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Uncertain cost consensus modeling regarding individual behavior constraints and its application

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Abstract

Group decision-making (GDM) mainly solves unstructured decision problems, involving subjective participation of various experts. In general, when solving GDM problems, decision-makers (DMs) eventually form a clear support or objection (i.e., consensus) via multiple rounds of negotiations with consensus cost. However, factors affecting the consensus reaching process (CRP) normally include the DMs' preference structures, decision environment, the influence of particular decision roles and etc., making the GDM full of uncertainty and unable to accurately predict the outcome in advance. Thus, a moderator on behalf of the collective interest is often introduced to increase the speed and efficiency of the GDM. Inspired by the minimum cost consensus (MCC) in the literature, this thesis aims to construct a series of new consensus optimization models to address real-life GDM problems from two perspectives of either minimizing the moderator's total cost or maximizing the individual DM's total revenue. In building these new models, we also incorporate diverse behavioral constraints, such as non-cooperation, trade-off of interest and equity or unbalanced adjustment willing. Specifically, we conduct threefold discussions as follows:

(1) Introduce uncertainty theory into the optimal consensus modeling to address unreliable results yielded when the reliability of decisions is determined only by experts due to the absence of sufficient historical data. To do that, we use uncertainty distribution and belief degree as a whole to fit individual preferences, and further discuss five scenarios of uncertain chance-constrained MCC models (MCCMs) from the angles of the moderator, individual DMs and non-cooperators. Besides, we provide consensus reaching conditions and the analytic formulae of the minimum total cost through deductions. Finally, the new models are verified as an extension of the traditional crisp number or interval preference-based MCCMs with the application of carbon quota negotiation.

(2) Extend uncertain MCCMs into the CRP framework by incorporating DM's unbalanced willing of modifying preference and designing a feedback mechanism on both preferences and weights due to democratic consensus. To do that, we build two new consensus optimization models based on the uncertain distance measure: one is to obtain a MCC on account of asymmetric costs, aggregation function and consensus measure; while the other provides a more flexible way to solve GDM problems without presetting a consensus level threshold. Moreover, binary variables are used to reduce the calculation complexity resulted from piecewise functions in the new multi-coefficient goal programming models and the feasibility of the new proposal is verified by illustrative examples.

(3) Inspired by the maximum compensation consensus models transformed from the MCCMs, we build several new consensus optimization models to obtain flexible (e.g., optimal or fair) carbon quota allocation schemes within a closed-loop trading system. To solve these new models, a relaxation method based on the PSO algorithm is proposed. Moreover, since the inability to perform real-life GDM usually stems from conflicts of interest based on the DMs' mutual competition, we further suggest two strategies to address the unfairness. Numerical results show that sufficient interactions among the DMs are of great significance in achieving fairness within a trading system.

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Chapter I

PhD dissertation

1 Introduction

Group decision-making (GDM) mainly solves unstructured decision problems, involving subjective participation of various experts [ZKP19]. In the GDM, through communication and multiple rounds of effective feedback/adjustment, decision-makers (DMs) eventually form a clear support or objection towards a certain issue, meaning that a relatively consistent opinion (also known as consensus) is reached [ERDS07]. In fact, consensus is a prerequisite for the effective GDM and widely exists in our daily lives, such as the online P2P lending [ZKP19] or trans-boundary water pollution control [CZC$^+$18]. Generally speaking, factors affecting the consensus reaching process (CRP) include the DMs' preference structures or psychological expectation, convergence rules, decision environment, and leaders' or non-cooperators' influence. As [HVCKP14] concluded that consensus boils down to cooperation, while the most GDM boils down to competition. Moreover, Urda and Loch [UL13] pointed out that individual behaviors in the GDM are driven by both their own economically rational deliberation and decision biases and social preferences (e.g. status achievement, reciprocal relations, or group identity). Thus, a moderator [GZF$^+$15] on behalf of the collective interest that has prominent skills in leadership and negotiation, is often introduced to persuade or tempt the DMs to gradually adjust their opinions into consensus via different effective means (collectively referred to as "consensus cost"), thereby increasing the speed and efficiency of the CRP.

The concept of the minimum cost consensus (MCC) was originally proposed by Ben-Arieh and Easton [BAE07] for solving the single and multi-criteria GDM problems via linear-time algorithms. Later, a quadratic cost function was adopted to discuss the influence of different factors (e.g., cost, opinion elasticity, the number of adjusted experts) on the consensus [BAEE09]. Meanwhile, the optimization-based consensus models [DXLF10], recognized as the minimum adjustment consensus models (MACMs), aim to maximize the retention of the DM's original preference, rather than pursuing a minimum resource consumption. Up to now, although abundant studies have been conducted, most are based on traditional preference structures (e.g., crisp numbers, intervals or linguistic information), neglecting the characteristics of stochastic distribution in the DM's preference. In contrast, linear uncertainty distributions with belief degrees provide a feasible way to better simulate the DM's uncertainty and ambiguous behaviors in the actual GDM [GGHV$^+$20].

In fact, even if there exists a moderator acting as a leader in the GDM, the DMs involved still cannot account for all factors; besides, diversity widely exists in individuals' research background, knowledge reserve, and the amount of private information. In a nutshell, the GDM is full of uncertainty, making it unable to accurately predict the outcome in advance. That is, the GDM essentially includes providing decision support for solving uncertainty. To date, theoretical methods for dealing with uncertainty include probability theory, interval analysis, fuzzy sets, rough sets and grey systems. However, obtaining a precise probability for a natural state in the real-life GDM is not easy, especially when little information is available for evaluating probabilities (i.e., usable information is insufficient), or when several information sources conflict with each other [AP14]; then, the reliability (or probability) that certain event will occur is primarily determined by experts. To handle such dilemmas where the reliable prediction that one event would occur has to be determined by individual subjectivity due to the inability to obtain its actual frequency, uncertainty theory was proposed [Liu07, Liu10], which has been an important branch of mathematics and mainly deals with human beings' subjective reliability.

Apart from the above consensus modeling with a minimum cost/adjustment, this thesis was also partially inspired by the construction of consensus models that aim to maximize the total revenue. By introducing linear primal-dual theory, various MCCMs with specific preference structures were adopted as the primal models, and then their corresponding dual forms (i.e., the optimization-based maximum compensation consensus models) along with their economic significance were deeply explored in [GZF$^+$15, GXZ$^+$15] and [ZKP19]. Subsequently, on account of the essential architecture of Stackelberg's game, [ZDZP20] presented a bi-level optimization consensus model that depicts the interaction between the DMs and the moderator, and divided the DM's total return into a modification component (also known as external compensation) provided by the moderator for the DM's initial preference adjustment and a recognition component based on the similarity between the DM's original opinion and the final consensus. Hence, to construct consensus models from the perspective of maximizing the revenue is logical and reasonable.

Therefore, this thesis attempts to construct a series of consensus optimization models to address some real-life GDM problems from the following two perspectives:

- From the perspective of minimizing the overall consensus cost: provided that the GDM problems cannot be solved with those existing theories dealing with uncertainty due to the inability to obtain its actual frequency, how to combine Liu's uncertainty theory with the traditional consensus GDM theory, and how to accordingly construct and interpret the new consensus optimization models by incorporating different behavioral constraints? Meanwhile, what's the relationship between the novel MCCMs with those traditional ones?
- From the perspective of maximizing the individual compensation (revenue or return): given that various practical negotiation problems (e.g., demolition or pollution control) that involve no less than two decision roles, gain increasing attention in the field of the GDM, how to build corresponding mathematical models by designing particular market trading mechanisms so as to realize better resource reallocations? Furthermore, how to balance the overall benefits and equity so as to provide managerial insights from a theoretical angle?

Concerning the second perspective, it is well known that the market is profit-oriented (i.e., simultaneously pursuing the maximization of revenue and the minimization of costs) and its operating mechanism is mostly affected by pricing strategy, participants' competition, supply and demand, and etc. Therefore, in discussing the closed-loop trading mechanisms, the revenue maximization of either the whole group or a single DM is set as our objective function in this thesis with related constraints such as supply and demand or prices. Undoubtedly, fairness concerns are also critical for the GDM [DLL22], since participants are motivated by not only the final results, but also the fairness they feel compared with others [Ada63]. Under a fixed total carbon quota, due to the fact that a scientific resource allocation scheme directly involves the economic development rights of different regions, it is bound to be an arduous task and worth investigating.

Bearing all the above points in our mind, this thesis constructs a series of consensus optimization models from the moderator's perspective of minimizing the total cost or the DM's perspective of maximizing their revenues. Specifically, we conduct the following discussions:

(1) In the absence of sufficient historical data, reliability of decisions is mainly determined by experts rather than some prior probability distributions, easily causing unreliable decision results. Thus, we use belief degree and uncertainty distribution as a whole to fit individual preferences, and we further discuss five scenarios of uncertain chance-constrained MCCMs from multiple roles of the moderator, the individual DMs and the non-cooperators. Besides, we provide the reaching conditions of the consensus and the analytic formulae of the minimum total cost via theoretical derivations. Finally, we verify that the new consensus models are essentially an extension of the traditional crisp number or interval preference-based MCCMs through the application of carbon quota negotiation.

(2) Since the CRP can facilitate more effective consensus by considering human behaviors. We extend the uncertain MCCMs proposed in the first item, but consider asymmetric costs into the CRP framework, where the DM's preference and weight are both adjusted according to democratic consensus. Moreover, we build two novel consensus optimization models based on the uncertain distance measure: one is to obtain a MCC by simultaneously considering asymmetric costs, aggregation function and consensus measure; while the other provides a more flexible way to address the GDM problems without pre-setting a specific consensus level (CL) threshold. We further introduce binary variables to reduce the calculation complexity resulted from piecewise functions in the new multi-coefficient goal programming models. Finally, we reveal the feasibility and the superiority of the new methods via illustrative examples.

(3) Using optimization-based consensus modeling to design flexible carbon quota trading mechanisms is novelty, namely, we provide basic allocation schemes within a closed-loop trading system by taking both revenue and fairness into account. We construct a series of consensus optimization models from the perspective of maximizing the overall revenue, and obtain the optimal/fair carbon quota allocation schemes that include detailed trading information as transferred quantities, transaction prices and etc. Furthermore, we propose a relaxation method based on the particle swarm optimization (PSO) algorithm to solve these new models. The inability to conduct the real-life GDM usually stems from conflicts of interest based on the DMs' mutual competition, thus, we suggest two strategies to handle the resulting unfairness within the trading system. Finally, the numerical results show that sufficient interactions among the DMs are of great significance in achieving fairness within a trading system.

Overall, this PhD dissertation consists of two main parts: the former part elaborates the targeted GDM problems that need to be solved and the main results acquired from the proposed consensus optimization models. The latter part is a compilation of the main publications that are associated with this thesis.

The rest is arranged as follows: Section 2 introduces some preliminaries that support the subsequent analysis, including the traditional consensus optimization models, a general CRP framework along with consensus measures, and uncertainty theory. Section 3 provides the basic assumptions and challenges that justify this thesis. Next, the main objectives are presented in Section 4 and the specific methodology used throughout this thesis are described in Section 5. Next, Section 6 elaborates the construction of various consensus optimization models. Additionally, Section 7 discusses the results acquired from this thesis. Finally, the concluding remarks of this thesis are summarized in Section 8, while Section 9 outlines prospects for future research.

Introducción

La toma de decisiones en grupo (GDM, por sus siglas en inglés) resuelve principalmente problemas de decisión no estructurados, que implican la participación subjetiva de varios expertos [ZKP19]. En GDM, a través del debate y de múltiples rondas de retroalimentación, los expertos o responsables de la toma de decisiones (DM, por sus siglas en inglés) acaban formando una opinión clara de apoyo u objeción hacia una determinada cuestión, la cual suele alcanzarse por acuerdo (consenso) [ERDS07]. De hecho, la toma de decisiones por consenso es un requisito previo para la eficacia de la GDM, y es algo que sucede normalmente en nuestra vida cotidiana, como los préstamos P2P en línea [ZKP19] o el control transfronterizo de la contaminación del agua [CZC$^+$18]. En términos generales, los factores que afectan al proceso de consecución de consenso (CRP, por sus siglas en inglés) incluyen las estructuras de preferencias o expectativas psicológicas de los DM, las reglas de convergencia, el entorno de decisión y la influencia de los líderes o de los no cooperadores. En [HVCKP14] se concluyó que el consenso se reduce a la cooperación, mientras que la mayoría de los GDM se reducen a la competencia. Urda y Loch [UL13] señalaron que los comportamientos individuales en GDM están impulsados tanto por sus propios sesgos de deliberación y decisión económicamente racionales como por preferencias sociales (por ejemplo, logro de estatus, relaciones recíprocas o identidad de grupo). Así pues, a menudo se introduce un moderador [GZF$^+$15] en nombre del interés colectivo que posee destacadas aptitudes de liderazgo y negociación, para persuadir o tentar a los DM a que ajusten continuamente sus opiniones hacia el consenso a través de diferentes medios (denominados colectivamente "coste del consenso"), aumentando así la velocidad y la eficacia de la CRP.

El concepto de consenso de coste mínimo (MCC) fue propuesto originalmente por Ben-Arieh y Easton [BAE07] para resolver problemas de GDM de uno o varios criterios mediante algoritmos de tiempo lineal. Más tarde, se adoptó una función de coste cuadrático para discutir la influencia de diferentes factores (por ejemplo, el coste, la elasticidad de la opinión, el número de expertos que deben ajustar sus opiniones) en el consenso [BAEE09]. Al mismo tiempo, los modelos de consenso basados en la optimización [DXLF10], reconocidos como los modelos de consenso de ajuste mínimo (MACMs), tienen como objetivo maximizar la retención de la preferencia original del DM, en lugar de perseguir un consumo mínimo de recursos. Hasta ahora, aunque se han realizado abundantes estudios, la mayoría se basan en estructuras de preferencia tradicionales (por ejemplo, números exactos, intervalos o información ling'uística), descuidando las características de la distribución estocástica en la preferencia del DM. Por el contrario, las distribuciones lineales de incertidumbre con grados de creencia proporcionan una forma factible de simular mejor la incertidumbre y los comportamientos ambiguos de los DM en los problemas reales de GDM [GGHV$^+$20].

Realmente, aunque exista un moderador que actúe como líder en el GDM, los DM implicados no pueden tener en cuenta todos los factores; además, existe una gran diversidad en la formación investigadora de los individuos, sus conocimientos y la cantidad de información privada que poseen. Por tanto, la GDM está llena de incertidumbre, lo que implica que se muy difcil o imposible predecir con exactitud el resultado de antemano. En otras palabras, la GDM consiste esencialmente en proporcionar apoyo a la toma de decisiones para solventar la incertidumbre. Hasta la fecha, los métodos teóricos para tratar la incertidumbre incluyen la teoría de la probabilidad, el análisis de intervalos, los conjuntos difusos, los conjuntos aproximados y los sistemas grises. Sin embargo, obtener una probabilidad precisa para un estado natural en GDM en la vida real no es fácil, especialmente cuando se dispone de poca información para evaluar las probabilidades (es decir, la información disponible es insuficiente), o cuando varias fuentes de información entran en conflicto entre sí [AP14]; entonces, la fiabilidad (o probabilidad) de que ocurra cierto suceso la determinan principalmente los expertos. Para hacer frente a estos dilemas, en los que la predicción fiable de que se producirá un suceso debe determinarse mediante la subjetividad individual debido a la imposibilidad de obtener su frecuencia real, se propuso la teoría de la incertidumbre [Liu07, Liu10], que ha sido una importante rama de las matemáticas y se ocupa principalmente de la fiabilidad subjetiva de los seres humanos.

Aparte de los anteriores modelos de consenso con un coste/ajuste mínimo, esta tesis también se inspiró parcialmente en la construcción de modelos de consenso que pretenden maximizar los ingresos totales. Mediante la introducción de la teoría lineal primal-dual, se adoptaron varios MCCM con estructuras de preferencias específicas como modelos primales y, a continuación, se exploraron en profundidad sus formas duales correspondientes (es decir, los modelos de consenso de compensación máxima basados en la optimización) junto con su importancia económica en [GZF+15, GXZ+15] y [ZKP19]. Posteriormente, teniendo en cuenta la arquitectura esencial del juego de Stackelberg, [ZDZP20] presentó un modelo de consenso de optimización de dos niveles que describe la interacción entre los DM y el moderador, y dividió el rendimiento total del DM en un componente de modificación (también conocido como compensación externa) proporcionado por el moderador para el ajuste de la preferencia inicial del DM y un componente de reconocimiento basado en la similitud entre la opinión original del DM y el consenso final. Por lo tanto, construir modelos de consenso desde la perspectiva de maximizar los ingresos es lógico y razonable.

De esta forma, esta tesis intenta construir una serie de modelos de optimización consensuados para abordar algunos problemas reales de GDM desde las dos perspectivas siguientes:

- Desde la perspectiva de la minimización del coste global del consenso. Siempre que los problemas de GDM no puedan resolverse con las teorías existentes que tratan la incertidumbre debido a la incapacidad de obtener su frecuencia real, ¿cómo combinar la teoría de la incertidumbre de Liu con la teoría tradicional de GDM por consenso? y ¿cómo construir e interpretar en consecuencia los nuevos modelos de optimización por consenso incorporando diferentes restricciones de comportamiento? Por otra parte, ¿cuál es la relación entre los nuevos MCCM con los tradicionales?

- Desde la perspectiva de la maximización de la compensación individual (ingreso o rendimiento). Dado que varios problemas prácticos de negociación (por ejemplo, la demolición o el control de la contaminación) que implican no menos de dos funciones de decisión, atraen cada vez más atención en el campo de la GDM, ¿cómo construir los modelos matemáticos correspondientes mediante el diseño de mecanismos particulares de negociación de mercado con el fin de realizar mejores reasignaciones de recursos? Por otra parte, ¿cómo equilibrar los beneficios globales y la equidad para aportar ideas de gestión desde un ángulo teórico?

En cuanto a la segunda perspectiva, es bien sabido que el mercado está orientado a la obtención de beneficios (es decir, persigue simultáneamente la maximización de los ingresos y la minimización de los costes) y su mecanismo de funcionamiento se ve afectado principalmente por la estrategia de precios, la competencia de los participantes, la oferta y la demanda, etc. Por lo tanto, al analizar los mecanismos de negociación en bucle cerrado, la maximización de los ingresos de todo el grupo o de un único gestor es nuestra función objetivo en esta tesis, con restricciones relacionadas como la oferta y la demanda o los precios. Sin duda, las cuestiones de equidad también son críticas para el GDM [DLL22], ya que los participantes están motivados no solo por los resultados finales, sino también por la equidad que sienten en comparación con los demás [Ada63]. En el marco de una cuota total fija de carbono, debido al hecho de que un esquema científico de asignación de recursos implica directamente los derechos de desarrollo económico de las distintas regiones, está llamado a ser una tarea ardua y digna de investigación.

Teniendo en cuenta todos los puntos anteriores, esta tesis construye los modelos de optimización de consenso desde la perspectiva del moderador de minimizar el coste total o desde la perspectiva del DM de maximizar sus ingresos. Así, llevamos a cabo las siguientes discusiones:

(1) En ausencia de suficientes datos históricos, la fiabilidad de las decisiones viene determinada principalmente por los expertos y no por algunas distribuciones de probabilidad previas, provocando fácilmente resultados de decisión poco fiables. Así pues, utilizamos el grado de creencia y la distribución de incertidumbre como un todo para ajustar las preferencias individuales, y analizamos además cinco escenarios de MCCM inciertos y limitados por el azar desde múltiples roles del moderador, los DMs y los no cooperantes. Además, proporcionamos las condiciones para alcanzar el consenso y las fórmulas analíticas del coste total mínimo mediante una derivación teórica. Por último, verificamos que los nuevos modelos de consenso son esencialmente una extensión de los MCCM tradicionales basados en números exactos o en preferencias de intervalo mediante la aplicación de la negociación de cuotas de carbono.

(2) Dado que los CRP pueden facilitar un consenso más eficaz al tener en cuenta los comportamientos humanos, ampliamos los MCCM inciertos propuestos en el primer punto, pero consideramos los costes asimétricos en el marco de los CRP, donde la preferencia y el peso del DM se ajustan de acuerdo con el consenso democrático. Además, construimos dos nuevos modelos de optimización del consenso basados en la medida de distancia incierta: uno consiste en obtener un MCC considerando simultáneamente los costes asimétricos, la función de agregación y la medida de consenso; mientras que el otro proporciona una forma más flexible de abordar los problemas de GDM sin preestablecer un umbral de nivel de consenso (CL, por sus siglas en inglés) específico. Además, introducimos variables binarias 0-1 para reducir la complejidad de cálculo derivada de las funciones a trozos en los nuevos modelos de programación de objetivos multieficientes. Por último, revelamos la viabilidad y la superioridad del nuevo método mediante ejemplos ilustrativos.

(3) El uso de modelos óptimos consensuados para diseñar mecanismos flexibles de comercio de cuotas de carbono es una novedad, en concreto, proporcionamos esquemas básicos de asignación dentro de un sistema de comercio de bucle cerrado teniendo en cuenta tanto los ingresos como la equidad. Construimos una serie de modelos de optimización consensuados desde la perspectiva de la maximización de los ingresos globales, y obtenemos esquemas de asignación de cuotas de carbono óptimos/justos que incluyen información comercial detallada como cantidades transferidas, precios de transacción, etc. Además, proponemos un método de relajación basado en el algoritmo de optimización de enjambre de partículas para resolver estos nuevos modelos. La imposibilidad de llevar a cabo GDM en la vida real suele deberse a conflictos de intereses basados en la competencia mutua de los DM, por lo que sugerimos dos estrategias para gestionar la injusticia resultante dentro del sistema de comercio. Por último, los resultados numéricos muestran que las interacciones suficientes entre los gestores son de gran importancia para lograr la equidad en un sistema comercial.

En conjunto, esta tesis consta de dos partes principales: en la primera se exponen los problemas de GDM que deben resolverse y los principales resultados obtenidos a partir de los modelos de optimización por consenso propuestos. La segunda parte es una recopilación de las principales publicaciones asociados con esta tesis.

El resto de esta memoria se organiza como sigue: la Sección 2 introduce algunos preliminares que apoyan el análisis posterior, incluyendo los modelos tradicionales de optimización del consenso, un marco general de CRP junto con medidas de consenso, y la teoría de la incertidumbre. La

Sección 3 presenta los supuestos básicos y los retos que justifican esta tesis. A continuación, en la Sección 4 se presentan los objetivos principales y en la Sección 5 se describe la metodología específica utilizada a lo largo de esta tesis. A continuación, la Sección 6 desarrolla la construcción de varios modelos de optimización de consenso. Además, en la Sección 7 se discuten los resultados obtenidos en esta investigación. Por último, en la Sección 8 se resumen las conclusiones obtenidas, mientras que en la Sección 9 se esbozan las perspectivas para futuras investigaciones.

2 Preliminaries

Some relevant knowledge about the subsequent optimal modeling in this PhD dissertation are given in this section. Specifically, Section 2.1 reviews different types of the traditional consensus optimization models, Section 2.2 recalls widely-used methods to measure the consensus level, and a general CRP framework proposed in [HVHC02], meanwhile, Section 2.3 introduces the uncertainty theory, including its basic concepts and calculation principles.

2.1 Traditional consensus optimization models

Suppose m DMs participate in a GDM problem, and a finite set $D = \{d_1, d_2, \cdots, d_m\}$ denotes all individuals. Let $o_i \in \mathcal{R}$ denote the i-th DM's (i.e., d_i's) original preference, $\bar{o}_i$ be d_i's adjusted preference, and o^c be their reached consensus. In addition, $c_i \in \mathcal{R}^+$ denotes the unit cost of adjusting d_i's preference closer to the consensus, while $w_i \in \mathcal{R}^+$ reflects d_i's importance degree (i.e., weight) with $w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$, $i \in M = \{1, 2, \cdots, m\}$.

2.1.1 Traditional MCCMs or MACMs

The DMs are normally willing to change opinions after repetitive negotiation efforts, though it escalates the costs of reaching a consensus. Adopting the p-norm distance measure (i.e., $|| \, ||_p, p \geq 1$), Ben-Arieh and Easton [BAE07] provided a linear-time algorithm to seek the optimal MCC o^{c*} by minimizing the weighted total cost $f_c(o^c) = \sum_{i=1}^{m} w_i c_i ||o^c - o_i||_p$. Later, they discussed such scenarios with/without an ε-consensus (denoted as $|o^c - \bar{o}_i| \leq \varepsilon, \varepsilon > 0$) [BAEE09]. For brevity, Ref. [ZDXL11] develops an optimization model to represent their ideas.

$$\begin{aligned} &\min \ \phi = \sum_{i=1}^{m} c_i |\bar{o}_i - o_i| \\ &\text{s.t.} \ \begin{cases} |o^c - \bar{o}_i| \leq \varepsilon_i, \ i \in M \\ o^c \in \mathcal{R}, \ \bar{o}_i \in \mathcal{R} \end{cases} \end{aligned} \tag{I.1}$$

Solving the Model (I.1) yields the optimal consensus o^{c*} and the DM's optimal adjusted preference $\bar{o}_i{}^*$. The first constraint denotes a tolerance behavior, and if $\varepsilon_i = 0$ ($\forall i \in M$), a hard consensus is achieved (i.e., $\bar{o}_i{}^* = o^{c*}$), which is unrealistic and uneconomical for the most GDM problems [HVCKP14]. However, as Cheng et al. [CYW$^+$22] argued that these two kinds of consensus measures correspond to different applicable scenarios, for instance, the hard consensus type is suitable for merger negotiations between any two companies to obtain an outcome of either a success (full consensus) or a failure (null consensus).

Meanwhile, Dong et al. [DXLF10] utilized the ordered weighted averaging (OWA) operator and a deviation measure to handle the consensus problems under a 2-tuple fuzzy linguistic environment, so as to preserve the DMs' original preferences as much as possible. Similarly, their main ideas can be mathematically described as follows.

$$\begin{aligned} &\min \ \phi = \sum_{i=1}^{m} d(\bar{o}_i, o_i) \\ &\text{s.t.} \ \begin{cases} d(o^c, \bar{o}_i) \leq \varepsilon, \ i \in M \\ o^c = F_w(\bar{o}_1, \bar{o}_2, \cdots, \bar{o}_m) \end{cases} \end{aligned} \tag{I.2}$$

where $d(\cdot)$ represents the rectilinear or Euclidean deviation measure, F_w denotes an aggregation function, and the objective function is to minimize all DMs' adjustments.

Although the Model (I.1) and the Model (I.2) were proposed according to different consensus mechanisms, Zhang et al. [ZDXL11] later verified that these two models could actually be merged into the following unified form.

$$\begin{aligned} & \min \phi = \sum_{i=1}^{m} c_i * d(\bar{o_i}, o_i) \\ & \text{s.t.} \left\{ \begin{array}{l} d(o^c, \bar{o_i}) \leq \varepsilon, \ i \in M \\ o^c = F_w(\bar{o_1}, \bar{o_2}, \cdots, \bar{o_m}) \end{array} \right. \end{aligned} \tag{I.3}$$

It turns out that once F_w takes the OWA operator as $(\frac{1}{2} \cdots 0 \cdots \frac{1}{2})^{\mathrm{T}}$, the Model (I.3) reduces to the Model (I.1); and if the unit costs of adjusting the DMs' opinions satisfy $c_i = c_j, \forall i, j \in M$, then the Model (I.3) equals the Model (I.2). In fact, this section provides the most original theoretical basis for the research conducted in this thesis.

2.1.2 Traditional maximum compensation consensus models

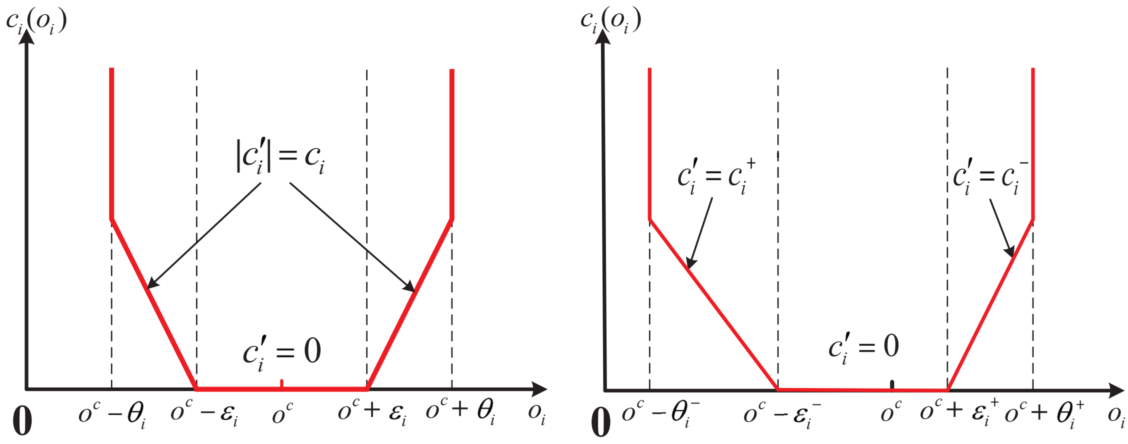
Explicitly, both the MCCMs and the MACMs aim to achieve a minimum cost/adjustment from a holistic perspective. However, from any DM's individual perspective, they are always economically rational by expecting a maximum compensation (or return, gain) due to their compromise via adjusting their preferences. In this regard, Gong et al. [GZF+15, GXZ+15] and Zhang et al. [ZKP19] explored the dual forms of different variants of Ben-Arieh and Easton's original MCC problem [BAE07] based on the linear primal-dual programming theory, so as to acquire the maximum compensation for all DMs. Particularly, a concise form of the maximum compensation consensus models is provided by Zhang et al. [ZKP19], shown as the Model (I.4).

$$\begin{aligned} & \text{Max} \quad \psi = \sum_{i=1}^{m} y_i * ||o^c - o_i||_p \\ & \text{s.t.} \left\{ \begin{array}{l} \sum_{i=1}^{m} y_i = 0 \\ |y_i| \leq w_i, \quad i \in M \end{array} \right. \end{aligned} \tag{I.4}$$

Solving the Model (I.4) yields the optimal consensus o^{c*}, and the optimal unit return y_i^* expected by d_i, and the maximum total returns by all DMs ψ^*, where w_i is the weight assigned to d_i and o_i is his/her original preference. Clearly, an optimal consensus o^{c*} can always be reached in terms of the minimum cost (e.g., the Model (I.3)) or the maximum return (e.g., the Model (I.4)). Moreover, taking game theory into account, Zhang et al. [ZDZP20] later specified the total returns into a modification part due to a DM's preference adjustment and a recognition part based on the similarity between his/her initial preference and the final consensus. To be noted, the theoretical modeling ideas of the third part in this thesis are inspired from this section, which indicates that consensus negotiation problems can also be discussed from the perspective of revenue maximization.

2.1.3 Consensus models with asymmetric costs

Cheng et al. [CZC+18, CYW+22] explored the GDM problems with cost constraints based on the DM's different adjustment directions, further extending the Model (I.3). In specific, Fig. 1 depicts their cost functions under a symmetric or asymmetric scenario by considering the DM's tolerance and limited compromise behaviors, where the horizontal axis is the DM's original preference (i.e., o_i), and the vertical axis represents a unit cost (i.e., c_i).



(a) The symmetric cost scenario (b) The asymmetric cost scenario

Figure 1: Cost functions with tolerance and compromise

For simplicity, only the asymmetric cost scenario is reported hereafter [CZC+18]. That is, once taking the tolerance and the compromise limit into account, d_i's optimal adjusted preference is obtained as

$$\bar{o_i}^* = \begin{cases} o^c - \varepsilon_i^-, & \text{if } o_i \in [o^c - \theta_i^-, o^c - \varepsilon_i^-) \\ o_i, & \text{if } o_i \in [o^c - \varepsilon_i^-, o^c + \varepsilon_i^+] \\ o^c + \varepsilon_i^+, & \text{if } o_i \in (o^c + \varepsilon_i^+, o^c + \theta_i^+] \end{cases} \tag{I.5}$$

The total cost of d_i to adjust his/her preference becomes

$$c_i(o_i) = \begin{cases} c_i^+(o^c - \varepsilon_i^- - o_i), & \text{if } o_i \in [o^c - \theta_i^-, o^c - \varepsilon_i^-) \\ 0, & \text{if } o_i \in [o^c - \varepsilon_i^-, o^c + \varepsilon_i^+] \\ c_i^-(o_i - o^c - \varepsilon_i^+), & \text{if } o_i \in (o^c + \varepsilon_i^+, o^c + \theta_i^+] \end{cases} \tag{I.6}$$

where c_i^- denotes d_i's unit cost with a downward adjustment, and c_i^+ conversely indicates the unit cost of an upward adjustment. In addition, ε_i measures d_i's tolerance of the consensus, while θ_i reflects d_i's compromise limit.

To be more specific, Fig. 1(b) shows that once d_i's original preference is located at $[o^c - \varepsilon_i^-, o^c + \varepsilon_i^+]$, a tolerance behavior exists (also known as the soft consensus [HVCKP14]), namely, any preference within this subinterval is acceptable, thereby requiring no adjustments nor yielding any costs. Moreover, any preference located at $[o^c - \theta_i^-, o^c - \varepsilon_i^-)$ or $(o^c + \varepsilon_i^+, o^c + \theta_i^+]$ corresponds to a compromise limit behavior, inducing a total cost of $c_i^+(o^c - \varepsilon_i^- - o_i)$ or $c_i^-(o_i - o^c - \varepsilon_i^+)$, respectively. In fact, too many preference adjustments go against the DM's willingness, and incur unnecessary extra costs, thus, those original preferences smaller than $o^c - \theta_i^-$ or larger than $o^c + \theta_i^+$ won't be discussed, however, the DMs can refresh their preferences to rejoin the GDM process. In this regard, the objective function of the Model (I.3) is revised into

$$\min \phi = \sum_{i:\ o_i \in [o^c - \theta_i^-, o^c - \varepsilon_i^-)} c_i^+(o^c - \varepsilon_i^- - o_i) + \sum_{i:\ o_i \in (o^c + \varepsilon_i^+, o^c + \theta_i^+]} c_i^-(o_i - o^c - \varepsilon_i^+) \tag{I.7}$$

2.2 A general CRP framework based on consensus measures

Methods to obtain the consensus level (CL) are reviewed in Section 2.2.1, while a general framework of the consensus reaching process (CRP) is recalled in Section 2.2.2.

2.2.1 Consensus measure

Consensus level (CL), the current level of unanimous within a group, is often calculated by distance functions [dMCTHV18] and generally measured in two ways [LLRM20]:

- The distance between the DM's preference and the consensus, shown as the Eq. (I.8).

$$CL(o_1, \ldots, o_m) = 1 - f_2(f_1(d(o_i, o^c))) \geq \beta \tag{I.8}$$

- The distance between two arbitrarily chosen individual preferences, shown as the Eq. (I.9).

$$CL(o_1, \ldots, o_m) = 1 - g_2(g_1(d(o_i, o_j))) \geq \beta, i \neq j \tag{I.9}$$

where β is a preset CL threshold, $d(\cdot)$ represents the distance measure, $f_1 : \mathcal{R}^+ \to \mathcal{R}^+$, $f_2 : \mathcal{R}^+ \to [0, 1]$, $g_1 : \mathcal{R}^+ \to \mathcal{R}^+$, $g_2 : \mathcal{R}^+ \to [0, 1]$ are mapping functions, and the remaining notation is defined in Section 2.1.

Since the DM's influence is directly reflected by the weight w_i with $w_i \geq 0$ and $\sum w_i = 1$, Ref. [LLRM20] incorporated individual weights into the calculation of the CL, that is,

$$CL(\bar{o_1}, \ldots, \bar{o_m}) = \sum_{i=1}^{m} w_i |\bar{o_i} - o^c| \leq \gamma \tag{I.10}$$

$$CL(\bar{o_1}, \ldots, \bar{o_m}) = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{w_i + w_j}{m-1} |\bar{o_i} - \bar{o_j}| \leq \gamma \tag{I.11}$$

where $\gamma = 1 - \beta \in [0, 1]$. In fact, both the Eq. (I.10) and the Eq. (I.11) emphasize the greater contribution of the more important DM to the CL, but only the Eq. (I.10) is used in our research.

2.2.2 A general CRP framework

Several basic steps constitute a CRP framework for solving the consensus GDM problems, that is, preference expression, preference aggregation, consensus measure, preference adjustment and selection. In addition, Herrera-Viedma et al. [HVHC02] provided a general consensus framework, shown as Fig. 2, to address the GDM problems with heterogeneous preference structures.

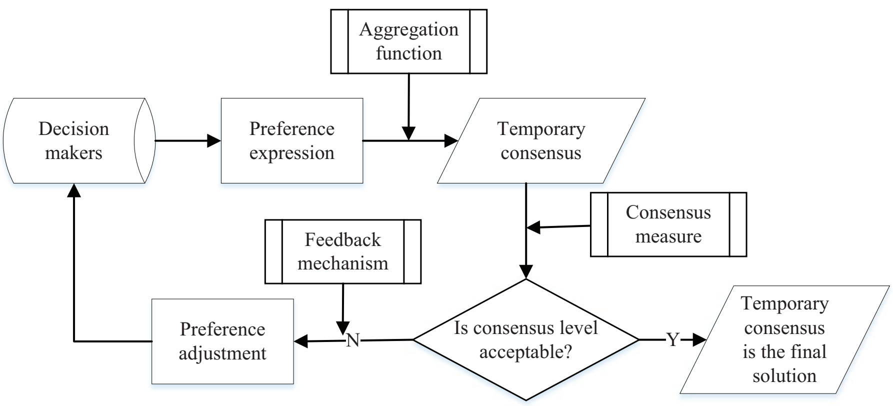


Figure 2: The general CRP framework in Ref. [HVHC02]

Regarding all the above mentioned steps, feedback mechanisms can be initiated based on diverse principles, such as the minimum deviation or cost [ZWD+21], the maximum number of experts adjusted under a limited budget [BAEE09], the minimum number of the adjusted DMs [ZDCY19] or some optimization-based rules [ZDZP20]. In our second topic, a CL threshold is used to initiate the DMs' preference modification process. Furthermore, it is worth noting that Section 2.1.3 and Section 2.2 both make a significant contribution to the second topic of this thesis.

2.3 Uncertainty theory

When no samples are available or only poor information obtained from historical data, the estimated distribution function will deviate far from the actual frequency, causing the law of large numbers invalid, and further obtaining some counterintuitive results. Thus, some domain experts are invited to evaluate the belief degree that certain events will happen. Distinguished from probability theory dealing with randomness of frequency, uncertainty theory was proposed to address the uncertainty of belief degrees.

2.3.1 Uncertain variable and uncertainty distribution

Let Γ be a nonempty set (sometimes referred as the universal set), and a collection $\mathcal{L}$ consisting of subsets of Γ is an algebra over Γ, if it meets the following three conditions: (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^C \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, we have $\bigcup_{i=1}^{n} \Lambda_i \in \mathcal{L}$, where, if condition (c) is replaced by closure under countable union, that is, if $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, we obtain $\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}$, then $\mathcal{L}$ is referred as a σ-algebra over Γ. Element Λ in $\mathcal{L}$ is called a measurable set, which also can be interpreted as an event in uncertainty theory. M is defined as an uncertain measure over the σ-algebra $\mathcal{L}$. Without loss of generality, real number $M\{\Lambda_i\}$ corresponds to event Λ_i one by one, representing the belief degree with which we belief event Λ_i will occur. There exist no doubt that such assignment is not arbitrary, and the uncertain measure M satisfies the following four axioms [Liu07, Liu09].

***Axiom* 1.** *(Normality Axiom): $M\{\Gamma\} = 1$ holds for the universal set Γ.*

***Axiom* 2.** *(Duality Axiom): $M\{\Lambda\} + M\{\Lambda^c\} = 1$ holds for any event Λ.*

***Axiom* 3.** *(Subadditivity Axiom): For every countable sequence of event Λ_1, Λ_2, $\cdots$, we have:*

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}$$

***Axiom* 4.** *(Product Axiom): Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertain space for $k \in N^+$, then the product of uncertain measure M is still an uncertain measure, and satisfies:*

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M\{\Lambda_k\}$$

where Λ_k are events arbitrarily chosen from $\mathcal{L}_k, (k \in N^+)$, respectively.

***Definition* 1.** *[Liu07] An uncertain variable ξ is a function from an uncertain space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers, and $\{\xi \in B\}$ is an event for any Borel set B of real numbers. For any real number x, the uncertainty distribution Φ of an uncertain variable ξ can be defined as: $\Phi(x) = M\{\xi \leq x\}$.*

$M\{\xi \leq x\}$ is the belief degree for the event $\xi \leq x$ may occur, and it is denoted as α, where $0 \leq \alpha \leq 1$. In other words, we have $\Phi(x) = M\{\xi \leq x\} = \alpha$. According to Axiom 2, we obtain $M\{\xi > x\} = 1 - \Phi(x) = 1 - \alpha$.

***Definition* 2.** *[Liu10] An uncertainty distribution $\Phi(x)$ is regular if it is a continuous and strictly increasing function with regard to x at which $0 < \Phi(x) < 1$, and satisfies*

$$\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1 \tag{I.12}$$

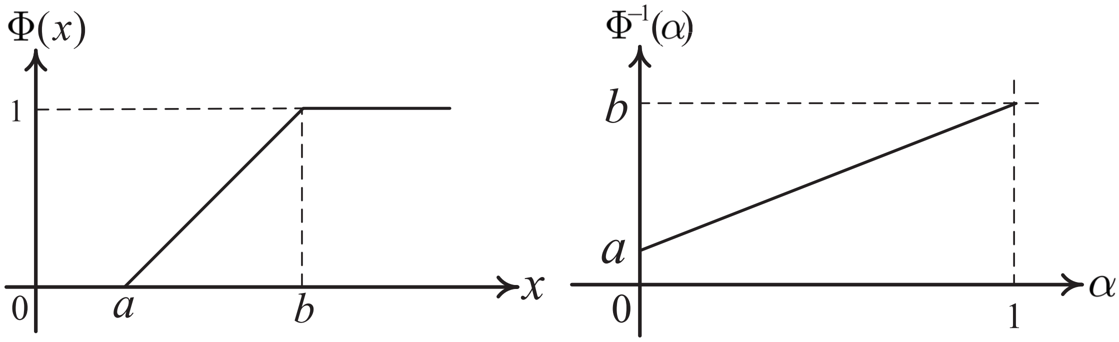
An uncertainty distribution Φ is regular if and only if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$.

***Example* 1.** *An uncertain variable ξ is called linear if it has a linear uncertainty distribution $\Phi(x)$ (see Fig. 3(a)), denoted by $\xi \sim \mathcal{L}(a, b)$, where real numbers a and b satisfy $a < b$.*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ \dfrac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } x \geq b \end{cases} \tag{I.13}$$

And its inverse uncertainty distribution (see Fig. 3(b)) is

$$\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b \tag{I.14}$$



(a) The linear uncertainty distribution (b) The inverse linear uncertainty distribution

Figure 3: Schematic diagram of a linear uncertain variable

2.3.2 Basic properties of uncertain variables

***Theorem* 1.** *[Liu10] Let ξ be an uncertain variable with an uncertainty distribution Φ, then for any real number x (i.e., $x \in \mathcal{R}$), we have*

$$M\{\xi \leq x\} = \Phi(x), \quad M\{\xi > x\} = 1 - \Phi(x) \tag{I.15}$$

To be noted, when the uncertainty distribution $\Phi(x)$ is a continuous function, we have $M\{\xi \leq x\} = M\{\xi < x\} = \Phi(x)$, and $M\{\xi > x\} = M\{\xi \geq x\} = 1 - \Phi(x)$.

***Theorem* 2.** *[Liu10] Let $\xi_1, \xi_2 \cdots \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2 \cdots \Phi_n$. If $f(\xi_1 \cdots \xi_m, \xi_{m+1} \cdots \xi_n)$ strictly increases with $\xi_1 \cdots \xi_m$ and decreases with $\xi_{m+1} \cdots \xi_n$, then $f(\xi_1 \cdots \xi_m, \xi_{m+1} \cdots \xi_n)$ has an inverse uncertainty distribution $f(\Phi_1^{-1}(\alpha) \cdots \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha) \cdots \Phi_n^{-1}(1-\alpha))$.*

***Example* 2.** *Let ξ_1 and ξ_2 be independent uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Then the inverse uncertainty distribution of $\xi_1 - \xi_2$ is*

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1-\alpha) \tag{I.16}$$

***Theorem* 3.** *[Liu07] Let ξ be an uncertain variable with its inverse uncertainty distribution denoted as $\Phi^{-1}(\alpha)$, if and only if $\Phi^{-1}(\alpha) \leq c$, then $M\{\xi \leq c\} \geq \alpha$, where α, c are constants within* $[0, 1]$.

***Theorem* 4.** *[Liu15] Let ξ and ς be independent uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Then the distance between ξ and ς is*

$$d(\xi, \varsigma) = \int_0^1 |\Phi^{-1}(\alpha) - \Psi^{-1}(1-\alpha)| d\alpha. \tag{I.17}$$

***Example* 3.** *Let ξ and ς be independent uncertain variables obeying linear uncertainty distributions as $\xi \sim \mathcal{L}(a, b)$ and $\varsigma \sim \mathcal{L}(c, d)$, where $a \leq b$ and $c \leq d$. Based on Example 1, the distance between ξ and ς becomes*

$$d(\xi, \varsigma) = \int_0^1 |(b + d - a - c)\alpha + a - d| d\alpha \tag{I.18}$$

To date, the efficiencies and advantages of uncertainty theory in solving the GDM problems have been elaborated in the literature [GGHV$^+$20, GGXHV20] through a comparative analysis with the existing well-known theories dealing with indeterminacy. In a nutshell, in addition to the traditional consensus decision-making theory, Section 2.3 lays a solid theoretical foundation for the optimal consensus modeling in the first two topics of this thesis.

3 Justification

Since all the relevant theories regarding this thesis are reviewed in the previous section, we then explain in detail the reasons and the basic assumptions to justify that the two aspects of the research conducted in this thesis, including 1) the construction of the new MCCMs combined with uncertainty theory; and 2) the explorations of theoretical innovations of the optimal consensus modeling in designing new trading mechanisms, are of great significance to extend the existing consensus decision-making theory. The justifications of each topic are described as follows.

3.1 The research on the MCCMs under linear uncertain-constrained scenarios

Upon conducting the state-of-art review of the relevant studies on the construction of the traditional consensus models, we find out that:

- Participants' preferences in the MCCMs/MACMs usually fit by crisp numbers instead of being fit by random distributions, but it is difficult for individuals to provide exact values as their preference, especially in the complex GDM contexts [WDC$^+$18, PMH14, DDMH17]. Therefore, the DMs are more likely to present their decisions by intervals with upper and lower bounds or various uncertainty distributions (e.g., linear uncertainty distribution).
- Previous studies focused on either the role combination of the moderator and the individual DMs [ZKP19, CZC$^+$18, BAE07, GXZ$^+$15] or independent decision status as the moderator [DDMH17], the individual DM [DXLF10] or the non-cooperators [PMH14]. In short, few extant contributions have built the MCCMs by considering all three roles simultaneously.

Given the above research gaps, we utilize uncertainty distributions to denote participants' decision preferences, and by discussing five scenarios from multiple decision-making roles (i.e., moderator, individual DMs and non-cooperators), we attempt to investigate a more general form of the classic MCCMs proposed in Section 2.1. However, in order to proceed with the first topic in this thesis, we give the following basic assumptions as the premise of this study:

(1) A moderator on behalf of the collective interest, who has skills in leadership and negotiation, and can persuade or tempt the DMs to continually adjust their opinions towards consensus, is introduced in the involved GDM problems;

(2) All the participants are independent of each other due to uncertainty theory, and since we aim to derive the consensus reaching conditions through deduction, we propose a simplest form of MCCMs by neglecting the process of preference aggregation;

(3) The consensus reached at the end of the GDM is a hard consensus, that is, all DMs' optimal adjusted preferences equal to the moderator's final preference. In a mathematical sense, we have $\bar{o_i}^* = o^{c*}$ due to the symbols defined in Section 2;

(4) Once non-cooperators are considered, they have absolute power over the GDM problems. Therefore, moderator's budget is mainly used to persuade these non-cooperators for compromising, corresponding to the decision rule of the minority being subordinate to the majority.

On account of all the above considerations, we later construct various uncertain MCCMs with the help of uncertainty theory, and we then transform all the non-linear consensus models into the equivalent linear forms by introducing the linear uncertainty distributions along with belief degrees. Afterwards, we propose detailed theorems in terms of both the consensus reaching conditions and their specific analytic formulae under each scenario through theoretical derivations. Finally, all these theoretical findings are verified by data analysis under a background of the carbon quota negotiation that involves the government and four heavily polluting enterprises located in the same administrative province.

3.2 The research on the CRP in the GDM with linear uncertain preferences and asymmetric costs

Through a literature review, we acknowledge that existing studies greatly contributed to the development of the GDM theory, but none has comprehensively explored the CRP framework combined with uncertain MCCMs, asymmetric costs, aggregation function, consensus measure and feedback mechanism. In other words, there still exist some research gaps that need to be investigated:

- Prior uncertain MCCMs do not consider asymmetric costs, nor the dynamic characteristic of the GDM, such as our first topic or [GGX$^+$21]. That is, they primarily focused on the optimal consensus modeling while neglecting the DM's unbalanced willingness to adjust.
- Extant MCCMs with asymmetric costs do not consider aggregation functions nor feedback mechanisms [CZC$^+$18, LJGQ21]. Namely, they don't aggregate the DMs' choices into a collective wisdom [ZDXL11], thereby failing to portray social choices or individual values.
- Current studies on the CRPs do not take into account the DMs' changeable influence during neither the consensus measure nor the feedback mechanism [ZLGZ18, ZL21], making the importance of different individuals unable to be fully demonstrated.

