $\mathfrak{D}(A, B) = 1$.

Furthermore,

$\mathfrak{D}((0, 0), (\frac{1}{2}, \frac{1}{2})) = \mathfrak{D}((0, 0), (-\frac{1}{2}, \frac{1}{2})) = 1, 0 = \mathfrak{D}((0, 0), (0, 0)) < \mathfrak{D}((\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2})) = 2.$

We can conclude that (A, B) satisfies the weak $P-$property but not the $P-$property.

Definition 14 ([26])**.** *Let $(X, \mathfrak{D})$ be an $\mathcal{F}-$metric space and (A, B) be a pair of nonempty closed subsets of X. An $\mathcal{F}-$metric space $(X, \mathfrak{D})$ has the $R-$property with respect to the pair (A, B), that is, if x_n is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\lim_{n \to +\infty} x_n = x^* \in A$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all $k \in \mathbb{N}$. If $A = B = X$, then we say that $(X, \mathfrak{D})$ satisfies property $(\mathcal{A})$.*

2. Common Best Proximity Points for New Proximal Contractions

First, we introduce the following auxiliary functions [27] that will be used in the main discussions.

Let Ψ denote the family of all functions $\psi : [0, +\infty) \to [0, +\infty)$, where

(1) ψ is increasing and continuous;
(2) $t \leq \psi(t)$ and $\psi(0) = 0$;
(3) $\psi(x + y) \leq \psi(x) + \psi(y), \forall x, y \in [0, +\infty)$.

Let Υ denote the family of all functions $\varphi : [0, +\infty) \to [0, +\infty)$, where

(1) φ is non-decreasing and continuous;
(2) $\varphi(t) < t, \forall t > 0$;
(3) $\sum_{n=1}^{+\infty} \varphi^n(t) < +\infty, \forall t > 0$.

Let Λ denote the family of all functions $\lambda : [0, +\infty) \to [0, 1)$, where

(1) λ is non-decreasing;
(2) $\lambda(t_n) \to 1 \Rightarrow t_n \to 0$.

Next, we will introduce the notions of proximally $\lambda - \psi-$dominated contraction, generalized $\eta_\beta^\gamma-$proximal contraction and Berinde-type weak proximal contraction in the setting of an $\mathcal{F}-$metric space as follows.

Definition 15. *Let A and B be two nonempty subsets of an $\mathcal{F}-$metric space $(X, \mathfrak{D})$. The pair (T, S) of non-self mappings $T, S : A \to B$ is said to be a proximally $\lambda - \psi-$dominated contraction if there exist $\lambda \in \Lambda$ and $\psi \in \Psi$ such that for all $u_1, u_2, v_1, v_2, x_1, x_2 \in A$*

$$\mathfrak{D}(u_1, Sx_1) = \mathfrak{D}(v_1, Tx_1) = \mathfrak{D}(A, B) = \mathfrak{D}(u_2, Sx_2) = \mathfrak{D}(v_2, Tx_2) \\ \Rightarrow \psi(\mathfrak{D}(u_1, u_2)) \leq \lambda(\mathfrak{D}(v_1, v_2))\psi(\mathfrak{D}(v_1, v_2)).$$

Definition 16. *Let A and B be two nonempty subsets of an $\mathcal{F}-$metric space $(X, \mathfrak{D})$. The pair (T, S) of non-self mappings $T, S : A \to B$ is said to be a generalized $\eta^{\gamma}_{\beta}-$proximal contraction if there exist $\eta \in (0, 1)$ and $\beta, \gamma \in [0, 1)$ with $\eta + \beta + \gamma < 1$ such that for all $x, y \in A$*

$$\mathfrak{D}(Tx, Ty) \leq \eta\mathfrak{D}(Sx, Sy) + \beta\mathfrak{D}(Sx, Tx) + \gamma\mathfrak{D}(Sy, Ty).$$

Definition 17. *Let A and B be two nonempty subsets of an $\mathcal{F}-$metric space $(X, \mathfrak{D})$. The pair (T, S) of non-self mappings $T, S : A \to B$ is said to be Berinde-type weak proximal contraction, if there exist $\xi \in (0, 1)$ and $L \geq 0$ such that for all $u_1, u_2, v_1, v_2, x_1, x_2 \in A$*

$$\mathfrak{D}(u_1, Sx_1) = \mathfrak{D}(v_1, Tx_1) = \mathfrak{D}(A, B) = \mathfrak{D}(u_2, Sx_2) = \mathfrak{D}(v_2, Tx_2) \Rightarrow \\ \mathfrak{D}(u_1, u_2) \leq \xi \max\{\mathfrak{D}(u_1, v_1), \mathfrak{D}(u_1, u_2)\} + L \min\{\mathfrak{D}(u_2, Sx_2) - \mathfrak{D}(A, B)\mathfrak{D}(u_1, v_1)\}.$$

Definition 18. *Let $(X, \mathfrak{D})$ be an $\mathcal{F}-$metric space and (A, B) be a pair of nonempty subsets of X. A mapping $T : A \to B$ is said to be a generalized $\alpha - \phi-$proximal contraction if there exists $\phi \in \Upsilon$ such that for all $x, y \in A$*

$$\alpha(x, y)\mathfrak{D}(Tx, Ty) \leq \phi(\mathfrak{D}(x, y)),$$

where $\alpha : A \times A \to [0, \infty)$ is a mapping.

Theorem 1. *Let (A, B) be a pair of nonempty subsets of an $\mathcal{F}-$complete metric space $(X, \mathfrak{D})$. Assume that A_0 is a nonempty and closed subset of X. Suppose that the pair (T, S) of non-self mappings $T, S : A \to B$ satisfies the following conditions:*

(1) the pair (T, S) of non-self mappings is a proximally $\lambda - \psi-$dominated contraction;
(2) T and S proximally commute;
(3) T and S are continuous;
(4) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.

Then the pair (T, S) admits a unique common best proximity point.

Proof. Let x_0 be a fixed element in A_0. Since $S(A_0) \subseteq T(A_0)$, there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Repeating this process, having chosen $x_n \in A_0$, we can find an element $x_{n+1} \in A_0$ satisfying

$$Sx_n = Tx_{n+1}, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{1}$$

Further, since $S(A_0) \subseteq B_0$, correspondingly, there exists an element $u_n \in A_0$ such that

$$\mathfrak{D}(u_n, Sx_n) = \mathfrak{D}(A, B), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{2}$$

Further, it follows from the choice of x_n and u_n and from (1) and (2) that for all $n \in \mathbb{N}$,

$$\mathfrak{D}(u_{n+1}, Sx_{n+1}) = \mathfrak{D}(u_n, Tx_{n+1}) = \mathfrak{D}(u_{n-1}, Tx_n) = \mathfrak{D}(A, B). \tag{3}$$

Since the pair (T, S) is a proximally $\lambda - \psi-$dominated contraction, from (1) to (3), we have

$$\psi(\mathfrak{D}(u_n, u_{n+1})) \leq \lambda(\mathfrak{D}(u_{n-1}, u_n))\psi(\mathfrak{D}(u_{n-1}, u_n)) < \psi(\mathfrak{D}(u_{n-1}, u_n)). \tag{4}$$

Since ψ is increasing, $\{\mathfrak{D}(u_{n-1}, u_n)\}$ is non-increasing and bounded. Therefore, $\lim_{n\to+\infty} \mathfrak{D}(u_{n-1}, u_n)$ exists.

