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Uncertain probabilistic linguistic multi-criteria decision-making models: Applications in evaluation systems

MEMORIA PRESENTADA POR

Wanying Xie

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DIRECTORES

Enrique Herrera-Viedma y Zeshui Xu

Departamento de Ciencias de la Computación
e Inteligencia Artificial

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1 Introduction

Generally, multi-criteria [HXD10] evaluation means that the decision makers (DMs) do evaluations based on the multiple criteria which cannot be substituted for each other. In the current complex and changeable decision-making environment, the DMs are often hard to quantify the criterion accurately. Therefore, it is necessary to invite the relevant experts to qualitatively analyze and hierarchically semi-quantitative describe those criteria that are difficult to be quantified or can't be quantified in the process of evaluating alternatives. Meanwhile, there are usually multiple evaluators involved in the evaluation process. The combination of the uncertainty of objective things, the limitation of human cognitive level and the ambiguity of thinking mode has caused a situation that the experts cannot always provide the well-measured evaluation information. Hence, how to realize the inter-conversion between qualitative and quantitative and reflect the ability of soft reasoning in language expression always has been a hot topic in the evaluation of uncertain systems and decision-making areas.

Probabilistic linguistic term sets (PLTSs) [PWX16], which can adopt the qualitative and quantitative form to show the decision-making information, are suitable for dealing with the evaluation problems of uncertain system in the decision-making process. Moreover, considering the advantages of the PLTSs, Lin et al. [LXZY17] proposed the probabilistic uncertain linguistic term sets (PULTSs) constructed by the PLTSs and the uncertain linguistic variables [Xu06b]. PULTSs inherit the good properties of both. From the angle of the composition of the elements, it keeps the non-determinacy of the uncertain linguistic variables. Combined with the homologous proportions of the given uncertain linguistic variables, it fully demonstrates the intricacy of the decision-making environment and the uncertainty of the DMs. Since its inception, its research mainly focuses on the basic measure [LXZY17] and the consensus [XRXW18], the other research is very few, such as the research for the consistency for the probabilistic uncertain linguistic preference relations (PULPRs), the research for incomplete probabilistic uncertain linguistic preference relations (IPULPRs) and so on.

Choosing the applicable linguistic evaluation scale is the foundation of making fuzzy linguistic decision-making. The two most commonly used linguistic evaluation scale are additive linguistic evaluation scale [BFP97, Xu04b, Xu06b, Xu05, Xu15] and the multiplicative linguistic evaluation scale [Xu04a, Xu06a]. In general, the most commonly used additive linguistic evaluation scale is a zero-centered symmetric linguistic evaluation scale [Xu05] and its subscript is almost uniformly distributed. The multiplicative linguistic evaluation scale is a locally heterogeneous linguistic evaluation scale. This thesis based on the multiplicative linguistic evaluation scale [Xu06a] defines the probabilistic uncertain multiplicative linguistic term sets (PUMLTSs) and the corresponding probabilistic uncertain multiplicative linguistic preference relations (PUMLPRs). As we all know, for a variety of reasons, DMs are impossible to grasp all the involved knowledge of the decision-making problem. Therefore, the incomplete decision-making phenomenon is common. This thesis studies the management of incomplete probabilistic uncertain multiplicative linguistic preferences in decision-making.

Moreover, based on the idea of dual hesitant fuzzy sets [ZXX12], the PLTSs is extended to the dual probabilistic linguistic term sets (DPLTSs) [XXR17]. The DPLTSs that can contain both the membership degree and non-membership degree. While the membership degree represents the epistemic certainty, and the non-membership degree represents the epistemic uncertainty. It can make the DMs flexibly give their suggestions and reduce the irresolution of the DMs for one thing or another when it is hard for them to reach a final agreement to some extent. For now, the research of DPLTSs is still in the new stage. Whether the basic concepts or the decision-making methods has a large research space. Therefore, it is necessary to improve existing uncertain probabilistic linguistic multi-criteria methods or develop new methods on the application of uncertain evaluation system.

In terms of measures, the superiority of correlation coefficient is to demonstrate the inter-relationship of the variables. This thesis studies the correlation coefficient between the DPLTSs, the weighted correlation coefficient, the entropy, comparable degree and distance measure. Moreover, as for decision-making methods, based on the defined weighted correlation coefficient, the thesis provides the multi-attribute decision-making method. As for the comparable degree, it is essentially similar to the distance measure. On the foundation of comparable degree, the thesis constructs the dual probabilistic linguistic grey relational analysis multi-criteria decision-making method. After that, with regard to preference relations, this thesis defines the dual probabilistic linguistic preference relations (DPLPRs), then based on the defined distance measure studies the consistency of the DPLPRs. Furthermore, this thesis based on the comparable degree studies the consensus of DPLPRs.

Furthermore, in order to demonstrate the validity of proposed theories and methods, this thesis extracts decision problems from recent high-profile events, such as 5G, artificial intelligence, cloud computing and so on. Then this thesis applies the proposed theories and methods to those decision-making problems and verifies the effectiveness and feasibility.

Overall, for the research of probabilistic uncertain linguistic decision-making, the innovation points of the thesis can be summarized as follows: (1) Redefine the possibility degree between the PULTSs for acquiring the priority; (2) Define the PUMLTSs on the multiplicative linguistic label and the PUMLPRs; (3) Consider the incomplete PUMLPRs; (4) Put forward pertinently the corresponding repairing method to obtain complete PUMLPRs; (5) Probe the consistency of the PUMLPRs.

For the reserach of dual probabilistic linguistic decision-making, the innovation points of the thesis can be divided as follows:

1) It defines the dual probabilistic linguistic correlation coefficient and the weight dual probabilistic linguistic correlation coefficient. Then describes the multi-attribute group decision-making problem under the dual probabilistic linguistic context, divides the weight vector into the subjective and objective forms, defines the entropy measure for the DPLTSs for the sake of obtaining the final comprehensive weight vector, and introduces the complete dual probabilistic linguistic multi-attribute group decision-making process. Moreover, it uses a simulation experiment related

to the influence evaluations for AI to clarify the feasibility and practicality of the dual probabilistic linguistic multi-attribute group decision-making process.

2) It defines the dual probabilistic multiplicative linguistic term sets (DPMLTSs), the basic operations among the DPMLTSs, the comparable degree between the individual dual probabilistic multiplicative linguistic preference relations (DPMLPRs), and study the consistency, consensus of the DPMLPRs. Then it computes the weights of criteria, introduces the expanding grey relational analysis (EGRA) method, and the integrated multi-criteria decision-making procedure. Moreover, it utilizes a simulation case relevant to the cloud computing industry to clarify the potential and reality of the dual probabilistic multiplicative linguistic multi-criteria group decision-making procedure.

3) It constructs a new multi-criteria decision model based on the incomplete dual probabilistic linguistic preference relations (IDPLPRs). It first proposes a step-by-step repairing method to repair the linguistic section and probabilistic section of IDPLPRs separately. The superiority is that this step-by-step method conforms to the principle of element generation. After that, the consistency index based on the distance measure between the DPLPRs is defined to check and improve the consistency of DPLPRs. Then the weights of criteria can be obtained by information fusion. Moreover, it constructs optimistic and pessimistic data envelopment analysis models under the dual probabilistic linguistic environment to do the sorting process. Optimistic and pessimistic data envelopment analysis models can demonstrate the efficiency of each decision-making unit (DMU) from the perspective of the most and least favorable. Finally, it simulates a cased of 5G industry market to help enterprises choose appropriate 5G partners by using proposed methods.

Introducción

En general, en los problema de toma de decisión con múltiples criterios [HXD10] se asume que los decisores/expertos que dan sus evaluaciones de acuerdo a diferentes criterios no puedan sustituir unos criterios por otros. En los actuales contextos de decisión aparecen variables complejas y cambiantes que dificultan a los expertos expresar sus preferencias de forma precisa. Por lo tanto, se require invitar a expertos relevantes para que analicen quantitativamente y describan jerarquicamente aquellos criterios que son difíciles de cuantificar o que no se pueden valorar adecuadamente en el proceso de evaluación de alternativas. Mientras tanto, generalmente hay múltiples evaluadores involucrados en el proceso de evaluación. La combinación de la incertidumbre de las cosas objetivas, la limitación del nivel cognitivo humano y la ambigüedad del modo de pensar ha provocado una situación en la que los expertos no siempre pueden proporcionar la información de evaluación bien medida. Por lo tanto, cómo realizar la conversión entre cualitativo y cuantitativo y reflejar la capacidad del razonamiento suave en la expresión del lenguaje siempre ha sido un tema candente en la evaluación de sistemas inciertos y áreas de toma de decisiones.

Los conjuntos de términos lingüísticos probabilísticos (PLTS) [PWX16], que pueden adoptar la forma cualitativa y cuantitativa para mostrar la información de toma de decisiones, son adecuados para tratar los problemas de evaluación del sistema incierto en el proceso de toma de decisiones. Además, considerando las ventajas de los PLTS, Lin et al. [LXZY17] propuso los conjuntos de términos lingüísticos inciertos probabilísticos (PULTS) construidos por los PLTS y las variables lingüísticas inciertas [Xu06b]. Los PULTS heredan las buenas propiedades de ambos. Desde el ángulo de la composición de los elementos, mantiene la no determinación de las variables lingüísticas inciertas. Combinado con las proporciones homólogas de las variables lingüísticas inciertas dadas, demuestra plenamente la complejidad del entorno de toma de decisiones y la incertidumbre de los DM. Desde su inicio, su investigación se enfoca principalmente en la medida básica [LXZY17] y el consenso [XRXW18], la otra investigación es muy poca, como la investigación para la consistencia de las relaciones probabilísticas inciertas de preferencia lingüística incierta (PULPRs), la investigación de relaciones de preferencias lingüísticas inciertas probabilísticas incompletas (IPULPR), etc.

La elección de la escala de evaluación lingüística aplicable es la base para tomar decisiones en ambiente linguístico difuso. Las dos escalas de evaluación lingüística más utilizadas son la escala de evaluación lingüística aditiva [BFP97, Xu04b, Xu06b, Xu05, Xu15] y la escala de evaluación lingüística multiplicativa [Xu04a,Xu06a]. En general, la escala de evaluación lingüística aditiva más utilizada es una escala de evaluación lingüística simétrica centrada en cero [Xu05] y su subíndice está distribuido de manera casi uniforme. La escala de evaluación lingüística multiplicativa es una escala de evaluación lingüística localmente heterogénea. Esta tesis basada en la escala de evaluación lingüística multiplicativa [Xu06a] define los conjuntos de términos lingüísticos multiplicativos inciertos probabilísticos (PUMLTS) y las correspondientes relaciones de preferencia lingüística multiplicativa incierta probabilística (PUMLPRs). Como todos sabemos, por una variedad de razones, a los expertos le es difícil comprender todos los aspectos a ser considerados en el problema de toma de decisiones. Por lo tanto, la aparición de información incompleta es algo normal en los problemas de decisión. Así pues, en esta tesis nos centramos en estudiar los problemas de toma de decisiones asumiendo el uso de preferencias lingüística multiplicativa incierta probabilística incompletas.

Además, según la idea de conjuntos difusos dubitativos duales [ZXX12], los PLTS se extienden a los conjuntos de términos lingüísticos probabilísticos duales (DPLTS) [XXR17]. Los DPLTS que pueden contener tanto el grado de pertenencia como el de no pertenencia. Mientras que el grado de pertenencia representa la certeza epistémica, el grado de no pertenencia representa la incertidumbre epistémica. De este mode se facilita a los expertos la expresión de sus preferencias y se reduce las posibilidades de fracaso en los procesos de toma de decisión. Por ahora, la investigación en DPLTS es todavia incipiente. Por ello creemos que se hace necesario investigar en toma de decisión que trate con problemas multicriterio bajo DPLTS y crear nuevos métodos para resolver semejantes problemas que implican el manejo de preferencias no precisas.

En términos de medidas, la superioridad del coeficiente de correlación es demostrar la interrelación de las variables. Esta tesis estudia el coeficiente de correlación entre los DPLTS, el coeficiente de correlación ponderado, la entropía, el grado comparable y la medida de distancia. Además, en cuanto a los métodos de toma de decisiones, basados en el coeficiente de correlación ponderado definido, la tesis proporciona el método de toma de decisiones con múltiples atributos. En cuanto al grado comparable, es esencialmente similar a la medida de distancia. Sobre la base de un grado comparable, la tesis construye el método de toma de decisiones de criterios múltiples de análisis relacional gris lingüístico probabilístico dual. Después de eso, con respecto a las relaciones de preferencia, esta tesis define las relaciones de preferencias lingüísticas probabilísticas duales (DPLPR), luego, basándose en la medida de distancia definida, estudia la consistencia de los DPLPR. Además, esta tesis basada en el grado comparable estudia el consenso de los DPLPR.

Además, para demostrar la validez de las teorías y métodos propuestos, esta tesis extrae problemas de decisión de eventos recientes de alto perfil, como 5G, inteligencia artificial, computación en la nube, etc. Luego, esta tesis aplica las teorías y métodos propuestos a esos problemas de toma de decisiones y verifica la efectividad y la factibilidad.

En general, para la investigación de la toma de decisiones lingüísticas inciertas probabilísticas, los puntos de innovación de la tesis se pueden resumir de la siguiente manera: (1) Redefinir el grado de posibilidad entre los PULTS para adquirir la prioridad; (2) Definir las BOMBAS en la etiqueta lingüística multiplicativa y las BOMBAS; (3) Considere las PUMLPR incompletas; (4) Presentar de manera pertinente el método de reparación correspondiente para obtener PUMLPR completos; (5) Probar la consistencia de los PUMLPRs.

Para la investigación de la toma de decisiones lingüísticas probabilísticas duales, los puntos de innovación de la tesis se pueden dividir de la siguiente manera:

1) Define el coeficiente de correlación lingüística probabilística dual y el coeficiente de correlación lingüística probabilística dual ponderal. Luego describe el problema de toma de decisiones grupales de múltiples atributos en el contexto lingüístico probabilístico dual, divide el vector de peso en las formas subjetiva y objetiva, define la medida de entropía para los DPLTS en aras de obtener el vector de peso integral final e introduce el Proceso de toma de decisiones grupal de múltiples atributos lingüísticos probabilísticos completos. Además, utiliza un experimento de simulación relacionado con las evaluaciones de influencia para IA para aclarar la viabilidad y la practicidad del proceso de toma de decisiones grupal de múltiples atributos lingüísticos probabilísticos.

2) Define los conjuntos de términos lingüísticos multiplicativos probabilísticos dobles (DPMLTS), las operaciones básicas entre los DPMLTS, el grado comparable entre las relaciones individuales de preferencias lingüísticas multiplicativas probabilísticas dobles (DPMLPR) y estudia la consistencia y el consenso de los DPMLPR. Luego, calcula los pesos de los criterios, introduce el método de análisis relacional gris expansivo (EGRA) y el procedimiento integrado de toma de decisiones con criterios múltiples. Además, utiliza un caso de simulación relevante para la industria de la computación en la nube para aclarar el potencial y la realidad del procedimiento de toma de decisiones grupal de criterios múltiples, lingüísticos, multiplicativos, probabilísticos y probabilísticos.

3) Construye un nuevo modelo de decisión de criterios múltiples basado en las relaciones de preferencias lingüísticas probabilísticas dobles incompletas (IDPLPR). Primero propone un método

de reparación paso a paso para reparar la sección lingüística y la sección probabilística de IDPLPR por separado. La superioridad es que este método paso a paso se ajusta al principio de generación de elementos. Después de eso, el índice de consistencia basado en la medida de distancia entre los DPLPR se define para verificar y mejorar la consistencia de los DPLPR. Entonces, los pesos de los criterios se pueden obtener mediante fusión de información. Además, construye modelos de análisis de envolvente de datos optimistas y pesimistas bajo el entorno lingüístico probabilístico dual para realizar el proceso de clasificación. Los modelos de análisis de envolvente de datos optimistas y pesimistas pueden demostrar la eficiencia de cada unidad de toma de decisiones (DMU) desde la perspectiva de los más favorables y menos favorables. Finalmente, simula una caja de mercado de la industria 5G para ayudar a las empresas a elegir socios 5G apropiados utilizando los métodos propuestos.

2 Preliminaries

Some basic knowledges about the PLTSs, the PULTSs, the normalized PULTSs and the DPLTSs are introduced in this section.

2.1 Probabilistic linguistic term sets

Definition 1 *(PLTSs). Let $S_1 = \{s_\alpha \left| \alpha = 0, \ldots, \tau \right.\}$ be a linguistic term set [BFP97], the general form [PWX16] of a PLTS can be expressed as:*

$$L(p) = \left\{ L^{(k)}\left(p^{(k)}\right) \left| L^{(k)} \in S_1, p^{(k)} \geq 0, \sum\nolimits_{k=1}^{\#L(p)} p^{(k)} \leq 1 \right. \right\}$$

where $k = 1, 2, \ldots, \#L(p)$, $\#L(p)$ is the number of elements of the PLTS $L(p)$, $L^{(k)}\left(p^{(k)}\right)$ is the k th element of the PLTS, the possibility of the k th linguistic term $L^{(k)}$ is $p^{(k)}$.

Definition 2 *(The max and min element of the PLTSs). Based on the Definition 1, the max and min element [XXR17] of the PLTS $L(p)$ can be defined as follows:*
1) $\max\{L(p)\} = \max\left\{ L^{(k)}\left(p^{(k)}\right) \left| k = 1, 2, \ldots, \#L(p) \right. \right\}$
$= \max\left\{ s_{r^{(k)}}\left(p^{(k)}\right) \left| k = 1, 2, \ldots, \#L(p) \right. \right\} = L^{+}\left(p^{(k)}\right)$
2) $\min\{L(p)\} = \min\left\{ L^{(k)}\left(p^{(k)}\right) \left| k = 1, 2, \ldots, \#L(p) \right. \right\}$
$= \min\left\{ s_{r^{(k)}}\left(p^{(k)}\right) \left| k = 1, 2, \ldots, \#L(p) \right. \right\} = L^{-}\left(p^{(k)}\right)$
where $r^{(k)}p^{(k)}$ are permuted in descending order, $r^{(k)}$ is the subscript of k th linguistic term $L^{(k)}$.

2.2 Probabilistic uncertain linguistic term sets

Definition 3 *(PULTSs). With the combination of the PLTS [PWX16] and uncertain linguistic variable [Xu06b], Lin et al. [LXZY17] proposed the coming equation to state the PULTS:*

$$U(p) = \left\{ \left\langle \left[M^l, N^l\right], p^l \right\rangle \left| p^l \geq 0, l = 1, 2, \ldots, \#U(p), \sum_{l=1}^{\#U(p)} p^l \leq 1 \right. \right\}$$

where $\left\langle \left[M^l, N^l\right], p^l \right\rangle$ stands for the uncertain linguistic variable $\left[M^l, N^l\right]$ affiliated to its probability p^l, M^l and N^l are the linguistic terms on the additive linguistic term set $S_2 = \{s_\alpha \left| \alpha \in [-\tau, \tau] \right.\}$, $M^l \leq N^l$, τ is a non-negative integer [Xu06b], and $\#U(p)$ is the cardinality of $U(p)$.

In addition to that, without special directions, all the PULTSs are ordered PULTSs in the following section. Moreover, in the cause of removing the difference in the number of elements between two PULTSs, Lin et al. [PWX16] devised the following steps to standardize the PULTSs as follows:

For two different PULTSs $U_1(p) = \left\{ \left\langle \left[M_1^l, N_1^l\right], p_1^l \right\rangle \left| l = 1, 2, \ldots, \#U_1(p) \right. \right\}$ and $U_2(p) = \left\{ \left\langle \left[M_2^l, N_2^l\right], p_2^l \right\rangle \left| l = 1, 2, \ldots, \#U_2(p) \right. \right\}$, the standardizing process can be summarized as:
1) If $0 < \sum\limits_{l=1}^{\#U_i(p)} p_i^l < 1$, $i = 1, 2$, then they standardize the possibility of $U_i(p)$ via $p_i^{Nl} = p_i^l / \sum\limits_{l=1}^{\#U_i(p)} p_i^l$.
2) If $\#U_1(p) \neq \#U_2(p)$, $\#U_1(p) > \#U_2(p)$, then they add $\#U_1(p) - \#U_2(p)$ uncertain linguistic variables to $U_2(p)$ for the purpose of the elements of the PULTSs $U_1(p)$ and $U_2(p)$ own the identical quantity. The additional elements are the smallest one(s) in $U_2(p)$, and the corresponding

probabilities of the additional elements are zero. Moreover, the two disparate component elements $\langle [M^i, N^i], p^i \rangle$ and $\langle [M^j, N^j], p^j \rangle]$ of the PULTS are confronted with the possibility degree of $[p^i \times M^i, p^i \times N^i]$ over $[p^j \times M^j, p^j \times N^j]$.

2.3 Dual probabilistic linguistic term sets

Definition 4 *(DPLTSs). Let X be a fixed set, Xie et al. [XXR17] defined the DPLTS on X as:*

$$D = \{\langle x, L(p), U(p) \rangle, x \in X\}$$

where

$$L(p) = \left\{ L^{(i)}\left(p^{(i)}\right) \middle| L^{(i)} \in S_2, p^{(i)} \geq 0, \sum_{i=1}^{\#L(p)} p^{(i)} \leq 1 \right\},$$

$$U(p) = \left\{ U^{(j)}\left(p^{(j)}\right) \middle| U^{(j)} \in S_2, p^{(j)} \geq 0, \sum_{j=1}^{\#U(p)} p^{(j)} \leq 1 \right\}.$$

$L(p)$ and $U(p)$ stand for the possible membership and non-membership degrees to the $x \in X$. Moreover, $s_{-\tau} \leq L^+ \oplus U^+ \leq s_\tau, s_{-\tau} \leq L^- \oplus U^- \leq s_\tau$, where L^+ and L^- are the linguistic terms of the max and min element of the PLTS $L(p)$, U^+ and U^- are the linguistic terms of the max and min element of the PLTS $U(p)$. Besides, the pair $D = \langle L(p), U(p) \rangle$ is named as the dual probabilistic linguistic term element (DPLTE).

3 Justification

Due to the complexity, uncertainty and ambiguity of human thinking, many problems in reality are difficult to be described quantitatively with precise numbers, but expressed in probabilistic uncertain linguistic term sets or dual probabilistic linguistic term sets. Based on this background, this thesis mainly studies multi-criteria decision-making methods and their applications based on probabilistic uncertain linguistic environment and dual probabilistic linguistic environment. Combined with the existing research foundation, the multi-criteria evaluation method based on probabilistic uncertain linguistic information and the multi-criteria evaluation method based on dual probabilistic linguistic information are deeply studied and applied to multi-criteria decision-making problems, and efforts are made to enrich multi-criteria decision theories and methods in the relevant fuzzy environment.

3.1 Probabilistic uncertain linguistic decision-making

PULTSs were first proposed by Lin et al. [LXZY17], which is the combination of PLTSs [PWX16] and uncertain linguistic variables [Xu06b]. PULTSs inherit the good properties of both. From the angle of the composition of the elements, it keeps the non-determinacy of the uncertain linguistic variables. Combined with the homologous proportions of the given uncertain linguistic variables, it fully demonstrates the intricacy of the decision-making environment and the uncertainty of the DMs. By drawing lessons from Xu [Xu00], in the practical application, different linguistic scales may lead to different sorting. Moreover, the performance of multiplicative linguistic scale is better than the additive linguistic scale. Therefore, this thesis extends the traditional PULTSs raised by Lin et al. [LXZY17] to the PUMLTSs and defines the homologous PUMLPRs.

For the complexity of its construction, the PUMLPRs is constructed by the combination of uncertain multiplicative linguistic variables and their respective probabilities. With regard to the decision-making procedure, the DMs that are invited to do the decision are not possible to prefect themselves in all the relevant information. Hence, owing to the various external and internal conditions, the DMs are not always give their completely-specified preference information. The incomplete phenomenon can be found everywhere indeed. This paper divides the incomplete preference information into two aspects by considering the components of PUMLPRs. On the one hand, owing to the lack of the familiarity with the evaluated problem, the DMs cannot determine the corresponding uncertain multiplicative linguistic variables. On the other hand, due to the deletion of the uncertain multiplicative linguistic variables, the corresponding probabilities cannot be determined, either. Then based on those two aspects, the thesis proposes step-by-step method to repair the incomplete preference informaton.

3.2 Dual probabilistic linguistic decision-making

Although the PLTSs not only can reflect the hesitation of the DMs, but also reflect the weights of the DMs, it cannot reflect the non-membership degree of the policy-making information. Therefore, Ref [XXR17, MBRA18] proposed the DPLTSs that can contain both the membership degree and non-membership degree later. While the membership degree represents the epistemic certainty, and the non-membership degree represents the epistemic uncertainty. It can make the DMs flexibly give their suggestions and reduce the irresolution of the DMs for one thing or another when it is hard for them to reach a final agreement to some extent. From this perspective, the DPLTSs are more suited than PLTSs for handling uncertainty and fuzziness. Since the DPLTSs were proposed, its researches mainly focus on the basic operations [XXR17] and the closeness coefficient [MBRA18], while the other relevant research has not been found yet.

(1) The correlation coefficient for DPLTSs

Since Karl Pearson first proposed the correlation coefficient in 1895, and the good nature of correlation coefficient is in measuring the interrelationship among the variables in statistical analysis, the research for correlation coefficient has been studied a lot, especially in the fuzzy area. For the sake of learning about the joint relationship among the fuzzy data, many researches have chosen the correlation coefficient as the tool to explore the internal relationship between the fuzzy data with different methods. Therefore, considering the advantages of DPLTSs and the importance of researching the correlation coefficient that can research the degree of linear correlation between variables, this thesis is to research the correlation coefficient between the DPLTSs.

Moreover, in order to apply the correlation coefficient to the practical application and the goal being studied may be allocated with different weights, this thesis further defines the weighted correlation coefficient for the DPLTSs. In addition to that, all sorts of objective factors and subjective factors may lead to the differences in the importance of the goals being studied. Hence, the weights of the goals under study are not always same. For example, one is going to buy a new apartment. He may need to consider many factors, such as the price, distance, daylight and so on. Obviously, not all the factors that need to be considered are the more the better. If let the buyer provide the respective weights of these influencing factors, the specific proportions must be different. For the practical application process, the relevant influencing factors that need to be considered are more and more complicated. Then the weights of the goals under study is difficult to be determined. Therefore, it is necessary to find the appropriate method to determine the weight. Different from the traditional research for weight correlation coefficient, either just considering the objective weight and ignoring the subjective weights of the DMs, or just considering the subjective weight, this thesis divides the weight vector into the subjective form and the objective form in the theis.

(2) The comparable degree for DPMLTSs

After studying the basic measures and properties of the DPLTSs, in order to further apply this set, similar to the proposition of the PUMLTSs, this thesis proposes the DPMLPRs on the multiplicative linguistic scale. Similar to the research for majority preference relations, the consistency of preference relations is the premise that it can be used to make decisions. Different from the majority of studies, this thesis defines the comparable degree between the DPMLPRs and utilizes it as the measure to judge the consistency of the DPMLPRs. The reason why the thesis uses the comparable degree is that the intrinsic quality between the comparable degree and the distance measure is same. Moreover, because of the structure of the operator itself, the computation of the comparable degree is also separated into two angles: the membership viewpoint and the non-membership viewpoint.

The crucial intention of decision-making is to judge the sort of the alternatives. For the multi-criteria decision-making, the research for weights has been done a lot. Most of them are divided into the following types: partially known, fully known and total unknown. The weights of criteria in this thesis is belong to the third type that is total unknown. On the foundation of classic arithmetic averaging method [SK17], this thesis considers the structural characteristics of DPMLTSs and designs the modified arithmetic averaging method to calculate the weights for criteria. After that, the grey relational analysis (GRA) [Den89] as one of the more common multi-criteria decision-making method, its superiority lies in that it does not require too many quantities involved in the decision-making. Moreover, it does not require that the quantities to be determined conform to a typical distribution. The amount of calculation is relatively small, and the results agree well with the qualitative analysis. So the GRA has been expanded in this thesis by merging with the proposed comparable degree to calculate the relational coefficient. The GRA based upon the comparable degree is named as expanding GRA (EGRA). Together with the weights of the criteria, the final priority of the alternatives is able to be procured at length.

(3) The multi-criteria decision model based on incomplete DPLPRs

Because of various subjective and objective reasons, the DMs are not always give their completely-specified preference information, the DPLPRs cannot be determined completely and directly. How to repair efficaciously the IDPLPRs is the problem that we want to settle in the thesis. This thesis studies multi-criteria decision making with incomplete preference information. Generally, this incomplete information often leads to the inability to fully determine the weight of the criterion. The weights of criteria for multi-criteria decision-making problem that we want to solve are set to unknown. This thesis chooses to determine the weight of the criterion based on the complete preference information. The repairing of the incomplete PRs under the dual probabilistic linguistic situation is divided into two steps: the linguistic section and the probabilistic section.

Moreover, data envelopment analysis (DEA) [CCR78] is an efficient evaluation method for multiple decision units with the ratio of multiple inputs to multiple outputs. Because of those multiple inputs and multiple outputs usually involve in many factors in the decision-making procedure, many scholars who commit to the research for the uncertain decision-making choose it as the research tool. Generally, in the practical application, the input and output data in the traditional DEA is specific number. To some extent, this situation may not always be effective in practical applications. That is to say, the situations that inputs and outputs are imprecise often exist. The imprecision in the input/output data can be presented in the form of fuzzy numbers, interval numbers, intuitional fuzzy numbers, hesitant fuzzy numbers, and so on.

In this thesis, by considering the feature of the DPLTEs, we regard the DPLTEs as the stochastic variable, and expand the DEA into the dual probabilistic linguistic environment. Moreover, this thesis assumes that these inputs and outputs are all DPLTEs, which means that these inputs and outputs are both stochastic variables. Then we build the model to measure the efficiency. FurtherMore, because that optimistic and pessimistic efficiencies can reflect the efficiency of each DMU from the most and least favorable situations, respectively. This thesis divides the model into two categories: one is the optimistic situation, and the other is the pessimistic situation.

For those two kinds of models, this thesis solves it by studying respective distributions of these inputs and outputs. The difference is that these inputs and outputs in this dual probabilistic linguistic DEA model are two dimensional discrete random variables. In order to obtain the final decision-making consequence, we use the score function of these discrete random variables as the inputs, the accuracy function of these dimensional discrete random variables as the outputs to obtain the ultima decision-making result. The advantage of this method is that the model with the stochastic variable can be converted into the model which does not contain the stochastic variable but the specific value.

4 Objectives

In the thesis, no matter the research for PULTSs or the research for DPLTSs, our objectives are based on theoretical research to meet the needs of solving practical problems.

- **At the theoretical level:** For the PULTSs, firstly, it has been extended to the multiplicative linguistic scale, and the PUMLTSs is defined. Then the PUMLPRs is constructed by the combination of uncertain multiplicative linguistic variables and their respective probabilities. Moreover, based on the fact that the DMs who are invited to do the decision are not possible to prefect themselves in all the relevant information in the decision-making procedure, this thesis divides the incomplete preference information into two aspects by considering the components of PUMLPRs. On the one hand, owing to the lack of the familiarity with the evaluated problem, the DMs cannot determine the corresponding uncertain multiplicative linguistic variables. On the other hand, due to the deletion of the uncertain multiplicative linguistic variables, the corresponding probabilities cannot be determined, either. While for the DPLTSs, this thesis firstly defines the basic operational laws between the DPLTEs. Then it defines the DPLPRs. Moreover, this thesis studies the situation that if the involved DMs are not completely with the evaluated problem, the incomplete preference will occur in the decision-making procedure. Hence, it defines the incomplete DPLPRs and looks for the suitable mean to restore the IDPLPRs. Otherwise, the thesis extends the DPLTSs to the multiplicative linguistic scale and defines the DPMLTSs. Then based on the DPMLTSs, it defines the DPMLPRs. Moreover, it defines the comparable degree between the DPMLTSs and the comparable degree between the DPMLPRs. Then based on the defined comparable degree, the thesis proposes the expanding ERA method to the uncertain decision-making. Furthermore, considering the superiority of correlation coefficient is to demonstrate the interrelationship of the variables, the thesis defines the weighted correlation coefficient as the measure to do the multi-attribute decision-making.