In this regard, our second topic attempts to integrate the dynamic features in the CRP (see Section 2.2) into the established uncertain MCCMs. In a same vein, to help with the new proposal, we first present several basic assumptions that listed as follows:

(1) The DMs' preferences are independent, so the consensus can be derived using aggregation functions, where the variables (e.g., preference and weight) are bounded within [0,1];

(2) The DMs are more sensitive to losses than gains (see prospect theory [KT13]), so let $|c_i^-| > |c_i^+|$ reflect the DM's adjusting willingness, namely, the changing trend of the downward-adjusting subinterval is steeper than that of the upward one (see Fig. 1(b) in Section 2.1.3);

(3) The DM's influence changes with the GDM procedure [LXGH22]: the DMs initially have equal weights, but their influence later diversifies due to their own contributions to the CL;

(4) All DMs are committed to reaching a consensus by completely following the given suggestions. In other words, the non-cooperative behavior [PMH14] is neglected.

Upon figuring out all the aforementioned gaps and assumptions, our second topic extends the established uncertainty theory-based MCCMs into a general CRP framework. Specifically, we build new optimization-based consensus models by simultaneously considering aggregation functions, asymmetric costs and consensus measure. Then, we present a novel CRP framework by respecting individual values with democratic consensus and simultaneously pursuing a minimum resource consumption based on uncertain MCCMs. Meanwhile, we introduce binary variables to reduce the computational complexity of piecewise functions in the new multi-coefficient programming models. Finally, we also verify these new consensus models by numerical analyses.

3.3 The research on the consensus modeling of a closed-loop trading mechanism regarding revenue and fairness

Through a literature review regarding the optimal consensus modeling along with the extant contributions on carbon issues, we realize that:

- Although many scholars have investigated the carbon issues, there has been few studies on designing the carbon quota trading mechanisms.
- Previous studies mainly discussed the fairness of carbon quota allocations at the global level, ignoring the interest-driven issues based on the individual or regional perspective.
- Consensus decision-making theory has not been adopted to deal with the design of carbon trading mechanisms and their resulting unfairness issues.

Therefore, the analysis of carbon trading mechanisms through optimal consensus modeling with all participators' interests taken into account is of great significance. To be noted, our third topic attempts to depict the most essential trading behaviors within a carbon quota market via the consensus modeling. Meanwhile, to reduce the computational complexity of the subsequent models, we simplify the problem to the greatest extent by clarifying the following basic assumptions:

(1) The carbon quota market discussed remains stable during a certain period, and the DMs can freely participate in the trading system;

(2) The unit price variables (e.g., the unit selling/buying/transaction price) are static, indicating that they do not fluctuate with time, supply and demand, and etc.

(3) Unit revenue of the individual DM's carbon quota is a constant, which is only determined by their own inherent characteristics rather than their initial holding quotas, meaning that the standard law of diminishing return assumption is not considered;

(4) Factors of costs within the profit-oriented trading system are implicit in the DM's initial unit revenue, so we can conduct analysis from the single perspective of revenue maximization.

Explicitly, the third topic of this thesis utilizes the consensus optimization models to assist the DMs in exchanging carbon quotas. In more detail, we present a benchmark consensus model that aims to maximize the overall revenue to derive an optimal carbon quota allocation scheme. Then, by building a two-stage programming model, new allocation schemes are acquired that focus on different single DM's revenue maximization, so as to gain detailed trading information. Next, on the basis of the newly defined individual or group development index, we propose two strategies to deal with the unfairness within the trading system. Finally, the feasibility of these new consensus models is verified via the numerical analyses of a carbon trading problem involving five regions.

4 Objectives

Since the overall objective of this thesis is to develop new consensus optimization models to solve the real-life GDM problems, we base our research on the following two aspects: 1) constructing new uncertain consensus models from the MCC perspective with the help of uncertainty theory; and 2) obtaining new resource reallocation schemes from the maximum compensation perspective with the help of the consensus decision-making theory. In specific, we attempt:

- **To construct the uncertain MCCMs from multiple decision roles.** Distinguished from the traditional MCCMs, we adopt the linear uncertainty distribution along with belief degree to represent the participant's preference structures. Meanwhile, we divide all the decision roles into the moderator, the individual DM and the non-cooperators. Then, we build five uncertain MCCMs where the uncertain preferences are used in different combinations of decision roles, so as to cater for a wide range of applications in real-life.
- **To verify that the GDM combined with uncertainty theory becomes a more inclusive theory.** The aim of introducing uncertainty theory into the traditional GDM theory, is to deal with common dilemmas where the reliability of decisions is mainly determined by experts rather than some prior probability distributions due to the insufficient historical data, and to verify that the GDM combined with uncertainty theory becomes a more inclusive theory by building a bridge between the deterministic and indeterministic GDM. Thus, detailed proofs of both the relevant theorems derived from theoretical deductions and the data analysis results corresponding to various decision scenarios are provided.
- **To propose a general CRP framework based on uncertainty theory and behavioral constraints.** As an essential part of the GDM, the CRP can facilitate more effective consensus by taking human behaviors into account. Thus, concerning the uncertain MCCMs in this thesis, we then incorporate prospect theory to rationalize the setting of asymmetric costs, adopt aggregation functions to portray social choice and individual values, and consider the DM's changeable influence in both the consensus measure and the feedback mechanism with the concept of democratic consensus [LXGH22].
- **To obtain flexible carbon quota trading mechanisms with the help of consensus modeling.** Since the final reallocation of resources within a close-loop trading system can be regarded as a consensual state in the GDM, it is reasonable to design a market trading mechanism using consensus models. Referring to the conventional market trading mechanism (i.e., the assumption of economic rationality), we establish new consensus models from the perspective of revenue maximization to derive the optimal carbon quota allocation scheme.
- **To put forward practical strategies to deal with fairness concerns within trading systems.** The inability to perform the real-life GDM usually stems from conflicts of interest due to mutual competition. Therefore, we define the individual or group development index, and we then accordingly provide the identification and the adjustment rules, similar as the CRP in the GDM, for the discordant DMs who display too much or too little revenue growth based on the results obtained from the single DM's revenue maximization models. In contrast, we also build a consensus optimization model with the fairness constraint to directly achieve an equilibrium state within the trading system.

5 Methodology

This section describes the methodology used throughout this thesis. Bearing our main objectives in mind, that is, to construct novel consensus optimization models with different individual behavioral constraints and to adopt them to address some real-life GDM problems so as to verify their effectiveness. A general procedure to conduct all the aforementioned studies is given as:

1. **Hypotheses formulation.** Hypotheses provide solid theoretical premises for the consensus modeling in this thesis. Intuitively, the GDM problems are destined to be large-scale, full of uncertainty and complexity with the rapid development of technology, thereby making the formulation of reasonable hypotheses become a compulsory choice to explore some particular GDM phenomena. Although we have elaborated the basic assumptions of each topic in Section 3, some general hypotheses deserve to be emphasized: 1) we only focus on the small-scale GDM problems with their highly abstracted forms, due to the fact that we attempt to obtain some interesting theorems through deductions; and 2) we haven't explored the non-additive consensus problems (i.e., we only adopt the prerequisite that the DMs are independent of each other), where the DMs' mutual interactions surely affect the final decisions.

2. **Model construction.** Originated from the concept of the MCC [BAE07], this thesis establishes a series of novel consensus optimization models by combining various considerations. For example, we construct the uncertain static MCCMs based on multiple decision roles that include the moderator, the individual DMs or the non-cooperators. Later, we further propose the uncertain MCCMs under the CRP framework by incorporating consensus measures, asymmetric cost setting based on prospect theory, and aggregation functions. Lastly, distinguished from the modeling idea of the MCC, we also develop new models that aim to maximize the DMs' revenues or to address fairness concerns during the theoretical explorations in the closed-loop trading systems, to acquire the optimal or fair resource allocation schemes.

3. **Model validation.** The validation process usually involves case studies, numerical examples or simulation experiments, to show the feasibility and the effectiveness of the new proposals. For instance, we present a case study in our first contribution to demonstrate how these newly proposed models work in practice. Specifically, a negotiation highly abstracted from the real-life GDM problem is given, over the carbon emission quota reallocation that conducted between the government and four local heavily polluting enterprises, to illustrate the validity of the five uncertain chance-constrained MCCMs and their associated theorems. In addition, the relationships between these uncertain MCCMs with the traditional ones are also deeply investigated through data analyses.

4. **Further discussion.** This final process normally involves a comparative analysis with some most related studies, or distinctive scenario analyses (e.g., parametric behavioral analysis or sensitivity analysis), so as to demonstrate the superiority and the advantages of the new proposals. For example, we present five cases in the carbon quota negotiation to discuss various uncertain-constrained MCCMs in the first topic, thereby better reflecting the government's humanized management where uncertain indicators are set instead of some deterministic and fixed ones, which could also be understood as the practical significance of the uncertain constraints in this thesis. Similarly, we also perform detailed comparative analyses of different parameters in the new models and of the existing literature, thus further highlighting the characteristics of the new methods.

6 Summary

This section presents the summary of the proposals included in this thesis, where the main contents along with the obtained results associated with the journal publications are provided. Note that, all the research carried out in this thesis and the results acquired from each topic are collected into the following published papers:

- Z.W. Gong, X.X. Xu, W.W. Guo, E. Herrera-Viedma, F.J. Cabrerizo. Minimum cost consensus modelling under various linear uncertain-constrained scenarios, Information Fusion, 2021, 66: 1-17.
- X.X. Xu, Z.W. Gong, E. Herrera-Viedma, G. Kou, F.J. Cabrerizo. Consensus reaching in group decision making with linear uncertain preferences and asymmetric costs, IEEE Transactions on Systems, Man, and Cybernetics: Systems, 2022, doi: 10.1109/TSMC.2022.3220837.
- X.X. Xu, Z.W. Gong, W.W. Guo, Z.M. Wu, E. Herrera-Viedma, F.J. Cabrerizo. Optimization consensus modeling of a closed-loop carbon quota trading mechanism regarding revenue and fairness, Computers & Industrial Engineering, 2021, 161: 107611.

The remainder of this section is organized by the sequence of the above publication list. In each subsection, by combining the objectives mentioned in Section 4, we provide the explanations in detail with particular research contents, important indicators defined, main research steps along with particular algorithms, in order to better understand each topic investigated in this thesis.

6.1 The research on the MCCMs under linear uncertain-constrained scenarios

To date, a large number of variants of the MCCMs or the MACMs mentioned in Section 2.1 have been proposed in the literature to meet the needs of different decision environments or applications, however, the first topic of this thesis is to achieve the goal of minimizing the total consensus cost instead of keeping the original preference information as much as possible. Without loss of generality, let ω_i denote the cost of adjusting d_i's original preference o_i towards the consensus o' one unit. If we normalize all these unit costs, they become the weighted arithmetic mean operators, which can also be understood as each individual's influence on the CRP [GXZ+15]. In reality, too many uncertain factors need to be considered in the GDM, making the above parameters difficult to quantify, hence, ω_i is subjectively determined in the follow-up discussion.

Herein, to obtain a unified form of the traditional consensus models, we have the following model abstracted from [BAE07] as

$$\begin{array}{l} Min\ \phi = \sum\limits_{i=1}^{n} \omega_i f_i(o') \\ s.t. \left\{ \begin{array}{l} f_i(o') = |o' - o_i| \\ |o' - o_i| \leq \varepsilon_i,\ i \in N \end{array} \right. \end{array} \tag{I.19}$$

where ϕ represents the total consensus cost for the whole GDM, and ε_i is the upper bound of the deviation (i.e., distance measure) between d_i's opinion and the optimal collective opinion, implying that we want to obtain an acceptable consensus.

Given that when no samples are available or only poor information obtained from historical data, the estimated distribution function will deviate far from the actual frequency, causing the law of large numbers invalid, and further obtaining some counterintuitive results. Then, some domain experts are invited to evaluate the belief degree that certain events will happen. Distinguished from probability theory dealing with randomness of frequency, uncertainty theory was proposed to address the uncertainty of belief degrees. To introduce uncertainty theory into the traditional consensus models, we first present the Model (I.20) that further abstracted from the Model (I.19).

$$\begin{aligned} & Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\ & s.t.\ \{\ |o' - o_i| \leq \varepsilon_i, \varepsilon_i \geq 0,\ i \in N \end{aligned} \tag{I.20}$$

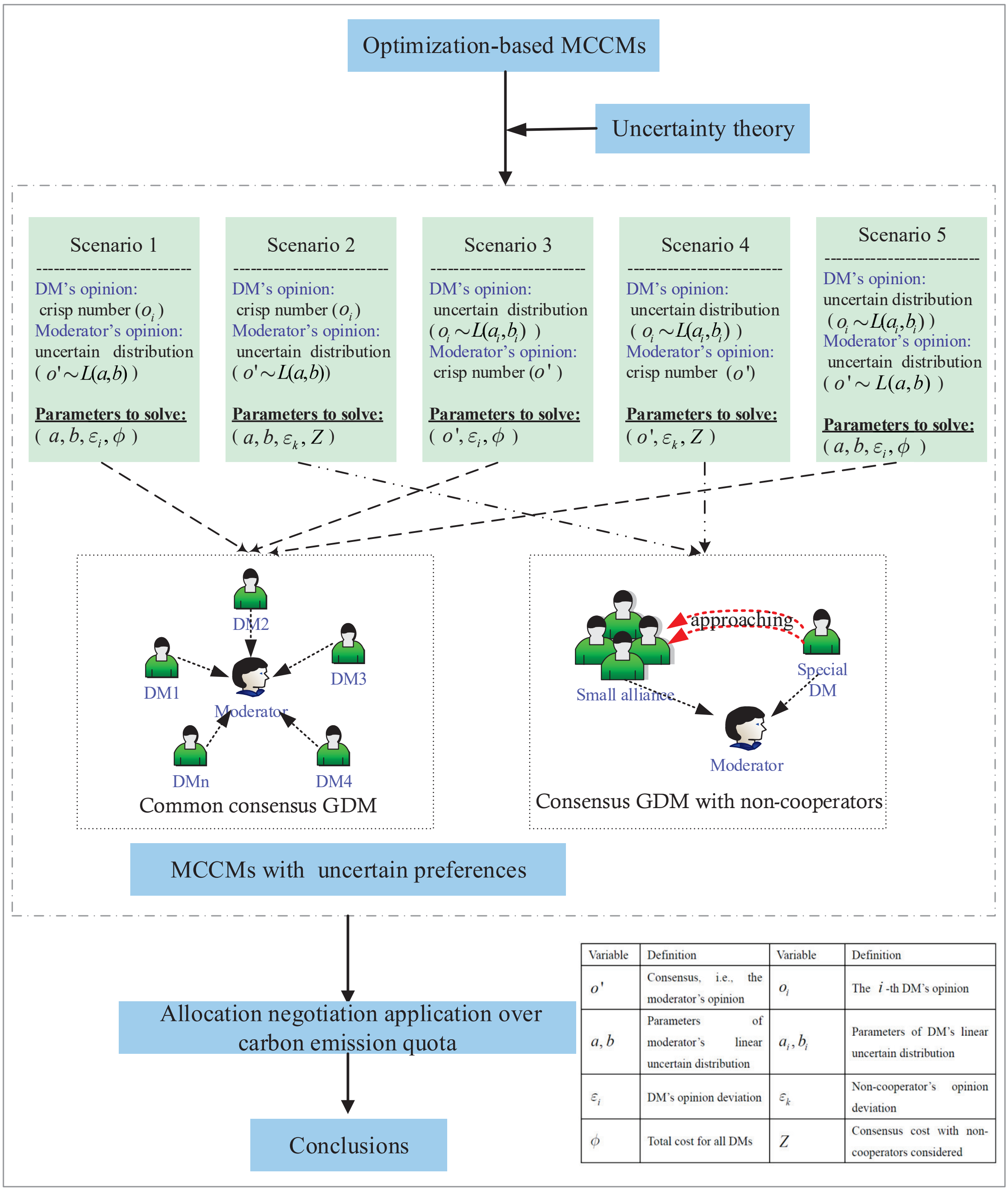
Note that decision variable in the Model (I.19) only includes o', while ε_i is a pre-defined threshold over the distance measure between o' and o_i, $i \in N$. In contrast, decision variables in the Model (I.20) include both o' and ε_i, where ε_i is bounded by the deterministic threshold given in the Model (I.19) under the premise that these parameters are set as same in both models. Obviously, the feasible domain of the solution set of the Model (I.20) is larger than that of the Model (I.19), making the optimal value of the objective function in the Model (I.20) be no larger than that of the Model (I.19). As a result, although the Model (I.20) is simpler in terms of form, its scope of application is wider than that of the Model (I.19). Thus, the Model (I.20) becomes the basis of our subsequent consensus models with uncertain variables.

That is, the new uncertain MCCMs are proposed based on the above Model (I.20), which are specified into five scenarios: 1) moderator with uncertain preference; 2) non-cooperators considered and moderator with uncertain preference; 3) the DMs with uncertain preferences; 4) non-cooperators considered and the DMs with uncertain preferences; and 5) the moderator and the DMs with uncertain preferences. For clarity, a flowchart of the first topic is given as Fig. 4, and the relationships between these five GDM scenarios are summarized in detail as follows.

Considering that the participants usually have disparate standpoints or interests when facing the real-life GDM, so uncertain preferences will be accordingly expressed by different roles under various decision contexts. Therefore, Scenario 1 and 2 assume that moderator's opinion is fit by uncertain preferences (denoted by linear uncertainty distribution along with belief degree) while the DMs present crisp number-based preferences, and then preference structures of those two roles are reversed in Scenario 3 and 4. Finally, in Scenario 5, all the participants involved present their judgements by uncertain preferences. For more in line with the real-life GDM problems, we also deeply explored the influence of non-cooperators on the uncertain MCCMs in Scenario 2 and 4, simultaneously aiming to correlate with the previous MCCMs in [GXZ+15].

Regarding the construction of the consensus optimization models, we take Scenario 1 as an example, where the moderator needs to consider many uncertain factors for the final convergent opinion, we assume that the moderator's opinion o' obeys an uncertainty distribution. Based on Liu's uncertainty theory, if the deviation between the consensus o' and the individual opinion o_i is no more than ε_i under the belief degree α, then it can be denoted as $M\{o' - o_i \leq \varepsilon_i\} \geq \alpha$ and $M\{o' - o_i \geq -\varepsilon_i\} \geq \alpha$, where M represents the uncertain measure in uncertainty theory, and the variable $\alpha \in [0, 1]$ indicates the belief degree of holding the constraint $|o' - o_i| \leq \varepsilon_i$, $i \in N$. As a result, a relevant uncertain MCCM is constructed due to the above Model (I.20).

Under each GDM scenario, we subsequently obtain a non-linear goal programming model based on Section 2.3, where the belief degree α can be discussed into two cases: α is a pre-determined value or α is a parameter to be determined. As in the latter case, the variable α solved by the new goal programming model will be an optimal belief degree in the discussed GDM. Moreover, since the linear uncertainty distributions can be easily transformed and can reduce the complexity of understanding and calculation, we replace the above uncertain constraints with corresponding



Variable	Definition	Variable	Definition
o'	Consensus, i.e., the moderator's opinion	o_i	The i-th DM's opinion
a, b	Parameters of moderator's linear uncertain distribution	a_i, b_i	Parameters of DM's linear uncertain distribution
ε_i	DM's opinion deviation	ε_k	Non-cooperator's opinion deviation
ϕ	Total cost for all DMs	Z	Consensus cost with non-cooperators considered

Figure 4: Flowchart of the MCCMs with uncertain preferences

linear type and finally derive relevant theorems about the analytic formulae of the optimal consensus cost and the consensus reaching conditions from their corresponding equivalent linear models.

The journal article corresponding to this section is:

- Z.W. Gong, X.X. Xu, W.W. Guo, E. Herrera-Viedma, F.J. Cabrerizo. Minimum cost consensus modelling under various linear uncertain-constrained scenarios, Information Fusion, 2021, 66: 1-17.

6.2 The research on the CRP in the GDM with linear uncertain preferences and asymmetric costs

Clearly, some important characteristics in the classic MCCMs or MACMs are neglected in our first topic, such as setting an aggregation function over the adjusted preferences to obtain a consensus [DXLF10, ZDXL11], using the CL to measure the efficiency of the CRP [ZKP19], or considering the asymmetric characteristic of unit costs [CZC+18]. Thus, to address those shortcomings, our second topic aims to consider more behavioral constraints, and to extend the uncertain MCCMs into a general CRP framework, so as to be more consistent with the real GDM situations.

To do that, we first transform the Eq. (I.7) into the following Eq. (I.21) based on Section 2.1.3 and Section 3.2. That is, the CL is bounded within [0, 1], meanwhile, the GDM discussed is further simplified: boundary values of the DM's preference in Fig. 1(b) are preset as $o^c - \theta_i^- = 0$ and $o^c + \theta_i^+ = 1$. In addition, d_i's tolerance behavior is no longer distinguished as ε_i^+ or ε_i^-, that is, only one parameter (i.e., ε_i) is used to denote d_i's tolerance to the consensus o^c.

$$\min \phi = \sum_{o_i \in [0, o^c - \varepsilon_i)} c_i^+ (o^c - \varepsilon_i - o_i) + \sum_{o_i \in (o^c + \varepsilon_i, 1]} c_i^- (o_i - o^c - \varepsilon_i) \tag{I.21}$$

To establish the new uncertain MCCMs with asymmetric cost, we first introduce the following Theorem (5) proposed in [GGX+21] to clarify that the arithmetic mean aggregation (AMA) operator still works in the new consensus models.

***Theorem* 5.** *Let $o_{i,i\in M}$ be an independent uncertain variable obeying a linear uncertainty distribution as $o_i \sim \mathcal{L}(a_i, b_i)$ with $a_i \le b_i$. Then $\sum w_i o_i$ obeys a linear uncertainty distribution as $\sum w_i o_i \sim \mathcal{L}(\sum w_i a_i, \sum w_i b_i)$.*

Hence, a new uncertain MCCM based on the linear uncertainty distributions is built by comprehensively considering asymmetric costs, aggregation function and consensus measure.

$$\begin{array}{l} \min \phi = \sum_{i=1}^{m} \{c_i^+, c_i^-\} * d(\bar{o}_i, o_i) \\ \text{s.t.} \begin{cases} d(\bar{o}_i, o^c) \le \varepsilon_i & (I.22-1) \\ o^c = \sum_{i=1}^{m} w_i \bar{o}_i & (I.22-2) \\ \sum_{i=1}^{m} w_i * d(\bar{o}_i, o^c) \le \gamma & (I.22-3) \\ o_i \sim \mathcal{L}(a_i, b_i), \bar{o}_i \sim \mathcal{L}(\bar{a}_i, \bar{b}_i), o^c \sim \mathcal{L}(a^c, b^c) & (I.22-4) \\ 0 \le \bar{a}_i \le \bar{b}_i \le 1, 0 \le a^c \le b^c \le 1, i \in M & (I.22-5) \end{cases} \end{array} \tag{I.22}$$

Solving the Model (I.22) yields the minimum cost ϕ^*, the optimal consensus o^{c*}, and d_i's optimal adjusted preference $\bar{o}_i{}^*$. Note that, $d(\bar{o}_i, o_i)$ is the distance measure between d_i's adjusted preference $\bar{o}_i$ and original preference o_i; the expression $\{c_i^+, c_i^-\}$ means only one coefficient is taken due to d_i's adjustment direction that corresponds the Eq. (I.21); (I.22-1) reflects d_i's tolerance behavior; (I.22-2) uses the AMA operator to fuse all adjusted preferences; (I.22-3) is the consensus measure, and (I.22-4) indicates that the DM's preference obeys a linear uncertainty distribution under (I.22-5).

Here, referring to Theorem (4) introduced in Section 2.3.2, [GGX+21] derived the following Theorem (6) to deal directly with the distance measure between any two uncertain variables.

***Theorem* 6.** *Distance between any two independent variables with linear uncertainty distributions, denoted as $\xi \sim \mathcal{L}(a,b)$ and $\varsigma \sim \mathcal{L}(c,d)$ with $a \leq b, c \leq d$, can be transformed into a piecewise function as*

$$d(\xi,\varsigma) = \begin{cases} \dfrac{a+b-c-d}{2}, & \text{if } a > d \\ \dfrac{c+d-a-b}{2}, & \text{if } b < c \\ \dfrac{(d-a)^2}{b+d-a-c+\epsilon} + \dfrac{a+b-c-d}{2}, & \text{otherwise} \end{cases} \tag{I.23}$$

where ϵ is the non-Archimedean infinitesimal. In the second topic of this thesis, we take $\epsilon = 10^{-6}$ to ensure that a rare case of $a = b$ and $c = d$ still holds in the Eq. (I.23), and then the two uncertain variables essentially degenerate to two real numbers.

Basically, piecewise functions seldom exist in the final models due to calculation complexity, thus, we use the big M method to transform the Eq. (I.23) into a hybrid 0-1 programming model (i.e., the Model (I.24)). Here, let U be a sufficiently large positive number and we take $U = 10^6$ in the subsequent analysis.

$$\begin{aligned} & d(\xi,\varsigma) = z_3 * \tfrac{(d-a)^2}{b+d-a-c+\epsilon} + (0.5 - z_2) * (a+b-c-d) \\ & \text{s.t.} \begin{cases} -U(1-z_2)(1-z_3) \leq d-a < U(1-z_1) \\ -U(1-z_1)(1-z_3) \leq b-c < U(1-z_2) \\ -U(1-z_1) < \tfrac{a+b-c-d}{2} < U(1-z_2) \\ -U(1-z_2) < \tfrac{c+d-a-b}{2} < U(1-z_1) \\ \tfrac{(d-a)^2}{b+d-a-c+\epsilon} + \tfrac{a+b-c-d}{2} > -U * z_2 \\ z_1 + z_2 + z_3 = 1 \\ z_1, z_2, z_3 \in \{0,1\} \end{cases} \end{aligned} \tag{I.24}$$

where z_1, z_2, z_3 are binary variables with one and only one value of 1. For example, if $z_1 = 1$, then $z_2 = 0$, $z_3 = 0$, we get the first case of $a > b$; and if $z_3 = 1$, then $z_1 = 0$, $z_2 = 0$, we have $d \geq a$ and $b \geq c$, which corresponds to the third case. Detailed transformation from the Eq. (I.23) to the Model (I.24), which focuses on the relative positions of the four parameters (i.e., a, b, c and d), is omitted here to save space. In addition, another 0-1 variable x_i is introduced to handle the multi-coefficient problem [CCZ12] of the Model (I.22) (i.e., $\{c_i^+, c_i^-\}$).

Through all these operations, we finally develop a new uncertain MCCM based on the Model (I.22) by simultaneously considering asymmetric costs, aggregation operators and consensus measures. However, inappropriate CL thresholds easily lead to the failure of reaching a consensus, so some maximum CL models are built to aid the DM's preference adjustment. That is, solving the above new uncertain consensus model may not yield the MCC due to inappropriate CL thresholds. Hence, building another consensus model that considers both the CL and the budget is of great necessity, where the CL threshold no longer needs to be predetermined. In this regard, we build a more flexible model to solve the GDM problems by introducing a trade-off coefficient λ ($\lambda \in [0,1]$).

Meanwhile, we also present a feedback mechanism with the concept of democratic consensus [LXGH22] based on the general CRP framework that recalled in Section 2.2.2, where the DMs are first assigned with equal weights to protect the interest of minorities, then their influence is updated with their own contribution to the CL (see Definition 3). To be noted that only one DM needs to be adjusted at each iteration due to the less modified DMs the better, thereby avoiding unnecessary adjustments and preserving the DMs' original preferences at most.

***Definition* 3.** *The DM d_k's contribution to the CL is measured by the deviation between the overall CL and the CL reached by the remaining $m-1$ DMs (i.e., $CL_{\bar{k}}$), thus,*

$$CL_k = CL - CL_{\bar{k}}, k \in M \tag{I.25}$$

Our new proposed CRP framework combines the idea of democratic consensus and the optimization-based uncertain MCCM with the CL threshold being its constraint. Essentially, the procedure, prior to performing the new uncertain consensus model with a CL determined by initial preference information, is an intra-group self-adjustment with only once, that fully respects individual values by adjusting only one DM's preference but updating all weights based on their contributions to the CL. Meanwhile, if the required CL is still not reached after the model takes effect, all the adjusted preferences optimized by the model will be directly used to start the next iteration. Note that whenever any DM has some changes in either the preference or the weight, a new CL is calculated, so as to minimize the resource consumption. Next, the identification rule (IR) and the direction rule (DR) in the new CRP are defined as follows.

- IR: the DM with a minimum contribution to the CL is chosen as the one to be adjusted, denoted as d_k. If there exist more than two DMs with a same value, then d_k is randomly determined.
- DR: the adjustment of the d_k's preference and all DMs' weight reallocation are considered in the modification process.
  - The DM d_k's updated preference is expressed as the Eq. (I.26), where $\delta \in [0, 1]$ is the parameter reflecting d_k's self-confidence, and the larger δ, the less he/she is willing to make revisions.
  $$\bar{o_k} = \delta * o_k + (1-\delta) * o^c \tag{I.26}$$
  - The weights are updated by the Eq. (I.27), where η is the variable that controls the impact of d_i's consensus contribution CL_i^t on the weight w_i^{t+1}. Next, all the new weights are normalized by the Eq. (I.28).
  $$w_i^{t+1} = w_i^t * (1 + CL_i^t)^\eta \tag{I.27}$$
  $$\bar{w_i} = \frac{w_i^{t+1}}{\sum_{i=1}^m w_i^{t+1}} \tag{I.28}$$

Explicitly, the larger the parameter η, the stronger modification of the DM d_i [XZW15]. In summary, our new CRP is proposed by incorporating the new uncertain MCCMs with the concept of democratic consensus, and is implemented as follows, where the ideal CL threshold is predetermined as $CL^* = 0.85$ with a maximum number of iterations of 5.

(1) Calculate an initial CL to determine whether there exists one DM (i.e., d_k) to be adjusted. If yes, go to Step 2; otherwise, terminate the CRP.

(2) Recalculate and check whether the new temporary consensus meets the threshold CL^* by adjusting d_k's preference and updating all DMs' weights. If yes, terminate the CRP; otherwise, move to Step 3.

(3) Solve the new uncertain MCCM with the initial CL value obtained from Step 1 to produce optimal solutions, then check if the obtained optimal CL meets CL^*. If yes, terminate the CRP; otherwise, return to Step 1 with acquired adjusted preferences.

The journal article relevant to this section is:

- X.X. Xu, Z.W. Gong, E. Herrera-Viedma, G. Kou, F.J. Cabrerizo. Consensus reaching in group decision making with linear uncertain preferences and asymmetric costs, IEEE Transactions on Systems, Man, and Cybernetics: Systems, 2022, doi: 10.1109/TSMC.2022.3220837.

6.3 The research on the consensus modeling of a closed-loop trading mechanism regarding revenue and fairness

Consensus modeling aims to obtain a collective agreement through the GDM, generally by building mathematical models. This section describes the use of the optimal consensus modeling to explore theoretical innovations regarding flexible carbon quota trading mechanisms, where basic allocation schemes are provided within a closed-loop trading system by simultaneously taking revenue and fairness into account. To do that, we develop a series of consensus optimization models from the perspective of maximizing the corresponding revenue that originated from the ideas in Section 2.1.2, and then obtain the optimal or fair carbon quota allocation schemes that include detailed trading information. Moreover, we further propose a relaxation method based on the PSO algorithm to solve the above models. Meanwhile, considering that the inability to conduct the real-life GDM usually stems from conflicts of interest based on the DMs' mutual competition, thus, we put forward two practical strategies to deal with the resulting unfairness within the trading system.

Based on the assumptions given in Section 3.3, we further present the following two goals that need to be met in discussing the closed-loop trading systems using the consensus GDM theory.

- Goal 1: each DM's total revenue derived from the trading is no less than his initial fixed one;
- Goal 2: the sum of all DMs' revenues acquired from the trading system should be maximized.

Goal 1 is set from the DM's perspective, and aims to maximize each DM's economic benefit. All DMs are assumed to be rational (that is, once the carbon quota trading is conducted, they must benefit themselves); otherwise, the transactions are invalid. This corresponds to real-life market trading and can be understood as the effectiveness of the trading mechanisms. On the contrary, Goal 2 is set from the collective angle. In general, the representative of the collective benefit is the participant who determines the initial carbon quotas for all DMs, and also the one who plays the role as a moderator in the GDM problems. Clearly, the primary goal of those representatives is to maximize the overall revenue.

To realize Goal 1, we have the following constraints: (1) $p_i \geq r_i$, (2) $q_i \leq r_i$, where p_i denotes the unit selling price, q_i represents the unit buying price, and r_i is the original fixed revenue for one unit of d_i's carbon quota. In addition, let the quantity transferred from d_i to d_j be I_{ij}, and their final unit transaction price be T_{ij}. Then, the following statement holds: if $p_i \leq q_j$, then the one-way carbon quota transaction from d_i to d_j can be realized. At this point, d_i can sell carbon quotas to d_j with $I_{ij} \geq 0$, and the unit transaction price $T_{ij} \in [p_i, q_j]$, which indicates that there is a negotiable space in the trading process between d_i and d_j. Meanwhile, we derive $I_{ji} = 0$, since $I_{ij} * I_{ji} = 0$ holds under the premise of one-way transactions.

As [CS06] indicated that the purpose of designing a trading mechanism is to provide a method for ensuring that the allocation decisions and pricing decisions in decision-making processes result in the desired outcomes. They also stated that once the allocation principle is set in a truthful mechanism, the prices are determined; similarly, once the pricing rule is determined, the allocation is settled. As a result, we take the maximization of the overall revenue or a single DM's revenue as the objective function, and uses the optimal consensus modeling to determine the allocation scheme (i.e., the determination of the DMs' updated quotas with specific transferred quantities) and the pricing scheme (i.e., the determination of variables p_i, q_i, T_{ij}) in the trading system.

In these regards, we develop a benchmark carbon trading consensus model with overall revenue maximization, where the objective function of maximizing the sum of all DMs' revenues within the system is subjected to a constraint set that includes the expression of each DM's final holding quotas based on their original ones and the trading behaviors with others, the constraint of the DM's unit price variables with their unit revenue that mentioned above, the achievable conditions to conduct a carbon trading shown as the following Theorem 7, and a constraint that the DMs' final quotas should be located in their own expected intervals provided initially as $[o_i^-, o_i^+]$. Most importantly, we prove that such consensus models always have optimal solutions with a unique maximum value of the objective function by identifying one special DM who meets Theorem 8.

Theorem 7. *The achievable constraints of the carbon quota trading mechanism are determined by d_i's unit selling price p_i and d_j's unit buying price q_j as:*

$$\begin{cases} I_{ij} \geq 0, & \text{if } p_i \leq q_j \text{ and } i < j,\ i,j \in N \\ I_{ij} = 0, & \text{otherwise} \end{cases}$$

which is equivalent to

$$\begin{cases} I_{ij} \leq \dfrac{|q_j - p_i| + q_j - p_i}{\delta}, & i < j, i, j \in N \\ I_{ij} = 0, & \text{otherwise} \end{cases} \tag{I.29}$$

where δ is a sufficiently small positive value approaching zero.

Theorem 7 states the achievable conditions of a closed-loop trading system under the basic hypothesis that all DMs are arranged in order based on the relationships among their original fixed unit revenues, that is, $r_1 \leq r_2 \leq ... \leq r_n$. In other words, the transferred quantity of the carbon quota is not only affected by the DM's location index, but also by the size of the DM's fixed unit revenue. In fact, carbon quota trading can only be achieved when the unit selling price of one DM with a small location index is no greater than the unit buying price of another DM with a large location index; otherwise, their carbon quota transaction fails.

Theorem 8. *There must exist an m-th DM such that $\sum_{i=1}^{m-1} o_i^- + \bar{o}_m + \sum_{i=m+1}^{n} o_i^+ = \sum_{i=1}^{n} o_i$ and $o_m^- \leq \bar{o}_m \leq o_m^+$. By then, the optimal value of the objective function in the consensus model that maximizes the overall revenue is $\sum_{i=1}^{m-1} r_i o_i^- + r_m \bar{o}_m + \sum_{i=m+1}^{n} r_i o_i^+$ and the optimal solution is $\bar{o}_i = o_i^- (1 \leq i \leq m-1), \bar{o}_m = \sum_{i=1}^{n} o_i - \sum_{i=1}^{m-1} o_i^- - \sum_{i=m+1}^{n} o_i^+, \bar{o}_i = o_i^+ (m+1 \leq i \leq n)$.*

As the competition mechanism refers to the struggle among market practitioners to maximize their own economic benefits, it focuses more on individual standpoints than the collective perspective. Hence, we accordingly change the above objective function into a new one of maximizing each DM's revenue, which includes the revenue of holding the new carbon quotas and the revenue of their trading behaviors by selling or buying quotas. Thus, a constraint of the unit transaction price between d_i and d_j denoted by $T_{ij} \in [p_i, q_j]$ is added into the new model. However, the individual revenue maximization model should be built based on the benchmark model, which indicates that maximizing a single DM's revenue is not unconstrained; instead, it should be carried out within the context of maximizing the overall revenue for the whole group.

Once all DMs pursue the maximization of their own revenues, it inevitably results in unfairness (e.g., the unbalanced growth of the DMs' revenues). During the CRP, if the DMs' improper initial parameters can be modified as early as possible, systemic losses (e.g., cost, time) will be significantly reduced [LDD+19]. Therefore, we examine the potential to achieve a relatively balanced state within the closed-loop trading system by adjusting some DMs' initial parameters. Once fairness is achieved, the DMs with too much revenue growth or too little revenue growth should no longer exist. In this regard, we accordingly give the definitions of the following two indices.

***Definition* 4.** *An individual development index is defined as a relative proportion of the DM's final revenue obtained through the carbon quota trading process with respect to their initial fixed revenue, that is,*

$$H_i = \frac{r_i\bar{o}_i + \sum\limits_{j=1, j\neq i}^{n} T_{ij}I_{ij} - \sum\limits_{j=1, j\neq i}^{n} T_{ji}I_{ji}}{r_i o_i}, i \in N$$

***Definition* 5.** *The group development index is defined as a relative proportion of the final total revenue obtained through the carbon quota trading process with respect to the initial fixed total revenue of the group, that is,*

$$\bar{H} = \frac{\sum\limits_{i=1}^{n} r_i\bar{o}_i}{\sum\limits_{i=1}^{n} r_i o_i}$$

This thesis follows the idea of fair development of all DMs in the trading system. By default, the difference between the individual development index H_i and the group development index $\bar{H}$ should be within a certain range, otherwise the DMs will be identified as the discordant DMs with too much or too little revenue growth. These two indices mainly depend on the DM's final carbon quota $\bar{o}_i$, which further depends on the endpoints of the expected interval $[o_i^-, o_i^+]$ provided by the DM d_i. Here, we choose interval values instead of crisp numbers to denote d_i's expected carbon quota quantity due to various uncertainties. By adjusting the expected carbon quota range $[o_i^-, o_i^+]$ of the discordant DMs, an equilibrium state with a minimum loss can be achieved within the trading system (see Fig. 5(c)). Let a discordant DM be $d_k, k \in \{0, 1, \cdots, n\}$, and his expected final carbon quota be adjusted from $[o_k^-, o_k^+]$ to $[o_k^{'-}, o_k^{'+}]$ through the following adjustment rules.

- When $H_k << \bar{H}$ and $|H_k - \bar{H}| > \gamma$, where γ is a pre-determined threshold and $<<$ denotes far less than, d_k is identified as a discordant DM with too little revenue growth. This DM is located in the unbalanced state shown in Fig. 5(a), and his adjustment rules are:
  - If $k > m$, then the amount purchased is too little, and so o_k^+ needs to be increased;
  - If $k < m$, then the amount sold is too little, and so o_k^- needs to be further decreased;
  - If $k = m$, then the current expected interval is improperly set, and we need to simultaneously reduce o_k^- and increase o_k^+.

- When $H_k >> \bar{H}$ and $|H_k - \bar{H}| > \gamma$, where γ is a pre-determined threshold and $>>$ means far more than, d_k is identified as a discordant DM with too much revenue growth. This DM is located in the unbalanced state shown in Fig. 5(b), and his adjustment rules are:
  - If $k > m$, then the quantity purchased is too great, and so o_k^+ needs to be decreased;
  - If $k < m$, then the amount sold is too great, and so o_k^- should be increased;
  - If $k = m$, then the current interval of the DM's expected carbon quota is inappropriate, and we need to increase o_k^- and decrease o_k^+ at the same time.

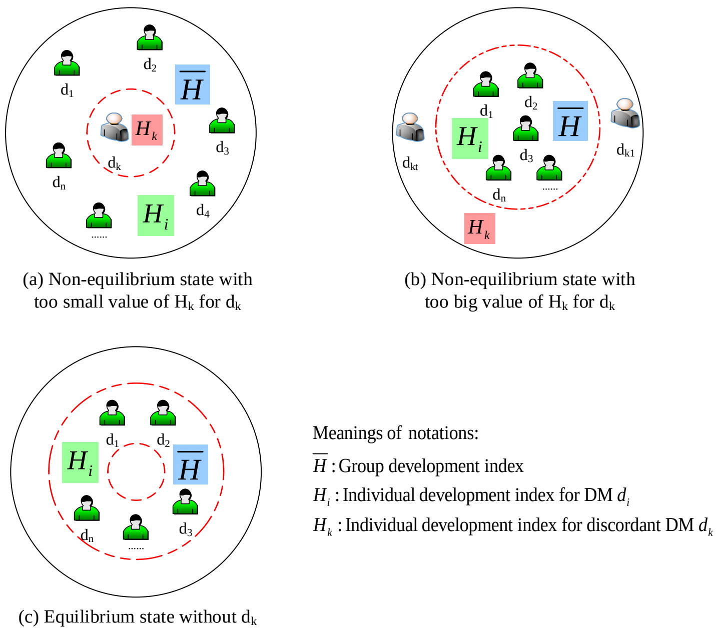


Figure 5: Identification of non-equilibrium states in the closed-loop trading system

On account of the above adjustment rules, a set of the updated trading information regarding all DMs can always be acquired. According to the definitions of the individual or group development index, we obtain the values of all $|H_i - \bar{H}|$ based on the n carbon trading consensus models with single DM's revenue maximization, so as to determine the threshold of the variable γ, as well as the difference value $|H_i - H_j|$ between any two DMs. However, the identification parameter γ needs to be manually set, and the specific adjustment ranges of the discordant DMs cannot be accurately specified, that is, we cannot determine by how much each discordant DM needs to adjust the upper and/or lower limits of their initial expected carbon quota intervals. To overcome these deficiencies, we further introduce a fairness measure variable α expressed as $|H_i - H_j| \leq \alpha(\alpha \geq 0, i < j, i, j \in N)$, into the previous benchmark consensus trading model, so as to directly obtain the optimal carbon quota allocation scheme due to the fairness consideration.

During the establishment of the new consensus trading models, we find out that our proposed models that maximizes the individual DM's revenue is essentially a non-convex optimization problem with too many decision variables to be determined. After realizing that two kinds of decision variables actually have no effect on the objective function, we first remove them and obtain the corresponding relaxation models, which are still non-linear optimization models. As a result, we present the following relaxation method based on the well-known PSO algorithm (see Algorithm 1) to solve those non-linear consensus optimization models.

Algorithm 1 Relaxation method based on the PSO algorithm for solving the single DM's revenue maximization model.

Input: Number of the DMs, N; d_i's initial carbon quota, o_i; d_i's initial fixed unit revenue, r_i; d_i's expected carbon quota interval, $[o_i^-, o_i^+]$; the maximum overall revenue obtained from the benchmark consensus model, Z_1; the maximal number of iterations, *limit*; population size, M.

Output: d_i's final carbon quota, $\bar{o}_i$; d_i's unit selling and buying prices, p_i, q_i; the transferred quantity, I_{ij}; the unit transaction price, T_{ij}; the specific DM's maximum total revenue, Z_2.

Step 1: Remove decision variables p_i, q_i that irrelevant with the objective function to obtain a relaxation optimization model;

Step 2: Use the PSO algorithm to solve the relaxed model;

Set current iteration as $t = 0$;
for each particle i **do**
    Initialize velocity V_i and position X_i for particle i;
    Evaluate particle i by the defined fitness function and set $pBesti = X_i$;
end for
$gBest$=min $\{pBesti\}$;
while $t < limit$ **do**
    for $i = 1$ to M **do**
        Update the velocity and position of particle i;
        Evaluate particle i by the defined fitness function;
        if $fit(X_i) < fit(pBesti)$ **then return** $pBesti$=X_i;
        end if
        if $fit(pBesti) < fit(gBest)$ **then return** $gBest = pBesti$;
        end if
    end for
end while
Z_2= $-fit(gBest)$;
return The optimal solution of $\bar{o}_i, I_{ij}, T_{ij}, Z_2$.

Step 3: Derive the optimal values of p_i, q_i based on the original relaxation constraints.

For clarity, we next present the following specific procedure to demonstrate the modeling ideas proposed in the third topic of this thesis.

(1) Referring to Section 2.1.2, a carbon trading optimization model is built to achieve the overall revenue maximization, i.e., to obtain the optimal carbon quota allocation scheme for different regions from the collective perspective. Specifically, the carbon quota quantities transferred among all DMs and the maximum value of the final total revenue of the system are acquired.

(2) Using the maximum overall revenue obtained in Step 1, and by adding the constraint of the unit transaction price, a series of consensus optimization models are built based on the benchmark model. Hence, a total of n allocation schemes are derived by maximizing each DM's own revenue, and detailed information such as d_i's unit buying and selling prices, transferred quantities, and unit transaction prices is obtained.

(3) Through a comparison of the individual/group development indices, it can be determined whether all DMs have developed fairly or not. If not, some discordant DMs are identified by the pre-defined threshold γ, then their initial parameters are adjusted accordingly. Next, repeat Step 1 and 2 until the allocation scheme satisfies the fairness requirement.

(4) Introduce the fairness measure variable α to build a new consensus model based on the initial benchmark one, so as to directly obtain fair allocation schemes for all the DMs with a maximum overall revenue, quantities of carbon quota transferred, and the unit transaction

prices. Additionally, a sensitivity analysis is applied to α to provide flexible suggestions for the moderator involved in the trading system.

(5) Conduct a comparison and discussion based on the results obtained in each step.

The journal article associated with this section is:

- X.X. Xu, Z.W. Gong, W.W. Guo, Z.M. Wu, E. Herrera-Viedma, F.J. Cabrerizo. Optimization consensus modeling of a closed-loop carbon quota trading mechanism regarding revenue and fairness, Computers & Industrial Engineering, 2021, 161: 107611.

7 Discussion of results

This section briefly discusses the results obtained in each topic of this PhD dissertation.

7.1 The research on the MCCMs under linear uncertain-constrained scenarios

Due to serious deterioration of the global environment, the reduction of carbon emission has become a key measure to improve the ecological system, so we choose the application of the carbon quota negotiation to verify the feasibility of the proposed models. Results show that the calculated values correspond to the analytic formulae of the optimal solutions under each scenario, verifying the correctness of the theorems obtained by theoretical deductions. Moreover, the results of the application indicate that traditional crisp number- or interval preference-based MCCMs are some special cases of the new uncertain MCCMs, suggesting that uncertainty theory can build a bridge between the deterministic and indeterministic GDM. Finally, we find that once the belief degree, set for the deviation of polluters' and government's quota indexes, is larger than the critical value of 0.5, then the optimal carbon quota consensus will be crisp numbers instead of uncertainty distributions. The above conclusion implies that only the belief degree is large enough, the GDM can achieve a deterministic consensus and the carbon quota negotiation can then be effectively conducted.

To show the novelties of our research, we also conduct a comparative analysis with existing studies: distinguished from previous research, we build the consensus models from three decision roles, by introducing non-cooperators into traditional MCCMs. Meanwhile, we first introduce Liu's uncertainty theory into consensus modeling, by adopting belief degree and uncertainty distribution as a whole to fit individual preferences, and find out the relations between the deterministic and indeterministic GDM through theoretical deductions. Finally, we apply the proposed models into the carbon emission quota allocation negotiation problem to verify their feasibility.

Inspired by the fact that flexible management has been a premiere goal pursued by the Chinese government, in order to encourage high-quality development of enterprises, the negotiation over the carbon emission quota allocation problem is chosen as our case background. In fact, when setting carbon emission reduction quotas for different enterprises with similar scales, it can better reflect the government's humanized management by setting uncertain indicators rather than some deterministic and fixed ones, which may also be understood as the practical significance of the uncertain constraints in this thesis. Without doubt, our newly proposed uncertain MCCMs can provide significant managerial implications for moderators to deal with those real-life GDM problems with flexible requirements, such as targeted recommendation system purchasing based on advertisers' market share, and second-hand housing selection bargain from different agencies.

7.2 The research on the CRP in the GDM with linear uncertain preferences and asymmetric costs

Herein, we utilize the trans-boundary water pollution negotiation of five cities located in the Yangtze River Delta under the governance of the Ministry of Water Resources (MWR) of China as our case background. Performing the CRP proposed in Section 6.2, results show that both the achieved consensus and the DM's optimal adjusted preferences degenerate to real numbers when the CRP is terminated by meeting the preset CL threshold. In other words, given the constructed form of the linear uncertainty distributions, all the final preferences have the same upper and lower bounds. Such findings are consistent with our first topic and the extant study on the uncertain MCCMs [GGX^{+}21]. That is, once certain conditions are met (e.g., the belief degree is no less than 0.5), the original linear uncertainty distributions degenerate to crisp numbers.