Let $\lim_{n\to+\infty} \mathfrak{D}(u_{n-1}, u_n) = r \geq 0$. Assume that $r > 0$. Then from (4) we have

$$\frac{\psi(\mathfrak{D}(u_n, u_{n+1}))}{\psi(\mathfrak{D}(u_{n-1}, u_n))} \leq \lambda(\mathfrak{D}(u_{n-1}, u_n)).$$

Since ψ is continuous, the above inequality yields

$$\lim_{n\to+\infty} \lambda(\mathfrak{D}(u_{n-1}, u_n)) = 1,$$

which implies that $r = 0$, that is,

$$\lim_{n\to+\infty} \mathfrak{D}(u_{n-1}, u_n) = 0.$$

Keeping in mind that $\{\mathfrak{D}(u_{n-1}, u_n)\}$ is non-increasing and λ is a non-decreasing function, we have

$$\lambda(\mathfrak{D}(u_0, u_1)) \geq \lambda(\mathfrak{D}(u_1, u_2)) \geq \ldots \geq \lambda(\mathfrak{D}(u_{n-1}, u_n)).$$

Further, from inequality (4), we have

$$\begin{aligned}
\psi(\mathfrak{D}(u_{n+1}, u_n)) &< \psi(\mathfrak{D}(u_{n-1}, u_n)) \\
&\leq \lambda(\mathfrak{D}(u_{n-2}, u_{n-1}))\psi(\mathfrak{D}(u_{n-2}, u_{n-1})) \\
&\leq \lambda(\mathfrak{D}(u_{n-2}, u_{n-1}))\lambda(\mathfrak{D}(u_{n-3}, u_{n-2}))\psi(\mathfrak{D}(u_{n-3}, u_{n-2})) \\
&\leq \lambda(\mathfrak{D}(u_{n-2}, u_{n-1}))\lambda(\mathfrak{D}(u_{n-3}, u_{n-2})) \cdots \lambda(\mathfrak{D}(u_1, u_2))\lambda(\mathfrak{D}(u_0, u_1))\psi(\mathfrak{D}(u_0, u_1)) \\
&\leq \lambda(\mathfrak{D}(u_0, u_1))\lambda(\mathfrak{D}(u_0, u_1)) \cdots \lambda(\mathfrak{D}(u_0, u_1))\lambda(\mathfrak{D}(u_0, u_1))\psi(\mathfrak{D}(u_0, u_1)) \\
&\leq \lambda^n(\mathfrak{D}(u_0, u_1))\psi(\mathfrak{D}(u_0, u_1)) \\
&= \mu^n \psi(\mathfrak{D}(u_0, u_1)),
\end{aligned}$$

where $\mu = \lambda(\mathfrak{D}(u_0, u_1)) \in [0, 1)$. Therefore,

$$\psi(\mathfrak{D}(u_n, u_{n+1})) < \mu^n \psi(\mathfrak{D}(u_0, u_1)),$$

which yields that

$$\sum_{i=n}^{m-1} \psi(\mathfrak{D}(u_i, u_{i+1})) < \frac{\mu^n}{1-\mu}\psi(\mathfrak{D}(u_0, u_1)). \tag{5}$$

Next, we will prove that $\{u_n\}$ is an $\mathcal{F}-$Cauchy sequence.

Without a loss in the generality, we may suppose that $\mathfrak{D}(u_0, u_1) > 0$. Otherwise, from the construction of $\{u_n\}$, we can conclude that $u_n = u_0$ for all $n \in \mathbb{N}$ and so $\{u_n\}$ is an $\mathcal{F}-$Cauchy.

First, let $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ be such that $(\mathfrak{D}_3)$ is satisfied. For any given $\epsilon > 0$, by $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha. \tag{6}$$

Since

$$\lim_{n\to+\infty} \frac{\mu^n}{1-\mu}\psi(\mathfrak{D}(u_0, u_1)) = 0,$$

then for $\delta > 0$, there exists some $N \in \mathbb{N}$ such that

$$0 < \frac{\mu^n}{1-\mu}\psi(\mathfrak{D}(u_0, u_1)) < \delta, \quad \forall n \geq N. \tag{7}$$

In addition, since ψ satisfies that $t \leq \psi(t)$, we have

$$\sum_{i=n}^{m-1} \mathfrak{D}(u_i, u_{i+1}) \leq \sum_{i=n}^{m-1} \psi(\mathfrak{D}(u_i, u_{i+1})) < \frac{\mu^n}{1-\mu}\psi(\mathfrak{D}(u_0, u_1)). \tag{8}$$

Hence, by (7)–(8) and $(\mathcal{F}_1)$, we have

$$f\left(\sum_{i=n}^{m-1} \mathfrak{D}(u_i, u_{i+1})\right) \leq f\left(\frac{\mu^n}{1-\mu}\psi(\mathfrak{D}(u_0, u_1))\right) < f(\epsilon) - \alpha, \quad \forall m > n \geq N. \tag{9}$$

By $(\mathfrak{D}_3)$ and inequality (9), we have

$$\mathfrak{D}(u_n, u_m) > 0, m > n > N \Rightarrow f(\mathfrak{D}(u_n, u_m)) \leq f\left(\sum_{i=n}^{m-1} \mathfrak{D}(u_i, u_{i+1})\right) + \alpha < f(\epsilon),$$

which implies by $(\mathcal{F}_1)$ that

$$\mathfrak{D}(u_n, u_m) < \epsilon, \quad \forall m > n \geq N.$$

This proves that $\{u_n\}$ is an $\mathcal{F}-$Cauchy sequence. Since $(X, \mathfrak{D})$ is a complete $\mathcal{F}-$metric space and A_0 is closed, there exists some $u \in A_0$ such that $\lim_{n\to+\infty} u_n = u$. Because of the fact that the mappings S and T proximally commute and from inequalities (2) and (3), we have

$$Tu_n = Su_{n-1}, \quad \forall n \in \mathbb{N}.$$

Therefore, the continuity of the mappings S and T ensures that

$$Tu = \lim_{n\to+\infty} Tu_n = \lim_{n\to+\infty} Su_{n-1} = Su.$$

Since $S(A_0) \subseteq B_0$, there exists an element $x \in A_0 \subseteq A$ such that

$$\mathfrak{D}(x, Su) = \mathfrak{D}(x, Tu) = \mathfrak{D}(A, B),$$

It follows from Definition 7 that $Sx = Tx$. Again, since $S(A_0) \subseteq B_0$, there exists $z \in A$ such that

$$\mathfrak{D}(z, Sx) = \mathfrak{D}(z, Tx) = \mathfrak{D}(A, B).$$

Since the pair of mappings (T, S) is $\lambda - \psi-$dominate proximally contractive, we have

$$\psi(\mathfrak{D}(x, z)) \leq \lambda(\mathfrak{D}(x, z))\psi(\mathfrak{D}(x, z)),$$

which implies $x = z$. Thus, it follows that

$$\mathfrak{D}(x, Sx) = \mathfrak{D}(z, Sx) = \mathfrak{D}(A, B) = \mathfrak{D}(x, Tx) = \mathfrak{D}(z, Tx).$$

Therefore, x is a common best proximity point of the mappings S and T.

Suppose that y is another common best proximity point of the mappings S and T such that $x \neq y$. We have

$$\begin{aligned} \mathfrak{D}(x, Sx) = \mathfrak{D}(A, B) = \mathfrak{D}(x, Tx). \\ \mathfrak{D}(y, Sy) = \mathfrak{D}(A, B) = \mathfrak{D}(y, Ty). \end{aligned} \tag{10}$$

By the definition of $\lambda - \psi-$dominate proximal contractivity and (10), we have

$$\psi(\mathfrak{D}(x, y)) \leq \lambda(\mathfrak{D}(x, y))\psi(\mathfrak{D}(x, y)) < \psi(\mathfrak{D}(x, y)),$$

which implies that $x = y$. Hence, x is a unique common best proximity point that satisfies $\mathfrak{D}(x, Sx) = \mathfrak{D}(A, B) = \mathfrak{D}(x, Tx)$. $\square$

Example 3. *Let $X^* = [0,1] \times [0,1]$ be endowed with a metric $\mathfrak{D} : [0,1]^2 \times [0,1]^2 \to [0, \infty)$ defined by for all $x = (x_1, x_2), y = (y_1, y_2) \in X^*$*

$$\mathfrak{D}(x, y) = \begin{cases} \exp^{\sum_{i=1}^{2}|x_i - y_i|}, & \text{if} \quad x_i \neq y_i, \\ 0, & \text{if} \quad x_i = y_i. \end{cases}$$

It can be easily checked that $\mathfrak{D}$ is an $\mathcal{F}-$metric with $f(t) = -\frac{1}{t} \in \mathcal{F}$ and $\alpha = 1$.