- **At the methodological level:** In the context of probabilistic uncertain linguistic, based on the defined PUMLPRs, the incomplete PUMLPRs are defined. For incomplete PUMLPRs, the thesis divides the elements into two parts and proposes the step-by-step method to repair them separately. It constructs a multi-objective programming model to repair the lacking uncertain multiplicative linguistic variables. While for the lacking probabilistic section, it puts forward a linear programming model to obtain the loss of probability. As for the dual probabilistic linguistic decision-making, on the one hand, based on a series of defined DPMLTSs, DPMLPRs and comparable degrees, expanding GRA method, expanding TODIM method and expanding VIKOR method are proposed to do the multi-criteria decision-making. On the other hand, in the case that the DPLPRs are not complete, the thesis also proposes a step-by-step method to separately repair the linguistic section and probabilistic section of incomplete DPLPRs. Through the construction of the linear programming model, the thesis can get the complete linguistic membership. By using the feature of additive consistency for PLPRs, the remaining incomplete probability part can be added completely. The same method is also applied to the remaining non-membership. Then the incomplete DPLPRs can be repaired completely. After that, because that the thesis aims to study the dual probabilistic linguistic multi-criteria decision-making problem with the unknown weight of the criterion and use dual probabilistic linguistic DEA to do the sorting process, it regards the DPLTEs as the stochastic variable, and expand the DEA into the dual probabilistic linguistic environment. In the dual probabilistic linguistic DEA model, by considering the feature of the DPLTEs, there are two dimensional discrete random variables. In order to obtain the final decision-making consequence, the thesis adopts the score function of these discrete random variables as the inputs, the accuracy function of these dimensional discrete random variables as the outputs to obtain the ultima decision-making result. The advantage of this method is that the model with the stochastic variable can be converted into the model which does not contain the stochastic variable but the specific value.

- **At the application level:** Combining with the current events that have attracted a lot attention, such as the artificial intelligence, the cloud computing, the online public opinion under big data, the 5G and so on, this thesis puts those proposed theories and methods into practice, verifies therir feasibility and effectiveness and solve related problems.

5 Methodology

This section introduces the methodologies used in the thesis.

1. **Construt the step-by-step method to repair the incomplete probabilistic uncertain multiplicative linguistic preference relations.** Considering the complexity of the structure for PUMLPRs, the repairing process for the incomplete PUMLPRs is separated into two steps. For the first one, this thesis studies the method to repair the missing parts that repair the uncertain multiplicative linguistic preference relations (UMLPRs). Moreover, for the sake of better repairing the incomplete UMLPRs, in this thesis, the uncertain multiplicative linguistic variables is separated into the two parts: the left and right of the interval. Then the whole UMLPRs can be divided into two linguistic preference relations (LPRs). This is a well-known fact for those who devote to researching for decision-making that consistency is the fundamental condition for researching preference relations (PRs). The repairing process uses the consistency of the LPRs as the basic to construct a multi-objective programming model and calculates the lacking uncertain multiplicative linguistic variables. For the second step, owing to its loss of probability, the thesis puts forward a linear programming model to obtain the loss of probability. Hence, the incomplete PUMLPRs are repaired completely.

2. **Calculate the correlation coefficient in two parts: the angle of membership and the angle of non-membership.** Considering the advantages of DPLTSs and the importance of researching the correlation coefficient that can research the degree of linear correlation between variables, this thesis is to research the correlation coefficient among the DPLTSs. Owing to the fact that the DPLTSs not only contain the membership degree but the non-membership degree, this thesis calculates the correlation coefficient in two parts. Firstly, it calculates the corresponding correlation coefficient between two membership degrees of the two DPLTSs that is the respective PLTSs of the two DPLTSs. Then it calculates the correlation coefficient between the two non-membership degrees of the two DPLTSs, averages the two obtained correlation coefficient, and gets the final correlation coefficient of the two DPLTSs.

3. **Propose the expanding grey relational analysis method.** As one of the more common multi-criteria decision-making methods, the superiority of GRA lies in that it does not require too many quantities involved in the decision-making. Moreover, it does not require that the quantities to be determined conform to a typical distribution. The amount of calculation is relatively small, and the results agree well with the qualitative analysis. Due to the reality that the comparable degree is similar to the distance measure in physical significance, So the GRA has been expanded in this thesis by merging with the proposed comparable degree to calculate the relational coefficient. The GRA based upon the comparable degree is named as EGRA.

4. **Propose the dual probabilistic linguistic data envelopment analysis method.** DEA is an efficient evaluation method for multiple decision units with the ratio of multiple inputs to multiple outputs. In this thesis, by considering the feature of the DPLTEs, it regards the DPLTEs as the stochastic variable, and expands the DEA into the dual probabilistic linguistic environment. Then it assumes that these inputs and outputs are all DPLTEs, which means that these inputs and outputs are both stochastic variables. Moreover, because that optimistic and pessimistic efficiencies can reflect the efficiency of each DMU from the most and least favorable situations, respectively. This thesis divides the model into two

categories: one is the optimistic situation, and the other is the pessimistic situation. For those two kinds of models, this thesis solves it by studying respective distributions of these inputs and outputs. The difference is that these inputs and outputs in this dual probabilistic linguistic DEA model are two dimensional discrete random variables. In order to obtain the final decision-making consequence, this thesis uses the score function of these discrete random variables as the inputs, the accuracy function of these dimensional discrete random variables as the outputs to obtain the ultima decision-making result. The advantage of this method is that the model with the stochastic variable can be converted into the model which does not contain the stochastic variable but the specific value.

6 Summary

In this section, a summary of the proposals included in this thesis is presented, describing the main contents along with the obtained results associated with the journal publication are provided. The research carried out for this thesis and the results obtained in each case are collected into the following published papers:

- W.Y. Xie, Z.S. Xu, Z.L. Ren, E. Herrera-Viedma, Restoring incomplete PUMLPRs for evaluatingthe management way of online public opinion, Information Sciences, 516, 72-88, 2020.
- W.Y. Xie, Z.S. Xu, Z.L. Ren, E. Herrera-Viedma, The probe for the weighted dual probabilistic linguistic correlation coefficient to invest an artificial intelligence project, Soft Computing, DOI: 10.1007/s00500-020-04873-0, 2020.
- W.Y. Xie, Z.S. Xu, Z.L. Ren, E. Herrera-Viedma, Expanding grey relational analysis with the comparable degree for dual probabilistic multiplicative linguistic term sets and its application on the cloud enterprise, IEEE Access, 2019, 7: 75041-75057.
- W.Y. Xie, Z.S. Xu, Z.L. Ren, E. Herrera-Viedma, A new multi-criteria decision model based on incomplete dual probabilistic linguistic preference relations, Applied Soft Computing, DOI: 10.1016/j.asoc.2020.106237, 2020.

The remainder of this section is organized into four sections. Section 6.1 restores incomplete PUMLPRs for evaluating the management way of online public opinion. Section 6.2 probes the weighted dual probabilistic linguistic correlation coefficient to invest an artificial intelligence project. Section 6.3 performs expanding grey relational analysis with the comparable degree for dual probabilistic multiplicative linguistic term sets and applies it to the cloud enterprise. A new multi-criteria decision model based on incomplete dual probabilistic linguistic preference relations is given in Section 6.4.

6.1 Restoring incomplete PUMLPRs for evaluating the management way of online public opinion

Assume that there are n alternatives $A = \{a_1, a_2, \ldots, a_n\}$, t DMs $E = \{e_1, e_2, \ldots, e_t\}$, for the decision maker (DM) e_κ, $\kappa = 1, 2, \ldots, t$, he/she displays his/her preference information of the alternative a_i over the alternative a_j through the matrix $U_\kappa = \left(U_{ij}^{\kappa}(p)\right)_{n\times n}$, $i, j = 1, 2, \ldots, n$. The PUMLPR can be ruled as follows:

A PUMLPR U is a matrix $U = (U_{ij}(p))_{n\times n}$, where $i, j = 1, 2, \ldots, n$, $U_{ij}(p) = \left(U_{ij}^{l}\left(p_{ij}^{l}\right)\right) = \left\{\left\langle\left[M_{ij}^{l}, N_{ij}^{l}\right], p_{ij}^{l}\right\rangle \middle| p_{ij}^{l} \geq 0, l = 1, 2, \ldots, \#U_{ij}(p), \sum_{l=1}^{\#U_{ij}(p)} p_{ij}^{l} \leq 1\right\}$ are the PUMLTSs on the multiplicative linguistic term set [Xu06a] $S = \{s_\alpha \,|\alpha \in [1/q, q]\}$, $\#U_{ij}(p)$ is the quantity of the uncertain multiplicative linguistic variables in $U_{ij}(p)$, $U_{ij}(p)$ reveals the preference degree of the alternative a_i over the alternative a_j, and fulfills the following qualifications:

$$M_{ij}^{l} \otimes N_{ji}^{l} = M_{ji}^{l} \otimes N_{ij}^{l} = s_1,\ p_{ij}^{l} = p_{ji}^{l},\ M_{ii}^{l} = N_{ii}^{l} = s_1,\ \#U_{ij}(p) = \#U_{ji}(p)$$

and $U_{ij}^{l+1}\left(p_{ij}^{l+1}\right) \succ U_{ij}^{l}\left(p_{ij}^{l}\right)$, $U_{ji}^{l}\left(p_{ji}^{l}\right) \prec U_{ji}^{l+1}\left(p_{ji}^{l+1}\right)$, $U_{ij}^{l}\left(p_{ij}^{l}\right)$ is the lth component element of the PULTS $U_{ij}(p)$.

As for the PUMLPR $U = (U_{ij}(p))_{n\times n}$, if all the component elements of the PUMLTSs in the upper triangular matrix are sorted in upward sequence and all the PUMLTSs in the upper triangular matrix are normalized in conformity to NPUMLTSs. Then, we name the PUMLPR $U = (U_{ij}(p))_{n\times n}$ the NPUMLPR, denoted as $U^N = \left(U_{ij}^N(p)\right)_{n\times n}$.

Actually, under the practical decision-making procedure, due to a variety of reasons, the decision-making information is not always complete. For example, if the DMs are not familiar enough with the question that they need to evaluate, so they can not provide the exact information. Provided that they choose the PUMLPRs as the tool to express their preference information, but because some of them don't familar with the evaluated problem. Then the uncertain multiplicative linguistic variables can not be given. Hence, the corresponding probabilities of the uncertain multiplicative linguistic variables cannot be provided, either. In order to show this possible situation more clearly, this thesis uses a definition to illustrate in detail.

Definition 1. For a PUMLPR $U = (U_{ij}(p))_{n\times n}$, where $i, j = 1, 2, \dots, n$, $U_{ij}(p) = \left(U_{ij}^l\left(p_{ij}^l\right)\right) = \left\{\left\langle\left[M_{ij}^l, N_{ij}^l\right], p_{ij}^l\right\rangle \middle| p_{ij}^l \geq 0, l = 1, 2, \dots, \#U_{ij}(p), \sum_{l=1}^{\#U_{ij}(p)} p_{ij}^l \leq 1\right\}$, then U is called an incomplete probabilistic uncertain multiplicative linguistic preference relation (IPUMLPR), if some of its uncertain multiplicative linguistic variables and its corresponding probabilities cannot be provided by the DMs, which it can be indicated as the unknown variable "$\left[M_{x_l}, N_{y_l}\right], z_{ij}^l$", and the others specified by the DMs satisfy

$$M_{ij}^l \otimes N_{ji}^l = M_{ji}^l \otimes N_{ij}^l = s_1,\ p_{ij}^l = p_{ji}^l,\ M_{ii}^l = N_{ii}^l = s_1,\ \#U_{ij}(p) = \#U_{ji}(p)$$

and $U_{ij}^{l+1}\left(p_{ij}^{l+1}\right) \succ U_{ij}^l\left(p_{ij}^l\right)$, $U_{ji}^l\left(p_{ji}^l\right) \prec U_{ji}^{l+1}\left(p_{ji}^{l+1}\right)$, $U_{ij}^l\left(p_{ij}^l\right)$ is the lth component element of the PULTS $U_{ij}(p)$, where $U_{ij}(p) \in \Omega_U$ and Ω_U is the set of all the known elements in U.

Considering the complexity of the structure for PUMLPRs, the repairing process for the incomplete PUMLPRs is also separated into two steps. For the first one, by learning from Ref. [XZW19], this thesis studies the method to repair the missing parts that repair the UMLPRs. Moreover, for the sake of better repairing the incomplete UMLPRs, in this paper, the uncertain multiplicative linguistic variables is separated into two parts: the left and the right of the interval. Then the whole UMLPRs can be divided into two linguistic preference relations (LPRs). As it is known that consistency is the fundamental condition for researching PRs. The repairing process uses the consistency of the LPRs as the basic to construct a multi-objective programming model and calculates the lacking uncertain multiplicative linguistic variables. For the second step, owing to its loss of probability, in the light of Ref. [XZW19], this thesis puts forward a linear programming model to obtain the loss of probability. Therefore, the incomplete PUMLPRs are repaired completely.

Just as we said before, the consistency is the necessary requirement for obtaining appropriate decision-making result. The discussion for the obtained complete PUMLPRs is also necessary. In this thesis, in the cause of researching the consistency of the PUMLPRs, through the study of Ref. [ZX17], thsi thesis defines the geometric PRs (GPRs) for the PUMLPRs. Moreover, it is easy to see that the GPRs is the UMLPRs. Then this thesis can use the interval consistency condition [DW08] to establish multi-objective programming model and to obtain the PUMLPRs with the acceptable consistency.

After obtaining the consistent PUMLPRs, the group PUMLPR can be calculated by the proposed probabilistic uncertain linguistic weighted geometric aggregated (PULWGA) operator and the weights of the DMs. In addition to that, on behalf of obtaining the final evaluation consequence, this thesis defines the possibility degree for the PUMLTSs. Then the possibility degree matrix can be calculated according to the group PUMLPR. Then with the method of weight calculation [Xu15], the final priority can be acquired directly. Furthermore, for the purpose of demonstrating the valid of the proposed theory, this thesis applies the proposed theory to the case mentioned before and help to identify the valid way to manage the online public opinion.

The journal paper with respect to this part is:

- W.Y. Xie, Z.S. Xu, Z.L. Ren, E. Herrera-Viedma, Restoring incomplete PUMLPRs for evaluatingthe management way of online public opinion, Information Sciences, 516, 72-88, 2020.

6.2 The probe for the weighted dual probabilistic linguistic correlation coefficient to invest an artificial intelligence project

Owing to the fact that the correlation coefficient plays an important role during the practical application process. This thesis proposes the definition of the correlation coefficient for DPLTSs in the following section. Similar to the proposed DPLTSs, the computation of the correlation coefficient is executed by the following two parts: one is to calculate the correlation coefficient of two matching memberships, and the other is to figure out the correlation coefficient of two non-memberships. Then this thesis puts them together as the final correlation coefficient for the DPLTSs. Let $X = \{x_1, x_2, ..., x_n\}$ be a fixed set, $D_A = \{\langle x_i, L_A(p), U_A(p)\rangle, x_i \in X\}$ and $D_B = \{\langle x_i, L_B(p), U_B(p)\rangle, x_i \in X\}$ be two DPLTSs on the linguistic term set [Xu05] $S = \{s_\alpha | \alpha \in [-q, q]\}$, then their correlation coefficient between the two DPLTSs can be defined as the following equation:

$$\rho = \frac{\rho_1 + \rho_2}{2}$$

where

$$\rho_1 = \frac{\sum\limits_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum\limits_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m(\Gamma_A(p))\right)\left(\frac{1}{\#\Gamma_B(p)}\sum\limits_{\mu_B=1}^{\#\Gamma_B(p)} I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)} - m(\Gamma_B(p))\right)}{\left[\sum\limits_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum\limits_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m(\Gamma_A(p))\right)^2\right]^{1/2}\left[\sum\limits_{s=1}^{n}\left(\frac{1}{\#\Gamma_B(p)}\sum\limits_{\partial_B=1}^{\#\Gamma_B(p)} I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)} - m(\Gamma_B(p))\right)^2\right]^{1/2}}$$

$$\rho_2 = \frac{\sum\limits_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_A(p)}\sum\limits_{\Im_A=1}^{\#\mathbb{N}_A(p)} I\left(_A^{(\Im_A)}\right)p_A^{(\Im_A)} - m(\mathbb{N}_A(p))\right)\left(\frac{1}{\#\mathbb{N}_B(p)}\sum\limits_{\Im_B=1}^{\#\mathbb{N}_B(p)} I\left(\mathbb{N}_B^{(\Im_B)}\right)p_B^{(\Im_B)} - m(\mathbb{N}_B(p))\right)}{\left[\sum\limits_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_A(p)}\sum\limits_{\Im_A=1}^{\#\mathbb{N}_A(p)} I\left(\mathbb{N}_A^{(\Im_A)}\right)p_A^{(\Im_A)} - m(\mathbb{N}_A(p))\right)^2\right]^{1/2}\left[\sum\limits_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_B(p)}\sum\limits_{\Im_B=1}^{\#\mathbb{N}_B(p)} I\left(\mathbb{N}_B^{(\Im_B)}\right)p_B^{(\Im_B)} - m(\mathbb{N}_B(p))\right)^2\right]^{1/2}}$$

$$m(\Gamma_A(p)) = \frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)}\right),$$
$$m(\Gamma_B(p)) = \frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)} I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)}\right),$$

$$m(\mathbb{N}_A(p)) = \frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_A(p)}\sum_{\Im_A=1}^{\#\mathbb{N}_A(p)} I\left(\mathbb{N}_A^{(\Im_A)}\right)p_A^{(\Im_A)}\right),$$
$$m(\mathbb{N}_B(p)) = \frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_B(p)}\sum_{\Im_B=1}^{\#\mathbb{N}_B(p)} I\left(\mathbb{N}_B^{(\Im_B)}\right)p_B^{(\Im_B)}\right).$$

Then the correlation coefficient between the two DPLTSs D_A and D_B satisfies the following properties:
1) $\rho(D_A, D_B) = \rho(D_B, D_A)$;
2) $\rho(D_A, D_B) = 1$, if $D_A = D_B$;
3) $|\rho(D_A, D_B)| \leq 1$.

It is easy to see the correlation coefficient is calculated by two parts: the corresponding correlation coefficient between two membership degrees of the two DPLTSs and the correlation coefficient between the two non-membership degrees of the two DPLTSs, then the two obtained correlation coefficients are averaged to get the final correlation coefficient of the two DPLTSs.

The aim of studying the correlation coefficient is to use it as the measure to do the selection process. This thesis further defines the weighted correlation coefficient for the DPLTSs. In addition to that, all sorts of objective factors and subjective factors may lead to the differences in the importance of the goals under study. Hence, the weights of the goals under study are not always same. For example, one is going to buy a new apartment. He may need to consider many factors, such as the price, distance, daylight and so on. Obviously, not all the factors that need to be considered are the more the better. If let the buyer provide the respective weights of these influencing factors, the specific proportions must be different. For the practical application process, the relevant influencing factors that need to be considered are more and more complicated. Then the weights of the goals being studied is more difficult to determine. Therefore, it is necessary to find the appropriate method to determine the weight. Without loss of generality, the research of the determination of the weight for the weight correlation coefficient is also necessary.

The journal paper with respect to this part is:

- W.Y. Xie, Z.S. Xu, Z.L. Ren, E. Herrera-Viedma, The probe for the weighted dual probabilistic linguistic correlation coefficient to invest an artificial intelligence project, Soft Computing, DOI: 10.1007/s00500-020-04873-0, 2020.

6.3 Expanding grey relational analysis with the comparable degree for dual probabilistic multiplicative linguistic term sets and its application on the cloud enterprise

For any two DPMLTEs $D_1 = \langle \wp_1(p), \Upsilon_1(p) \rangle$ and $D_2 = \langle \wp_2(p), \Upsilon_2(p) \rangle$, the comparable degree between two DPMLTEs can be calculated as follows:

$$C(D_1, D_2) = \frac{1}{2}\left(\left|\log e_1^{\wp} - \log e_2^{\wp}\right| + \left|\log e_1^{\Upsilon} - \log e_2^{\Upsilon}\right|\right)$$

where for a DPMLTE $D = \langle \wp(p), \Upsilon(p) \rangle$, the expected value of the DPMLTE is

$$E_D = \left\langle \sum_{i=1}^{\#\wp(p)} p^{(i)} I\left(\wp^{(i)}\right), \sum_{j=1}^{\#\Upsilon(p)} p^{(j)} I\left(\Upsilon^{(j)}\right) \right\rangle = \left\langle e_{\wp(p)}, e_{\Upsilon(p)} \right\rangle$$

A DPMLPR on the mentioned set [Xu06a] $S = \{s_\alpha | \alpha \in [1/q, q]\}$ is defined as the matrix $D = (d_{ij})_{n \times n}$, $d_{ij} = \langle \wp_{ij}(p), \Upsilon_{ij}(p) \rangle$, which meets

$$\wp_{ij}(p) = \Upsilon_{ji}(p),\ \Upsilon_{ij}(p) = \wp_{ji}(p),\ i \neq j,\ i, j = 1, 2, \ldots, n$$

Moreover, if $i = j$, then $\wp_{ii}(p) = \Upsilon_{ii}(p) = \langle \{s_1(1)\}, \{s_1(1)\} \rangle$.

Furthermore, for two different DPMLPRs $D_1 = \left(d_{ij}^1\right)_{n\times n} = \left(\left\langle \wp_{ij}^1(p), \Upsilon_{ij}^1(p)\right\rangle\right)_{n\times n}$ and $D_2 = \left(d_{ij}^2\right)_{n\times n} = \left(\left\langle \wp_{ij}^2(p), \Upsilon_{ij}^2(p)\right\rangle\right)_{n\times n}$, the comparable degree of D_1 and D_1 can be defined as:

$$C(D_1, D_2) = \frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\log e_{ij1}^{\wp} - \log e_{ij2}^{\wp}\right| + \left|\log e_{ij1}^{\Upsilon} - \log e_{ij2}^{\Upsilon}\right|\right)\right]$$

where $E_{D_1} = \left\langle e_{ij1}^{\wp}, e_{ij1}^{\Upsilon}\right\rangle$ and $E_{D_2} = \left\langle e_{ij2}^{\wp}, e_{ij2}^{\Upsilon}\right\rangle$ are the homologous expected value of the different DPMLPRs D_1 and D_2, respectively.

Then on the foundation of the established comparable degree between the individual DPMLPRs and the group DPMLPR, the group consensus can be checked directly. Moreover, if the consensus cannot be satisfied in the decision-making procedure, then the DMs need to adjust their PRs, until the consensus is satisfied in the end, and the checking is over.
The journal paper with respect to this part is:

- W.Y. Xie, Z.S. Xu, Z.L. Ren, E. Herrera-Viedma, Expanding grey relational analysis with the comparable degree for dual probabilistic multiplicative linguistic term sets and its application on the cloud enterprise, IEEE Access, 2019, 7: 75041-75057.

6.4 A new multi-criteria decision model based on incomplete dual probabilistic linguistic preference relations

On the linguistic term set [Xu15] $S = \{s_\alpha | \alpha \in [0, 2q]\}$, if the matrix $D = (d_{ij}(p))_{n\times n} = (\langle \mathbb{C}_{ij}(p), \mathbb{Q}_{ij}(p)\rangle)_{n\times n}$ fulfills these coming qualifications:

$$\mathbb{C}_{ij}(p) = \mathbb{Q}_{ji}(p),\ \mathbb{Q}_{ij}(p) = \mathbb{C}_{ji}(p),\ \mathbb{C}_{ii}(p) = \mathbb{Q}_{ii}(p) = \{s_q\}$$

Then we call the matrix $D = (d_{ij}(p))_{n\times n}$ a DPLPR.

Moreover, due to some reasons that either the boundedness of the knowledge of the DMs or the intricacies of the factors involved in policy making, these constructed elements in DPLPRs are not always given completely, which leads to the produce of the IDPLPR. They can be shown as follows:

If some of the elements of the matrix $D = (d_{ij}(p))_{n\times n} = (\langle \mathbb{C}_{ij}(p), \mathbb{Q}_{ij}(p)\rangle)_{n\times n}$ are missing, then the matrix is named as IDPLPR, where $\mathbb{C}_{ij}(p) = \mathbb{Q}_{ji}(p)$, $\mathbb{Q}_{ij}(p) = \mathbb{C}_{ji}(p)$, $s_0 \leq \mathbb{C}_{ij}(p) \oplus \mathbb{Q}_{ij}(p) \leq s_{2q}$, $\mathbb{C}_{ij}(p) = \left\{\mathbb{C}_{ij}^{(l)}\left(p_{ij}^{(l)}\right) | l = 1, 2, \ldots, \neq \mathbb{C}_{ij}(p)\right\}$, $\mathbb{C}_{ij}^{(l)}$ is the lth linguistic term in $\mathbb{C}_{ij}(p)$, $p_{ij}^{(l)}$ is the possibility of the linguistic term $\mathbb{C}_{ij}^{(l)}$, $\mathbb{Q}_{ij}(p) = \left\{\mathbb{Q}_{ij}^{(\ell)}\left(p_{ij}^{(\ell)}\right) | \ell = 1, 2, \ldots, \neq {}_{ij}(p)\right\}$, $\mathbb{Q}_{ij}^{(\ell)}$ is the ℓth linguistic term in $\mathbb{Q}_{ij}(p)$, $p_{ij}^{(\ell)}$is the possibility of the linguistic term $\mathbb{Q}_{ij}^{(\ell)}$. $d_{ij}(p) \in \Omega_D$, Ω_D is the set of all the known elements. To facilitate the application, in all of the following sections, we set $l = \ell$, which means that the membership part and the non-membership part have the same number of elements.

Furthermore, if each unknown element of the IDPLPR can be acquired by its adjacent known elements, then the IDPLPR is called acceptable, where the adjacent known elements mean that for the two elements d_{ij} and d_{st} in the IDPLPR $D = (d_{ij}(p))_{n\times n}$, if $(i, j) \cap (s, t) \neq \emptyset$, the elements d_{ij} and d_{st} are adjacent. For convenience, all the IDPLPRs in the remaining paper are acceptable.

For those constituent elements in the incomplete DPLPRs are several memberships and several non-memberships, there is no applicable method to repair the IDPLPRs. Therefore, based on the unique feature of DPLTSs, this thesis makes the following improvement to the method of Ref. [MTH19]. For the first step, it uses the coming model to repair the lacking linguistic section:

$$\Phi = \min \sum_{i=1}^{j-1} \sum_{j=i+1}^{n} \left(u_{ij}^{+} + u_{ij}^{-} + v_{ij}^{+} + v_{ij}^{-} \right)$$

$$s.t. \begin{cases} \theta_{ij}^{(l)} \left[\begin{array}{l} (n-2)\left(\eta_{ij}^{(l)} f\left(\mathbb{C}_{ij}^{(l)}\right) + \left(1-\eta_{ij}^{(l)}\right)\left(2q - f\left(\mathbb{Q}_{ij}^{(l)}\right)\right) + q\right) - \\ \sum\limits_{k=1,k\neq i,j}^{n} \left(\eta_{ik}^{(l)} f\left(\mathbb{C}_{ik}^{(l)}\right) + \left(1-\eta_{ik}^{(l)}\right)\left(2q - f\left(\mathbb{Q}_{ik}^{(l)}\right)\right)\right) \\ +\eta_{kj}^{(l)} f\left(\mathbb{C}_{kj}^{(l)}\right) + \left(1-\eta_{kj}^{(l)}\right)\left(2q - f\left(\mathbb{Q}_{kj}^{(l)}\right)\right) - u_{ij}^{(l)+} + u_{ij}^{(l)-} \end{array} \right] = 0 \\ \theta_{ij}^{(l)} \left[\begin{array}{l} (n-2)\left(\eta_{ij}^{(l)} f\left(\mathbb{Q}_{ij}^{(l)}\right) + \left(1-\eta_{ij}^{(l)}\right)\left(2q - f\left(\mathbb{C}_{ij}^{(l)}\right)\right) + q\right) - \\ \sum\limits_{k=1,k\neq i,j}^{n} \left(\eta_{ik}^{(l)} f\left(\mathbb{Q}_{ik}^{(l)}\right) + \left(1-\eta_{ik}^{(l)}\right)\left(2q - f\left(\mathbb{C}_{ik}^{(l)}\right)\right)\right) \\ +\eta_{kj}^{(l)} f\left(\mathbb{Q}_{kj}^{(l)}\right) + \left(1-\eta_{kj}^{(l)}\right)\left(2q - f\left(\mathbb{C}_{kj}^{(l)}\right)\right) - v_{ij}^{(l)+} + v_{ij}^{(l)-} \end{array} \right] = 0 \\ 0 \le f\left(\mathbb{C}_{ij}^{(l)}\right) \le 2q - f\left(\mathbb{Q}_{ij}^{(l)}\right), f\left(\mathbb{C}_{ij}^{(l)}\right) \in \mho_{\mathbb{C}} \wedge f\left(\mathbb{Q}_{ij}^{(l)}\right) \notin \mho_{\mathbb{Q}} \\ 0 \le f\left(\mathbb{Q}_{ij}^{(l)}\right) \le 2q - f\left(\mathbb{C}_{ij}^{(l)}\right), f\left(\mathbb{C}_{ij}^{(l)}\right) \notin \mho_{\mathbb{C}} \wedge f\left(\mathbb{Q}_{ij}^{(l)}\right) \in \mho_{\mathbb{Q}} \\ 0 \le f\left(\mathbb{C}_{ij}^{(l)}\right), f\left(\mathbb{Q}_{ij}^{(l)}\right), f\left(\mathbb{C}_{ij}^{(l)}\right) + f\left(\mathbb{Q}_{ij}^{(l)}\right) \le 2q, f\left(\mathbb{C}_{ij}^{(l)}\right) \in \mho_{\mathbb{C}} \wedge f\left(\mathbb{Q}_{ij}^{(l)}\right) \in \mho_{\mathbb{Q}} \\ \eta_{ij}^{(l)} = 0 \vee 1, i,j = 1,2,\ldots,n \\ u_{ij}^{(l)+}, u_{ij}^{(l)-}, v_{ij}^{(l)+}, v_{ij}^{(l)-} \ge 0, i,j = 1,2,\ldots,n, i<j \end{cases}$$

where $\mho_{\mathbb{C}} = \left\{ f\left(\mathbb{C}_{ij}^{(l)}\right) \middle| \mathbb{C}_{ij}^{(l)} \text{ is missing for all } i,j = 1,2,\ldots,n \text{ with } i<j \right\}$
Next, based on the principle of additive consistency [ZXWL16] in PLPRs, it constructs the following model to repair these lacking probabilities for the incomplete PLPRs $\mathbb{C}' = (\mathbb{C}'_{ij}(p))_{n\times n}$:

$$Min \;\; \varepsilon_{ij} = \frac{1}{n} \sum_{k=1}^{n} \left| S\left(\mathbb{C}'_{ij}(p)\right) - S\left(\mathbb{C}'_{ik}(p) \oplus \mathbb{C}'_{kj}(p)\right) \right|$$

$$\text{s.t.} \begin{cases} \sum\limits_{\eta=1}^{\neq \mathbb{C}'_{ij}(p)} p^{(\eta)} = 1 \\ 0 \le p^{(\eta)} \le 1 \end{cases}$$

where $\mathbb{C}'_{ij}(p) = \left\{ \mathbb{C}'^{(\eta)}_{ij}\left(p^{(\eta)}\right) \middle| \mathbb{C}'^{(\eta)}_{ij} \in S, p^{(\eta)} \ge 0, \sum\limits_{\eta=1}^{\#\mathbb{C}'_{ij}(p)} p^{(\eta)} \le 1 \right\}$. So far, if $\varepsilon_{ij} = 0$, then the lacking of membership part for the IDPLPRs can be repaired completely. Similarly, the the lacking non-membership part for the IDPLPRs can also be repaired completely. Hnece, the IDPLPRs can be repaired completely, and the whole repairing procedure is over.

The research contributions of the thesis can be summarized as follows: (1) Because the DPLPRs can reveal the decision-making information through the association of the membership part and the non-membership part, this thesis chooses the DPLPRs to reflect the preference information in the procedure of dealing with the uncertain decision-making problem. (2) In view of various subjective and objective reasons, the DPLPRs can't always be obtained fully. Then, this thesis studies the case of incomplete preference information that is the IDPLPRs. (3) Complete preference information is the precondition to make a decision. Therefore, how to repair the incomplete preference information is the third contribution we make in this thesis. (4) For the sake of obtaining the relatively meaningful decision-making results, this thesis checks and improves the consistency of completed preference information. (5) All work on PRs is to determine the criteria for the multi-criteria decision-making problem under study. Based on those PRs that already satisfy consistency, this thesis can obtain the criteria for the decision issue to be addressed. (6) The final aim of making a decision is to get the final decision-making result by looking for the applicable method. For the features of dual probabilistic linguistic preference information, thesis constructs optimistic and pessimistic dual probabilistic DEA model to make final decision.