Besides, we perform parametric behavior analysis of the trade-off coefficient λ between the budget and the CL in the second new model (see Fig. (6)), and the controlling variable η of the DM's contribution to the CL on their weights using the first model (see Fig. (7)) proposed in Section 6.2 (i.e., the new uncertain MCCM with a preset CL threshold).

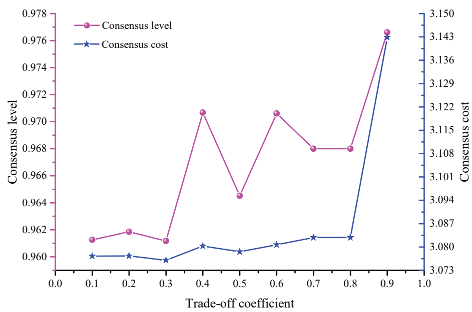


Figure 6: Parametric behavior analysis of λ

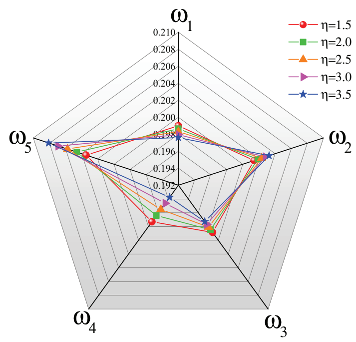


Figure 7: Changes of weights with η

Fig. 6 shows that the final range of the CL is [0.9612, 0.9766] and the total cost is [3.0760, 3.1429] in light of λ. Once the two objectives (i.e., the total cost and the CL) are considered separately, that is, if only the resource is required to be least consumed or the maximum CL becomes the priority of the discussed GDM, the obtained results of these two extreme scenarios are far away from the rest, so they are omitted in Fig. 6. Overall, there are sharp fluctuations when $\lambda \in [0.3, 0.7]$, thus CL is of great necessity for the GDM, where detailed reasons remain to be further explored; although such fluctuations are negligible once taking the numerical scale into account. Besides, the consistent volatility trends of the CL and the cost suggest that increasing budget helps improve CL, but their conflict within [0.6, 0.7] might due to the complete transformation of linear uncertainty distributions into crisp numbers with $\lambda = 0.7$ or factors ignored in this thesis.

Fig. 7 shows that once the controlling variable η of the DM's CL contribution to their new weight gets larger, the more obvious the differences in weight distribution. In other words, the smaller the value of η, the more even of all DMs' weights, and the less differences among individual influence on the final decision. Based on the final numerical results, only minor changes exist in the optimal values of the consensus cost and the CL, thus, the impact of η on the final results is actually negligible here.

Finally, using the original data and directly setting the CL threshold with the final expected value for the first new model, namely, feedback mechanisms are no longer considered, we find out that the uncertain MCCM still works without feedback mechanisms involved. However, the cost required to achieve the same CL is less with the CRP considered, implying that the feedback mechanism can promote a higher cost-effective consensus.

7.3 The research on the consensus modeling of a closed-loop trading mechanism regarding revenue and fairness

To verify the rationality and effectiveness of the proposed models in the third topic, we have considered the example of carbon quota trading among five regions. The novel consensus models can derive the optimal allocation scheme from the global perspective (i.e., the moderator's perspective in the GDM), and can also obtain allocation schemes from different DM's perspectives, in which the maximization of each region's revenue is the modeling goal. We derive the following findings based on the obtained results as:

- Consensus modeling to maximize the overall revenue can obtain the optimal allocation scheme for the whole group, but cannot identify specific pricing decisions. Moreover, the final carbon quotas of different regions obtained from the models that maximize each region's revenue are the same as those obtained from the former modeling mechanism. That is, the optimal values of the DM's final holding quotas are fixed. However, detailed trading information (e.g., the trading regions involved) change with the specific region being studied.
- The unit selling and buying prices of each region derived from the proposed consensus optimization models do not change according to which region's revenue is being maximized and do not depend on the value of the fairness measure variable, implying indirectly that the carbon quota trading mechanism discussed in this thesis is robust to some extent.
- For the two strategies proposed to solve the unfairness in the trading system, adjusting the initial parameters of discordant regions is effective, but complicated in practice. In addition, the parameter γ for identifying discordant regions, the adjustment range for each region, and whether the final allocation scheme meets the GDM requirements are all subjective. In contrast, the strategy of directly introducing the fairness measure variable α is convenient and effective, and further sensitivity analysis enables feasible allocation schemes to be obtained.
- The introduction of the fairness measure variable increases the number of trading paths among different regions, meaning that absolute fairness within the closed-loop system is realized only when carbon quotas are fully traded among different regions. Thus, sufficient interactions among participators are highly significant in achieving consensus or the pursuit of the DMs' balanced development during a GDM process.

8 Concluding remarks

In this section, we present the main conclusions obtained from each topic carried out in this PhD dissertation. All our research follows a common goal of building new consensus optimization models with various behavioral constraints to solve the GDM problems in real-life. More specifically,

The first subject is to adopt the linear uncertainty distribution to fit individual judgements, and to propose a series of uncertain MCCMs. To do that, we build the optimization-based MCCMs from multiple decision roles (i.e., the moderator, the individual DMs, and the non-cooperators). Among which, belief degree and uncertainty distributions are used as a whole to simulate the DMs' preference structure, making the new models more feasible than those traditional ones (i.e., crisp number- or interval preference-based MCCMs), better avoiding the paradox in interval operations (e.g. $[1,3]-[1,3]=[-2,2]\neq[0,0]$), and maintaining the integrity of decision information by analyzing individual uncertain opinions as a whole instead of only endpoints being considered. Meanwhile, based on the transformed equivalent linear programming models, the analytic formulae of the optimal consensus and minimum total cost under each scenario are given via mathematical deduction. It is verified that the uncertain preference-based MCCMs are more inclusive than those traditional ones due to the basic conclusions of the crisp number- or interval preference-based models are some special cases of the uncertain MCCMs under different belief degrees, meaning that the MCCMs combined with uncertainty theory is more flexible in the actual GDM.

The second topic aims to extend the above uncertain MCCMs into a new CRP framework. In specific, two new consensus models are built by considering asymmetric costs, aggregation function and consensus measure, where the DM's preference is fit by the linear uncertainty distribution and the setting of asymmetric costs is further rationalized based on prospect theory. Moreover, a new CRP is designed by respecting the DM's values with democratic consensus and minimizing resources with new uncertain MCCMs. To avoid the calculation complexity from piecewise functions in the uncertain distance measure, binary variables are introduced to transform the multi-coefficient goal programming models in view of the big M method. Furthermore, we find out that (1) the new consensus models exclude moderator's influence by setting the CL threshold with a benchmark from initially provided information, or by providing a full relationship between the CL and the cost via the second new model; (2) the CRP helps promote a higher cost-effective consensus; and (3) once certain conditions are met, the DMs' preferences fit by linear uncertainty distributions degenerate into crisp numbers, which is consistent with previous findings. Note that in addition to trans-boundary pollution management, our method is also feasible to handle other GDM problems characterized by non-randomness and non-fuzziness, such as urban demolition negotiation, trust evaluation in social networks or emergency management for natural disasters.

Our third contribution describes the use of the optimal consensus modeling theory to explore theoretical innovations regarding flexible carbon trading mechanisms. Specifically, we investigate essential carbon quota allocation schemes within a closed-loop trading system with the aim of ensuring both revenue maximization and fairness. First, the optimal carbon quota allocation scheme is derived by maximizing the overall revenue through a benchmark consensus model. Then, its analytical formula and the achievable conditions for successful trading are provided through theoretical deduction. Next, simultaneously taking the group revenue maximization and the competition mechanism into account, we build the new models to obtain the optimal allocation schemes by maximizing each individual's revenue. Since conflicts of interest are the main reasons for the failure of the GDM in the real world, we give the definitions of the individual or group development index, and further present two strategies to solve the unfair problems, where the former is based on calculating the difference between the development indices, with fairness achieved through the identification of the discordant DMs and the adjustment of their initial parameters; and the latter introduces a fairness measure variable, allowing fair allocation schemes to be directly obtained.

Finally, it is worth clarifying that the validity and the feasibility of all the consensus optimization models, theorems and methods proposed in this thesis are demonstrated by numerical examples and comparative analyses.

Conclusiones

En esta sección, presentamos las principales conclusiones obtenidas de cada uno de los temas llevados a cabo en esta tesis doctoral. Todas nuestras investigaciones persiguen el objetivo común de construir nuevos modelos de optimización consensuada con diversas restricciones de comportamiento para resolver problemas de GDM en la vida real. Concretamente:

El primer tema es adoptar la distribución lineal de incertidumbre para ajustar los juicios individuales, y proponer una serie de MCCM inciertos. Para ello, construimos los MCCM basados en la optimización a partir de múltiples roles de decisión (es decir, el moderador, los DM individuales y los no cooperadores). Entre ellos, el grado de creencia y las distribuciones de incertidumbre se utilizan como un todo para simular la estructura de preferencias de los DM, haciendo que los nuevos modelos sean más factibles que los tradicionales (es decir, los MCCM basados en números exactos o en intervalos de preferencia), evitando mejor la paradoja en las operaciones de intervalo (por ejemplo, $[1, 3] - [1, 3] = [-2, 2] \neq [0, 0]$), y manteniendo la integridad de la información de la decisión mediante el análisis de las opiniones individuales inciertas como un todo en lugar de considerar solo los puntos finales. Mientras tanto, a partir de los modelos de programación lineal equivalentes transformados, se obtienen por deducción matemática las fórmulas analíticas del consenso óptimo y del coste total mínimo en cada escenario. Se verifica que los MCCM basados en preferencias inciertas son más inclusivos que los tradicionales debido a que las conclusiones básicas de los modelos basados en números exactos o en intervalos de preferencias son algunos casos especiales de los MCCM inciertos bajo diferentes grados de creencia, lo que significa que los MCCM combinados con la teoría de la incertidumbre son más flexibles en el GDM real.

El segundo tema pretende ampliar los MCCM inciertos anteriores a un nuevo marco de CRP. En concreto, se construyen dos nuevos modelos de consenso teniendo en cuenta los costes asimétricos, la función de agregación y la medida de consenso, donde la preferencia de la DM se ajusta a la distribución lineal de incertidumbre y el establecimiento de costes asimétricos se racionaliza aún más sobre la base de la teoría de la prospectiva. Además, se diseña un nuevo CRP respetando los valores del DM con un consenso democrático y minimizando los recursos con nuevos MCCM inciertos. Para evitar la complejidad de cálculo de las funciones a trozos en la medida de distancia incierta, se introducen variables binarias para transformar los modelos de programación por objetivos multicoeficientes teniendo en cuenta el método de la gran M. Además, descubrimos que (1) los nuevos modelos de consenso excluyen la influencia del moderador estableciendo el umbral de CL con un punto de referencia a partir de la información proporcionada inicialmente, o proporcionando una relación completa entre la CL y el coste a través del segundo modelo nuevo; (2) la CRP ayuda a promover un consenso más rentable; y (3) una vez que se cumplen ciertas condiciones, las preferencias de los DMs ajustadas por distribuciones lineales de incertidumbre degeneran en números exactos, lo que concuerda con hallazgos anteriores. Cabe señalar que, además de la gestión de la contaminación transfronteriza, nuestro método también permite abordar otros problemas de GDM caracterizados por la ausencia de aleatoriedad y de incertidumbre, como la negociación de la demolición urbana, la evaluación de la confianza en las redes sociales o la gestión de emergencias en caso de catástrofes naturales.

Nuestra tercera contribución en esta tesis describe el uso de la teoría de modelos de consenso de optimización para explorar innovaciones teóricas relativas a mecanismos flexibles de comercio de carbono. En concreto, investigamos esquemas esenciales de asignación de cuotas de carbono dentro de un sistema de comercio de ciclo cerrado con el objetivo de garantizar tanto la maximización de los ingresos como la equidad. En primer lugar, se deriva el esquema óptimo de asignación de cuotas de carbono maximizando los ingresos globales a través de un modelo de consenso de referencia. A continuación, se deducen teóricamente su fórmula analítica y las condiciones necesarias para el éxito del comercio. Posteriormente, teniendo en cuenta simultáneamente la maximización de los ingresos del grupo y el mecanismo de competencia, construimos los nuevos modelos para obtener los esquemas óptimos de asignación maximizando los ingresos de un individuo. Dado que los conflictos de intereses son las principales razones del fracaso de la GDM en el mundo real, damos las definiciones del índice de desarrollo individual o de grupo, y además presentamos dos estrategias para resolver los problemas injustos, donde la primera se basa en el cálculo de la diferencia entre los índices de desarrollo, lográndose la equidad mediante la identificación de DM discordantes y el ajuste de sus parámetros iniciales; y la segunda introduce una variable de medida de la equidad, permitiendo obtener directamente esquemas de asignación justos.

Por último, conviene aclarar que la validez y la viabilidad de todos los modelos, teoremas y métodos de optimización del consenso propuestos en esta tesis se demuestran mediante ejemplos numéricos y análisis comparativos.

9 Future works

After presenting the main contributions of this PhD dissertation, several new topics to further extend our research come out. In what follows, several interesting research directions are suggested as follows, which are worth exploring in the near future.

9.1 Parameters rationalization in the consensus optimization models

This thesis explores some real-life GDM problems via the consensus optimization models originated from the MCC concept in [BAE07], such as the carbon quota allocation between the government and the heavily polluting enterprises, the trans-boundary water pollution negotiation, or the close-loop carbon emission rights reallocation trading. It is undeniable that, we focus only on the essential mechanisms during all the optimal modeling, that is, we construct those optimization models with some highly abstracted forms to reduce the complexity of modeling and computation. For example, the unit costs or the unit prices attached to each DM are set as constants, even in our second topic to consider the characteristics of asymmetry due to the DM's preference adjustment directions, which are slightly contrary to the reality. Thus, in the future, we may adopt some robust methods, such as game theory [ZDZP20] or stochastic programming theory [LJGQ21], to assure those parameters to be more reasonable. For example, we could set the variable unit costs with time constraints for the DMs to deal with the phenomena under which tiered pricing is set for heavily polluting enterprises after overrun, so as to be more in line with practical situations.

9.2 Optimization modeling under the complex GDM environments

In this thesis, the consensus reaching conditions are only explored under the homogenous GDM background, for example, the DMs express their preferences with only linear uncertainty distributions with belief degrees in our first two topics or crisp numbers in the last topic, however, real-life decision is rather complex and changeable, making it highly possible for involved participants to simultaneously present completely different preference structures. Thus, in the future, we need to deal with the heterogeneous GDM problems [HVHC02, CHVP13] by taking other common regular uncertainty distributions (e.g., zigzag uncertainty distribution or normal uncertainty distribution) or traditional preference structures (e.g., intervals, utility values, linguistic preferences) into account, although it brings about another topic that we should explore the axiomatic approach of the conversion between the uncertain and traditional preference structures, thereby further extending our research in this thesis. Meanwhile, due to the rapid development of technology, decision problems tend to be more and more complex, thus, combining our existing research to address the large-scale GDM problems that featured by social interactions or opinion dynamics [DPW$^+$20], will also be an important direction of our subsequent research.

9.3 Practical application of the consensus optimization models

So far, we have conducted all our studies under some particular case background to facilitate the understanding of those proposed consensus optimization models, thus, how to integrate well-known decision technology (e.g., survey, data mining) or empirical approaches to practically interpret our findings with some real datasets is worth investigating. In addition, the optimal consensus modeling or the trading mechanisms should focus on more behavioral constraints, such as the participant's risk attitudes or utility expectations [HCK15], rather than only considering the reallocation schemes or the pricing decisions from the MCC perspective or the revenue maximization perspective. Clearly, our newly proposed uncertain MCCMs can provide significant managerial implications for moderators to deal with the real-life GDM problems with flexible requirements, such as targeted recommendation system purchasing based on advertisers' market share, and second-hand housing selection bargain from different agencies. Thus, to better show the flexibility and feasibility of our new proposals, except carbon quota negotiation or the trans-boundary water pollution negotiation mentioned in this thesis, the application of our consensus optimization models in more real GDM problems needs to be studied and discussed in the future.

Chapter II

Publications: Published Papers

1 Minimum cost consensus modelling under various linear uncertain-constrained scenarios

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Minimum Cost Consensus Modeling under Various Linear Uncertain-constrained Scenarios

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Abstract

Group decision-making combined with uncertainty theory is verified as a more conclusive theory, by building a bridge between deterministic and indeterministic group decision-making in this paper. Due to the absence of sufficient historical data, reliability of decisions are mainly determined by experts rather than some prior probability distributions, easily leading to the problem of subjectivity. Thus, belief degree and uncertainty distribution are used in this paper to fit individual preferences, and five scenarios of uncertain chance-constrained minimum cost consensus models are further discussed from the perspectives of the moderator, individual decision-makers and non-cooperators. Through deduction, reaching conditions for consensus and analytic formulas of the minimum total cost are both theoretically given. Finally, with the application in carbon quota negotiation, the proposed models are demonstrated as a further extension of the crisp number or interval preference-based minimum cost consensus models. In other words, the basic conclusions of the traditional models are some special cases of the uncertain minimum cost consensus models under different belief degrees.

Keywords: Group decision-making; Minimum cost consensus model (MCCM); Uncertainty theory; Linear uncertainty distribution; Belief degree

1. Introduction

Group decision-making (GDM) mainly solves unstructured decision-making problems, involving subjective participation of various experts [1, 2]. In GDM, through communication and multiple rounds of effective feedback/adjustment, decision-makers (DMs) eventually form a clear support or objection towards a certain issue. Then, a relatively consistent consensus is reached [3]. Consensus

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decision-making is a prerequisite for effective GDM and widely exists in our daily lives, such as online P2P lending [2], emergency decision support [4], and trans-boundary water pollution control [5, 6]. In general, factors affecting the consensus reaching process (CRP) include DMs' preference structures or psychological expectation [6, 7, 8, 9], convergence rules [10, 11], decision environment [12, 13, 14, 15, 16, 17], and leaders' [2, 18, 19, 20] or non-cooperators' influence [1, 21, 22, 23, 24]. Urda and Loch [25] indicated that individual behaviours in GDM are driven by both their own economically rational deliberation and decision biases and social preferences (e.g. status achievement, reciprocal relations, or group identity). Thus, a moderator [2, 19, 20], on behalf of collective interest, is often introduced to improve the speed and efficiency of CRP. He/she possesses prominent skills in leadership and negotiation, and can persuade/tempt DMs to continually adjust their opinions into consensus through different effective means (collectively referred to as "consensus cost").

The concept of minimum cost consensus model (MCCM) was first proposed by Ben-Arieh and Easton [20], to explore a single and a multi-criteria decision consensus problem with a linear cost using linear-time algorithms. Afterward, they built models based on quadratic cost functions by taking account of consensus cost, opinion elasticity and the maximum number of experts [26]. Meanwhile, Dong et al. [10] investigated the internal relations of several OWA-based linguistic operators based on position indexes, and originally presented the optimisation-based minimum adjustment consensus models (MACMs). Subsequently, Zhang et al. [27] proposed a new framework for consensus models under aggregation operators, and illustrated that a link existed between MCCMs and MACMs. To further explore the original MCCMs, Gong et al. [19, 28] and Zhang et al. [2] adopted the linear prime-dual theory and presented the economic interpretations of their new consensus models. Wu et al. [29] discussed the scheme recommendation and users' trust measure using the feedback mechanism in MCCMs with social network analysis. Meanwhile, considering that the cost coefficients are asymmetric due to the adjustment direction of DMs' opinions, Cheng et al. [5] analysed the impact of individual limited compromises and tolerance behaviours on MCCMs. Research paradigms about the MCCMs/MACMs with feedback mechanism during the last decade were concluded by Zhang et al. [30], and they further pointed out new directions for the future research. So far, most extant MCCMs/MACMs assume DMs' preferences denoted by crisp numbers or intervals, making the stochastic distribution for DMs' opinions seldom considered. Thus, uncertainty distributions are used to fit DM's preferences in this paper.

Actually, even if there exists a moderator acting as a leader in GDM, the DMs involved still cannot account for all factors; besides, diversity widely exists in individuals' research background, knowledge reserve, and the amount of private information. Thus, GDM is full of uncertainty, making it unable to accurately predict the outcome in advance. GDM essentially includes providing decision support for solving uncertainty. Without loss of generality, theoretical methods for dealing with uncertainty include probability theory, interval analysis, fuzzy sets, rough sets and grey systems. However, it is often difficult to obtain a precise probability for a natural state in real-life GDM, especially when there is little information available for evaluating probabilities, usable information is insufficient, or when several information sources conflict with each other [31]; then, the reliability (or probability) that certain event will occur is primarily determined by experts. To handle situations where the reliable prediction that one event would occur has to be determined by individual subjectivity due to the inability to obtain its actual frequency [32], uncertainty theory was proposed by Liu [33], which gradually extended into a systematic subject, from a theoretical perspective [34, 35, 36, 37, 38] and an application perspective [39, 40]. As an important branch of mathematics [41], uncertainty theory is mainly used to deal with human beings' subjective reliability and has been successfully applied into trust measure in social networks [42]. To the best of the authors' knowledge, compared with multiple prominent theories dealing with indeterminacy, efficiencies and advantages of uncertainty theory in GDM are concluded in [43, 44].

Due to the uncertainty in DMs' opinions, traditional probability and statistics methods are no longer suitable for the preference analysis of individual behaviours involved in GDM, because frequency distribution, probability distribution, and density function for individual opinions are difficult to obtain. However, we can always grasp a certain degree of certainty, such as 95% confidence/belief, to achieve consensus, and when the consensus is reached with a certain degree of belief, CRP is more consistent with actual GDM situations. Therefore, this paper introduces the belief degree and uncertain variables to simulate DMs' judgement behaviours, and by combining the MCCMs and uncertainty theory, this paper extends traditional MCCMs into five scenarios from diverse roles as moderator, individual DMs, and non-cooperators. In GDM, decisions are always made before the realisation of individual preferences (i.e., random variables), so we suppose that the belief degree of the constraints satisfied is no less than a specified value. Such problems can be solved by chance-constrained goal programming [45]. As a stochastic programming method, chance-constrained problems can always be transformed into an equivalent deterministic mathematical model, making it convenient to obtain Pareto optimal solutions toward the original problems. In short, our main contributions are:

- Uncertain MCCMs are discussed from the perspectives of multiple roles, such as the moderator, individual DMs and non-cooperators;
- Since interval preference-based MCCMs take only endpoints into account, belief degree and uncertainty distribution are introduced as a whole to fit individual judgements, making the proposed models more feasible;
- Analytic formulas for both the optimal consensus and the total cost (i.e., the optimal solutions) under each scenario are presented, through linear transformation of the uncertain MCCMs.
- Feasibility of the new uncertain MCCMs is verified by the carbon emission quota negotiation conducted between the heavily polluting enterprises and the local government.

The rest of the paper is organised as follows. Section 2 recalls preliminaries on traditional consensus models (i.e., MCCMs or MACMs) and uncertainty theory. Inspired by the consensus modeling in [10, 19, 28], Section 3 adopts belief degree and uncertain variables to characterise DMs' preferences. In addition, by discussing five GDM scenarios, a series of optimisation-based consensus models are developed. General reaching conditions for the consensus under each scenario are also provided in this section through theoretical deduction. Subsequently, Section 4 verifies the feasibility of the proposed models through the optimal carbon quota allocation negotiation between heavily polluting enterprises and the local government. Finally, concluding remarks and future research directions are presented in Section 5.

2. Preliminaries

2.1. Consensus models with a minimum cost or adjustment

Suppose there exist n DMs participating in GDM, $o_i \in R$ is the original opinion of DM d_i, $i \in N = \{1, 2, \cdots, n\}$ and o' is the collective opinion reached by the whole group (i.e., consensus). Let $f_i(o') = |o' - o_i|$ be the rectilinear distance measure between d_i's original opinion and the consensus [20]. Generally, reaching a consensus depends largely on behaviours of DMs [46], meanwhile high-impact moderators [2] or opinion leaders [18] can effectively promote the speed and efficiency of CRP. Particularly, by exercising significant leadership skills or scheduling limited resources (e.g., human, material, or financial resources), moderators are capable of guiding or coordinating with DMs to change individual inconsistent opinions towards a relatively consistent group opinion. Based on the above distance measure, Ben-Arieh and Easton [20] first put forward the concept of the minimum cost consensus, aiming at minimizing resources consumption during decision-making process; meanwhile, Dong et al. [10] initially proposed consensus models with minimum preference adjustment (i.e., MACMs) by introducing aggregation operators, aiming to preserve DMs' original preference information as much as possible. Subsequently, the two aforementioned modeling ideas become an important foundation of most extant consensus works (e.g., [2, 5, 6, 23, 27, 29, 30, 46]).

This paper mainly pursue the goal of minimizing the total consensus cost instead of keeping the original preference information as much as possible. Without loss of generality, let ω_i denote the cost for moving d_i's original opinion o_i towards the consensus o' one unit. In fact, the main difference between MCCMs and MACMs lies in whether considering the unit cost or not. Mathematically, if we normalize these unit costs, then they become the weighted arithmetic mean operators, which can also be understood as each individual's influence on CPR [28]. In reality, too many uncertain factors need to be considered in GDM, making the above parameters difficult to quantify, hence, ω_i is subjectively determined in the follow-up discussion of this paper. Zhang et al. [46] presented a bi-level optimization model to describe the interaction behaviors within CRP based on Stackelberg game, and further provided an optimal unit cost from a pre-defined reasonable range rather than assuming ω_i as a known parameter. Anyway, $\omega_i f_i(o')$ and $\sum\limits_{i=1}^{n} \omega_i f_i(o')$ indicate the costs paid by the moderator for persuading individual d_i and all DMs to change their inconsistent opinions during GDM, respectively.

$$
\begin{aligned}
& Min\ \phi = \sum_{i=1}^{n} \omega_i f_i(o') \\
& s.t. \begin{cases} f_i(o') = |o' - o_i| \\ |o' - o_i| \le \varepsilon_i,\ i \in N \end{cases}
\end{aligned}
\tag{1}
$$

Since the less the total cost the better, a MCCM based on the above principles is built as Model (1) [19, 20, 27], where ϕ represents the total consensus cost for the whole GDM, and ε_i is the upper bound of the deviation (i.e., distance measure) between d_i's opinion and the optimal collective opinion, implying that we want to obtain an acceptable consensus (i.e., soft consensus [27, 47]).

2.2. Uncertainty theory

Uncertainty widely exists in real-life GDM, for instance, when faced with emergency, human beings usually cannot determine the occurrence frequency of certain events due to the absence of historic data, making it difficult to accurately estimate the probability distribution of such events. Aiming at the above limitations in classical probability theory, uncertainty theory proposed by Liu [33, 48] is an important and useful mathematical instrument to handle uncertain phenomenon with non-randomness and non-fuzziness. Next, some basic concepts in uncertainty theory are introduced.

Let Γ be a nonempty set (sometimes referred as universal set), and a collection $\mathcal{L}$ consisting of subsets of Γ is an algebra over Γ, if it meets the following three conditions: (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^C \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, we have $\bigcup_{i=1}^{n} \Lambda_i \in \mathcal{L}$, where, if condition (c) is replaced by closure under countable union, that is, if $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, we obtain $\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}$, then $\mathcal{L}$ is referred as a σ-algebra over Γ. Element Λ in $\mathcal{L}$ is called a measurable set, which also can be interpreted as an event in uncertainty theory. M is defined as an uncertain measure over the σ-algebra $\mathcal{L}$. Without loss of generality, real number $M\{\Lambda_i\}$ corresponds to event Λ_i one by one, representing the belief degree with which we belief event Λ_i will occur. There exist no doubt that such assignment is not arbitrary, and the uncertain measure M satisfies the following four axioms [33, 48].

***Axiom* 1.** *(Normality Axiom): $M\{\Gamma\} = 1$ holds for the universal set Γ.*

***Axiom* 2.** *(Duality Axiom): $M\{\Lambda\} + M\{\Lambda^c\} = 1$ holds for any event Λ.*

***Axiom* 3.** *(Subadditivity Axiom): For every countable sequence of event Λ_1, Λ_2, $\cdots$, we have:*

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}$$

***Axiom* 4.** *(Product Axiom): Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertain space for $k \in N^+$, then the product of uncertain measure M is still an uncertain measure, and satisfies:*

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M\{\Lambda_k\}$$

where Λ_k are events arbitrarily chosen from $\mathcal{L}_k, (k \in N^+)$, respectively.

***Definition* 1.** [33] An uncertain variable ξ is a function from an uncertain space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers, and $\{\xi \in B\}$ is an event for any Borel set B of real numbers. For any real number x, the uncertainty distribution Φ of an uncertain variable ξ can be defined as: $\Phi(x) = M\{\xi \leq x\}$.

$M\{\xi \leq x\}$ is the belief degree for the event $\xi \leq x$ may occur, and it is denoted as α, where $0 \leq \alpha \leq 1$. In other words, we have $\Phi(x) = M\{\xi \leq x\} = \alpha$. According to Axiom 2, we obtain $M\{\xi > x\} = 1 - \Phi(x) = 1 - \alpha$.

***Definition* 2.** [41] An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and satisfies $\lim_{x\to-\infty} \Phi(x) = 0$ and $\lim_{x\to+\infty} \Phi(x) = 1$.

Note that, linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution and lognormal uncertainty distribution are all common regular uncertainty distributions.

***Theorem* 1.** *[41] Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distribution $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \cdots, \xi_m$, and strictly decreasing with respect to $\xi_{m+1}, \cdots, \xi_n$, then $f(\xi_1, \cdots, \xi_n)$ has an inverse uncertainty distribution of $\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))$.*

***Theorem* 2.** *[33] Let ξ be an uncertain variable with its inverse uncertainty distribution denoted as $\Phi^{-1}(\alpha)$, if and only if $\Phi^{-1}(\alpha) \leq c$, then $M\{\xi \leq c\} \geq \alpha$, where α, c are constants within $[0, 1]$.*

***Theorem* 3.** *Let uncertain variables ξ_1 and ξ_2 be independent with inverse uncertainty distribution Φ_1 and Φ_2, respectively, then the inverse uncertainty distribution for the difference between these two variables (denoted by $\xi_1 - \xi_2$) can be defined as: $\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1-\alpha)$.*

Actually, uncertain measure can be understood as DMs' personal belief degree (not frequency) of an event may occur, so the real meanings of belief degree and uncertain measure appear to be the same. Generally, regular uncertainty distributions include linear uncertainty distribution, normal uncertainty distribution and so on. Hereafter, we only discuss the linear type since it can be easily transformed when the analytic formulas of the proposed models are to be obtained.

***Definition* 3.** [33] Uncertain variable ξ satisfies a linear uncertainty distribution (see Fig. 1), denoted as $\xi \sim \mathcal{L}(a, b)$, where a, b are both real numbers and $a < b$, then linear uncertainty distribution function is presented as:

$$\Phi(x) = \begin{cases} 0, & if\, x \leq a \\ \dfrac{x-a}{b-a}, & if\, a \leq x \leq b \\ 1, & if\, x \geq b \end{cases}$$

***Definition* 4.** [33] An uncertain variable ξ satisfies $\xi \sim \mathcal{L}(a, b)$, then its inverse uncertainty distribution function (see Fig. 2) is expressed as:

$$\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b$$

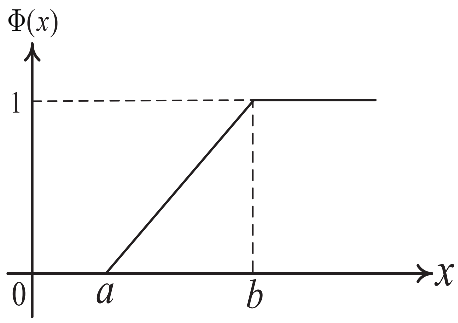


Fig. 1 Linear uncertainty distribution

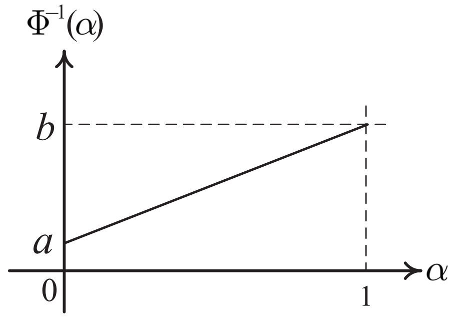


Fig. 2 Inverse linear uncertainty distribution

3. MCCMs with uncertain preferences

Participants' preferences in original MCCMs or MACMs are usually denoted by crisp numbers, without taking into account that their opinions fit by random distributions. In fact, it is often difficult for individuals to provide exact values as their preference, especially in some complex GDM contexts (e.g. social network GDM [6, 22, 29], large-scale GDM [1, 12, 14, 24] or GDM with dynamic opinions [18, 49]). Thus, DMs are more likely to present their decisions by intervals with upper and lower bounds or various uncertainty distributions (e.g. uniform uncertainty distribution or normal uncertainty distribution). Previous research focus on either role combination with moderator and individual DMs [2, 5, 20, 28] or independent decision-making status as moderator [18, 50], individual DM [10] or non-cooperators [1, 22, 23, 24]. Few extant works have build MCCMs by simultaneously taking account on three roles altogether. Given the above points, we utilise uncertainty distributions to denote participants' decision preferences, and by discussing five scenarios from multiple decision-making roles (i.e., moderator, individual DMs and non-cooperators), we aim to investigate a more general form of Model (1).

$$\begin{aligned} & Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\ & s.t.\ \{\ |o' - o_i| \leq \varepsilon_i, \varepsilon_i \geq 0,\ i \in N \end{aligned} \tag{2}$$

To introduce uncertainty theory into soft consensus decision-making, we obtain a further abstracted form from Model (1), which is denoted as Model (2). In specific, decision variable in Model (1) only includes o', while ε_i is a pre-defined threshold set over the distance measure between o' and o_i, $i \in N$. Meanwhile, decision variables in Model (2) include both o' and ε_i, and ε_i is bound by the deterministic threshold given in Model (1) under the premise that these parameters are set as same in both models. Obviously, the feasible domain of the solution set of Model (2) is larger than that of Model (1), making the optimal value of the objective function in Model (2) be no larger than that in Model (1). As a result, although the form of Model (2) is simpler, its scope of application is wider than Model (1). Furthermore, Model (2) becomes the basis of the following consensus models with uncertain variables.

3.1. Moderator with uncertain preference

Assume the original opinion o_i is a known crisp number presented by individual d_i and ω_i is a pre-defined unit cost paid by the moderator for d_i's change amount towards consensus o', $i \in N$. Since the moderator needs to consider many uncertain factors for the final convergent opinion, we assume that the moderator's opinion o' obeys uncertainty distribution. Based on Liu's uncertainty theory, if the deviation between the consensus o' and the individual opinion o_i is no more than ε_i under the belief degree α, then it can be denoted as $M\{o' - o_i \leq \varepsilon_i\} \geq \alpha$ and $M\{o' - o_i \geq -\varepsilon_i\} \geq \alpha$, where M represents the uncertain measure in uncertainty theory, and the variable $\alpha \in [0, 1]$ indicates the belief degree of the constraint of $|o' - o_i| \leq \varepsilon_i$ holding, $i \in N$. Accordingly, an MCCM with uncertain chance constraints can be constructed as follows:

$$
\begin{aligned}
& Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\
& s.t. \begin{cases} M\{o' \leq o_i + \varepsilon_i\} \geq \alpha \\ M\{o' \geq o_i - \varepsilon_i\} \geq \alpha \\ \varepsilon_i \geq 0,\ i \in N \end{cases}
\end{aligned}
\tag{3}
$$

***Theorem* 4.** *Model (3) is equal to the non-linear goal programming Model (4).*

$$
\begin{aligned}
& Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\
& s.t. \begin{cases} \varepsilon_i \geq \Phi^{-1}(\alpha) - o_i & (4-1) \\ \varepsilon_i \geq -\Phi^{-1}(1-\alpha) + o_i & (4-2) \\ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i \in N & (4-3) \end{cases}
\end{aligned}
\tag{4}
$$

where ϕ is the total budget for the consensus reached; ω_i is the unit-persuading cost paid by the moderator to DM d_i; the consensus o' obeys a linear uncertainty distribution as $o' \sim \mathcal{L}(a, b)$, where a and b are decision variables obeying an uncertainty distribution; and constraints (4-1) and (4-2) mean that the deviation between individual original opinion o_i and consensus o' is no more than ε_i under the premise of no less than an uncertain belief degree α. Clearly, the belief variable α ($\alpha \in [0, 1]$) can be a pre-determined fixed value or a decision variable to be solved.

Thus, if we reconsider the uncertain belief degree α, Model (3) or Model (4) essentially includes two issues: α is a pre-determined value or α is a parameter to be determined. As for the latter situation, the variable α solved by Model (3) or Model (4) will be an optimal belief degree in GDM. Besides, when $o' \sim \mathcal{L}(a, b)$, Model (4) can be further transformed into a linear programming model as in Corollary 1.

***Corollary* 1.** *Assuming that the consensus opinion obeys a linear uncertainty distribution as $o' \sim \mathcal{L}(a, b)$, Model (4) is equivalent to the following optimisation model:*

$$
\begin{aligned}
& Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\
& s.t. \begin{cases} \varepsilon_i \geq (1-\alpha)a + \alpha b - o_i \\ \varepsilon_i \geq -\alpha a - (1-\alpha)b + o_i \\ a \leq b,\ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i \in N \end{cases}
\end{aligned}
\tag{5}
$$

***Corollary* 2.** *If and only if $\alpha = 1$ and o' is a crisp number (i.e. $a = b$), Model (5) degenerates into Model (2), and under this situation, the two models have identical values of the optimal consensus and minimum consensus cost.*

With the constraint of $o' \sim \mathcal{L}(a, b)$, we are to discuss when the total budget and consensus opinion are exactly the same as that solved by Model (2), and to gain the threshold of the belief degree α once the consensus is reached. In fact, by conducting sensitivity analysis on variable α, we obtain the analytical formulas of the optimal solutions for Model (5), namely, we aim to explore the conditions under which Model (5) and Model (2) have identical optimal consensus and total budget.

***Theorem* 5.** *Assume DM's original opinions in Model (5) are arranged in order (i.e. $o_1 \leq o_2 \leq \cdots \leq o_n$), weights attached to each DM (i.e. $\omega_{i,\ i \in N}$) are different, and moderator's opinion obeys a linear uncertainty distribution as $o' \sim \mathcal{L}(a, b)$, where a and b are decision variables (see Section 2.2). Once the belief degree satisfies $\alpha \geq 0.5$, the optimal objective value and consensus reached conditions for Model (5) are:*

$$\phi^* = min \sum_{i=1}^{n} \omega_i \varepsilon_i = \begin{cases} \sum\limits_{i=m+1}^{n} \omega_i o_i - \sum\limits_{i=1}^{m} \omega_i o_i, \ a = b \in [o_m, o_{m+1}] \\ \qquad iff \ \sum\limits_{i=1}^{m} \omega_i = \sum\limits_{i=m+1}^{n} \omega_i \\ \sum\limits_{i=m+1}^{n} \omega_i (o_i - o_m) + \sum\limits_{i=1}^{m} \omega_i (o_m - o_i), \ a = b = o_m \\ \qquad iff \ \sum\limits_{i=1}^{m-1} \omega_i < \sum\limits_{i=m}^{n} \omega_i, \sum\limits_{i=1}^{m} \omega_i > \sum\limits_{i=m+1}^{n} \omega_i \end{cases}$$

Proof. See Appendix A. □

Remark 1. Theorem 5 shows that once $\alpha \geq 0.5$, the optimal consensus and total cost will be constants and irrelevant with the belief degree any more. By then, Model (5) with linear uncertain preferences is equivalent to Model (2) with preferences denoted by crisp numbers. That is, the two models have identical minimum budget and optimal collective opinions. Above findings verify that the uncertain MCCMs proposed do have practical meanings.

3.2. Non-cooperators considered and moderator with uncertain preference

So far, non-cooperators' impact on MCCMs has gradually become an intriguing topic [23], particularly under some complex GDM contexts [1, 21, 22, 24], and most of those research are analysed by theoretical modeling and simulation experiments. Thus, without loss of generality, suppose multiple individuals have similar preferences or interest in GDM, while some non-cooperators insist on their own opinions for certain reasons, who may have authority power within industries or districts, making the moderator unable to ignore their demands. Under this scenario, moderator's budget is mainly used to persuade these non-cooperators for compromising. MCCMs discussed here correspond to the decision rule of minority being subordinate to majority. For example, a certain district is stepping into the final stage of China's urban demolition process, a large amount of local citizens have agreed to move while few nail-house holders insist to stay put, probably for more compensation from the government or for some stuff hard to let go. Then, the government has to schedule some extra budget to pursue better development for the whole district. Such phenomenon can be modeled as:

$$Min\ Z = \sum_{i=1}^{t} \omega_k \varepsilon_k$$
$$s.t. \begin{cases} f_k(o') \leq \varepsilon_k, k \in \{1, 2, \ldots, t\} & (6-1) \\ f_i(o') \leq \varepsilon_i, i \in N \backslash k & (6-2) \\ \varepsilon_i \geq 0,\ i \in N & (6-3) \end{cases} \tag{6}$$

Model (6) assumes there exist a total of t non-cooperators (denoted as d_k). Once a consensus is reached, the change amount of d_k's opinion is $f_k(o') = |o' - o_k|$ and his/ her unit cost paid by the moderator is ω_k, then the total consensus cost for this GDM scenario is Z. Note, $i \in N \backslash k$ means that excluding those non-cooperators, individuals belong to a small alliance where they may have similar interest or have already reached a temporary consensus. Model (6) is a general form of GDM with non-cooperators, however, situations with only one non-cooperator is discussed hereafter (i.e., Model (7)), for simplicity and for easy to obtain the analytic formulas of the uncertain MCCMs. In fact, when there exist no less than two non-cooperators, the modeling mechanism is similar and the optimal solutions can be easy to get by using softwares such as MATLAB.

$$Min\ Z = \omega_k \varepsilon_k$$
$$s.t. \begin{cases} f_k(o') \leq \varepsilon_k & (7-1) \\ f_i(o') \leq \varepsilon_i, i \in N, i \neq k & (7-2) \\ \varepsilon_i \geq 0,\ i \in N & (7-3) \end{cases} \tag{7}$$

By introducing uncertain chance constraints based on the removal of the absolute value symbols, (7-1) can be transformed as $M\{o' - o_k \leq \varepsilon_k\} \geq \beta$ and $M\{o' - o_k \geq -\varepsilon_k\} \geq \beta$, and (7-2) becomes the uncertain constraints as $M\{o' - o_i \leq \varepsilon_i\} \geq \alpha$ and $M\{o' - o_i \geq -\varepsilon_i\} \geq \alpha$, where β and α are the belief degrees imposed on d_k's and other DMs' opinion deviations with the consensus, respectively. Obviously, constraints (7-1) and (7-2) simultaneously define the threshold of the variable o'. Next, we obtain an equivalent non-linear consensus model.

$$Min\ Z = \omega_k \varepsilon_k$$
$$s.t. \begin{cases} \Phi^{-1}(\beta) \leq \varepsilon_k + o_k \\ \Phi^{-1}(1-\beta) \geq -\varepsilon_k + o_k \\ \Phi^{-1}(\alpha) \leq \varepsilon_i + o_i, i \in N, i \neq k \\ \Phi^{-1}(1-\alpha) \geq -\varepsilon_i + o_i, i \in N, i \neq k \\ 0 \leq \alpha,\ \beta \leq 1,\ \varepsilon_i \geq 0,\ i \in N \end{cases} \tag{8}$$

If the consensus obeys a linear uncertainty distribution with unknown parameters of a and b, namely $o' \sim \mathcal{L}(a, b)$, then Model (8) can be further extended as Model (9). For the convenience of comparative analysis with [19], this paper sets Model (9) as an MCCM with a soft-consensus constraint. That is, except for the non-cooperator, all other threshold constraints $\varepsilon_{i, i \in N, i \neq k}$ imposed on DMs' opinions and the final consensus are pre-determined. To make up for the deficiency of hard consensus [51], soft consensus, which allows for a certain range between individual opinions and the collective opinion, is proposed [2, 47, 20]. Generally, soft consensus can be measured by consensus level [22, 51]. Therefore,

Z, a, b and ε_k are all decision variables in Model (9).

$$
\begin{aligned}
& Min\ Z = \omega_k \varepsilon_k \\
& s.t. \begin{cases}
(1-\beta)a + \beta b - o_k \leq \varepsilon_k & (9-1) \\
-\beta a + (\beta - 1)b + o_k \leq \varepsilon_k & (9-2) \\
(1-\alpha)a + \alpha b - o_i \leq \varepsilon_i, i \in N, i \neq k & (9-3) \\
-\alpha a + (\alpha - 1)b + o_i \leq \varepsilon_i, i \in N, i \neq k & (9-4) \\
a \leq b,\ 0 \leq \alpha,\ \beta \leq 1, \varepsilon_i \geq 0,\ i \in N & (9-5)
\end{cases}
\end{aligned}
\tag{9}
$$

Constraints (9-1)-(9-4) create bounds on the parameters of a and b, which may lead to an empty solution space, that is, a feasible solution maybe no longer exist in Model (9). However, this situation makes sense in real-life GDM. For example, if a non-cooperator is no longer rational enough, then the urban demolition negotiation may bring to an end. Furthermore, we should note that once no feasible solution exists, then the roles of different DMs will change. Specifically, DMs other than d_k now have a veto power, and in fact are then more powerful than d_k, which may result in a new iteration for reaching a consensus. Currently, such scenarios haven't been analysed in this paper, but it will be an interesting topic in our future research. However, using conclusions in Theorem 6, we can always set certain pre-defined parameters in Model (9) to guarantee that a feasible solution exist.

***Theorem* 6.** *When belief degrees α and β in Model (9) satisfy the constraint (10). Then, if and only if*

$$
\frac{1}{2} \leq \beta \leq 1 \begin{cases}
0 \leq \alpha \leq \frac{1}{2} \begin{cases}
o_k > min\ (o_i + \varepsilon_i) \begin{cases}
\varepsilon_k^* = min\{(\beta - \frac{1}{2})(b-a), o_k - min(o_i + \varepsilon_i)\} \\
if\ \varepsilon_k^* = (\beta - \frac{1}{2})(b-a): \\
\quad a = \frac{min(o_i+\varepsilon_i) - 2\alpha o_k}{1-2\alpha},\ b = \frac{(2-2\alpha)o_k - min(o_i+\varepsilon_i)}{1-2\alpha} \\
if\ \varepsilon_k^* = o_k - min(o_i + \varepsilon_i): \\
\quad a = b = min(o_i + \varepsilon_i)
\end{cases} \\
max\ (o_i - \varepsilon_i) \leq o_k \leq min\ (o_i + \varepsilon_i) \begin{cases}
a = b = o_k \\
\varepsilon_k^* = 0
\end{cases} \\
o_k < max\ (o_i - \varepsilon_i) \begin{cases}
\varepsilon_k^* = min\{(\beta - \frac{1}{2})(b-a), max(o_i - \varepsilon_i) - o_k\} \\
if\ \varepsilon_k^* = (\beta - \frac{1}{2})(b-a): \\
\quad a = \frac{max(o_i-\varepsilon_i) + (2-2\alpha)o_k}{1-2\alpha},\ b = \frac{max(o_i-\varepsilon_i) - 2\alpha o_k}{1-2\alpha} \\
if\ \varepsilon_k^* = max(o_i - \varepsilon_i) - o_k: \\
\quad a = b = max(o_i - \varepsilon_i)
\end{cases}
\end{cases} \\
\frac{1}{2} \leq \alpha \leq 1 \begin{cases}
o_k > min\ (o_i + \varepsilon_i) \begin{cases}
a = b = min\ (o_i + \varepsilon_i) \\
\varepsilon_k^* = o_k - min\ (o_i + \varepsilon_i)
\end{cases} \\
max\ (o_i - \varepsilon_i) \leq o_k \leq min\ (o_i + \varepsilon_i) \begin{cases}
a = b = o_k \\
\varepsilon_k^* = 0
\end{cases} \\
o_k < max\ (o_i - \varepsilon_i) \begin{cases}
a = b = max\ (o_i - \varepsilon_i) \\
\varepsilon_k^* = max\ (o_i - \varepsilon_i) - o_k
\end{cases}
\end{cases}
\end{cases}
\tag{10}
$$

$a = b$, Model (9) degenerates into the $P_k(\varepsilon)$ problem in [19] (i.e. Model (11)), meaning that Model (9)

and Model (11) have identical optimal solutions, then the final collective opinion (i.e. the consensus) for Model (9) is also obtained.

$$
\begin{array}{l}
P_k(\varepsilon): \ Min \ Z = \omega_k |o' - o_k| \\
s.t. \begin{cases} |o' - o_i| \le \varepsilon_i, i \in N, i \ne k \\ o' \ge 0 \end{cases}
\end{array} \tag{11}
$$

Proof. See Appendix B. □

Remark 2. Theorem 6 provides consensus reaching conditions for MCCMs in light of the non-cooperator d_k and the consensus o' obeying a linear uncertainty distribution. And when $\alpha = \beta = 1$, Model (9) is equivalent to Model (7).