Let $A = \{(0, x^*) : 0 \leq x^* \leq 1\}$ and $B = \{(1, y^*) : 0 \leq y^* \leq 1\}$. Notice that we can have that $A_0 = A, B_0 = B$ and $\mathfrak{D}(A, B) = e$. Consider the mappings $T, S : A \to B$ defined by $S(0, x^*) = (1, \ln(1 + x^*))$, and $T(0, x^*) = (1, x^*)$.

Now, we claim that the pair (T, S) is a $\lambda - \psi-$dominate proximal contraction. Consider $\lambda \in \Lambda$ and $\psi \in \Psi$ defined by $\lambda(t) = 1 - \frac{\ln^2 t}{2t}$, $\forall t \in (0, \infty)$ and $\psi(t) = t$, $\forall t \in [0, \infty)$. Choosing $u_1 = (0, u), v_1 = (0, v), w_1 = (0, w), s_1 = (0, s), x_1 = (0, x), y_1 = (0, y)$ in A satisfying

$$\mathfrak{D}(u_1, Sx_1) = \mathfrak{D}(v_1, Sy_1) = \mathfrak{D}(A, B) = \mathfrak{D}(w_1, Tx_1) = \mathfrak{D}(s_1, Ty_1) = e.$$

We have $w = x, s = y$ and $u = \ln(1 + x), v = \ln(1 + y)$.

Thus, we can write

$$\begin{aligned} \psi(\mathfrak{D}(u_1, v_1)) &= e^{|\ln(1+x) - \ln(1+y)|} \\ &= e^{|\ln(1+w) - \ln(1+s)|} \\ &\leq e^{\ln(1+|w-s|)} \\ &< e^{|w-s|} - \frac{1}{2}|w - s|^2 \\ &= e^{|w-s|}\left(1 - \frac{\ln^2 e^{|w-s|}}{2e^{|w-s|}}\right) \\ &= \lambda(\mathfrak{D}(w_1, s_1))\psi(\mathfrak{D}(w_1, s_1)), \end{aligned}$$

which shows that the pair (T, S) commutes proximally and is a $\lambda - \psi-$dominate proximal contraction. Therefore, all conditions of Theorem 1 are satisfied. From the conclusion of Theorem 1, (T, S) has a unique common best proximity point, which is $(0, 0) \in A$.

Theorem 2. *Suppose that (A, B) is a pair of nonempty subsets of a complete $\mathcal{F}-$metric space $(X, \mathfrak{D})$ that satisfies the $P-$property. Assume that A_0 is a nonempty and closed subset of A. Further, suppose that $S, T : A \to B$ are continuous mappings, where T and S proximally commute. Further, assume that the pair (T, S) is a generalized $\eta_\beta^\gamma-$proximal contraction satisfying $S(A_0) \subseteq T(A_0)$ and $S(A_0) \subseteq B_0$. Then the pair (T, S) admits a unique common best proximity point.*

Proof. Let x_0 be a fixed element in A_0. Since $S(A_0) \subseteq T(A_0)$, there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Repeating this process, having chosen $x_n \in A_0$, we can find an element $x_{n+1} \in A_0$ satisfying

$$Tx_n = Sx_{n+1}, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{11}$$

Further, since $S(A_0) \subseteq B_0$, correspondingly, there exists an element $u_n \in A_0$ such that

$$\mathfrak{D}(u_n, Tx_n) = \mathfrak{D}(A, B), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{12}$$

Further, it follows from the choice of x_n and u_n and from (11) and (12) that

$$\mathfrak{D}(u_{n+1}, Tx_{n+1}) = \mathfrak{D}(u_n, Sx_{n+1}) = \mathfrak{D}(u_{n-1}, Sx_n) = \mathfrak{D}(A, B), \quad \forall n \in \mathbb{N}. \tag{13}$$

Since the pair (A, B) satisfies the $P-$property, and it is a generalized $\eta_\beta^\gamma-$proximal contraction, we have

$$\begin{aligned}\mathfrak{D}(u_n, u_{n+1}) &= \mathfrak{D}(Tx_n, Tx_{n+1}) \\ &\leq \eta\mathfrak{D}(Sx_n, Sx_{n+1}) + \beta\mathfrak{D}(Sx_n, Tx_n) + \gamma\mathfrak{D}(Sx_{n+1}, Tx_{n+1}) \\ &= \eta\mathfrak{D}(u_{n-1}, u_n) + \beta\mathfrak{D}(u_{n-1}, u_n) + \gamma\mathfrak{D}(u_n, u_{n+1}).\end{aligned}$$

After simplification, we have

$$\begin{aligned}\mathfrak{D}(u_n, u_{n+1}) - \gamma\mathfrak{D}(u_n, u_{n+1}) &\leq \eta\mathfrak{D}(u_{n-1}, u_n) + \beta\mathfrak{D}(u_{n-1}, u_n) \\ (1-\gamma)\mathfrak{D}(u_n, u_{n+1}) &\leq (\eta+\beta)\mathfrak{D}(u_{n-1}, u_n) \\ \mathfrak{D}(u_n, u_{n+1}) &\leq \frac{\eta+\beta}{1-\gamma}\mathfrak{D}(u_{n-1}, u_n).\end{aligned}$$

Further, it can be written as

$$\mathfrak{D}(u_n, u_{n+1}) \leq \rho\mathfrak{D}(u_{n-1}, u_n), \quad \forall n \in \mathbb{N}, \tag{14}$$

where $\rho = \frac{\eta+\beta}{1-\gamma} \in (0,1)$, due to $\eta + \beta + \gamma < 1$.

Now, we will prove that $\{u_n\}$ is an $\mathcal{F}-$Cauchy sequence.

Reasoning as in the proof of Theorem 1, we can assume, without a loss in generality, that $\mathfrak{D}(u_0, u_1) > 0$. From (14), we have

$$\begin{aligned}\mathfrak{D}(u_n, u_{n+1}) &\leq \rho\mathfrak{D}(u_{n-1}, u_n) \\ &\leq \rho^2\mathfrak{D}(u_{n-2}, u_{n-1}) \\ &\vdots \\ &\leq \rho^n\mathfrak{D}(u_0, u_1), \quad \forall n \in \mathbb{N}.\end{aligned}$$

It follows that

$$\sum_{i=n}^{m-1}\mathfrak{D}(u_i, u_{i+1}) \leq \frac{\rho^n}{1-\rho}\mathfrak{D}(u_0, u_1), \quad \forall m \geq n. \tag{15}$$

This further implies that

$$\lim_{n\to+\infty}\frac{\rho^n}{1-\rho}\mathfrak{D}(u_0, u_1) = 0. \tag{16}$$

Let $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ be such that $(\mathfrak{D}_3)$ is satisfied.

For any given $\epsilon > 0$, by $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha. \tag{17}$$

It follows from $(\mathcal{F}_1)$ and (15)–(17) that there exists $N \in \mathbb{N}$ such that

$$f\left(\sum_{i=n}^{m-1}\mathfrak{D}(u_i, u_{i+1})\right) \leq f\left(\frac{\rho^n}{1-\rho}\mathfrak{D}(u_0, u_1)\right) < f(\epsilon) - \alpha, \quad \forall m > n \geq N.$$

This implies that

$$f\left(\sum_{i=n}^{m-1}\mathfrak{D}(u_i, u_{i+1})\right) < f(\epsilon) - \alpha, \quad \forall m > n \geq N. \tag{18}$$

Using $(\mathfrak{D}_3)$ and (18), we obtain

$$\mathfrak{D}(u_n, u_m) > 0, m > n > N \Rightarrow f(\mathfrak{D}(u_n, u_m)) \leq f\left(\sum_{i=n}^{m-1} \mathfrak{D}(u_i, u_{i+1})\right) + \alpha < f(\epsilon),$$

which implies by $(\mathcal{F}_1)$ that $\mathfrak{D}(u_n, u_m) \leq \epsilon, \forall m > n \geq N$.