The journal paper with respect to this part is:

- W.Y. Xie, Z.S. Xu, Z.L. Ren, E. Herrera-Viedma, A new multi-criteria decision model based on incomplete dual probabilistic linguistic preference relations, Applied Soft Computing, DOI: 10.1016/j.asoc.2020.106237, 2020.

7 Discussion of results

In this section, we make a discussion about the results obtained in each stage of the thesis. The discussions of results are divided into four sections, please see below for details.

7.1 Restoring incomplete PUMLPRs for evaluating the management way of online public opinion

In the age of big data explosion, the management of online public opinion has encountered great challenges. Which way can effectively manage online public opinion has become a decision-making question for us to think about. PUMLPRs are a remarkable instrument to solve uncertain evaluation problems. This paper uses the PUMLPRs to assess the management ways of the online public opinion. Owing to the intricacy of decision-making domain, the PUMLPRs are not always complete. The repairing process for the incomplete PUMLPRs is also separated into two steps. For the first one, this thesis studies the method to repair the missing parts that repair the UMLPRs.For the second step, owing to its loss of probability, this thesis puts forward a linear programming model to obtain the loss of probability. Hence, the incomplete PUMLPRs are repaired completely.

The consistency is the necessary requirement for obtaining appropriate decision-making result. this thesis defines the GPRs for the PUMLPRs. Moreover, it is easy to see that the GPRs is the UMLPRs. Then it can use the interval consistency condition to establish multi-objective programming model and to obtain the PUMLPRs with the acceptable consistency. After obtaining the consistent PUMLPRs, the group PUMLPR can be calculated by the proposed PULWGA operator and the weights of the DMs. In addition to that, on behalf of obtaining the final evaluation consequence, it defines the possibility degree for the PUMLTSs. Then the possibility degree matrix can be calculated according to the group PUMLPR. Then with the method of weight calculation, the final priority can be acquired directly. Furthermore, for the purpose of demonstrating the valid of the proposed theory, it applies the proposed theory to the case mentioned before and help to identify the valid way to manage the online public opinion.

The experimental results are the same regardless of the environment, the means of human resource management is the relatively more suitable way to manage the online public opinion under the big data circumstance. On the strength of the above analysis, the restoring method mentioned in this thesis can be used in the probabilistic uncertain linguistic context, the probabilistic linguistic circumstance and the hesitant fuzzy linguistic environment. The reason for this phenomenon may be that the dimension of the data being tested is not large enough. The difference is that the elements of these three decision-making instruments have different structures. In the procedure of making a decision, the same method applied in these three contexts, the computation of the PUMLPRs is more complicated than the PMLPRs and the HFLPRs. To some extent, the proposed method can also be applied to similar tasks. But from the point of fully expressing the decision-making information, the PUMLPRs is better. Except for the validation of the method, the contrastive analysis also demonstrates the extension of the restoring method. But owing to the process for restoring incomplete PRs is not direct. That is to say, the restoring process is relatively complex. It has completely repaired the incomplete PUMLPRs indeed. Moreover, the consistent procedure is performed in the derived UMLPRs, not the original PUMLPRs. Although it can get the final priorities, but it cannot get the consistent PUMLPRs.

7.2 The probe for the weighted dual probabilistic linguistic correlation coefficient to invest an artificial intelligence project

Since the computer "dark blue" of the IBM company defeated the world chess champion of mankind in 2016, AI has aroused the extensive attention of the public. Therefore, many scientific and technical corporations have begun to join the AI market war in succession. In order to better develop the AI industry, before determining the specific project, they need to do the evaluation for the influence of the AI.

For the sake of making rational decisions, the DMs need to choose the appropriate decision-making tool. The DPLTSs can make the DMs flexibly give their suggestions and reduce the irresolution of the DMs for one thing or another when it is hard for them to reach a final agreement to some extent. Considering the advantages of DPLTSs and the importance of researching the correlation coefficient that can research the degree of linear correlation between variables, this thesis is to research the correlation coefficient among the DPLTSs.

This thesis firstly defines the dual probabilistic linguistic correlation coefficient and the weight dual probabilistic linguistic correlation coefficient. Then describes the multi-attribute group decision-making problem under the dual probabilistic linguistic context, divides the weight vector into the subjective and objective forms, defines the entropy measure for the DPLTSs for the sake of obtaining the final comprehensive weight vector, and introduces the complete dual probabilistic linguistic multi-attribute group decision-making process. Moreover, it uses a simulation experiment related to the influence evaluations for AI to clarify the feasibility and practicality of the dual probabilistic linguistic multi-attribute group decision-making process.

Besides, In order to analyze the final decision-making result, it utilizes the devised distance measure to calculate the closeness coefficient as the basis to obtain the policy-making result, and contradistinguish the differences between two methods. No matter how to use the correlation coefficient or the closeness coefficient as the benchmark to get the ultimate decision-making consequence, the final best alternative is same. In addition, compared with the closeness coefficient, the calculation of correlation coefficient does not need to adjust the number of elements for the membership degree and non-membership degree in DPLTSs and get the uniform cardinality. The decision-making information can retain the original as soon as possible. Whereas, the process for computing the correlation coefficient is relative complexity than the computation process of closeness coefficient.

7.3 Expanding grey relational analysis with the comparable degree for dual probabilistic multiplicative linguistic term sets and its application on the cloud enterprise

Under the cloud trend of enterprises, how do traditional businesses get on the cloud becomes a worth pondering question. To help those traditional businesses that have no experience to dispel the clouds and see the sun as soon as possible, this thesis is planing to choose one corporation with rich experience to take them into cloud market. The quintessence of DPLTSs is that it uses the combination of several linguistic terms and their proportions to reveal decision information by opposite angles. This thesis proposes the DPMLPRs based upon the DPMLTSs. Then it defines the comparable degree between the DPMLPRs and studies the consensus of the group DPMLPR. Moreover, it probes the EGRA process under the proposed comparable degree between the DPMLTSs. After that, one example of choosing the experienced cloud cooperative partner is simulated under the dual probabilistic linguistic circumstance. Besides, the comparative analysis is performed by considering the similarity among the EGRA, TODIM and VIKOR.

Apparently, the obtained optimal decisions by three different methods are different. For the EGRA method, the optimal alternative is a_3. Usually, it is based on the degree of similarity or dissimilarity between the development trends of factors, that is, the "grey correlation degree", as a method to measure the degree of association between factors. It considers the relative comparable degree between the ideal solution and the alternative. It has the advantage of being simple to calculate. For the ETODIM method, the optimal alternative is a_4. It is a typical decision-making method considering the mental behavior of DMs based on the prospect theory. It sorts and optimizes the solution by calculating the dominance of the alternatives over other scenarios. The salient features of it are that it not only accelerates the risk factor in the system, but also enriches the range of decision-making procedure. Moreover, it provides a chance for us to check gains and losses for any two alternatives with regard to any criteria. While for the EVIKOR method, the optimal alternative is a_3. If there is a conflict between the indicators, it sorts the scheme according to a certain method, so as to obtain an optimal solution. Because it maximizes group benefits and minimizes individual losses, it leads to a compromise solution that can be acknowledged by DMs. Moreover, the compromise solution is the optimal solution in the solution space.

7.4 A new multi-criteria decision model based on incomplete dual probabilistic linguistic preference relations

The use of DPLTSs to represent the use's preferences in decision making can reflect the decision maker's cognitive certainty and uncertainty. Additionally, the appearance of incomplete preferences is a recurring phenomenon that must be taken into account if you want to make a successful decision. This thesis presents a new multi-criteria decision model based on the IDPLPRs. It first proposes a step-by-step repairing method to repair the linguistic section and probabilistic section of IDPLPRs separately. The superiority is that this step-by-step method conforms to the principle of element generation. After that, the consistency index based on the distance measure between the DPLPRs is defined to check and improve the consistency of DPLPRs. Then the weights of criteria can be obtained by information fusion. Moreover, it constructs optimistic and pessimistic data envelopment analysis models under the dual probabilistic linguistic environment to do the sorting process. Optimistic and pessimistic data envelopment analysis models can demonstrate the efficiency of each DMU from the perspective of the most and least favorable. Finally, it simulates a cased of 5G industry market to help enterprises choose appropriate 5G partners by using proposed methods.

In general, the implementation of the method can be summarized as follows: Step1. Categorizing the PRs: the complete DPLPRs and the incomplete DPLPRs; Step 2. Repairing the incomplete DPLPRs: the repairing for linguistic section and the repairing for probabilistic section; Step 3. Checking and improving the consistency of the complete DPLPRs; Step 4. Aggregating all the consistent DPLPRs into the group DPLPR and determine the weight vector of the criterion; Step 5. To fuse all the decision-making information with these weights of criteria and get the group decision-making matrix; Step 6. Building respectively optimistic and pessimistic efficiency models based on the group decision-making matrix and complete the sorting process.

Our work is mainly to solve the uncertain multi-criteria decision-making problem. The two keys to solving this problem are to choose the right decision-making tools and determine the weights of criteria. In this thesis, in light of the complexity of decision-making problems in real-world applications and the limitations of the available knowledge to DMs, the preference information is not always fully available. Therefore, it is reasonable to consider an uncertain multi-criteria decision-making problem with incomplete preference information in this thesis. To some extent, this consideration is close to real-world application. Moreover, now that the preference information is not complete, we think about to deal with the decision-making problem with tools that can give the DMs as much information as possible. The DPLTSs can reveal the decision-making information through the association of the membership part and the non-membership part. Moreover, whether it is the membership element or the non-membership element, both are composed by several linguistic terms and the corresponding probabilities. From this point of view, the DPLTSs can reflect the decision information as fully as possible. Hence, this thesis chooses to study the incomplete dual probabilistic linguistic multi-criteria decision-making problem. Based on the proposed method, the weights of criteria can be determined by a series of steps. Moreover, combining the consideration of incomplete decision and the choice of decision tools, the proposed method meets the needs of most practical problems. The proposed method is of great importance and practical applications for solving uncertain multi-criteria decision-making problem.

8 Concluding remarks

Given that the intricacy of the real decision-making situations, not all the PRs are always complete. This paper has researched the incomplete PUMLPRs. Owing to the complexity of the structure of the PUMLPRs, the incomplete PUMLPRs cannot be restored directly. Therefore, this thesis has divided the restored process into two steps: one is to restore the uncertain linguistic variables; the other is to restore the corresponding probabilities. Moreover, due to the necessity of the consistency, the probe for the consistency of the obtained complete PUMLPRs has also been discussed later. Then this thesis has assessed the management way of the online opinion under the big data context. Indeed, the utilization of the case has illustrated the validity of the series of the suggested decision-making procedure. In addition to that, so as to further give evidence of the efficacy of the suggested decision-making procedure, this thesis has also applied it to the probabilistic linguistic surrounding. Obviously, the procedure that we have provided can also solve the incomplete probabilistic linguistic decision-making problem.

Moreover, this thesis has enriched the basic theory of the DPLTSs by the following directions: first defined the complement of the DPLTSs, and then defined the different distance measures for the DPLTSs with the same cardinalities. Then considering the importance of the defined correlation coefficient in the policy-making area, this thesis has proposed the correlation coefficient between the DPLTSs. Moreover, for the sake of applying the suggested correlation coefficient to the practical policy-making problem, this thesis has proposed the weighted correlation coefficient. In addition, in order to get the utmost out of the decision-making content, this thesis has divided the weight vector into the subjective form and objective vector, not only considered the subjective cognition from the perspectives of the DMs, but also considered the objectivity of the attributes. On the side, this thesis has defined the entropy for the DPLTEs to calculate the comprehensive weight. After that, this thesis has applied the weighted correlation coefficient to the specific problem and helped to choose the best project for AI industry. Finally, the specific execution of the example has demonstrated the effective of the proposed theory. Besides, one comparative analysis by using the closeness coefficient based upon the distance measure has been performed to highlight the advantages and disadvantages of the correlation coefficient; the other comparative analysis has been compared with probabilistic linguistic decision-making matrices, which further demonstrates the differences of the two kinds of decision-making information.

Furthermore, this thesis has enriched the basic theory of the DPLTSs by putting forward the DPMLTSs and the DPMLPRs, separately. Then it has considered the importance of the consistency of the PRs in the procedure of obtaining the logical decision result, and probed the consistency of the DPMLPRs. Moreover, on the foundation of the proposed comparable degree between the DPMLPRs, it has researched the consensus of the group DPMLPR. In addition, in order to obtain the final decision result, it has proposed the EGRA method. On the side, we have also developed the ETODIM method and the EVIKOR method based upon the comparable degrees. After that, it has applied the proposed method to settle the problem and helped to choose the best cooperative enterprise for cloud enterprise. Finally, the specific execution of the example has demonstrated the effective of the proposed theory. Besides, two comparative analyses have been utilized to highlight the advantages and disadvantages of the proposed method.

Besides, this thesis has enriched the basic theory of the DPLTSs by these following directions: firstly, it has defined the DPLPRs, and then defined the IDPLPRs. Moreover, for the sake of obtaining the logical decision-making result, it has constructed different linear programming models to repair the missing linguistic portion and the probabilistic portion, and then, it has probed the consistency of the DPLPRs. Furthermore, it has built the dual probabilistic linguistic DEA model to make decisions. After that, it has applied the proposed method to solve the problem and helped to choose the best project for 5G enterprise. The research result shows that the enterprise should choose Huawei as a partner to develop the 5G industry. To some extent, the decision-making result is in line with the current status of 5G industry development.

Conclusiones

Dado que la complejidad de las situaciones reales de toma de decisiones, no todos los RP siempre están completos. Este artículo ha investigado los PUMLPR incompletos. Debido a la complejidad de la estructura de los PUMLPR, los PUMLPR incompletos no se pueden restaurar directamente. Por lo tanto, esta tesis ha dividido el proceso restaurado en dos pasos: uno es restaurar las variables lingüísticas inciertas; el otro es restaurar las probabilidades correspondientes. Además, debido a la necesidad de la consistencia, la sonda para la consistencia de los PUMLPR completos obtenidos también se ha discutido más adelante. Luego, esta tesis ha evaluado la forma de gestión de la opinión en línea en el contexto de big data. De hecho, la utilización del caso ha ilustrado la validez de la serie del procedimiento de toma de decisiones sugerido. Además de eso, para dar más evidencia de la eficacia del procedimiento de toma de decisiones sugerido, esta tesis también lo ha aplicado al entorno lingüístico probabilístico. Obviamente, el procedimiento que hemos proporcionado también puede resolver el problema incompleto de toma de decisiones lingüísticas probabilísticas.

Además, esta tesis ha enriquecido la teoría básica de los DPLTS en las siguientes direcciones: primero definió el complemento de los DPLTS y luego definió las diferentes medidas de distancia para los DPLTS con las mismas cardinalidades. Luego, considerando la importancia del coeficiente de correlación definido en el área de formulación de políticas, esta tesis ha propuesto el coeficiente de correlación entre los DPLTS. Además, en aras de aplicar el coeficiente de correlación sugerido al problema práctico de formulación de políticas, esta tesis ha propuesto el coeficiente de correlación ponderado. Además, para sacar el máximo provecho del contenido de la toma de decisiones, esta tesis ha dividido el vector de peso en la forma subjetiva y el vector objetivo, no solo consideró la cognición subjetiva desde las perspectivas de los DM, sino que también consideró la objetividad de los atributos. Por otro lado, esta tesis ha definido la entropía para que los DPLTE calculen el peso integral. Después de eso, esta tesis ha aplicado el coeficiente de correlación ponderado al problema específico y ayudó a elegir el mejor proyecto para la industria de IA. Finalmente, la ejecución específica del ejemplo ha demostrado la efectividad de la teoría propuesta. Además, se realizó un análisis comparativo utilizando el coeficiente de cercanía basado en la medida de distancia para resaltar las ventajas y desventajas del coeficiente de correlación; El otro análisis comparativo se ha comparado con matrices probabilísticas de toma de decisiones lingüísticas, lo que demuestra aún más las diferencias de los dos tipos de información de toma de decisiones.

Además, esta tesis ha enriquecido la teoría básica de los DPLTS al presentar los DPMLTS y los DPMLPR por separado. Luego, consideró la importancia de la consistencia de los RP en el procedimiento para obtener el resultado de la decisión lógica y probó la consistencia de los DPMLPR. Además, sobre la base del grado comparable propuesto entre los DPMLPR, ha investigado el consenso del grupo DPMLPR. Además, para obtener el resultado final de la decisión, ha propuesto el método EGRA. Por otro lado, también hemos desarrollado el método ETODIM y el método EVIKOR basado en grados comparables. Después de eso, aplicó el método propuesto para resolver el problema y ayudó a elegir la mejor empresa cooperativa para la empresa en la nube. Finalmente, la ejecución específica del ejemplo ha demostrado la efectividad de la teoría propuesta. Además, se han utilizado dos análisis comparativos para resaltar las ventajas y desventajas del método propuesto.

Además, esta tesis ha enriquecido la teoría básica de los DPLTS mediante estas instrucciones: en primer lugar, ha definido los DPLPR y luego los IDPLPR. Además, en aras de obtener el resultado lógico de toma de decisiones, ha construido diferentes modelos de programación lineal para reparar la porción lingüística faltante y la porción probabilística, y luego, ha probado la consistencia de los DPLPR. Además, ha construido el modelo de DEA lingüístico probabilístico dual para tomar decisiones. Después de eso, aplicó el método propuesto para resolver el problema y ayudó a elegir el mejor proyecto para la empresa 5G. El resultado de la investigación muestra que la empresa debería elegir a Huawei como socio para desarrollar la industria 5G. Hasta cierto punto, el resultado de la toma de decisiones está en línea con el estado actual del desarrollo de la industria 5G.

9 Future works

For the defined PULPRs, in addition to the distance measure and possibility degree, we can also consider using inclusion, entropy, etc. as the basis for research. We can also consider studying the basic properties such as additive consistency, multiplicative consistency, order consistency, expectation consistency, etc., or studying new decision-making methods. In addition, to a certain extent, although the step-by-step repairing method solves the problem of incomplete information on the PUMLPRs, the repairing process is not straightforward and simple, and the repair process is based on its uncertain multiplicative linguistic variable and corresponding probability. Therefore, in the future, we can try to study more suitable repairing methods based on the nature of the preference relationship itself, and repair the missing elements of the incomplete probabilistic uncertain multiplicative linguistic preference relationship in a holistic manner.

There are many measures that can reflect the relationship between elements. In the future, we can study more coefficients between DPLTSs, measures, and explore the internal connections between elements. Different measures have different meanings and the resulting decision methods. We can also consider studying other decision-making methods in the dual probabilistc linguistic environment, the consistency and consensus of the DPLPRs, and further enrich the dual probabilistc linguistic decision-making theories and methods.

Although the repairing method for incomplete DPLPRs is reasonable, but the procedure is relatively complex. Moreover, the dual probabilistic linguistic data envelopment analysis is performed on score functions and accuracy functions, not the original DPLTSs. Therefore, in the future, we can make further research from the following two perspectives: (1) Looking for simpler and more straightforward repairing methods. (2) Using the original data for data envelopment analysis as soon as possible.

In addition, the cases in this thesis are simulation experiments. In the future, we can consider combining with actual decision-making problems, collecting data, and collating, so as to achieve the purpose of actually solving practical problems. In addition, because of the explosion of the information society, on the one hand, the increase in data information and the increase in scale require us to deal with large-scale decision-making problems. On the other hand, things are changing faster and faster, and we need to propose new decision-making tools to study such dynamic changing decision-making problems.

sec:Intro

Chapter II

Publications: Published Papers

1 Restoring incomplete PUMLPRs for evaluating the management way of online public opinion

- W.Y. Xie, Z.S. Xu, Z.L. Ren, E. Herrera-Viedma, Restoring incomplete PUMLPRs for evaluating the management way of online public opinion. Information Sciences, 516 (2020) 72-88.
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Restoring incomplete PUMLPRs for evaluating the management way of online public opinion

Wanying Xie [a,c], Zeshui Xu [a,b,*], Zhiliang Ren [a], Enrique Herrera Viedma [c,*]

[a] *School of Economics and Management, Southeast University, Nanjing, Jiangsu 211189, China*

[b] *Business School, Sichuan University, Chengdu, Sichuan, 610064, China*

[c] *Andalusian Research Institute in Data Science and Computational Intelligence, University of Granada, E-18071 Granada, Spain*

Abstract

In the age of big data explosion, the management of online public opinion has encountered great challenges. Which way can effectively manage online public opinion has become a decision-making question for us to think about. Probabilistic uncertain multiplicative linguistic preference relations (PUMLPRs) are a remarkable instrument to solve uncertain evaluation problems. This paper uses the PUMLPRs to assess the management ways of the online public opinion. Owing to the intricacy of decision-making domain, the PUMLPRs are not always complete. We get the complete PUMLPRs in two steps: the repair for uncertain multiplicative linguistic variables and the repair for probability. Moreover, the consistency of the complete PUMLPRs is researched. Then the final priorities are obtained by the proposed possibility degree formula. After that, the numerical example that helps assess the valid way to manage online public opinion is performed to check the feasibility of the proposed decision-making procedure.

Keywords: Online public opinion; IPUMLPRs; Repairing; Consistency; Possibility degree.

[*] Corresponding author. E-mail addresses: xie_wanyinglai@163.com (W.Y. Xie), xuzeshui@263.net (Z.S. Xu), ren_zhi_liang@126.com (Z.L. Ren), viedma@decsai.ugr.es (E. Herrera Viedma).

1. Introduction

In a certain social space, through the internet and around the development and change of mediating social events, online public opinion [47, 18] is social attitudes, beliefs and values produced and owned by the citizens to the public issues and social managers. The concept of big data [20, 27, 28] was first proposed in the 1980s. In 2011, the consulting company of McKinsey released its research results “big data: the next innovation, competition and productivity”, which enabled the concept to be widely promoted. The big data has four characters: volume, variety, velocity and value.

Under these four characters, the online public opinion has the following features: (1) The information is retransmitted by just copying and pasting on the internet. Compared with the limited distribution of traditional media, the online public opinion has the potential to spread infinitely. This feature of the network makes it easy to pass through the blockade, leaving the regulatory authorities at a loss. (2) The characteristics of virtuality, anonymity, borderless and instant interaction in the network society make the online public opinion diverse and non-mainstream in terms of value delivery and interest appeal. (3) The network breaks the boundaries between time and space. While major news events become the focus of attention on the internet, they also quickly become hot spots of public opinion. At present, the way of public opinion hype is mainly released by traditional media, then reprints on the again, forms a network public opinion, and finally feedbacks to traditional media. The network can be updated in real time, enabling network public opinion to spread at the fastest speed. (4) Legal ethics in cyberspace lacks regulatory restrictions and effective supervision. Due to the concealed identity of the speaker, the speeches expressed often lack objectivity and cannot be equated with the positions of the public.

According to the peculiarities, the corresponding management methods can be summarized as the following aspects: Social, Governmental, Technological and Human. Which way can effectively manage online public opinion has become a decision-making question for us to think about. In order to solve the question, we choose the PUMLPRs as the decision-making instrument. Probabilistic uncertain linguistic term sets (PULTSs) were first proposed by Lin et al. [17], which is the combination of probabilistic linguistic term sets (PLTSs) [9] and uncertain linguistic variables [38]. PULTSs inherit the good properties of both. From the angle of the composition of the elements, it keeps the non-determinacy of the uncertain linguistic variables. Combined with the homologous proportions of the given uncertain linguistic variables, it fully demonstrates the intricacy of the decision-making environment and the uncertainty of the DMs.

In this paper, we extend the traditional PULTSs raised by Lin et al. [17] to the probabilistic uncertain multiplicative linguistic term sets (PUMLTSs) by drawing lessons from Xu [36], and define the homologous PUMLPRs. For the complexity of its construction, the PUMLPRs is constructed by the combination of uncertain multiplicative linguistic variables and their respective probabilities. With regard to the decision-making procedure, the DMs that are invited to do the decision are not possible to prefect themselves in all the relevant information. Hence, owing to the various external and internal conditions, the DMs are not always give their completely-specified preference information. The incomplete phenomenon can be found everywhere [15, 10, 26, 37, 33, 23, 43, 24, 7, 31, 32] indeed. This paper divides the incomplete preference information into two aspects by considering the components of PUMLPRs. On the one hand, owing to the lack of the familiarity with the evaluated problem, the DMs cannot determine the corresponding uncertain multiplicative linguistic variables. On the other hand, due to the deletion of the uncertain multiplicative linguistic variables, the corresponding probabilities cannot be determined, either.

Therefore, considering the complexity of the structure for PUMLPRs, the repairing process for the incomplete PUMLPRs is also separated into two steps. For the first one, by learning from Ref. [34], we study the method to repair the missing parts that repair the uncertain multiplicative linguistic preference relations (UMLPRs) [44, 45, 39]. Moreover, for the sake of better repairing the incomplete UMLPRs, in this paper, the uncertain multiplicative linguistic variables is separated into the two parts: the left and right of the interval. Then the whole UMLPRs can be divided into two linguistic preference relations (LPRs) [11, 35]. As is known that who devotes to the research of decision-making, consistency [14, 2, 30, 19, 42] is the fundamental condition for the researched preference relations (PRs). The repairing process uses the consistency of the LPRs as the basic to construct a multi-objective programming model and calculates the lacking uncertain multiplicative linguistic variables. For the second step, owing to its loss of probability, in the light of Ref. [34], we put forward a linear programming model to obtain the loss of probability. Hence, the incomplete PUMLPRs are repaired completely.

Just as we said before, the consistency is the necessary requirement for obtaining appropriate decision-making result. The discussion for the obtained complete PUMLPRs is also integrant. In this paper, in the cause of researching the consistency of the PUMLPRs, through the study of Ref. [46], we define the geometric PRs (GPRs) for the PUMLPRs. Moreover, it is easy to see that the GPRs is the UMLPRs. Then we can use the interval consistency condition [6] to establish multi-objective programming model and to obtain the PUMLPRs with the acceptable consistency.

After obtaining the consistent PUMLPRs, the group PUMLPR can be calculated by the proposed probabilistic uncertain linguistic weighted geometric aggregated (PULWGA) operator and the weights of the DMs. In addition to that, on behalf of obtaining the final evaluation consequence, we define the possibility degree for the PUMLTSs. Then the possibility degree matrix can be calculated according to the group PUMLPR. Then with the method of weight calculation [40], the final priority can be acquired directly. Furthermore, for the purpose of demonstrating the valid of the proposed theory, we apply the proposed theory to the case mentioned before, to help identify the valid way to manage the online public opinion.

The experimental result shows that the mean of human resource management is a relatively appropriate measure than three other means. To some extent, the experimental result is rational. For the online public opinion, the source of the problem is mainly human. Therefore, the best method is to start from the source and prevent small mistakes.

Broadly speaking, the innovation points of the paper can be summarized as follows: (1) Redefine the possibility degree between the PULTSs for acquiring the priority; (2) Define the PUMLTSs on the multiplicative linguistic label and the PUMLPRs; (3) Consider the incomplete PUMLPRs; (4) Put forward pertinently the corresponding repairing method to obtain complete PUMLPRs; (5) Probe the consistency of the PUMLPRs.

The surplus of the paper is formed as follows: Section 2 describes some fundamental notions involving the PULTSs. Section 3 regulates the possibility degree, PUMLPRs and the incomplete PUMLPRs. Section 4 is separated into two proportions: one is to propose repairing methods for incomplete PUMLPRs. The other is the discussion of consistency for the obtained complete PUMLPRs. Section 5 applies the proposed decision-making procedure to the precise online public opinion case, compares and analyzes the diversities between the incomplete probabilistic uncertain multiplicative linguistic decision-making and the incomplete probabilistic multiplicative linguistic decision-making consequences. Section 6 concludes the paper with some conclusions.

2. Preliminaries

For this part, we are going to momently list some inevitable notions with reference to the uncertain multiplicative linguistic variable and the PULTS.

2.1. The uncertain multiplicative linguistic variable

Provided that $S=\{s_\alpha | \alpha\in[1/q,q]\}$ is a successive multiplicative linguistic label set [39], and q is a fully big positive integer. Moreover, if $\alpha>\beta$, $s_\alpha>s_\beta$; $rec(s_\alpha)=s_\beta$, $\alpha\beta=1$; peculiarly, $rec(s_1)=s_1$. In view of the set S mentioned before, Xu [39] advanced the uncertain multiplicative linguistic variables and defined a few of essential operation rules for them as follows:

$\Delta=[s_{\alpha_1},s_{\beta_1}]$ and $\nabla=[s_{\alpha_2},s_{\beta_2}]$ are any two uncertain multiplicative linguistic variables, and the parameter $\xi\in[0,1]$, the coming fundamental operational laws are satisfied:

1) $\Delta\otimes\nabla=[s_{\alpha_1},s_{\beta_1}]\otimes[s_{\alpha_2},s_{\beta_2}]=[\max\{s_{1/q},\min\{s_{\alpha_1\alpha_2},s_q\}\},\max\{s_{1/q},\min\{s_{\beta_1\beta_2},s_q\}\}]$;

2) $\Delta^\xi=[s_{\alpha_1},s_{\beta_1}]^\xi=[s_{\alpha_1^\xi},s_{\beta_1^\xi}]$;

3) $(\Delta\otimes\nabla)^\xi=\Delta^\xi\otimes\nabla^\xi$;

4) $\Delta^\xi\otimes\Delta^\zeta=\Delta^{\xi+\zeta}$, $\zeta\in[0,1]$;

5) $\Delta=\nabla$, if $\alpha_1=\alpha_2$, $\beta_1=\beta_2$.

where $s_{\alpha_i},s_{\beta_i}\in S$, $i=1,2$, s_{α_i} and s_{β_i} are the lower and upper limits, respectively.

Additionally, Xu [36] raised the following way to calculate the possibility degree that can distinguish the diverse uncertain multiplicative linguistic variables:

Provided that $\Delta=[s_{\alpha_1},s_{\beta_1}]$ and $\nabla=[s_{\alpha_2},s_{\beta_2}]$ are two any various uncertain multiplicative linguistic variables, the function $I(\cdot)$ satisfies $I(s_{\alpha_1})=\alpha_1$, then the calculation formula of possibility degree of $\Delta\geq\nabla$ is ruled as:

$$p(\Delta\geq\nabla)=\frac{\max\{0,I(s_{\beta_1})-I(s_{\alpha_1})+I(s_{\beta_2})-I(s_{\alpha_2})-\max\{\beta_2-\alpha_1,0\}\}}{I(s_{\beta_1})-I(s_{\alpha_1})+I(s_{\beta_2})-I(s_{\alpha_2})} \tag{1}$$

Then $0\leq p(\Delta\geq\nabla)\leq1$, $p(\Delta\geq\nabla)+p(\nabla\geq\Delta)=1$, $p(\Delta\geq\nabla)=p(\nabla\geq\Delta)=1/2$.

2.2. Probabilistic uncertain linguistic term set

With the combination of the PLTS [21] and uncertain linguistic variable [38], Lin et al. [17] proposed the coming equation to state the PULTS:

$$U(p)=\left\{\left\langle\left[M^{l},N^{l}\right],p^{l}\right\rangle \mid p^{l}\geq 0,l=1,2,\ldots,\#U(p),\sum_{l=1}^{\#U(p)}p^{l}\leq 1\right\} \tag{2}$$

where $\left\langle\left[M^{l},N^{l}\right],p^{l}\right\rangle$ stands for the uncertain linguistic variable $\left[M^{l},N^{l}\right]$ affiliated to its probability p^{l}, M^{l} and N^{l} are the linguistic terms on the additive linguistic term set $S_1=\left\{s_\alpha \middle| \alpha\in[-\tau,\tau]\right\}$, $M^{l}\leq N^{l}$, τ is a non-negative integer [5, 41], and $\#U(p)$ is the cardinality of $U(p)$.