3.3. DMs with uncertain preferences

Suppose individual opinion $o_i = [a_i, b_i]$ obeys an uncertainty distribution, while the random distribution characteristics of the consensus is not considered (i.e., o' denoted as a crisp number). Similar to the aforementioned research idea, deviation between o_i and o' can be expressed using uncertain measure based on the removal of the absolute value symbols as $M\{o' - o_i \le \varepsilon_i\} \ge \alpha$ and $M\{o' - o_i \ge -\varepsilon_i\} \ge \alpha$, $(i \in N)$. Therefore, an optimisation-based MCCM with uncertain preferences is built as follows.

$$
\begin{array}{l}
Min \ \phi = \sum\limits_{i=1}^{n} \omega_i \varepsilon_i \\
s.t. \begin{cases} \Phi_i^{-1}(\alpha) \le o' + \varepsilon_i, \ i \in N \\ \Phi_i^{-1}(1-\alpha) \ge o' - \varepsilon_i, \ i \in N \\ o' \ge 0, \ 0 \le \alpha \le 1, \ \varepsilon_i \ge 0, \ i \in N \end{cases}
\end{array} \tag{12}
$$

If an individual opinion specifically obeys a linear uncertainty distribution, denoted as $o_i \sim \mathcal{L}(a_i, b_i)$, where a_i and b_i are predetermined parameters of d_i's original uncertain preference, exhibiting certain extent of indetermination, $i \in N$. Other variables are similarly defined as in Section 3.1. Model (12) then equals to Model (13).

$$
\begin{array}{l}
Min \ \phi = \sum\limits_{i=1}^{n} \omega_i \varepsilon_i \\
s.t. \begin{cases} (1-\alpha)a_i + \alpha b_i - o' \le \varepsilon_i, \ i \in N & (13-1) \\ o' - \alpha a_i - (1-\alpha) b_i \le \varepsilon_i, \ i \in N & (13-2) \\ o' \ge 0, \ 0 \le \alpha \le 1, \ \varepsilon_i \ge 0, \ i \in N & (13-3) \end{cases}
\end{array} \tag{13}
$$

Theorem 7. *Suppose original individual opinions in GDM satisfy linear uncertainty distributions as $o_i \sim \mathcal{L}(a_i, b_i)$, $i \in N$. If and only if $\frac{1}{2} \le \alpha \le 1$ and all opinions are organised in order as $\frac{a_1+b_1}{2} \le$*

$\frac{a_2+b_2}{2} \le \cdots \le \frac{a_n+b_n}{2}$, *analytic formulas of the objective function and the consensus are obtained as:*

$$\phi^* = \begin{cases} \sum\limits_{i=m+1}^{n} \omega_i[(1-\alpha)a_i + \alpha b_i] - \sum\limits_{i=1}^{m} \omega_i[\alpha a_i + (1-\alpha)b_i], \ where\ o' \in [\frac{a_m+b_m}{2}, \frac{a_{m+1}+b_{m+1}}{2}], \\ \qquad iff\ \sum\limits_{i=1}^{m} \omega_i = \sum\limits_{i=m+1}^{n} \omega_i \\ (\sum\limits_{i=1}^{m} \omega_i - \sum\limits_{i=m+1}^{n} \omega_i)\frac{a_m+b_m}{2} + \sum\limits_{i=m+1}^{n} \omega_i[(1-\alpha)a_i + \alpha b_i] - \sum\limits_{i=1}^{m} \omega_i[\alpha a_i + (1-\alpha)b_i], \ where\ o' = \frac{a_m+b_m}{2}, \\ \qquad iff\ \sum\limits_{i=1}^{m-1} \omega_i < \sum\limits_{i=m}^{n} \omega_i, \sum\limits_{i=1}^{m} \omega_i > \sum\limits_{i=m+1}^{n} \omega_i \end{cases}$$

Theorem 7 can be verified by a similar mechanism as for Theorem 5, thereby its relevant proof is omitted here due to space limitation. Note that when $0 \le \alpha \le \frac{1}{2}$, there is no general conclusion for Model (13). In addition, from practical perspective, if the belief degree belongs to the threshold of $[0, \frac{1}{2}]$, the CRP discussed makes no sense.

Remark 3. Theorem 7 indicates that once the value of the belief degree α is large enough, the consensus in Section 3.3 is only related to the mean values of individual opinions expressed by linear uncertainty distributions. Essentially, Theorem 7 and Theorem 5 are equivalent in forms.

3.4. Non-cooperators considered and DMs with uncertain preferences

Assume there exist a total of t non-cooperators in GDM process (referred to as d_k), and all the individual opinions $o_{i,\ i\in N}$ obey uncertainty distributions while the consensus is presented as a crisp number. Since DMs other than $d_k, k \in \{1, 2, \ldots, t\}$ are like-minded and form a small alliance, then the whole group will mostly emphasize on d_k's interest, thus, an optimisation-based uncertain MCCM is constructed as

$$\begin{aligned} & Min\ Z = \sum_{i=1}^{t} \omega_k \varepsilon_k \\ & s.t. \begin{cases} M\{o' - o_k \le \varepsilon_k\} \ge \beta, M\{o' - o_k \ge -\varepsilon_k\} \ge \beta, k \in \{1, 2, \ldots, t\} & (14-1) \\ M\{o' - o_i \le \varepsilon_i\} \ge \alpha, M\{o' - o_i \ge -\varepsilon_i\} \ge \alpha, i \in N \backslash k & (14-2) \\ o' \in O, o' \ge o, \varepsilon_k \ge 0, k \in \{1, 2, \ldots, t\} & (14-3) \end{cases} \end{aligned} \tag{14}$$

In Model (14), d_k, ($k \in \{1, 2, \ldots, t\}$) is non-cooperated with the small alliance in GDM. That is, other $(n-t)$ DMs have basically reached a temporary consensus, or the $(n-t)$ DMs may have similar interest or like-minded, so this scenario aims to minimise the total consensus cost (i.e. Z) on d_k for adjusting their opinions. Constraint (14-1) denotes the uncertain measure for those non-cooperators with belief degree β, while constraint (14-2) represents other individual opinions obeying an uncertainty distribution under the belief degree of α. Consensus o' belongs to the feasible set of O, and all opinions are greater than zero by default. For the logical consistency of this paper and easy to obtain the analytic formulas of the uncertain MCCMs, hereafter, we still discuss the GDM scenario with only one non-cooperator considered, then, the above uncertain MCCM is further transformed as Model (15).

Namely, if individual opinions satisfy linear uncertainty distributions as $o_i \sim \mathcal{L}(a_i, b_i)$, then Model (14) with only one non-cooperator considered is equivalent to Model (15), where $\forall i \in N$, a_i and b_i are pre-determined. Similar in Section 3.2, $\varepsilon_{i,\ i\neq k}$ are some known soft-consensus thresholds, $\alpha,\ \beta$ are belief degrees for different DMs, and o', ε_k are decision variables. Similarly, when GDM situation involves more than two non-cooperators, corresponding optimization models with linear uncertain preferences can be easily solved by software as MATLAB, however, the analytic formulas of their optimal solutions will be difficult to obtain then, thus, this paper mainly focuses on the simplest GDM context.

$$
\begin{aligned}
& Min\ Z = \omega_k \varepsilon_k \\
& s.t. \begin{cases}
(1-\beta)a_k + \beta b_k - o' \le \varepsilon_k & (15-1) \\
o' - [\beta a_k + (1-\beta) b_k] \le \varepsilon_k & (15-2) \\
(1-\alpha)a_i + \alpha b_i - o' \le \varepsilon_i, i \in N, i \neq k & (15-3) \\
o' - [\alpha a_i + (1-\alpha) b_i] \le \varepsilon_i, i \in N, i \neq k & (15-4) \\
o' \in O, o' \ge o, \varepsilon_k \ge 0 & (15-5)
\end{cases}
\end{aligned}
\tag{15}
$$

Theorem 8. *If d_i's opinion ($i \in N$) obeys a linear uncertainty distribution as $o_i \sim \mathcal{L}(a_i, b_i)$, the analytic formulas of the objective function and the consensus in Model (15) satisfy constraint (16).*

$$
\begin{cases}
0 \le \beta \le \frac{1}{2} \begin{cases}
\beta a_k + (1-\beta) b_k] < G: & \varepsilon_k^* = o' - \beta a_k - (1-\beta) b_k,\ o' = G \\
(1-\beta) a_k + \beta b_k > H: & \varepsilon_k^* = (1-\beta) a_k + \beta b_k - o',\ o' = H \\
Otherwise: & \varepsilon_k^* = 0,\ o' \in [(1-\beta)a_k + \beta b_k, \beta a_k + (1-\beta) b_k] \bigcap [G, H]
\end{cases} \\
\frac{1}{2} \le \beta \le 1 \begin{cases}
\frac{a_k+b_k}{2} < G: & \varepsilon_k^* = o' - \beta a_k - (1-\beta) b_k,\ o' = G \\
\frac{a_k+b_k}{2} \in [G, H]: & \varepsilon_k^* = (\beta - \frac{1}{2})(b_k - a_k),\ o' = \frac{a_k+b_k}{2} \\
\frac{a_k+b_k}{2} > H: & \varepsilon_k^* = (1-\beta) a_k + \beta b_k - o',\ o' = H
\end{cases}
\end{cases}
\tag{16}
$$

where $G = max\{(1-\alpha)a_i + \alpha b_i - \varepsilon_i\}$ *and* $H = min\{\alpha a_i + (1-\alpha) b_i + \varepsilon_i\}$, $i \in N, i \neq k$.

Proof. See Appendix C. □

3.5. Moderator and DMs with uncertain preferences

Suppose all participants' opinions (including moderator and individual DMs) obey uncertainty distributions. Once individuals in the group obey diverse uncertainty distributions, the MCCM constructed aims to solve heterogeneous GDM problems [13, 14]. However, this is not the focus we intend to explore, in other words, this paper assumes that all participants obey the same type of uncertainty distribution. Therefore, a corresponding CRP can be mathematically constructed as:

$$
\begin{aligned}
& Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i \\
& s.t. \begin{cases}
M\{o' - o_i \le \varepsilon_i\} \ge \alpha \\
M\{o' - o_i \ge -\varepsilon_i\} \ge \alpha \\
0 \le \alpha \le 1,\ \varepsilon_i \ge 0,\ i \in N
\end{cases}
\end{aligned}
\tag{17}
$$

As both individual opinion o_i and consensus o' obey uncertainty distributions, then based on Theorem 3, Model (17) can be further extended as

$$Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i$$
$$s.t. \begin{cases} \Phi_{o'}^{-1}(\alpha) - \Phi_{o_i}^{-1}(1-\alpha) \leq \varepsilon_i & (18-1) \\ \Phi_{o_i}^{-1}(\alpha) - \Phi_{o'}^{-1}(1-\alpha) \leq \varepsilon_i & (18-2) \\ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i \in N & (18-3) \end{cases} \tag{18}$$

Specifically, suppose DM's opinion $o_i = [a_i, b_i]$ obeys a linear uncertainty distribution (denoted as $o_i \sim \mathcal{L}(a_i, b_i)$), and moderator's opinion, on behalf of the interest of the whole group, also obeys a linear uncertainty distribution by default as $o' \sim \mathcal{L}(a, b)$. Where a_i and b_i are known parameters of d_i's uncertain preference, while a and b are unknowns to be solved. Model (18) is equivalent to

$$Min\ \phi = \sum_{i=1}^{n} \omega_i \varepsilon_i$$
$$s.t. \begin{cases} a + (b-a)\alpha + (b_i - a_i)\alpha - b_i \leq \varepsilon_i \\ (b_i - a_i)\alpha + (b-a)\alpha - b + a_i \leq \varepsilon_i \\ a \leq b,\ \ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i \in N \end{cases} \tag{19}$$

***Theorem* 9.** *Assume all individual DMs' opinions satisfy $o_i \sim \mathcal{L}(a_i, b_i)$ and moderator's opinion satisfies $o' \sim \mathcal{L}(a, b)$, and adjust individual original opinions in sequence as $\frac{a_1+b_1}{2} \leq \frac{a_2+b_2}{2} \leq \cdots \leq \frac{a_n+b_n}{2}$, then if and only if $\frac{1}{2} \leq \alpha \leq 1$, the optimal solution for Model (19) exists, which satisfies the following conditions:*

$$\phi^* = \begin{cases} \sum_{i=m+1}^{n} \omega_i[(1-\alpha)a_i + \alpha b_i] - \sum_{i=1}^{m} \omega_i[\alpha a_i + (1-\alpha)b_i], \\ \quad where\ a = b \in [\frac{a_m+b_m}{2}, \frac{a_{m+1}+b_{m+1}}{2}],\ iff\ \sum_{i=1}^{m} \omega_i = \sum_{i=m+1}^{n} \omega_i \\ (\sum_{i=1}^{m} \omega_i - \sum_{i=m+1}^{n} \omega_i)\frac{a_m+b_m}{2} + \sum_{i=m+1}^{n} \omega_i[(1-\alpha)a_i + \alpha b_i] - \sum_{i=1}^{m} \omega_i[\alpha a_i + (1-\alpha)b_i], \\ \quad where\ a = b = \frac{a_m+b_m}{2},\ iff\ \sum_{i=1}^{m-1} \omega_i < \sum_{i=m}^{n} \omega_i, \sum_{i=1}^{m} \omega_i > \sum_{i=m+1}^{n} \omega_i \end{cases}$$

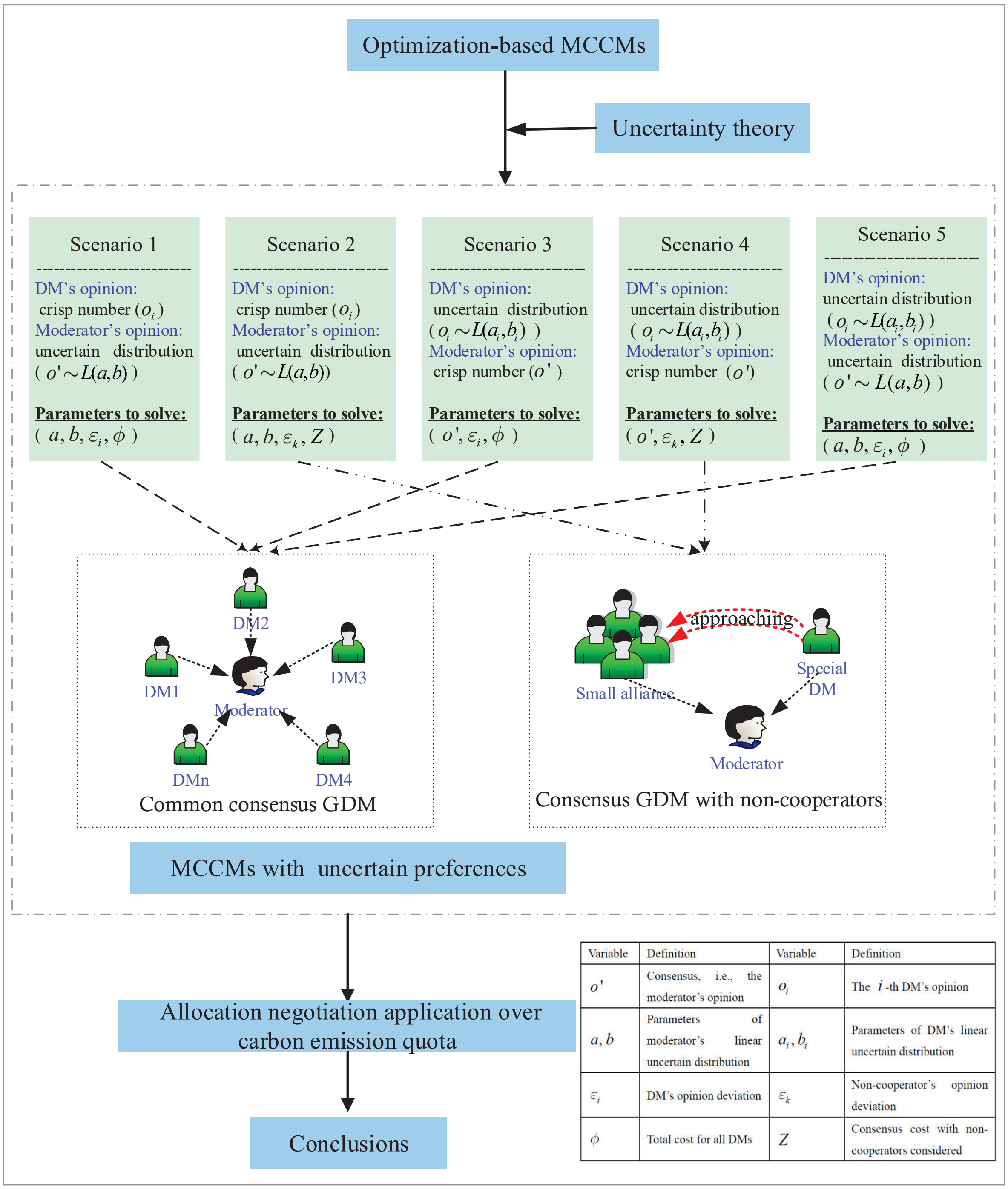
Theorem 9 can be proved by a same mechanism as for Theorem 5, therefore, its relevant proof is omitted.

Remark 4. Theorem 9 verifies that once the participants' opinions obey linear uncertainty distributions in GDM, the final consensus is only related to the weight allocation and the mean values of initial opinions for all DMs. Besides, Theorem 9, Theorem 7 and Theorem 5 are formally equivalent.

3.6. Flowchart of MCCMs with uncertain preferences

For clarity, a flowchart of this paper is given as Fig. 3, and the relations between the aforementioned five GDM scenarios are also summarized in detail.

In specific, we differentiate all the GDM participants into three roles as moderator, individual DMs, and non-cooperators. Considering that participants usually have disparate standpoints or interests when facing real-life GDM, uncertain preferences will be accordingly expressed by different roles under various decision contexts. Thus, Section 3.1 and 3.2 assume that moderator's opinion is expressed as uncertain



Variable	Definition	Variable	Definition
o'	Consensus, i.e., the moderator's opinion	o_i	The i-th DM's opinion
a, b	Parameters of moderator's linear uncertain distribution	a_i, b_i	Parameters of DM's linear uncertain distribution
ε_i	DM's opinion deviation	ε_k	Non-cooperator's opinion deviation
ϕ	Total cost for all DMs	Z	Consensus cost with non-cooperators considered

Fig. 3 Flowchart of MCCMs with uncertain preferences

preference (denoted by belief degree and uncertainty distribution) while individuals present crisp number preferences, and then preference structures of those two roles are reversed in Section 3.3 and 3.4. Finally, in Section 3.5, participants involved in GDM all present their judgements by uncertain preferences. For more in line with real-life GDM problems, we also deeply explored the influence of non-cooperators in uncertain MCCMs in Section 3.2 and 3.4, simultaneously aiming to conduct an association research with previous MCCMs in [19].

4. Application in carbon quota negotiation

A negotiation abstracted from real-life GDM is conducted in this section, over the carbon emission quota issue between the government and four local heavily polluting enterprises, so as to further illustrate the validity of the above five uncertain chance-constrained MCCMs and the proposed theorems. In addition, this section also deeply investigates the relations between the newly constructed models with the traditional MCCMs through data analysis.

4.1. Research background

The fifth assessment report of the intergovernmental panel on climate change (IPCC) clearly states that global warming is intensified according to the observable data over the global surface temperatures and the rising sea levels, which is largely due to human activities [52]. Therefore, how to reduce the impact of human activities on the environment through greenhouse gas emission reduction has become a priority for entire mankind. Jiang et al. [53] proved that the most cost-efficient way to deal with global warming is to build a carbon market under which the key problem becomes to allocate carbon emission quota (hereafter carbon quota). "Carbon emission right" refers to the right of enterprises to legally discharge greenhouse gases such as carbon dioxide into the atmosphere according to relevant laws, and "carbon quota" refers to the legal amount for each enterprise within a certain period through some bargains with the government. Based on the concept of the carbon market, if the actual emission amounts of enterprises are more than their quotas, then they need to pay for extra quotas to make up their illegal amounts; conversely, if the actual emission amounts are smaller, then the balance can be sold out in the carbon market (http://www.tanpaifang.com). Given the above background, a negotiation is usually conducted over specific carbon quotas between the government and different enterprises, and thus, the allocation of carbon quotas for various enterprises is essentially a CRP.

How to allocate carbon quotas? Take a certain region as an example, local government can always provide a rough carbon quota allocation scheme for each enterprise, through comprehensively considering their historical emission data, advanced emission reduction measures, and future development strategies. Then, through bargain or negotiation, the government and all enterprises can reach a carbon quota consensus. In fact, enterprises are mostly profit-oriented and usually believe that environmental actions lead to financial cost increasing, because their proactive huge investment in green technology may not pay off for decades [54]. Therefore, it is relatively difficult for enterprises to provide an exact emission index. Although an exact emission number sometimes needs to be given by enterprises, it is highly likely to have some deception in that index from the view of enterprises' interest [49]. Acting as a macro-moderator, government needs to take into account both economic and social benefits and always stick in line with the principle of fairness and effectiveness, so as to help enterprises more accurately determine carbon quotas through multiple means (e.g. game, negotiation, and implementation of relevant administrative orders or incentives). Obviously, such carbon quota negotiations involving the government (i.e. moderator) and enterprises (i.e. DMs) can constitute a cost consensus GDM problem. As mentioned earlier, we will not discuss the economic benefits of enterprises resulting from their subsequent carbon quota transferring (i.e. trading behaviour in carbon market), which can be viewed as post-consensus decision-making problems [55], such as how to use tiered pricing after overrun for different heavily polluting enterprises.

4.2. Numerical discussion and sensitivity analysis

Assume four heavily polluting enterprises located in different regions within a same province, with similar qualifications and scales in the same industry, denoted as d_i, $i \in N = \{1, 2, 3, 4\}$. o_i is their original carbon quota (unit: 10,000 tons/year). We assume that to facilitate unified management, the provincial government needs to set a unified standard (i.e. an optimal carbon quota) for these enterprises. The optimal carbon quota negotiated above is not only the consensus reached but also the final value expected by the government, which can be marked as o'. For simplicity, we conduct comparative analysis with the data in [19, 28] (see Table 1). If there exists a non-cooperator, we might as well assume that $d_k = d_3$, and it holds a special position different from the others (e.g. d_3 is a pillar industry within its region, receiving special support from the government, while others aren't). Note that, by considering the moderator's preference on some specific factors, we can always easily identify such non-cooperators in real-life GDM.

Table 1 Summary of original decision information

Cases	(1)		(2)			(3)		(4)			(5)	
	$o' \sim \mathcal{L}(a,b)$		$o' \sim \mathcal{L}(a,b)$			$o_i \sim \mathcal{L}(a_i,b_i)$		$o_i \sim \mathcal{L}(a_i,b_i)$			$o' \sim \mathcal{L}(a,b)$;$o_i \sim \mathcal{L}(a_i,b_i)$	
Variables	o_i	ω_i	o_i	ω_i	ε_i	o_i	ω_i	o_i	ω_i	ε_i	o_i	ω_i
d_1	0	1	0	-	5	[14,37]	1	[14,37]	-	12	[14,37]	1
d_2	3	2	3	-	4	[22,30]	2	[22,30]	-	5	[22,30]	2
d_3	6	3	6	3	-	[64,153]	3	[64,153]	3	-	[64,153]	3
d_4	10	1	10	-	6	[8,61]	1	[8,61]	-	36	[8,61]	1
Unknown	$a, b, \varepsilon_i, \phi$		a, b, ε_3, Z			o', ε_i, ϕ		o', ε_3, Z			$a, b, \varepsilon_i, \phi$	

Note: ϕ indicates the total consensus cost for all plants d_i, ($i \in N$); Z indicates the consensus cost for the non-cooperator d_k.
Unit for o_i, o': 10,000 tons/year; unit for w_i: 10,000 yuan/ton.

Case 1. Assume that the original carbon quotas required by four enterprises are $o_1 = 0$, $o_2 = 3$, $o_3 = 6$, and $o_4 = 10$ (unit: 10,000 tons/year). To promote the allocation of optimal carbon quotas, unit costs that the provincial government is willing to pay are: $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 3$, and $\omega_4 = 1$ (unit:10,000 yuan/ton). Here, we suppose that the optimal quota expected by the government (i.e. o')

obeys a linear uncertainty distribution, represented as $o' \sim \mathcal{L}(a,b)$, where a and b are unknown.

$$Model(4-1): \ Min\ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4$$
$$s.t. \begin{cases} M\{o'-0 \le \varepsilon_1\} \ge \alpha, M\{o'-0 \ge -\varepsilon_1\} \ge \alpha \\ M\{o'-3 \le \varepsilon_2\} \ge \alpha, M\{o'-3 \ge -\varepsilon_2\} \ge \alpha, \\ M\{o'-6 \le \varepsilon_3\} \ge \alpha, M\{o'-6 \ge -\varepsilon_3\} \ge \alpha, \\ M\{o'-10 \le \varepsilon_4\} \ge \alpha, M\{o'-10 \ge -\varepsilon_4\} \ge \alpha \\ o' \sim \mathcal{L}(a,b),\ \varepsilon_i \ge 0,\ i=1,2,3,4 \end{cases}$$

Taking the negotiation cost initiated by the government to minimise as our main goal, an MCCM based on the carbon quota is constructed as Model (4-1), which is finally transformed as Model (4-11) (see Appendix D). Without regard to the random distribution for the consensus opinion, optimal solution obtained by [19] is $o^* = 6$ and $\phi^* = 16$. Table 2 provides the sensitivity results for Model (4-1) when the step length for the belief degree α is 0.1. In addition, Table 2 indicates that if and

Table 2 Sensitivity results for Case 1

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
ε_1	0	1.11	2.5	3	4	6	6	6	6	6	6
ε_2	0	0	0	0	1	3	3	3	3	3	3
ε_3	0	0	0	0	0	0	0	0	0	0	0
ε_4	0	0	0	3	4	4	4	4	4	4	4
$o^* = [a,b]$	[0,10]	[0,11.11]	[0,12.5]	[0,10]	[0,10]	6	6	6	6	6	6
ϕ^*	0	1.11	2.5	6	10	16	16	16	16	16	16

only if $\alpha \ge 0.5$, the optimal carbon quota o^* reached by polluters and the government evolves from an uncertainty distribution to a single real value 6, and then, the consensus budget ϕ^* also reaches a stable level of 16. Moreover, when we take the belief degree α as an unknown variable, then its optimal solution solved by Model (4-1) is $\alpha^* = [0.5, 1]$. Compared to the results in [19], Corollary 2 holds. As polluters' original carbon quotas have already been ranked in an ascending order, and due to $\omega_1 + \omega_2 < \omega_3 + \omega_4$, $\omega_1 + \omega_2 + \omega_3 > \omega_4$, the optimal quota for minimising the objective function is $o^* = o_3 = 6$, so Theorem 5 is verified.

Combined with the research background, when the belief degree is rather low (i.e. $\alpha < 0.5$), the provincial government can only obtain a threshold for the optimal carbon quota. However, when the belief degree is no less than 0.5, the optimal carbon quota is constant at 60,000 tons/year, meaning that once the government's belief degree reaches a critical value (i.e. 0.5), the optimal quota will stabilize to a fixed value. Next, we economically explain the changes in carbon quotas for each polluter. Enterprises d_1 and d_2 require a relatively low quota at the beginning, possibly due to the overconfidence or lack of comprehensive verification of their emission capacity, but the government believes that they should get 60,000 tons of quota per year from the perspective of their previous emission situation or future development demands. As for d_4, a rather high quota is given on the account of excessive conservatism or the desire to obtain more economic subsidies from the government. However, d_4 should finally lower its emission standard to balance the environmental and economic benefits.

As we stated earlier, if the actual emission amounts of enterprises are less than the allocated quotas, the balance can produce certain economic benefits in the carbon market, which can be regarded as an incentive to promote emission reduction. Conversely, for polluters whose real emission amounts exceed the allocated quota, only by purchasing extra quotas, can they complete their business targets, and then, the transaction can be considered as a negative incentive. In short, irrespective of how much the polluters actually emit, the government can always achieve the emission reduction target by setting an optimal carbon quota.

Case 2. As shown in Table 1, enterprise d_3 acts as a non-cooperator, and the government provides it a unit negotiation cost as $\omega_3 = 3$ (unit: 10,000 yuan/ton). Meanwhile, to make the CRP more flexible, final optimal carbon quotas for other three polluters have soft- consensus thresholds as $\varepsilon_1 = 5$, $\varepsilon_2 = 4$ and $\varepsilon_4 = 6$ (unit: 10,000 tons/year). Assume the optimal quota o' obeys a linear uncertainty distribution as $o' \sim \mathcal{L}(a, b)$ by default. Aiming to minimise the total budget, an uncertain carbon quota MCCM is built as Model (4-2) and its linear equivalent form is Model (4-21) (see Appendix D). Ref [19] provided an optimal solution as $o^* = 5, Z^* = 3$ without considering DMs' opinions characterizing random distributions.

$$
Model(4-2): \ Min\ Z = 3 * \varepsilon_3
$$
$$
s.t. \begin{cases} M\{o' - 6 \leq \varepsilon_3\} \geq \beta, M\{o' - 6 \geq -\varepsilon_3\} \geq \beta \\ M\{o' - 0 \leq \varepsilon_1\} \geq \alpha, M\{o' - 0 \geq -\varepsilon_1\} \geq \alpha \\ M\{o' - 3 \leq \varepsilon_2\} \geq \alpha, M\{o' - 3 \geq -\varepsilon_2\} \geq \alpha \\ M\{o' - 10 \leq \varepsilon_4\} \geq \alpha, M\{o' - 10 \geq -\varepsilon_4\} \geq \alpha \\ 0 \leq \varepsilon_1 \leq 5,\ 0 \leq \varepsilon_2 \leq 4,\ 0 \leq \varepsilon_4 \leq 6,\ \varepsilon_3 \geq 0 \\ o' \sim \mathcal{L}(a, b), a \leq b,\ 0 \leq \alpha,\ \beta \leq 1 \end{cases}
$$

In fact, Case 2 introduces the soft-consensus constraints based on Case 1, and analyses the consensus GDM with only considering some non-cooperators instead of all DMs. Table 3 provides sensitivity results for the variable α in Model (4-2) when another belief degree set for the non-cooperator d_3 is fixed as $\beta = 0.6$. For detailed analysis, we identify the consensus reaching conditions with all DMs' carbon quotas being crisp numbers, while the emission index for the local government obeys a linear uncertainty distribution. Namely, we draw conclusions by adapting the belief degree α within the interval of [0,1] for the other three polluters during the carbon quota negotiation. Through calculation, we find that once the soft-consensus thresholds are given in advance, the optimal values for variables $\varepsilon_{i,\ i\neq k,\ i\in N}$ always take the upper limits as $\varepsilon_1^* = 5, \varepsilon_2^* = 4$ and $\varepsilon_4^* = 6$, so these values are omitted in Table 3.

Table 3 Sensitivity results for Case 2.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
β	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
ε_3	0.2	0.25	0.33	0.5	1	1	1	1	1	1	1
$o^* = [a, b]$	[5,7]	[4.75,7.25]	[4.33,7.67]	[3.5,8.5]	[1,11]	5	5	5	5	5	5
Z^*	0.6	0.75	1	1.5	3	3	3	3	3	3	3

Table 3 indicates that when $\alpha \geq 0.5$, the optimal carbon quota becomes a fixed constant from an uncertainty distribution. $\forall i \in N, i \neq k$, $max(o_i - \varepsilon_i) = 4$, $min(o_i + \varepsilon_i) = 5$, we have $o_{k=3} = 6 > min(o_i + \varepsilon_i)$. Therefore, when $\alpha \leq 0.5$, we obtain the optimal value for ε_k^* and further obtain accurate values for a and b, by comparing the sizes of $(\beta - 0.5)(b - a)$ and $o_k - min(o_i + \varepsilon_i)$. Taking the situation of $\alpha = 0.3, \beta = 0.6$ as an example, we have $a = [min(o_i + \varepsilon_i) - 2\alpha * o_k]/(1 - 2\alpha) = (5 - 2*0.3*6)/(1 - 2*0.3) = 3.5$ and $b = [(2 - 2\alpha)o_k - min(o_i + \varepsilon_i)]/(1 - 2\alpha) = [(2 - 2*0.3)*6 - 5]/(1 - 2*0.3) = 8.5$. Thus, $\varepsilon_k^* = min\{(\beta - 0.5)(b - a), o_k - min(o_i + \varepsilon_i)\} = 0.5$. Meanwhile if $\alpha \geq 0.5$, for $o_k = 6 > min(o_i + \varepsilon_i)$ always holds, thereby $a^* = b^* = 5$, $Z^* = \omega_3 * (o_3 - min(o_i + \varepsilon_i)) = 3$. Obviously, above calculations are in accordance with the data in Table 3, so Theorem 6 holds.

Owing to the randomness of data selection, Case 2 only validates the conclusion of $o_k > min(o_i + \varepsilon_i)$. By adjusting d_3's original carbon quota, the rest of Theorem 6 can always be validated by a similar mechanism.

Case 3. Initial emission quotas of the four polluters are listed in Table 1. The local government, for obtaining an optimal allocation with unified standards, provides each enterprise a unit cost as $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 3$, and $\omega_4 = 1$ (unit:10,000 yuan/ton). Here, the final collective carbon quota is defaulted as a crisp number, while the original emission indexes for the four polluters obey linear uncertainty distributions. Thus, an uncertain MCCM is constructed as Model (4-3), whose optimal solution is solved by Model (4-31) (see Appendix D).

$$
\begin{aligned}
&Model(4-3): \ Min\ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4 \\
&s.t. \begin{cases}
M\{o' - o_i \leq \varepsilon_i\} \geq \alpha, \ i = 1,2,3,4 \\
M\{o' - o_i \geq -\varepsilon_i\} \geq \alpha, \ i = 1,2,3,4 \\
o_1 \sim \mathcal{L}(14,37), o_2 \sim \mathcal{L}(22,30), \\
o_3 \sim \mathcal{L}(64,153), o_4 \sim \mathcal{L}(8,61) \\
o' \geq 0, \ 0 \leq \alpha \leq 1, \ \varepsilon_i \geq 0, \ i = 1,2,3,4
\end{cases}
\end{aligned}
$$

With no uncertain chance-constraints, an optimal solution presented in [28] is $o^* = [37, 61], \phi^* = 95$. Previous work only regards DMs' opinions as intervals, without considering the characteristics of opinions obeying random distributions.

Table 4 Sensitivity results for Case 3.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
ε_1	0	0	0	0	1.4	9	11.3	13.6	15.9	18.2	20.5
ε_2	7	5.5	4	2.5	2.4	8.5	9.3	10.1	10.9	11.7	12.5
ε_3	27	38.2	49.4	60.6	70.4	74	82.9	91.8	100.7	109.6	118.5
ε_4	0	0	0	0	0	0	5.3	10.6	15.9	21.2	26.5
o^*	[37,61]	[34.7,55.7]	[32.4,50.4]	[30.1,45.1]	[29.2,39.8]	34.5	34.5	34.5	34.5	34.5	34.5
ϕ^*	95	125.6	156.2	186.8	217.4	248	283.9	319.8	355.7	391.6	427.5

When belief degree α is an unknown parameter, we obtain $\alpha^* = 0$ and $o^* = [37, 61], \phi^* = 95$ (see Column 2 in Table 4), corresponding to the optimal solution in [28]. Note that, keeping $\alpha^* = 0, \phi^* = 95$ constant, the variable o^* can be any optimal value within the interval [37,61] and variables $\varepsilon_{i,\ i \in N}$ will change with the specific value of o^*. Due to $\varepsilon_{i,\ i\in N}$ have no effect on the final results, relevant analysis is omitted in the paper.

Table 4 intuitively shows that once the uncertain belief degree of the carbon quota negotiation satisfies $\alpha \geq 0.5$, the optimal consensus expected by the moderator becomes a real value from an uncertainty distribution. As original emission indexes of the four DMs haven't been ranked in an ascending order, Table 5 gives the updated values to conveniently verify the effectiveness of relevant theorems. Since $\omega_{(1)} + \omega_{(2)} < \omega_{(3)} + \omega_{(4)}$, $\omega_{(1)} + \omega_{(2)} + \omega_{(3)} > \omega_{(4)}$, the optimal carbon quota is obtained as $o^* = \frac{a_{(3)}+b_{(3)}}{2} = \frac{69}{2} = 34.5$. Then, using the analytic formula of the objective function in Theorem 7, values of ϕ^* calculated are identical with the data in Table 4. Thus, Theorem 7 holds.

Table 5 Updated opinions in Case (3-5) with an ascending order

Updated	Origin	a_i	b_i	$a_i + b_i$	weight
$o_{(1)}$	o_1	14	37	51	1
$o_{(2)}$	o_2	22	30	52	2
$o_{(3)}$	o_4	8	61	69	1
$o_{(4)}$	o_3	64	153	217	3

Note: o_i represents the opinion for the original i-th DM;
$o_{(i)}$ represents the opinion for the updated i-th DM in an ascending order.

Case 4. Similar as in Case 2, d_3 is assumed as the non-cooperating enterprise. Data of polluters' original carbon quotas, unit cost for d_3 as well as the soft-consensus thresholds for the other three enterprises are all listed in Table 1. Here, the consensus o' obtained for the government is defaulted as a crisp number, while DMs' preferences obey linear uncertainty distributions. Then, Model (4-4) is built with great emphasis on d_3's interest. Table 6 and Table 7 are obtained by solving Model (4-41) (see Appendix D).

$$
Model(4-4):\ Min\ Z = 3 * \varepsilon_3
$$
$$
s.t. \begin{cases} M\{o' - o_3 \leq \varepsilon_3\} \geq \beta, M\{o' - o_3 \geq -\varepsilon_3\} \geq \beta \\ M\{o' - o_i \leq \varepsilon_i\} \geq \alpha, M\{o' - o_i \geq -\varepsilon_i\} \geq \alpha,\ i = 1, 2, 4 \\ o_1 \sim \mathcal{L}(14, 37), o_2 \sim \mathcal{L}(22, 30), \\ o_3 \sim \mathcal{L}(64, 153), o_4 \sim \mathcal{L}(8, 61) \\ 0 \leq \varepsilon_1 \leq 12,\ 0 \leq \varepsilon_2 \leq 5,\ 0 \leq \varepsilon_4 \leq 36 \\ o' \geq o,\ \varepsilon_3 \geq 0 \end{cases}
$$

Table 6 provides the sensitivity results over the belief degree α, which is set for the small alliance (including polluters d_1, d_2 and d_4), meantime, the belief degree imposed on o_3 and o' is fixed as $\beta = 0.75$. Table 7 gives the sensitivity results over the belief degree β for d_3. Similar in Case 2, once $\varepsilon_{i,i\in N,i\neq k}$ are pre-defined, their optimal values are exactly the pre-determined upper limits, so values of $\varepsilon_1^*, \varepsilon_2^*, \varepsilon_4^*$ are omitted in Table 6 and Table 7.

Values of the total cost and the consensus calculated by the analytic formula in Theorem 8 are exactly the same as the data in Table 6 and Table 7. However, detailed analysis for Case 4 is omitted here due to space limitation. Note, Case 4 only validates part of the conclusions in Theorem 8, but by adjusting the value of o_3, the remaining parts can also be verified.

Table 6 Sensitivity results for Case 4 on α when $\beta = 0.75$.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
β	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75
ε_3	95.75	96.55	97.35	98.15	98.95	99.75	100.55	101.35	102.15	102.95	104.75
o^*	35	34.2	33.4	32.6	31.8	31	30.2	29.4	28.6	27.8	26
z^*	287.25	289.65	292.05	294.45	296.85	299.25	301.65	304.05	306.45	308.85	314.25
$i = 1$	2	4.3	6.6	8.9	11.2	13.5	15.8	18.1	20.4	22.7	25
$i = 2$	17	17.8	18.6	19.4	20.2	21	21.8	22.6	23.4	24.2	25
$i = 4$	-28	-22.7	-17.4	-12.1	-6.8	-1.5	3.8	9.1	14.4	19.7	25
$G = Max$	17	17.8	18.6	19.4	20.2	21	21.8	22.6	23.4	24.2	25
$i = 1$	49	46.7	44.4	42.1	39.8	37.5	35.2	32.9	30.6	28.3	26
$i = 2$	35	34.2	33.4	32.6	31.8	31	30.2	29.4	28.6	27.8	27
$i = 4$	97	91.7	86.4	81.1	75.8	70.5	65.2	59.9	54.6	49.3	44
$H = Min$	35	34.2	33.4	32.6	31.8	31	30.2	29.4	28.6	27.8	26

Note: $G = Max\{(1-\alpha) * a_i + \alpha * b_i - \varepsilon_i\}$, $H = Min\{\alpha * a_i + (1-\alpha) * b_i + \varepsilon_i\}$, $i \in N, i \neq k$

Table 7 Sensitivity results for Case 4 on β when $\alpha = 0.86$.

α	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86
β	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
ε_3	35.88	44.78	53.68	62.58	71.48	80.38	89.28	98.18	107.08	115.98	124.88
o^*	28.12	28.12	28.12	28.12	28.12	28.12	28.12	28.12	28.12	28.12	28.12
Z^*	107.64	134.34	161.04	187.74	214.44	241.14	267.84	294.54	321.24	347.94	374.64

Case 5. Assume all participants involved in this negotiation obey linear uncertainty distributions. Relevant data is provided as Table 1, so an uncertain chance-constrained MCCM is constructed as Model (4-5) and its equivalent linear transformation is Model (4-51) (see Appendix D).

$$Model(4-5): \ Min \ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4$$
$$s.t. \begin{cases} M\{o' - o_i \leq \varepsilon_i\} \geq \alpha, \ i = 1,2,3,4 \\ M\{o' - o_i \geq -\varepsilon_i\} \geq \alpha, \ i = 1,2,3,4 \\ o' \sim \mathcal{L}(a,b), o_1 \sim \mathcal{L}(14,37), o_2 \sim \mathcal{L}(22,30), \\ o_3 \sim \mathcal{L}(64,153), o_4 \sim \mathcal{L}(8,61) \\ a \leq b, \ \ 0 \leq \alpha \leq 1, \ \varepsilon_i \geq 0, \ i = 1,2,3,4 \end{cases}$$

By solving Model (4-51), optimal solutions under different belief degrees are obtained as Table 8. Simultaneously, Table 8 provides the changes for both the optimal carbon quota o^* and the optimal total cost ϕ^* with the variable α.

Table 8 Sensitivity results for Case 5.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
ε_1	0	0	0	8.77	38.6	9	11.3	13.6	15.9	18.2	20.5
ε_2	0	0	0	11.27	39.6	8.5	9.3	10.1	10.9	11.7	12.5
ε_3	0	0	0	0	0	74	82.9	91.8	100.7	109.6	118.5
ε_4	0	0	0	0	26.6	0	5.3	10.6	15.9	21.2	26.5
$o^* = [a,b]$	[0,64]	[0,81]	[0,102.25]	[0,129.57]	[0,166]	34.5	34.5	34.5	34.5	34.5	34.5
ϕ^*	0	0	0	31.31	144.4	248	283.9	319.8	355.7	391.6	427.5

Once $\alpha \geq 0.5$, the optimal allocation values evolves from an uncertainty distribution to the real value of 34.5. Similar as the analysis in Case 3, $o^* = \frac{a_{(3)}+b_{(3)}}{2} = \frac{69}{2} = 34.5$ is obtained by referring to Table 5. Negotiation costs calculated by the analytic expression of ϕ^* are in accordance with the data in Table 8. Taking $\alpha = 0.7$ as an example, $\phi^* = (\omega_{(1)} + \omega_{(2)} + \omega_{(3)} - \omega_{(4)}) * 34.5 + \omega_{(4)} * (0.3*64 + 0.7*153) - \omega_{(1)} * (0.7*14 + 0.3*37) - \omega_{(2)} * (0.7*22 + 0.3*30) - \omega_{(3)} * (0.7*8 + 0.3*61) = 319.8$. So Theorem 9 holds.

4.3. Comparison and discussion

Due to serious deterioration of the global environment, the reduction of carbon emission has become a key measure to improve the ecological system, so we choose the application in carbon quota negotiation to verify the feasibility of the proposed models. Results show that the calculated values correspond to the analytic formulas of the optimal solutions under each scenario, verifying the correctness of the theorems obtained by theoretical deduction. Moreover, findings in the application indicate that traditional crisp number- or interval preference-based MCCMs are some special cases of the new uncertain MCCMs, suggesting that uncertainty theory can build a bridge between deterministic and indeterministic GDM. Finally, we find that once the belief degree, set for the deviation of polluters' and government's quota indexes, is larger than the critical value of 0.5, then the optimal carbon quota consensus will be crisp numbers and no longer obey uncertainty distributions. The above conclusion implies that only belief degree is large enough, GDM can achieve a deterministic consensus and the carbon quota negotiation can then be effectively conducted to some extent.

To illustrate the novelty of our research, we conduct a comparative analysis (see Table 9). Distinguished from previous research, we build the consensus models from three decision roles, by introducing non-cooperators into traditional MCCMs. Meanwhile, we first introduce Liu's uncertainty theory into consensus modeling, by adopting belief degree and uncertainty distribution as a whole to fit individual preferences, and find out the relations between the deterministic and indeterministic GDM through theoretical deduction. Finally, we apply the proposed models into the carbon emission quota allocation negotiation problem to verify their feasibility. However, it is undeniable that some important contributions in relevant MCCMs/ MACMs may be neglected in this paper, such as setting an aggregation function over the adjusted individual opinions to obtain a consensus [2, 10, 27], using consensus level to measure the efficiency of CRP [2, 10], or considering the asymmetric characteristic of unit costs [5].

Table 9 Comparative analysis on relevant MCCMs/ MACMs.

Consensus models	**Decision roles**	**DM's preference**	**Application**
This paper	Moderator; individual DMs; non-cooperators	Uncertainty distributions and belief degree	Carbon emission quota allocation
Ben-Arieh and Easton [20]	Moderator; individual DMs	Crisp numbers	None-numerical examples
Dong et al. [10]	Individual DMs	Linguistic preferences	None-numerical examples
Zhang et al. [27]	Individual DMs	Crisp numbers	Apartment selection
Gong et al. [28]	Moderator; individual DMs	Interval preferences	None-numerical examples
Gong et al. [19]	Moderator; individual DMs	Crisp numbers	None-numerical examples
Zhang et al. [2]	Moderator; individual DMs	Crisp numbers	Loan consensus problems in Online P2P lending
Cheng et al. [5]	Moderator; individual DMs	Crisp numbers	Trans-boundary pollution control

Inspired by the fact that flexible management has been a premiere goal pursued by Chinese government, in order to encourage high-quality development of enterprises, the negotiation over the carbon emission quota allocation problem is chosen as our case background. In fact, when setting carbon emission reduction quotas for different enterprises with similar scales, it can better reflect the government's humanized management by setting uncertain indicators rather than some deterministic and fixed ones, which may also be understood as the practical significance of the uncertainty constraints in this paper. Without doubt, our newly proposed uncertain MCCMs can provide significant managerial implications for moderators to deal with real-life GDM problems with flexible requirements, such as targeted recommendation system purchasing based on advertisers' market share, and second-hand housing selection bargain from different agencies.

5. Conclusion

Compared to traditional deterministic preferences, fitting DMs' preferences with uncertainty distributions is more suitable for real-life decision-making contexts, especially for complex GDM. In this paper, linear uncertainty distributions are adopted to fit individual judgements, and a series of uncertain MCCMs are proposed. Through transformation into equivalent linear programming models, the analytic formulas of the optimal consensus and minimum total cost under each scenario are given in the paper. We find out that the uncertain preference-based MCCMs are more inclusive than those traditional ones, in other words, the basic conclusions of the crisp number- or interval preference-based models are some special cases of uncertain MCCMs under different belief degrees, thus our research is more flexible in actual GDM. In addition, optimal solution of each uncertain chance-constrained MCCM is theoretically provided, and the feasibility of the proposed models are further verified with the application in carbon quota negotiation between enterprises and the local government.

Main contributions of this paper are as follows. Firstly, this paper builds the optimisation-based MCCMs from multiple decision roles (i.e., the moderator, individual DMs, and non-cooperators). Secondly, belief degree and uncertainty distributions are used as a whole to simulate DMs' preference structure, making the new models more feasible than those traditional ones (i.e., crisp number- or interval preference-based MCCMs), better avoiding the paradox in interval operations (e.g. $[1, 3] - [1, 3] = [-2, 2] \neq [0, 0]$), and maintaining the integrity of decision information by analyzing individual uncertain opinions as a whole instead of only endpoints being considered. Thirdly, consensus reaching conditions under different GDM scenarios are presented through mathematical deduction. By taking the application in carbon quota negotiation, the proposed models are verified as a more general paradigm of the traditional MCCMs.

This paper explores consensus reaching conditions in homogenous GDM, but real-life decision is rather complex and changeable, making it highly possible for involved participants to simultaneously present completely different preference structures. So, in the future, we may deal with heterogeneous GDM problems [14, 56] by modeling non-linear uncertain chance-constrained MCCMs. At present, unit costs attached to DMs are subjectively given in the paper, afterwards, we may adopt some robust methods, such as game theory [46], to assure those parameters to be more reasonable. In specific, we may need to set variable unit costs for DMs to deal with the situation under which tiered pricing is set for heavily polluting enterprises after overrun. Finally, in this paper, we aim to figure out how an optimal consensus can be reached within each certain stage of the whole GDM process under the uncertain chance constraints, neglecting the dynamic characteristics for the whole process, which are definitely of great significance for GDM, so our subsequent research may also focus on dynamic uncertain MCCMs with feedback mechanism [6, 30, 57] or social interactions [12, 29, 49].

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Appendix A. Proof of Theorem 5.