Therefore, the sequence $\{u_n\}$ is an $\mathcal{F}-$Cauchy sequence. Since $(X, \mathfrak{D})$ is an $\mathcal{F}-$complete metric space and A_0 is closed, there exists some $u \in A_0$ such that $\lim_{n\to+\infty} u_n = u$. Since the mappings S and T proximally commute from (11) and (12), we have

$$Su_n = Tu_{n-1}, \quad \forall n \in \mathbb{N}.$$

Since S and T are continuous mappings, therefore,

$$Su = \lim_{n\to+\infty} Su_n = \lim_{n\to+\infty} Tu_{n-1} = Tu.$$

Since $T(A_0) \subseteq B_0$, there exists an element $x \in A$ such that

$$\mathfrak{D}(x, Tu) = \mathfrak{D}(A, B) = \mathfrak{D}(x, Su). \tag{19}$$

It follows from Definition 7 that $Sx = Tx$. Again, since $T(A_0) \subseteq B_0$, there exists $z \in A$ such that

$$\mathfrak{D}(z, Tx) = \mathfrak{D}(A, B) = \mathfrak{D}(z, Sx). \tag{20}$$

To prove that $z = x$, suppose that they are distinct.

Since the pair of subsets (A, B) satisfies the $P-$property and from Equations (19) and (20), we have

$$\begin{aligned}
\mathfrak{D}(x, z) &= \mathfrak{D}(Tu, Tx) \\
&\leq \eta\mathfrak{D}(Su, Sx) + \beta\mathfrak{D}(Su, Tu) + \gamma\mathfrak{D}(Sx, Tx) \\
&= \eta\mathfrak{D}(x, z) + \beta\mathfrak{D}(x, x) + \gamma\mathfrak{D}(z, z) \\
&= \eta\mathfrak{D}(x, z) \\
&< \mathfrak{D}(x, z),
\end{aligned}$$

which is a contradiction. It can be easily seen from (20) that

$$\mathfrak{D}(x, Tx) = \mathfrak{D}(A, B) = \mathfrak{D}(x, Sx). \tag{21}$$

Then x is a common best proximity point of the mappings T and S.

Suppose y is another common best proximity point of the mappings T and S. We have

$$\mathfrak{D}(y, Ty) = \mathfrak{D}(A, B) = \mathfrak{D}(y, Sy). \tag{22}$$

As the pair (A, B) satisfies the $P-$property, from (21) and (22), we have

$$\begin{aligned}
\mathfrak{D}(x, y) &= \mathfrak{D}(Tx, Ty) \\
&\leq \eta\mathfrak{D}(Sx, Sy) + \beta\mathfrak{D}(Sx, Tx) + \gamma\mathfrak{D}(Sy, Ty) \\
&\leq \eta\mathfrak{D}(x, y) + \beta\mathfrak{D}(x, x) + \gamma\mathfrak{D}(y, y) \\
&\leq \eta\mathfrak{D}(x, y) \\
&< \mathfrak{D}(x, y),
\end{aligned}$$

which is a contradiction. Hence, $x = y$; that is, x is the unique common best proximity point of the mappings S and T such that $\mathfrak{D}(x, Tx) = \mathfrak{D}(A, B) = \mathfrak{D}(x, Sx)$. □

Example 4. *Let $X = \mathbb{R}^2$ be endowed with an $\mathcal{F}-$metric $\mathfrak{D} : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ defined by $\mathfrak{D}((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Suppose that $A = \{(1, x), x \in \mathbb{R}\}$ and $B = \{(0, x), x \in \mathbb{R}\}$. For given A and B, we have $A_0 = A, B_0 = B$ and $\mathfrak{D}(A, B) = 1$. Obviously, A_0 is closed and nonempty.*

Define $T, S : A \to B$ by $S(1, x) = (0, x)$, $T(1, x) = (0, 2x)$ for all $x \in \mathbb{R}$. After a simple calculation, we can see that T and S are continuous and $S(A_0) \subseteq T(A_0)$ and $S(A_0) \subseteq B_0$. Choosing any $(1, a), (1, b), (1, c) \in A$ such that $\mathfrak{D}((1, a), T(1, c)) = \mathfrak{D}((1, b), S(1, c)) = \mathfrak{D}(A, B) = 1$, we have $a = 2c$ and $c = b$. Thus, we have $T(1, a) = (0, a) = S(1, b) = (0, 2b)$, which demonstrates that T, S proximally commute.

Moreover, for $\forall (1, x), (1, y) \in A$, there exist $\eta = \frac{1}{2} \in (0, 1)$ and $\beta = \gamma = \frac{1}{8} \in [0, 1)$ with $\eta + \beta + \gamma < 1$ such that

$$\begin{aligned}
|x - y| &= \mathfrak{D}(T(1, x), T(1, y)) \\
&\leq \frac{1}{2}\mathfrak{D}(S(1, x), S(1, y)) + \beta\mathfrak{D}(S(1, x), T(1, x)) + \gamma\mathfrak{D}(S(1, y), T(1, y)) \\
&= |x - y| + \frac{1}{8}(|x| + |y|),
\end{aligned}$$

which shows that (T, S) is a generalized $\eta_\beta^\gamma-$proximal contraction.

Hence, all the assumptions of Theorem 2 are satisfied. Therefore, T, S have a unique common best proximity point, which is $(1, 0)$.

Theorem 3. *Let (A, B) be a pair of nonempty subsets of a complete $\mathcal{F}-$metric space $(X, \mathfrak{D})$ satisfying the $P-$property. Assume that A_0 is a nonempty closed subset of A. Further, suppose that $S, T : A \to B$ proximally commute, where T and S commute proximally. Further, assume that the pair (T, S) is a Berinde-type weak proximal contraction verifying $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$. Then the pair (T, S) admits a unique common best proximity point.*

Proof. Let x_0 be a fixed element in A_0. Following the process stated in the proof of Theorem 1, we can find two sequences $\{x_n\}, \{u_n\}$ in A_0 such that

$$Sx_n = Tx_{n+1}, \quad \forall n \in \mathbb{N} \tag{23}$$

and

$$\mathfrak{D}(u_n, Sx_n) = \mathfrak{D}(u_{n+1}, Sx_{n+1}) = \mathfrak{D}(u_n, Tx_{n+1}) = \mathfrak{D}(u_{n-1}, Tx_n) = \mathfrak{D}(A, B), \quad \forall n \in \mathbb{N}. \tag{24}$$

Since the pair (T, S) is a Berinde-type weak proximal contraction, from Equation (24), we have

$$\begin{aligned}
\mathfrak{D}(u_n, u_{n+1}) &\leq \lambda^* \max\{\mathfrak{D}(u_{n-1}, u_n), \mathfrak{D}(u_n, u_{n+1})\} \\
&\quad + L \min\{\mathfrak{D}(u_{n+1}, Sx_{n+1}) - \mathfrak{D}(A, B), \mathfrak{D}(u_{n-1}, u_n)\} \\
&= \lambda^* \max\{\mathfrak{D}(u_{n-1}, u_n), \mathfrak{D}(u_n, u_{n+1})\},
\end{aligned}$$

which can be written as

$$\mathfrak{D}(u_n, u_{n+1}) \leq \lambda^* \max\{\mathfrak{D}(u_{n-1}, u_n), \mathfrak{D}(u_n, u_{n+1})\}. \tag{25}$$

If $\max\{\mathfrak{D}(u_{n-1}, u_n), \mathfrak{D}(u_n, u_{n+1})\} = \mathfrak{D}(u_n, u_{n+1})$ for some $n \in \mathbb{N}$, then from (25), we have

$$\mathfrak{D}(u_n, u_{n+1}) \leq \lambda^*\mathfrak{D}(u_n, u_{n+1}) < \mathfrak{D}(u_n, u_{n+1}),$$

which is a contradiction. Therefore, we have

$$\mathfrak{D}(u_n, u_{n+1}) \leq \lambda^*\mathfrak{D}(u_{n-1}, u_n), \quad \forall n \in \mathbb{N}. \tag{26}$$

Next, we will prove that $\{u_n\}$ is an $\mathcal{F}-$Cauchy sequence.