For a PULTS $U(p)=\left\{\left\langle\left[M^{l},N^{l}\right],p^{l}\right\rangle \mid l=1,2,\ldots,\#U(p)\right\}$, if all the elements in $U(p)$ are placed in rising sequence, then we name it an ordered PULTS. Two disparate component elements $\left\langle\left[M^{s},N^{s}\right],p^{s}\right\rangle$ and $\left\langle\left[M^{t},N^{t}\right],p^{t}\right\rangle$ of the PULTS are confronted with the design formula of possibility degree of $\left[p^{s}\times M^{s},p^{s}\times N^{s}\right]$ over $\left[p^{t}\times M^{t},p^{t}\times N^{t}\right]$.

In addition to that, without special directions, all the PULTSs are ordered PULTSs in the following section. Moreover, in the cause of removing the difference in the number of elements between two PULTSs, Lin et al. [17] devised the following steps to standardize the PULTSs as follows:

For two different PULTSs $U_1(p)=\left\{\left\langle\left[M_1^{l},N_1^{l}\right],p_1^{l}\right\rangle \mid l=1,2,\ldots,\#U_1(p)\right\}$ and $U_2(p)=\left\{\left\langle\left[M_2^{l},N_2^{l}\right],p_2^{l}\right\rangle \mid l=1,2,\ldots,\#U_2(p)\right\}$, the standardizing process can be summarized as:

1) If $0<\sum_{l=1}^{\#U_i(p)}p_i^{l}<1$, $i=1,2$, then we standardize the possibility of $U_i(p)$ via $\dot{p}_i^{l}=p_i^{l}/\sum_{l=1}^{\#U_i(p)}p_i^{l}$.

2) If $\#U_1(p)\neq\#U_2(p)$, $\#U_1(p)>\#U_2(p)$, then we add $\#U_1(p)-\#U_2(p)$ uncertain linguistic variables to $U_2(p)$ for the purpose of the elements of the PULTSs $U_1(p)$ and $U_2(p)$ own the identical quantity. The additional elements are the smallest one(s) in $U_2(p)$, and the corresponding probabilities of the additional elements are zero. Moreover, the two disparate component elements $\left\langle\left[M^{i},N^{i}\right],p^{i}\right\rangle$ and

$\langle [M^j, N^j], p^j \rangle$ of the PULTS are confronted with the possibility degree of $[p^i \times M^i, p^i \times N^i]$ over $[p^j \times M^j, p^j \times N^j]$.

In this paper, on account of the mentioned uncertain multiplicative linguistic variables, the PULTSs are extended into the following form:

$$U(p) = \left\{ \langle [M^l, N^l], p^l \rangle \mid p^l \geq 0, l = 1,2,...,\#U(p), M^l \in S \quad and \quad N^l \in S, \sum_{l=1}^{\#U(p)} p^l \leq 1 \right\} \tag{3}$$

which means that the uncertain linguistic variables defined on the multiplicative linguistic label, then we call the extended PULTSs the probabilistic uncertain multiplicative linguistic term sets (PUMLTSs).

Then we define the basic operation for the PUMLTSs as follows:

Let $U_1(p) = \left\{ \langle [M_1^l, N_1^l], p_1^l \rangle \mid l = 1,2,...,\#U_1(p) \right\}$ and $U_2(p) = \left\{ \langle [M_2^l, N_2^l], p_2^l \rangle \mid l = 1,2,...,\#U_2(p) \right\}$ be two normalized and ordered PUMLTSs, $\#U_1(p) = \#U_2(p)$, and the parameter $\wp \geq 0$, then multiplicative operation can be defined as:

$$U_1(p) \otimes U_2(p) = \cup_{\langle [M_1^l, N_1^l], p_1^l \rangle \in U_1(p), \langle [M_2^l, N_2^l], p_2^l \rangle \in U_2(p)} \left\{ \langle [M_1^l, N_1^l] \otimes [M_2^l, N_2^l], p_1^l p_2^l \rangle \right\} \tag{4}$$

$$(U_1(p))^{\wp} = \cup_{\langle [M_1^l, N_1^l], p_1^l \rangle \in U_1(p)} \left\{ \langle [M_1^l, N_1^l]^{\wp}, (p_1^l)^{\wp} \rangle \right\} \tag{5}$$

where

$$\begin{aligned} U_1(p) \otimes U_2(p) &= \cup_{\langle [M_1^l, N_1^l], p_1^l \rangle \in U_1(p), \langle [M_2^l, N_2^l], p_2^l \rangle \in U_2(p)} \left\{ \langle [M_1^l, N_1^l] \otimes [M_2^l, N_2^l], p_1^l p_2^l \rangle \right\} \\ &= \cup_{\langle [M_2^l, N_2^l], p_2^l \rangle \in U_2(p), \langle [M_1^l, N_1^l], p_1^l \rangle \in U_1(p)} \left\{ \langle [M_2^l, N_2^l] \otimes [M_1^l, N_1^l], p_1^l p_2^l \rangle \right\} \\ &= U_2(p) \otimes U_1(p) \end{aligned} \tag{6}$$

$$\begin{aligned} &(U_1(p) \otimes U_2(p))^{\wp} \\ &= \left(\cup_{\langle [M_1^l, N_1^l], p_1^l \rangle \in U_1(p), \langle [M_2^l, N_2^l], p_2^l \rangle \in U_2(p)} \left\{ \langle [M_1^l, N_1^l] \otimes [M_2^l, N_2^l], p_1^l p_2^l \rangle \right\} \right)^{\wp} \\ &= \cup_{\langle [M_1^l, N_1^l], p_1^l \rangle \in U_1(p), \langle [M_2^l, N_2^l], p_2^l \rangle \in U_2(p)} \left\{ \langle [M_1^l, N_1^l]^{\wp}, (p_1^l)^{\wp} \rangle \otimes \langle [M_2^l, N_2^l]^{\wp}, (p_2^l)^{\wp} \rangle \right\} \\ &= (U_1(p))^{\wp} \otimes (U_2(p))^{\wp} \end{aligned} \tag{7}$$

Then, the probabilistic uncertain linguistic weighted geometric (PULWG) operator can be defined as follows:

Given n PUMLTSs $U_i(p)=\left\{\left\langle\left[M_i^l,N_i^l\right],p_i^l\right\rangle \mid l=1,2,...,\#U_i(p)\right\}$, $(i=1,2,\ldots,n)$, the weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$, $\omega_i\in[0,1]$, $\sum_{i=1}^{n}\omega_i=1$, we call

$$\begin{aligned}
&PUWGA(U_1(p),U_2(p),\ldots,U_n(p))=(U_i(p))^{\omega_i}\otimes(U_2(p))^{\omega_2}\otimes\ldots\otimes(U_n(p))^{\omega_n}\\
&=\cup_{\left\langle\left[M_1^l,N_1^l\right],p_1^l\right\rangle\in U_1(p)}\left\{\left\langle\left[M_1^l,N_1^l\right]^{\omega_1},\left(p_1^l\right)^{\omega_1}\right\rangle\right\}\otimes\cup_{\left\langle\left[M_2^l,N_2^l\right],p_2^l\right\rangle\in U_2(p)}\left\{\left\langle\left[M_2^l,N_2^l\right]^{\omega_2},\left(p_2^l\right)^{\omega_2}\right\rangle\right\}\\
&\otimes\cdots\otimes\cup_{\left\langle\left[M_n^l,N_n^l\right],p_n^l\right\rangle\in U_n(p)}\left\{\left\langle\left[M_n^l,N_n^l\right]^{\omega_n},\left(p_n^l\right)^{\omega_n}\right\rangle\right\}
\end{aligned}\tag{8}$$

probabilistic uncertain linguistic weighted geometric averaging (PULWGA) operator.

3. Probabilistic uncertain multiplicative linguistic preference relations

After the introduction of the PUMLTSs, in this section, on behalf of putting better the PUMLTSs into use, the possibility degree [13, 9, 29, 1] between the PUMLTSs is first introduced. Moreover, the PUMLPRs are also introduced. For the possibility degree, it can realize the comparison of the two PULTSs, while the PUMLPRs can realize the comparisons of the considered objects.

3.1. Possibility degree between PULTSs

With regard to two random PUMLTSs $U_1(p)=\left\{\left\langle\left[M_1^l,N_1^l\right],p_1^l\right\rangle \mid l=1,2,...,\#U_1(p)\right\}$ and $U_2(p)=\left\{\left\langle\left[M_2^l,N_2^l\right],p_2^l\right\rangle \mid l=1,2,...,\#U_2(p)\right\}$, the possibility degree $p(U_1(p)\geq U_2(p))$ is ruled as:

$$p(U_1(p)\geq U_2(p))=\frac{1}{\#U_1(p)}\sum_{l=1}^{\#U_1(p)}p\left(\left\langle\left[M_1^l,N_1^l\right],p_1^l\right\rangle\geq\left\langle\left[M_2^l,N_2^l\right],p_2^l\right\rangle\right)\tag{9}$$

where

$$p\left(\left\langle\left[M_1^l,N_1^l\right],p_1^l\right\rangle\geq\left\langle\left[M_2^l,N_2^l\right],p_2^l\right\rangle\right)=p\left(p_1^l\left[M_1^l,N_1^l\right]\geq p_2^l\left[M_2^l,N_2^l\right]\right)\tag{10}$$

Moreover, if $p(U_1(p)\geq U_2(p))>0.5$, then $U_1(p)\succ U_2(p)$; if $p(U_1(p)\geq U_2(p))=0.5$, then $U_1(p)\sim U_2(p)$; if $p(U_1(p)\geq U_2(p))<0.5$, $U_1(p)\prec U_2(p)$; if $p(U_1(p)\geq U_2(p))=1$, then $U_1(p)$ is absolutely superior to $U_2(p)$; if $p(U_1(p)\geq U_2(p))=0$, then $U_2(p)$ is absolutely superior to $U_1(p)$.

Then we utilize an concrete example to display the calculating procedure to gain the possibility degree.

Example 1. For two diverse PUMLTSs $U_1(p)=\{\langle[s_3,s_4],0.1\rangle,\langle[s_2,s_3],0.2\rangle,\langle[s_4,s_5],0.7\rangle\}$ and $U_2(p)=\{\langle[s_1,s_2],0\rangle,\langle[s_1,s_2],0.6\rangle,\langle[s_2,s_3],0.4\rangle\}$, then according to Eq. (9), the possibility degree of $p(U_1(p)\geq U_2(p))$ can be calculated as follows:

$$
\begin{aligned}
p(U_1(p)\geq U_2(p)) &= \frac{1}{3}\begin{pmatrix} p(\langle[s_3,s_4](0.1)\rangle\geq\langle[s_1,s_2](0)\rangle)+p(\langle[s_2,s_3](0.2)\rangle\geq\langle[s_1,s_2](0.6)\rangle) \\ +p(\langle[s_4,s_5](0.7)\rangle\geq\langle[s_2,s_3](0.4)\rangle) \end{pmatrix} \\
&= \frac{1}{3}\left(p(0.1[s_3,s_4]\geq 0[s_1,s_2])+p(0.2[s_2,s_3]\geq 0.6[s_1,s_2])+p(0.7[s_4,s_5]\geq 0.4[s_2,s_3])\right) \\
&= \frac{1}{3}\left(p([s_{0.3},s_{0.4}]\geq[s_0,s_0])+p([s_{0.4},s_{0.6}]\geq[s_{0.6},s_{1.2}])+p([s_{2.8},s_{3.5}]\geq[s_{0.8},s_{1.2}])\right) \\
&= \frac{1}{3}(1+0+1)=0.6667>0.5
\end{aligned}
$$

then $U_1(p)\succ U_2(p)$.

3.2. PUMLPRs

Assume that there are n alternatives $A=\{a_1,a_2,\ldots,a_n\}$, t DMs $E=\{e_1,e_2,\ldots,e_t\}$, and the weight vector of the DMs is $\pi=(\pi_1,\pi_2,\ldots,\pi_t)$, $\pi_\kappa\in[0,1]$, $\sum_{\kappa=1}^{t}\pi_\kappa=1$. For the decision maker (DM) e_κ, $\kappa=1,2,\ldots,t$, he/she displays his/her preference information of the alternative a_i over the alternative a_j through the PUMLPR $U_\kappa=(U_{ij}^{\kappa}(p))_{n\times n}$, $i,j=1,2,\ldots,n$. The PUMLPR can be ruled as follows:

A PUMLPR U is a matrix $U=(U_{ij}(p))_{n\times n}$, where $i,j=1,2,\ldots,n$, $U_{ij}(p)=(U_{ij}^{l}(p_{ij}^{l}))=\left\{\langle[M_{ij}^{l},N_{ij}^{l}],p_{ij}^{l}\rangle \middle| p_{ij}^{l}\geq 0, l=1,2,\ldots,\#U_{ij}(p), \sum_{l=1}^{\#U_{ij}(p)}p_{ij}^{l}\leq 1\right\}$ are the PUMLTSs on the mentioned set $S=\{s_\alpha|\alpha\in[1/q,q]\}$, $\#U_{ij}(p)$ is the quantity of the uncertain multiplicative linguistic variables in $U_{ij}(p)$, $U_{ij}(p)$ reveals the preference degree of the alternative a_i over the alternative a_j, and fulfills the following qualifications:

$$M_{ij}^{l}\otimes N_{ji}^{l}=M_{ji}^{l}\otimes N_{ij}^{l}=s_1,\ p_{ij}^{l}=p_{ji}^{l},\ M_{ii}^{l}=N_{ii}^{l}=s_1,\ \#U_{ij}(p)=\#U_{ji}(p) \tag{11}$$

and $U_{ij}^{l+1}\left(p_{ij}^{l+1}\right)\succ U_{ij}^{l}\left(p_{ij}^{l}\right)$, $U_{ji}^{l}\left(p_{ji}^{l}\right)\prec U_{ji}^{l+1}\left(p_{ji}^{l+1}\right)$, $U_{ij}^{l}\left(p_{ij}^{l}\right)$ is the lth component element of the PULTS $U_{ij}(p)$.

Resembled the NPULTSs, next, the normalized PUMLPR (NPUMLPR) is ruled as follows:

As for the PUMLPR $U=\left(U_{ij}(p)\right)_{n\times n}$, if all the component elements of the PUMLTSs in the upper triangular matrix are sorted in upward sequence and all the PUMLTSs in the upper triangular matrix are normalized in conformity to NPUMLTSs. Then, we name the PUMLPR $U=\left(U_{ij}(p)\right)_{n\times n}$ the NPUMLPR, denoted as $U^{N}=\left(U_{ij}^{N}(p)\right)_{n\times n}$.

Example 2. Let $S=\left\{s_{\alpha}\,\middle|\,\alpha\in[1/9,9]\right\}$ be a previously-mentioned multiplicative set. A PUMLPR $U=\left(U_{ij}(p)\right)_{n\times n}$ can be shown below:

$$U=\begin{pmatrix} [s_1,s_1] & \{\langle[s_3,s_4],0.1\rangle,\langle[s_4,s_5],0.7\rangle\} & \{\langle[s_4,s_5],0.5\rangle,\langle[s_5,s_6],0.5\rangle\} & \{\langle[s_3,s_4],0.2\rangle,\langle[s_4,s_5],0.8\rangle\} \\ \{\langle[s_{1/4},s_{1/3}],0.1\rangle,\langle[s_{1/5},s_{1/4}],0.7\rangle\} & [s_1,s_1] & \{\langle[s_4,s_7],0.6\rangle\} & \{\langle[s_1,s_2],0.6\rangle,\langle[s_2,s_3],0.4\rangle\} \\ \{\langle[s_{1/5},s_{1/4}],0.5\rangle,\langle[s_{1/6},s_{1/5}],0.5\rangle\} & \{\langle[s_{1/7},s_{1/4}],0.6\rangle\} & [s_1,s_1] & \{\langle[s_8,s_9],0.3\rangle\} \\ \{\langle[s_{1/4},s_{1/3}],0.2\rangle,\langle[s_{1/5},s_{1/4}],0.8\rangle\} & \{\langle[s_{1/2},s_1],0.6\rangle,\langle[s_{1/3},s_{1/2}],0.4\rangle\} & \{\langle[s_{1/9},s_{1/8}],0.3\rangle\} & [s_1,s_1] \end{pmatrix}$$

According to the NPUMLPR, all the component elements of the PUMLTSs in the upper triangular matrix are sorted in upward sequence and all the PUMLTSs in the upper triangular matrix are normalized. Additionally, on account of the characters $p_{ij}^{l}=p_{ji}^{l}$ and $U_{ij}^{l}=rec\left(U_{ji}^{l}\right)$ of the PUMLPR, all the PUMLTSs in the lower triangular matrix can be acquired lightly. Then its NPUMLPR is:

$$U^N=\begin{pmatrix} [s_1,s_1] & \{\langle[s_3,s_4],0.125\rangle,\langle[s_4,s_5],0.875\rangle\} & \{\langle[s_4,s_5],0.5\rangle,\langle[s_5,s_6],0.5\rangle\} & \{\langle[s_3,s_4],0.2\rangle,\langle[s_4,s_5],0.8\rangle\} \\ \{\langle[s_{1/4},s_{1/3}],0.125\rangle,\langle[s_{1/5},s_{1/4}],0.875\rangle\} & [s_1,s_1] & \{\langle[s_4,s_7],0\rangle,\langle[s_4,s_7],1\rangle\} & \{\langle[s_1,s_2],0.6\rangle,\langle[s_2,s_3],0.4\rangle\} \\ \{\langle[s_{1/5},s_{1/4}],0.5\rangle,\langle[s_{1/6},s_{1/5}],0.5\rangle\} & \{\langle[s_{1/7},s_{1/4}],0\rangle,\langle[s_{1/7},s_{1/4}],1\rangle\} & [s_1,s_1] & \{\langle[s_8,s_9],0\rangle,\langle[s_8,s_9],1\rangle\} \\ \{\langle[s_{1/4},s_{1/3}],0.2\rangle,\langle[s_{1/5},s_{1/4}],0.8\rangle\} & \{\langle[s_{1/2},s_1],0.6\rangle,\langle[s_{1/3},s_{1/2}],0.4\rangle\} & \{\langle[s_{1/9},s_{1/8}],0\rangle,[s_{1/9},s_{1/8}],1\} & [s_1,s_1] \end{pmatrix}$$

3.3. Incomplete probabilistic uncertain multiplicative linguistic preference relations

Actually, under the practical decision-making procedure, due to a variety of reasons, the decision-making information is not always complete [4, 25, 3, 16, 12, 22]. For example, the DMs are not familiar enough with the question that they need to evaluate, so they could not provide the exact information that is the uncertain multiplicative linguistic variables. Hence, the corresponding probabilities of the uncertain multiplicative linguistic variables cannot be provided, either. In order to show this possible situation more clearly, next we use a definition to illustrate in detail.

Definition 1. For a PUMLPR $U=\left(U_{ij}(p)\right)_{n\times n}$, where $i,j=1,2,\dots,n$, $U_{ij}(p)=\left(U_{ij}^l\left(p_{ij}^l\right)\right)=\left\{\left\langle\left[M_{ij}^l,N_{ij}^l\right],p_{ij}^l\right\rangle\middle|p_{ij}^l\geq 0,l=1,2,\dots,\#U_{ij}(p),\sum_{l=1}^{\#U_{ij}(p)}p_{ij}^l\leq 1\right\}$, then U is called an incomplete probabilistic uncertain multiplicative linguistic preference relation (IPUMLPR), if some of its uncertain multiplicative linguistic variables and its corresponding probabilities cannot be provided by the DMs, which we signify it by the unknown variable "$\left[M_{x_l},N_{y_l}\right],z_{ij}^l$", and the others specified by the DMs satisfy

$$M_{ij}^l\otimes N_{ji}^l=M_{ji}^l\otimes N_{ij}^l=s_1,\quad p_{ij}^l=p_{ji}^l,\quad M_{ii}^l=N_{ii}^l=s_1,\quad \#U_{ij}(p)=\#U_{ji}(p) \tag{12}$$

$U_{ij}^{l+1}\left(p_{ij}^{l+1}\right) \succ U_{ij}^{l}\left(p_{ij}^{l}\right)$, $U_{ji}^{l}\left(p_{ji}^{l}\right) \prec U_{ji}^{l+1}\left(p_{ji}^{l+1}\right)$, $U_{ij}^{l}\left(p_{ij}^{l}\right)$ is the lth element of the PUMLTS $U_{ij}(p)$, where $U_{ij}(p) \in \Omega_U$, and Ω_U is the set of all the known elements in U.

Example 3. Let U be an IPUMLPR as shown below:

$$U = \begin{pmatrix} [s_1, s_1] & \left\{\left\langle\left[s_{x_{12}^1}, s_{y_{12}^1}\right], z_{12}^1\right\rangle, \left\langle\left[s_{x_{12}^2}, s_{y_{12}^2}\right], z_{12}^2\right\rangle\right\} & \left\{\left\langle[s_4, s_5], 0.5\right\rangle, \left\langle[s_5, s_6], 0.5\right\rangle\right\} & \left\{\left\langle\left[s_{x_{14}^1}, s_{y_{14}^1}\right], z_{14}^1\right\rangle, \left\langle\left[s_{x_{14}^2}, s_{y_{14}^2}\right], z_{14}^2\right\rangle\right\} \\ \left\{\left\langle\left[s_{1/y_{21}^1}, s_{1/x_{21}^1}\right], z_{21}^1\right\rangle, \left\langle\left[s_{1/y_{21}^2}, s_{1/x_{21}^2}\right], z_{21}^2\right\rangle\right\} & [s_1, s_1] & \left\{\left\langle[s_4, s_7], 0.6\right\rangle\right\} & \left\{\left\langle[s_1, s_2], 0.6\right\rangle, \left\langle[s_2, s_3], 0.4\right\rangle\right\} \\ \left\{\left\langle\left[s_{1/5}, s_{1/4}\right], 0.5\right\rangle, \left\langle\left[s_{1/6}, s_{1/5}\right], 0.5\right\rangle\right\} & \left\{\left\langle\left[s_{1/7}, s_{1/4}\right], 0.6\right\rangle\right\} & [s_1, s_1] & \left\{\left\langle[s_8, s_9], 0.3\right\rangle\right\} \\ \left\{\left\langle\left[s_{1/y_{41}^1}, s_{1/x_{41}^1}\right], z_{41}^1\right\rangle, \left\langle\left[s_{1/y_{41}^2}, s_{1/x_{41}^2}\right], z_{41}^2\right\rangle\right\} & \left\{\left\langle\left[s_{1/2}, s_1\right], 0.6\right\rangle, \left\langle\left[s_{1/3}, s_{1/2}\right], 0.4\right\rangle\right\} & \left\{\left\langle\left[s_{1/9}, s_{1/8}\right], 0.3\right\rangle\right\} & [s_1, s_1] \end{pmatrix}$$

Then, similar to the normalization of the complete PUMLPRs, we can get a normalized IPUMLPR as:

$$U^N = \begin{pmatrix} [s_1, s_1] & \left\{\left\langle\left[s_{x_{12}^1}, s_{y_{12}^1}\right], z_{12}^1\right\rangle, \left\langle\left[s_{x_{12}^2}, s_{y_{12}^2}\right], z_{12}^2\right\rangle\right\} & \left\{\left\langle[s_4, s_5], 0.5\right\rangle, \left\langle[s_5, s_6], 0.5\right\rangle\right\} & \left\{\left\langle\left[s_{x_{14}^1}, s_{y_{14}^1}\right], z_{14}^1\right\rangle, \left\langle\left[s_{x_{14}^2}, s_{y_{14}^2}\right], z_{14}^2\right\rangle\right\} \\ \left\{\left\langle\left[s_{1/y_{21}^1}, s_{1/x_{21}^1}\right], z_{21}^1\right\rangle, \left\langle\left[s_{1/y_{21}^2}, s_{1/x_{21}^2}\right], z_{21}^2\right\rangle\right\} & [s_1, s_1] & \left\{\left\langle[s_4, s_7], 0\right\rangle, \left\langle[s_4, s_7], 1\right\rangle\right\} & \left\{\left\langle[s_1, s_2], 0.6\right\rangle, \left\langle[s_2, s_3], 0.4\right\rangle\right\} \\ \left\{\left\langle\left[s_{1/5}, s_{1/4}\right], 0.5\right\rangle, \left\langle\left[s_{1/6}, s_{1/5}\right], 0.5\right\rangle\right\} & \left\{\left\langle\left[s_{1/7}, s_{1/4}\right], 0\right\rangle, \left\langle\left[s_{1/7}, s_{1/4}\right], 1\right\rangle\right\} & [s_1, s_1] & \left\{\left\langle[s_8, s_9], 0\right\rangle, \left\langle[s_8, s_9], 1\right\rangle\right\} \\ \left\{\left\langle\left[s_{1/y_{41}^1}, s_{1/x_{41}^1}\right], z_{41}^1\right\rangle, \left\langle\left[s_{1/y_{41}^2}, s_{1/x_{41}^2}\right], z_{41}^2\right\rangle\right\} & \left\{\left\langle\left[s_{1/2}, s_1\right], 0.6\right\rangle, \left\langle\left[s_{1/3}, s_{1/2}\right], 0.4\right\rangle\right\} & \left\{\left\langle\left[s_{1/9}, s_{1/8}\right], 0\right\rangle, \left[s_{1/9}, s_{1/8}\right], 1\right\} & [s_1, s_1] \end{pmatrix}$$

4. The repairing for probabilistic uncertain incomplete multiplicative linguistic preference relations

The suitable decision-making needs to be made on the foundation of the complete PRs. Therefore, we need to make up for all the incomplete PRs, study the consistency of the complete PRs, and make a final appropriate decision-making. This section proposes the repairing methods for incomplete PUMLRs. Please see the following section for the details.

4.1. The specific repairing process for IPUMLPRs

Definition 2. Let U be an IPUMLPR and $U^N=\left(U_{ij}^N\left(p\right)\right)_{n\times n}$ be the corresponding NIPUMLPR, if $\left(i,j\right)\cap\left(s,t\right)\neq\varnothing$, then we call that the elements $U_{ij}^N\left(p\right)$ and $U_{st}^N\left(p\right)$ are adjacent. For a missing element $U_{ij}^N\left(p\right)$, if there have two conterminal known elements $U_{is}^N\left(p\right)$ and $U_{sj}^N\left(p\right)$, then we call that $U_{ij}^N\left(p\right)$ is available.

For this kind of incomplete PRs, considering its complexity of the structure of PUMLPRs in essence, the repairing process is divided into two steps. The first one is to repair the incomplete uncertain multiplicative linguistic variables. Moreover, for the sake of acquiring the ultima decision-making consequence, similar to most references, consistency is the essential presupposition for repairing the incomplete PRs. Therefore, without loss of generality, by considering the consistency of the incomplete PRs, we repair the incomplete PRs by following procedures:

For the incomplete uncertain multiplicative linguistic variables, by virtue of Ref. [34], we use the similar way to study the consistency of the uncertain multiplicative linguistic preference relations [39] (UMLPRs) to repair the incomplete portion of the PRs.

For a UMLPR $\hat{U}=\left(u_{ij}\right)_{n\times n}$, $u_{ij}=\left[u_{ij}^L,u_{ij}^R\right]$, $i,j=1,2,\ldots,n$, satisfies that $s_{1/q}\le u_{ij}^L\le u_{ij}^R\le s_q$, $u_{ij}^L\otimes u_{ji}^R=u_{ij}^R\otimes u_{ji}^L=s_1$, $u_{ii}^L=u_{ii}^R=s_1$. $\hat{U}$ is called a consistent UMLPR, if two multiplicative linguistic preference relations (MLPRs) $Q=\left(q_{ij}\right)_{n\times n}$ and $\Re=\left(r_{ij}\right)_{n\times n}$ are consistent, that is

$$q_{ij}=q_{ik}\otimes q_{kj},\ \ r_{ij}=r_{ik}\otimes r_{kj},\qquad \forall i,j,k=1,2,\ldots,n,\ \ q_{ij},r_{ij}\in\left[s_{1/q},s_q\right] \tag{13}$$

and

$$q_{ij}=\begin{cases}u_{ij}^{R}, & i<j\\ s_1, & i=j\\ u_{ij}^{L}, & i>j\end{cases},\quad r_{ij}=\begin{cases}u_{ij}^{L}, & i<j\\ s_1, & i=j\\ u_{ij}^{R}, & i>j\end{cases}.$$

Moreover, we propose the coming goal-programming model to repair the IPUMLPR that is to repair the UMLPR:

$$\begin{aligned}\varepsilon_{ijq}^{k}&=\left|I\left(q_{ij}\right)-I\left(q_{ik}\right)\times I\left(q_{kj}\right)\right|\\ \varepsilon_{ijr}^{k}&=\left|I\left(r_{ij}\right)-I\left(r_{ik}\right)\times I\left(r_{kj}\right)\right|\end{aligned}\tag{14}$$

Then a multi-objective programming model can be expressed below:

$$\begin{aligned}&\min\quad \varepsilon=\sum_{i,j\in\Theta_U}\sum_{k\neq i,j}\left(\varepsilon_{ijq}^{k}+\varepsilon_{ijr}^{k}\right)\\ &s.t.\quad\begin{cases}1/q\leq I\left(r_{ij}\right)\leq I\left(q_{ij}\right)\leq q, & i<j,\\ 1/q\leq I\left(q_{ij}\right)\leq I\left(r_{ij}\right)\leq q, & i>j.\end{cases}\end{aligned}\tag{15}$$

where Θ_U indicates the unknown elements in the IPUMLPRs. Then the missing ingredient that the incomplete uncertain multiplicative linguistic variables can be repaired completely by the model (15).

While for the sake of calculating the loss of the probability, next we define the geometric value of the PUMLTs as follows:

If $U(p)=\left\{\left\langle\left[M^{l},N^{l}\right],p^{l}\right\rangle\mid l=1,2,...,\#U(p)\right\}$ is a PUMLTS, then its geometric value is:

$$G\left(U(p)\right)=\overset{\#U(p)}{\underset{l=1}{\otimes}}\left[M^{l},N^{l}\right]^{p^{l}}=g\tag{16}$$

Example 4. For a PUMLTS $U(p)=\left\{\left\langle[s_1,s_2],0.6\right\rangle,\left\langle[s_2,s_3],0.4\right\rangle\right\}$, its geometric value can be calculated as:

$$G\left(U(p)\right)=\left[s_1,s_2\right]^{0.6}\otimes\left[s_2,s_3\right]^{0.4}=\left[s_{1^{0.6}},s_{2^{0.6}}\right]\otimes\left[s_{2^{0.4}},s_{3^{0.4}}\right]=\left[s_{1^{0.6}\times2^{0.4}},s_{2^{0.6}\times3^{0.4}}\right]$$

Then for a PUMLPR $U=\left(U_{ij}\left(p\right)\right)_{n\times n}$, where $i,j=1,2,\ldots,n$, $U_{ij}\left(p\right)=\left(U_{ij}^{l}\left(p_{ij}^{l}\right)\right)=\left\{\left\langle\left[M_{ij}^{l},N_{ij}^{l}\right],p_{ij}^{l}\right\rangle\middle|p_{ij}^{l}\geq 0,l=1,2,\ldots,\#U_{ij}\left(p\right),\sum_{l=1}^{\#U_{ij}(p)}p_{ij}^{l}\leq 1\right\}$, a geometric preference relation with respect to U is defined as $G_U=\left(g_{ij}\right)_{n\times n}$, where $g_{ij}=\overset{\#U(p)}{\underset{l=1}{\otimes}}U_{ij}^{l\left(p_{ij}^{l}\right)}$.

Moreover, it is obvious to see that the geometric preference relation is a UMLPR, where

$$g_{ij}\otimes g_{ji}=\overset{\#U(p)}{\underset{l=1}{\otimes}}U_{ij}^{l\left(p_{ij}^{l}\right)}\otimes\overset{\#U(p)}{\underset{l=1}{\otimes}}U_{ji}^{l\left(p_{ij}^{l}\right)}=\overset{\#U(p)}{\underset{l=1}{\otimes}}\left(U_{ij}^{l\left(p_{ij}^{l}\right)}\otimes U_{ji}^{l\left(p_{ij}^{l}\right)}\right)=\overset{\#U(p)}{\underset{l=1}{\otimes}}\left(\left(U_{ij}^{l}\otimes U_{ji}^{l}\right)^{\left(p_{ij}^{l}\right)}\right)=\overset{\#U(p)}{\underset{l=1}{\otimes}}\left(\left[s_1,s_1\right]^{\left(p_{ij}^{l}\right)}\right)=\left[s_1,s_1\right]$$

and $g_{ii}=\overset{\#U(p)}{\underset{l=1}{\otimes}}U_{ii}^{l\left(p_{ij}^{l}\right)}=\left[s_1,s_1\right]$.