Proof. Without loss of generality, we suppose n DMs participate in GDM, and their decisions satisfy $o_1 \leq o_2 \leq \cdots \leq o_m \leq o_{m+1} \leq \cdots \leq o_n$. Furthermore, let $o_m \leq \frac{a+b}{2} \leq o_{m+1}$, then when $\alpha \geq 0.5$, the constraints of Model (5) are simplified as

$$\Phi^{-1}(\alpha) - o_i \geq \tfrac{a+b}{2} - o_i \geq 0 \quad (i = 1, 2, \cdots m);$$

$$-\Phi^{-1}(1-\alpha) + o_i \geq -\tfrac{a+b}{2} + o_i \geq 0 \quad (i = m+1, \cdots n).$$

At this point, the objective function satisfies

$$\begin{aligned} min\sum_{i=1}^{n}\omega_i\varepsilon_i &= \omega_1(\Phi^{-1}(\alpha) - o_1) + \cdots + \omega_m(\Phi^{-1}(\alpha) - o_m) \\ &\quad +\omega_{m+1}(-\Phi^{-1}(1-\alpha) + o_{m+1}) + \cdots \\ &\quad +\omega_n(-\Phi^{-1}(1-\alpha) + o_n) \\ &\geq \sum_{i=m+1}^{n}\omega_i o_i - \sum_{i=1}^{m}\omega_i o_i + \tfrac{a+b}{2}(\sum_{i=1}^{m}\omega_i - \sum_{i=m+1}^{n}\omega_i) \end{aligned}$$

If and only if $\Phi^{-1}(\alpha) = \Phi^{-1}(1-\alpha) = \frac{a+b}{2}$, the above inequality takes the mark of equality, then we obtain $a = b$ and $min\sum_{i=1}^{n}\omega_i\varepsilon_i = \sum_{i=m+1}^{n}\omega_i o_i - \sum_{i=1}^{m}\omega_i o_i + a(\sum_{i=1}^{m}\omega_i - \sum_{i=m+1}^{n}\omega_i)$.

Next, the optimal analytic expression of the objective function is derived by comparing $\sum_{i=1}^{m}\omega_i$ and $\sum_{i=m+1}^{n}\omega_i$.

- If $\sum_{i=1}^{m}\omega_i = \sum_{i=m+1}^{n}\omega_i$, then $a = b$ can be any value in the interval of $[o_m, o_{m+1}]$, and $\phi^* = \sum_{i=m+1}^{n}\omega_i o_i - \sum_{i=1}^{m}\omega_i o_i$.
- If $\sum_{i=1}^{m-1}\omega_i < \sum_{i=m}^{n}\omega_i$ and $\sum_{i=1}^{m}\omega_i > \sum_{i=m+1}^{n}\omega_i$, because $\omega_{i,i\in N}$ are positive constants, the objective function ϕ first decreases and then increases with the variable a. Thus, when $a = b = o_m$, the optimal value for the objective function will be $\phi^* = \sum_{i=m+1}^{n}\omega_i(o_i - o_m) + \sum_{i=1}^{m}\omega_i(o_m - o_i)$.

This completes the proof for Theorem 5. □

Appendix B. Proof of Theorem 6.

Proof. Theorem 6 is derived in two steps: (1) Determination of the analytic formula of the objective function $min\ Z = \omega_k \varepsilon_k(a,b)$. As the parameter ω_k is pre-defined, only the formula of $\varepsilon_k^* = min\ \varepsilon_k(a,b)$ actually needs solving. Then (2), determination of the optimal solutions for variables a, b and ε_k^*.

Part 1. Determination of $\varepsilon_k^* = min\ \varepsilon_k(a,b)$.

As the value of ε_k only depends on the constraints of (9-1) and (9-2), let $A = (1-\beta)a + \beta b - o_k$ and $B = -\beta a + (\beta - 1)b + o_k$. Compared to the sizes of A and B, the following three situations are discussed:

- Case 1: If $A = B$, then $a + b = 2o_k$, making $\varepsilon_k \geq A = (\beta - \frac{1}{2})(b-a)$ hold.
  - If $\frac{1}{2} \leq \beta \leq 1$, then $\varepsilon_k^* = (\beta - \frac{1}{2})(b-a)$;
  - If $0 \leq \beta \leq \frac{1}{2}$, then $(\beta - \frac{1}{2})(b-a) \leq 0$; also due to $\varepsilon_k \geq 0$, then $\varepsilon_k^* = 0$ is obtained.
- Case 2: if $A > B$, we obtain $a + b > 2o_k$.
  - If $\frac{1}{2} \leq \beta \leq 1$, then $\varepsilon_k = A \geq \frac{a+b}{2} - o_k$, so if and only if $a = b$, the above inequality takes the mark of equality, making $\varepsilon_k^* = \frac{a+b}{2} - o_k$;
  - If $0 \leq \beta \leq \frac{1}{2}$, we obtain $\varepsilon_k = A \leq \frac{a+b}{2} - o_k$. However, due to $\varepsilon_k \geq 0$ and $\frac{a+b}{2} - o_k > 0$, we get $0 \leq \varepsilon_k \leq \frac{a+b}{2} - o_k$. Thus, $\varepsilon_k^* = 0$.
- Case 3: If $A < B$, we obtain $a + b < 2o_k$.
  - If $\frac{1}{2} \leq \beta \leq 1$, we obtain $\varepsilon_k = B \geq o_k - \frac{a+b}{2}$, so if and only if $a = b$, the above inequality takes the mark of equality, then $\varepsilon_k^* = o_k - \frac{a+b}{2}$ holds;
  - If $0 \leq \beta \leq \frac{1}{2}$, we get $\varepsilon_k = B \leq o_k - \frac{a+b}{2}$; by taking both $\varepsilon_k \geq 0$ and $o_k - \frac{a+b}{2} > 0$ into consideration, we obtain $0 \leq \varepsilon_k \leq o_k - \frac{a+b}{2}$. Thus, $\varepsilon_k^* = 0$.

Above all,

$$\varepsilon_k^* = \begin{cases} (\beta - \frac{1}{2})(b-a), iff\ \beta \in [\frac{1}{2}, 1], a + b = 2o_k & (B1-1) \\ \frac{a+b}{2} - o_k, iff\ \beta \in [\frac{1}{2}, 1], a = b, a + b > 2o_k & (B1-2) \\ o_k - \frac{a+b}{2}, iff\ \beta \in [\frac{1}{2}, 1], a = b, a + b < 2o_k & (B1-3) \\ 0, iff\ \beta \in [0, \frac{1}{2}] & (B1-4) \end{cases} \tag{B1}$$

Part 2. Determination of optimal solutions for a, b and ε_k^*.

As the analytic formula for $min\ \varepsilon_k$ is determined in Part 1 and the original opinion o_k for the non-cooperator d_k with greater influence is known in advance, the value of ε_k^* mainly depends on those of $a + b$ and $b - a$. Without loss of generality, let $a + b = m$ and $b - a = n$, then we have $a = \frac{m-n}{2}$ and $b = \frac{m+n}{2}$. By substituting constraints (9-3) and (9-4), we get

$$\begin{cases} (1-\alpha) \cdot \frac{m-n}{2} + \alpha \cdot \frac{m+n}{2} - o_i \leq \varepsilon_i \\ -\alpha \cdot \frac{m-n}{2} + (\alpha - 1) \cdot \frac{m+n}{2} + o_i \leq \varepsilon_i \end{cases} \tag{B2}$$

After simplifying the inequality (B2), $\forall i \in N, i \neq k$, the range of m is solved as

$$(2\alpha - 1)n + 2(o_i - \varepsilon_i) \leq m \leq 2(o_i + \varepsilon_i) - (2\alpha - 1)n \tag{B3}$$

By fully considering the values of o_i, ε_i, and n, the above formula is equivalent to

$$min\ [max\ 2(o_i - \varepsilon_i) + (2\alpha - 1)n] \leq m \leq max\ [min\ 2(o_i + \varepsilon_i) - (2\alpha - 1)n] \tag{B4}$$

Taking both the actual GDM and the construction mechanism of uncertainty theory into account, opinions of the non-cooperator d_k with great influence cannot be ignored, so the belief degree β for d_k's original opinion o_k and the finally reached consensus o' should satisfy the condition of $\beta \geq \frac{1}{2}$. Then, the CRP makes sense. Based on the conclusion derived from Part 1, the optimal value of the objective function is always equal to zero when $0 \leq \beta \leq \frac{1}{2}$. Thus, only the situation of $\frac{1}{2} \leq \beta \leq 1$ will be discussed below. For simplicity, let $E = max\ 2(o_i - \varepsilon_i) + (2\alpha - 1)n$ and $F = min\ 2(o_i + \varepsilon_i) - (2\alpha - 1)n$.

Situation 1: When $0 \leq \alpha \leq \frac{1}{2}$ and $\frac{1}{2} \leq \beta \leq 1$, we get $2\alpha - 1 \leq 0$, considering $n = b - a \geq 0$, so E monotonically decreases with respect to n, while F monotonically increases with respect to n. On account of inequality constraints (B4), if $\exists a, b,\ 0 \leq a \leq b$ such that $a + b = 2o_k$, then $min\ \varepsilon_k$ exists, satisfying $\varepsilon_k^* = (\beta - \frac{1}{2})(b - a)$. In view of α, the optimal values of a and b are gained from three scenarios.

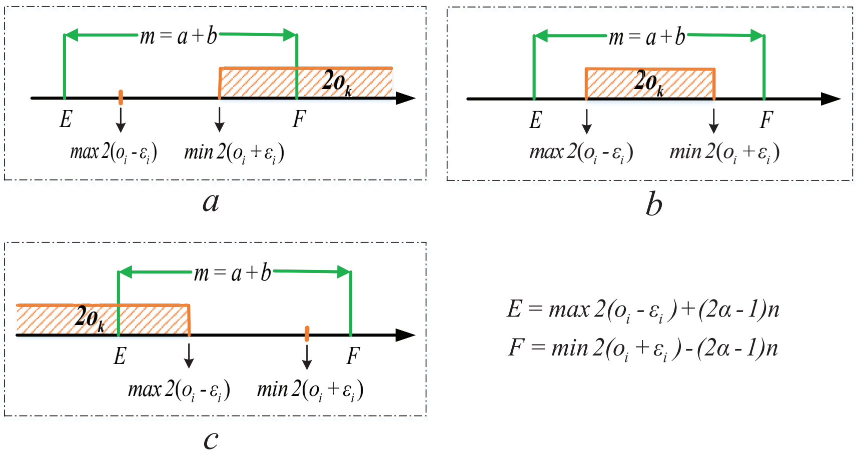

Fig. B1 Discussion on $0 \leq \alpha \leq \frac{1}{2}$ and $\frac{1}{2} \leq \beta \leq 1$

- As shown in Fig. B1(a), when $o_k > min\ (o_i + \varepsilon_i)$, then $2o_k > min\ 2(o_i + \varepsilon_i)$ is obtained.

  1. If $2o_k \leq F$, then $min2(o_i + \varepsilon_i) < 2o_k \leq min2(o_i + \varepsilon_i) - (2\alpha - 1)n$, that is, when $n \geq \frac{2o_k - min2(o_i+\varepsilon_i)}{1-2\alpha}$, then $a + b = 2o_k$ holds; thus, $min\varepsilon_k = (\beta - \frac{1}{2})(b - a) = (\beta - \frac{1}{2})n$ is derived. Obviously, ε_k is a monotonically increasing function of n, so once n takes the minimum value, namely $n = b - a = \frac{2o_k - min2(o_i+\varepsilon_i)}{1-2\alpha}$, ε_k^* exists. On the basis of the formulas of $a + b$ and $b - a$,

we get

$$\begin{cases} a = \frac{min\ (o_i+\varepsilon_i)-2\alpha o_k}{1-2\alpha} \\ b = \frac{(2-2\alpha)o_k - min\ (o_i+\varepsilon_i)}{1-2\alpha} \\ \varepsilon_k^* = (\beta - \frac{1}{2})(b-a) \end{cases}$$

2. If $2o_k > F$, then $a+b < 2o_k$, such that $min\varepsilon_k = o_k - \frac{a+b}{2}$. Obviously, ε_k monotonically decreases with the variable of $(a+b)$. Therefore, once $a+b$ takes the maximum value, namely $a+b = m = F = min\ 2(o_i+\varepsilon_i) - (2\alpha-1)n$, then ε_k^* exists and $a = b$ holds. In other words, $n = 0$ is obtained. Thus, if $a = b = min\ (o_i+\varepsilon_i)$, the optimal objective function will be $\varepsilon_k^* = o_k - min\ (o_i+\varepsilon_i)$.

As a result, when $o_k > min\ (o_i+\varepsilon_i)$, we obtain $\varepsilon_k^* = min\ \{(\beta-\frac{1}{2})(b-a), o_k - min\ (o_i+\varepsilon_i)\}$, where if $\varepsilon_k^* = (\beta-\frac{1}{2})(b-a)$, then $a = \frac{min\ (o_i+\varepsilon_i)-2\alpha o_k}{1-2\alpha}$ and $b = \frac{(2-2\alpha)o_k - min\ (o_i+\varepsilon_i)}{1-2\alpha}$; if $\varepsilon_k^* = o_k - min\ (o_i+\varepsilon_i)$, then $a = b = min\ (o_i+\varepsilon_i)$ holds.

- As shown in Fig. B1(b), $max(o_i-\varepsilon_i) \le o_k \le min(o_i+\varepsilon_i)$, for $n = b-a \ge 0$, so $2o_k \in [max2(o_i-\varepsilon_i), min2(o_i+\varepsilon_i)] \subseteq [E, F]$. Clearly, $\exists m$ such that $m = a+b = 2o_k$. That is, when $min\varepsilon_k = (\beta-\frac{1}{2})(b-a) = (\beta-\frac{1}{2})n$, then $n = b-a = 0$ holds. Thus, if $a = b = o_k$, we obtain $\varepsilon_k^* = 0$.

- As shown in Fig. B1(c), $o_k < max(o_i-\varepsilon_i)$, so $2o_k < max2(o_i-\varepsilon_i)$.

1. If $2o_k \ge E$, then $max2(o_i-\varepsilon_i) + (2\alpha-1)n \le 2o_k < max(o_i-\varepsilon_i)$, and we have $n \ge \frac{max2(o_i-\varepsilon_i)-2o_k}{1-2\alpha}$. Because $min\varepsilon_k = (\beta-\frac{1}{2})(b-a) = (\beta-\frac{1}{2})n$ can be obtained when $a+b = 2o_k$, obviously, once n takes the minimum value, namely when $n = \frac{max2(o_i-\varepsilon_i)-2o_k}{1-2\alpha} = b-a$, ε_k^* exists. Due to the formulas of $a+b$ and $b-a$, we obtain

$$\begin{cases} a = \frac{max\ (o_i-\varepsilon_i)+(2-2\alpha)o_k}{1-2\alpha} \\ b = \frac{max\ (o_i-\varepsilon_i)-2\alpha o_k}{1-2\alpha} \\ \varepsilon_k^* = (\beta - \frac{1}{2})(b-a) \end{cases}$$

2. If $2o_k < E$, then $a+b > 2o_k$, so we get $min\ \varepsilon_k = \frac{a+b}{2} - o_k$. Obviously, ε_k is increasing with $(a+b)$. Thus, if $a+b$ takes the minimum value, that is, $a+b = m = E = max\ 2(o_i-\varepsilon_i) + (2\alpha-1)n$, ε_k^* exists. Based on the constraint of (B1-2), $a = b$ holds (i.e. $n = 0$). Therefore, once $a = b = max\ (o_i-\varepsilon_i)$, we always have $\varepsilon_k^* = max\ (o_i-\varepsilon_i) - o_k$.

As a result, if $o_k < max\ (o_i-\varepsilon_i)$, $\varepsilon_k^* = min\ \{(\beta-\frac{1}{2})(b-a), max\ (o_i-\varepsilon_i) - o_k\}$, where if $\varepsilon_k^* = (\beta-\frac{1}{2})(b-a)$, we have $a = \frac{max\ (o_i-\varepsilon_i)+(2-2\alpha)o_k}{1-2\alpha}$ and $b = \frac{max\ (o_i-\varepsilon_i)-2\alpha o_k}{1-2\alpha}$; if $\varepsilon_k^* = max\ (o_i-\varepsilon_i) - o_k$, we get $a = b = max\ (o_i-\varepsilon_i)$.

Situation 2: When $\frac{1}{2} \le \alpha \le 1$ and $\frac{1}{2} \le \beta \le 1$, we obtain $2\alpha - 1 \ge 0$, as $n = b-a \ge 0$, making E monotonically increase with respect to n and F monotonically decrease with respect to n. Then, we divide a similar discussion into three scenarios.

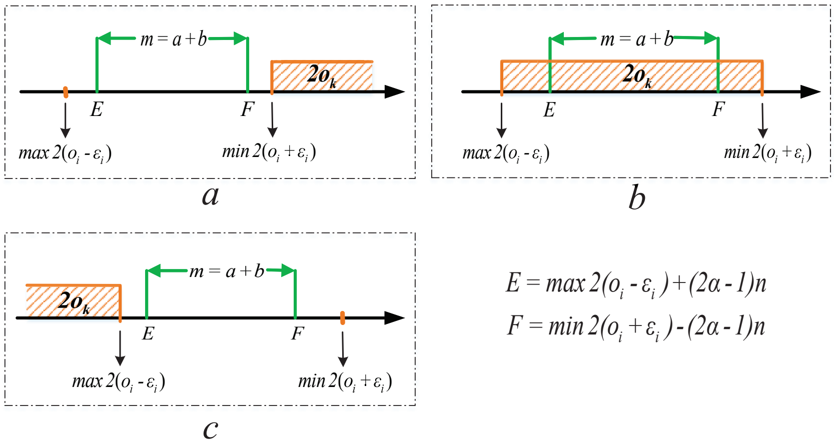

Fig. B2 Discussion on $\frac{1}{2} < \alpha \leq 1$ and $\frac{1}{2} \leq \beta \leq 1$.

- As shown in Fig. B2(a), if $o_k > min\ (o_i + \varepsilon_i)$, then $2o_k > min\ 2(o_i + \varepsilon_i)$; thus, $m = a + b \leq min\ 2(o_i + \varepsilon_i) - (2\alpha - 1)n \leq min\ 2(o_i + \varepsilon_i) < 2o_k$ and then $min\ \varepsilon_k = o_k - \frac{a+b}{2}$. Obviously, ε_k is a monotonically decreasing function of $(a + b)$. In view of formula (B1-3) in Part 1, as $a = b$ and $a + b = min\ 2(o_i + \varepsilon_i)$, we have $a = b = min\ (o_i + \varepsilon_i)$ and $\varepsilon_k^* = o_k - min\ (o_i + \varepsilon_i)$.

- As shown in Fig. B2(b), if $max\ (o_i - \varepsilon_i) \leq o_k \leq min\ (o_i + \varepsilon_i)$, then $max\ 2(o_i - \varepsilon_i) \leq 2o_k \leq min\ 2(o_i + \varepsilon_i)$; if and only if $n = 0$, then $m = a + b = 2o_k$ holds. Thus, $a = b = o_k$ and $\varepsilon_k^* = (\beta - \frac{1}{2})(b - a) = 0$.

- As shown in Fig. B2(c), if $o_k < max\ (o_i - \varepsilon_i)$, then $2o_k < max\ 2(o_i - \varepsilon_i) \leq max\ 2(o_i - \varepsilon_i) + (2\alpha - 1)n \leq m = a + b$. Considering the formula (B1-2) in Part 1, we obtain $a = b$ and $min\ \varepsilon_k = \frac{a+b}{2} - o_k$. Obviously, ε_k increases with $(a + b)$, so if $a = b = max\ (o_i - \varepsilon_i)$, $\varepsilon_k^* = max\ (o_i - \varepsilon_i) - o_k$ holds.

Above all, conditions of the existence for ε_k^* and o^* in Model (9) are obtained as Theorem 6. □

Appendix C. Proof of Theorem 8.

Proof. From constraints (15-3) and (15-4), we have $(1 - \alpha)a_i + \alpha b_i - \varepsilon_i \leq o' \leq \alpha a_i + (1 - \alpha)b_i + \varepsilon_i$, where $i \in N, i \neq k$. The above inequalities are equivalent to $max\{(1-\alpha)a_i + \alpha b_i - \varepsilon_i\} \leq o' \leq min\{\alpha a_i + (1 - \alpha)b_i + \varepsilon_i\}$. For simplicity, let $G = max\{(1 - \alpha)a_i + \alpha b_i - \varepsilon_i\}$ and $H = min\{\alpha a_i + (1 - \alpha)b_i + \varepsilon_i\}$, $(i \in N, i \neq k)$, such that $G \leq o' \leq H$.

From constraints (15-1) and (15-2), we have

- If $(1 - \beta)a_k + \beta b_k - o' = o' - [\beta a_k + (1 - \beta)b_k]$, we have $a_k + b_k = 2o'$, then $min\ \varepsilon_k = (1 - \beta)a_k + \beta b_k - \frac{a_k+b_k}{2} = (\beta - \frac{1}{2})(b_k - a_k)$. When $\frac{1}{2} \leq \beta \leq 1$, then $\varepsilon_k^* = (\beta - \frac{1}{2})(b_k - a_k) \geq 0$ holds, and when $0 \leq \beta \leq \frac{1}{2}$, $(\beta - \frac{1}{2})(b_k - a_k) \leq 0$, for $\varepsilon_k \geq 0$, we have $\varepsilon_k^* = 0$.

- If $(1 - \beta)a_k + \beta b_k - o' < o' - [\beta a_k + (1 - \beta)b_k]$, we have $a_k + b_k < 2o'$, and if and only if $\frac{1}{2} \leq \beta \leq 1$, $\varepsilon_k^* = o' - [\beta a_k + (1 - \beta)b_k] > 0$ holds; when $0 \leq \beta \leq \frac{1}{2}$, for $\varepsilon_k \in [0, o' - \frac{a_k+b_k}{2}]$; thus, $\varepsilon_k^* = 0$.

- If $(1-\beta)a_k+\beta b_k-o' > o'-[\beta a_k+(1-\beta)b_k]$, we have $a_k+b_k > 2o'$, and if and only if $\frac{1}{2} \le \beta \le 1$, $\varepsilon_k^* = (1-\beta)a_k+\beta b_k - o' > 0$ holds; when $0 \le \beta \le \frac{1}{2}$, then $\varepsilon_k \in [0, \frac{a_k+b_k}{2} - o']$; thus, $\varepsilon_k^* = 0$.

If $\frac{1}{2} \le \beta \le 1$, we should discuss the sizes of (a_k+b_k) and $2o'$, so as to obtain the existing conditions for o'.

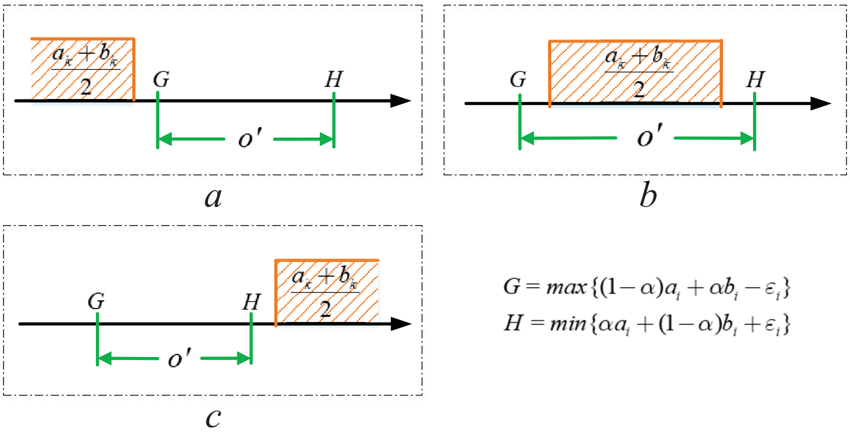

Fig. C1 Comparative analysis between o' and $\frac{a_k+b_k}{2}$.

- As shown in Fig. C1(a), if $\frac{a_k+b_k}{2} < G$, namely $a_k+b_k < 2o'$, then $min\varepsilon_k = o'-[\beta a_k+(1-\beta)b_k]$ is a monotonically increasing function with respect to o'. Therefore, when $o' = G = max\{(1-\alpha)a_i+\alpha b_i-\varepsilon_i\}$, ε_k^* exists, and $\varepsilon_k^* = max\{(1-\alpha)a_i+\alpha b_i-\varepsilon_i\} - \beta a_k-(1-\beta)b_k$.

- As shown in Fig. C1(b), if $G \le \frac{a_k+b_k}{2} \le H$, then $a_k+b_k = 2o'$ holds; thus, when $o' = \frac{a_k+b_k}{2}$, $\varepsilon_k^* = (\beta-\frac{1}{2})(b_k-a_k)$.

- As shown in Fig. C1(c), if $\frac{a_k+b_k}{2} > H$, namely $a_k+b_k > 2o'$, then $min\varepsilon_k = (1-\beta)a_k+\beta b_k-o'$ decreases with o'. Therefore, when $o' = H = min\{\alpha a_i+(1-\alpha)b_i+\varepsilon_i\}$, ε_k^* exists and $\varepsilon_k^* = (1-\beta)a_k+\beta b_k - min\{\alpha a_i+(1-\alpha)b_i+\varepsilon_i\}$.

When $0 \le \beta \le \frac{1}{2}$, due to the constraints of (15-1) and (15-2), we have $o' \in [\frac{a_k+b_k}{2}, \beta a_k+(1-\beta)b_k]$ or $o' \in [(1-\beta)a_k+\beta b_k, \frac{a_k+b_k}{2}]$. Then, $o' \in [(1-\beta)a_k+\beta b_k, \beta a_k+(1-\beta)b_k]$ and $\varepsilon_k^* = 0$. However, because of $G \le o' \le H$, we need to comprehensively discuss the final threshold of o'.

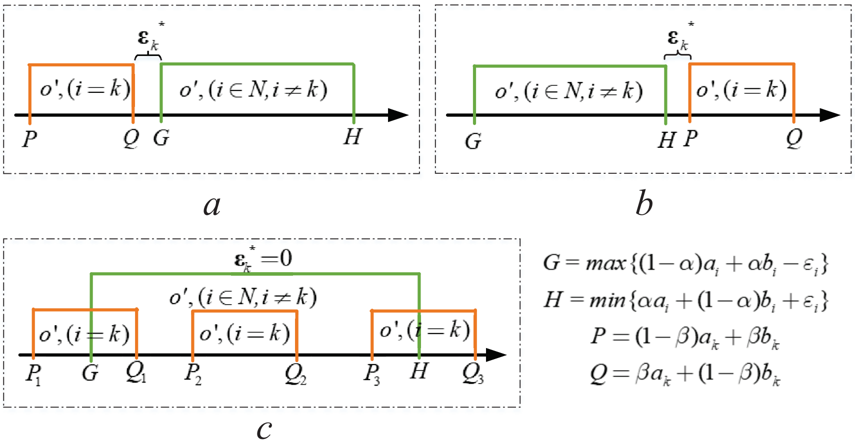

Fig. C2 Discussion on the analytic formulas of the objective function depends on o'.

- As shown in Fig. C2(a), if $\beta a_k + (1-\beta)b_k < G$, then $\varepsilon_k^* = o' - [\beta a_k + (1-\beta)b_k]$, $o' = G$.
- As shown in Fig. C2(b), if $(1-\beta)a_k + \beta b_k > H$, then $\varepsilon_k^* = (1-\beta)a_k + \beta b_k - o'$, $o' = H$.
- As shown in Fig. C2(c), if other circumstances are met, then $\varepsilon_k^* = 0$ and $o' \in [(1-\beta)a_k + \beta b_k, \beta a_k + (1-\beta)b_k] \bigcap [G, H]$.

Thus, this completes the proof for Theorem 8. □

Appendix D. Equivalent forms of carbon quota MCCMs in Case (1-5)

Table D1 Equivalent forms of carbon quota MCCMs in Case (1-5)

Cases	Models
Case 1	$Model(4-11):\ Min\ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4$ $s.t. \begin{cases} \varepsilon_1 \geq (1-\alpha)a + \alpha b - 0,\ \varepsilon_1 \geq -(1-\alpha)b - \alpha a + 0 \\ \varepsilon_2 \geq (1-\alpha)a + \alpha b - 3,\ \varepsilon_2 \geq -(1-\alpha)b - \alpha a + 3 \\ \varepsilon_3 \geq (1-\alpha)a + \alpha b - 6,\ \varepsilon_3 \geq -(1-\alpha)b - \alpha a + 6 \\ \varepsilon_4 \geq (1-\alpha)a + \alpha b - 10,\ \varepsilon_4 \geq -(1-\alpha)b - \alpha a + 10 \\ a \leq b,\ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i = 1,2,3,4 \end{cases}$
Case 2	$Model(4-21):\ Min\ Z = 3*\varepsilon_3$ $s.t. \begin{cases} (1-\beta)a + \beta b - 6 \leq \varepsilon_3,\ -\beta a + (\beta-1)b + 6 \leq \varepsilon_3 \\ (1-\alpha)a + \alpha b - 0 \leq \varepsilon_1,\ -\alpha a + (\alpha-1)b + 0 \leq \varepsilon_1 \\ (1-\alpha)a + \alpha b - 3 \leq \varepsilon_2,\ -\alpha a + (\alpha-1)b + 3 \leq \varepsilon_2 \\ (1-\alpha)a + \alpha b - 10 \leq \varepsilon_4,\ -\alpha a + (\alpha-1)b + 10 \leq \varepsilon_4 \\ 0 \leq \varepsilon_1 \leq 5,\ 0 \leq \varepsilon_2 \leq 4,\ 0 \leq \varepsilon_4 \leq 6,\ \varepsilon_3 \geq 0 \\ a \leq b,\ 0 \leq \alpha,\ \beta \leq 1 \end{cases}$
Case 3	$Model(4-31):\ Min\ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4$ $s.t. \begin{cases} 14(1-\alpha) + 37\alpha - o' \leq \varepsilon_1,\ o' - 14\alpha - 37(1-\alpha) \leq \varepsilon_1 \\ 22(1-\alpha) + 30\alpha - o' \leq \varepsilon_2,\ o' - 22\alpha - 30(1-\alpha) \leq \varepsilon_2 \\ 64(1-\alpha) + 153\alpha - o' \leq \varepsilon_3,\ o' - 64\alpha - 153(1-\alpha) \leq \varepsilon_3 \\ 8(1-\alpha) + 61\alpha - o' \leq \varepsilon_4,\ o' - 8\alpha - 61(1-\alpha) \leq \varepsilon_4 \\ o' \geq 0,\ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i = 1,2,3,4 \end{cases}$
Case 4	$Model(4-41):\ Min\ Z = 3*\varepsilon_3$ $s.t. \begin{cases} 64(1-\beta) + 153\beta - o' \leq \varepsilon_3, o' - [64\beta + 153(1-\beta)] \leq \varepsilon_3 \\ 14(1-\alpha) + 37\alpha - o' \leq \varepsilon_1,\ o' - [14\alpha + 37(1-\alpha)] \leq \varepsilon_1 \\ 22(1-\alpha) + 30\alpha - o' \leq \varepsilon_2,\ o' - [22\alpha + 30(1-\alpha)] \leq \varepsilon_2 \\ 8(1-\alpha) + 61\alpha - o' \leq \varepsilon_4,\ o' - [8\alpha + 61(1-\alpha)] \leq \varepsilon_4 \\ 0 \leq \varepsilon_1 \leq 12,\ 0 \leq \varepsilon_2 \leq 5,\ 0 \leq \varepsilon_4 \leq 36 \\ o' \geq o,\ \varepsilon_3 \geq 0 \end{cases}$
Case 5	$Model(4-51):\ Min\ \phi = 1*\varepsilon_1 + 2*\varepsilon_2 + 3*\varepsilon_3 + 1*\varepsilon_4$ $s.t. \begin{cases} a + (b-a)\alpha + (37-14)\alpha - 37 \leq \varepsilon_1,\ (37-14)\alpha + (b-a)\alpha - b + 14 \leq \varepsilon_1 \\ a + (b-a)\alpha + (30-22)\alpha - 30 \leq \varepsilon_2,\ (30-22)\alpha + (b-a)\alpha - b + 22 \leq \varepsilon_2 \\ a + (b-a)\alpha + (153-64)\alpha - 153 \leq \varepsilon_3,\ (153-64)\alpha + (b-a)\alpha - b + 64 \leq \varepsilon_3 \\ a + (b-a)\alpha + (61-8)\alpha - 61 \leq \varepsilon_4,\ (61-8)\alpha + (b-a)\alpha - b + 8 \leq \varepsilon_4 \\ a \leq b,\ \ 0 \leq \alpha \leq 1,\ \varepsilon_i \geq 0,\ i = 1,2,3,4 \end{cases}$

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2 Consensus reaching in group decision making with linear uncertain preferences and asymmetric costs

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Consensus Reaching in Group Decision Making with Linear Uncertain Preferences and Asymmetric Costs

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***Abstract*—Consensus reaching process (CRP), as an essential part of group decision making (GDM), can facilitate more effective consensus by taking human behaviors into account. Extending the established research on uncertain minimum cost consensus models (MCCMs), this paper continues to adopt linear uncertainty distributions (LUDs) to represent decision-maker's (DM's) preference, but considers asymmetric costs into a new framework of CRP, where DM's preference and weight are both adjusted according to democratic consensus. Moreover, in light of the uncertain distance measure, two novel optimization-based consensus models are built in this paper: one is to obtain a minimum cost consensus by simultaneously considering asymmetric costs, aggregation function and consensus measure; while the other provides a more flexible way to address GDM problems without pre-setting a specific consensus level (CL) threshold. Some 0-1 binary variables are further introduced to reduce the calculation complexity resulted from piecewise functions in the new multi-coefficient goal programming models. Finally, an illustrative example and further discussion reveal the feasibility and superiority of our new method.**

***Index Terms*—Group decision making (GDM); Consensus reaching process (CRP); Asymmetric costs; Uncertainty theory; Feedback mechanism.**

I. INTRODUCTION

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DECISION making is important for human daily life, featured by the motivation to select an optimal solution from at least two alternatives [1] or achieve a unanimous agreement where decision-makers (DMs) express various feelings as to the values in question [2]. Practically speaking, multiple DMs participate in a dynamic and negotiable process so as to derive a common solution on behalf of the collective interest, thereby constituting a group decision making (GDM) problem [3]. In other words, eliminating disagreements among DMs becomes necessary to obtain an agreed solution, which is widely appreciated by all stakeholders in real-life scenarios [4]. Without loss of generality, a consensus reaching process (CRP) introduced in GDM can effectively bring DMs' views closer to each other through limited rounds of compromises and adjustments [5]. In fact, Herrera-Viedma et al. [2] concluded that consensus boils down to cooperation, while most GDM boils down to competition.

Generally, a CRP consists of three steps as consensus measure, feedback process and selection [6], while an effective CRP relies heavily on its feedback mechanism, where identification rule (IR) and direction rule (DR) are exploited [7]. To date, developing an automatic feedback mechanism that excludes the moderator's influence has been increasingly crucial to consensus [2]. On the other hand, several indicators are commonly used to measure the CRP performance, such as the minimum deviation or cost [8], the maximum number of experts adjusted under a limited budget [9] or the minimum number of adjusted DMs [10]. Moreover, Labella et al. [11] proposed a new metric to comprehensively evaluate the performance. Given the significance of deriving a consensus for GDM, see [2], [5] for more studies on feedback mechanisms.

The concept of the minimum cost consensus (MCC) was originally proposed by Ben-Arieh and Easton [12] for solving single and multi-criteria GDM problems via linear-time algorithms. Later, a quadratic cost function was adopted to discuss the influence of different factors (e.g., cost, opinion elasticity, the number of adjusted experts) on the consensus [9]. At the same period, Dong et al.'s optimization-based consensus models [13], recognized as the minimum adjustment consensus models (MACMs), aim to maximize the retention of DM's original preference, rather than pursuing a minimum resource consumption. Afterwards, although abundant studies have been conducted [14]–[16], most are based on traditional preference structures (e.g., crisp numbers, intervals or linguistic information), neglecting the characteristics of stochastic distribution in DM's preference. In contrast, linear uncertainty distributions

(LUDs) with belief degrees provide a feasible way to better simulate DM's uncertainty and ambiguous behaviors in actual GDM problems [17].

When building the minimum cost consensus models (MCCMs), unit costs (i.e., the resources used to persuade DMs to adjust their preference one unit towards the consensus) are usually preset as fixed values [9], [12], which have been criticized due to over-idealization. Therefore, under the premise of unit adjustment costs being intervals, Li et al. [7] facilitated a consensus through optimization modeling, while Zhang et al. [8] further took Stackelberg game into consideration. Moreover, [18] associated the variable unit costs with the incompleteness degree of the DM's linguistic distribution assessments. In contrast, Cheng et al. [19] analyzed the impact of experts' compromise limit and tolerance behaviors on MCCMs based on one hypothesis that cost coefficients are asymmetric due to individual different adjustment directions. Subsequently, the work in [19] was extended through stochastic programming [20] or data-driven robust optimization [21]. As a result, how to make the best use of unit costs to portray DM's psychological behavior in GDM is worth investigating.

In addition to the above unit costs, weights also play a vital role in the CRP, measuring the relative importance of each criterion/DM [22]. Traditionally, weights attached to DMs in the MCCMs and MACMs are either assigned directly by the moderator or automatically preset according to their preferences [23]. However, individuals involved in GDM prefer their influence (i.e., the weight coefficients) to change over time as they compromise to achieve the consensus, which raises a new question of how to determine these coefficients to avoid strategic manipulation [24]. Hence, we adopt the idea that important DMs contribute more to the consensus [11]; meanwhile, the concept of democratic consensus [25] is also incorporated into our feedback mechanism during the CRP, where DMs' weights are variable with the proceeding of GDM. Specifically, DMs' initial weights are equally assigned, but once an initial consensus derived, their weights will be automatically reallocated based on their own contribution to the consensus level (CL). Briefly, a new feedback mechanism excluding the moderator is proposed, by sufficiently respecting individual values via DM's established preference information.

In other words, the final results obtained from GDM are always expected to be as objective as possible; but it's tough to accurately predict the outcomes in advance, due to inherent subjectivity, imprecision and vagueness in DM's preference articulation [2]. Pragmatically, valid information in real-life GDM is insufficient to derive the probability, not to mention that information sources frequently conflict with each other [26], leaving the reliability of the occurrence of certain events to be determined by domain experts. As a result, uncertainty theory proposed by Liu [27] becomes a new mathematical tool to address uncertain phenomenon characterized by non-randomness and non-fuzziness. So far, Liu's theory has been verified feasible to solve GDM problems with low frequency or small sample [17], where the probability cannot be fitted by frequency. In fact, the LUD provides a more inclusive and richer expression form for individuals in GDM [28]. Hence, this paper continues to adopt the LUD to fit DM's preference under a new framework of CRP.

Existing studies greatly contributed to the development of GDM theory, but none has comprehensively explored the CRP framework combined with uncertain MCCMs, asymmetric costs, aggregation function, consensus measure and feedback mechanism. Specifically,

- Prior uncertain MCCMs do not consider asymmetric costs nor the dynamic characteristic of GDM [28], [29]. In other words, they primarily focused on optimization-based consensus modeling while neglecting DM's unbalanced willingness to adjust.
- Extant MCCMs with asymmetric costs do not consider aggregation functions nor feedback mechanisms [19], [20]. Namely, they don't aggregate DMs' choices into a collective wisdom [14], thereby failing to portray social choices or individual values.
- Current studies on the CRP do not take into account DMs' changeable influence during neither the consensus measure nor the feedback mechanism [15], [18], making the importance of different individuals unable to be fully demonstrated.

To conclude, this paper extends the established uncertainty theory-based MCCMs into a general CRP framework. Specifically, our main contributions are threefold: (1) new optimization-based consensus models are built by simultaneously considering aggregation functions, asymmetric costs and consensus measure; (2) a novel CRP framework is proposed by respecting individual values with democratic consensus and simultaneously pursuing a minimum resource consumption based on uncertain MCCMs; and (3) binary variables are introduced to reduce the computational complexity of piecewise functions in the new multi-coefficient programming models.

The rest is organized as follows. Section II recalls preliminaries regarding uncertainty theory and traditional consensus GDM theory. Section III reviews methods of consensus measure and a general CRP framework. Section IV develops two new uncertain consensus models with asymmetric costs. Moreover, Section V gives a numerical example, while further discussion is provided in Section VI. Finally, Section VII gives concluding remarks and directions for subsequent research.

II. Preliminaries

This section recalls basic knowledge of uncertainty theory (see II-A) and classic consensus models (see II-B).

A. Uncertainty theory

When no samples are available or only poor information obtained from historical data, the estimated distribution function will deviate far from the actual frequency, causing the law of large numbers invalid, and further obtaining some counterintuitive results. Thus, some domain experts are invited to evaluate the belief degree that certain events will happen. Distinguished from probability theory dealing with randomness of frequency, uncertainty theory was proposed to address the uncertainty of belief degrees.

1) Uncertain variable and uncertainty distribution:

Definition *1:* [27] Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $M : \mathcal{L} \to [0,1]$ is called an uncertainty measure if and only if (i) $M\{\Gamma\} = 1$; (ii) $M\{\Lambda\} + M\{\Lambda^c\} = 1$; (iii) $M\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}$; and (iv) $M\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} M\{\Lambda_k\}$.

For an uncertain variable ξ, its uncertainty distribution Φ is defined as $\Phi(x) = M\{\xi \leq x\}$, where $M\{\xi \leq x\}$ is the belief degree of the event $\{\xi \leq x\}$. Meanwhile, the belief degree is also measured by α, $\alpha \in [0,1]$. Thus, $\Phi(x) = M\{\xi \leq x\} = \alpha$ holds for any real number x.

Definition *2:* [30] An uncertainty distribution $\Phi(x)$ is regular if it is a continuous and strictly increasing function with regard to x at which $0 < \Phi(x) < 1$, and satisfies

$$\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1 \tag{1}$$

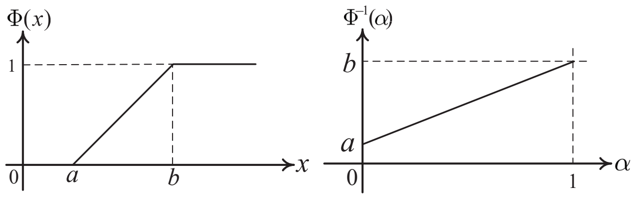
An uncertainty distribution Φ is regular iff its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$.

Example *1:* An uncertain variable ξ is called linear if it has a LUD $\Phi(x)$ (see Fig. 1(a)), denoted by $\xi \sim \mathcal{L}(a,b)$, where real numbers a and b satisfy $a < b$.

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ \dfrac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } x \geq b \end{cases} \tag{2}$$

And its inverse uncertainty distribution (see Fig. 1(b)) is

$$\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b \tag{3}$$



(a) The linear uncertainty distribution (b) The inverse linear uncertainty distribution

Fig. 1: Schematic diagram of a linear uncertain variable

2) Basic properties of uncertain variables:

Theorem *1:* [30] Let ξ be an uncertain variable with an uncertainty distribution Φ, then for any real number x (i.e., $x \in \mathcal{R}$), we have

$$M\{\xi \leq x\} = \Phi(x), \quad M\{\xi > x\} = 1 - \Phi(x) \tag{4}$$

To be noted, when the uncertainty distribution $\Phi(x)$ is a continuous function, we have $M\{\xi \leq x\} = M\{\xi < x\} = \Phi(x)$, and $M\{\xi > x\} = M\{\xi \geq x\} = 1 - \Phi(x)$.

Theorem *2:* [30] Let $\xi_1, \xi_2 \cdots \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2 \cdots \Phi_n$. If $f(\xi_1 \cdots \xi_m, \xi_{m+1} \cdots \xi_n)$ strictly increases with $\xi_1 \cdots \xi_m$ and decreases with $\xi_{m+1} \cdots \xi_n$, then $f(\xi_1 \cdots \xi_m, \xi_{m+1} \cdots \xi_n)$ has an inverse uncertainty distribution $f(\Phi_1^{-1}(\alpha) \cdots \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha) \cdots \Phi_n^{-1}(1-\alpha))$.

Example *2:* Let ξ_1 and ξ_2 be independent uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Then the inverse uncertainty distribution of $\xi_1 - \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1-\alpha) \tag{5}$$

Theorem *3:* [31] Let ξ and ς be independent uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Then the distance between ξ and ς is

$$d(\xi, \varsigma) = \int_0^1 |\Phi^{-1}(\alpha) - \Psi^{-1}(1-\alpha)| d\alpha. \tag{6}$$

Example *3:* Let ξ and ς be independent uncertain variables obeying LUDs as $\xi \sim \mathcal{L}(a,b)$ and $\varsigma \sim \mathcal{L}(c,d)$, where $a \leq b$ and $c \leq d$. Based on Example 1, the distance between ξ and ς becomes

$$d(\xi, \varsigma) = \int_0^1 |(b+d-a-c)\alpha + a - d| d\alpha \tag{7}$$

B. Traditional consensus GDM theory

Suppose m DMs participate in a GDM problem, and a finite set $D = \{d_1, d_2, \cdots, d_m\}$ denotes all individuals. Let $o_i \in \mathcal{R}$ denote d_i's original preference, $\bar{o_i}$ be d_i's adjusted preference, and o^c be their reached consensus. In addition, $c_i \in \mathcal{R}^+$ denotes the unit cost of adjusting d_i's preference closer to the consensus, while $w_i \in \mathcal{R}^+$ reflects d_i's importance degree (i.e., weight), $i \in M = \{1, 2, \cdots, m\}$.

1) Traditional consensus models: DMs are normally willing to change opinions after repetitive negotiation efforts, though it escalates the costs of reaching a consensus. Adopting the p-norm distance measure (i.e., $|| \; ||_p, p \geq 1$), Ben-Arieh and Easton [12] provided a linear-time algorithm to seek the optimal MCC o^{c*} by minimizing the weighted total cost $f_c(o^c) = \sum_{i=1}^m w_i c_i ||o^c - o_i||_p$. Later, they discussed such scenarios with/without an ε-consensus (denoted as $|o^c - \bar{o_i}| \leq \varepsilon, \varepsilon > 0$) [9]. For brevity, Ref. [14] develops an optimization model to represent their ideas.

$$\begin{aligned} &\min \ \phi = \sum_{i=1}^{m} c_i |\bar{o_i} - o_i| \\ &\text{s.t.} \ \begin{cases} |o^c - \bar{o_i}| \leq \varepsilon_i, \ i \in M \\ o^c \in \mathcal{R}, \ \bar{o_i} \in \mathcal{R} \end{cases} \end{aligned} \tag{8}$$

Solving Model (8) yields the optimal consensus o^{c*} and DM's optimal adjusted preference $\bar{o_i}^*$. The first constraint denotes a tolerance behavior, and if $\varepsilon_i = 0$ ($\forall i \in M$), a hard consensus is achieved (i.e., $\bar{o_i}^* = o^{c*}$), which is unrealistic and uneconomical for most GDM problems [2].

Meanwhile, Dong et al. [13] utilized the ordered weighted averaging (OWA) operator and a deviation measure to handle the consensus problems under a 2-tuple fuzzy linguistic environment, so as to preserve DMs' original preferences as much as possible. Similarly, their main ideas can be mathematically described as follows.

$$\begin{aligned} &\min \ \phi = \sum_{i=1}^{m} d(\bar{o_i}, o_i) \\ &\text{s.t.} \ \begin{cases} d(o^c, \bar{o_i}) \leq \varepsilon, \ i \in M \\ o^c = F_w(\bar{o_1}, \bar{o_2}, \cdots, \bar{o_m}) \end{cases} \end{aligned} \tag{9}$$

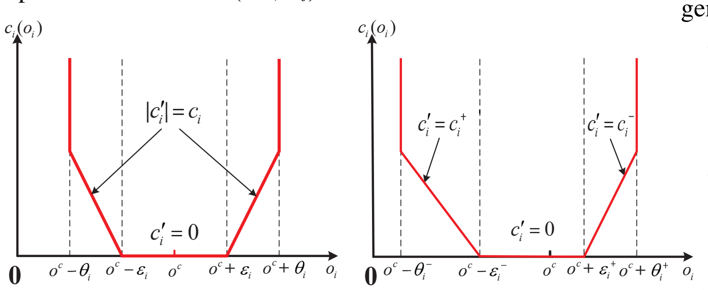
where $d(\cdot)$ represents the rectilinear or Euclidean deviation measure, F_w denotes an aggregation function, and the objective function is to minimize all DMs' adjustments.

Although Model (8) and Model (9) were proposed according to different consensus mechanisms, Zhang et al. [14] later argued that these two models could actually be merged as

$$\min \phi = \sum_{i=1}^{m} c_i * d(\bar{o}_i, o_i)$$
$$\text{s.t.} \begin{cases} d(o^c, \bar{o}_i) \le \varepsilon, \ i \in M \\ o^c = F_w(\bar{o}_1, \bar{o}_2, \cdots, \bar{o}_m) \end{cases} \tag{10}$$

It turns out that once F_w takes the OWA operator as $(\frac{1}{2} \cdots 0 \cdots \frac{1}{2})^{\mathrm{T}}$, Model (10) reduces to Model (8); and if the unit costs of adjusting DMs' opinions satisfy $c_i = c_j, \forall i, j \in M$, then Model (10) equals Model (9).

2) Consensus models with asymmetric costs: Cheng et al. [19], [32] explored GDM problems with cost constraints based on DM's different adjustment directions, further extending Model (10). In specific, Fig. 2 depicts their cost functions under a symmetric or asymmetric scenario by considering DM's tolerance and compromise behaviors, where the horizontal axis is DM's original preference (i.e., o_i), and the vertical axis represents a unit cost (i.e., c_i).



(a) The symmetric cost scenario (b) The asymmetric cost scenario

Fig. 2: Cost functions with tolerance and compromise

Due to space limitations, only the asymmetric cost scenario is reported hereafter [19]. That is, once taking the tolerance and compromise limit into account, d_i's optimal adjusted preference is obtained as

$$\bar{o}_i^* = \begin{cases} o^c - \varepsilon_i^-, & \text{if } o_i \in [o^c - \theta_i^-, o^c - \varepsilon_i^-) \\ o_i, & \text{if } o_i \in [o^c - \varepsilon_i^-, o^c + \varepsilon_i^+] \\ o^c + \varepsilon_i^+, & \text{if } o_i \in (o^c + \varepsilon_i^+, o^c + \theta_i^+] \end{cases} \tag{11}$$

The total cost for d_i to adjust preference becomes

$$c_i(o_i) = \begin{cases} c_i^+(o^c - \varepsilon_i^- - o_i), & \text{if } o_i \in [o^c - \theta_i^-, o^c - \varepsilon_i^-) \\ 0, & \text{if } o_i \in [o^c - \varepsilon_i^-, o^c + \varepsilon_i^+] \\ c_i^-(o_i - o^c - \varepsilon_i^+), & \text{if } o_i \in (o^c + \varepsilon_i^+, o^c + \theta_i^+] \end{cases} \tag{12}$$

where c_i^- denotes d_i's unit cost with a downward adjustment, and c_i^+ conversely indicates the unit cost of an upward adjustment. In addition, ε_i measures d_i's tolerance of the consensus, while θ_i reflects d_i's compromise limit.