Reasoning as in the proof of Theorem 1, we can assume, without a loss in generality, that $\mathfrak{D}(u_0, u_1) > 0$.

It follows from (26) that for all $n \in \mathbb{N}$,

$$\begin{aligned} \mathfrak{D}(u_n, u_{n+1}) &\leq \lambda^* \mathfrak{D}(u_{n-1}, u_n) \\ &\leq \lambda^*(\lambda^* \mathfrak{D}(u_{n-2}, u_{n-1})) \\ &= \lambda^{*2} \mathfrak{D}(u_{n-2}, u_{n-1}) \\ &\vdots \\ &\leq \lambda^{*n} \mathfrak{D}(u_0, u_1). \end{aligned}$$

It follows that

$$\sum_{i=n}^{m-1} \mathfrak{D}(u_i, u_{i+1}) \leq \frac{\lambda^{*n}}{1-\lambda^*} \mathfrak{D}(u_0, u_1), \quad \forall m \geq n. \tag{27}$$

This further implies that

$$\lim_{n \to +\infty} \frac{\lambda^{*n}}{1-\lambda^*} \mathfrak{D}(u_0, u_1) = 0. \tag{28}$$

Given $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that $(\mathfrak{D}_3)$ holds. For any given $\epsilon > 0$, by $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha. \tag{29}$$

It follows from $(\mathcal{F}_1)$ and (27)–(29) that there exists $N \in \mathbb{N}$ such that

$$f\left(\sum_{i=n}^{m-1} \mathfrak{D}(u_i, u_{i+1})\right) \leq f\left(\frac{\lambda^{*n}}{1-\lambda^*} \mathfrak{D}(u_0, u_1)\right) < f(\epsilon) - \alpha, \quad \forall m > n \geq N.$$

This implies

$$f\left(\sum_{i=n}^{m-1} \mathfrak{D}(u_i, u_{i+1})\right) \leq f(\epsilon) - \alpha, \quad \forall m > n \geq n_0. \tag{30}$$

Using $(\mathfrak{D}_3)$ and (30), we obtain

$$\mathfrak{D}(u_n, u_m) > 0, m > n > N \Rightarrow f(\mathfrak{D}(u_n, u_m)) \leq f\left(\sum_{i=n}^{m-1} \mathfrak{D}(u_i, u_{i+1})\right) + \alpha \leq f(\epsilon),$$

which implies by $(\mathcal{F}_1)$

$$\mathfrak{D}(u_n, u_m) < \epsilon, \quad \forall m > n \geq N.$$

Therefore, sequence $\{u_n\}$ is an $\mathcal{F}-$Cauchy sequence.

Since $(X, \mathfrak{D})$ is a complete $\mathcal{F}-$metric space and A_0 is closed, there exists $u \in A_0$ such that $\lim_{n \to +\infty} u_n = u$. Because of the fact that the mappings T and S proximally commute and from (23) and (24), we have

$$Su_{n-1} = Tu_n, \quad \forall n \in \mathbb{N}.$$

Therefore, the continuity of the mappings T and S ensures that

$$Tu = \lim_{n \to +\infty} Tu_n = \lim_{n \to +\infty} Su_{n-1} = Su.$$

Since $S(A_0) \subseteq B_0$, there exists an $x \in A$ such that

$$\mathfrak{D}(x, Su) = \mathfrak{D}(A, B) = \mathfrak{D}(x, Tu). \tag{31}$$

As T and S proximally commute, $Tx = Sx$. Further, $S(A_0) \subseteq B_0$, there exists $z \in A$ such that

$$\mathfrak{D}(z, Sx) = \mathfrak{D}(A, B) = \mathfrak{D}(z, Tx). \tag{32}$$

If we suppose that x is distinct from z, it follows from (31), (32) and the $P-$property that

$$\begin{aligned}\mathfrak{D}(x,z) &\leq \lambda^* \max\{\mathfrak{D}(x,x), \mathfrak{D}(x,z)\} + L \min\{\mathfrak{D}(z,Sx) - \mathfrak{D}(A,B), \mathfrak{D}(x,x)\} \\ &\leq \lambda^* \mathfrak{D}(x,z) \\ &< \mathfrak{D}(x,z),\end{aligned}$$

which is a contradiction; hence $x = z$. It follows from (32) that

$$\mathfrak{D}(x, Sx) = \mathfrak{D}(A, B) = \mathfrak{D}(x, Tx). \tag{33}$$

Therefore, x is a common best proximity point of the pair (T, S).

Suppose $y \in X$ is a distinct common best proximity point of the mappings T and S. We have

$$\mathfrak{D}(y, Sy) = \mathfrak{D}(A, B) = \mathfrak{D}(y, Ty). \tag{34}$$

Since the pair (T, S) is a Berinde-type weak proximal contraction, from (33) and (34), we have

$$\mathfrak{D}(x,y) \leq \lambda^* \max\{\mathfrak{D}(x,x), \mathfrak{D}(x,y)\} + L \min\{\mathfrak{D}(y,Sy) - \mathfrak{D}(A,B), \mathfrak{D}(x,x)\}.$$

Therefore, we deduce that

$$\mathfrak{D}(x,y) \leq \lambda^* \mathfrak{D}(x,y) < \mathfrak{D}(x,y),$$

which is a contradiction. Therefore, x is the unique common best proximity point of the pair of mappings (T, S). □

Example 5. *Let $X = \mathbb{R}$ be endowed with metric $\mathfrak{D} : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ defined as $\mathfrak{D}(x,y) = |x - y|$, for all $x, y \in X$. We can see that $(X, \mathfrak{D})$ is an $\mathcal{F}-$metric space with $f(t) = \ln t \in \mathcal{F}$ and $\alpha = 0$. Suppose that $A = [-2, 0]$ and $B = [1, +\infty)$. We have that $A_0 = \{0\}, B_0 = \{1\}$ and $\mathfrak{D}(A, B) = 1$. Obviously, A_0 is closed and nonempty, and A, B satisfy the $P-$property.*

Define $T, S : A \to B$ by $T(x) = x^2 + 1$, $S(x) = -2x + 1$. After a simple computation, we can see that T, S are continuous and $S(A_0) \subseteq T(A_0)$ and $S(A_0) \subseteq B_0$. Choosing any $x, y, z \in A$ such that $\mathfrak{D}(x, Tz) = \mathfrak{D}(y, Sz) = \mathfrak{D}(A, B) = 1$, we have $x = z = y = 0$. Thus, we have $Ty = Sx$, which verifies that T, S commute proximally.

Moreover, let $\xi \in (0, 1)$ and $L \geq 0$, and assume that for some $u_1, u_2, v_1, v_2, x_1, x_2 \in A$ the following equations hold

$$\mathfrak{D}(u_1, Sx_1) = \mathfrak{D}(v_1, Tx_1) = \mathfrak{D}(u_2, Sx_2) = \mathfrak{D}(v_2, Tx_2) = \mathfrak{D}(A, B) = 1.$$

Then we have $x_1 = x_2 = u_1 = u_2 = v_1 = v_2 = 0$. Thus, (T, S) is a Berinde-type weak proximal contraction. Hence, all the conditions of Theorem 3 are satisfied. Therefore, (T, S) has a unique common best proximity point, which is 0.