On account of Refers. [46, 6] and the defined geometric value, we study the geometric consistency of the PUMLPRs as follows:

If $G_U=\left(g_{ij}\right)_{n\times n}=\left(g_{ij}^{L},g_{ij}^{R}\right)_{n\times n}$ is a UMLPR, and there exists a weight vector $\omega=\left(\omega_1,\omega_2,\ldots,\omega_n\right)^T$, such that $I\left(g_{ij}^{L}\right)\leq\frac{\omega_i}{\omega_j}\leq I\left(g_{ij}^{R}\right)$, for all $i,j=1,2,\ldots,n$, then we say that G_U is a consistent UMLPR.

Analogously, based on the premise of consistency, though considering the relationships of weights and the elements in PRs, we utilize the following linear programming models to figure out the loss probabilities:

$$J=\min\sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_{ij}^{L}+d_{ij}^{R}\right)$$

$$s.t.\begin{cases}\omega_i\geq I\left(g_{ij}^{L}\right)\omega_j-d_{ij}^{L}, & i,j=1,2,\ldots,n-1,\quad j=1,2,\ldots,n;\\ \omega_i\leq I\left(g_{ij}^{R}\right)\omega_j+d_{ij}^{R}, & i=1,2,\ldots,n-1,\quad j=1,2,\ldots,n;\\ \sum_{i=1}^{n}\omega_i=1,\quad\omega_i\geq 0; & \\ \sum_{k=1}^{\#S(p)}p_{ij}^{k}=1,\quad p_{ij}^{k}\in\left[0,1\right]; & \\ d_{ij}^{L},\quad d_{ij}^{R}\geq 0,\quad i=1,2,\ldots,n-1,\quad j=1,2,\ldots,n. & \end{cases}\tag{17}$$

Moreover, Eq. (17) can be overwritten as the coming manner:

$$J=\min\sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_{ij}^{L}+d_{ij}^{R}\right)$$

$$
s.t.\begin{cases}
\omega_i \geq I\left(\left\lceil \overset{\#U(p)}{\underset{l=1}{\otimes}} U_{ij}^{l\left(p_{ij}^{l}\right)} \right\rceil\right)\omega_j - d_{ij}^{L}, \quad i=1,2,\ldots,n-1, \quad j=1,2,\ldots,n; \\
\omega_i \leq I\left(\left\lfloor \overset{\#U(p)}{\underset{l=1}{\otimes}} U_{ij}^{l\left(p_{ij}^{l}\right)} \right\rfloor\right)\omega_j + d_{ij}^{R}, \quad i=1,2,\ldots,n-1, \quad j=1,2,\ldots,n; \\
\sum_{i=1}^{n} \omega_i = 1, \quad \omega_i \geq 0; \\
\sum_{l=1}^{\#U(p)} p_{ij}^{l} = 1, \quad p_{ij}^{l} \in [0,1]; \\
d_{ij}^{L}, \quad d_{ij}^{R} \geq 0, \quad i=1,2,\ldots,n-1, \quad j=1,2,\ldots,n.
\end{cases} \tag{18}
$$

Remark 1. $\lceil \bullet \rceil$ is a function that obtains the lower limits of the uncertain multiplicative linguistic variable, while $\lfloor \bullet \rfloor$ is a function that obtain the upper limits of the uncertain multiplicative linguistic variable. For example, $\left\lceil [s_\alpha, s_\beta] \right\rceil = s_\alpha$, $\left\lfloor [s_\alpha, s_\beta] \right\rfloor = s_\beta$.

Obviously, by figuring out the model (18), we can acquire the loss probabilities and repair the IPUMLPR as PUMLPR.

Next, we utilize a straightforward algorithm to introduce the detailed process to repair the IPUMLPRs as follows:

Algorithm 1

Input: Individual IPUMLPR U

Output: The complete PUMLPR

Step 1. Base on Eq. (15), repair the loss of the uncertain multiplicative linguistic variables in the IPUMLPR.

Step 2. In the light of Eq. (18), figure out the loss of probabilities with respect to the IPUMLPR.

Step 3. Obtain the complete PUMLPR.

Step 4. End.

4.2. The consistency for PUMLPRs

When the elements are compared in pairs by experts, it is difficult to avoid inconsistencies, so it is necessary to detect the consistency of the PRs. Moreover, consistency is the primary requirement for acquiring the logical decision-making consequence. For Subsection 4.1, after obtaining the complete PUMPLPRs, we still need to check the consistency of the PUMLPRs. Just like the introduction of Subsection 4.1., the geometric preference relations obtained from the PUMLPRs are the UMLPRs.

Then, the consistent UMLPR $G=\left(g_{ij}\right)_{n\times n}$ should satisfy $I\left(g_{ij}^{L}\right)\leq\frac{\omega_i}{\omega_j}\leq I\left(g_{ij}^{R}\right)$, where $g_{ij}=\left[g_{ij}^{L},g_{ij}^{R}\right]$, $i,j=1,2,\ldots,n$, $g_{ij}=\overset{\#U(p)}{\underset{l=1}{\otimes}}U_{ij}^{l\left(p_{ij}^{l}\right)}$, then we build the coming model:

$$J=\min\sum_{i=1}^{n}\sum_{j=1}^{n}\left(d_{ij}^{L}+d_{ij}^{R}\right)$$

$$s.t.\begin{cases}\omega_i\geq I\left(\left[\overset{\#U(p)}{\underset{l=1}{\otimes}}U_{ij}^{l\left(p_{ij}^{l}\right)}\right]\right)\omega_j-d_{ij}^{L}, & i=1,2,\ldots,n-1,\quad j=1,2,\ldots,n;\\ \omega_i\leq I\left(\left[\overset{\#U(p)}{\underset{l=1}{\otimes}}U_{ij}^{l\left(p_{ij}^{l}\right)}\right]\right)\omega_j+d_{ij}^{R}, & i=1,2,\ldots,n-1,\quad j=1,2,\ldots,n;\\ \sum_{i=1}^{n}\omega_i=1,\quad \omega_i\geq 0;\\ \sum_{l=1}^{\#U(p)}p_{ij}^{l}=1,\quad p_{ij}^{l}\in[0,1];\\ d_{ij}^{L},\quad d_{ij}^{R}\geq 0,\quad i=1,2,\ldots,n-1,\quad j=1,2,\ldots,n.\end{cases}\tag{19}$$

If $J=0$, then the UMLPR is consistent, thus we can get the optimal deviation values d_{ij}^{L} and d_{ij}^{R}, $i=1,2,\ldots,n-1,\quad j=1,2,\ldots,n$. Then, we let $\tilde{g}_{ij}=\left[\tilde{g}_{ij}^{L},\tilde{g}_{ij}^{R}\right]$ be the improved UMLPR $\tilde{G}=\left(\tilde{g}_{ij}\right)_{n\times n}$, where

$$\tilde{g}_{ij}^{L}=I^{-1}\left(I\left(g_{ij}^{L}\right)-d_{ij}^{L}\right),\quad \tilde{g}_{ij}^{R}=I^{-1}\left(I\left(g_{ij}^{R}\right)+d_{ij}^{R}\right).$$

Obviously, the improved UMLPR $\tilde{G}=\left(\tilde{g}_{ij}\right)_{n\times n}$ is a consistent UMLPR.

Moreover, based on Eq. (8), we utilize the PULWGA operator to aggregate all the consistent

PUMLPRs into the group PUMLPR. Then with Eq. (9), we are able to earn a possibility degree matrix $P=\left(p_{ij}\right)_{n\times n}$. Furthermore, in terms of Ref. [40], the priority vector can be calculated by the coming equation:

$$\widehat{\omega}_i = \frac{1}{n(n-1)}\left(\sum_{j=1}^{n} p_{ij} + \frac{n}{2} - 1\right) \tag{20}$$

Then the assorting sequence of the alternatives can be resolved, too. So far, the whole decision-making process is over.

4.3. The whole group decision-making procedure for PUIMLPRs

The whole group decision-making process based upon the PUIMLPRs can be shown by the following figure:

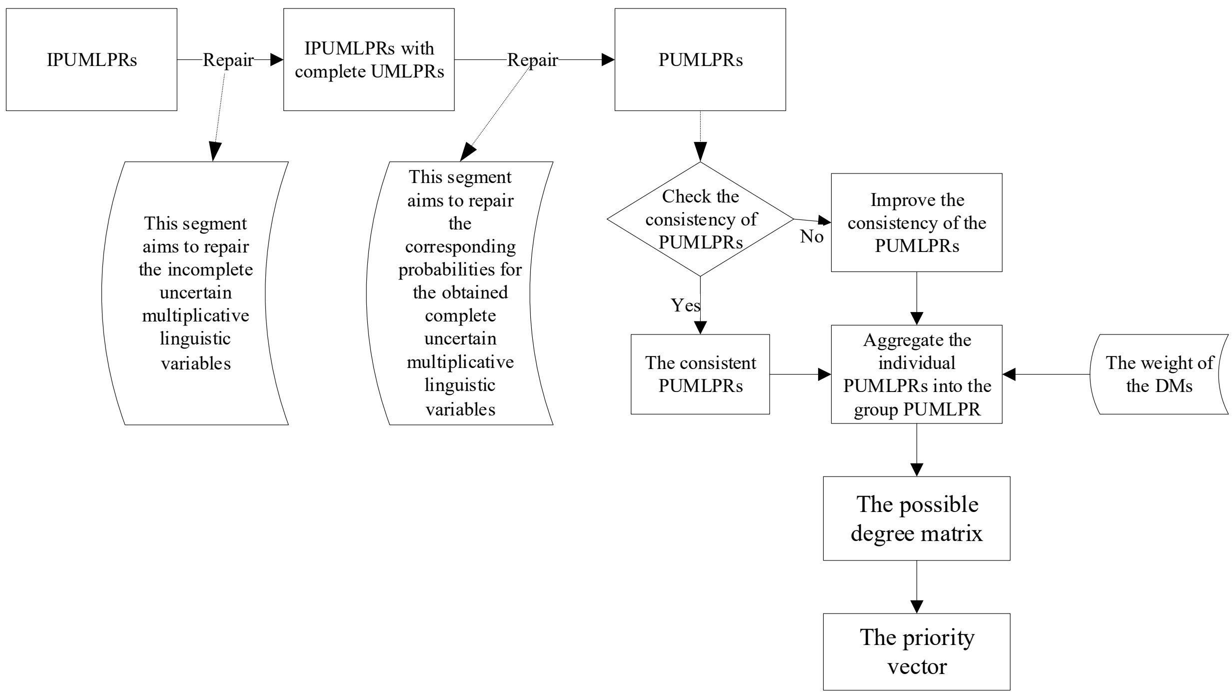

Figure 1. The whole group decision-making procedure based upon the IPUMLPRs

5. Case study

As noted above, the paper aims to evaluate the way of online public opinion management under big data. Therefore, in this section, we recommend briefly the way of online public opinion management under big data. Moreover, considering the superiority of the PUMLPRs, the DMs choose it as the instrument to evaluate the management way. In addition to that, the DMs are not always familiar with all the things that need to evaluate, so the incomplete PUMLPRs appear in the decision-making process. Hence, in this section, we state a decision-making procedure to demonstrate the specific repairing process for incomplete PUMLPRs.

5.1. The application to the public opinion management under big data

In the background of big data, the online public opinion takes on the following new features: (1) The cycle of events that are concerned is shortened. In the area of big data, with the proliferation of online information, it is difficult for people to focus their attention on a specific event for a long time. Onlookers will disband immediately after engaging in clustering emotional attention and evaluation, and they will switch to other hot spots, so they cannot form a rational consensus. Such a time period is not conducive to the resolution of public opinion events. (2) The types of the events are complex. In an open environment, the network information is complicated, and the data forms and sources are also diverse. It covers different types of data, such as text, audio, pictures, and video, which makes the network public information more complicated. (3) An extreme volume of data makes the online public opinion more prosperous. People use various new media tools to express their opinions on the internet, expound their opinions, and make many social topics become the focus of street talks for a period of time.

Big data proposes new requirements for network public opinion management. The first is to shift from focusing on individual cases to overall control. Traditional online public opinion management focuses on the management of major public opinion cases, and big data can better grasp the overall situation of network public opinion development. The second is the transition from passive response to active prediction. The core of big data is forecasting. Through analysis in massive data, we find the subtle relationships hidden behind, so as to predict future trends and deploy preventive responses in advance. The third is the transition from qualitative management to quantitative management. All relevant information, including netizen comments, emotional changes, social relations, etc., are quantified into standard data for calculation and analysis, and calculated by data model to analyze the situation and trend of public opinion.

Then on this kind of online public opinion, the management can be divided into the following four aspects:

1) Social means, online public opinion is essentially the embodiment of social situation and public opinion. Strengthening online public opinion management means strengthening social governance. It is necessary to use the powerful "correlation analysis" capability of big data to build a "cube" of online public opinion data, integrate all aspects of online and offline data, analyze and explore the deep relationship behind network public opinion and social dynamics, and realize network public opinion management and close coordination of social governance and simultaneous advancement.

2) Governmental means, combine big data and online government information openly and tightly to enhance the government's credibility. At present, the US government has established a unified data opening portal, and provides an interface for the community to develop applications to use the data of various departments. This action impels the disclosure of governmental affairs disclosure from the "information level" to the "data level", opening up the new path of the disclosure of governmental information.

3) Technological means, construct a data processing integration entity for public opinion information, and forming an exchange and interaction mode of information data between various branches of the public opinion industry. Public opinion services and research objects are extensive, and it is inevitable to process and analyze information data across industries and regions. The database distributed on the network and its affiliated units are equivalent to a unit on the network. Through the technical means to interconnect these

units, the information processing and interaction center to make information allocation, data processing integration play its "commander" and "traffic police" functions, mobilize the flow of command information data, while maintaining data exchange order and security.

4) The means of human resource management, it needs to establish a professional team of public opinion analysis services. In the era of "data explosion", mastering the ability of data capture and lyric interpretation, and achieving "value-added" data through "processing" will be an essential skill for future public opinion analysis. At present, many lyric services organizations do not have a dedicated data management department and a professional analysis team. Analysts have to improve the discriminative power and control of the information. The establishment of a team of public opinion analysis service talents depends on coordinating the power of universities, research institutes, media organizations and government departments. A training system is also developed from the fields of public opinion collection, data mining, and information analysis to realize the interconnection and cooperation between media organizations and universities and research institutes.

Owing to the new features of online public opinion, they require the management styles satisfy the following characters: Immediacy, Globality, Substantivity and Availability. Moreover, based on those characters, we can decide which one is the more appropriate way to manage and control online public opinion.

Hence, with an eye to these characters, and to find the more adequate way to manage the online public opinion, four DMs $\left(e_{\kappa}, \kappa=1,2,3,4\right)$ who derive from the decision-making ministries are asked to give their preference information over the alternative companies by making comparisons among them. Nonetheless, because of the sophistication of the evaluation question, the DMs cannot give the detailed evaluation value in precise figures. Additionally, the DMs may have diverse preference degrees for those companies due to the unlike awareness and experience of the DMs. As a result, we opt for the PUMLTSs as an implement for the DMs to deliver favor information in this GDM issue.

As far as this decision-making question, we utilize the alternatives $x_i\left(i=1,2,3,4\right)$ to represent four management styles: Social, Governmental, Technological and Human in relation to four criteria: Immediacy, Globality, Substantivity and Availability, and appraise each pair of styles on the foundation of the mentioned set $S=\left\{s_\alpha \middle| \alpha\in[1/9,9]\right\}$. Then the κ DM furnishes the appraisal information with the PUMLPR $U_\kappa=\left(U_{ij}\left(p\right)_\kappa\right)_{m\times n}$ with respect to the alternative x_i over the criterion c_j, and the weight vector of the DMs is assumed as $\pi=\left(0.20,0.30,0.15,0.35\right)^T$. Moreover, assume that the PUMLPRs are as follows:

$$U_1=\begin{pmatrix} [s_1,s_1] & \left\{\left\langle\left[M_{12}^1,N_{12}^1\right],z_{12}^1\right\rangle,\left\langle\left[M_{12}^2,N_{12}^2\right],z_{12}^2\right\rangle\right\} & \left\{\left\langle[s_{0.25},s_{0.4}],0.7\right\rangle,\left\langle[s_{0.4},s_{0.7143}],0.3\right\rangle\right\} & \left\{\left\langle[s_{0.1429},s_{0.25}],0.5\right\rangle,\left\langle[s_{0.2857},s_{0.5556}],0.4\right\rangle\right\} \\ \left\{\left\langle\left[M_{21}^1,N_{21}^1\right],z_{21}^1\right\rangle,\left\langle\left[M_{21}^2,N_{21}^2\right],z_{21}^2\right\rangle\right\} & [s_1,s_1] & \left\{\left\langle[s_{0.75},s_{0.8}],0.6\right\rangle,\left\langle[s_{0.6},s_{0.8571}],0.4\right\rangle\right\} & \left\{\left\langle[s_{0.4286},s_{0.5}],0.4\right\rangle,\left\langle[s_{0.4286},s_{0.6667}],0.2\right\rangle\right\} \\ \left\{\left\langle[s_{2.5},s_4],0.7\right\rangle,\left\langle[s_{1.4},s_{2.5}],0.3\right\rangle\right\} & \left\{\left\langle[s_{1.25},s_{1.3333}],0.6\right\rangle,\left\langle[s_{1.1667},s_{1.6667}],0.4\right\rangle\right\} & [s_1,s_1] & \left\{\left\langle[s_{0.5714},s_{0.6250}],0.2\right\rangle,\left\langle[s_{0.7143},s_{0.7778}],0.2\right\rangle\right\} \\ \left\{\left\langle[s_4,s_7],0.5\right\rangle,\left\langle[s_{1.8},s_{3.5}],0.4\right\rangle\right\} & \left\{\left\langle[s_2,s_{2.3333}],0.4\right\rangle,\left\langle[s_{1.5},s_{2.3333}],0.2\right\rangle\right\} & \left\{\left\langle[s_{1/9},s_{1/8}],0.5\right\rangle\right\} & [s_1,s_1] \end{pmatrix}$$

$$U_2=\begin{pmatrix} [s_1,s_1] & \left\{\left\langle[s_{0.3333},s_{0.75}],0.4\right\rangle,\left\langle[s_{0.75},s_{0.8333}],0.2\right\rangle\right\} & \left\{\left\langle[s_{0.25},s_{0.6}],0.3\right\rangle,\left\langle[s_{0.5},s_{0.7143}],0.2\right\rangle\right\} & \left\{\left\langle[s_{0.125},s_{0.4286}],0.6\right\rangle,\left\langle[s_{0.4286},s_{0.5556}],0.3\right\rangle\right\} \\ \left\{\left\langle[s_{1.3333},s_3],0.4\right\rangle,\left\langle[s_{1.2},s_{1.3333}],0.2\right\rangle\right\} & [s_1,s_1] & \left\{\left\langle\left[M_{23}^1,N_{23}^1\right],z_{23}^1\right\rangle,\left\langle\left[M_{23}^2,N_{23}^2\right],z_{23}^2\right\rangle\right\} & \left\{\left\langle[s_{0.3750},s_{0.5714}],0.4\right\rangle,\left\langle[s_{0.5714},s_{0.6667}],0.5\right\rangle\right\} \\ \left\{\left\langle[s_{1.6667},s_4],0.3\right\rangle,\left\langle[s_{1.4},s_2],0.2\right\rangle\right\} & \left\{\left\langle\left[M_{32}^1,N_{32}^1\right],z_{32}^1\right\rangle,\left\langle\left[M_{32}^2,N_{32}^2\right],z_{32}^2\right\rangle\right\} & [s_1,s_1] & \left\{\left\langle[s_{0.5},s_{0.7143}],0.1\right\rangle,\left\langle[s_{0.7778},s_{0.8571}],0.2\right\rangle\right\} \\ \left\{\left\langle[s_{2.3333},s_8],0.6\right\rangle,\left\langle[s_{1.8},s_{2.3333}],0.3\right\rangle\right\} & \left\{\left\langle[s_{1.75},s_{2.6667}],0.4\right\rangle,\left\langle[s_{1.5},s_{1.75}],0.5\right\rangle\right\} & \left\{\left\langle[s_{1.4},s_2],0.1\right\rangle,\left\langle[s_{1.1667},s_{1.2857}],0.2\right\rangle\right\} & [s_1,s_1] \end{pmatrix}$$

$$U_3=\begin{pmatrix} [s_1,s_1] & \{\langle[s_{0.3333},s_{0.75}],0.2\rangle,\langle[s_{0.8},s_{0.8333}],0.7\rangle\} & \{\langle[s_{0.2},s_{0.6}],0.5\rangle,\langle[s_{0.6667},s_{0.7143}],0.4\rangle\} & \{\langle[s_{0.1429},s_{0.5}],0.2\rangle,\langle[s_{0.5556},s_{0.5714}],0.6\rangle\} \\ \{\langle[s_{1.3333},s_3],0.2\rangle,\langle[s_{1.2},s_{1.25}],0.7\rangle\} & [s_1,s_1] & \{\langle[s_{0.6},s_{0.8}],0.4\rangle,\langle[s_{0.8333},s_{0.8571}],0.6\rangle\} & \{\langle[M_{24}^1,N_{24}^1],z_{24}^1\rangle,\langle[M_{24}^2,N_{24}^2],z_{24}^2\rangle\} \\ \{\langle[s_{1.6667},s_5],0.5\rangle,\langle[s_{1.4},s_{1.5}],0.4\rangle\} & \{\langle[s_{1.25},s_{1.6667}],0.4\rangle,\langle[s_{1.1667},s_{1.2}],0.6\rangle\} & [s_1,s_1] & \{\langle[s_{0.7143},s_{0.8333}],0.3\rangle,\langle[s_{0.7778},s_{0.8571}],0.7\rangle\} \\ \{\langle[s_2,s_7],0.2\rangle,\langle[s_{1.75},s_{1.8}],0.6\rangle\} & \{\langle[M_{42}^1,N_{42}^1],z_{42}^1\rangle,\langle[M_{42}^2,N_{42}^2],z_{42}^2\rangle\} & \{\langle[s_{1.2},s_{1.4}],0.3\rangle,\langle[s_{1.1667},s_{1.2857}],0.7\rangle\} & [s_1,s_1] \end{pmatrix}$$

$$U_4=\begin{pmatrix} [s_1,s_1] & \{\langle[s_{0.25},s_{0.6667}],0.2\rangle,\langle[s_{0.6},s_{2.6667}],0.5\rangle\} & \{\langle[s_{0.1667},s_{0.3333}],0.2\rangle,\langle[s_{0.4286},s_{1.6}],0.2\rangle\} & \{\langle[s_{0.1429},s_{0.2857}],0.7\rangle,\langle[s_{0.3750},s_{1.1429}],0.2\rangle\} \\ \{\langle[s_{1.5},s_4],0.2\rangle,\langle[s_{0.375},s_{1.6667}],0.5\rangle\} & [s_1,s_1] & \{\langle[s_{0.5},s_{0.6667}],0.1\rangle,\langle[s_{0.6},s_{0.7143}],0.9\rangle\} & \{\langle[s_{0.4286},s_{0.5714}],0.5\rangle,\langle[s_{0.4286},s_{0.6250}],0.1\rangle\} \\ \{\langle[s_3,s_6],0.2\rangle,\langle[s_{0.625},s_{2.3333}],0.2\rangle\} & \{\langle[s_{1.5},s_2],0.1\rangle,\langle[s_{1.4},s_{1.6667}],0.9\rangle\} & [s_1,s_1] & \{\langle[M_{34}^1,N_{34}^1],z_{34}^1\rangle,\langle[M_{34}^2,N_{34}^2],z_{34}^2\rangle\} \\ \{\langle[s_{3.5},s_7],0.7\rangle,\langle[s_{0.875},s_{2.6667}],0.2\rangle\} & \{\langle[s_{1.75},s_{2.3333}],0.5\rangle,\langle[s_{1.6},s_{2.3333}],0.1\rangle\} & \{\langle[M_{43}^1,N_{43}^1],z_{43}^1\rangle,\langle[M_{43}^2,N_{43}^2],z_{43}^2\rangle\} & [s_1,s_1] \end{pmatrix}$$

Then, in the light of normalization procedure, we can get three NPUMLPRs $U_\kappa^N=\left(U_{ij}^N(p)_\kappa\right)_{n\times n}$, $\kappa=1,2,3,4$, respectively.

$$U_1^N=\begin{pmatrix} [s_1,s_1] & \{\langle[M_{12}^1,N_{12}^1],z_{12}^1\rangle,\langle[M_{12}^2,N_{12}^2],z_{12}^2\rangle\} & \{\langle[s_{0.25},s_{0.4}],0.7\rangle,\langle[s_{0.4},s_{0.7143}],0.3\rangle\} & \{\langle[s_{0.1429},s_{0.25}],0.5556\rangle,\langle[s_{0.2857},s_{0.5556}],0.4444\rangle\} \\ \{\langle[M_{21}^1,N_{21}^1],z_{21}^1\rangle,\langle[M_{21}^2,N_{21}^2],z_{21}^2\rangle\} & [s_1,s_1] & \{\langle[s_{0.75},s_{0.8}],0.6\rangle,\langle[s_{0.6},s_{0.8571}],0.4\rangle\} & \{\langle[s_{0.4286},s_{0.5}],0.6667\rangle,\langle[s_{0.4286},s_{0.6667}],0.3333\rangle\} \\ \{\langle[s_{2.5},s_4],0.7\rangle,\langle[s_{1.4},s_{2.5}],0.3\rangle\} & \{\langle[s_{1.25},s_{1.3333}],0.6\rangle,\langle[s_{1.1667},s_{1.6667}],0.4\rangle\} & [s_1,s_1] & \{\langle[s_{0.5714},s_{0.6250}],0.2\rangle,\langle[s_{0.7143},s_{0.7778}],0.2\rangle\} \\ \{\langle[s_4,s_7],0.5556\rangle,\langle[s_{1.8},s_{3.5}],0.4444\rangle\} & \{\langle[s_2,s_{2.3333}],0.6667\rangle,\langle[s_{1.5},s_{2.3333}],0.3333\rangle\} & \{\langle[s_{1/9},s_{1/8}],0.5\rangle\} & [s_1,s_1] \end{pmatrix}$$

$$
U_2^N = \begin{pmatrix}
[s_1,s_1] & \{\langle[s_{0.3333},s_{0.75}],0.6667\rangle,\langle[s_{0.75},s_{0.8333}],0.3333\rangle\} & \{\langle[s_{0.25},s_{0.6}],0.6\rangle,\langle[s_{0.5},s_{0.7143}],0.4\rangle\} & \{\langle[s_{0.125},s_{0.4286}],0.6667\rangle,\langle[s_{0.4286},s_{0.5556}],0.3333\rangle\} \\
\{\langle[s_{1.3333},s_3],0.6667\rangle,\langle[s_{1.2},s_{1.3333}],0.3333\rangle\} & [s_1,s_1] & \{\langle[M_{23}^1,N_{23}^1],z_{23}^1\rangle,\langle[M_{23}^2,N_{23}^2],z_{23}^2\rangle\} & \{\langle[s_{0.3750},s_{0.5714}],0.4444\rangle,\langle[s_{0.5714},s_{0.6667}],0.5556\rangle\} \\
\{\langle[s_{1.6667},s_4],0.6\rangle,\langle[s_{1.4},s_2],0.4\rangle\} & \{\langle[M_{32}^1,N_{32}^1],z_{32}^1\rangle,\langle[M_{32}^2,N_{32}^2],z_{32}^2\rangle\} & [s_1,s_1] & \{\langle[s_{0.5},s_{0.7143}],0.3333\rangle,\langle[s_{0.7778},s_{0.8571}],0.6667\rangle\} \\
\{\langle[s_{2.3333},s_8],0.6667\rangle,\langle[s_{1.8},s_{2.3333}],0.3333\rangle\} & \{\langle[s_{1.75},s_{2.6667}],0.4444\rangle,\langle[s_{1.5},s_{1.75}],0.5556\rangle\} & \{\langle[s_{1.4},s_2],0.3333\rangle,\langle[s_{1.1667},s_{1.2857}],0.6667\rangle\} & [s_1,s_1]
\end{pmatrix}
$$

$$
U_3^N = \begin{pmatrix}
[s_1,s_1] & \{\langle[s_{0.3333},s_{0.75}],0.2222\rangle,\langle[s_{0.8},s_{0.8333}],0.7778\rangle\} & \{\langle[s_{0.2},s_{0.6}],0.5556\rangle,\langle[s_{0.6667},s_{0.7143}],0.4444\rangle\} & \{\langle[s_{0.1429},s_{0.5}],0.25\rangle,\langle[s_{0.5556},s_{0.5714}],0.75\rangle\} \\
\{\langle[s_{1.3333},s_3],0.2222\rangle,\langle[s_{1.2},s_{1.25}],0.7778\rangle\} & [s_1,s_1] & \{\langle[s_{0.6},s_{0.8}],0.4\rangle,\langle[s_{0.8333},s_{0.8571}],0.6\rangle\} & \{\langle[M_{24}^1,N_{24}^1],z_{24}^1\rangle,\langle[M_{24}^2,N_{24}^2],z_{24}^2\rangle\} \\
\{\langle[s_{1.6667},s_5],0.5556\rangle,\langle[s_{1.4},s_{1.5}],0.4444\rangle\} & \{\langle[s_{1.25},s_{1.6667}],0.4\rangle,\langle[s_{1.1667},s_{1.2}],0.6\rangle\} & [s_1,s_1] & \{\langle[s_{0.7143},s_{0.8333}],0.3\rangle,\langle[s_{0.7778},s_{0.8571}],0.7\rangle\} \\
\{\langle[s_2,s_7],0.25\rangle,\langle[s_{1.75},s_{1.8}],0.75\rangle\} & \{\langle[M_{42}^1,N_{42}^1],z_{42}^1\rangle,\langle[M_{42}^2,N_{42}^2],z_{42}^2\rangle\} & \{\langle[s_{1.2},s_{1.4}],0.3\rangle,\langle[s_{1.1667},s_{1.2857}],0.7\rangle\} & [s_1,s_1]
\end{pmatrix}
$$

$$
U_4^N = \begin{pmatrix}
[s_1,s_1] & \{\langle[s_{0.25},s_{0.6667}],0.2857\rangle,\langle[s_{0.6},s_{2.6667}],0.7143\rangle\} & \{\langle[s_{0.1667},s_{0.3333}],0.5\rangle,\langle[s_{0.4286},s_{1.6}],0.5\rangle\} & \{\langle[s_{0.1429},s_{0.2857}],0.7778\rangle,\langle[s_{0.3750},s_{1.1429}],0.2222\rangle\} \\
\{\langle[s_{1.5},s_4],0.2857\rangle,\langle[s_{0.375},s_{1.6667}],0.7143\rangle\} & [s_1,s_1] & \{\langle[s_{0.5},s_{0.6667}],0.1\rangle,\langle[s_{0.6},s_{0.7143}],0.9\rangle\} & \{\langle[s_{0.4286},s_{0.5714}],0.8333\rangle,\langle[s_{0.4286},s_{0.6250}],0.1667\rangle\} \\
\{\langle[s_3,s_6],0.5\rangle,\langle[s_{0.625},s_{2.3333}],0.5\rangle\} & \{\langle[s_{1.5},s_2],0.1\rangle,\langle[s_{1.4},s_{1.6667}],0.9\rangle\} & [s_1,s_1] & \{\langle[M_{34}^1,N_{34}^1],z_{34}^1\rangle,\langle[M_{34}^2,N_{34}^2],z_{34}^2\rangle\} \\
\{\langle[s_{3.5},s_7],0.7778\rangle,\langle[s_{0.875},s_{2.6667}],0.2222\rangle\} & \{\langle[s_{1.75},s_{2.3333}],0.8333\rangle,\langle[s_{1.6},s_{2.3333}],0.1667\rangle\} & \{\langle[M_{43}^1,N_{43}^1],z_{43}^1\rangle,\langle[M_{43}^2,N_{43}^2],z_{43}^2\rangle\} & [s_1,s_1]
\end{pmatrix}
$$