To be more specific, Fig. 2(b) shows that once d_i's original preference is located at $[o^c - \varepsilon_i^-, o^c + \varepsilon_i^+]$, a tolerance behavior exists (also known as the soft consensus [2]), namely, any preference within this subinterval is acceptable, thereby requiring no adjustments nor yielding any costs. Moreover, any preference located at $[o^c - \theta_i^-, o^c - \varepsilon_i^-)$ or $(o^c + \varepsilon_i^+, o^c + \theta_i^+]$ corresponds to a compromise limit behavior, inducing a total cost of $c_i^+(o^c - \varepsilon_i^- - o_i)$ or $c_i^-(o_i - o^c - \varepsilon_i^+)$, respectively. In fact, too many preference adjustments go against DM's willingness, and incur unnecessary extra costs, thus, those original preferences smaller than $o^c - \theta_i^-$ or larger than $o^c + \theta_i^+$ won't be discussed, however, DMs can refresh their preferences to rejoin the GDM process. In this regard, the objective function of Model (10) is revised into

$$\begin{aligned} \min \phi = & \sum_{i:\ o_i \in [o^c - \theta_i^-, o^c - \varepsilon_i^-)} c_i^+(o^c - \varepsilon_i^- - o_i) \\ & + \sum_{i:\ o_i \in (o^c + \varepsilon_i^+, o^c + \theta_i^+]} c_i^-(o_i - o^c - \varepsilon_i^+) \end{aligned} \tag{13}$$

III. General CRP Framework Regarding GDM Problems

Methods to obtain the CL are reviewed in III-A, while a general framework of CRP is recalled in III-B.

A. Consensus measure

Consensus level (CL), the current level of unanimous within a group, is often calculated by distance functions [4] and generally measured in two ways [11]:

- The distance between DM's preference and the collective preference, shown as Eq. (14);

$$CL(o_1, \ldots, o_m) = 1 - f_2(f_1(d(o_i, o^c))) \ge \beta \tag{14}$$

- The distance between two arbitrarily chosen individual preferences, shown as Eq. (15).

$$CL(o_1, \ldots, o_m) = 1 - g_2(g_1(d(o_i, o_j))) \ge \beta, i \ne j \tag{15}$$

where β is a preset CL threshold, $d(\cdot)$ represents the distance measure, $f_1 : \mathcal{R}^+ \to \mathcal{R}^+$, $f_2 : \mathcal{R}^+ \to [0, 1]$, $g_1 : \mathcal{R}^+ \to \mathcal{R}^+$, $g_2 : \mathcal{R}^+ \to [0, 1]$ are mapping functions, and the remaining notation is defined in Section II-B.

Since DM's influence is directly reflected by the weight w_i with $w_i \ge 0$ and $\sum w_i = 1$, Ref. [11] incorporated individual weights into the calculation of the CL, that is,

$$CL(\bar{o}_1, \ldots, \bar{o}_m) = \sum_{i=1}^{m} w_i |\bar{o}_i - o^c| \le \gamma \tag{16}$$

$$CL(\bar{o}_1, \ldots, \bar{o}_m) = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{w_i + w_j}{m-1} |\bar{o}_i - \bar{o}_j| \le \gamma \tag{17}$$

where $\gamma = 1 - \beta \in [0, 1]$. In fact, both Eqs. (16) and (17) emphasize the greater contribution of the more important DM to the CL, but only Eq. (16) is used in our subsequent analysis.

B. A general CRP framework

Several basic steps constitute a CRP framework for solving consensus GDM problems, that is, preference expression, preference aggregation, consensus measure, preference adjustment and selection. In addition, Herrera-Viedma et al. [6] provided a general consensus framework, shown as Fig. 3, to address GDM problems with heterogeneous preference structures.

Regarding all the above mentioned steps, feedback mechanisms can be initiated based on diverse principles, such as the minimum deviation or cost [33], the maximum number of experts adjusted under a limited budget [9], the minimum number of adjusted DMs [10] or some optimization-based rules [8]. In this paper, a CL threshold is used to initiate DMs' preference modification process.

Table I: Notation and their meanings

Notation	Description
d_i	The i-th DM and $i \in M = \{1, 2, \cdots, m\}$
o_i	d_i's original preference with $o_i \sim \mathcal{L}(a_i, b_i)$ and $0 \le a_i \le b_i \le 1$
$\bar{o_i}$	d_i's adjusted preference with $\bar{o_i} \sim \mathcal{L}(\bar{a_i}, \bar{b_i})$ and $0 \le \bar{a_i} \le \bar{b_i} \le 1$
o^c	Consensus with $o^c \sim \mathcal{L}(a^c, b^c)$ and $0 \le a^c \le b^c \le 1$
α	Belief degree with $\alpha \in [0, 1]$
Φ, $\Phi^{-1}(\alpha)$	Uncertainty distribution and inverse uncertainty distribution
c_i^+, c_i^-	Unit cost of d_i's upward/downward preference adjustment
ε_i, θ_i	d_i's tolerance/compromise limit threshold over consensus
w_i, $\bar{w_i}$	d_i's original/updated weight with $w_i, \bar{w_i} \ge 0$ and $\sum w_i = 1$, $\sum \bar{w_i} = 1$
η	Adjusted parameter of weights attached to CL (e.g., $\eta = 1.5$)
γ, γ_0, $\Delta\gamma$	Thresholds of the overall preference deviation (Dev) with γ=γ_0-$\Delta\gamma$ and $\Delta\gamma \ge 0$
β	Predefined threshold of CL with $\beta = 1 - \gamma$
λ	Trade-off coefficient between CL and the total cost with $\lambda \in [0, 1]$
δ	d_i's preference adjustment willingness with $\delta \in [0, 1]$
z_{i1}, z_{i2}, z_{i3}, y_{i1}, y_{i2}, y_{i3}, x_i	0-1 binary variable

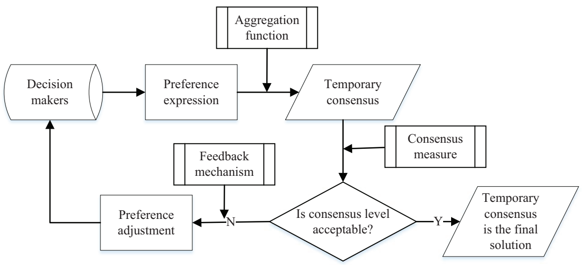


Fig. 3: The general CRP framework in Ref. [6]

IV. Uncertain consensus models considering asymmetric costs and consensus measure

Traditional preference structures (e.g., crisp numbers [9], [12] or linguistic information [13]) are used in Model (8) and Model (9), however, providing exact values is rather difficult. By contrast, DMs are more likely to express their decisions through LUDs, which give sufficient tolerance to their inherent uncertainty and hesitation [17]. Therefore, the LUD is continuously adopted to simulate DM's preference in this paper. For brevity, basic notation is listed in Table I, and four assumptions are given as follows:

- Individual preferences are independent of each other, implying that consensus can be derived through aggregation functions, where variables (e.g., preference and weight) are bounded within [0,1].
- DMs are more sensitive to losses than gains (see prospect theory [34]), so let $|c_i^-| > |c_i^+|$ reflect DM's adjusting willingness, namely, the changing trend of the downward-adjusting subinterval is steeper than that of the upward one (see Fig. 2(b)).
- DM's influence changes with the procedure of GDM [25]. In specific, DMs initially have equal weights, then their influence diversifies due to their own contribution to the CL during the CRP.
- All DMs are committed to reaching a consensus by completely following feedback suggestions, in other words, the non-cooperation behavior [3] is neglected.

Since CL is bounded within [0, 1], the GDM problem discussed can be further simplified: boundary values of DM's preference in Fig. 2(b) are preset as $o^c - \theta_i^- = 0$ and $o^c + \theta_i^+ = 1$. In addition, d_i's tolerance behavior is no longer distinguished as ε_i^+ or ε_i^-, that is, only one parameter (i.e., ε_i) is used to denote d_i's tolerance to the consensus o^c. As a result, Eq. (13) evolves into

$$\min \phi = \sum_{o_i \in [0, o^c - \varepsilon_i)} c_i^+ (o^c - \varepsilon_i - o_i) + \sum_{o_i \in (o^c + \varepsilon_i, 1]} c_i^- (o_i - o^c - \varepsilon_i) \tag{18}$$

A. Uncertain MCCM with asymmetric costs

Theorem 4: [29] Let $o_{i,i \in M}$ be an independent uncertain variable obeying a linear uncertainty distribution (LUD) as $o_i \sim \mathcal{L}(a_i, b_i)$ with $a_i \le b_i$. Then $\sum w_i o_i$ obeys a LUD as $\sum w_i o_i \sim \mathcal{L}(\sum w_i a_i, \sum w_i b_i)$.

Cost is always a crucial metric to measure the GDM quality and efficiency, so we take the total cost minimization as our priority. As stated before, Eq. (16) is used to obtain the CL. Hence, a new uncertain MCCM is built by comprehensively considering asymmetric costs, aggregation function and consensus measure.

$$\begin{aligned} &\min \phi = \textstyle\sum_{i=1}^m \{c_i^+, c_i^-\} * d(\bar{o_i}, o_i) \\ &\text{s.t.} \begin{cases} d(\bar{o_i}, o^c) \le \varepsilon_i & (19-1) \\ o^c = \sum_{i=1}^m w_i \bar{o_i} & (19-2) \\ \sum_{i=1}^m w_i * d(\bar{o_i}, o^c) \le \gamma & (19-3) \\ o_i \sim \mathcal{L}(a_i, b_i), \bar{o_i} \sim \mathcal{L}(\bar{a_i}, \bar{b_i}), o^c \sim \mathcal{L}(a^c, b^c) & (19-4) \\ 0 \le \bar{a_i} \le \bar{b_i} \le 1, 0 \le a^c \le b^c \le 1, i \in M & (19-5) \end{cases} \end{aligned} \tag{19}$$

Solving Model (19) yields the minimum cost ϕ^*, the optimal consensus o^{c*}, and d_i's optimal adjusted preference $\bar{o_i}^*$. Note that, $d(\bar{o_i}, o_i)$ is the distance measure between d_i's adjusted preference $\bar{o_i}$ and original preference o_i; the expression $\{c_i^+, c_i^-\}$ means only one coefficient is taken due to d_i's adjustment direction, corresponding to Eq. (18); (19-1) reflects d_i's tolerance behavior; (19-2) uses the arithmetic weighted averaging (AWA) operator to fuse DMs' preferences; (19-3) is the consensus measure, and (19-4) indicates that all involved preferences obey the LUDs under (19-5).

Theorem 5: [29] Distance between any two independent variables with LUDs, denoted as $\xi \sim \mathcal{L}(a, b)$ and $\varsigma \sim \mathcal{L}(c, d)$ with $a \le b, c \le d$, can be transformed into a piecewise function as

$$d(\xi,\varsigma)=\begin{cases}\dfrac{a+b-c-d}{2}, & \text{if } a>d\\ \dfrac{c+d-a-b}{2}, & \text{if } b<c\\ \dfrac{(d-a)^2}{b+d-a-c+\epsilon}+\dfrac{a+b-c-d}{2}, & \text{otherwise}\end{cases} \tag{20}$$

where ϵ is the non-Archimedean infinitesimal. Hereafter, we take $\epsilon=10^{-6}$ to ensure that a rare case of $a=b$ and $c=d$ still holds in Eq. (20), and then the two uncertain variables essentially degenerate to two real numbers.

Basically, piecewise functions seldomly exist in final models due to calculation complexity, meanwhile, distinguished from the method in [29], which remains the form of absolute values, we use the big M method to transform Eq. (20) into a hybrid 0-1 programming model (i.e., Model (21)). Here, let U be a sufficiently large positive number and we take $U=10^6$ in the subsequent analysis.

$$\begin{aligned}&d(\xi,\varsigma)=z_3*\tfrac{(d-a)^2}{b+d-a-c+\epsilon}+(0.5-z_2)*(a+b-c-d)\\ &\text{s.t.}\begin{cases}-U(1-z_2)(1-z_3)\le d-a<U(1-z_1)\\ -U(1-z_1)(1-z_3)\le b-c<U(1-z_2)\\ -U(1-z_1)<\frac{a+b-c-d}{2}<U(1-z_2)\\ -U(1-z_2)<\frac{c+d-a-b}{2}<U(1-z_1)\\ \frac{(d-a)^2}{b+d-a-c+\epsilon}+\frac{a+b-c-d}{2}>-U*z_2\\ z_1+z_2+z_3=1\\ z_1,z_2,z_3\in\{0,1\}\end{cases}\end{aligned} \tag{21}$$

where z_1,z_2,z_3 are binary variables with one and only one value of 1. For example, if $z_1=1$, then $z_2=0$, $z_3=0$, we get the first case of $a>b$; and if $z_3=1$, then $z_1=0$, $z_2=0$, we have $d\ge a$ and $b\ge c$, which corresponds to the third case. Detailed transformation from Eq. (20) to Model (21), which focuses on the relative positions of the four parameters (i.e., a,b,c and d), is omitted here due to limited space. In addition, another 0-1 variable x_i is introduced to handle the multi-coefficient problem [35] of Model (19) (i.e., $\{c_i^+,c_i^-\}$), we then have Model (22).

$$\begin{aligned}&\min\ \phi=\textstyle\sum_{i=1}^m[x_ic_i^++(1-x_i)c_i^-]*[z_{i3}*\frac{(b_i-\bar{a_i})^2}{\bar{b_i}+b_i-\bar{a_i}-a_i+\epsilon}+(0.5-z_{i2})*(\bar{a_i}+\bar{b_i}-a_i-b_i)]\\ &\text{s.t.}\begin{cases}-U(1-z_{i2})(1-z_{i3})\le b_i-\bar{a_i}<U(1-z_{i1}) & (22-1)\\ -U(1-z_{i1})(1-z_{i3})\le \bar{b_i}-a_i<U(1-z_{i2}) & (22-2)\\ -U(1-z_{i1})<\frac{\bar{a_i}+\bar{b_i}-a_i-b_i}{2}<U(1-z_{i2}) & (22-3)\\ -U(1-z_{i2})<\frac{a_i+b_i-\bar{a_i}-\bar{b_i}}{2}<U(1-z_{i1}) & (22-4)\\ \frac{(b_i-\bar{a_i})^2}{\bar{b_i}+b_i-\bar{a_i}-a_i+\epsilon}+\frac{\bar{a_i}+\bar{b_i}-a_i-b_i}{2}>-U*z_{i2} & (22-5)\\ y_{i3}*\frac{(\sum w_i\bar{b_i}-\bar{a_i})^2}{\bar{b_i}+\sum w_i\bar{b_i}-\bar{a_i}-\sum w_i\bar{a_i}+\epsilon}+(0.5-y_{i2})(\bar{a_i}+\bar{b_i}-\sum w_i\bar{a_i}-\sum w_i\bar{b_i})\le\varepsilon_i & (22-6)\\ -U(1-y_{i2})(1-y_{i3})\le\sum w_i\bar{b_i}-\bar{a_i}<U(1-y_{i1}) & (22-7)\\ -U(1-y_{i1})(1-y_{i3})\le\bar{b_i}-\sum w_i\bar{a_i}<U(1-y_{i2}) & (22-8)\\ -U(1-y_{i1})<\frac{\bar{a_i}+\bar{b_i}-\sum w_i\bar{a_i}-\sum w_i\bar{b_i}}{2}<U(1-y_{i2}) & (22-9)\\ -U(1-y_{i2})<\frac{\sum w_i\bar{a_i}+\sum w_i\bar{b_i}-\bar{a_i}-\bar{b_i}}{2}<U(1-y_{i1}) & (22-10)\\ \frac{(\sum w_i\bar{b_i}-\bar{a_i})^2}{\bar{b_i}+\sum w_i\bar{b_i}-\bar{a_i}-\sum w_i\bar{a_i}+\epsilon}+\frac{\bar{a_i}+\bar{b_i}-\sum w_i\bar{a_i}-\sum w_i\bar{b_i}}{2}>-U*y_{i2} & (22-11)\\ z_{i1}+z_{i2}+z_{i3}=1,\ y_{i1}+y_{i2}+y_{i3}=1 & (22-12)\\ a^c=\sum w_i\bar{a_i},\ b^c=\sum w_i\bar{b_i} & (22-13)\\ \sum_{i=1}^m w_i*[y_{i3}*\frac{(\sum w_i\bar{b_i}-\bar{a_i})^2}{\bar{b_i}+\sum w_i\bar{b_i}-\bar{a_i}-\sum w_i\bar{a_i}+\epsilon}+(0.5-y_{i2})(\bar{a_i}+\bar{b_i}-\sum w_i\bar{a_i}-\sum w_i\bar{b_i})]\le\gamma & (22-14)\\ o_i\sim\mathcal{L}(a_i,b_i),\bar{o_i}\sim\mathcal{L}(\bar{a_i},\bar{b_i}),o^c\sim\mathcal{L}(a^c,b^c) & (22-15)\\ z_{i1},z_{i2},z_{i3},y_{i1},y_{i2},y_{i3},x_i\in\{0,1\},0\le\bar{a_i}\le\bar{b_i}\le1,0\le a^c\le b^c\le1,i\in M & (22-16)\end{cases}\end{aligned} \tag{22}$$

Solving Model (22) yields d_i's optimal adjusted preference $\bar{o_i}^*$ (i.e., $\bar{a_i}^*$, $\bar{b_i}^*$), the optimal consensus o^{c*} (i.e., a^{c*}, b^{c*}), and the minimum cost ϕ^*. Here, DM's preference is fitted by the LUD as $o_i\sim\mathcal{L}(a_i,b_i)$, with known parameters a_i,b_i satisfying $0\le a_i\le b_i\le 1$. The objective function is constituted by two parts: the former $x_ic_i^++(1-x_i)c_i^-$ determines the exact value from the known asymmetric costs c_i^+,c_i^- due to d_i's adjustment direction; while the latter $z_{i3}*\frac{(b_i-\bar{a_i})^2}{\bar{b_i}+b_i-\bar{a_i}-a_i+\epsilon}+(0.5-z_{i2})*(\bar{a_i}+\bar{b_i}-a_i-b_i)$ is transformed from the distance measure $d(\bar{o_i},o_i)$ due to Model (21), satisfying (22-1)-(22-5) and (22-12). In specific, if a DM adjusts the preference upwards, the asymmetric cost takes c_i^+ with $x_i=1$; otherwise, the unit cost is c_i^- with $x_i=0$, where $|c_i^-|>|c_i^+|$ corresponds to the previous second assumption. Similarly, the expression $y_{i3}*\frac{(\sum w_i\bar{b_i}-\bar{a_i})^2}{\bar{b_i}+\sum w_i\bar{b_i}-\bar{a_i}-\sum w_i\bar{a_i}+\epsilon}+(0.5-y_{i2})(\bar{a_i}+\bar{b_i}-\sum w_i\bar{a_i}-\sum w_i\bar{b_i})$ in (22-6) and (22-14) is transformed from $d(\bar{o_i},o^c)$ under (22-7)-(22-12). In addition, (22-6) reflects individual tolerance behavior, where ε_i is predetermined due to each individual DM's background, knowledge and experience, while the unit costs c_i^+ and c_i^- accordingly preset by the moderator; (22-13) is derived from Theorem 4, and (22-14) is the consensus measure (i.e., Eq. (16)). Generally, the higher the CL the better, so let Dev denote the overall preference deviation, rather than the CL defined by [11]. Since $Dev=1-CL$, the smaller Dev the better.

Once all DMs provide initial preferences with LUDs, an initial consensus is reached immediately based on Theorem (4), thereby obtaining the initial Dev (denoted by γ_0) based on Eq. (16) and Eq. (20), which can be viewed as the Dev benchmark of Model (22). Generally, the Dev threshold (i.e., γ) in Model (22) should be smaller than γ_0 to improve the CL, thus $\gamma=\gamma_0-\Delta\gamma$ with $\Delta\gamma\ge 0$.

B. Uncertain consensus model with asymmetric costs

Inappropriate CL thresholds easily lead to the failure of reaching a consensus, so some maximum CL models are built to aid DM's preference adjustment [15], [16]. Similarly, solving Model (22) may not yield the MCC due to inappropriate γ thresholds. Hence, building a new model that considers both the CL and the budget is of great necessity [36], where the Dev (or CL) threshold no longer needs to be pre-determined. By introducing a trade-off coefficient λ ($\lambda \in [0,1]$), a more flexible model to handle GDM problems is built (i.e., Model (23)), where variables and constraints are defined same as in Model (22). Furthermore, since the existence theorem of optimal solutions [37] (i.e., a single-objective programing model with non-empty and bounded feasible region must have an optimal solution), Model (23) surely has optimal solutions.

In fact, if $\lambda = 0$, Model (23) reduces to Model (22); and if $\lambda = 1$, Model (23) is the maximum CL model [16]. By comparison, Model (23) is more objective since it doesn't need to pre-determine a Dev threshold; besides, it better supports GDM by providing a full relationship between the CL and the total cost with regard to the trade-off coefficient λ.

$$
\begin{array}{l}
\min \lambda * \gamma + (1-\lambda) * \phi \\
\text{s.t.} \left\{
\begin{array}{ll}
\phi = \sum_{i=1}^{m} [x_i c_i^{+} + (1-x_i) c_i^{-}] * [z_{i3} * \frac{(b_i - \bar{a_i})^2}{\bar{b_i} + b_i - \bar{a_i} - a_i + \epsilon} + (0.5 - z_{i2}) * (\bar{a_i} + \bar{b_i} - a_i - b_i)] & (23-1) \\
-U(1-z_{i2})(1-z_{i3}) \leq b_i - \bar{a_i} < U(1-z_{i1}) & (23-2) \\
-U(1-z_{i1})(1-z_{i3}) \leq \bar{b_i} - a_i < U(1-z_{i2}) & (23-3) \\
-U(1-z_{i1}) < \frac{\bar{a_i} + \bar{b_i} - a_i - b_i}{2} < U(1-z_{i2}) & (23-4) \\
-U(1-z_{i2}) < \frac{a_i + b_i - \bar{a_i} - \bar{b_i}}{2} < U(1-z_{i1}) & (23-5) \\
\frac{(b_i - \bar{a_i})^2}{\bar{b_i} + b_i - \bar{a_i} - a_i + \epsilon} + \frac{\bar{a_i} + \bar{b_i} - a_i - b_i}{2} > -U * z_{i2} & (23-6) \\
y_{i3} * \frac{(\sum w_i \bar{b_i} - \bar{a_i})^2}{\bar{b_i} + \sum w_i \bar{b_i} - \bar{a_i} - \sum w_i \bar{a_i} + \epsilon} + (0.5 - y_{i2})(\bar{a_i} + \bar{b_i} - \sum w_i \bar{a_i} - \sum w_i \bar{b_i}) \leq \varepsilon_i & (23-7) \\
-U(1-y_{i2})(1-y_{i3}) \leq \sum w_i \bar{b_i} - \bar{a_i} < U(1-y_{i1}) & (23-8) \\
-U(1-y_{i1})(1-y_{i3}) \leq \bar{b_i} - \sum w_i \bar{a_i} < U(1-y_{i2}) & (23-9) \\
-U * (1-y_{i1}) < \frac{\bar{a_i} + \bar{b_i} - \sum w_i \bar{a_i} - \sum w_i \bar{b_i}}{2} < U(1-y_{i2}) & (23-10) \\
-U(1-y_{i2}) < \frac{\sum w_i \bar{a_i} + \sum w_i \bar{b_i} - \bar{a_i} - \bar{b_i}}{2} < U(1-y_{i1}) & (23-11) \\
\frac{(\sum w_i \bar{b_i} - \bar{a_i})^2}{\bar{b_i} + \sum w_i \bar{b_i} - \bar{a_i} - \sum w_i \bar{a_i} + \epsilon} + \frac{\bar{a_i} + \bar{b_i} - \sum w_i \bar{a_i} - \sum w_i \bar{b_i}}{2} > -U * y_{i2} & (23-12) \\
z_{i1} + z_{i2} + z_{i3} = 1, \ y_{i1} + y_{i2} + y_{i3} = 1 & (23-13) \\
a^c = \sum w_i \bar{a_i}, \ b^c = \sum w_i \bar{b_i} & (23-14) \\
\sum_{i=1}^{m} w_i * [y_{i3} * \frac{(\sum w_i \bar{b_i} - \bar{a_i})^2}{\bar{b_i} + \sum w_i \bar{b_i} - \bar{a_i} - \sum w_i \bar{a_i} + \epsilon} + (0.5 - y_{i2})(\bar{a_i} + \bar{b_i} - \sum w_i \bar{a_i} - \sum w_i \bar{b_i})] \leq \gamma & (23-15) \\
o_i \sim \mathcal{L}(a_i, b_i), \bar{o_i} \sim \mathcal{L}(\bar{a_i}, \bar{b_i}), o^c \sim \mathcal{L}(a^c, b^c) & (23-16) \\
z_{i1}, z_{i2}, z_{i3}, y_{i1}, y_{i2}, y_{i3}, x_i \in \{0,1\}, 0 \leq \bar{a_i} \leq \bar{b_i} \leq 1, 0 \leq a^c \leq b^c \leq 1, \gamma \in [0,1], i \in M & (23-17)
\end{array}
\right.
\end{array}
\tag{23}
$$

C. Feedback mechanism based on democratic consensus

Feedback mechanisms insist the less modified DMs the better, avoiding unnecessary adjustments and preserving DMs' original preferences to the fullest extent [36]. Assume only one DM needs to be adjusted at each iteration, and democratic consensus is implemented during the CRP, which ensures DMs' effective participation and satisfaction by realizing a soft consensus [25]. Specifically, DMs are first assigned with equal weights to protect the interest of minorities, then their influence is updated with their own contribution to the CL.

Definition 3: DM d_k's contribution to the CL is measured by the deviation between the overall CL and the CL reached by the remaining $m-1$ DMs (i.e., $CL_{\bar{k}}$), thus,

$$CL_k = CL - CL_{\bar{k}}, k \in M \tag{24}$$

Modifying the preference with a maximum deviation from the consensus and moderately decreasing the weight can significantly improve the CL [38]. Next, define the identification rule (IR) and direction rule (DR) as

- IR: DM with a minimum contribution to CL is identified as the one to be adjusted, denoted as d_k. If there exist more than two DMs with the same contribution value, then d_k is randomly chosen.
- DR: Preference adjustment and weight reallocation are both incorporated in the modification process.
  - The DM d_k's updated preference is expressed as Eq. (25), where $\delta \in [0,1]$ is the parameter reflecting d_k's self-confidence, and the larger δ, the less he/she is willing to make revisions.

$$\bar{o_k} = \delta * o_k + (1-\delta) * o^c \tag{25}$$

  - The weights are updated by Eq. (26), where η is the variable that controls the impact of d_i's consensus contribution CL_i^t on the weight w_i^{t+1}. Next, all the new weights are normalized by Eq. (27).

$$w_i^{t+1} = w_i^t * (1 + CL_i^t)^{\eta} \tag{26}$$

$$\bar{w_i} = \frac{w_i^{t+1}}{\sum_{i=1}^{m} w_i^{t+1}} \tag{27}$$

Explicitly, the larger the parameter η, the stronger modification of the DM d_i [38]. Fig. 4 provides a new CRP framework that combines democratic consensus and the optimization-based uncertain MCCMs. Essentially, the procedure prior to Model (22) is an intra-group self-adjustment that fully respects individual values by adjusting only one DM's preference but updating all weights based on each DM's contribution to the CL. Note that if the required CL threshold is still not reached after Model (22) takes effect, all the adjusted preferences optimized by Model (22) will be used to start the next iteration; meanwhile, whenever any DM has some changes in either the preference or the weight, a new CL is calculated, so as to

Table II: Original example data of Section V

DM d_i	Initial weight w_i	Original preference o_i	Upward cost c_i^+	Downward cost c_i^-	Tolerance ε_i
d_1	0.2	$\mathcal{L}(0.24, 0.86)$	6.0	7.4	0.10
d_2	0.2	$\mathcal{L}(0.25, 0.75)$	7.1	8.1	0.12
d_3	0.2	$\mathcal{L}(0.29, 0.90)$	2.8	9.3	0.20
d_4	0.2	$\mathcal{L}(0.05, 0.46)$	1.5	8.8	0.03
d_5	0.2	$\mathcal{L}(0.55, 0.80)$	6.2	9.6	0.08

minimize the overall resource consumption.

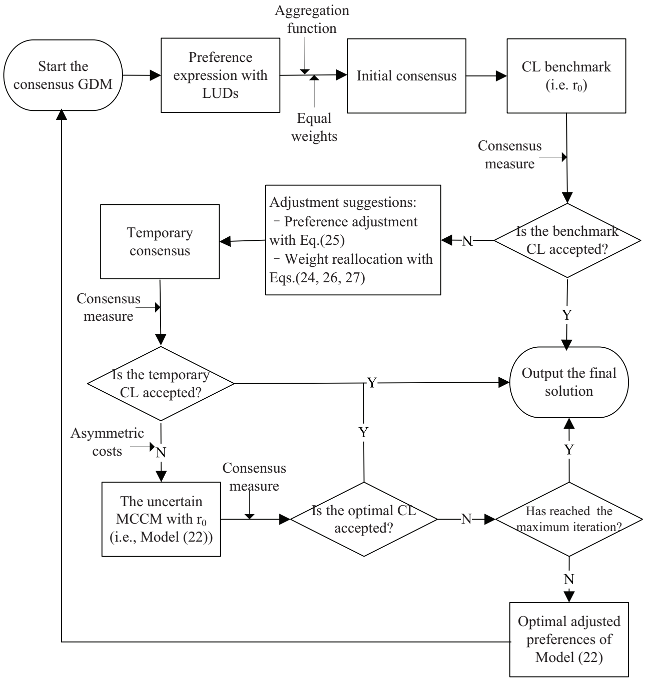


Fig. 4: The new CRP framework of this paper

V. Illustrative example

Suppose five cities (DMs) located in the Yangtze River Delta participate in the trans-boundary water pollution negotiation under the governance of the Ministry of Water Resources (MWR) of China (i.e., the moderator) [19], denoted as d_i, $i \in M = \{1, 2, 3, 4, 5\}$. Considering the differences and flexibility of the environmental capacity, population density, annual financial goals and historical pollution data, the original data about the five cities are shown as Table II, including d_i's initial preference as their desired pollution index o_i obeying the LUD (i.e., $o_i \sim \mathcal{L}(a_i, b_i)$), the asymmetric costs c_i^+, c_i^- satisfying $|c_i^-| > |c_i^+|$, and each tolerance ε_i to the final index (i.e., o^c). In pursuit of democracy, the weight of each city is initially assigned with 0.2. Normally, an ideal CL threshold is pre-determined due to specific GDM problem, hereafter let $CL^* = 0.85$ and the maximum iteration be 5 [16]. The new CRP is implemented step by step as follows.

Step 1. Calculate an initial CL to determine whether there exists one DM (i.e., d_k) to be adjusted. If yes, go to Step 2; otherwise, terminate the CRP.

As the weight vector is $(0.2, 0.2, 0.2, 0.2, 0.2)^{\mathrm{T}}$, we initially get the consensus $o^c \sim \mathcal{L}(0.28, 0.75)$ with Theorem 4. Next, Eq. (20) is adopted to obtain an initial CL by computing variables in Table III, including the uncertain distance (i.e., $d(o_i, o^c)$), the overall Dev/CL except d_i (i.e., $Dev_{\bar{i}}$ and $CL_{\bar{i}}$), and d_i's contribution to the reached consensus (i.e., CL_i). Hence, we get the benchmark Dev $\gamma_0 = 0.263$, and the benchmark $CL = 0.737 < CL^*$, indicating that d_k needs to be identified.

Table III: Results calculated from original data

d_i	$d(o_i, o^c)$	$Dev_{\bar{i}}$	$CL_{\bar{i}}$	CL_i
d_1	0.276	0.260	0.740	-0.003
d_2	0.244	0.268	0.732	0.005
d_3	0.278	0.259	0.741	-0.004
d_4	0.299	0.254	0.746	-0.009
d_5	0.217	0.274	0.726	0.011

Specifically, taking d_1 as an example to obtain Table III, since $o_1 \sim \mathcal{L}(0.24, 0.86)$ and $o^c \sim \mathcal{L}(0.28, 0.75)$, we get $a = 0.24 < d = 0.75$ and $b = 0.86 > c = 0.28$, then the third case of Eq. (20) is employed to obtain their distance as $\frac{(0.75-0.24)^2}{0.86+0.75-0.24-0.28+10^{-6}} + \frac{0.24+0.86-0.28-0.75}{2} = 0.276$. To be noted, when computing $Dev_{\bar{i}}$, the remaining four DMs' weights are changed from 0.2 into 0.25 for $\tilde{w_j} = \frac{w_j}{1-w_i}$, $j \neq i$, $\forall j \in M$. Comparing all the values in the last column of Table III, we have $d_k = d_4$ due to his/her minimum contribution $CL_4 = -0.009$.

Step 2. Recalculate and check whether the new temporary consensus meets the threshold CL^* by adjusting d_k's preference and updating all DMs' weights. If yes, terminate the CRP; otherwise, move to Step 3.

Let the parameter of d_k's adjustment willingness be $\delta = 0.5$, and the controlling parameter in Eq. (26) be $\eta = 1.5$ [25]. Therefore, d_4's adjusted preference is $\bar{o_4} \sim \mathcal{L}(0.165, 0.605)$, and the weight vector is updated as $(0.199, 0.201, 0.199, 0.197, 0.203)^{\mathrm{T}}$ (see Table IV). Here, the last five columns of Table IV are the weights of the remaining four DMs except d_i, which are used to calculate $Dev_{\bar{i}}$. For example, the last value of 0.248 is derived by $\tilde{w_4} = \frac{0.197}{1-0.203} = 0.248$. Moreover, using d_3 as an instance, the new weight $\bar{w_3} = 0.199$ is obtained by first calculating Eq. (26) (i.e., $w_3^1 = 0.2 * (1 - 0.004)^{1.5} = 0.1989$), then being normalized with Eq. (27). Repeating the calculation in Step 1, we have $CL = 0.75$, which still fails to meet the threshold of 0.85, so proceed to Step 3.

Step 3. Solve Model (22) with γ_0 from Step 1 to produce optimal solutions, then check if the obtained optimal CL meets CL^*. If yes, terminate the CRP; otherwise, return to Step 1 with derived adjusted preferences.

Use the Dev benchmark in Step 1 (i.e., $\gamma_0 = 0.263$), instead of any smaller values, to pre-determine γ in Model (22),

Table IV: Updated weights due to DM's variable influence

d_i	w_i	$\bar{w_i}$	$w_{\bar{1}}$	$w_{\bar{2}}$	$w_{\bar{3}}$	$w_{\bar{4}}$	$w_{\bar{5}}$
d_1	0.2	0.199		0.249	0.248	0.248	0.250
d_2	0.2	0.201	0.251		0.251	0.251	0.253
d_3	0.2	0.199	0.248	0.249		0.248	0.250
d_4	0.2	0.197	0.246	0.247	0.246		0.248
d_5	0.2	0.203	0.254	0.255	0.254	0.253	

so as to verify if Model (22) is effective. Moreover, solve Model (22) by adopting updated information in Step 2, to fully demonstrate individual influence on the CRP.

Table V: Optimal solution of Model (22) with $\gamma = 0.263$

d_i	$\bar{o_i}^*$	$d(\bar{o_i}^*, o^{c*})$	$Dev_{\bar{i}}$	$CL_{\bar{i}}$	CL_i
d_1	$\mathcal{L}(0.5690, 0.5690)$	0.000	0.051	0.949	0.010
d_2	$\mathcal{L}(0.4991, 0.4991)$	0.070	0.033	0.967	-0.007
d_3	$\mathcal{L}(0.5906, 0.5906)$	0.022	0.045	0.955	0.005
d_4	$\mathcal{L}(0.5390, 0.5390)$	0.030	0.043	0.957	0.003
d_5	$\mathcal{L}(0.6490, 0.6490)$	0.080	0.031	0.969	-0.010

Finally, the reached consensus from Model (22) is $o^{c*} \sim \mathcal{L}(0.569, 0.569)$, and the minimum cost is $\phi^* = 2.8982$. Meanwhile, Table V provides d_i's optimal adjusted preference along with relevant indicators. Obviously, both the achieved consensus and DM's optimal adjusted preferences degenerate to real numbers. In other words, given the constructed form of the LUD, all the final preferences have the same upper and lower bounds. Such findings are consistent with existing studies on uncertain MCCMs [28], [29]. That is, once certain conditions are met (e.g., the belief degree is no less than 0.5 [28]), the original LUDs degenerate to crisp numbers. The obtained final Dev is $\gamma^* = 0.04$, and the optimal $CL = 0.96$ exceeds its threshold of 0.85, so CRP is terminated at this point, which implies that Model (22) works in the new CRP framework. Provided that the current CL still fails to meet its threshold, all the optimal adjusted preferences from Model (22) will be used for the next iteration.

VI. Further discussion

Parametric behavior analysis regarding the two new models is conducted in VI-A, while a comparative analysis with other studies is given in VI-B.

A. Parametric behavior analysis

1) Parametric behavior analysis of λ: The influence of λ (i.e., the trade-off coefficient between the budget and the CL) on the consensus is discussed with a step of 0.1 due to $\lambda \in [0, 1]$. Note that initial adjustments of weights or preference aren't involved in this section. Namely, results are obtained by solving Model (23) with data in Table II. Fig. 5 provides the changing trends of the CL and the consensus cost with λ; and Table VI exhibits the λ-related optimal solutions, including the reached consensus and DMs' optimal adjusted preferences.

Fig. 5 shows that the final range of the CL is [0.9612, 0.9766] and the total cost is [3.0760, 3.1429] in light of λ. Once the two objectives (i.e., γ and ϕ) are considered separately, that is, if $\lambda = 0$ (i.e., only the resource is required to

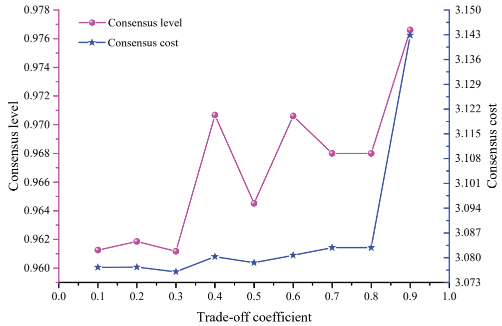


Fig. 5: Parametric behavior analysis of λ in Model (23)

be least consumed), the optimal cost is 3.0743 with CL being 0; by contrast, if $\lambda = 1$ (i.e., the maximum CL becomes the priority of the discussed GDM), then CL reaches the maximum value 1 with a cost of 16.596. Since the results of these two extreme scenarios are far away from the rest, they are omitted in Fig. 5. Table VI verifies most optimal preferences end up being LUDs with identical endpoint values, where the optimized preferences still strictly obeying the LUD with two different endpoints are underlined. Moreover, results show that once the trade-off parameter reaches a certain threshold (e.g., 0.7 in Table VI), the GDM involved focuses more on CL rather than the total cost, and all preferences become crisp numbers.

Overall, there are sharp fluctuations when $\lambda \in [0.3, 0.7]$, thus CL is of great necessity for GDM, where detailed reasons remain to be further explored; although such fluctuations are negligible once taking the numerical scale into account. Besides, the consistent volatility trends of the CL and the cost suggest that increasing budget helps improve CL, but their conflict within [0.6, 0.7] might due to the complete transformation of LUDs into crisp numbers with $\lambda = 0.7$ or factors ignored in this paper.

2) Parametric behavior analysis of η: Since DM's weight plays an important role during CRP, this section conducts the parametric behavior analysis of η from 1.5 to 3.5 with a step of 0.5. Fig. 6 provides the changes of weights regarding η, and the optimal solutions of Model (22) are shown in Table VII, where the original data except w_i come from Table II.

Fig. 6 shows that once the controlling variable η of DM's CL contribution to their new weight gets larger, the more obvious the differences in weight distribution, where specific data are shown as the 2nd column of Table VII. In other words, the smaller the value of η, the more even of all DMs' weights, and the less differences among individual influence on the final decision. Results in Table VII show that Model (22) works, since all CLs are significantly improved from the benchmark $CL = 0.737$ from Step 1 in Section V. Here, only d_4's optimal adjusted preference $\bar{o_4}^*$ is listed in Table VII due to limited space and the fact that other four DMs' preferences have all degenerated into crisp numbers. Since only minor changes exist in all the optimal values of ϕ^* and CL, the impact of η on the final results is actually negligible here. Note that the parameter behavior analysis of δ in Eq. (25), which reflects

Table VI: Optimal results of Model (23)

λ	o^{c*}	$\bar{o_1}^*$	$\bar{o_2}^*$	$\bar{o_3}^*$	$\bar{o_4}^*$	$\bar{o_5}^*$
0.0	$\mathcal{L}(0.5599, 0.5699)$	$\mathcal{L}(0.5499, 0.5499)$	$\mathcal{L}(0.4999, 0.4999)$	$\mathcal{L}(0.5948, 0.5948)$	$\mathcal{L}(0.5099, 0.5599)$	$\mathcal{L}(0.6449, 0.6449)$
0.1	$\mathcal{L}(0.5638, 0.5638)$	$\mathcal{L}(0.5638, 0.5638)$	$\mathcal{L}(0.4969, 0.4969)$	$\mathcal{L}(0.5807, 0.5807)$	$\mathcal{L}(0.5338, 0.5338)$	$\mathcal{L}(0.6438, 0.6438)$
0.2	$\mathcal{L}(0.5642, 0.5642)$	$\mathcal{L}(0.5642, 0.5642)$	$\mathcal{L}(0.4988, 0.4988)$	$\mathcal{L}(0.5795, 0.5795)$	$\mathcal{L}(0.5342, 0.5342)$	$\mathcal{L}(0.6442, 0.6442)$
0.3	$\mathcal{L}(0.5584, 0.5684)$	$\mathcal{L}(0.5584, 0.5584)$	$\mathcal{L}(0.5013, 0.5013)$	$\mathcal{L}(0.5805, 0.5805)$	$\mathcal{L}(0.5084, 0.5584)$	$\mathcal{L}(0.6434, 0.6434)$
0.4	$\mathcal{L}(0.5572, 0.5705)$	$\mathcal{L}(0.5572, 0.5572)$	$\mathcal{L}(0.5138, 0.5138)$	$\mathcal{L}(0.5705, 0.5705)$	$\mathcal{L}(0.5004, 0.5673)$	$\mathcal{L}(0.6438, 0.6438)$
0.5	$\mathcal{L}(0.5592, 0.5650)$	$\mathcal{L}(0.5592, 0.5592)$	$\mathcal{L}(0.5063, 0.5063)$	$\mathcal{L}(0.5708, 0.5708)$	$\mathcal{L}(0.5176, 0.5465)$	$\mathcal{L}(0.6421, 0.6421)$
0.6	$\mathcal{L}(0.5583, 0.5713)$	$\mathcal{L}(0.5583, 0.5583)$	$\mathcal{L}(0.5148, 0.5148)$	$\mathcal{L}(0.5713, 0.5713)$	$\mathcal{L}(0.5023, 0.5674)$	$\mathcal{L}(0.6448, 0.6448)$
0.7	$\mathcal{L}(0.5607, 0.5607)$	$\mathcal{L}(0.5607, 0.5607)$	$\mathcal{L}(0.5107, 0.5107)$	$\mathcal{L}(0.5607, 0.5607)$	$\mathcal{L}(0.5307, 0.5307)$	$\mathcal{L}(0.6407, 0.6407)$
0.8	$\mathcal{L}(0.5607, 0.5607)$	$\mathcal{L}(0.5607, 0.5607)$	$\mathcal{L}(0.5107, 0.5107)$	$\mathcal{L}(0.5607, 0.5607)$	$\mathcal{L}(0.5307, 0.5307)$	$\mathcal{L}(0.6407, 0.6407)$
0.9	$\mathcal{L}(0.5650, 0.5650)$	$\mathcal{L}(0.5650, 0.5650)$	$\mathcal{L}(0.5364, 0.5364)$	$\mathcal{L}(0.5650, 0.5650)$	$\mathcal{L}(0.5350, 0.5350)$	$\mathcal{L}(0.6235, 0.6235)$
1.0	$\mathcal{L}(0.0500, 0.0500)$	$\mathcal{L}(0.0500, 0.0500)$	$\mathcal{L}(0.0500, 0.0500)$	$\mathcal{L}(0.0500, 0.0500)$	$\mathcal{L}(0.0500, 0.0500)$	$\mathcal{L}(0.0500, 0.0500)$

Table VII: Optimal results of Model (22)

η	$\bar{W}$	$\bar{o_4}^*$	o^{c*}	ϕ^*	CL
1.5	$(0.1990, 0.2014, 0.1989, 0.1973, 0.2034)^{\mathrm{T}}$	$\mathcal{L}(0.5342, 0.5342)$	$\mathcal{L}(0.5642, 0.5642)$	3.0774	0.9605
2.0	$(0.1986, 0.2019, 0.1985, 0.1964, 0.2046)^{\mathrm{T}}$	$\mathcal{L}(0.5349, 0.5350)$	$\mathcal{L}(0.5650, 0.5650)$	3.0774	0.9603
2.5	$(0.1983, 0.2023, 0.1981, 0.1955, 0.2058)^{\mathrm{T}}$	$\mathcal{L}(0.5347, 0.5347)$	$\mathcal{L}(0.5647, 0.5647)$	3.0773	0.9605
3.0	$(0.1980, 0.2028, 0.1977, 0.1946, 0.2069)^{\mathrm{T}}$	$\mathcal{L}(0.5102, 0.5604)$	$\mathcal{L}(0.5604, 0.5702)$	3.0743	0.9555
3.5	$(0.1976, 0.2033, 0.1973, 0.1938, 0.2081)^{\mathrm{T}}$	$\mathcal{L}(0.5102, 0.5605)$	$\mathcal{L}(0.5605, 0.5702)$	3.0743	0.9554

DM's adjustment willingness [33] is no longer discussed.

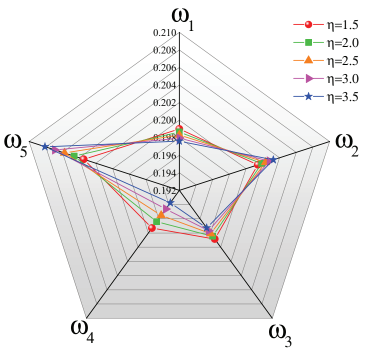


Fig. 6: Changes of weights with regard to η

B. Comparative analysis

1) Comparison between models with/without feedback mechanism: Using data from Table II and directly setting $\gamma = 0.15$ for Model (22), namely, feedback mechanisms are no longer considered in this section. Results show that all adjusted preferences degenerate into crisp numbers (see Table VIII), the minimum cost is 3.077, and the consensus reached is $o^{c*} \sim \mathcal{L}(0.564, 0.564)$.

The final $CL = 0.96$ achieves the target 0.85, which means that the uncertain MCCM still works without feedback mechanisms. However, the cost required to achieve the same CL (i.e., $CL = 0.96$) is less in Section V ($\phi^* = 2.8982$) than here ($\phi^* = 3.077$), implying that the feedback mechanism does promote a higher cost-effective consensus.

Table VIII: Optimal results of Model (22) with $\gamma = 0.15$

d_i	$\bar{o_i}^*$	$d(\bar{o_i}^*, o^{c*})$	$Dev_{\bar{i}}$	$CL_{\bar{i}}$	CL_i
d_1	$\mathcal{L}(0.564, 0.564)$	0.000	0.049	0.951	0.010
d_2	$\mathcal{L}(0.496, 0.496)$	0.068	0.032	0.968	-0.007
d_3	$\mathcal{L}(0.582, 0.582)$	0.018	0.045	0.955	0.005
d_4	$\mathcal{L}(0.534, 0.534)$	0.030	0.042	0.958	0.002
d_5	$\mathcal{L}(0.644, 0.644)$	0.080	0.029	0.971	-0.010

2) Comparison with existing approaches: Table IX provides a comparison of existing research with our new method from several perspectives, such as individual preference structure, unit cost setting, aggregation function, feedback mechanism and solving method. In fact, Gong et al. [28] also looked into the influence of non-cooperation behaviors on the uncertain MCCMs; while Guo et al. [29] incorporated the utility function into uncertain MCCMs, and presented the relationships between their new models with traditional MCCMs; Similarly, Cheng et al. [32] also took the satisfaction functions into account, and explored the effect of individual behaviors (i.e., tolerance and compromise limit) on the final decision; besides, [20], [21] adopted stochastic or robust optimization theory to deepen Cheng et al.'s research with asymmetric costs. Meanwhile, Refs. [7], [8] adopted optimization-based models with CRP to locate specific unit costs from pre-determined intervals. Concisely, this paper extends the MCCMs with LUDs into a new CRP framework by integrating asymmetric costs, aggregation function and consensus measure, which further generalizes the classic GDM theory by considering uncertainty theory and individual behaviors.