In our next result, we proved the existence and uniqueness of the best proximity point for a modified Berinde-type weak proximal contractive mapping T in a complete $\mathcal{F}-$metric space $(X, \mathfrak{D})$.

Corollary 1. *Let (A, B) be a pair of nonempty subsets of a complete $\mathcal{F}-$metric space $(X, \mathfrak{D})$ that satisfy the $P-$property. Assume that A_0 is a nonempty and closed subset of A, and $T : A \to B$ is a continuous mapping with $T(A_0) \subseteq B_0$ as well as satisfies that there exist $\lambda^* \in (0, 1)$ and $L \geq 0$ such that for all $v_1, v_2, x_1, x_2 \in A$,*

$$\begin{aligned}&\mathfrak{D}(v_1, Tx_1) = \mathfrak{D}(A, B) = \mathfrak{D}(v_2, Tx_2)\\ &\Rightarrow \mathfrak{D}(v_1, v_2) \leq \lambda^* \max\{\mathfrak{D}(v_1, x_1), \mathfrak{D}(v_2, x_2)\}\\ &\quad + L\min\{\mathfrak{D}(v_1, x_1), \mathfrak{D}(v_1, x_2), \mathfrak{D}(v_2, x_1), \mathfrak{D}(v_1, x_2)\}.\end{aligned} \tag{35}$$

Then mapping T has a unique best proximity point.

Proof. Let x_0 be a fixed element in A_0. Following the process stated in the proof of Theorem 1, we can find a sequence $\{x_n\}$ in A_0 such that

$$\mathfrak{D}(x_{n+1}, Tx_n) = \mathfrak{D}(A, B) = \mathfrak{D}(x_n, Tx_{n-1}), \quad \forall n \in \mathbb{N}. \tag{36}$$

Further, it follows from inequality (36) that

$$\begin{aligned}\mathfrak{D}(x_n, x_{n+1}) &\leq \lambda^* \max\{\mathfrak{D}(x_n, x_{n-1}), \mathfrak{D}(x_{n+1}, x_n)\}\\ &\quad + L\min\{\mathfrak{D}(x_n, x_{n-1}), \mathfrak{D}(x_n, x_n), \mathfrak{D}(x_{n+1}, x_{n-1}), \mathfrak{D}(x_n, x_n)\}\\ &\leq \lambda^* \max\{\mathfrak{D}(x_n, x_{n-1}), \mathfrak{D}(x_{n+1}, x_n)\}.\end{aligned}$$

If $\mathfrak{D}(x_{n+1}, x_n) > \mathfrak{D}(x_n, x_{n-1})$, then the above inequality can be written as

$$\mathfrak{D}(x_n, x_{n+1}) \leq \lambda^* \mathfrak{D}(x_{n+1}, x_n) < \mathfrak{D}(x_{n+1}, x_n),$$

which is a contradiction. Therefore, we have $\mathfrak{D}(x_{n+1}, x_n) < \mathfrak{D}(x_n, x_{n-1})$.

In addition, we also have

$$\begin{aligned}\mathfrak{D}(x_n, x_{n+1}) &\leq \lambda^* \mathfrak{D}(x_{n-1}, x_n)\\ &\leq \lambda^{*2} \mathfrak{D}(x_{n-2}, x_{n-1})\\ &\ \ \vdots\\ &\leq \lambda^{*n} \mathfrak{D}(x_0, x_1),\end{aligned}$$

which further implies that $\lim_{n\to\infty} \mathfrak{D}(x_n, x_{n+1}) = 0$. The rest of the proof runs as the one in Theorem 3. □

Next, we present the best proximity result for a generalized $\alpha - \varphi-$proximal contraction in an $\mathcal{F}-$complete metric space $(X, \mathfrak{D})$.

Theorem 4. *Assume that $(X, \mathfrak{D})$ is an $\mathcal{F}-$metric space and (A, B) be a pair of nonempty closed subsets of X. Suppose that the following conditions are satisfied:*

(1) $(X, \mathfrak{D})$ is $\mathcal{F}-$complete;
(2) A_0 and B_0 are nonempty subsets of A and B satisfying the $P-$property;
(3) a non-self mapping $T : A \to B$ is $\alpha-$proximal admissible and also is a generalized $\alpha - \varphi-$proximal contraction satisfying $T(A_0) \subseteq B_0$;
(4) there exist elements $x_0, x_1 \in A_0$ such that $\mathfrak{D}(x_1, Tx_0) = \mathfrak{D}(A, B)$ and $\alpha(x_0, x_1) \geq 1$;
(5) $(X, \mathfrak{D})$ has the $R-$property with respect to the pair (A, B).

Then T has the best proximity point $x^* \in A_0$ such that $\mathfrak{D}(x^*, Tx^*) = \mathfrak{D}(A, B)$.

Proof. Let $x_0, x_1 \in A_0$ such that $\mathfrak{D}(x_1, Tx_0) = \mathfrak{D}(A, B)$ and $\alpha(x_0, x_1) \geq 1$. As $Tx_0 \in T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $\mathfrak{D}(x_2, Tx_1) = \mathfrak{D}(A, B)$ and $\alpha(x_1, x_2) \geq 1$.

Inductively, we can construct a sequence $\{x_n\} \subset A_0$ such that

$$\mathfrak{D}(x_{n+1}, Tx_n) = \mathfrak{D}(A, B), \quad \alpha(x_n, x_{n+1}) \geq 1, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{37}$$

We claim that $\{x_n\}$ is an $\mathcal{F}-$Cauchy sequence. Using the $P-$property, we deduce from (37) that

$$\left.\begin{array}{l} \mathfrak{D}(x_n, Tx_{n-1}) = \mathfrak{D}(A, B) \\ \mathfrak{D}(x_{n+1}, Tx_n) = \mathfrak{D}(A, B) \end{array}\right\} \Rightarrow \mathfrak{D}(x_n, x_{n+1}) = \mathfrak{D}(Tx_{n-1}, Tx_n), \quad \forall n \in \mathbb{N}. \tag{38}$$

Since T is generalized $\alpha - \varphi-$proximally contractive, there exists a function $\varphi \in \Lambda$ such that

$$\mathfrak{D}(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n)\mathfrak{D}(Tx_{n-1}, Tx_n) \leq \varphi(\mathfrak{D}(x_{n-1}, x_n)),$$

which implies

$$\mathfrak{D}(x_n, x_{n+1}) = \mathfrak{D}(Tx_{n-1}, Tx_n) \leq \varphi(\mathfrak{D}(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}. \tag{39}$$

Since the mapping φ is increasing, inequality (39) becomes

$$\begin{aligned} \mathfrak{D}(x_n, x_{n+1})) &\leq \varphi^2(\mathfrak{D}(x_{n-2}, x_{n-1})) \\ &\leq \varphi^3(\mathfrak{D}(x_{n-3}, x_{n-2})) \\ &\vdots \\ &\leq \varphi^n(\mathfrak{D}(x_0, x_1)), \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

The second property of φ yields that

$$\begin{aligned} &\sum_{i=n}^{m-1} \mathfrak{D}(u_i, u_{i+1}) \\ &= \mathfrak{D}(x_n, x_{n+1}) + \mathfrak{D}(x_{n+1}, x_{n+2}) + \mathfrak{D}(x_{n+2}, x_{n+3}) + \cdots + \mathfrak{D}(x_{m-1}, x_m) \\ &\leq \varphi^n(\mathfrak{D}(x_0, x_1)) + \varphi^{n+1}(\mathfrak{D}(x_0, x_1)) + \varphi^{n+2}(\mathfrak{D}(x_0, x_1)) + \cdots + \varphi^{m-1}(\mathfrak{D}(x_0, x_1)) \\ &= \varphi^n(\mathfrak{D}(x_0, x_1))(1 + \varphi(\mathfrak{D}(x_0, x_1)) + \varphi^2(\mathfrak{D}(x_0, x_1)) + \cdots + \varphi^{m-n-1}(\mathfrak{D}(x_0, x_1)) \\ &< \varphi^n(\mathfrak{D}(x_0, x_1))\left(1 + \frac{1}{1 - \varphi(\mathfrak{D}(x_0, x_1))}\right). \end{aligned}$$