According to Algorithm 1, we can get a complete PUMLPR for $U_i^N\ (i=1,2,3,4)$ as follows:

$$\tilde{U}_1^N = \begin{pmatrix} [s_1,s_1] & \{\langle[s_{0.3333},s_{0.5}],0.5\rangle,\langle[s_{0.6667},s_{0.8333}],0.5\rangle\} & \{\langle[s_{0.25},s_{0.4}],0.7\rangle,\langle[s_{0.4},s_{0.7143}],0.3\rangle\} & \{\langle[s_{0.1429},s_{0.25}],0.5556\rangle,\langle[s_{0.2857},s_{0.5556}],0.4444\rangle\} \\ \{\langle[s_2,s_3],0.5\rangle,\langle[s_{1.2},s_{1.5}],0.5\rangle\} & [s_1,s_1] & \{\langle[s_{0.75},s_{0.8}],0.6\rangle,\langle[s_{0.6},s_{0.8571}],0.4\rangle\} & \{\langle[s_{0.4286},s_{0.5}],0.6667\rangle,\langle[s_{0.4286},s_{0.6667}],0.3333\rangle\} \\ \{\langle[s_{2.5},s_4],0.7\rangle,\langle[s_{1.4},s_{2.5}],0.3\rangle\} & \{\langle[s_{1.25},s_{1.3333}],0.6\rangle,\langle[s_{1.1667},s_{1.6667}],0.4\rangle\} & [s_1,s_1] & \{\langle[s_{0.5714},s_{0.6250}],0.2\rangle,\langle[s_{0.7143},s_{0.7778}],0.2\rangle\} \\ \{\langle[s_4,s_7],0.5556\rangle,\langle[s_{1.8},s_{3.5}],0.4444\rangle\} & \{\langle[s_2,s_{2.3333}],0.6667\rangle,\langle[s_{1.5},s_{2.3333}],0.3333\rangle\} & \{\langle[s_{1/9},s_{1/8}],0.5\rangle\} & [s_1,s_1] \end{pmatrix}$$

$$\tilde{U}_2^N = \begin{pmatrix} [s_1,s_1] & \{\langle[s_{0.3333},s_{0.75}],0.6667\rangle,\langle[s_{0.75},s_{0.8333}],0.3333\rangle\} & \{\langle[s_{0.25},s_{0.6}],0.6\rangle,\langle[s_{0.5},s_{0.7143}],0.4\rangle\} & \{\langle[s_{0.125},s_{0.4286}],0.6667\rangle,\langle[s_{0.4286},s_{0.5556}],0.3333\rangle\} \\ \{\langle[s_{1.3333},s_3],0.6667\rangle,\langle[s_{1.2},s_{1.3333}],0.3333\rangle\} & [s_1,s_1] & \{\langle[s_{0.75},s_{0.8}],0.5\rangle,\langle[s_{0.7007},s_{0.8175}],0.5\rangle\} & \{\langle[s_{0.3750},s_{0.5714}],0.4444\rangle,\langle[s_{0.5714},s_{0.6667}],0.5556\rangle\} \\ \{\langle[s_{1.6667},s_4],0.6\rangle,\langle[s_{1.4},s_2],0.4\rangle\} & \{\langle[s_{1.25},s_{1.3333}],0.5\rangle,\langle[s_{1.2233},s_{1.4272}],0.5\rangle\} & [s_1,s_1] & \{\langle[s_{0.5},s_{0.7143}],0.3333\rangle,\langle[s_{0.7778},s_{0.8571}],0.6667\rangle\} \\ \{\langle[s_{2.3333},s_8],0.6667\rangle,\langle[s_{1.8},s_{2.3333}],0.3333\rangle\} & \{\langle[s_{1.75},s_{2.6667}],0.4444\rangle,\langle[s_{1.5},s_{1.75}],0.5556\rangle\} & \{\langle[s_{1.4},s_2],0.3333\rangle,\langle[s_{1.1667},s_{1.2857}],0.6667\rangle\} & [s_1,s_1] \end{pmatrix}$$

$$\tilde{U}_3^N = \begin{pmatrix} [s_1,s_1] & \{\langle[s_{0.3333},s_{0.75}],0.2222\rangle,\langle[s_{0.8},s_{0.8333}],0.7778\rangle\} & \{\langle[s_{0.2},s_{0.6}],0.5556\rangle,\langle[s_{0.6667},s_{0.7143}],0.4444\rangle\} & \{\langle[s_{0.1429},s_{0.5}],0.25\rangle,\langle[s_{0.5556},s_{0.5714}],0.75\rangle\} \\ \{\langle[s_{1.3333},s_3],0.2222\rangle,\langle[s_{1.2},s_{1.25}],0.7778\rangle\} & [s_1,s_1] & \{\langle[s_{0.6},s_{0.8}],0.4\rangle,\langle[s_{0.8333},s_{0.8571}],0.6\rangle\} & \{\langle[s_{0.4286},s_{0.6667}],0.5\rangle,\langle[s_{0.6713},s_{0.7102}],0.5\rangle\} \\ \{\langle[s_{1.6667},s_5],0.5556\rangle,\langle[s_{1.4},s_{1.5}],0.4444\rangle\} & \{\langle[s_{1.25},s_{1.6667}],0.4\rangle,\langle[s_{1.1667},s_{1.2}],0.6\rangle\} & [s_1,s_1] & \{\langle[s_{0.7143},s_{0.8333}],0.3\rangle,\langle[s_{0.7778},s_{0.8571}],0.7\rangle\} \\ \{\langle[s_2,s_7],0.25\rangle,\langle[s_{1.75},s_{1.8}],0.75\rangle\} & \{\langle[s_{1.5},s_{2.3333}],0.5\rangle,\langle[s_{1.408},s_{1.4897}],0.5\rangle\} & \{\langle[s_{1.2},s_{1.4}],0.3\rangle,\langle[s_{1.1667},s_{1.2857}],0.7\rangle\} & [s_1,s_1] \end{pmatrix}$$

$$\tilde{U}_4=\begin{pmatrix}
[s_1,s_1] & \{\langle[s_{0.25},s_{0.6667}],0.2857\rangle,\langle[s_{0.6},s_{2.6667}],0.7143\rangle\} & \{\langle[s_{0.1667},s_{0.3333}],0.5\rangle,\langle[s_{0.4286},s_{1.6}],0.5\rangle\} & \{\langle[s_{0.1429},s_{0.2857}],0.7778\rangle,\langle[s_{0.3750},s_{1.1429}],0.2222\rangle\} \\
\{\langle[s_{1.5},s_4],0.2857\rangle,\langle[s_{0.375},s_{1.6667}],0.7143\rangle\} & [s_1,s_1] & \{\langle[s_{0.5},s_{0.6667}],0.1\rangle,\langle[s_{0.6},s_{0.7143}],0.9\rangle\} & \{\langle[s_{0.4286},s_{0.5714}],0.8333\rangle,\langle[s_{0.4286},s_{0.6250}],0.1667\rangle\} \\
\{\langle[s_3,s_6],0.5\rangle,\langle[s_{0.625},s_{2.3333}],0.5\rangle\} & \{\langle[s_{1.5},s_2],0.1\rangle,\langle[s_{1.4},s_{1.6667}],0.9\rangle\} & [s_1,s_1] & \{\langle[s_{0.8571},s_{0.8671}],0.5\rangle,\langle[s_{0.7946},s_{0.7946}],0.5\rangle\} \\
\{\langle[s_{3.5},s_7],0.7778\rangle,\langle[s_{0.875},s_{2.6667}],0.2222\rangle\} & \{\langle[s_{1.75},s_{2.3333}],0.8333\rangle,\langle[s_{1.6},s_{2.3333}],0.1667\rangle\} & \{\langle[s_{1.1667},s_{1.1667}],0.5\rangle,\langle[s_{1.2584},s_{1.2584}],0.5\rangle\} & [s_1,s_1]
\end{pmatrix}$$

Moreover, for the obtained complete PUMLPRs $\widetilde{U}_i^N\ (i=1,2,3,4)$, according to Eq. (21), the consistency UMLPRs can be obtained as follows:

$$\tilde{G}_1=\begin{pmatrix}
[s_1,s_1] & \{[s_{0.4714},s_{0.6455}]\} & \{[s_{0.2879},s_{0.4760}]\} & \{[s_{0.2290},s_{0.3952}]\} \\
\{[s_{1.5490},s_{2.1210}]\} & [s_1,s_1] & \{[s_{0.6860},s_{0.8224}]\} & \{[s_{0.6015},s_{0.6988}]\} \\
\{[s_{2.1009},s_{3.4740}]\} & \{[s_{1.2160},s_{1.4578}]\} & [s_1,s_1] & \{[s_{0.8359},s_{0.8657}]\} \\
\{[s_{2.5301},s_{4.3669}]\} & \{[s_{1.4310},s_{1.6626}]\} & \{[s_{1.1552},s_{1.19631}]\} & [s_1,s_1]
\end{pmatrix}$$

$$\tilde{G}_2=\begin{pmatrix}
[s_1,s_1] & \{[s_{0.6084},s_{0.8594}]\} & \{[s_{0.5743},s_{0.8021}]\} & \{[s_{0.2227},s_{0.5042}]\} \\
\{[s_{1.1636},s_{1.6438}]\} & [s_1,s_1] & \{[s_{0.7249},s_{0.8087}]\} & \{[s_{0.5106},s_{0.6527}]\} \\
\{[s_{1.2468},s_{1.7411}]\} & \{[s_{1.2366},s_{1.3795}]\} & [s_1,s_1] & \{[s_{0.8873},s_{0.9376}]\} \\
\{[s_{1.9832},s_{4.4900}]\} & \{[s_{1.5320},s_{1.9584}]\} & \{[s_{1.0666},s_{1.1270}]\} & [s_1,s_1]
\end{pmatrix}$$

$$\tilde{G}_3=\begin{pmatrix}
[s_1,s_1] & \{[s_{0.6867},s_{0.8310}]\} & \{[s_{0.3803},s_{0.6771}]\} & \{[s_{0.4762},s_{0.6223}]\} \\
\{[s_{1.2034},s_{1.4563}]\} & [s_1,s_1] & \{[s_{0.7307},s_{0.8338}]\} & \{[s_{0.5364},s_{0.6881}]\} \\
\{[s_{1.4770},s_{2.6298}]\} & \{[s_{1.1993},s_{1.3685}]\} & [s_1,s_1] & \{[s_{0.7582},s_{0.8499}]\} \\
\{[s_{1.6070},s_{2.0998}]\} & \{[s_{1.4533},s_{1.8644}]\} & \{[s_{1.1766},s_{1.3190}]\} & [s_1,s_1]
\end{pmatrix}$$

$$\tilde{G}_4=\begin{pmatrix}
[s_1,s_1] & \{[s_{0.5870},s_{1.5058}]\} & \{[s_{0.5899},s_{0.8819}]\} & \{[s_{0.2105},s_{0.4273}]\} \\
\{[s_{0.6641},s_{1.7035}]\} & [s_1,s_1] & \{[s_{0.5892},s_{0.7094}]\} & \{[s_{0.6015},s_{0.7212}]\} \\
\{[s_{1.1340},s_{1.6952}]\} & \{[s_{1.4097},s_{1.6973}]\} & [s_1,s_1] & \{[s_{0.8253},s_{0.8253}]\} \\
\{[s_{2.3402},s_{4.7508}]\} & \{[s_{1.3865},s_{1.6626}]\} & \{[s_{1.2117},s_{1.2117}]\} & [s_1,s_1]
\end{pmatrix}$$

Then with the aggregated operator (8), the group UMLPR can be obtained as:

$$\tilde{G}=\begin{pmatrix} [s_1,s_1] & \{[s_{0.5814},s_{0.9826}]\} & \{[s_{0.4746},s_{0.7282}]\} & \{[s_{0.2461},s_{0.4678}]\} \\ \{[s_{1.0177},s_{1.7199}]\} & [s_1,s_1] & \{[s_{0.6676},s_{0.7786}]\} & \{[s_{0.5629},s_{0.6907}]\} \\ \{[s_{1.3732},s_{2.1068}]\} & \{[s_{1.2844},s_{1.4980}]\} & [s_1,s_1] & \{[s_{0.8349},s_{0.8695}]\} \\ \{[s_{2.1379},s_{4.0635}]\} & \{[s_{1.4479},s_{1.7766}]\} & \{[s_{1.1500},s_{1.1978}]\} & [s_1,s_1] \end{pmatrix}$$

Moreover, with the calculation formula of possibility degree (9), we can construct a possibility matrix for the group UMLPR as follows:

$$P=\begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 1 & 0.5 & 0 & 0 \\ 1 & 1 & 0.5 & 0 \\ 1 & 1 & 1 & 0.5 \end{pmatrix}$$

Furthermore, with the priority weight vector (22), the priorities of the alternatives can be calculated as follows:

$$\widehat{\omega}_1=0.5,\ \ \widehat{\omega}_2=1.5,\ \ \widehat{\omega}_3=2.5,\ \ \widehat{\omega}_4=3.5$$

$$\widehat{\omega}_4>\widehat{\omega}_3>\widehat{\omega}_2>\widehat{\omega}_1$$

5.2. Comparative analysis with incomplete probabilistic multiplicative linguistic preference relations

In the cause of further displaying the validity of the proposed decision-making procedure, we apply it to the probabilistic linguistic context. Supposed that the incomplete probabilistic multiplicative linguistic preference relations (PMLPRs) are as follows:

$$D_1=\begin{pmatrix} s_1 & \{M_{11}^1(z_{12}^1),M_{11}^2(z_{12}^2)\} & \{s_{0.4}(0.5),s_{0.4286}(0.2)\} & \{s_{0.25}(0.7),s_{0.3333}(0.3)\} \\ \{M_{21}^1(z_{21}^1),M_{21}^2(z_{21}^2)\} & s_1 & \{s_{0.8571}(0.3),s_1(0.4)\} & \{s_{0.5}(0.1),s_{0.5556}(0.4)\} \\ \{s_{2.333}(0.2),s_{2.5}(0.5)\} & \{s_1(0.4),s_{1.1667}(0.3)\} & s_1 & \{s_{0.625}(0.2),s_{0.7778}(0.7)\} \\ \{s_3(0.3),s_4(0.7)\} & \{s_{1.8}(0.4),s_2(0.1)\} & \{s_{1.2857}(0.7),s_{1.6}(0.2)\} & s_1 \end{pmatrix}$$

$$D_2=\begin{pmatrix} s_1 & \{s_{0.5}(0.8),s_{0.5}(0.2)\} & \{M_{13}^1(z_{13}^1),M_{13}^2(z_{13}^2)\} & \{s_{0.25}(0.4),s_{0.3333}(0.5)\} \\ \{s_2(0.2),s_2(0.8)\} & s_1 & \{s_{0.6667}(0.3),s_{0.8}(0.3)\} & \{s_{0.5}(0.6),s_{0.5714}(0.3)\} \\ \{M_{31}^1(z_{31}^1),M_{31}^2(z_{31}^2)\} & \{s_{1.25}(0.3),s_{1.5}(0.3)\} & s_1 & \{s_{0.7143}(0.1),s_{0.75}(0.3)\} \\ \{s_3(0.5),s_4(0.4)\} & \{s_{1.75}(0.3),s_2(0.6)\} & \{s_{1.3333}(0.3),s_{1.4}(0.1)\} & s_1 \end{pmatrix}$$

$$D_3 = \begin{pmatrix} s_1 & \{s_{0.4}(0.3), s_{0.75}(0.3)\} & \{s_{0.3333}(0.2), s_{0.4286}(0.5)\} & \{s_{0.2222}(0.1), s_{0.3750}(0.7)\} \\ \{s_{1.3333}(0.3), s_{2.5}(0.3)\} & s_1 & \{M_{23}^1(z_{23}^1), M_{23}^2(z_{23}^2)\} & \{s_{0.5}(0.4), s_{0.7143}(0.2)\} \\ \{s_{2.3333}(0.5), s_3(0.2)\} & \{M_{32}^1(z_{32}^1), M_{32}^2(z_{32}^2)\} & s_1 & \{s_{0.6667}(0.5), s_{0.8750}(0.2)\} \\ \{s_{2.6667}(0.7), s_{4.5}(0.1)\} & \{s_{1.4}(0.2), s_2(0.4)\} & \{s_{1.1429}(0.2), s_{1.5}(0.5)\} & s_1 \end{pmatrix}$$

$$D_4 = \begin{pmatrix} s_1 & \{s_{0.25}(0.2), s_{0.8}(0.2)\} & \{s_{0.1667}(0.5), s_{0.5714}(0.1)\} & \{M_{14}^1(z_{14}^1), M_{14}^2(z_{14}^2)\} \\ \{s_{1.25}(0.2), s_4(0.2)\} & s_1 & \{s_{0.6667}(0.5), s_{0.7143}(0.5)\} & \{s_{0.5556}(0.3), s_{0.6667}(0.4)\} \\ \{s_{1.75}(0.1), s_6(0.5)\} & \{s_{1.4}(0.5), s_{1.5}(0.5)\} & s_1 & \{s_{0.7778}(0.5), s_{0.8571}(0.5)\} \\ \{M_{41}^1(z_{41}^1), M_{41}^2(z_{41}^2)\} & \{s_{1.5}(0.4), s_{1.8}(0.3)\} & \{s_{1.1667}(0.5), s_{1.2857}(0.5)\} & s_1 \end{pmatrix}$$

Then, with the similar normalized method and the restoring method, we can get the complete PMLPRs as follows:

$$\tilde{D}_1 = \begin{pmatrix} s_1 & \{s_{0.4833}(0.5622), s_{0.5143}(0.4378)\} & \{s_{0.4}(0.5), s_{0.4286}(0.2)\} & \{s_{0.25}(0.7), s_{0.3333}(0.3)\} \\ \{s_{1.9444}(0.4378), s_{2.0690}(0.5622)\} & s_1 & \{s_{0.8571}(0.3), s_1(0.4)\} & \{s_{0.5}(0.1), s_{0.5556}(0.4)\} \\ \{s_{2.333}(0.2), s_{2.5}(0.5)\} & \{s_1(0.4), s_{1.1667}(0.3)\} & s_1 & \{s_{0.625}(0.2), s_{0.7778}(0.7)\} \\ \{s_3(0.3), s_4(0.7)\} & \{s_{1.8}(0.4), s_2(0.1)\} & \{s_{1.2857}(0.7), s_{1.6}(0.2)\} & s_1 \end{pmatrix}$$

$$\tilde{D}_2 = \begin{pmatrix} s_1 & \{s_{0.5}(0.8), s_{0.5}(0.2)\} & \{s_{0.3417}(0.3267), s_{0.4221}(0.6733)\} & \{s_{0.25}(0.4), s_{0.3333}(0.5)\} \\ \{s_2(0.2), s_2(0.8)\} & s_1 & \{s_{0.6667}(0.3), s_{0.8}(0.3)\} & \{s_{0.5}(0.6), s_{0.5714}(0.3)\} \\ \{s_{2.3693}(0.6733), s_{2.9268}(0.3267)\} & \{s_{1.25}(0.3), s_{1.5}(0.3)\} & s_1 & \{s_{0.7143}(0.1), s_{0.75}(0.3)\} \\ \{s_3(0.5), s_4(0.4)\} & \{s_{1.75}(0.3), s_2(0.6)\} & \{s_{1.3333}(0.3), s_{1.4}(0.1)\} & s_1 \end{pmatrix}$$

$$\tilde{D}_3 = \begin{pmatrix} s_1 & \{s_{0.4}(0.3), s_{0.75}(0.3)\} & \{s_{0.3333}(0.2), s_{0.4286}(0.5)\} & \{s_{0.2222}(0.1), s_{0.3750}(0.7)\} \\ \{s_{1.3333}(0.3), s_{2.5}(0.3)\} & s_1 & \{s_{0.7917}(0.7529), s_{0.6939}(0.2471)\} & \{s_{0.5}(0.4), s_{0.7143}(0.2)\} \\ \{s_{2.3333}(0.5), s_3(0.2)\} & \{s_{1.4412}(0.2471), s_{1.2632}(0.7529)\} & s_1 & \{s_{0.6667}(0.5), s_{0.8750}(0.2)\} \\ \{s_{2.6667}(0.7), s_{4.5}(0.1)\} & \{s_{1.4}(0.2), s_2(0.4)\} & \{s_{1.1429}(0.2), s_{1.5}(0.5)\} & s_1 \end{pmatrix}$$

$$\tilde{D}_4 = \begin{pmatrix} s_1 & \{s_{0.25}(0.2), s_{0.8}(0.2)\} & \{s_{0.1667}(0.5), s_{0.5714}(0.1)\} & \{s_{0.1340}(0.82), s_{0.5116}(0.18)\} \\ \{s_{1.25}(0.2), s_4(0.2)\} & s_1 & \{s_{0.6667}(0.5), s_{0.7143}(0.5)\} & \{s_{0.5556}(0.3), s_{0.6667}(0.4)\} \\ \{s_{1.75}(0.1), s_6(0.5)\} & \{s_{1.4}(0.5), s_{1.5}(0.5)\} & s_1 & \{s_{0.7778}(0.5), s_{0.8571}(0.5)\} \\ \{s_{1.9546}(0.18), s_{7.4654}(0.82)\} & \{s_{1.5}(0.4), s_{1.8}(0.3)\} & \{s_{1.1667}(0.5), s_{1.2857}(0.5)\} & s_1 \end{pmatrix}$$

Moreover, after checking the consistency, we can get the group PRs for information fusion:

$$\tilde{D} = \begin{pmatrix} s_1 & \{s_{0.5006}\} & \{s_{0.3371}\} & \{s_{0.2566}\} \\ \{s_{2.2260}\} & s_1 & \{s_{0.7576}\} & \{s_{0.5617}\} \\ \{s_{3.4860}\} & \{s_{1.3366}\} & s_1 & \{s_{0.7595}\} \\ \{s_{4.4825}\} & \{s_{1.7902}\} & \{s_{1.3220}\} & s_1 \end{pmatrix}$$

Finally, the priorities for the assessed alternatives can be obtained as follows:

$$\widehat{\omega}_1 = 0.4065,\ \widehat{\omega}_2 = 0.8785,\ \widehat{\omega}_3 = 1.2246,\ \widehat{\omega}_4 = 1.6098$$

$$\widehat{\omega}_4 > \widehat{\omega}_3 > \widehat{\omega}_2 > \widehat{\omega}_1$$

5.3. Comparative analysis with incomplete hesitant fuzzy linguistic preference relations

For further highlighting the advantage of the proposed method, in this subsection, we choose hesitant fuzzy linguistic term sets on the multiplicative linguistic scale as the decision-making instrument to assess the management ways of the online public opinion. Supposed that the incomplete hesitant fuzzy linguistic preference relations (HFLPRs) are as follows:

$$H_1 = \begin{pmatrix} s_1 & \{M_{11}^1, M_{11}^2\} & \{s_{0.4}, s_{0.4286}\} & \{s_{0.25}, s_{0.3333}\} \\ \{M_{21}^1, M_{21}^2\} & s_1 & \{s_{0.8571}, s_1\} & \{s_{0.5}, s_{0.5556}\} \\ \{s_{2.333}, s_{2.5}\} & \{s_1, s_{1.1667}\} & s_1 & \{s_{0.625}, s_{0.7778}\} \\ \{s_3, s_4\} & \{s_{1.8}, s_2\} & \{s_{1.2857}, s_{1.6}\} & s_1 \end{pmatrix}$$

$$H_2 = \begin{pmatrix} s_1 & \{s_{0.5}, s_{0.5}\} & \{M_{13}^1, M_{13}^2\} & \{s_{0.25}, s_{0.3333}\} \\ \{s_2, s_2\} & s_1 & \{s_{0.6667}, s_{0.8}\} & \{s_{0.5}, s_{0.5714}\} \\ \{M_{31}^1, M_{31}^2\} & \{s_{1.25}, s_{1.5}\} & s_1 & \{s_{0.7143}, s_{0.75}\} \\ \{s_3, s_4\} & \{s_{1.75}, s_2\} & \{s_{1.3333}, s_{1.4}\} & s_1 \end{pmatrix}$$

$$H_3 = \begin{pmatrix} s_1 & \{s_{0.4}, s_{0.75}\} & \{s_{0.3333}, s_{0.4286}\} & \{s_{0.2222}, s_{0.3750}\} \\ \{s_{1.3333}, s_{2.5}\} & s_1 & \{M_{23}^1(z_{23}^1), M_{23}^2(z_{23}^2)\} & \{s_{0.5}, s_{0.7143}\} \\ \{s_{2.3333}, s_3\} & \{M_{32}^1, M_{32}^2\} & s_1 & \{s_{0.6667}, s_{0.8750}\} \\ \{s_{2.6667}, s_{4.5}\} & \{s_{1.4}, s_2\} & \{s_{1.1429}, s_{1.5}\} & s_1 \end{pmatrix}$$

$$H_4 = \begin{pmatrix} s_1 & \{s_{0.25}, s_{0.8}\} & \{s_{0.1667}, s_{0.5714}\} & \{M_{14}^1, M_{14}^2\} \\ \{s_{1.25}, s_4\} & s_1 & \{s_{0.6667}, s_{0.7143}\} & \{s_{0.5556}, s_{0.6667}\} \\ \{s_{1.75}, s_6\} & \{s_{1.4}, s_{1.5}\} & s_1 & \{s_{0.7778}, s_{0.8571}\} \\ \{M_{41}^1, M_{41}^2\} & \{s_{1.5}, s_{1.8}\} & \{s_{1.1667}, s_{1.2857}\} & s_1 \end{pmatrix}$$

Then, with the similar normalized method and the restoring method, we can get the complete HFLPRs as follows:

$$\tilde{H}_1=\begin{pmatrix} s_1 & \{M_{11}^1,M_{11}^2\} & \{s_{0.4},s_{0.4286}\} & \{s_{0.25},s_{0.3333}\} \\ \{M_{21}^1,M_{21}^2\} & s_1 & \{s_{0.8571},s_1\} & \{s_{0.5},s_{0.5556}\} \\ \{s_{2.333},s_{2.5}\} & \{s_1,s_{1.1667}\} & s_1 & \{s_{0.625},s_{0.7778}\} \\ \{s_3,s_4\} & \{s_{1.8},s_2\} & \{s_{1.2857},s_{1.6}\} & s_1 \end{pmatrix}$$

$$\tilde{H}_2=\begin{pmatrix} s_1 & \{s_{0.5},s_{0.5}\} & \{M_{13}^1,M_{13}^2\} & \{s_{0.25},s_{0.3333}\} \\ \{s_2,s_2\} & s_1 & \{s_{0.6667},s_{0.8}\} & \{s_{0.5},s_{0.5714}\} \\ \{M_{31}^1,M_{31}^2\} & \{s_{1.25},s_{1.5}\} & s_1 & \{s_{0.7143},s_{0.75}\} \\ \{s_3,s_4\} & \{s_{1.75},s_2\} & \{s_{1.3333},s_{1.4}\} & s_1 \end{pmatrix}$$

$$\tilde{H}_3=\begin{pmatrix} s_1 & \{s_{0.4},s_{0.75}\} & \{s_{0.3333},s_{0.4286}\} & \{s_{0.2222},s_{0.3750}\} \\ \{s_{1.3333},s_{2.5}\} & s_1 & \{M_{23}^1(z_{23}^1),M_{23}^2(z_{23}^2)\} & \{s_{0.5},s_{0.7143}\} \\ \{s_{2.3333},s_3\} & \{M_{32}^1,M_{32}^2\} & s_1 & \{s_{0.6667},s_{0.8750}\} \\ \{s_{2.6667},s_{4.5}\} & \{s_{1.4},s_2\} & \{s_{1.1429},s_{1.5}\} & s_1 \end{pmatrix}$$

$$\tilde{H}_4=\begin{pmatrix} s_1 & \{s_{0.25},s_{0.8}\} & \{s_{0.1667},s_{0.5714}\} & \{M_{14}^1,M_{14}^2\} \\ \{s_{1.25},s_4\} & s_1 & \{s_{0.6667},s_{0.7143}\} & \{s_{0.5556},s_{0.6667}\} \\ \{s_{1.75},s_6\} & \{s_{1.4},s_{1.5}\} & s_1 & \{s_{0.7778},s_{0.8571}\} \\ \{M_{41}^1,M_{41}^2\} & \{s_{1.5},s_{1.8}\} & \{s_{1.1667},s_{1.2857}\} & s_1 \end{pmatrix}$$

Moreover, after checking the consistency, we can get the group PRs for information fusion:

$$\tilde{H}=\begin{pmatrix} s_1 & \{s_{0.5103}\} & \{s_{0.3746}\} & \{s_{0.2860}\} \\ \{s_{2.1949}\} & s_1 & \{s_{0.7573}\} & \{s_{0.5660}\} \\ \{s_{3.0079}\} & \{s_{1.3359}\} & s_1 & \{s_{0.7599}\} \\ \{s_{3.9456}\} & \{s_{1.7794}\} & \{s_{1.3259}\} & s_1 \end{pmatrix}$$

Finally, the priorities for the assessed alternatives can be obtained as follows:

$$\widehat{\omega}_1=0.1035,\ \widehat{\omega}_2=0.2177,\ \widehat{\omega}_3=0.2906,\ \widehat{\omega}_4=0.3882$$

$$\widehat{\omega}_4>\widehat{\omega}_3>\widehat{\omega}_2>\widehat{\omega}_1$$

Obviously, the experimental results are the same regardless of the environment, the means of human resource management is the relatively more suitable way to manage the online public opinion under the big data circumstance. On the strength of the above analysis, the restoring method mentioned in this paper can be used in the probabilistic uncertain linguistic context, the probabilistic linguistic circumstance and the hesitant fuzzy linguistic environment. The reason for this phenomenon may be that the dimension of the data being tested is not large enough. The difference is that the elements of these three decision-making instruments have different structures. In the procedure of making a decision, the same method applied in these three contexts, the computation of the PUMLPRs is more complicated than the PMLPRs and the HFLPRs. To some extent, the proposed method can also be applied to similar tasks. But from the point of fully expressing the decision-making information, the PUMLPRs is better. Except for the validation of the method, the contrastive analysis also demonstrates the extension of the restoring method. But owing to the process for restoring incomplete PRs is not direct. That is to say, the restoring process is relatively complex. It has completely repaired the incomplete PUMLPRs indeed. Moreover, the consistent procedure is performed in the derived UMLPRs, not the original PUMLPRs. Although we can get the final priorities, but we cannot get the consistent PUMLPRs.

6. Conclusions

Given that the intricacy of the real decision-making situations, not all the PRs are always complete. This paper has researched the incomplete PUMLPRs. Owing to the complexity of the structure of the PUMLPRs, the incomplete PUMLPRs cannot be restored directly. Therefore, this paper has divided the restored process into two steps: one is to restore the uncertain linguistic variables; the other is to restore the corresponding probabilities. Moreover, due to the necessity of the consistency, the probe for the consistency of the obtained complete PUMLPRs has also been discussed later. Then, combined with the case mentioned at the beginning of the paper, this paper has assessed the management way of the online opinion under the big data context. Indeed, the utilization of the case has illustrated the validity of the series of the suggested decision-making procedure. In addition to that, so as to further give evidence of the efficacy of the suggested decision-making procedure, we have also applied it to the probabilistic linguistic surrounding. Obviously, the procedure that we have provided can also solve the incomplete probabilistic linguistic decision-making problem.

In the future, we can also study the incomplete PULPRs under other situations, the more convenient and direct restoring method and so on.

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2 The probe for the weighted dual probabilistic linguistic correlation coefficient to invest an artificial intelligence project

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The probe for the weighted dual probabilistic linguistic correlation coefficient to invest an artificial intelligence project

Wanying Xie [a,c], Zeshui Xu [a,b,*], Zhiliang Ren [a], Enrique Herrera Viedma [c,d,*]

[a] *School of Economics and Management, Southeast University, Nanjing, Jiangsu 211189, China*

[b] *Business School, Sichuan University, Chengdu, Sichuan, 610064, China*

[c] *Andalusian Research Institute in Data Science and Computational Intelligence, University of Granada, E-18071 Granada, Spain*

[d] *Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah, 21589, Saudi Arabia*

Abstract

As one of the burgeoning decision-making instruments, the integrity of dual probabilistic linguistic term sets (DPLTSs) is to express the decision information in terms of cognitive certainty and uncertainty. The superiority of correlation coefficient is to demonstrate the inter-relationship of the variables. This paper aims to give full play to the advantages of the above two. Firstly, it defines the dual probabilistic linguistic correlation coefficient. Then it based on the proposed entropy for DPLTSs calculates the comprehensive weight vector. Moreover, combined with the proposed correlation coefficient, it further defines the weighted correlation coefficient as a measure for the application about artificial intelligence (AI). Besides, it uses the dual probabilistic linguistic closeness coefficient as the reference to compare the pros and cons. Finally, a specific numeric simulation is utilized to demonstrate the feasibility of the two different measures.