VII. Conclusion

Uncertainty theory well depicts DM's inherent subjectivity, imprecision and vagueness, so this paper adopts LUDs to extend the uncertain MCCMs into a new CRP framework. In specific, two new consensus models are built by considering asymmetric costs, aggregation function and consensus measure, where DM's preference is fit by the LUD and the setting of asymmetric costs is further rationalized based on prospect theory. Moreover, a new CRP is designed by respecting DM's values with democratic consensus and minimizing resources with new uncertain MCCMs. To avoid the calculation complexity from piecewise functions in the uncertain distance measure, binary variables are introduced to transform the multi-coefficient goal programming models in view of the big M method. Furthermore, we found that (i) the new consensus

Table IX: Comparison with existing consensus studies

Refs	Preferences	Unit cost	Aggregation	Feedback mechanism	Solving method
[28]	LUD	Single fixed value	—	—	OM
[29]	LUD	Single fixed value	AWA	—	OM
[19]	Crisp number	Asymmetric costs	—	—	OM
[32]	Crisp number	Asymmetric costs	AWA	—	OM
[7]	Crisp number	Interval value	OWA	Adjustment of preference & unit costs	OM
[8]	Crisp number	Interval value	AWA	Adaptive differential evolution algorithm	Stackelberg game & OM
[20], [21]	Crisp number	Asymmetric costs	—	—	Stochastic/Robust OM
This paper	LUD	Asymmetric costs	AWA	Preference adjustment & weight reallocation	OM

Note: "—" means not involved; and "OM" is short for optimization-based modeling.

models exclude moderator's influence by setting the Dev threshold of Model (22) with a benchmark from initially provided information, or by providing a full relationship between the CL and the cost via Model (23); (ii) CRP helps promote a higher cost-effective consensus (see Section VI-B); and (iii) once certain conditions are met, DMs' preferences fit by LUDs degenerate into crisp numbers, which is consistent with previous findings [28], [29]. Note that in addition to trans-boundary pollution management, our method is also feasible to handle other GDM problems characterized by non-randomness and non-fuzziness, such as urban demolition negotiation, trust evaluation in social networks or emergency management for natural disasters [39].

It's widely accepted that heterogeneous preferences [6], [23] attract more concerns in real-life situations, so transformation from the LUD to the traditional expression forms needs to be explored. Moreover, interactions among DMs are truly non-negligible, thereby building uncertain MCCMs combined with social network analysis [36] will be our next focus. Finally, how to integrate well-known decision technology (e.g., survey, datamining) [39], [40] to practically interpret our findings is also worth investigating.

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3 Optimization consensus modeling of a closed-loop carbon quota trading mechanism regarding revenue and fairness

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Optimization consensus modeling of a closed-loop carbon quota trading mechanism regarding revenue and fairness

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Abstract

Consensus modeling aims to obtain collective agreement through group decision-making (GDM), generally by building mathematical programming models. This paper describes the use of optimization consensus modeling to explore theoretical innovations regarding flexible carbon quota trading mechanisms, with basic allocation schemes provided within a closed-loop trading system by simultaneously taking revenue and fairness into account. A series of optimization consensus models are constructed from the perspective of maximizing the corresponding revenue, resulting in optimal/fair carbon quota allocation schemes that include detailed trading information, e.g., participating individuals, transferred quantities, and unit transaction prices. To solve these models, a relaxation method based on particle swarm optimization is proposed. The inability to conduct real-life GDM usually stems from conflicts of interest based on the decision-makers' mutual competition, thus, two practical strategies are presented to deal with the resulting unfairness within the trading system. Finally, a numerical example incorporating five regions demonstrates the effectiveness of the proposed trading mechanisms. The results show that sufficient interactions among decision-makers are of great significance in achieving fairness within a trading system.

Keywords: Group decision-making (GDM); Consensus; Revenue and fairness; Carbon quota trading mechanism; Allocation scheme

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1. Introduction

Group decision-making (GDM) refers to a process in which multiple individuals participate in decision-making analysis and make a final choice based on their collective wisdom: Clark & Stephenson (1995) have pointed out that GDM represents a collective recall of information. Generally, communication and negotiation effectively promote the interactions among decision-makers (DMs) (Hirokawa & Poole, 1996) and the flow of information within the group. Moreover, technological innovations have significantly updated the means of group communication and decision-making (Kiesler & Sproull, 1992). Without loss of generality, three stable states of fragmentation, polarization, or consensus may finally be achieved by rational DMs considering their own interests (Hegselmann & Krause, 2002; Liang et al., 2020; Zhao et al., 2016). Among them, consensus usually requires multiple rounds of communication, coordination, preference modification, and even concessions or compromises within the group. Only in this way can a relatively consistent collective agreement be obtained (Cabrerizo et al., 2014; Liu et al., 2019; Wu & Chiclana, 2014; Wu et al., 2018; Zhang et al., 2020a,b). For example, if a new allocation scheme of resources is obtained through GDM within a trading system, which is widely accepted by the whole group, then a consensus is reached. Liang et al. (2020) clarified that the consensus-reaching process (CRP) does not mean that an optimal solution must be achieved. Instead, CRP is more like a decision tool or a synthesizing process that assists DMs in building connections and communicating with each other, thereby providing a more effective way for the group to find unity on how to proceed (Susskind et al., 1999).

Considering that cost, which may be embodied as human, material, financial, time and other resources, is an important influencing factor in GDM, Ben-Arieh & Easton (2007) first proposed the concept of minimum cost consensus, and acquired the optimal collective opinion with a linear/quadratic cost function (Ben-Arieh et al., 2009). Since then, other scholars have made further extensions to their minimum cost consensus models (MCCMs) by taking various factors into account, such as uncertain preference structures (Gong et al., 2021; Guo et al., 2021), aggregation rules (Zhang et al., 2011), measurement of consensus effectiveness (Labella et al., 2020) or parameter improvements of initial models (Cheng et al., 2018; Lu et al., 2021; Zhang et al., 2020a). Since unit costs are difficult to objectively determine in advance, and DMs' opinions are hard to modify during GDM, Dong et al. (2010) proposed minimum adjustment consensus models (MACMs) with an ordered weighted average operator, which preserve the DMs' initial preference information as much as possible. Similarly, their modeling idea has also been widely explored (del Moral et al., 2018; Dong et al., 2016; Gong et al., 2020; Yu et al., 2021; Zhang et al., 2018), especially under social networks (Cheng et al., 2020; Wu et al., 2018) or opinion evolution contexts (Chen et al., 2021; Liang et al., 2020). Moreover, Zhang et al. (2020b) summarized the original and basic consensus models based on feedback mechanisms with a minimum cost/adjustment and reviewed diverse consensus modeling under some complicated GDM scenarios.

Different from the above consensus modeling with a minimum cost/adjustment, this paper was partially inspired by the construction of consensus models that aim to maximize the total revenue. By introducing linear primal-dual theory, various MCCMs (including hard and soft consensus (Herrera-Viedma et al., 2014; Zhang et al., 2011)) with specific preference structures (e.g., DM's opinion denoted by crisp numbers or interval values) were adopted as the primal models, and then their corresponding dual forms (i.e., the optimization maximum compensation consensus models) along with their economic significance were deeply explored by Gong et al. (2015a,b) and Zhang et al. (2019). Subsequently, taking the essential architecture of Stackelberg's game into account, Zhang et al. (2020a) presented a bi-level optimization consensus model that depicts the interaction between DMs and the moderator, and divided the DM's total return into a modification component (also known as external compensation) provided by the moderator for the DM's initial preference adjustment and a recognition component based on the similarity between the DM's original opinion and the final consensus. It is well known that the market is profit-oriented (i.e., simultaneously pursuing the maximization of revenue and the minimization of costs) and its operating mechanism is mostly affected by pricing strategy, participants' competition, supply and demand, and etc. (Lamba et al., 2019; Ruidas et al., 2021; Zhou et al., 2020b; Zou et al., 2021). Therefore, in discussing closed-loop trading mechanisms, the revenue maximization of either the whole group or a single DM is set as our objective function in this paper, and constraints such as supply and demand or prices are introduced. A series of optimization consensus models are then constructed as a means of deriving the optimal resource allocation schemes within a trading system.

Rapid industrialization and economic growth have led to significant increases in emissions of carbon dioxide and other greenhouse gases, and have rendered environmental pollution and extreme weather events increasingly serious and frequent, resulting in severe negative impacts on economic development and human health (Wang et al., 2017). Therefore, mitigating the impact of human activities on the environment through reductions in carbon emissions has gradually become a global consensus. Diaz-Rainey & Tulloch (2018) conducted the first empirical analysis of New Zealand's carbon trading scheme using allowance importation and exportation data, and found that the imports of offsets are the major carbon price determinant, with small trading systems able to reap benefits from imposing quantitative import restrictions. Aiming at developing sustainable supply chain, joint decisions were made under various carbon emission regulatory policies, with respect to different influence factors, such as inventory, pricing, financing and ordering (Ruidas et al., 2021; Zhou et al., 2020b; Zou et al., 2021). Furthermore, carbon issues combined with decision-making technology has also been investigated (Gong et al., 2021; Huang & Xu, 2020; Lamba et al., 2019). For instance, Lamba et al. (2019) proposed a mixed-integer nonlinear program for supplier selection and the right lot-sizes determination under a dynamic background with multiple periods, products and suppliers, and evaluated different costs of carbon emissions under three regulating policies (viz. cap-and-trade, strict cap on emissions and carbon tax) using big data technology. Huang & Xu (2020) constructed a bi-level multi-objective programming model to solve the carbon emission quota allocation problem with co-combustion of coal and sewage sludge, and formulated the interaction between authorities and coal-fired power plants before examining a real case demonstrating the trade-off between economic development, energy conservation, and renewable energy utilization.

Setting targets for carbon emissions in different countries/regions (i.e., operating collective schemes for optimal carbon quota allocation) is one of the main obstacles to reaching a comprehensive agreement on global warming. This is exacerbated by long-term tensions between industrialized and developing countries regarding unfairness issues on burden-sharing, with industrialized countries pleading special circumstances and seeking differentiation in their obligations (Rose et al., 1998). Fairness concerns, gained widespread attention in the supply chain management (Liu et al., 2021; Zheng et al., 2019), are also critical for GDM (Du et al., 2021), because participants are motivated by not only the final results, but also the fairness they feel compared with others (Adams, 1963). Under a fixed total carbon quota, the scientific allocation of binding carbon allowances for different regions is a complex and arduous task, because it directly involves the economic development rights of each region. In general, the fairness of carbon emissions quotas is measured using the Atkinson index (Hedenus & Azar, 2005), Theil index (Duro & Padilla, 2006), and Gini coefficient (Chen et al., 2017). The traceability method, which uses historical carbon emissions as the relevant feature of the initial carbon quota allocation (i.e., the free distribution principle), has been criticized by Fromm & Hansjürgens (1996) and Sijm et al. (2007) for being inconsistent with the "polluter pays" principle and lacking fairness from the perspective of society as a whole. In addition, Van Steenberghe (2004) found that the so-called fair rule to allocate greenhouse gas emission permits is not beneficial for all nations, with some countries being worse off under global agreement than under non-cooperative contexts. Under the framework of the Kyoto Protocol, Gomes & Lins (2008) adopted the zero-sum gains data envelopment analysis method to provide a fair carbon emissions allocation plan for various countries, which not only stabilizes the concentration of greenhouse gases in the atmosphere, but also achieves carbon quota trading with no impact on global emissions. The above studies have mostly considered the fairness of carbon quota allocations at the global level, ignoring the interest-driven issues of individual/regional perspectives. Therefore, the analysis of carbon trading mechanisms through consensus modeling with all participators' interests taken into account is of great significance.

Although many studies have investigated carbon issues, there has been few research on carbon quota trading mechanisms, and consensus decision-making theory has not been adopted to deal with the design of carbon trading mechanisms and their resulting unfairness issues. That is, using optimization consensus models to assist DMs in exchanging carbon quotas and the development of fair connections among them within a closed-loop trading system are neglected. Hence, the main contributions of this study are as follows: (i) By referring to conventional market trading mechanisms, a benchmark consensus model with the aim of overall revenue maximization is presented to derive the optimal carbon quota allocation scheme. (ii) By building a two-stage programming model, new allocation schemes are acquired that focus on different single DM's revenue maximization, allowing detailed trading information such as the transferred quantities, DM's unit selling and buying prices, and unit transaction prices to be acquired. (iii) Two strategies based on individual/group development indices are proposed to deal with the unfairness issue within the trading system. (iv) A relaxation method based on particle swarm optimization (PSO) (Kennedy & Eberhart, 1995) is proposed to solve the above consensus models. And (v) numerical analysis of a trading system composed of five regions is conducted to verify the effectiveness of the proposed models.

The rest of this paper is organized as follows. Section 2 briefly reviews the optimization consensus models, then Section 3 presents some assumptions of the trading mechanisms, and justifies the rationality of the hypothesis through theoretical deduction. Section 4 constructs a series of new consensus models from which optimal/fair allocation schemes are obtained within the closed-loop trading system, and further proposes an optimization algorithm to solve these models. A numerical example is reported in Section 5 to demonstrate the feasibility of the proposed mechanisms. Finally, Section 6 gives some concluding remarks and identifies future research directions.

2. Preliminaries on optimization consensus modeling

To better understand the subsequent construction of optimization closed-loop carbon trading consensus models, this section briefly reviews theoretical GDM models for obtaining the optimal consensus. However, before introducing the basic consensus models, we define some related notation. Let $D = \{d_1, d_2, \cdots, d_n\}$ be the set of all DMs, where d_i denotes the i-th DM and $i \in N = \{1, 2, \cdots, n\}$. Let $O = \{o_1, o_2, \cdots, o_n\}$ and $O' = \{o'_1, o'_2, \cdots, o'_n\}$ be the sets of initial and final preferences (i.e., opinions, judgements) of the group, where o_i, o'_i denote d_i's initial and final opinions, respectively. The existing forms of expressions for DMs include, but are not limited to, linear uncertainty preferences (Gong et al., 2020, 2021), linguistic preferences (Cabrerizo et al., 2013; Wu et al., 2018; Yu et al., 2021), fuzzy preference (Herrera-Viedma et al., 2014; Wu & Chiclana, 2014; Zhang et al., 2018). Nevertheless, aiming to solve real-life GDM problems, we adopt traditional forms, i.e., positive and real numbers, to denote DM's opinions in this paper. Let ω_i denote the unit cost provided by the moderator for d_i adjusting his opinions, $i \in N$. In fact, the modeling mechanisms are similar for both MCCM (Ben-Arieh & Easton, 2007; Ben-Arieh et al., 2009) and MACM (Dong et al., 2016, 2010). If all DMs' unit costs satisfy $w_i = w_j, \forall i, j \in N, i \neq j$, then the former reduces to the latter (Zhang et al., 2020b). A general framework of the minimum cost/adjustment consensus model provided by Zhang et al. (2011) can be introduced as:

$$
\begin{aligned}
&\min \sum_{i=1}^{n} w_i * d(o'_i, o_i) \\
&\text{s.t.} \begin{cases} o^c = F(o'_1, o'_2, \cdots, o'_n) & (1-1) \\ CD(o'_i, o^c) \leq \alpha, \forall i \in N & (1-2) \end{cases}
\end{aligned}
\tag{1}
$$

In Model (1), $d(o'_i, o_i)$ represents the distance or deviation between d_i's initial and final (or adjusted) opinions (del Moral et al., 2018), which is generally given by the Manhattan distance (Ben-Arieh & Easton, 2007) or Euclidean distance (Ben-Arieh et al., 2009). Constraint (1-1) means that the collective opinion (i.e., consensus) o^c should be obtained by the aggregation function F over all DMs' final opinions $\{o'_1, o'_2, \cdots, o'_n\}$, which corresponds to various social selections; and constraint (1-2) measures the consensus level CD attached to d_i's adjusted opinion o'_i and the consensus o^c, where α is a pre-defined threshold that is usually employed when solving soft consensus problems (Herrera-Viedma et al., 2014; Zhang et al., 2011, 2019).

The above model is an optimization consensus model with a minimum cost/adjustment from the moderator's perspective. However, individuals in GDM always expect some compensation for adjusting their opinion, the more the better. Hence, introducing linear primal-dual theory, Gong et al. (2015a,b) and Zhang et al. (2019) explored the dual forms of Model (1) in specific contexts so as to obtain the maximum compensation for all DMs. In particular, Zhang et al. (2019) provided a concise form of the maximum compensation consensus models (i.e., Model (2)), where R means the set of real numbers, and y_i is the unit compensation expected by d_i. As discussed earlier, Zhang et al. (2020a) divided the objective function of Model (2) into a modification return provided by the moderator for the DM's opinion adjustment and a recognition return based on the similarity between the DM's initial opinion and the final consensus. However, their model is omitted here due of space limitations.

$$\begin{array}{ll} \max & \sum_{i=1}^{n} y_i * (o_i - o^c) \\ \text{s.t.} & y_i \in R,\ i \in N \end{array} \tag{2}$$

The optimal collective opinion o^c can always be obtained, regardless from the minimum cost perspective (i.e., Model (1)) or the maximum compensation perspective (i.e., Model (2)). Therefore, the idea of discussing the closed-loop carbon quota trading mechanism with an objective function that maximizes the overall revenue is feasible. In addition, the above two models obtain the optimal collective opinion o^c, whereas this paper aims to derive all DMs' optimal adjusted opinions (i.e., the set of $O^{'}$) during the trading process. Thus, in the following discussion, we introduce some influential factors into the conventional market trading mechanisms and build a series of optimization consensus models that provide optimal or fair carbon quota allocations within a closed-loop trading system.

3. Assumptions for carbon quota trading mechanisms

This paper explores how to develop a satisfactory carbon quota allocation scheme under the goal of maximizing the revenue for either the whole group or a single DM through market trading mechanisms. To facilitate a better understanding, Table 1 presents the main notation used in this paper. Suppose that multiple DMs (e.g., companies, regions, nations) form a closed-loop trading system with a fixed total carbon quota. Let r_i be d_i's initial fixed unit revenue and $r_1 \leq r_2 \leq ... \leq r_n$, where r_i is determined by each DM's unique qualities, such as social and economic development, natural conditions, resource endowments, industrial structures, and energy usage rates.

Table 1 Summary of notation used in this paper

Notation	Meaning	Notation	Meaning
d_i	The i-th DM	I_{ij}	Quantity transferred from d_i to d_j
r_i	Initial fixed unit revenue of d_i's CQ	T_{ij}	Unit transaction price between d_i and d_j
p_i	Unit selling price of d_i's CQ	δ	Non-archimedean infinitesimal
q_i	Unit buying price of d_i's CQ	γ	Fairness threshold
o_i	d_i's initial CQ	α	Fairness measure variable
$o_i^{'}$	d_i's final CQ	Z_1	Obj to maximize overall revenue
o_i^-	Lower limit of d_i's IECQI	Z_2	Obj to maximize a specific DM's revenue
o_i^+	Upper limit of d_i's IECQI	Z_3	Obj regarding revenue and fairness
H_i	Individual development index	$\bar{H}$	Group development index

Note: CQ, IECQI and Obj are short for carbon quota, initially expected carbon quota interval and the objective function, respectively.

To be noted, this paper aims to depict the most essential trading behavior within a carbon quota market by consensus modeling. Meanwhile, in order to reduce the computational complexity of the subsequent models, we currently simplify the problem to the greatest extent. Therefore, several basic assumptions need to be clarified as:

1. The carbon quota market discussed remains stable during a certain period, and DMs can freely participate in the trading system;
2. Variables of unit prices (i.e., p_i, q_i, T_{ij}) are static, meaning that they don't fluctuate with time, supply and demand, and etc.;
3. Unit revenue of d_i's carbon quota (i.e., r_i) is a constant, which is only determined by d_i's own inherent characteristics rather than o_i, meaning that the standard law of diminishing returns assumption is not considered;
4. Factors regarding costs within the profit-oriented trading system are implicit in d_i's initial unit revenue, which means we only need to conduct analysis from the perspective of revenue maximization.

Actually, assumptions listed above are all to reduce the complexity of our GDM problem, and each point could be an interesting topic in our subsequent research. Anyway, the final results obtained from the closed-loop trading system through consensus modeling should satisfy two main objectives:

- Goal 1: Each DM's total revenue derived from the trading is no less than his initial fixed total revenue;
- Goal 2: The sum of all DMs' revenue acquired from the closed-loop trading system should be maximized.

Goal 1 is set from the DM's perspective, and aims to maximize each DM's economic benefits. All DMs are assumed to be rational (that is, once the carbon quota trading is conducted, they must benefit themselves); otherwise, the transactions are invalid. This corresponds to real-life market trading and can be understood as the effectiveness of the trading mechanisms. On the contrary, Goal 2 is set from the collective angle. In general, the representative for the collective benefit is the participant who determines the initial carbon quota for all DMs, and also the one who plays the role as a moderator in GDM problems (Ben-Arieh & Easton, 2007; Gong et al., 2021), such as local governments or world organizations. For those representatives, the primary goal is to maximize the overall revenue.

To realize Goal 1, we have the following constraints: (1) $p_i \geq r_i$, (2) $q_i \leq r_i$, where p_i denotes the unit selling price, q_i represents the unit buying price, and r_i is the original fixed revenue for one unit of d_i's carbon quota. Let the quantity transferred from d_i to d_j be I_{ij}, and their final unit transaction price be T_{ij}. Then, the following statement holds: If $p_i \leq q_j$, then the one-way carbon quota transaction from d_i to d_j can be realized, that is, d_i can sell a carbon quota to d_j, and so $I_{ij} \geq 0$ and the unit transaction price $T_{ij} \in [p_i, q_j]$, which indicates there is a negotiable space in the trading process between d_i and d_j. At the same time, we derive $I_{ji} = 0$, since $I_{ij} * I_{ji} = 0$ holds under the premise of one-way trading.

The above constraint indicates that there is a directionality in the carbon quota trading between any two DMs. Specifically, once a carbon transaction occurs between d_i and d_j, the transferred quantity sold by d_i to d_j is I_{ij}, and we get $r_i \leq p_i \leq q_j \leq r_j$. Moreover, because the unit transaction price satisfies $p_i \leq T_{ij} \leq q_j$, we have that $r_i \leq p_i \leq T_{ij} \leq q_j \leq r_j$. Thus, d_i's revenue is $T_{ij}I_{ij} - r_iI_{ij} \geq 0$, whereas d_j's revenue is $r_jI_{ij} - T_{ij}I_{ij} \geq 0$. This trading mechanism guarantees that every carbon transaction that occurs is profitable for both parties, implying that each DM's final revenue after the carbon trading is no less than their initial total fixed revenue. Thereby, Goal 1 is always met.

Theorem 1. *$I_{ij} * I_{ji} = 0$ and $I_{ij} \geq 0$, $I_{ji} \geq 0$ $(i \neq j, i, j \in N)$, if and only if $p_i = q_i = p_j = q_j = r_i = r_j$, $I_{ij} \geq 0$, and $I_{ji} \geq 0$ hold simultaneously. At this time, the unit selling and buying prices, as well as the initial fixed unit revenue for both d_i and d_j, are equal. In this case, the transaction does not bring about a change in revenue, so it has no economic significance.*

Proof. As $p_i \geq r_i$ and $q_i \leq r_i$, we have $p_i \geq r_i \geq q_i$. When $I_{ij} \geq 0$ and $I_{ji} \geq 0$ hold simultaneously, $p_i \leq q_j$ and $p_j \leq q_i$ are obtained, that is, $r_i \leq p_i \leq q_j \leq r_j \leq p_j \leq q_i \leq r_i$. So when $p_i = q_i = p_j = q_j = r_i = r_j$, both $I_{ij} \geq 0$ and $I_{ji} \geq 0$ hold. Under other situations, if $I_{ij} \geq 0$, we have $I_{ji} = 0$; on the contrary, if $I_{ji} \geq 0$, we get $I_{ij} = 0$. To sum up, based on the aforementioned four assumptions, once DM d_i buys (sells) carbon quota from (to) d_j, he/she will no longer sell (buy) carbon quota to (from) d_j. □

Theorem 1 guarantees that the transactions between any two DMs in the closed-loop carbon quota trading system are one-way. When the initial parameters provided by the two DMs (including unit buying and selling prices as well as their initial fixed unit revenue) are all equal, their transaction has no direction constraint. However, any transaction realized under these conditions cannot increase the DMs' revenue, so it has no economic value.

Theorem 2. *Suppose r_i is d_i's initial fixed unit revenue and $r_1 \leq r_2 \leq ... \leq r_n$, if $i \leq j$, $I_{ij} \geq 0$ holds; if $i > j$ and $r_i \neq r_j$, $I_{ij} = 0$ holds; and if $i > j$ and $r_i = r_j$, $I_{ij} \geq 0$ holds.*

Proof. If $i \leq j$, we have $r_i \leq r_j$, and because $p_i \geq r_i, q_j \leq r_j$, there must exist p_i, q_j such that $r_i \leq p_i \leq q_j \leq r_j$, then $I_{ij} \geq 0$. Besides, if $i > j$ and $r_i \neq r_j$, then $r_i > r_j$, and since $p_i \geq r_i, q_j \leq r_j$, that is, $p_i \geq r_i > r_j \geq q_j$, thus there exist no p_i, q_j such that $p_i \leq q_j$, thereby we have $I_{ij} = 0$. Similarly, if $i > j$ and $r_i = r_j$, then $p_i \geq r_i = r_j \geq q_j$. Clearly, only if $p_i = r_i = r_j = q_j$, $I_{ij} \geq 0$ holds, otherwise, we have $I_{ij} = 0$. □

Theorem 2 takes the basic hypothesis of this paper into consideration: all DMs are arranged in order based on the relationships among their original fixed unit revenues, that is, $r_1 \leq r_2 \leq ... \leq r_n$. The quantity of the carbon quota that is transferred is not only affected by the DM's location index, but also by the size of the DM's fixed unit revenue. This theorem implies that carbon quota trading can only be carried out from one DM with a smaller fixed unit revenue to another with a larger unit revenue. Therefore, DMs with small fixed unit revenues have to sell their carbon quota to increase their total revenue, because $p_i \geq r_i$. On the contrary, DMs with large unit revenues can only improve their revenue by purchasing carbon quotas, because $q_i \leq r_i$.

Theorem 3. *Let d_i's final carbon quota be $o_i^{'}$. Considering that some uncertainty exists during the trading process, the above final carbon quota is represented by an interval value, denoted as $[o_i^-, o_i^+]$, whose endpoints satisfy:*

$$\sum_{i=1}^{n} o_i^- \leq \sum_{i=1}^{n} o_i \leq \sum_{i=1}^{n} o_i^+$$

Proof. Since $o_i^- \leq o_i^{'} \leq o_i^+$, we have $\sum_{i=1}^{n} o_i^- \leq \sum_{i=1}^{n} o_i^{'} \leq \sum_{i=1}^{n} o_i^+$. Meanwhile, because the total carbon quota in the closed-loop trading system is fixed, namely $\sum_{i=1}^{n} o_i^{'} = \sum_{i=1}^{n} o_i$, then $\sum_{i=1}^{n} o_i^- \leq \sum_{i=1}^{n} o_i \leq \sum_{i=1}^{n} o_i^+$.

□

Theorem 3 is based on the assumption that the total carbon quota in the closed-loop trading system is fixed, which complies with the provisions of the clean development mechanism. That is, under the premise of fixed global carbon emission levels, high-emission countries can finance some projects in low-emission countries to reach their established limit (i.e., compensatory reduction) (Gomes & Lins, 2008). In short, the so-called "carbon market" can reduce the economic impact on high-emission countries and achieve the overall goal of reducing carbon emissions. In addition, for rational DMs, threshold constraints attached to their final carbon quota can better exhibit the uncertainties during the trading process (Ruidas et al., 2021); for the moderator, there is no need to grasp all transaction details, namely, the moderator only needs to have overall control of the total amount, that is, the lower limit of the final total carbon quota is no greater than the initial total amount, while the upper limit should be no less than the sum of all DMs' original carbon quotas.

An example of carbon quota trading conducted by three regions is presented below to preliminarily clarify our modeling ideas. Initial information is listed in Table 2, while the trading results, including the final carbon quota, and the corresponding revenue, are shown in Table 3. Meanwhile, the specific trading process is exhibited in Fig. 1. Note that the elaborated example only corresponds to the aforementioned basic assumptions, and does not really involve the consensus modeling in the next section.

Table 2 Example of the initial information provided by three regions

	d_1	d_2	d_3
o_i	10	10	10
r_i	50	80	120
$r_i o_i$	500	800	1200

Table 3 Example of the final carbon quotas through the trading conducted by three regions

	d_1	d_2	d_3
$o_i^{'}$	3	11	16
r_i	50	80	120
$r_i o_i^{'}$	150	880	1920
Trading revenue	530	-35	-495
Total revenue	680	845	1425

Table 2 provides the initial carbon quota (i.e., o_i) allocated to each region along with its fixed unit revenue (i.e., r_i), from which the initial total revenue (i.e., $r_i o_i$) of each region can be obtained. As d_1 has the smallest unit revenue r_1, this DM can only increase his revenue by selling a carbon quota; as d_3 has the largest unit revenue r_3, this DM can only increase his total revenue by purchasing a carbon quota. For d_2, revenue may be increased by selling, purchasing, or combining both trading behavior (see Fig. 1).

To make the trading mechanism effective and feasible, DM's unit selling price should be no less than his initial unit revenue (i.e., $p_i \geq r_i$), while the unit buying price should be no larger than the fixed unit revenue (i.e., $q_i \leq r_i$).

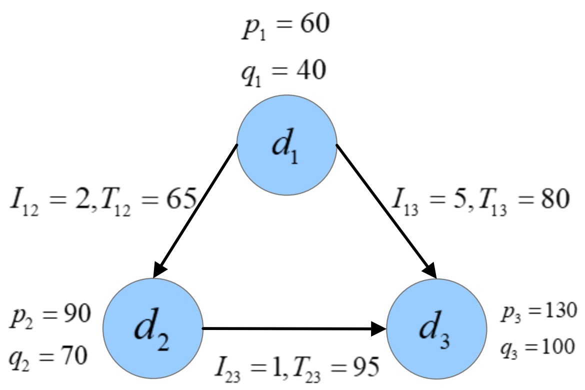

Fig. 1 Schematic diagram of carbon quota trading among three regions

Take d_2 as an example for detailed analysis: the total revenue for d_2's initial carbon quota is $10 * 80 = 800$, and suppose through optimization consensus modeling, d_2's unit selling and buying prices are derived as $p_2 = 90$ and $q_2 = 70$, respectively. The parameters for other regions see Fig. 1. Since $T_{ij} \in [p_i, q_j]$, here we might as well let the unit transaction price be $T_{ij} = \frac{p_i+q_j}{2}$, then we derive $T_{12} = 65, T_{23} = 95$, and the transferred carbon quota quantities related to d_2 are assumed to be obtained through mathematical modeling as $I_{12} = 2, I_{23} = 1$. As a result, d_2's total carbon quota is $10 + 2 - 1 = 11$, and the new fixed revenue for holding his carbon quota is $11 * 80 = 880$, while the transaction revenue (i.e., the difference between the income from selling carbon quotas and the cost of buying quotas) is $1 * 95 - 2 * 65 = -35$, making d_2's final total revenue of $880 - 35 = 845$ be larger than the initial total revenue of 800. Results in Tables 2 and 3 demonstrate that the final revenue of every region in the closed-loop trading system has increased with respect to their initial total revenue, indicating that the proposed trading mechanism is feasible.

4. Optimization consensus modeling concerning carbon quota trading mechanism

Chu & Shen (2006) indicated that the purpose of designing a trading mechanism is to provide a method for ensuring that the allocation decisions and pricing decisions in decision-making processes result in the desired outcomes. They also found that, once the allocation principle is set in a truthful mechanism, the prices are determined; similarly, once the pricing rule is determined, the allocation is settled. Different from the extant research on the carbon market (Diaz-Rainey & Tulloch, 2018; Gomes & Lins, 2008; Lamba et al., 2019; Ruidas et al., 2021; Van Steenberghe, 2004; Zhou et al., 2020b; Zou et al., 2021), this section takes the maximization of the overall revenue or a single DM's revenue as the objective function, and uses optimization consensus modeling to determine the allocation scheme (i.e., determination of variables $o_i^{'}, I_{ij}$) and the pricing scheme (i.e., determination of variables p_i, q_i, T_{ij}) in the carbon quota trading system.

4.1. Benchmark carbon trading consensus model with overall revenue maximization

To realize Goal 2 (as defined in Section 3), we build the following optimization consensus model (i.e., Model (3)) to maximize the sum of the revenues of all DMs within the closed-loop trading system as:

$$\max\ Z_1 = \sum_{i=1}^{n} r_i o_i^{'}$$
$$\text{s.t.} \begin{cases} o_i^{'} = o_i - \sum\limits_{j=1, j\neq i}^{n} I_{ij} + \sum\limits_{j=1, j\neq i}^{n} I_{ji}, i \in N & (3-1) \\ q_i \leq r_i \leq p_i, i \in N & (3-2) \\ \begin{cases} I_{ij} \geq 0, \text{if } p_i \leq q_j \text{ and } i < j,\ i, j \in N \\ I_{ij} = 0, \text{otherwise} \end{cases} & (3-3) \\ o_i^{-} \leq o_i^{'} \leq o_i^{+}, i \in N & (3-4) \\ p_i \geq 0, q_i \geq 0, I_{ij} \geq 0, i, j \in N & (3-5) \end{cases} \tag{3}$$

The objective function Z_1 in Model (3) attempts to maximize the final total revenue for all DMs within the carbon quota trading system. Constraint (3-1) is the expression of d_i's final quota, which is equal to the initial quantity minus all the sold quantities $\sum\limits_{j=1, j\neq i}^{n} I_{ij}$ and plus all the purchased quantities $\sum\limits_{j=1, j\neq i}^{n} I_{ji}$, where I_{ij} denotes the carbon quota quantity transferred from d_i to d_j. Since the sum of all transfer-out quantities equals to the sum of all transfer-in quantities, we can easily obtain $\sum_{i=1}^{n} o_i^{'} = \sum_{i=1}^{n} o_i$ through constraint (3-1), corresponding to the fact that the total carbon quota amount in the closed-loop trading system is fixed. Constraint (3-2) is the threshold constraint attached to the unit selling price p_i and the unit buying price q_i based on the pre-defined initial fixed unit revenue r_i. Constraint (3-3) specifies the achievable conditions of the carbon trading between any two DMs. Namely, only when the seller's location index is smaller than the purchaser's index, and the unit selling price p_i is no greater than the unit buying price q_j, will the transaction from d_i to d_j be achieved (i.e., $I_{ij} \geq 0$). Constraint (3-4) assumes that d_i's final quota is located in his own expected interval provided initially. Constraint (3-5) indicates that all variables are nonnegative. Hence, Model (3) explores the optimal carbon quota allocation problem under the maximization of the overall revenue of the trading system, where $Z_1, o_i^{'}, I_{ij}, p_i, q_i, (i \in N)$ are decision variables and r_i, o_i, o_i^-, o_i^+ are known parameters. In fact, due to insufficient constraints (e.g., the absence of specific transaction prices building connections with the unit price variables), only the ranges of p_i and q_i instead of their optimal values can be obtained through Model (3).

***Theorem* 4.** *There must exist an m-th DM such that* $\sum_{i=1}^{m-1} o_i^- + o_m^{'} + \sum_{i=m+1}^{n} o_i^+ = \sum_{i=1}^{n} o_i$ *and* $o_m^- \leq o_m^{'} \leq o_m^+$. *By then, the optimal value for the objective function of Model (3) is* $\sum_{i=1}^{m-1} r_i o_i^- + r_m o_m^{'} + \sum_{i=m+1}^{n} r_i o_i^+$ *and the optimal solution is* $o_i^{'} = o_i^- (1 \leq i \leq m-1), o_m^{'} = \sum_{i=1}^{n} o_i - \sum_{i=1}^{m-1} o_i^- - \sum_{i=m+1}^{n} o_i^+, o_i^{'} = o_i^+ (m+1 \leq i \leq n)$.

Proof. First, when $o_i^{'} = o_i^- (1 \leq i \leq m-1), o_m^{'} = \sum_{i=1}^{n} o_i - \sum_{i=1}^{m-1} o_i^- - \sum_{i=m+1}^{n} o_i^+, o_i^{'} = o_i^+ (m+1 \leq i \leq n)$, there exists no $I_{ij} > 0$ to further increase the objective function. That is, except d_m, all DMs have reached their critical points of their expected carbon quota intervals (i.e., $[o_i^-, o_i^+]$), either the lower limit of d_i $(1 \leq i \leq m-1)$ or the upper limit of d_i $(m+1 \leq i \leq n)$, making DMs with a location index smaller than m cannot further sell carbon quota while DMs with a location index larger than m cannot further buy carbon quota, based on the given condition as $r_1 \leq r_2 \leq ... \leq r_n$. In a nutshell, if $I_{ij} > 0$, the objective function of Model (3) increases to $f^* = f + (r_j - r_i) * I_{ij}$, where f is the total revenue before the transaction. Due to $r_j \geq r_i, (i < j)$, we get $f^* \geq f$, indicating that if and only if $I_{ij} = 0$, the value of the objective function no longer increases and becomes the optimal value. Thus, the solution at this point is exactly the optimal solution, and the objective function becomes $\sum_{i=1}^{m-1} r_i o_i^- + r_m o_m^{'} + \sum_{i=m+1}^{n} r_i o_i^+$.

Next, we prove that this critical DM with the m-th location index always exists. Because $o_i^- \leq o_i^{'} \leq o_i^+$, we have $\sum_{i=1}^{n} r_i o_i^- \leq \sum_{i=1}^{n} r_i o_i^{'} \leq \sum_{i=1}^{n} r_i o_i^+$. If $m = 1$, then $r_1 o_1^- + \sum_{i=2}^{n} r_i o_i^+ \leq \sum_{i=1}^{n} r_i o_i^{'} \leq \sum_{i=1}^{n} r_i o_i^+$. If $m = 2$, then $\sum_{i=1}^{2} r_i o_i^- + \sum_{i=3}^{n} r_i o_i^+ \leq \sum_{i=1}^{n} r_i o_i^{'} \leq r_1 o_1^- + \sum_{i=2}^{n} r_i o_i^+$. In the same way, if $m = n$, then $\sum_{i=1}^{n} r_i o_i^- \leq \sum_{i=1}^{n} r_i o_i^{'} \leq \sum_{i=1}^{n-1} r_i o_i^- + r_n o_n^+$. Therefore, once m takes a specific value within the set N, $\sum_{i=1}^{n} r_i o_i^{'}$ can take any value from the interval $[\sum_{i=1}^{n} r_i o_i^-, \sum_{i=1}^{n} r_i o_i^+]$, and so the known constraint $\sum_{i=1}^{n} r_i o_i^- \leq \sum_{i=1}^{n} r_i o_i^{'} \leq \sum_{i=1}^{n} r_i o_i^+$ means that d_m must exist such that $o_i^{'} = o_i^- (1 \leq i \leq m-1), o_m^{'} = \sum_{i=1}^{n} o_i - \sum_{i=1}^{m-1} o_i^- - \sum_{i=m+1}^{n} o_i^+, o_i^{'} = o_i^+ (m+1 \leq i \leq n)$ hold. □

***Theorem* 5.** *When Model (3) reaches its maximum value, we obtain* $\sum_{j=1, j \neq i}^{n} I_{ij} - \sum_{j=1, j \neq i}^{n} I_{ji} = o_i - o_i^-$ $(1 \leq i \leq m-1)$, $\sum_{j=1, j \neq m}^{n} I_{mj} - \sum_{j=1, j \neq m}^{n} I_{jm} = o_m - \sum_{i=1}^{n} o_i + \sum_{i=1}^{m-1} o_i^- + \sum_{i=m+1}^{n} o_i^+$, $\sum_{j=1, j \neq i}^{n} I_{ij} - \sum_{j=1, j \neq i}^{n} I_{ji} = o_i - o_i^+$ $(m+1 \leq i \leq n)$.

Proof. Theorem 4 implies that once Model (3) reaches its maximum value, and if $1 \leq i \leq m-1$, then $o_i^{'} = o_i^-$

holds, meantime, due to $o_i^{'} = o_i - \sum\limits_{j=1,j\neq i}^{n} I_{ij} + \sum\limits_{j=1,j\neq i}^{n} I_{ji}$, we have $\sum\limits_{j=1,j\neq i}^{n} I_{ij} - \sum\limits_{j=1,j\neq i}^{n} I_{ji} = o_i - o_i^{-} (1 \leq i \leq m-1)$. Similar analysis can be conducted for the rest situations. □

Theorems 4 and 5 indicate that the optimal solution of Model (3) and the maximum value of the objective function exist and are unique. Therefore, the optimal allocation for all DMs' carbon quotas is determined. In other words, by solving Model (3), we obtain all information about carbon quota transfers within the trading system. However, note that only the feasible regions can be obtained by Model (3), rather than the optimal values of the decision variables p_i, q_i.

***Theorem* 6.** *The achievable constraints of the carbon quota trading mechanism are determined by d_i's unit selling price p_i and d_j's unit buying price q_j as:*

$$\begin{cases} I_{ij} \geq 0, & \text{if } p_i \leq q_j \text{ and } i < j,\ i,j \in N \\ I_{ij} = 0, & \text{otherwise} \end{cases}$$

which is equivalent to

$$\begin{cases} I_{ij} \leq \dfrac{|q_j - p_i| + q_j - p_i}{\delta}, & i < j, i, j \in N \\ I_{ij} = 0, & \text{otherwise} \end{cases} \tag{4}$$

where δ is the non-Archimedean infinitesimal, viz. a sufficiently small positive value approaching zero (Charnes et al., 1994; Mehrabian et al., 2000).

Proof. If $i < j, i, j \in N$, then carbon quota trading between the seller d_i and the purchaser d_j is achievable, so $I_{ij} \geq 0$ holds. Next, we discuss the effect of prices on the transferred quantity: when $p_i < q_j$, according to Eq. (4), we have $I_{ij} < \dfrac{2(q_j - p_i)}{\delta}$, and because δ is the non-Archimedean infinitesimal, $I_{ij} < +\infty$, that is, $I_{ij} \geq 0$ holds; when $p_i \geq q_j$, based on Eq. (4), we have $I_{ij} = 0$. In addition, if $i \geq j, i, j \in N$, the one-way transaction from d_i to d_j cannot be achieved, so we have $I_{ij} = 0$. This completes the proof of Theorem 6. □

Theorem 6 states the achievable conditions for a closed-loop trading system. Specifically, carbon quota trading can only be achieved when the unit selling price of one DM with a small location index is no greater than the unit buying price of another DM with a large location index; otherwise, their carbon quota transaction fails.

4.2. Carbon trading consensus models with single DM's revenue maximization

The competition mechanism refers to the struggle among market practitioners to maximize their own economic benefits, so it focuses more on individual standpoints than the collective perspective. The model developed in Section 4.1 only maximizes the overall revenue of the trading process, and ignores the individual DM's interests and the resulting unfairness issues. This section considers individual DMs as the research object, and uses optimization consensus models to derive detailed information about the trading process, including the participating DMs, transferred quantities, and the final unit transaction prices. That is, when the group realizes its optimal allocation by considering every DM's revenue maximization, this section attempts to determine not only d_i's final carbon quota $o_i^{'}$ from its expected interval $[o_i^-, o_i^+]$, but also each DM's psychological expected unit selling and buying prices (i.e., p_i, q_i) and the transferred quantity I_{ij} along with the best achievable unit transaction price T_{ij}. Based on the above principles, a two-stage programming model is built as:

$$\max Z_2 = r_i o_i^{'} + \sum_{j=1, j\neq i}^{n} T_{ij} I_{ij} - \sum_{j=1, j\neq i}^{n} T_{ji} I_{ji}$$

$$\text{s.t.} \begin{cases} \begin{cases} p_i \le T_{ij} \le q_j, & \text{if } p_i \le q_j,\ i<j, i,j \in N \\ T_{ij} = 0, & \text{otherwise} \end{cases} & (5-1) \\ \begin{cases} Max \ \sum_{i=1}^{n} r_i o_i^{'} \\ o_i^{'} = o_i - \sum_{j=1, j\neq i}^{n} I_{ij} + \sum_{j=1, j\neq i}^{n} I_{ji}, i \in N \\ q_i \le r_i \le p_i, i \in N \\ \begin{cases} I_{ij} \le \dfrac{|q_j - p_i| + q_j - p_i}{\delta}, & i<j, i,j \in N \\ I_{ij} = 0, & \text{otherwise} \end{cases} \\ o_i^- \le o_i^{'} \le o_i^+,\ p_i \ge 0, q_i \ge 0, I_{ij} \ge 0, T_{ij} \ge 0, \delta > 0, i,j \in N \end{cases} & (5-2) \end{cases} \quad (5)$$

Model (5) introduces constraint (5-1) into Model (3), that is, adding the expression of the unit transaction price T_{ij} between DMs d_i and d_j, which is a range bounded by d_i's unit selling price p_i and d_j's unit buying price q_j. As stated in Section 3, only the location indices satisfy $i<j, i,j \in N$, and $p_i \le q_j$ holds, can the unit transaction price between d_i and d_j be denoted as $T_{ij} \in [p_i, q_j]$. Here, the unit transaction price T_{ij} obeys a uniform distribution by default, as each point within the interval $[p_i, q_j]$ can be selected with equal possibility, which makes it easy to

calculate, understand and be applied into real-life GDM. The objective function in Model (5) is the sum of d_i's final carbon quota holding revenue (i.e., $r_i o_i^{'}$) and the transaction revenue for selling or buying carbon quotas (i.e., $\sum_{j=1,j\neq i}^{n} T_{ij} I_{ij} - \sum_{j=1,j\neq i}^{n} T_{ji} I_{ji}$), and this value the larger the better. Model (5) indicates that maximizing a single DM's revenue is not unconstrained; instead, it should be carried out within the context of maximizing the overall revenue for the whole group (i.e., constraint (5-2)). Referring to Theorem 4, Model (5) can be further transformed into Model (6), where constraints (6-2)–(6-9) provide the analytical formula of constraint (5-2). The definitions of other variables and constraints in Model (6) are consistent with those in Models (3) and (5).

Theorem 4 states that the optimal solution of Model (3) exists and is unique. Thus, there must exist feasible solutions for Model (6). Actually, constraints (6-6)–(6-8) in Model (6) provide the analytical formula for the DM's final carbon quota $o_i^{'}$, and are acquired by solving Model (3). Hence, variables $Z_2, I_{ij}, p_i, q_i, T_{ij}$ and m in Model (6) are decision variables, while $r_i, o_i, o_i^-, o_i^+, \delta$ are known parameters. In short, under the premise of maximizing the overall revenue, and by further adding the expression of the unit transaction prices, Model (6) determines the optimal values for d_i's unit selling and buying prices (i.e., p_i and q_i), and further obtains detailed trading information including the quantity I_{ij} transferred from d_i to d_j and their corresponding unit transaction price T_{ij}.

$$
\begin{aligned}
&\max\ Z_2 = r_i o_i^{'} + \sum_{j=1,j\neq i}^{n} T_{ij} I_{ij} - \sum_{j=1,j\neq i}^{n} T_{ji} I_{ji} \\
&\text{s.t.}\left\{
\begin{array}{ll}
\begin{cases} p_i \le T_{ij} \le q_j, & \text{if } p_i \le q_j,\ i<j, i,j \in N \\ T_{ij} = 0, & \text{otherwise} \end{cases} & (6-1) \\
\sum_{i=1}^{n} r_i o_i^{'} = \sum_{i=1}^{m-1} r_i o_i^- + r_m o_m^{'} + \sum_{i=m+1}^{n} r_i o_i^+ & (6-2) \\
o_i^{'} = o_i - \sum_{j=1,j\neq i}^{n} I_{ij} + \sum_{j=1,j\neq i}^{n} I_{ji}, i \in N & (6-3) \\
q_i \le r_i \le p_i, i \in N & (6-4) \\
\begin{cases} I_{ij} \le \dfrac{|q_j - p_i| + q_j - p_i}{\delta}, & i<j, i,j \in N \\ I_{ij} = 0, & \text{otherwise} \end{cases} & (6-5) \\
o_i^{'} = o_i^-, 1 \le i \le m-1 & (6-6) \\
o_m^{'} = \sum_{i=1}^{n} o_i - \sum_{i=1}^{m-1} o_i^- - \sum_{i=m+1}^{n} o_i^+,\ o_m^- \le o_m^{'} \le o_m^+ & (6-7) \\
o_i^{'} = o_i^+, m+1 \le i \le n & (6-8) \\
p_i \ge 0,\ q_i \ge 0, I_{ij} \ge 0, T_{ij} \ge 0, \delta > 0, i,j,m \in N & (6-9)
\end{array}
\right.
\end{aligned}
\qquad (6)
$$

4.3. Identification and adjustment rules for discordant DMs

In Section 4.2, we considered the case in which every single DM pursues the maximization of his own revenue, which inevitably results in unfairness (e.g., the unbalanced growth of the DMs' revenue). Therefore, this section examines the potential to achieve a relatively balanced state within the closed-loop carbon quota trading system by adjusting some DMs' initial parameters. Once fairness is achieved, DMs with too much revenue growth or too little revenue growth should no longer exist in the final stage of carbon trading. Any such DMs are collectively referred to as **discordant DMs** in the trading system. During CRP, if the DMs' improper initial parameters can be modified as early as possible, systemic losses (e.g., cost, time) will be significantly reduced. In short, an earlier intervention during GDM is more advantageous (Liang et al., 2020). Compared with extant research adopting utility function (Du et al., 2021) or fuzzy theory (Liu et al., 2021) to characterize the fairness concerns, this paper defines two indicators to directly judge whether the GDM results are fair, so as to further identify discordant DMs and make some corresponding adjustments.