We conclude from taking the limit as $n \to +\infty$ in the above inequality that

$$\lim_{n \to +\infty} \varphi^n(\mathfrak{D}(x_0, x_1))(1 + \frac{1}{1 - \varphi(\mathfrak{D}(x_0, x_1))}) = 0. \tag{40}$$

Let $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ satisfy $(\mathfrak{D}_3)$ and $\varepsilon > 0$ be fixed. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha. \tag{41}$$

For the $\delta > 0$ in (41), by (40), there exists $N \in \mathbb{N}$ such that

$$0 < \varphi^n(\mathfrak{D}(x_0, x_1))(1 + \frac{1}{1 - \varphi(\mathfrak{D}(x_0, x_1))}) < \delta, \quad \forall m > n \geq N.$$

Hence, by (41) and $(\mathcal{F}_2)$, we have

$$f\left(\sum_{i=n}^{m-1}\mathfrak{D}(x_i,x_{i+1})\right) \leq f\left(\varphi^n(\mathfrak{D}(x_0,x_1))(1+\frac{1}{1-\varphi(\mathfrak{D}(x_0,x_1))}\right) \\ < f(\epsilon)-\alpha, \quad \forall m > n \geq N. \tag{42}$$

Using $(\mathfrak{D}_3)$ and (42), we have

$$\mathfrak{D}(x_n,x_m) > 0 \Rightarrow f(\mathfrak{D}(x_n,x_m)) \leq f\left(\sum_{i=n}^{m-1}\mathfrak{D}(u_i,u_{i+1})\right)+\alpha < f(\epsilon), \quad m > n \geq N.$$

This implies by $(\mathcal{F}_1)$ that

$$\mathfrak{D}(x_n,x_m) < \epsilon, \quad m > n \geq N.$$

Hence, the sequence $\{x_n\}$ is an $\mathcal{F}-$Cauchy sequence. Since $(X,\mathfrak{D})$ is a complete $\mathcal{F}-$metric space and A_0 is a closed subset of $(X,\mathfrak{D})$, there exists $x_* \in A_0$ such that x_n converges to x_*.

By $R-$Property, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x_*) \geq 1$ for all $k \in \mathbb{N}$. Since T is a generalized $\alpha-\varphi-$proximal contraction, we have

$$\mathfrak{D}(Tx_{n_k},Tx^*) \leq \alpha(x_{n_k},x^*)\mathfrak{D}(Tx_{n_k},Tx^*) \\ \leq \varphi(\mathfrak{D}(x_{n_k},x^*)), \quad \forall k \in \mathbb{N}. \tag{43}$$

We will prove that x^* is the best proximity point of T. Suppose that $\mathfrak{D}(x^*,Tx^*) > \mathfrak{D}(A,B)$. By $(\mathfrak{D}_3)$, we have

$$f(\mathfrak{D}(x_*,Tx_*)-\mathfrak{D}(A,B)) \leq f(\mathfrak{D}(x_*,Tx_{n_k})+\mathfrak{D}(Tx_{n_k},Tx_*)-\mathfrak{D}(A,B))+\alpha, \quad \forall k \in \mathbb{N}.$$

Together with (43) and $(\mathcal{F}_1)$, we have

$$f(\mathfrak{D}(x^*,Tx^*)) \leq f(\mathfrak{D}(x^*,Tx_{n_k})+\varphi(\mathfrak{D}(x_{n_k},x^*)))+\alpha.$$

Taking the limit as $k \to +\infty$ in the above inequality, it follows from $(\mathcal{F}_2)$ that

$$f(\mathfrak{D}(x^*,Tx^*)-\mathfrak{D}(A,B)) \leq \lim_{k\to+\infty} f(\mathfrak{D}(x^*,Tx_{n_k})+\varphi(\mathfrak{D}(x_{n_k},x^*)))+\alpha \\ = -\infty,$$

which implies that $\mathfrak{D}(x^*,Tx^*)-\mathfrak{D}(A,B)=0$. Hence x^* is the best proximity point of T. □

Remark 1. *The conclusions of Theorems 2, 3, and 4 and Corollary 1 still hold if we replace the $P-$property assumption imposed on (A,B) by the weak $P-$property.*

3. Coincidence Point Results in $\mathcal{F}-$Metric Spaces

In this section, we will discuss some coincidence point results for proximal contractions endowed with an $\mathcal{F}-$metric.

From Theorems 1–3, we can obtain the following coincidence/fixed point results by using our previous proximal contraction results.

Theorem 5. *Let $(X,\mathfrak{D})$ be a complete $\mathcal{F}-$metric space. Suppose that $T,S: X \to X$ are continuous and commuting; also the pair (T,S) is a $\lambda-\psi-$dominated contraction. Then (T,S) has a unique coincidence point.*

Proof. If we take $A = B = X$ in Theorem 1, then $\mathfrak{D}(A, B) = 0$. In addition to this, every proximally $\lambda - \psi-$dominated contraction becomes a $\lambda - \psi-$dominated contraction. Analysis similar to that in the proof of Theorem 1 shows that there exists $x \in X$ such that

$$\mathfrak{D}(x, Sx) = \mathfrak{D}(x, Tx) = \mathfrak{D}(A, B) = 0,$$

which implies that $Sx = Tx$. Hence, x is a coincidence point of the pair (T, S). Moreover, the uniqueness of the coincidence point can be deduced from the same arguments presented in the proof of Theorem 1. □

Theorem 6. *Let $(X, \mathfrak{D})$ be a complete $\mathcal{F}-$metric space. Suppose that $T, S : X \to X$ are continuous and commuting; also the pair (T, S) is a generalized $\eta_\beta^\gamma-$contraction. Then (T, S) has a unique coincidence point.*

Proof. The conclusion can be drawn by applying the same argument in the proof of Theorem 5 by replacing the $\lambda - \psi-$dominate proximal contraction with the generalized $\eta_\beta^\gamma-$contraction. □

Theorem 7. *Let $(X, \mathfrak{D})$ be a complete $\mathcal{F}-$metric space. Suppose that $T, S : X \to X$ are continuous and commuting; also the pair (T, S) is a Berinde-type weak contraction. Then (T, S) has a unique coincidence point.*

Proof. The conclusion can be drawn by applying the same argument in the proof of Theorem 5 by replacing the $\lambda - \psi-$dominate proximal contraction with the Berinde-type weak contraction. □

Definition 19. *A mapping $T : X \to X$ is said to be a generalized $\alpha - \varphi-$contraction, if*

$$\alpha(x, y)\mathfrak{D}(Tx, Ty) \leq \varphi(\mathfrak{D}(x, y)), \quad \forall x, y \in A,$$

where $\alpha : A \times A \to [0, +\infty)$, $\varphi \in \Upsilon$.

Taking $A = B = X$ in Theorem 4, we get the following fixed point result.