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[*] Corresponding author. E-mail addresses: xie_wanyinglai@163.com (W.Y. Xie), xuzeshui@263.net (Z.S. Xu), ren_zhi_liang@126.com (Z.L. Ren), viedma@decsai.ugr.es (E. Herrera Viedma).

Keywords: Dual probabilistic linguistic term sets; Weighted correlation coefficient; Distance measure; Closeness coefficient; Multi-attribute group decision making.

1. Introduction

Since the computer "dark blue" of the IBM company defeated the world chess champion of mankind in 2016, AI has aroused the extensive attention of the public. Actually, with the technological breakthrough of AI industry and its continuous integration with traditional industries, AI plays a huge role in promoting the transformation and upgrading of industries and improving production efficiency [1]. Moreover, its related technology has permeated many areas, such as economic [2-5], culture [6,7], education [8-10], medical treatment [11,12] and so on. Therefore, many scientific and technical corporations have begun to join the AI market war in succession. In order to better develop the AI industry, before determining the specific project, they need to do the evaluation for the influence of the AI.

For the sake of making rational decisions, the DMs need to choose the appropriate decision-making tool. As one of the fashionable decision-making implements for now, the research for probabilistic linguistic term sets (PLTSs) [13] has already been investigated in many directions [14-17]. Although the PLTSs not only can reflect the hesitation of the DMs, but also reflect the weights of the DMs, it cannot reflect the non-membership degree of the policy-making information. Therefore, Ref. [18, 19] proposed the DPLTSs that can contain both the membership degree and non-membership degree later. While the membership degree represents the epistemic certainty, and the non-membership degree represents the epistemic uncertainty. It can make the DMs flexibly give their suggestions and reduce the irresolution of the DMs for one thing or another when it is hard for them to reach a final agreement to some extent. From this perspective, the DPLTSs are more suited then PLTSs for handling uncertainty and fuzziness. For example, given the linguistic term set [20] $S=\{s_{-2},s_{-1},s_0,s_1,s_2\}$, two DMs are invited to do the evaluation for one project, then for the first DM, the acceptable level of this project is $\{s_1(0.55)\}$, while for the other DM is $\{s_2(0.45)\}$. Whereas for the first DM, the repulsive level of the project is $\{s_{-1}(0.4)\}$ and the other is $\{s_{-2}(0.6)\}$. Then the decision information of the two DMs can be expressed as $\left\langle\{s_1(0.55),s_2(0.45)\},\{s_{-1}(0.4),s_{-2}(0.6)\}\right\rangle$. Obviously, the DPLTSs can display the decision-making information better and more comprehensively compared with the PLTSs. Since the DPLTSs were proposed, its researches mainly focus on the basic operations [18] and the closeness coefficient [19], while the other relevant researches have not been found yet.

Since Karl Pearson first proposed the correlation coefficient in 1895, and the good nature of correlation coefficient in measuring the interrelationship among the variables in statistical analysis, the researches for correlation coefficient have occurred a lot, especially in the fuzzy area. For the sake of learning about the joint relationship among the fuzzy data, many researches have chosen the correlation coefficient as the tool to explore the internal relationship among the fuzzy data with different methods, such as the construction of correlation coefficient for fuzzy numbers by forming non-linear programs [21], the intuitionistic fuzzy correlation coefficient based on the entropy weight [22], the hesitant fuzzy correlation coefficient and its combination with TOPSIS [23], the correlation coefficient built on the dual hesitant fuzzy decision-making information [24], the correlation coefficient for probabilistic hesitant fuzzy elements by separately computing probability [25], the probabilistic linguistic correlation coefficient under the attribute weights are completely unknown [26] and so on.

Therefore, considering the advantages of DPLTSs and the importance of researching the correlation coefficient that can research the degree of linear correlation between variables, this paper is to research the correlation coefficient among the DPLTSs. In the beginning, the paper based on Ref. [18] and [27] proposes the correlation coefficient between the two DPLTSs, and divides the two DPLTSs into two parts: the angle of membership and the angle of non-membership. Then it calculates the corresponding correlation coefficient between two membership degrees of the two DPLTSs that is the respective PLTSs of the two DPLTSs. Moreover, it calculates the correlation coefficient between the two non-membership degrees of the two DPLTSs, averages the two obtained correlation coefficient, and gets the final correlation coefficient of the two DPLTSs.

Furthermore, in order to apply the correlation coefficient to the practical application and the goal of the study may be allocated with different weights [28, 29], we further define the weighted correlation coefficient for the DPLTSs. In addition to that, all sorts of objective factors and subjective factors may lead to the differences between the degrees of importance of the goal of the study. Hence, the weights of the goal of the study are not always same. For example, one is going to buy a new apartment. He may need to consider many factors, such as the price, distance, daylight and so on. Obviously, not all the factors that need to be considered are the more the better. If let the buyer provide the respective weights of these influencing factors, the specific proportions must be different. For the practical application process, the relevant influencing factors that need to be considered are more and more complicated. Then the weights of the goal of the study is more difficult to determine. Therefore, it is necessary to find the appropriate method to determine the weight. Without loss of generality, the research of the determination of the weight for the weight correlation coefficient is also necessary.

Different from the traditional researches for weight correlation coefficient, either just considering the objective weight [30-34] and ignoring the subjective weights of the DMs, or just considering the subjective weight [35-37], we divide the weight vector into the subjective form and the objective form by means of Ref. [38] in the paper. For the subjective weight vector, in accordance with the DMs' knowledge of the decision-making issue to be solved, it can be obtained from the DMs directly. For the objective weight vector, by learning from Ref. [38], we based on the proposed entropy measure for the DPLTSs to define it. Moreover, because of Ref. [39], the final comprehensive weight can be acquired.

Besides, the correlation coefficient that we have defined under the dual probabilistic linguistic environment have the following good characteristics: For one thing, its strength lies in the interval $\left[-1,1\right]$, which conforms to the basic requirement of the conventional statistics. That is to say, it can not only reflect the internal relationship between DPLTSs, but also display the positive or negative correlation. For another, the defined weighted correlation coefficient uses the integrated weight that both contains the subjective weight of the DMs and the objective weight of the policy-making event is good for taking full advantage of the data feedback to deal with the practical problem.

Presently, the application of correlation coefficient mainly involves in many fields, such as the data analysis and classification [40-43], pattern recognition [44, 45], decision-making [46-48] and so on. Obviously, the proposition of the dual probabilistic linguistic correlation coefficient aims to solve the problems raised at the beginning of the paper that is to choose a suitable AI project by considering the four influence factors. Here, we can regard economic, culture, education, and medical treatment as four attributes, and the selected projects that the company plan to invest as the selected alternatives, then construct a multi-attribute group decision-making problem [49-53]. Then by solving the multi-attribute group decision-making problem, the specific calculation process for the dual probabilistic linguistic correlation coefficient can be executed in detail, and the applicable AI project can be chosen, too.

Simulation result shows that the third project should be invested, that is the education project. Actually, the wave of AI+ education is in full swing, such as 51Talk, Vipkid and so on. Since the birth of AI and AI science in 1956, its research and application fields are closely related to education. AI is the study of science and technology that allows computers to receive education and improve intelligence. Whereas, the research results of AI are applied to every aspect of our daily lives and improve our lives.

The rest of this paper is organized as follows: Section 2 separates into two parts: one recalls some basic definitions, and the other defines the distance measures for the DPLTSs. Section 3 defines the dual probabilistic linguistic correlation coefficient and the weight dual probabilistic linguistic correlation coefficient. Section 4 describes the multi-attribute group decision-making problem under the dual probabilistic linguistic context, divides the weight vector into the subjective and objective forms, defines the entropy measure for the DPLTSs for the sake of obtaining the final comprehensive weight vector, and introduces the complete dual probabilistic linguistic multi-attribute group decision-making process. Section 5 uses a simulation experiment related to the influence evaluations for AI to clarify the feasibility and practicality of the dual probabilistic linguistic multi-attribute group decision-making process. Section 6 ends with some conclusions.

2. Preliminaries

In the chapter, some basic notions that contain the linguistic term, the probabilistic term set (PLTS), the dual probabilistic linguistic term set (DPLTS) and the normalized dual probabilistic linguistic term element (NDPLTE) are shown below:

Assume that $S=\left\{s_\alpha \middle| \alpha\in[-q,q]\right\}$ is a continual linguistic term set, where q is an adequately large positive integer. For any two linguistic terms s_α and s_β, the parameter $\lambda>0$, they satisfy the following basic operations [54]:

$$neg\left(s_\alpha\right)=s_{-\alpha},\ \lambda s_\alpha=s_{\lambda\alpha},\ s_\alpha\otimes s_\beta=\max\left\{s_{-q},\min\left\{s_{\alpha\beta},s_q\right\}\right\},$$

$$s_\alpha\oplus s_\beta=\max\left\{s_{-q},\min\left\{s_{\alpha+\beta},s_q\right\}\right\}$$

2.1. The PLTS

According to the thought of Pang et al. [13], the PLTS $\Gamma(p)$ is constructed by $\#\Gamma(p)$ combinations of $\Gamma^{(\kappa)}$ and $p^{(\kappa)}$. Concretely, it is usually depicted by the coming equation:

$$\Gamma(p)=\left\{\Gamma^{(\kappa)}(p^{(\kappa)}) \mid \Gamma^{(\kappa)}\in S,\, p^{(\kappa)}\geq 0, \kappa=1,2,...,\#\Gamma(p),\ \sum_{\kappa=1}^{\#\Gamma(p)} p^{(\kappa)}\leq 1\right\}, \tag{1}$$

Where S is the linguistic term set mentioned before, $\Gamma^{(\kappa)}$ is one of these elements in S and $p^{(\kappa)}$ is its matching weight.

Moreover, considering the weight of $\Gamma^{(\kappa)}$ are not always equal to 1, and different PLTSs are not always with the same number of elements, it may lead to some questions in the process of calculation, Pang et al. [13] proposed the following normalized PLTS (NPLTS) to reduce the produce of these computational problems.

For any two different PLTSs $\Gamma_1(p)$ and $\Gamma_2(p)$, where $\Gamma_1(p)=\left\{\Gamma_1^{(\kappa)}\left(p_1^{(\kappa)}\right)\middle|\kappa=1,2,\ldots,\#\Gamma_1(p)\right\}$ and $\Gamma_2(p)=\left\{\Gamma_2^{(\kappa)}\left(p_2^{(\kappa)}\right)\middle|\kappa=1,2,\ldots,\#\Gamma_2(p)\right\}$, $\#\Gamma_1(p)$ and $\#\Gamma_2(p)$ are the numbers of several elements in $\Gamma_1(p)$ and $\Gamma_2(p)$. Under normal conditions, $\#\Gamma_1(p)\neq\#\Gamma_2(p)$. Then, it is necessary to adjust the numbers of elements so that the numbers of elements in $\Gamma_1(p)$ and $\Gamma_2(p)$ are equal. In a general way, the adjusting process divides into two aspects: one is the quantity adjustment, the other is the weight adjustment. For the first aspect, the adjusting method is to add some linguistic terms that are the smallest one(s) in the shorter PLTS to the shorter one. For the second aspect, on behalf of preserving the original information as soon as possible, the probabilities of all the added linguistic terms are set to zero. Based on these two adjustments, the NPLTSs can be acquired directly.

Remark 1. The smallest one(s) can be obtained according to Ref. [55], here we will not take much space to give detailed introduction any more.

Then, in order to rank the elements of the PLTS, Ref. [18] defined the max element for the elements of PLTS as follows:

$$\diamond \quad \max\left(\Gamma(p)\right)=\max\left(\Gamma^{(\kappa)}\left(p^{(\kappa)}\right)\middle|\kappa=1,2,\ldots,\#\Gamma(p)\right)$$
$$=\left\{\Gamma^{(\kappa)}\left(p^{(\kappa)}\right)\middle|\max_{\kappa}\left(I\left(\Gamma^{(\kappa)}\right)p^{(\kappa)}\right)\middle|\kappa=1,2,\ldots,\#\Gamma(p)\right\}=\Gamma^{+}\left(p^{(\kappa)}\right),$$

where $I(\bullet)$ is a function that can obtain the subscripts of the matching linguistic term $\Gamma^{(\kappa)}$.

The min element for the elements of PLTS was stipulated as follows：

$$\diamond \quad \min\left(\Gamma(p)\right)=\min\left(\Gamma^{(\kappa)}\left(p^{(\kappa)}\right)\middle|\kappa=1,2,\ldots,\#\Gamma(p)\right)$$
$$=\left\{\Gamma^{(\kappa)}\left(p^{(\kappa)}\right)\middle|\min_{\kappa}\left(I\left(\Gamma^{(\kappa)}\right)p^{(\kappa)}\right)\middle|\kappa=1,2,\ldots,\#\Gamma(p)\right\}=\Gamma^{-}\left(p^{(\kappa)}\right).$$

To extend the superiorities of the PLTS, Xie et al. [18] defined the DPLTS that not only considers the perspective of membership degree, but also considers the perspective of non-membership.

2.2. The DPLTS

$$\Gamma(p)=\left\{\Gamma^{(s)}\left(p^{(s)}\right)\middle|\Gamma^{(s)}\in S, s=1,2,\ldots,\#\Gamma(p), p^{(s)}\geq 0, \sum_{s=1}^{\#\Gamma(p)} p^{(s)}\leq 1\right\}$$

and

$$\mathbb{N}(p)=\left\{\mathbb{N}^{(t)}\left(p^{(t)}\right)\middle|\mathbb{N}^{(t)}\in S, t=1,2,\ldots,\#\mathbb{N}(p), p^{(t)}\geq 0, \sum_{t=1}^{\#\mathbb{N}(p)} p^{(t)}\leq 1\right\}$$

are two PLTSs stipulated on the fixed set X. Based on the idea of Xie et al. [18], the DPLTS should contain the membership degree $\Gamma(p)$ and non-membership degree $\mathbb{N}(p)$ together, so they use the coming equation to depict the DPLTS:

$$D=\left\{\left\langle x,\Gamma(p),\mathbb{N}(p)\right\rangle, x\in X\right\}, \tag{2}$$

Where $\Gamma(p)$ and $\mathbb{N}(p)$ satisfy the conditions that $s_{-q}\leq\Gamma^{+}\oplus\mathbb{N}^{+}\leq s_{q}$, $s_{-q}\leq\Gamma^{-}\oplus\mathbb{N}^{-}\leq s_{q}$. Moreover, the pair $D=\left\langle\Gamma(p),\mathbb{N}(p)\right\rangle$ is named as dual probabilistic linguistic term element (DPLTE).

Furthermore, here in this paper, we provide the complement definition of the DPLTE as follows:

$$D^c=\begin{cases}\left\langle \mathbb{N}(p),\Gamma(p)\right\rangle, & if \quad \Gamma(p)\neq\varnothing \quad and \quad \mathbb{N}(p)\neq\varnothing,\\ \Gamma^c(p), & if \quad \Gamma(p)\neq\varnothing \quad and \quad \mathbb{N}(p)=\varnothing,\\ \mathbb{N}^c(p), & if \quad \Gamma(p)=\varnothing \quad and \quad \mathbb{N}(p)\neq\varnothing,\end{cases}$$

where the complement of the PLTS $\Gamma(p)$ is $\Gamma^c(p)=\left\{neg\left(\Gamma^{(s)}\right)\left(p^{(s)}\right)\middle|\Gamma^{(s)}\in S, p^{(s)}\geq 0, \sum_{s=1}^{\#\Gamma(p)} p^{(s)}\leq 1\right\}$,

and the complement of the PLTS $\mathbb{N}(p)$ is $\mathbb{N}(p)^c=\left\{neg\left(\mathbb{N}^{(t)}\right)\left(p^{(t)}\right)\middle|\mathbb{N}^{(t)}\in S, p^{(t)}\geq 0, \sum_{t=1}^{\#\mathbb{N}(p)} p^{(t)}\leq 1\right\}$.

Moreover, in the cause of cutting down the headache of the computation, Xie et al. [18] further proposed the following process to normalize the DPLTEs (NDPLTEs):

Let $D_A=\left\langle \Gamma_A(p),\mathbb{N}_A(p)\right\rangle$ and $D_B=\left\langle \Gamma_B(p),\mathbb{N}_B(p)\right\rangle$ be any two diverse DPLTEs. Firstly, similar to obtain the NPLTSs, we eliminate the difference in the number of the elements of two PLTSs $\Gamma_A(p)$ and $\Gamma_B(p)$, and obtain the two new PLTSs $\Gamma_A(p)$ and $\Gamma_B(p)$ with the same number of the elements, that is $\#\Gamma_A(p)=\#\Gamma_B(p)$. Secondly, we reschedule the PLTSs $\Gamma_A(p)$ and $\Gamma_B(p)$ in descending order, respectively. Moreover, we apply the same two steps to the PLTSs $\mathbb{N}_A(p)$ and $\mathbb{N}_B(p)$, then two new DPLTEs are acquired as $\dot{D}_A=\left\langle \dot{\Gamma}_A(p),\dot{\mathbb{N}}_A(p)\right\rangle$, $\dot{D}_B=\left\langle \dot{\Gamma}_2(p),\dot{\mathbb{N}}_B(p)\right\rangle$, where

$$\dot{\Gamma}_l(p)=\left\{\dot{\Gamma}_l^{(s)}\left(\dot{p}_l^{(s)}\right)\middle|\dot{\Gamma}_l^{(s)}\in S, s=1,2,\ldots,\#\dot{\Gamma}_l(p), \dot{p}_l^{(s)}\geq 0, \sum_{i=1}^{\#\dot{\Gamma}_l(p)} \dot{p}_l^{(s)}\leq 1\right\}$$

and

$$\dot{\mathbb{N}}_l(p)=\left\{\dot{\mathbb{N}}_l^{(t)}\left(\dot{p}_l^{(t)}\right)\middle|\dot{\mathbb{N}}_l^{(t)}\in S, t=1,2,\ldots,\#\dot{\mathbb{N}}_l(p), \dot{p}_l^{(t)}\geq 0, \sum_{t=1}^{\#\dot{\mathbb{N}}_l(p)} \dot{p}_l^{(t)}\leq 1\right\}$$

are displayed in descending order, $\#\dot{\Gamma}_A(p)=\#\dot{\Gamma}_B(p)$, $\#\dot{\mathbb{N}}_A(p)=\#\dot{\mathbb{N}}_B(p)$, $l=A,B$.

Moreover, Xie et al. [18] further defined the deviation degree between the two DPLTEs:

For any two DPLTEs D_A and D_B, the deviation degree between D_A and D_B is:

$$d_1\left(D_A,D_B\right)=\left[\frac{1}{2}\left(\left(\sum_{s=1}^{\#\dot{\Gamma}_A(p)}\left(\dot{p}_A^{\sigma(s)}I\left(\dot{\Gamma}_A^{\sigma(s)}\right)-\dot{p}_B^{\sigma(s)}I\left(\dot{\Gamma}_B^{\sigma(s)}\right)\right)^2\right)\Big/\#\dot{\Gamma}_A(p)\right.\right.$$
$$\left.\left.+\left(\sum_{t=1}^{\#\dot{\mathbb{N}}_A(p)}\left(\dot{p}_A^{\sigma(t)}I\left(\dot{\mathbb{N}}_A^{\sigma(t)}\right)-\dot{p}_B^{\sigma(t)}I\left(\dot{\mathbb{N}}_B^{\sigma(t)}\right)\right)^2\right)\Big/\#\dot{\mathbb{N}}_A(p)\right)\right]^{1/2} \quad l=A,B. \tag{3}$$

where $I\left(\dot{\Gamma}_l^{\sigma(s)}\right)$ are the subscripts of the matching linguistic term of the $\sigma(s)th$ largest element $\dot{\Gamma}_l^{\sigma(s)}\left(\dot{p}_l^{\sigma(s)}\right)$ in $\dot{\Gamma}_l(p)$ and $\dot{p}_l^{\sigma(s)}$ is the matching probability of linguistic term $\dot{\Gamma}_l^{\sigma(s)}$; $I\left(\dot{\mathbb{N}}_l^{\sigma(t)}\right)$ are the subscripts of the matching linguistic term of the $\sigma(t)th$ largest element $\dot{\mathbb{N}}_l^{\sigma(t)}\left(\dot{p}_l^{\sigma(t)}\right)$ in $\dot{\mathbb{N}}_l(p)$ and $\dot{p}_l^{\sigma(t)}$ is the matching probability of the linguistic term $\dot{\mathbb{N}}_l^{\sigma(t)}$. Besides, without special instructions, all the following DPLTEs are the ordered DPLTEs with the same cardinality, and the elements are ranked in descending order.

Actually, the physical significance for deviation degree is similar to the distance measure. Hence, without loss of generality, in the following, we extend it to the more general situation and define the distance measure between two DPLTSs:

Let $D_1=\left\{\left\langle x_s,L_1(p),U_1(p)\right\rangle,x_s\in X\right\}$ and $D_2=\left\{\left\langle x_s,L_2(p),U_2(p)\right\rangle,x_s\in X\right\}$ be two DPLTSs that are clearly defined in the mentioned set $S=\left\{s_\alpha\left|\alpha\in[-q,q]\right.\right\}$ and the established set $X=\{x_1,x_2,...,x_n\}$, where

$$\Gamma_1(p)=\left\{\Gamma_1^{(\partial_1)}\left(p_1^{(\partial_1)}\right)\middle|\Gamma_1^{(\partial_1)}\in S,\partial_1=1,2,\ldots,\#\Gamma_1(p),p_1^{(\partial_1)}\geq 0,\sum_{\partial_1=1}^{\#\Gamma_1(p)}p_1^{(\partial_1)}\leq 1\right\},$$

$$\mathbb{N}_1(p)=\left\{\mathbb{N}_1^{(\Im_1)}\left(p_1^{(\Im_1)}\right)\middle|\mathbb{N}_1^{(\Im_1)}\in S,\Im_1=1,2,\ldots,\#\mathbb{N}_1(p),p_1^{(\Im_1)}\geq 0,\sum_{\Im_1=1}^{\#\mathbb{N}_1(p)}p_1^{(\Im_1)}\leq 1\right\},$$

$$\Gamma_2(p)=\left\{\Gamma_2^{(\partial_2)}\left(p_2^{(\partial_2)}\right)\middle|\Gamma_2^{(\partial_2)}\in S,\partial_2=1,2,\ldots,\#\Gamma_2(p),p_2^{(\partial_2)}\geq 0,\sum_{\partial_2=1}^{\#\Gamma_2(p)}p_2^{(\partial_2)}\leq 1\right\},$$

$$\mathbb{N}_2(p)=\left\{\mathbb{N}_2^{(\Im_2)}\left(p_2^{(\Im_2)}\right)\middle|\mathbb{N}_2^{(\Im_2)}\in S,\Im_2=1,2,\ldots,\#\mathbb{N}_2(p),p_2^{(\Im_2)}\geq 0,\sum_{\Im_2=1}^{\#\mathbb{N}_2(p)}p_2^{(\Im_2)}\leq 1\right\}.$$

1) The generalized dual probabilistic linguistic normalized distance:

$$d_2\left(D_1,D_2\right)=\left[\frac{1}{2n}\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_1\left(p\right)}\sum_{\partial_1=1}^{\#\Gamma_1(p)}\left|\frac{I\left(\Gamma_1^{(\partial_1)}\right)}{q}\times p_1^{(\partial_1)}-\frac{I\left(\Gamma_2^{(\mu_2)}\right)}{q}\times p_2^{(\partial_2)}\right|^{\lambda}+\right.\right.$$

$$\left.\left.\frac{1}{\#\mathbb{N}_1(p)}\sum_{\Im_1=1}^{\#\mathbb{N}_1(p)}\left|\frac{I\left(\mathbb{N}_1^{(\Im_1)}\right)}{q}\times p_1^{(\Im_1)}-\frac{I\left(\mathbb{N}_2^{(\Im_2)}\right)}{q}\times p_2^{(\Im_2)}\right|^{\lambda}\right)\right]^{1/\lambda} \tag{4}$$

where $I\left(\bullet\right)$ is a function that can obtain the subscripts of the corresponding linguistic term, and if $\lambda=1$, then the generalized dual probabilistic linguistic normalized distance shall turn into the dual probabilistic linguistic normalized Hamming distance d_{NH}; if $\lambda=2$, the generalized dual probabilistic linguistic normalized distance shall turn into the dual probabilistic linguistic normalized Euclidean distance d_{NE}.

Furthermore, supposing the Hausdorff metric is applied to the distance, then

2) The general generalized dual probabilistic linguistic Hausdorff distance is defined as:

$$d_3\left(D_1,D_2\right)=\left[\frac{1}{n}\sum_{s=1}^{n}\max\left\{\max_{\partial_1}\frac{1}{\#\Gamma_1\left(p\right)}\left|\frac{I\left(\Gamma_1^{(\partial_1)}\right)}{q}\times p_1^{(\partial_1)}-\frac{I\left(\Gamma_2^{(\mu_2)}\right)}{q}\times p_2^{(\partial_2)}\right|^{\lambda},\right.\right.$$

$$\left.\left.\max_{\Im_1}\frac{1}{\#\mathbb{N}_1(p)}\left|\frac{I\left(\mathbb{N}_1^{(\Im_1)}\right)}{q}\times p_1^{(\Im_1)}-\frac{I\left(\mathbb{N}_2^{(\Im_2)}\right)}{q}\times p_2^{(\Im_2)}\right|^{\lambda}\right\}\right]^{1/\lambda} \tag{5}$$

If $\lambda=1$, then the generalized dual probabilistic linguistic Hausdorff distance becomes the dual probabilistic linguistic normalized Hamming–Hausdorff distance d_{NHH}; if $\lambda=2$, then the generalized dual probabilistic linguistic Hausdorff distance becomes the dual probabilistic linguistic normalized Euclidean-Hausdorff distance d_{NEH}.

Example 1. Let $S=\left\{s_\alpha\left|\alpha\in\left[-6,6\right]\right.\right\}$ be a linguistic term set, for two ordered DPLTSs with the same cardinalities: $D_1=\left\{\left\langle x_1,\left\{s_2\left(0.75\right),s_3\left(0.25\right)\right\},\left\{s_5\left(1\right),s_5\left(0\right)\right\}\right\rangle,\left\langle x_2,\left\{s_0\left(0.4\right),s_{-2}\left(0.6\right)\right\},\left\{s_2\left(0.3\right),s_1\left(0.5\right)\right\}\right\rangle\right\}$ and $D_2=\left\{\left\langle x_1,\left\{s_{-3}(0.1),s_{-4}(0.8)\right\},\left\{s_5(0.7),s_4(0.3)\right\}\right\rangle,\left\langle x_2,\left\{s_3(0.4),s_1(0.6)\right\},\left\{s_{-1}(0.5),s_{-2}(0.5)\right\}\right\rangle\right\}$, then the distance d_2 between D_1 and D_2 can be calculated as follows:

$$d_2\left(D_1,D_2\right)=\left[\frac{1}{2n}\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_1\left(p\right)}\sum_{\partial_1=1}^{\#\Gamma_1(p)}\left|\frac{I\left(\Gamma_1^{(\partial_1)}\right)}{q}\times p_1^{(\partial_1)}-\frac{I\left(\Gamma_2^{(\mu_2)}\right)}{q}\times p_2^{(\partial_2)}\right|^{\lambda}+\right.\right.$$

$$\left.\left.\frac{1}{\#\mathbb{N}_1(p)}\sum_{\Im_1=1}^{\#\mathbb{N}_1(p)}\left|\frac{I\left(\mathbb{N}_1^{(\Im_1)}\right)}{q}\times p_1^{(\Im_1)}-\frac{I\left(\mathbb{N}_2^{(\Im_2)}\right)}{q}\times p_2^{(\Im_2)}\right|^{\lambda}\right)\right]^{1/\lambda}$$

$$=\left[\frac{1}{8}\left(\left|\frac{1.5}{6}+\frac{0.3}{6}\right|^{\lambda}+\left|\frac{0.75}{6}+\frac{3.2}{6}\right|^{\lambda}+\left|\frac{0}{6}-\frac{1.2}{6}\right|^{\lambda}+\left|\frac{-1.2}{6}-\frac{0.6}{6}\right|^{\lambda}+\left|\frac{5}{6}-\frac{3.5}{6}\right|^{\lambda}+\left|\frac{0}{6}-\frac{1.2}{6}\right|^{\lambda}+\left|\frac{0.6}{6}+\frac{0.5}{6}\right|^{\lambda}+\left|\frac{0.5}{6}+\frac{1}{6}\right|^{\lambda}\right)\right]^{1/\lambda}$$

$$=\left[\frac{1}{8\times 6}\left(\left|1.8\right|^{\lambda}+\left|3.95\right|^{\lambda}+\left|1.2\right|^{\lambda}+\left|-1.8\right|^{\lambda}+\left|1.5\right|^{\lambda}+\left|-1.2\right|^{\lambda}+\left|1.1\right|^{\lambda}+\left|1.5\right|^{\lambda}\right)\right]^{1/\lambda}$$

If $\lambda=1$, then d_2 reduces to d_{NH}, and $d_{NH}=0.2927$; if $\lambda=2$, then d_2 reduces to d_{NE}, and $d_{NE}=0.1113$.

Similarly, the distance d_3 between D_1 and D_2 can be calculated as follows:

$$d_3\left(D_1,D_2\right)=\left[\frac{1}{4}\left(\left(\frac{3.95}{6}\right)^{\lambda}+\left(\frac{1.8}{6}\right)^{\lambda}\right)\right]^{1/\lambda}$$

If $\lambda=1$, then d_3 reduces to d_{NHH}, and $d_{NHH}=0.2396$; if $\lambda=2$, then d_3 reduces to d_{NEH}, and $d_{NEH}=0.1809$.

3. The correlation coefficient for DPLTSs

Owing to the fact that the correlation coefficient plays an important role during the practical application process. We propose the definition of the correlation coefficient for DPLTSs in the following section. Similar to the proposed DPLTSs, the computation of the correlation coefficient is executed by the following two parts: one is to calculate the correlation coefficient of two matching memberships, and the other is to figure out the correlation coefficient of two non-memberships. Then we put them together as the final correlation coefficient for the DPLTSs.

3.1. The matching correlation coefficient for DPLTSs

Let $X=\left\{x_1,x_2,...,x_n\right\}$ be a fixed set, $D_A=\left\{\left\langle x_i,L_A\left(p\right),U_A\left(p\right)\right\rangle,x_i\in X\right\}$ and $D_B=\left\{\left\langle x_i,L_B\left(p\right),U_B\left(p\right)\right\rangle,x_i\in X\right\}$ be two DPLTSs on the linguistic term set $S=\left\{s_\alpha \middle| \alpha\in[-q,q]\right\}$, then their correlation coefficient between the two DPLTSs can be defined as the following equation:

$$\rho=\frac{\rho_1+\rho_2}{2} \tag{6}$$

where

$$\rho_1=\frac{\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)}I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)}-m\left(\Gamma_A(p)\right)\right)\left(\frac{1}{\#\Gamma_B(p)}\sum_{\mu_B=1}^{\#\Gamma_B(p)}I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)}-m\left(\Gamma_B(p)\right)\right)}{\left[\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)}I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)}-m\left(\Gamma_A(p)\right)\right)^2\right]^{1/2}\left[\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)}I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)}-m\left(\Gamma_B(p)\right)\right)^2\right]^{1/2}}$$

$$\rho_2=\frac{\sum_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_A(p)}\sum_{\Im_A=1}^{\#\mathbb{N}_A(p)}I\left(\mathbb{N}_A^{(\Im_A)}\right)p_A^{(\Im_A)}-m\left(\mathbb{N}_A(p)\right)\right)\left(\frac{1}{\#\mathbb{N}_B(p)}\sum_{\Im_B=1}^{\#\mathbb{N}_B(p)}I\left(\mathbb{N}_B^{(\Im_B)}\right)p_B^{(\Im_B)}-m\left(\mathbb{N}_B(p)\right)\right)}{\left[\sum_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_A(p)}\sum_{\Im_A=1}^{\#\mathbb{N}_A(p)}I\left(\mathbb{N}_A^{(\Im_A)}\right)p_A^{(\Im_A)}-m\left(\mathbb{N}_A(p)\right)\right)^2\right]^{1/2}\left[\sum_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_B(p)}\sum_{\Im_B=1}^{\#\mathbb{N}_B(p)}I\left(\mathbb{N}_B^{(\Im_B)}\right)p_B^{(\Im_B)}-m\left(\mathbb{N}_B(p)\right)\right)^2\right]^{1/2}}$$

$m\left(\Gamma_A(p)\right)=\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)}I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)}\right)$, $m\left(\Gamma_B(p)\right)=\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)}I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)}\right)$,

$m\left(\mathbb{N}_A(p)\right)=\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_A(p)}\sum_{\Im_A=1}^{\#\mathbb{N}_A(p)}I\left(\mathbb{N}_A^{(\Im_A)}\right)p_A^{(\Im_A)}\right)$, $m\left(\mathbb{N}_B(p)\right)=\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_B(p)}\sum_{\Im_B=1}^{\#\mathbb{N}_B(p)}I\left(\mathbb{N}_B^{(\Im_B)}\right)p_B^{(\Im_B)}\right)$.