***Definition* 1.** An individual development index is defined as a relative proportion of the DM's final revenue obtained through the carbon quota trading process with respect to their initial fixed revenue, that is,

$$H_i = \frac{r_i o_i^{'} + \sum\limits_{j=1, j \neq i}^{n} T_{ij} I_{ij} - \sum\limits_{j=1, j \neq i}^{n} T_{ji} I_{ji}}{r_i o_i}, i \in N$$

***Definition* 2.** The group development index is defined as a relative proportion of the final total revenue obtained through the carbon quota trading process with respect to the initial fixed total revenue of the group, that is,

$$\bar{H} = \frac{\sum\limits_{i=1}^{n} r_i o_i^{'}}{\sum\limits_{i=1}^{n} r_i o_i}$$

This section follows the idea of fair development of all DMs in the trading system. By default, the difference between the individual development index H_i and the group development index $\bar{H}$ should be within a certain range, otherwise DMs will be identified as discordant DMs with too much or too little revenue growth. These two development indices mainly depend on the DM's final carbon quota $o_i^{'}$, which further depends on the endpoints of the expected interval $[o_i^-, o_i^+]$ provided by DM d_i. Here, we choose interval values instead of crisp numbers to denote d_i's expected carbon quota quantity due to various uncertainties (Ruidas et al., 2021). Hence, by adjusting the expected carbon quota range $[o_i^-, o_i^+]$ of discordant DMs, an equilibrium state with a minimum loss can be achieved within the trading system (see Fig. 2(c)). Let a discordant DM be $d_k, k \in \{0, 1, \cdots, n\}$, and his expected final carbon quota be adjusted from $[o_k^-, o_k^+]$ to $[o_k^{'-}, o_k^{'+}]$ through the following adjustment rules.

- When $H_k << \bar{H}$ and $|H_k - \bar{H}| > \gamma$, where γ is a pre-determined threshold and $<<$ denotes far less than, d_k is identified as a discordant DM with too little revenue growth. This DM is located in the unbalanced state shown in Fig. 2(a), and his adjustment rules are:
  - If $k > m$, then the amount purchased is too little, and so o_k^+ needs to be increased;
  - If $k < m$, then the amount sold is too little, and so o_k^- needs to be further decreased;
  - If $k = m$, then the current expected interval is improperly set, and we need to simultaneously reduce o_k^- and increase o_k^+.
- When $H_k >> \bar{H}$ and $|H_k - \bar{H}| > \gamma$, where γ is a pre-determined threshold and $>>$ means far more than, d_k is identified as a discordant DM with too much revenue growth. This DM is located in the unbalanced state shown in Fig. 2(b), and his adjustment rules are:
  - If $k > m$, then the quantity purchased is too great, and so o_k^+ needs to be decreased;
  - If $k < m$, then the amount sold is too great, and so o_k^- should be increased;
  - If $k = m$, then the current interval of the DM's expected carbon quota is inappropriate, and we need to increase o_k^- and decrease o_k^+ at the same time.

Through the above adjustment rules, a set of updated trading information for all DMs can always be acquired. Based on the individual/group development indices, we obtain the values of all $|H_i - \bar{H}|$ based on Model (6) so as to determine the threshold for the variable γ, as well as the difference value $|H_i - H_j|$ between any two DMs. By repeating the calculations of Models (3) and (6), it is then possible to verify whether the above adjustments are effective or not. The above rules are used to identify discordant DMs and provide the corresponding direction of adjustments. However, the identification parameter γ needs to be manually set, and the specific adjustment range for each DM cannot be accurately specified, that is, we cannot determine by how much each discordant DM

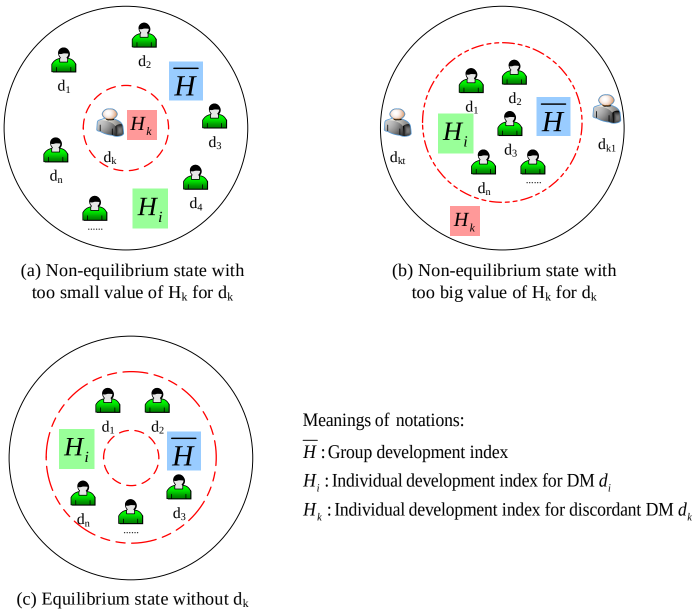

Fig. 2 Identification of non-equilibrium states in closed-loop carbon trading system

needs to adjust the upper and lower limits of their initial expected carbon quota intervals. To overcome these deficiencies, a fairness measure variable α is introduced in the next section, and the optimal carbon quota allocation scheme considering fairness is directly acquired through consensus modeling. Furthermore, by applying a sensitivity analysis to the variable α, we can obtain flexible allocation schemes according to specific GDM scenarios.

4.4. Carbon trading consensus model regarding fairness and revenue

When only a single DM's revenue is considered, the overall revenue cannot be maximized; moreover, when only the overall revenue is taken into account, there can be large gaps between the total revenue of different DMs, highlighting the unfairness issues. Thus, this section introduces a fairness constraint (that is, the difference between any two individual development indices should be within a certain acceptable threshold) under the premise of ensuring the maximization of the overall revenue. Specifically, the fairness constraint is expressed as $|H_i - H_j| \leq \alpha(\alpha \geq 0, i < j, i, j \in N)$, and the optimization carbon quota consensus model considering both revenue and fairness is built as follows:

$$
\max\ Z_3=\sum_{i=1}^{n} r_i o_i^{'}
$$

$$
\text{s.t.}\left\{\begin{array}{ll}
o_i^{'}=o_i-\sum\limits_{j=1,j\neq i}^{n} I_{ij}+\sum\limits_{j=1,j\neq i}^{n} I_{ji}, i\in N & (7-1)\\
q_i\leq r_i\leq p_i, i\in N & (7-2)\\
\left\{\begin{array}{ll} p_i\leq T_{ij}\leq q_j, & \text{if } p_i\leq q_j,\ i<j,i,j\in N\\ T_{ij}=0, & \text{otherwise}\end{array}\right. & (7-3)\\
\left\{\begin{array}{ll} I_{ij}\leq \dfrac{|q_j-p_i|+q_j-p_i}{\delta}, & i<j,i,j\in N\\ I_{ij}=0, & \text{otherwise}\end{array}\right. & (7-4)\\
H_i=\dfrac{r_i o_i^{'}+\sum\limits_{j=1,j\neq i}^{n} T_{ij}I_{ij}-\sum\limits_{j=1,j\neq i}^{n} T_{ji}I_{ji}}{r_i o_i}, i\in N & (7-5)\\
|H_i-H_j|\leq\alpha, i<j,i,j\in N & (7-6)\\
o_i^{-}\leq o_i^{'}\leq o_i^{+},\ q_i\geq 0,\ p_i\geq 0, I_{ij}\geq 0, T_{ij}\geq 0,\delta>0,\alpha\geq 0, i,j\in N & (7-7)
\end{array}\right. \tag{7}
$$

Z_3 in Model (7) aims to maximize the overall revenue after carbon quota trading under the premise that each DMs' revenue has been fairly developed. Constraint (7-1) is the expression of d_i's final carbon quota, which guarantees $\sum_{i=1}^{n} o_i^{'}=\sum_{i=1}^{n} o_i$. Constraint (7-2) sets d_i's optimal psychological expected unit selling price p_i and unit buying price q_i based on his own initial fixed unit revenue r_i. Constraint (7-3) denotes the unit transaction price between any two DMs, and (7-4) provides the achievable conditions for carbon quota trading considering both the DMs' location indices (i.e., i,j) and the relationships between p_i and q_j. Constraint (7-5) defines the individual development index (i.e., Definition 1), and (7-6) specifies the fairness constraints attached to different DMs, where $\alpha\geq 0$ is the fairness measure variable that is pre-determined from the differences among individual development indices (see Section 4.3). Finally, (7-7) provides the thresholds for all variables. Variables $Z_3, o_i^{'}, I_{ij}, p_i, q_i, T_{ij}, H_i, (i\neq j, i,j\in N)$ in Model (7) are to be solved, while $r_i, o_i, o_i^{-}, o_i^{+}, \alpha, \delta, (i\in N)$ are determined in advance.

4.5. Solution method to solve carbon trading consensus models

Clearly, Model (6) is a non-convex optimization problem with many decision variables to be determined. As the pricing decisions (i.e., variables p_i, q_i) have no direct effect on the objective function Z_2, we remove these two variables using constraints (6-1), (6-4), and (6-5), thus obtaining the following relaxation model:

$$
\begin{aligned}
&\max\ Z_2 = r_i o_i^{'} + \sum_{j=1,j\neq i}^{n} T_{ij} I_{ij} - \sum_{j=1,j\neq i}^{n} T_{ji} I_{ji} \\
&\text{s.t.}\begin{cases}
\begin{cases} T_{ij} = 0, & \text{if } i \geq j,\ i,j \in N \\ r_i \leq T_{ij} \leq r_j, & \text{if } i < j,\ i,j \in N \end{cases} & (8-1) \\
\sum_{i=1}^{n} r_i o_i^{'} = \max\ Z_1 & (8-2) \\
o_i^{'} = o_i - \sum_{j=1,j\neq i}^{n} I_{ij} + \sum_{j=1,j\neq i}^{n} I_{ji}, i \in N & (8-3) \\
\begin{cases} I_{ij} = 0, & \text{if } i \geq j, i,j \in N \\ I_{ij} \leq \dfrac{2(r_j - r_i)}{\delta}, & \text{if } i < j, i,j \in N \end{cases} & (8-4) \\
o_i^{-} \leq o_i^{'} \leq o_i^{+}, I_{ij} \geq 0, T_{ij} \geq 0, \delta > 0, i,j \in N & (8-5)
\end{cases}
\end{aligned}
\tag{8}
$$

where Z_1 is the maximum value obtained from Model (3), and definitions of other variables and constraints refer to Model (6). Without loss of generality, Model (6) is the original problem, and the relaxation model (i.e., Model (8)) is its sub-problem, thus the solution of Model (6) can be directly obtained after solving Model (8). In other words, as the submodel of Model (6), the solution of the relaxation Model (8) doesn't affect the results of Model (6).

To our knowledge, PSO algorithm was put forward to optimize nonlinear functions based on the initial point and stopping criteria (Kennedy & Eberhart, 1995), and has been proven to be an effective tool for streamlining decision making (Cabrerizo et al., 2013; Zhou et al., 2020a). In this paper, a relaxation method based on the PSO algorithm (i.e., Algorithm 1) is proposed for determining the optimal solution of Model (6). In specific, Algorithm 1 is proposed to solve the original problem (i.e., Model (6)), while PSO algorithm is used to solve its sub-problem (i.e., Model (8)). Note that, if the selection of parameters (e.g., initial points) is appropriate, a global optimal solution can be found (Campana et al., 2010; Sun et al., 2012). Generally, the above-mentioned non-convex models can be linearized and solved using standard exact solvers, but linearization only obtains an approximate solution, while our proposed relaxation method can derive the equivalent form of the original problem. By adopting similar principles, a fine-tuning algorithm can be used to solve Model (7), but it is omitted here due to space limitations.

Algorithm 1 Relaxation method based on PSO algorithm for solving Model (6).

Input: Number of DMs, N; d_i's initial carbon quota, o_i; d_i's initial fixed unit revenue, r_i; d_i's expected carbon quota interval, $[o_i^-, o_i^+]$; the maximum overall revenue obtained from Model (3), Z_1; the maximal number of iterations, $limit$; population size, M.

Output: d_i's final carbon quota, $o_i^{'}$; d_i's unit selling and buying prices, p_i, q_i; the transferred quantity, I_{ij}; the unit transaction price, T_{ij}; the specific DM's maximum total revenue, Z_2.

Step 1: Remove decision variables p_i, q_i to obtain a relaxation optimization model (see Model (8)), based on constraints (6-1), (6-4) and (6-5);

Step 2: Use PSO algorithm to solve Model (8);

Set current iteration as $t = 0$;
for each particle i **do**
    Initialize velocity V_i and position X_i for particle i;
    Evaluate particle i by the defined fitness function and set $pBesti = X_i$;
end for
$gBest$=min $\{pBesti\}$;
while t¡$limit$ **do**
    for $i = 1$ to M **do**
        Update the velocity and position of particle i;
        Evaluate particle i by the defined fitness function;
        if $fit(X_i)$ ¡ $fit(pBesti)$ **then return** $pBesti$=X_i;
        end if
        if $fit(pBesti)$ ¡ $fit(gBest)$ **then return** $gBest = pBesti$;
        end if
    end for
end while
Z_2= $-fit(gBest)$;
return Best solution of $o_i^{'}, I_{ij}, T_{ij}, Z_2$.

Step 3: Derive the optimal values of p_i, q_i based on relaxation constraints (6-1) and (6-5) .

5. Numerical analysis

To verify the feasibility of the optimization consensus models proposed in this paper, this section presents a numerical case study. Suppose there are five regions (d_1, d_2, d_3, d_4, d_5) in a closed-loop carbon quota trading system (i.e., $N = \{1, \cdots, 5\}$). The initial information provided by each region is summarized in Table 4.

Table 4 Summary of the initial trading information provided by five regions

Regions	r_i	o_i	o_i^-	o_i^+	$r_i o_i$
d_1	12	16	13	19	192
d_2	15	20	16	24	300
d_3	23	34	27	41	782
d_4	34	18	14	22	612
d_5	40	12	10	26	480
Total	—	100	80	132	2366

Note: d_i is the i-th region; r_i denotes the initial fixed unit revenue; o_i is d_i's initial carbon quota; o_i^- and o_i^+ are the lower and upper limit of d_i's initially expected interval, respectively; and $r_i o_i$ is d_i's initial carbon quota holding revenue.

5.1. Steps of the research on carbon quota trading mechanism

To clarify the construction mechanism described in this paper, five steps are presented below.

Step 1: Referring to Model (3), an optimization carbon trading model is built to achieve overall revenue maximization, i.e., to obtain the optimal carbon quota allocation scheme for different regions from the collective perspective. Specifically, the carbon quota quantities transferred among regions and the maximum value of the final total revenue of the system are acquired.

Step 2: Using the maximum overall revenue obtained in Step 1, and by adding the constraint of the unit transaction price, a series of optimization consensus models are built based on Model (6). Hence, a total of n allocation schemes are derived by maximizing each region's revenue, and detailed information such as d_i's unit buying and selling prices, transferred quantities, and unit transaction prices is obtained.

Step 3: Through a comparison of the individual/group development indices, it can be determined whether regions have developed fairly or not. If not, some discordant regions are identified by a pre-defined threshold γ, then their initial parameters are adjusted accordingly. Next, the calculations in Steps 1 and 2 are repeated until the allocation scheme satisfies the fairness requirement.

Step 4: Introduce the fairness measure variable α to build consensus models based on Model (7), so as to directly obtain fair carbon quota allocation schemes for the five regions in terms of the maximum overall revenue, quantities of carbon quota transferred, and the unit transaction prices. Additionally, a sensitivity analysis is applied to α to provide flexible suggestions for authorities involved in the trading system.

Step 5: Conduct a comparison and discussion based on the results obtained in each step.

5.2. Analysis of the overall revenue maximization model

Based on Model (3), we obtain a closed-loop carbon quota trading system involving the five regions listed in Table 4. Aiming to maximize the overall revenue, an optimization consensus model is constructed:

$$\max Z_1 = 12 * o_1^{'} + 15 * o_2^{'} + 23 * o_3^{'} + 34 * o_4^{'} + 40 * o_5^{'}$$
$$\text{s.t.} \begin{cases} \begin{cases} o_1^{'} = 16 - \sum\limits_{j=2}^{5} I_{1j} + \sum\limits_{j=2}^{5} I_{j1};\ o_2^{'} = 20 - \sum\limits_{j=1,j\neq 2}^{5} I_{2j} + \sum\limits_{j=1,j\neq 2}^{5} I_{j2} \\ o_3^{'} = 34 - \sum\limits_{j=1,j\neq 3}^{5} I_{3j} + \sum\limits_{j=1,j\neq 3}^{5} I_{j3};\ o_4^{'} = 18 - \sum\limits_{j=1,j\neq 4}^{5} I_{4j} + \sum\limits_{j=1,j\neq 4}^{5} I_{j4} \\ o_5^{'} = 12 - \sum\limits_{j=1}^{4} I_{5j} + \sum\limits_{j=1}^{4} I_{j5} \end{cases} & (9-1) \\ q_1 \leq 12 \leq p_1,\ q_2 \leq 15 \leq p_2, q_3 \leq 23 \leq p_3,\ q_4 \leq 34 \leq p_4,\ q_5 \leq 40 \leq p_5 & (9-2) \\ \begin{cases} I_{ij} \leq \dfrac{|q_j - p_i| + q_j - p_i}{\delta}, & i < j, i, j \in N \\ I_{ij} = 0, & \text{otherwise} \end{cases} & (9-3) \\ 13 \leq o_1^{'} \leq 19,\ 16 \leq o_2^{'} \leq 24,\ 27 \leq o_3^{'} \leq 41,\ 14 \leq o_4^{'} \leq 22,\ 10 \leq o_5^{'} \leq 26 & (9-4) \\ p_i \geq 0, q_i \geq 0, I_{ij} \geq 0, \delta > 0, i, j \in N & (9-5) \end{cases} \quad (9)$$

The objective function in Model (9) aims to maximize the total holding revenue for all five regions through the carbon quota trading process, where $o_i^{'}, i \in N$ is the final quota for the i-th region, which is restricted by both (9-1) and (9-4). Constraints (9-2)–(9-3) concern the unit selling and buying prices, and the transferred quantity for each region. δ in constraint (9-3) is a non-Archimedean infinitesimal, and hereafter it is set as $\delta = 10^{-6}$. The optimal solution of Model (9) is presented in Table 5.

Table 5 Optimal solution of Model (9) with overall revenue maximization

Regions	r_i	$o_i^{'}$	$r_i o_i^{'}$	I_{ij}	Value-I
d_1	12	13	156	(1,5)	3
d_2	15	16	240	(2,5)	4
d_3	23	27	621	(3,5)	7
d_4	34	18	612		
d_5	40	26	1040		
Total	—	100	2669	—	—

Note: d_i denotes the i-th region; r_i denotes the initial fixed unit revenue of carbon quota; $o_i^{'}$ is d_i's final carbon quota; $r_i o_i^{'}$ is d_i's final carbon quota holding revenue; and I_{ij} denotes the quantity transferred from d_i to d_j.

The results in Table 5 show that the maximum value of the objective function in Model (9) is 2669. According to Theorem 4, the critical region within the trading system is d_4, namely, $m = 4$. When $i < m$, the regions with small original fixed unit revenues are d_1, d_2, d_3. These regions can increase their revenue by selling carbon quotas, and their final quotas are the lower limit of their original expected intervals, namely 13, 16, and 27, respectively (i.e., o_i^- in Table 4). When $i > m$, i.e., for d_5, the only way to increase revenue is to purchase carbon quotas, and the final quota for this region is the upper limit of the original interval, namely 26 (i.e., o_i^+ in Table 4). Moreover, the final quota for d_4 is located in the initial range, and the data of I_{ij} show that region d_4 does not become involved in the trading. In summary, Theorem 4 has been verified. Region d_1 sold three carbon quota units to d_5; region d_5 bought three, four, and seven carbon quota units from regions d_1, d_2, d_3, respectively, making its total buying quantity $3 + 4 + 7 = o_5^{'} - o_5 = 26 - 12 = 14$. The transferred quantities for the remaining regions can be obtained in the same way. Thus, Theorem 5 has also been verified. Note that the revenue for each region in Table 5 only involves the fixed revenue for holding a certain carbon quota, while the transaction revenue from the trading of carbon quotas is not included.

5.3. Analysis of the single-region revenue maximization model

Model (9) can only provide feasible regions for variables $p_i, q_i, (i \in N)$, rather than their optimal values (see Section 4.1). Therefore, we construct Model (10) to acquire these optimal values under the objective of maximizing the revenue of individual regions, which follows the research ideas of Models (5) and (6). Obviously, we obtain five allocation schemes, one for each of the five regions taking part in the carbon quota trading process. For brevity, only the model that maximizes revenue for d_4 is illustrated here.

$$
\max Z_2 = 34 * o_4^{'} + \sum_{j=1,j\neq 4}^{5} T_{4j} I_{4j} - \sum_{j=1,j\neq 4}^{5} T_{j4} I_{j4}
$$

$$
\text{s.t.}\begin{cases}
\begin{cases} p_i \le T_{ij} \le q_j, & \text{if } p_i \le q_j,\ i<j, i,j\in N \\ T_{ij}=0, & \text{otherwise} \end{cases} & (10-1) \\
12 * o_1^{'} + 15 * o_2^{'} + 23 * o_3^{'} + 34 * o_4^{'} + 40 * o_5^{'} = 2669 & (10-2) \\
\begin{cases} o_1^{'} = 16 - \sum_{j=2}^{5} I_{1j} + \sum_{j=2}^{5} I_{j1};\ o_2^{'} = 20 - \sum_{j=1,j\neq 2}^{5} I_{2j} + \sum_{j=1,j\neq 2}^{5} I_{j2} \\ o_3^{'} = 34 - \sum_{j=1,j\neq 3}^{5} I_{3j} + \sum_{j=1,j\neq 3}^{5} I_{j3};\ o_4^{'} = 18 - \sum_{j=1,j\neq 4}^{5} I_{4j} + \sum_{j=1,j\neq 4}^{5} I_{j4} \\ o_5^{'} = 12 - \sum_{j=1}^{4} I_{5j} + \sum_{j=1}^{4} I_{j5} \end{cases} & (10-3) \\
q_1 \le 12 \le p_1,\ q_2 \le 15 \le p_2, q_3 \le 23 \le p_3,\ q_4 \le 34 \le p_4,\ q_5 \le 40 \le p_5 & (10-4) \\
\begin{cases} I_{ij} \le \dfrac{|q_j - p_i| + q_j - p_i}{\delta}, & i<j, i,j \in N \\ I_{ij} = 0, & \text{otherwise} \end{cases} & (10-5) \\
13 \le o_1^{'} \le 19,\ 16 \le o_2^{'} \le 24,\ 27 \le o_3^{'} \le 41,\ 14 \le o_4^{'} \le 22,\ 10 \le o_5^{'} \le 26 & (10-6) \\
p_i \ge 0, q_i \ge 0, I_{ij} \ge 0, T_{ij} \ge 0, \delta > 0, i \in N & (10-7)
\end{cases} \tag{10}
$$

The objective function Z_2 in Model (10) is the maximum total revenue that can be achieved by region d_4 through carbon trading. This is composed of fixed revenue for holding carbon quotas (i.e., $34 * o_4^{'}$) and transaction revenue for trading behavior (i.e., $\sum_{j=1,j\neq 4}^{5} T_{4j} I_{4j} - \sum_{j=1,j\neq 4}^{5} T_{j4} I_{j4}$). Constraint (10-2) ensures that the above trading is carried out under the premise of maximizing the overall revenue, where 2669 is the maximum value obtained by solving Model (9). Other definitions see Model (9). Using Algorithm 1, the optimal solution of Model (10) is presented in Table 6, while the results of maximizing the revenue for other regions see Table A1. Here, all the demand parameters in Algorithm 1 are set as $N = 5, Z_1 = 2669, limit = 5000$ and $M = 50$. In addition, the values for o_i, r_i, o_i^-, o_i^+ refer to Table 4 and the parameters regarding the PSO algorithm are set in Matlab R2016a by default.

The decision variable $o_i^{'}$ in Model (6) is directly given by constraints (6-6)–(6-8), but needs to be solved under constraints (10-3) and (10-6) in Model (10). The results in Tables 6 and A1 indicate that, regardless of which region's revenue is maximized, the optimal allocation scheme is fixed and consistent with the results obtained in Section 5.2, that is, $o_1^{'} = 13, o_2^{'} = 16, o_3^{'} = 27, o_4^{'} = 18, o_5^{'} = 26$. Moreover, the unit selling and buying prices of each region

Table 6 Optimal solution of Model (10) with d_4's revenue maximization

Regions	$o_i^{'}$	p_i	q_i	I_{ij}	Value-I	Value-T	H_i	$\lvert H_i - \bar{H} \rvert$	Z_2
d_1	13	12	[0,12]	(1,4)	3	12	1.0000	0.1281	
d_2	16	15	15	(2,4)	4	15	1.0000	0.1281	
d_3	27	23	23	(3,4)	7	23	1.0000	0.1281	915
d_4	18	34	34	(4,5)	14	40	1.4951	0.3670	
d_5	26	[40,+∞)	40				1.0000	0.1281	

Note: $o_i^{'}$ is d_i's final carbon quota; p_i, q_i are d_i's unit selling and buying prices; I_{ij} is the quantity transferred from d_i to d_j with Value-I as its specific value and Value-T as its corresponding unit transaction price; $H_i, \bar{H}$ are individual/group development index; and Z_2 is the optimal value of the objective function regarding single DM's revenue maximization.

are also consistent, although the transferred quantities, corresponding unit transaction prices, and the individual development indices differ in each model. Note that, the values of p_5, q_1 are intervals due to the reason that they are subjected to unilateral constraints of corresponding T_{ij}. In fact, these are auxiliary variables for realizing the trading process, because d_1 cannot purchase carbon quotas and d_5 cannot sell carbon quotas considering their fixed order of unit revenues. Optimal values of all $T_{ij}, (i, j \in N)$ are provided during the calculation, but most are omitted here because they don't affect our analysis on the results.

The relationship between the individual development index H_i and the group development index $\bar{H}$ is now analyzed to identify whether there exist some discordant regions with too much or too little revenue growth. First, based on $\sum_{i=1}^{5} r_i o_i = 2366$ in Table 4 and $\sum_{i=1}^{5} r_i o_i^{'} = 2669$ in Table 5, we derive the group development index as $\bar{H} = \dfrac{2669}{2366} = 1.128064$. Based on the data in Tables 4, 5, and 6, individual development indices for each region can then be computed. Taking d_4 as an example, $H_4 = \dfrac{r_4 o_4^{'} + T_{45} I_{45} - T_{14} I_{14} - T_{24} I_{24} - T_{34} I_{34}}{r_4 o_4} = \dfrac{34 * 18 + 14 * 40 - 3 * 12 - 4 * 15 - 7 * 23}{34 * 18} = 1.4951$. The individual development indices in Tables 6 and A1 can be derived using a similar calculation method.

5.4. Identification and parameter adjustment of discordant regions

Using the individual development indices H_i in Table A1, we obtain the absolute values of the differences in development indices between each region and the group (i.e., $|H_i - \bar{H}|$) or the absolute difference between any two regions (i.e., $|H_i - H_j|$). Generally, in actual GDM problems, we can always judge whether the development of different regions is balanced, namely, we can always pre-determine a threshold γ to identify discordant regions. To determine the value of the parameter γ, Table 7 summarizes various development indices based on Table A1, including the maximum, minimum, and mean for the abovementioned difference values. Numbers in bold font indicate relatively large values in each column, which require more attention.

Table 7 Summary of the development indices under the region's revenue maximization

Difference between individual and group $\lvert H_i - \bar{H}\rvert$			**Difference between individuals** $\lvert H_i - H_j\rvert$		
Maximum	Minimum	Average	Maximum	Minimum	Average
0.3094	0	0.1461	0.4375	0	0.2336
0.4853	0.0396	0.1692	**0.6133**	0	**0.2789**
0.2594	0.1281	0.1543	0.3875	0	0.1550
0.3670	0.1281	0.1759	0.4951	0	0.1980
0.5032	0.1281	**0.2031**	**0.6313**	0	**0.2525**

Referring to the adjustment rules designed in Section 4.3, the initial parameters provided by discordant regions, namely, their predetermined expected carbon quota interval $[o_i^-, o_i^+]$, will be adjusted accordingly. If we set $\gamma = 0.5$, then only d_5 is identified as a discordant region with too much revenue growth. However, if the difference between the development indices of any two regions is considered, the corresponding maximum values for d_2, d_5 should be considered, as they are both greater than 0.6. Thus, the threshold for the parameter is adjusted to $\gamma = 0.45$. Liang et al. (2020) concluded that the shorter the time required to reach a consensus, the more necessary it is to make greater adjustments to the initial opinions. Initially concluded from a phenomenon of 20% people possessing 80% of the wealth in the world, the 80/20 Rule (i.e., the Pareto principle) is now extended to a fact that an optimal ratio exists between the effort and gain. In other words, once we change 20% of the key factors, qualitative change will occur, implying that we can derive enough (like 80% of) expected results on that critical point. Therefore, we may wish to adjust the endpoints of the expected carbon quota interval by 20% of their initial values. Because d_2 sold too much of his quota, the quota interval is adjusted from $[16, 24]$ to $\underline{[20, 24]}$; and as d_5 purchased too much carbon quota, his expected range is adjusted from $[10, 26]$ to $\underline{[10, 23.6]}$. Here, taking d_2 as an instance for specific explanation. Acted as a seller, d_2 needs to decrease its sales volum to reduce its revenue growth, so d_2 increases its lower limit by adding 20% of its initial carbon quota (i.e., o_2), thus we derive the adjusted lower limit of d_2's expected interval as $16 + 20\% * 20 = 20$. Distinguished from d_2, the buyer d_4 should decrease its upper limit so as to possess less carbon quota at the end.

After repeating the calculations of Models (3) and (6), new allocation schemes are obtained. For brevity, the specific calculation models are omitted here. Using updated information, the new optimal allocation scheme for overall revenue maximization is as presented in Table 8; the schemes maximizing different region's revenue are presented in Table A2. Table 9 provides an updated summary of the development indices after adjusting the initial parameters of d_2, d_5 by 20% of their initial carbon quotas.

Table 8 Optimal solution for maximizing overall revenue after adjusting the initial parameters of d_2, d_5

Regions	r_i	$o_i^{'}$	$r_i o_i^{'}$	I_{ij}	Value-I
d_1	12	13	156	(1,2)	3
d_2	15	20	300	(2,5)	3
d_3	23	27	621	(3,5)	7
d_4	34	16.4	557.6	(4,5)	1.6
d_5	40	23.6	944		
Total	—	100	2578.6	—	—

Note: Definitions of notation see Table 5.

Table 9 Summary of the development indices under the region's revenue maximization after adjusting the initial parameters of d_2, d_5

Difference between individual and group $\lvert H_i - \bar{H} \rvert$			**Difference between individuals** $\lvert H_i - H_j \rvert$		
Maximum	Minimum	Average	Maximum	Minimum	Average
0.3476	0.0064	0.1211	**0.4375**	0	**0.2073**
0.1901	0.0064	0.0896	0.2800	0	0.1443
0.1697	0.0699	0.1018	0.2596	0	0.1078
0.2575	0.0899	0.1234	0.3474	0	0.1390
0.3531	0.0899	**0.1425**	**0.4429**	0	0.1772

Results in Tables 7 and 9 show that the unfairness in the system is ameliorated by adjusting the initial parameters of d_2, d_5. Specifically, the maximum difference between the individual and group development indices drops from 0.5032 to 0.3531, while the maximum difference between any two regions drops from 0.6313 to 0.4429. In fact, if policy-makers are not satisfied with the results in Table 9, they may repeat the above calculations. The maximum value of each region's revenue declines in most scenarios because the total transaction amount decreases as the overall revenue drops from 2669 to 2578.6 (see column Z_2 in Tables A1 and A2). Note that the identification of discordant regions, adjustment of their parameters, and fairness of the final result all depend on the experience of the policy-makers. In addition, the adjustment range of the initial parameters for those discordant regions has a significant influence on the number of adjustments and the final allocation scheme of the trading system. Obviously, the "fairness" reached through the above strategy is effective, but subjective and rather complicated.

5.5. Analysis regarding both fairness and revenue

Based on Table 4 and Model (7), this section considers the optimization consensus model (i.e., Model (11)) for obtaining a relatively fair carbon quota allocation scheme with the goal of maximizing the final overall revenue within the closed-loop trading system.

$$
\max Z_3 = 12 * o_1^{'} + 15 * o_2^{'} + 23 * o_3^{'} + 34 * o_4^{'} + 40 * o_5^{'}
$$

$$
\text{s.t.}\begin{cases}
\begin{cases}
o_1^{'} = 16 - \sum\limits_{j=2}^{5} I_{1j} + \sum\limits_{j=2}^{5} I_{j1};\ o_2^{'} = 20 - \sum\limits_{j=1,j\neq 2}^{5} I_{2j} + \sum\limits_{j=1,j\neq 2}^{5} I_{j2} \\
o_3^{'} = 34 - \sum\limits_{j=1,j\neq 3}^{5} I_{3j} + \sum\limits_{j=1,j\neq 3}^{5} I_{j3};\ o_4^{'} = 18 - \sum\limits_{j=1,j\neq 4}^{5} I_{4j} + \sum\limits_{j=1,j\neq 4}^{5} I_{j4} \\
o_5^{'} = 12 - \sum\limits_{j=1}^{4} I_{5j} + \sum\limits_{j=1}^{4} I_{j5}
\end{cases} & (11-1) \\
q_1 \leq 12 \leq p_1,\ q_2 \leq 15 \leq p_2, q_3 \leq 23 \leq p_3,\ q_4 \leq 34 \leq p_4,\ q_5 \leq 40 \leq p_5 & (11-2) \\
\begin{cases}
p_i \leq T_{ij} \leq q_j, \quad \text{if } p_i \leq q_j,\ i < j, i, j \in N \\
T_{ij} = 0, \qquad \text{otherwise}
\end{cases} & (11-3) \\
\begin{cases}
I_{ij} \leq \dfrac{|q_j - p_i| + q_j - p_i}{\delta}, \quad i < j, i, j \in N \\
I_{ij} = 0, \qquad \text{otherwise}
\end{cases} & (11-4) \\
13 \leq o_1^{'} \leq 19,\ 16 \leq o_2^{'} \leq 24,\ 27 \leq o_3^{'} \leq 41,\ 14 \leq o_4^{'} \leq 22,\ 10 \leq o_5^{'} \leq 26 & (11-5) \\
H_i = \dfrac{r_i o_i^{'} + \sum\limits_{j=1,j\neq i}^{n} T_{ij} I_{ij} - \sum\limits_{j=1,j\neq i}^{n} T_{ji} I_{ji}}{r_i o_i}, i \in N & (11-6) \\
|H_i - H_j| \leq \alpha, i < j, i, j \in N & (11-7) \\
p_i \geq 0, q_i \geq 0, I_{ij} \geq 0, T_{ij} \geq 0, H_i \geq 0, \delta > 0, \alpha \geq 0, i, j \in N & (11-8)
\end{cases}
\qquad (11)
$$

Z_3 in Model (11) maximizes the overall revenue of the carbon quota trading system. Constraint (11-1) describes the relationship between the final quotas and the carbon quotas transferred by each region, and $\sum\limits_{i=1}^{n} o_i^{'} = 100$. Constraints (11-2)–(11-4) concern the unit buying and selling prices, the unit transaction prices and transferred quantities, where δ is a pre-determined non-Archimedean infinitesimal. Constraint (11-5) is the threshold for decision variable $o_i^{'}$, and (11-6) defines the individual development index. Constraint (11-7) is the fairness restriction, where α is the pre-determined fairness measure variable. Other variables are consistent with those in Model (7).

Table 10 presents the solution set for Model (11) when the fairness measure variable $\alpha = 0$. At this time, the trading system achieves an absolutely fair state, that is, all individual development indices are equal to the group development index of 1.1281. The results of a sensitivity analysis of α are given in Table B1, and show that any value of α in the interval [0,0.5] gives an optimal value of the objective function of 2669. The final carbon quotas for all regions are also fixed to $o_1^{'} = 13, o_2^{'} = 16, o_3^{'} = 27, o_4^{'} = 18, o_5^{'} = 26$.

Table 10 Solutions to Model (11) when $\alpha = 0$

Regions	p_i	q_i	I_{ij}	Value-I	Value-T	I_{ij}	Value-I	Value-T	H_i
d_1	15	[0,12]	(1,2)	0.09	15	(2,5)	1.82	36.15	1.1281
d_2	15	15	(1,3)	1.54	23	(3,4)	5.36	34	1.1281
d_3	34	23	(1,4)	1.37	17.36	(3,5)	3.51	34	1.1281
d_4	36.15	34	(2,3)	0.32	15	(4,5)	8.68	36.15	1.1281
d_5	$[40,+\infty)$	36.15	(2,4)	1.95	15				1.1281

Note: Definitions of notation see Table 6.

Tables 10 and B1 show that, as the fairness measure variable α gradually decreases, although the final carbon quota of each region $o_i^{'}$ is fixed, the transaction frequency significantly increases, implying that carbon quotas are fully traded within the system. Besides, when α is greater than 0.1, the variables p_i, q_i for each region are fixed, but when $\alpha \leq 0.1$, those pricing decisions change. Overall, the introduction of the fairness measure changes the allocation schemes by increasing the number of trading paths in the system. Clearly, as the closed-loop carbon quota trading mechanism gradually complicates the transaction process, a state of absolute fairness is finally reached, namely, sufficient interactions among regions are achieved as the fairness measure variable decreases to zero.

5.6. Discussion

To verify the rationality and effectiveness of the proposed models in the paper, this section has considered the example of carbon quota trading among five regions. Our optimization consensus models can derive the optimal allocation scheme from the global perspective (i.e., the moderator's perspective in GDM), and can also obtain allocation schemes from different DM's perspectives, in which the maximization of each region's revenue is the modeling goal. The following findings can be elicited from our results:

- Consensus modeling to maximize the overall revenue can obtain the optimal allocation scheme for the whole group, but cannot identify specific pricing decisions. Moreover, the final carbon quotas of different regions obtained from the models that maximize each region's revenue are the same as those obtained from the former modeling mechanism. That is, the optimal values of $o_i^{'}$ are fixed. However, detailed trading information, such as the trading regions involved and the unit transaction prices, change with the specific region being studied (see Tables 5, A1, 8, and A2).

- The unit selling and buying prices of each region (i.e., variables p_i, q_i) derived from the proposed optimization consensus models do not change according to which region's revenue is being maximized (see Tables A1 and

A2) and do not depend on the value of the fairness measure variable (see Table B1). This indirectly implies that the carbon quota trading mechanism discussed in this paper is robust to some extent.

- For the two strategies proposed to deal with the unfairness issue within the trading system, adjusting the initial parameters of discordant regions is effective (see Tables 7 and 9), but complicated in practice. In addition, the parameter γ for identifying discordant regions, the adjustment range for each region, and whether the final allocation scheme meets the GDM requirements are all subjective (see Section 5.4). In contrast, the strategy of directly introducing the fairness measure variable α is convenient and effective, and further sensitivity analysis enables feasible allocation schemes to be obtained (see Tables 10 and B1).
- The introduction of the fairness measure variable increases the number of trading paths among different regions (see Tables 10 and B1), meaning that absolute fairness within the closed-loop system is realized only when carbon quotas are fully traded among different regions. Thus, sufficient interactions among participators are highly significant in achieving consensus or the pursuit of DMs' balanced development during a GDM process.

6. Conclusion

This paper has described the use of optimization consensus modeling theory to explore theoretical innovations regarding flexible carbon trading mechanisms. Specifically, we have investigated essential carbon quota allocation schemes within a closed-loop trading system with the aim of ensuring both revenue maximization and fairness. First, the optimal carbon quota allocation scheme was derived by maximizing the overall revenue through Model (3). Then, its analytical formula and the achievable conditions for successful trading were provided through theoretical deduction. Next, simultaneously taking the group revenue maximization and the competition mechanism into account, models for deriving the optimal allocation schemes by maximizing individual's revenues were constructed as Models (5) and (6). Since conflicts of interest are the main reasons for the failure of GDM in the real world, individual/group development indices were defined as Definitions (1) and (2), and two fairness strategies were further presented. The former is based on calculating the difference between the development indices, with fairness achieved through the identification of discordant DMs and the adjustment of their initial parameters. The latter introduces a fairness measure variable, allowing fair allocation schemes to be directly obtained from Model (7). Finally, a numerical example was conducted to demonstrate the performance of the proposed models.

The results show that the final carbon quotas of all regions can be determined through the proposed consensus models, but detailed trading information (including the participating regions and the unit transaction prices) can only be acquired through the models that focus on single-region revenue maximization. In addition, the strategies for dealing with the unfairness issue are both practical and effective, but the second strategy of directly introducing a fairness measure variable is more objective and easier to operate. Finally, the results of a sensitivity analysis of the fairness measure variable show that, as the variable decreases to zero, that is, when the group approaches the state of absolute fairness, the frequency of DMs' transactions within the group increases significantly, corresponding to the fact that reaching fairness within a group requires sufficient interactions among DMs.

In the future, some varaibles in our proposed models will be comprehensively determined to be more in line with real-life, for example, price variables are no longer static and could be accurately positioned by combining with game theory (Liu et al., 2021; Zheng et al., 2019). In addition, trading mechanisms should also focus on some critical factors, such as risk or utility (Zheng & Chang, 2021) in practical markets, rather than only considering the allocation and pricing decisions from the revenue maximization perspective. Moreover, with large-scale GDM problems (Dong et al., 2018; Zhang et al., 2017), especially under social network contexts (Liu et al., 2019; Wu et al., 2019), attracting increased attention, the use of artificial intelligence methods (Ding et al., 2020) to solve large-scale trading issues will also be a focus of our subsequent research.

Appendix A. Results with single region's revenue maximization

Based on Sections 4.2 and 4.3, Table A1 lists the optimal solutions (including $o_i^{'}, p_i, q_i, I_{ij}, T_{ij}$, and Z_2) to Model (6) and the values of the development indices (including H_i and $|H_i - \bar{H}|$) in the case of each region maximizing its revenue (note: the specific region discussed in Model (6) is marked with $\star$ in the first column in Table A1). Moreover, Table A2 exhibits the corresponding results after the initial parameters of d_2, d_5 have been adjusted by 20% of their initial carbon quotas.

Appendix B. Sensitivity analysis of the fairness measure variable

If the fairness measure variable α in Model (11) is decreased from 0.5 at intervals of 0.1, then the optimal solutions of the above optimization consensus model are as listed in Table B1.

Table A1 Optimal solution of Model (6) with different region's revenue maximization

Regions	$o_i^{'}$	p_i	q_i	I_{ij}	Value-I	Value-T	H_i	$\|H_i - \bar{H}\|$	Z_2
$\star d_1$	13	12	[0,12]	(1,5)	3.00	40	1.4375	0.3094	
d_2	16	15	15	(2,4)	2.10	15	1.0000	0.1281	
d_3	27	23	23	(2,5)	1.90	15	1.0000	0.1281	276
d_4	18	34	34	(3,4)	3.50	23	1.1280	0	
d_5	26	[40,+∞)	40	(3,5)	3.50	23	1.2930	0.1650	
–	–	–	–	(4,5)	5.60	34	–	–	
d_1	13	12	[0,12]	(1,2)	3	12	1.0000	0.1281	
$\star d_2$	16	15	15	(2,5)	7	40	1.6133	0.4853	
d_3	27	23	23	(3,4)	3.50	23	1.0000	0.1281	484
d_4	18	34	34	(3,5)	3.50	23	1.0629	0.0652	
d_5	26	[40,+∞)	40	(4,5)	3.50	34	1.1677	0.0396	
d_1	13	12	[0,12]	(1,3)	3	12	1.0000	0.1281	
d_2	16	15	15	(2,3)	4	15	1.0000	0.1281	
$\star d_3$	27	23	23	(3,5)	14	40	1.3875	0.2594	1085
d_4	18	34	34				1.0000	0.1281	
d_5	26	[40,+∞)	40				1.0000	0.1281	
d_1	13	12	[0,12]	(1,4)	3	12	1.0000	0.1281	
d_2	16	15	15	(2,4)	4	15	1.0000	0.1281	
d_3	27	23	23	(3,4)	7	23	1.0000	0.1281	915
$\star d_4$	18	34	34	(4,5)	14	40	1.4951	0.3670	
d_5	26	[40,+∞)	40				1.0000	0.1281	
d_1	13	12	[0,12]	(1,5)	3	12	1.0000	0.1281	
d_2	16	15	15	(2,5)	4	15	1.0000	0.1281	
d_3	27	23	23	(3,5)	7	23	1.0000	0.1281	783
d_4	18	34	34				1.0000	0.1281	
$\star d_5$	26	[40,+∞)	40				1.6313	0.5032	

Note: $o_i^{'}$ is d_i's final carbon quota; p_i, q_i are d_i's unit selling and buying prices; I_{ij} is the quantity transferred from d_i to d_j with Value-I as its specific value and Value-T as its corresponding unit transaction price; $H_i, \bar{H}$ are individual/group development index; and Z_2 is the optimal value of the objective function regarding single DM's revenue maximization.

Table A2 Optimal solution of Model (6) with different region's revenue maximization after adjusting the initial parameters of d_2, d_5

Regions	$o_i^{'}$	p_i	q_i	I_{ij}	Value-I	Value-T	H_i	$\lvert H_i - \bar{H}\rvert$	Z_2
$\star d_1$	13	12	[0,12]	(1,5)	3.00	40	1.4375	0.3476	
d_2	20	15	15	(3,4)	4.64	23	1.0000	0.0899	
d_3	27	23	23	(3,5)	2.36	23	1.0000	0.0899	276
d_4	16.4	34	34	(4,5)	6.24	34	1.0835	0.0064	
d_5	23.6	[40,+∞)	40				1.1615	0.0716	
d_1	13	12	[0,12]	(1,2)	3.00	12	1.0000	0.0899	
$\star d_2$	20	15	15	(2,5)	3.00	40	1.2800	0.1901	
d_3	27	23	23	(3,4)	4.64	23	1.0000	0.0899	384
d_4	16.4	34	34	(3,5)	2.36	23	1.0835	0.0064	
d_5	23.6	[40,+∞)	40	(4,5)	6.24	34	1.1615	0.0716	
d_1	13	12	[0,12]	(1,3)	3	12	1.0000	0.0899	
d_2	20	15	15	(3,5)	10	40	1.0000	0.0899	
$\star d_3$	27	23	23	(4,5)	1.6	34	1.2596	0.1697	985
d_4	16.4	34	34				1.0000	0.0899	
d_5	23.6	[40,+∞)	40				1.0200	0.0699	
d_1	13	12	[0,12]	(1,4)	3	12	1.0000	0.0899	
d_2	20	15	15	(3,4)	7	23	1.0000	0.0899	
d_3	27	23	23	(4,5)	11.6	40	1.0000	0.0899	824.6
$\star d_4$	16.4	34	34				1.3474	0.2575	
d_5	23.6	[40,+∞)	40				1.0000	0.0899	
d_1	13	12	[0,12]	(1,5)	3	12	1.0000	0.0899	
d_2	20	15	15	(3,5)	7	23	1.0000	0.0899	
d_3	27	23	23	(4,5)	1.6	34	1.0000	0.0899	692.6
d_4	16.4	34	34				1.0000	0.0899	
$\star d_5$	23.6	[40,+∞)	40				1.4429	0.3531	

Note: $o_i^{'}$ is d_i's final carbon quota; p_i, q_i are d_i's unit selling and buying prices; I_{ij} is the quantity transferred from d_i to d_j with Value-I as its specific value and Value-T as its corresponding unit transaction price; $H_i, \bar{H}$ are individual/group development index; and Z_2 is the optimal value of the objective function regarding single DM's revenue maximization.

Table B1 Sensitivity of the results to the fairness measure variable α

Regions	p_i	q_i	I_{ij}	Value-I	Value-T	H_i	α
d_1	12	[0,12]	(1,5)	3	40	1.4375	
d_2	15	15	(2,5)	4	40	1.3333	
d_3	23	23	(3,5)	7	40	1.1522	0.5
d_4	34	34				1.0000	
d_5	[40,+∞)	40				1.0000	
d_1	12	[0,12]	(1,3)	0.42	23	1.4000	
d_2	15	15	(1,5)	2.58	40	1.3333	
d_3	23	23	(2,5)	4	40	1.1614	0.4
d_4	34	34	(3,5)	7.42	40	1.0000	
d_5	[40,+∞)	40				1.0000	
d_1	12	[0,12]	(1,3)	1.55	23	1.3000	
d_2	15	15	(1,5)	1.45	40	1.3000	
d_3	23	23	(2,5)	4	37.5	1.1859	0.3
d_4	34	34	(3,5)	8.55	40	1.0000	
d_5	[40,+∞)	40				1.0208	
d_1	12	[0,12]	(1,3)	2.68	23	1.2000	
d_2	15	15	(1,5)	0.32	40	1.2000	
d_3	23	23	(2,5)	4	30	1.2000	0.2
d_4	34	34	(3,5)	9.68	39.15	1.0000	
d_5	[40,+∞)	40				1.1004	
d_1	15	[0,12]	(1,3)	1.15	23	1.0946	
d_2	23	15	(1,4)	1.85	15	1.1358	
d_3	34	23	(2,5)	4	25.19	1.1186	0.1
d_4	40	34	(3,4)	1.93	34	1.0946	
d_5	[40,+∞)	40	(3,5)	6.22	34.51	1.1946	
—	—	—	(4,5)	3.78	40	—	

Note: d_i denotes the i-th region; p_i, q_i are d_i's unit selling and buying prices; I_{ij} is the quantity transferred from d_i to d_j with Value-I as its specific value and Value-T as the corresponding unit transaction price; H_i is the individual development index; and α is the fairness measure variable.

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