Corollary 2. *Let A be a nonempty $\mathcal{F}-$closed subset of a complete $\mathcal{F}$-metric space $(X, \mathfrak{D})$. Suppose that an $\alpha-$admissible mapping $T : X \to X$ is a generalized $\alpha - \varphi-$contraction and there exist elements $x_0, x_1 \in X$ such that $\alpha(x_0, x_1) \geq 1$. Further, $(X, \mathfrak{D})$ satisfies property $(\mathcal{A})$. Then T has a unique fixed point $x^* \in A$.*

4. Completeness of $\mathcal{F}-$Metric Spaces via the Best Proximity Points

In Mathematics, the "Completeness Problem "is an essential issue that concerns when a space is complete. In such a case, a Cauchy sequence converges. The famous Banach Contraction Principle holds in complete metric spaces, but completeness is not a necessary condition; that is, there are incomplete metric spaces on which every contraction has a fixed point (see [28]). The Banach contraction principle does not characterize metric completeness. As every metric is $\mathcal{F}-$metric, the Banach contraction principle does not characterize the completeness of $\mathcal{F}-$metric spaces. The study of the characterization of the completeness of a metric space can be traced to Subrahmanyam [29] in 1975, who proved that Kannan's contraction characterizes the metric completeness; that is, a metric space (X, d) is complete if and only if every Kannan's contraction on X has a fixed point. For more on the Completeness Problem in various contexts, we refer the readers to [30,31] and the references therein. The "Completeness Problem" is equivalent to another problem in Behavioral Sciences known as the "End Problem" (see [32]). Completeness characterizations have further been studied in connection with best proximity points [33,34].

In this section, we obtain a completeness characterization of an $\mathcal{F}-$metric space via the best proximity points.

Definition 20. *Let $(X, \mathfrak{D})$ be an $\mathcal{F}-$metric space and (A, B) be a pair of nonempty closed subsets of X. A generalized $\alpha - \varphi-$proximal contraction $T : A \to B$ is an $\alpha - \varphi - SVV$ proximal contraction if*

(1) *A_0 is nonempty;*
(2) *A and B satisfy the $P-$property;*
(3) *$T(A_0) \subseteq B_0$;*
(4) *there exist elements $x_0, x_1 \in A_0$ such that $\mathfrak{D}(x_1, Tx_0) = \mathfrak{D}(A, B)$ and $\alpha(x_0, x_1) \geq 1$.*
(5) *T is $\alpha-$proximal admissible;*
(6) *$(X, \mathfrak{D})$ satisfies the $R-$property with respect to the pair (A, B).*

If, in the above definition, we set $A = B = X$, then we obtain the following.

Definition 21 ([26])**.** *Let $(X, \mathfrak{D})$ be an $\mathcal{F}-$metric space. A generalized $\alpha - \varphi-$contraction $T : X \to X$ is an $\alpha - \varphi - SVV$ contraction if*

(1) *there exist elements $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(2) *T is $\alpha-$admissible;*
(3) *$(X, \mathfrak{D})$ satisfies property $(\mathcal{A})$.*

Romaguera and Tirado [26] obtained the following characterization of completeness of metric spaces via $\alpha - \varphi-$contractions $T : X \to X$ as follows.

Theorem 8 ([26])**.** *A metric space is complete if and only if every $\alpha - \varphi - SVV$ contraction has a fixed point.*

Now we get the following characterization of $\mathcal{F}-$completeness of an $\mathcal{F}-$metric space $(X, \mathfrak{D})$.

Theorem 9. *Let $(X, \mathfrak{D})$ be an $\mathcal{F}-$metric space and (A, B) a pair of nonempty closed subsets of X. Then the following statements are equivalent:*

(i) *$(X, \mathfrak{D})$ is $\mathcal{F}-$complete.*
(ii) *Every $\alpha - \varphi - SVV$ proximal contraction $T : A \to B$ has the best proximity point in $(X, \mathfrak{D})$.*
(iii) *Every $\alpha - \varphi - SVV$ contraction $T : X \to X$ has a fixed point in $(X, \mathfrak{D})$.*

Proof. $(i) \implies (ii)$: It follows directly from Theorem 4.
$(ii) \implies (iii)$: This follows by setting $A = B = X$ in (ii).
$(iii) \implies (i)$: Suppose, on the contrary, that $(X, \mathfrak{D})$ is not $\mathcal{F}$-complete, that is, there is an $\mathcal{F}-$Cauchy sequence $\{w_n\}$ (of distinct points) in $(X, \mathfrak{D})$ that does not converge.

Set $B = \{w_n : n \in \mathbb{N}\}$. As $\mathfrak{D}(w_1, B \setminus \{w_1\}) > 0$, there exists $h_1 \in \mathbb{N}$ with $h_1 > 1$ such that

$$\mathfrak{D}(w_j, w_k) < \frac{1}{2}\mathfrak{D}(w_1, B \setminus \{w_1\})$$

for all $k \geq j \geq h_1$. Similarly, there exists $h_2 \in \mathbb{N}$ with $h_2 > \max\{2, h_1\}$ such that

$$\mathfrak{D}(w_j, w_k) < \frac{1}{2}\mathfrak{D}(w_2, B \setminus \{w_2\})$$

for all $k \geq j \geq h_2$. Repeating this argument, we get a subsequence $\{h_n\}$ of $\mathbb{N}$ such that $h_n > \max\{n, h_{n-1}\}$ and

$$\mathfrak{D}(w_j, w_k) < \frac{1}{2}\mathfrak{D}(w_n, B \setminus \{w_n\})$$

for all $k \geq j \geq h_n$. Define the mappings $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ as

$$Tw = \begin{cases} w_{h_n}, \text{ if } w = w_n \text{ for } n \in \mathbb{N} \\ w_1, \text{ if } w \in X \setminus B \end{cases}$$

and

$$\alpha(w,z) = \begin{cases} 1, \text{if } \quad w = w_n \text{ and } z = w_m \text{ for } m, n \in \mathbb{N} \text{ with } n < m, \\ 0, \text{otherwise.} \end{cases}$$

Note that $\alpha(w_1, Tw_1) = 1$ as $h_1 > 1$. If $\alpha(w,z) \geq 1$ for $w = w_n$, and $z = w_m$, then $\alpha(Tw, Tz) = \alpha(w_{h_n}, w_{h_m}) = 1$, as $h_m > h_n$. Hence T is $\alpha-$admissible. Further, $(X, \mathfrak{D})$ satisfies property $(\mathcal{A})$ because any $\mathcal{F}-$convergent sequence $\{z_n\}$ satisfying $\alpha(z_n, z_{n+1}) \geq 1$ is a constant sequence. Now for $\varphi(t) = \frac{t}{2}$ and from the construction of α, it is sufficient to check the $\alpha - \varphi-$ contraction condition for $w = w_n$ and $z = w_m$ with $n < m$. Thus

$$\begin{aligned} \alpha(w,z)\mathfrak{D}(Tw,Tz) &= \alpha(w_n,w_m)\mathfrak{D}(w_{h_n},w_{h_m}) \\ &< \frac{1}{2}\mathfrak{D}(w_n, B \setminus \{w_n\}) \\ &< \frac{1}{2}\mathfrak{D}(w_n,w_m) = \frac{1}{2}\mathfrak{D}(w,z). \end{aligned}$$

Hence T is an $\alpha - \varphi - SVV$ contraction, which does not have a fixed point. This is a contradiction. Hence $(X, \mathfrak{D})$ is $\mathcal{F}-$complete. This completes the proof. □

5. Conclusions and Future Work

This article proves the existence of common best proximity points for proximally $\lambda - \psi-$dominated contractions, generalized $\eta_\beta^\gamma-$contractions and Berinde-type weak contractions in the setting of $\mathcal{F}-$complete metric spaces. As an application, fixed point and coincidence point results for such generalized proximal contraction are obtained. Moreover, a completeness characterization of $\mathcal{F}-$metric spaces is obtained via the existence of the best proximity points of a certain proximal contraction. One can consider the results in this paper for further study in the setup of more general spaces such as metric-like spaces, quasi-metric spaces, fuzzy metric spaces and so on. Besides this, one can strive to obtain weaker conditions to ensure the existence of the best proximity points for some generalized proximal contractions in several classical metric spaces; for instance, the existence of a proximity point without $P-$property in a fuzzy metric space would be worth investigating. Moreover, Ghasab et al. [35] introduced $\mathcal{F}-$quasi-metric spaces, which is viewed as a generalization of $\mathcal{F}-$metric spaces. One could extend our main results to the $\mathcal{F}-$quasi-metric spaces for furnishing the best proximity theory.

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