Then the correlation coefficient between the two DPLTSs D_A and D_B satisfies the following properties:

1) $\rho\left(D_A,D_B\right)=\rho\left(D_B,D_A\right)$;

2) $\rho\left(D_A,D_B\right)=1$, if $D_A{=}D_B$;

3) $\left|\rho\left(D_A,D_B\right)\right|\le 1$.

Proof. 1) The proof is apparent;

2) If $D_A{=}D_B$, then

$$\rho_1 = \frac{\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m\left(\Gamma_A(p)\right)\right)\left(\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)} I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)} - m\left(\Gamma_B(p)\right)\right)}{\left[\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m\left(\Gamma_A(p)\right)\right)^2\right]^{1/2}\left[\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)} I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)} - m\left(\Gamma_B(p)\right)\right)^2\right]^{1/2}}$$

$$= \frac{\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m\left(\Gamma_A(p)\right)\right)^2}{\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m\left(\Gamma_A(p)\right)\right)^2} = 1$$

Analogously, if $\rho_2 = 1$, then $\rho = \frac{\rho_1+\rho_2}{2} = 1$.

3) According to the Cauchy inequality $\left|\sum_{s=1}^{n} u_s v_s\right|^2 \le \sum_{s=1}^{n}|u_s|^2 \cdot \sum_{s=1}^{n}|v_s|^2$, then

$$\left|\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m\left(\Gamma_A(p)\right)\right)\left(\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)} I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)} - m\left(\Gamma_B(p)\right)\right)\right|^2 \le$$

$$\sum_{s=1}^{n}\left|\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m\left(\Gamma_A(p)\right)\right|^2 \cdot \sum_{s=1}^{n}\left|\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)} I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)} - m\left(\Gamma_B(p)\right)\right|^2$$

Taking the square root of two flanks, this inequality can be decomposed into the following:

$$\left|\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m\left(\Gamma_A(p)\right)\right)\left(\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)} I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)} - m\left(\Gamma_B(p)\right)\right)\right| \le$$

$$\left[\sum_{s=1}^{n}\left|\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m\left(\Gamma_A(p)\right)\right|^2\right]^{1/2} \cdot \left[\sum_{i=1}^{n}\left|\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)} I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)} - m\left(\Gamma_B(p)\right)\right|^2\right]^{1/2}$$

Therefore, this inequality can be rewritten as:

$$\frac{\left|\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m\left(\Gamma_A(p)\right)\right)\left(\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)} I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)} - m\left(\Gamma_B(p)\right)\right)\right|}{\left[\sum_{s=1}^{n}\left|\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)} I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)} - m\left(\Gamma_A(p)\right)\right|^2\right]^{1/2} \cdot \left[\sum_{s=1}^{n}\left|\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)} I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)} - m\left(\Gamma_B(p)\right)\right|^2\right]^{1/2}} \le 1$$

That is, $|\rho_1| \le 1$. Similar to ρ_1, we can get $|\rho_2| \le 1$. Then $|\rho| = \left|\frac{\rho_1+\rho_2}{2}\right| = \frac{|\rho_1+\rho_2|}{2} \le \frac{|\rho_1|}{2} + \frac{|\rho_2|}{2} \le \frac{1}{2} + \frac{1}{2} = 1$.

Example 2. Let $S=\{s_\alpha \mid \alpha\in[-6,6]\}$ be a linguistic term set, for two DPLTSs D_A and D_B on the set $X=\{x_1,x_2\}$, where

$D_A=\{\langle x_1,\{s_1(0.75),s_2(0.25)\},\{s_4(1),s_4(0)\}\rangle,\langle x_2,\{s_0(0.4),s_{-2}(0.6)\},\{s_2(0.3),s_1(0.5)\}\rangle\}$ and

$D_B=\{\langle x_1,\{s_{-4}(0.4),s_{-3}(0.1)\},\{s_2(0.2),s_1(0.6)\}\rangle,\langle x_2,\{s_1(0.5),s_3(0.5)\},\{s_{-1}(0.5),s_{-3}(0.5)\}\rangle\}$.

Then based on Eq. (6), the correlation coefficient between the DPLTSs D_A and D_B can be calculated below:

$$m(\Gamma_A(p))=\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)}I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)}\right)=\frac{1}{2}\left(\frac{1}{2}(1\times0.75+2\times0.25)+\frac{1}{2}\left(0\times0.4+(-2)\times0.6\right)\right)=0.0125$$

$$m(\Gamma_B(p))=\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)}I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)}\right)=\frac{1}{2}\left(\frac{1}{2}((-4)\times0.4+(-3)\times0.1)+\frac{1}{2}(1\times0.5+3\times0.5)\right)=0.025$$

$$m(\mathbb{N}_A(p))=\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_A(p)}\sum_{\Im_A=1}^{\#\mathbb{N}_A(p)}I\left(\mathbb{N}_A^{(\Im_A)}\right)p_A^{(\Im_A)}\right)=\frac{1}{2}\left((4\times1)+\frac{1}{2}(2\times0.3+1\times0.5)\right)=2.275$$

$$m(\mathbb{N}_B(p))=\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{\#\mathbb{N}_B(p)}\sum_{\Im_B=1}^{\#\mathbb{N}_B(p)}I\left(\mathbb{N}_B^{(\Im_B)}\right)p_B^{(\Im_B)}\right)=\frac{1}{2}\left(\frac{1}{2}(2\times0.2+1\times0.6)+\frac{1}{2}((-1)\times0.5+(-3)\times0.5)\right)=-0.25$$

$\rho_1=-0.4540$, $\rho_2=1$, then $\rho=\dfrac{\rho_1+\rho_2}{2}=0.273$.

Besides, in the real application, the object maybe assigned different weights. Hence, for purpose of better applying the correlation coefficient to the real application, we further introduce the weighted correlation coefficient for DPLTSs in the following subsection.

3.2. The matching weighted correlation coefficient for DPLTSs

Assume that $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weighting vector of $x_s\,(s=1,2,\ldots,n)$ that satisfies $\omega_s\geq0$ and $\sum_{s=1}^{n}\omega_s=1$. As the extension of Eq. (6), the weighted correlation coefficient can be denoted as follows:

$$\rho_\omega=\frac{\rho_{\omega1}+\rho_{\omega2}}{2} \tag{7}$$

where

$$\rho_{\omega 1}=\frac{\sum_{s=1}^{n}\omega_s\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)}I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)}-m\left(\Gamma_A(p)\right)\right)\left(\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)}I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)}-m\left(\Gamma_B(p)\right)\right)}{\left[\sum_{s=1}^{n}\omega_s\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)}I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)}-m\left(\Gamma_A(p)\right)\right)^2\right]^{1/2}\left[\sum_{s=1}^{n}\omega_s\left(\frac{1}{\#\Gamma_B(p)}\sum_{\partial_B=1}^{\#\Gamma_B(p)}I\left(\Gamma_B^{(\partial_B)}\right)p_B^{(\partial_B)}-m\left(\Gamma_B(p)\right)\right)^2\right]^{1/2}}$$

$$\rho_{\omega 2}=\frac{\sum_{s=1}^{n}\omega_s\left(\frac{1}{\#\mathbb{N}_A(p)}\sum_{\Im_A=1}^{\#\mathbb{N}_A(p)}I\left(\mathbb{N}_A^{(\Im_A)}\right)p_A^{(\Im_A)}-m\left(\mathbb{N}_A(p)\right)\right)\left(\frac{1}{\#\mathbb{N}_B(p)}\sum_{\Im_B=1}^{\#\mathbb{N}_B(p)}I\left(\mathbb{N}_B^{(\Im_B)}\right)p_B^{(\Im_B)}-m\left(\mathbb{N}_B(p)\right)\right)}{\left[\sum_{s=1}^{n}\omega_s\left(\frac{1}{\#\mathbb{N}_A(p)}\sum_{\Im_A=1}^{\#\mathbb{N}_A(p)}I\left(\mathbb{N}_A^{(\Im_A)}\right)p_A^{(\Im_A)}-m\left(\mathbb{N}_A(p)\right)\right)^2\right]^{1/2}\left[\sum_{s=1}^{n}\omega_s\left(\frac{1}{\#\mathbb{N}_B(p)}\sum_{\Im_B=1}^{\#\mathbb{N}_B(p)}I\left(\mathbb{N}_B^{(\Im_B)}\right)p_B^{(\Im_B)}-m\left(\mathbb{N}_B(p)\right)\right)^2\right]^{1/2}}$$

It is easy to see that if $\omega_i=1/n(i=1,2,\ldots,n)$, then Eq. (7) turns into Eq. (6). Analogously, the weighted correlation coefficient also satisfies the following basic properties:

1) $\rho_\omega(D_A,D_B)=\rho_\omega(D_B,D_A)$;

2) $\rho_\omega(D_A,D_B)=1$, if $D_A=D_B$;

3) $|\rho_\omega(D_A,D_B)|\le 1$.

By analogy with the previous proof method, the proofs of properties (1)–(3) are omitted.

4. Multi-attribute group decision-making based on the correlation coefficient measure

This section aims to apply the proposed weighted correlation coefficient to the specific decision-making approach, not only check its rationality, but also help solve the decision-making problem.

4.1. The basic introduction for the multi-attribute group decision-making problem

This paper aims to handle a decision-making problem under the multi-attributes dual probabilistic linguistic circumstance. Assume that a set of definite alternatives is $\{a_1,a_2,\ldots,a_m\}$, and the set of attributes is $\{c_1,c_2,\ldots,c_n\}$. The oth ($o=1,2,\ldots,n$) DM provides the decision-making information for the alternatives $a_i(i=1,2,\ldots,m)$ in relation to the attributes $c_j(j=1,2,\ldots,n)$ by the DPLTEs

$D_{ij}^{o}=\left\langle L_{ij}^{o}\left(p\right),U_{ij}^{o}\left(p\right)\right\rangle$ and the weight ω^{o} ($o=1,2,\ldots,n$) of each DM satisfies ($\omega^{o}\in\left[0,1\right],\sum_{o=1}^{n}\omega^{o}=1$).

Therefore, the decision information of the oth DM is able to be expressed as $D_{o}=\left(D_{ij}^{o}\right)_{m\times n}$, where

$$D_{o}=\left(D_{ij}^{o}\right)_{m\times n}=\begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1n} \\ D_{21} & D_{22} & \cdots & D_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ D_{m1} & D_{m2} & \cdots & D_{mn} \end{pmatrix}, \text{ for } o=1,2,\ldots,n \tag{8}$$

4.2. The entropy for the DPLTSs

After the DMs provide their respective decision-making matrices, we based on the following aggregation operator to calculate the group decision-making matrix:

For a set of DPLTSs $D_{A}=\left\{\left\langle x_{i},\Gamma_{A}\left(p\right),\mathbb{N}_{A}\left(p\right)\right\rangle,x_{i}\in X\right\}$ on the set $X=\left\{x_{1},x_{2},...,x_{n}\right\}$, then

$$DPLA\left(D_{A_1},D_{A_2},\ldots,D_{A_n}\right)=\frac{1}{n}\left(D_{A_1}\oplus D_{A_1}\oplus\ldots\oplus D_{A_n}\right) \tag{9}$$

is named as dual probabilistic linguistic averaging (DPLA) operator.

Moreover, combined with the weight vector $\omega=\left(\omega_{1},\omega_{2},\ldots,\omega_{n}\right)^{T}$ of $D_{A_i}\left(i=1,2,\ldots,n\right)$ that satisfies the conditions: $\omega_{i}\in\left[0,1\right]$ and $\sum_{i=1}^{n}\omega_{i}=1$, then

$$DPLWA\left(D_{A_1},D_{A_2},\ldots,D_{A_n}\right)=\omega_{1}D_{A_1}\oplus\omega_{1}D_{A_1}\oplus\ldots\oplus\omega_{n}D_{A_n} \tag{10}$$

is named as dual probabilistic linguistic weighted averaging (DPLWA) operator.

Besides, if $\omega=\left(1/n,1/n,\ldots,1/n\right)^{T}$, the DPLWA operator will evolve into the DPLA operator, and

$$D_{A}\oplus D_{B}=\bigcup\nolimits_{\Gamma_{A}^{(\partial_{A})}\in\Gamma_{A},\mathbb{N}_{A}^{(\Im_{A})}\in\mathbb{N}_{A},\Gamma_{B}^{(\partial_{B})}\in\Gamma_{B},\mathbb{N}_{B}^{(\Im_{B})}\in\mathbb{N}_{B}}\left\{\left\{\Gamma_{A}^{(\partial_{A})}\oplus\Gamma_{B}^{(\partial_{B})}\left(p_{A}^{(\partial_{A})}p_{B}^{(\partial_{B})}\right)\right\},\left\{\mathbb{N}_{A}^{(\partial_{A})}\otimes\mathbb{N}_{B}^{(\partial_{B})}\left(p_{A}^{(\partial_{A})}p_{B}^{(\partial_{B})}\right)\right\}\right\}$$

$$\lambda D_{A}=\bigcup\nolimits_{\Gamma_{A}^{(\partial_{A})}\in\Gamma_{A},\mathbb{N}_{A}^{(\Im_{A})}\in\mathbb{N}_{A}}\left\{\left\{\lambda\Gamma_{A}^{(\partial_{A})}\left(p_{A}^{(\partial_{A})}\right)\right\},\left\{\lambda U_{A}^{(\Im_{A})}\left(p_{A}^{(\Im_{A})}\right)\right\}\right\}$$

Example 3. Let $S=\left\{s_{\alpha}\left|\alpha\in\left[-6,6\right]\right.\right\}$ be a concrete linguistic term set, regarding two DPLTEs D_{1} and D_{2}, where $D_{1}=\left\langle\left\{s_{-2}\left(0.6\right),s_{-1}\left(0.4\right)\right\},\left\{s_{4}\left(1\right),s_{4}\left(0\right)\right\}\right\rangle$ and $D_{2}=\left\langle\left\{s_{2}(0.5),s_{3}(0.5)\right\},\left\{s_{-1}(0.5),s_{-3}(0.5)\right\}\right\rangle$, $\lambda=0.5$, then

$$D_1 \oplus D_2 = \left\langle \left\{ s_0(0.3), s_1(0.3), s_1(0.2), s_2(0.2) \right\}, \left\{ s_{-4}(0.5), s_{-6}(0.5), s_{-4}(0), s_{-6}(0) \right\} \right\rangle$$
$$= \left\langle \left\{ s_2(0.2), s_1(0.2), s_1(0.3), s_0(0.3) \right\}, \left\{ s_{-4}(0.5), s_{-6}(0.5) \right\} \right\rangle$$
$$\lambda D_1 = \left\langle \left\{ s_{-1}(0.6), s_{-0.5}(0.4) \right\}, \left\{ s_2(1), s_2(0) \right\} \right\rangle = \left\langle \left\{ s_{-1}(0.6), s_{-0.5}(0.4) \right\}, \left\{ s_2(1) \right\} \right\rangle$$

Then in order to obtain objective and relative practical decision results, by learning from Ref. [38], we divide the weight into the subjective form and objective form. The subjective weight is given by the DMs directly. The objective weight that means the opposing moment of miscellaneous attributes without considering the DMs' preferences is obtained by the mathematical models. Among so many methods to obtain the objective weight [56-60], the entropy weight is one of the widely employed methods to settle the decision-making issues. Moreover, on the strength of the axiomatic definition of entropy measure in Ref. [61], here we define the entropy measure of the DPLTE as follows:

For a DPLTE $D = \left\langle \Gamma(p), \mathbb{N}(p) \right\rangle$, where $\Gamma(p) = \left\{ \Gamma^{(\partial)}(p^{(\partial)}) \middle| \Gamma^{(\partial)} \in S, p^{(\partial)} \geq 0, \sum_{\partial=1}^{\#\Gamma(p)} p^{(\partial)} \leq 1 \right\}$, $\partial = 1,2,...,\#\Gamma(p)$, and $\mathbb{N}(p) = \left\{ \mathbb{N}^{(\Im)}(p^{(\Im)}) \middle| \mathbb{N}^{(\Im)} \in S, p^{(\Im)} \geq 0, \sum_{\Im=1}^{\#\mathbb{N}(p)} p^{(\Im)} \leq 1 \right\}$, $\Im = 1,2,...,\#\mathbb{N}(p)$, then the following absolute-valued function E is called the entropy measure of a DPLTE and satisfies the axiomatic requirements:

a) $0 \leq E(D) \leq 1$;

b) $E(D)=0$, if and only if $\Delta = 1$ and $\Theta=0$;

c) $E(D)=1$, if and only if $\Delta = 0$ and $\Theta=1$;

d) $E(D^c)=E(D)$;

e) E is monotonic decreasing with regards to Δ and monotonic increasing with respect to Θ.

where

$$E(D) = \frac{1}{\#\Gamma(p)} \sum_{\partial=1}^{\#\Gamma(p)} \left(1 - \frac{\Delta^{\zeta} + (1-\Theta)^{\zeta}}{2} \right) \tag{11}$$

$\Delta=\left|\frac{I\left(\Gamma^{(\partial)}\right)}{q}\cdot p^{(\partial)}-\frac{I\left(\mathbb{N}^{(\Im)}\right)}{q}\cdot p^{(\Im)}\right|$, $\Theta=1-\left(\frac{I\left(\Gamma^{(\partial)}\right)}{q}\cdot p^{(\partial)}+\frac{I\left(\mathbb{N}^{(\Im)}\right)}{q}\cdot p^{(\Im)}\right)$, and $\zeta>0$.

Remark 2: Provided that $\#\Gamma(p)\neq\#\mathbb{N}(p)$, then we need to adjust the cardinality of the two PLTSs, and get the same cardinality. Similar to the normalization process of the PLTSs [13], we need to add some elements for the shorter one, the added linguistic terms are the smallest ones, and the probabilities of all the added linguistic terms are zero. Next, please see the following example for details:

Example 4. Let $D=\left\langle\left\{s_1(0.75),s_2(0.25)\right\},\left\{s_4(1)\right\}\right\rangle$ be a DPLTE on the specific set $S=\left\{s_\alpha \middle| \alpha\in[-6,6]\right\}$, $\zeta=1$. Obviously, $\#\Gamma(p)=2$, $\#\mathbb{N}(p)=1$, and $\#\Gamma(p)\neq\#\mathbb{N}(p)$, on behalf of calculating the entropy, it is necessary to add one element for the shorter one $\mathbb{N}(p)$, and get $D=\left\langle\left\{s_1(0.75),s_2(0.25)\right\},\left\{s_4(1),s_4(0)\right\}\right\rangle$, then

$$E(D)=\frac{1}{\#\Gamma(p)}\sum_{\partial=1}^{\#\Gamma(p)}\left(1-\frac{\Delta^{\zeta}+(1-\Theta)^{\zeta}}{2}\right)$$

$$=\frac{1}{2}\left(1-\frac{\left|\frac{1}{6}\times0.75-\frac{4}{6}\times1\right|+\left(1-\left(1-\left(\frac{1}{6}\times0.75+\frac{4}{6}\times1\right)\right)\right)}{2}+1-\frac{\left|\frac{2}{6}\times0.25-\frac{4}{6}\times0\right|+\left(1-\left(1-\left(\frac{2}{6}\times0.25+\frac{4}{6}\times0\right)\right)\right)}{2}\right)$$

$$=\frac{1}{2}\left(1-\frac{\left|\frac{1}{6}\times0.75-\frac{4}{6}\times1\right|+\left(\frac{1}{6}\times0.75+\frac{4}{6}\times1\right)}{2}+1-\frac{\left|\frac{2}{6}\times0.25-\frac{4}{6}\times0\right|+\left(\frac{2}{6}\times0.25+\frac{4}{6}\times0\right)}{2}\right)$$

$$=\frac{1}{2}\left(1-\frac{8/6}{2}+1-\frac{1/6}{2}\right)=\frac{1}{2}\left(2-\frac{9}{12}\right)=0.625$$

Then the entropy weight measure that determines the objective weight based on the dual probabilistic linguistic environment is able to be calculated below:

$$C_i\left(D(x)\right)=\frac{1-E\left(D(x_i)\right)}{n-\sum_{i=1}^{n}E\left(D(x_i)\right)} \tag{12}$$

where $D(x)$ is a DPLTS in the settled set $X=\{x_1,x_2,...,x_n\}$, and $E(D(x_i))$ is calculated by Eq. (11).

Then for a DPLTS

$$D(x)=\{\langle x_1,\{s_1(0.75),s_2(0.25)\},\{s_4(1),s_4(0)\}\rangle,\langle x_2,\{s_0(0.4),s_{-2}(0.6)\},\{s_2(0.3),s_1(0.5)\}\rangle\}$$

the entropy weight measure can be calculated as follows:

$$C_1(D(x))=\frac{1-E(D(x_1))}{n-\sum_{i=1}^{n}E(D(x_i))}=\frac{1-0.625}{2-(0.625+0.908)}=0.803$$

$$C_2(D(x))=\frac{1-E(D(x_2))}{n-\sum_{i=1}^{n}E(D(x_i))}=\frac{1-0.908}{2-(0.625+0.908)}=0.197$$

Property. The suggested entropy weight measure meets the following contents:

1) $C_i(D(x))\in[0,1]$;

2) $\sum_{i=1}^{n}C_i(D(x))=1$.

Proof.

1) Owing to that the entropy $E(D)$ satisfies $0\le E(D)\le 1$, so $n-\sum_{i=1}^{n}E(D(x_i))\in[0,1]$. Besides, $\left(1-E(D(x_i))\right)+\left(n-1+\sum_{j=1,j\neq i}^{n}E\left(D(x_j)\right)\right)=n-\sum_{i=1}^{n}E(D(x_i))\ge 0$, hence, $n-1+\sum_{j=1,j\neq i}^{n}E\left(D(x_j)\right)\ge 0$, that is to say, $n-\sum_{i=1}^{n}E(D(x_i))\ge 1-E(D(x_i))$, $C_i(D(x))=\dfrac{1-E(D(x_i))}{n-\sum_{i=1}^{n}E(D(x_i))}\in[0,1]$.

2) $$\sum_{i=1}^{n}C_i(D(x))=\sum_{i=1}^{n}\frac{1-E(D(x_i))}{n-\sum_{i=1}^{n}E(D(x_i))}=\frac{\sum_{i=1}^{n}1-E(D(x_i))}{n-\sum_{i=1}^{n}E(D(x_i))}=\frac{n-\sum_{i=1}^{n}E(D(x_i))}{n-\sum_{i=1}^{n}E(D(x_i))}=1$$

4.3. The general weighted process for settling multi-attribute group decision-making problem

Based on the subjective weight vector $\omega=\left(\omega_1,\omega_2,\ldots,\omega_n\right)^T$ of the DMs, Eq. (12) and Ref. [39], the final weight vector $W=\left(W_1,W_2,\ldots,W_n\right)^T$ can be obtained by the following equation:

$$W_i=\frac{\omega_i C_i\left(D(x)\right)}{\sum_{i=1}^{n}\omega_i C_i\left(D(x)\right)} \tag{13}$$

Generally, in the multi-attribute group decision-making environment, the ideal point can help recognize the best alternative in the given alternative set. In spite that the ideal alternative does not always exist in the actual world, it does offer an available academic formulation to assess alternatives.

Consequently, we consider to define each ideal DPLTE $D_j^*=\left\langle \Gamma_j^*(p),\mathbb{N}_j^*(p)\right\rangle$ in each ideal alternative $a^*=\left\{\left\langle c_j,D_j^*\right\rangle\right\}$ for $j=1,2,\ldots,n$. Given that all the DPLTEs have been handled with the same cardinality, and the definitions of positive ideal solution (PIS) and negative ideal solution (NIS) for PLTSs [13], the positive ideal solution of alternatives can be denoted below:

Let $D=\left(D_{ij}\right)_{m\times n}$ be the dual probabilistic linguistic decision-making matrix, where $D_{ij}=\left\langle \Gamma_{ij}(p),\mathbb{N}_{ij}(p)\right\rangle$, then $D^*=\left(D_1^*,D_2^*,\ldots,D_n^*\right)$ is called PIS of alternatives, where $D_j^*=\left\langle \Gamma_j(p)^*,\mathbb{N}_j(p)^*\right\rangle$, $\Gamma_j(p)^*=\left\{\Gamma_j^{(\partial)}\left(p_j^{(\partial)}\right)\middle|\partial=1,2,...,\#\Gamma_{ij}(p)\right\}$, $\mathbb{N}_j(p)^*=\left\{\mathbb{N}_j^{(\Im)}\left(p_j^{(\Im)}\right)\middle|\Im=1,2,...,\#\mathbb{N}_{ij}(p)\right\}$, $j=1,2,\ldots,n$, and the PIS needs to satisfy the following conditions:

$$PIS=\left\{\left\langle \Gamma_j(p)^*,\mathbb{N}_j(p)^*\right\rangle\middle|\max_i\left\{s_{D_{ij}}=\sum_{\partial=1}^{\#\Gamma_{ij}(p)}p_{ij}^{(\partial)}I\left(\Gamma_{ij}^{(\partial)}\right)-\sum_{\Im=1}^{\#\mathbb{N}_{ij}(p)}p_{ij}^{(\Im)}I\left(\mathbb{N}_{ij}^{(\Im)}\right)\right\}\right\} \tag{14}$$

Analogously, $D^-=\left(D_1^-,D_2^-,\ldots,D_n^-\right)$ is called the NIS of alternatives, where $D_j^-=\left\langle \Gamma_j(p)^-,\mathbb{N}_j(p)^-\right\rangle$, $\Gamma_j(p)^-=\left\{\Gamma_j^{(\partial)}\left(p_j^{(\partial)}\right)\middle|\partial=1,2,...,\#\Gamma_{ij}(p)\right\}$, $\mathbb{N}_j(p)^-=\left\{\mathbb{N}_j^{(\Im)}\left(p_j^{(\Im)}\right)\middle|\Im=1,2,...,\#\mathbb{N}_{ij}(p)\right\}$, $j=1,2,\ldots,n$, and the NIS needs to satisfy the following conditions:

$$NIS=\left\{\left\langle \Gamma_j(p)^-,\mathbb{N}_j(p)^-\right\rangle\middle|\min_i\left\{s_{D_{ij}}=\sum_{\partial=1}^{\#\Gamma_{ij}(p)}p_{ij}^{(\partial)}I\left(\Gamma_{ij}^{(\partial)}\right)-\sum_{\Im=1}^{\#\mathbb{N}_{ij}(p)}p_{ij}^{(\Im)}I\left(\mathbb{N}_{ij}^{(\Im)}\right)\right\}\right\} \tag{15}$$

Example 5. Continued with Example 4, for a column of DPLTEs $D_{11}=\left\langle\{s_{-2}(1)\},\{s_5(0.25),s_2(0.75)\}\right\rangle$, $D_{21}=\left\langle\{s_2(1)\},\{s_{-1}(0.4),s_{-2}(0.6)\}\right\rangle$ and $D_{31}=\left\langle\{s_4(1)\},\{s_1(0.5)\},s_{-3}(0.3)\right\rangle$, then

$$s_{D_{11}}=(-2)\times1-((0.25)\times5+(0.75)\times2)=-2-2.75=-4.75,$$

$$s_{D_{21}}=2\times1-(-1\times0.4+(-2)\times0.6)=2+1.6=3.6,$$

$$s_{D_{31}}=4\times1-(1\times0.5+(-3)\times0.3)=4+0.4=4.4.$$

It is easy to see that $s_{D_{31}}>s_{D_{21}}>s_{D_{11}}$, so the positive ideal element is D_{31}, the negative ideal element is D_{11}. Then the positive ideal element is $D^*=\left\langle\{s_4(1)\},\{s_1(0.5)\},s_{-3}(0.3)\right\rangle$, and the negative ideal element is $D^-=\left\langle\{s_{-2}(1)\},\{s_5(0.25),s_2(0.75)\}\right\rangle$.

Next, we use the weighted correlation coefficient to deal with the multi-attribute group decision-making problem under the dual probabilistic linguistic surrounding and the basic step can be depicted as follows:

Step 1. On account of the subjective weight of the DMs and the DPLWA operator, we obtain the group dual probabilistic linguistic decision-making matrix.

Step 2. Figure up the entropy measure of group dual probabilistic linguistic decision-making matrix. In view of Eq. (11), the entropy measure matrix can be calculated as follows:

$$E=\begin{pmatrix}E(D_{11}) & E(D_{12}) & \cdots & E(D_{1n})\\ E(D_{21}) & E(D_{22}) & \cdots & E(D_{2n})\\ \vdots & \vdots & \ddots & \vdots\\ E(D_{n1}) & E(D_{n2}) & \cdots & E(D_{nn})\end{pmatrix}$$

Step 3. Compute the objective entropy weight measure matrix C

$$C=\begin{pmatrix} C(D_{11}) & C(D_{12}) & \cdots & C(D_{1n}) \\ C(D_{21}) & C(D_{22}) & \cdots & C(D_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ C(D_{n1}) & C(D_{n2}) & \cdots & C(D_{nn}) \end{pmatrix}=\begin{pmatrix} \frac{1-E(D_{11})}{\sum_{i=1}^{n}(1-E(D_{1i}))} & \frac{1-E(D_{12})}{\sum_{i=1}^{n}(1-E(D_{1i}))} & \cdots & \frac{1-E(D_{1n})}{\sum_{i=1}^{n}(1-E(D_{1i}))} \\ \frac{E(D_{21})}{\sum_{i=1}^{n}(1-E(D_{2i}))} & \frac{E(D_{22})}{\sum_{i=1}^{n}(1-E(D_{2i}))} & \cdots & \frac{E(D_{2n})}{\sum_{i=1}^{n}(1-E(D_{2i}))} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{E(D_{n1})}{\sum_{i=1}^{n}(1-E(D_{ni}))} & \frac{E(D_{n2})}{\sum_{i=1}^{n}(1-E(D_{ni}))} & \cdots & \frac{E(D_{nn})}{\sum_{i=1}^{n}(1-E(D_{ni}))} \end{pmatrix}$$

Step 4. Calculate the final weight vector matrix W :

$$W=\begin{pmatrix} W(D_{11}) & W(D_{12}) & \cdots & W(D_{1n}) \\ W(D_{21}) & W(D_{22}) & \cdots & W(D_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ W(D_{n1}) & W(D_{n2}) & \cdots & W(D_{nn}) \end{pmatrix}=\begin{pmatrix} \frac{\omega_1 C_{11}}{\sum_{i=1}^{n}\omega_i C_{1i}} & \frac{\omega_2 C_{12}}{\sum_{i=1}^{n}\omega_i C_{1i}} & \cdots & \frac{\omega_n C_{1n}}{\sum_{i=1}^{n}\omega_i C_{1i}} \\ \frac{\omega_1 C_{21}}{\sum_{i=1}^{n}\omega_i C_{2i}} & \frac{\omega_2 C_{22}}{\sum_{i=1}^{n}\omega_i C_{2i}} & \cdots & \frac{\omega_n C_{2n}}{\sum_{i=1}^{n}\omega_i C_{2i}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\omega_1 C_{n1}}{\sum_{i=1}^{n}\omega_i C_{ni}} & \frac{\omega_2 C_{n2}}{\sum_{i=1}^{n}\omega_i C_{ni}} & \cdots & \frac{\omega_n C_{nn}}{\sum_{i=1}^{n}\omega_i C_{ni}} \end{pmatrix}$$

Step 5. Determine the positive ideal alternative a^*, and calculate the weighted correlation coefficient between an alternative a_i and the positive ideal alternative a^* through the use of the following equation:

$$\rho_\omega\left(a_i,a^*\right)=\rho_\omega=\frac{\rho_{\omega 1}+\rho_{\omega 2}}{2} \tag{16}$$

where

$$\rho_{\omega 1}=\frac{\sum_{j=1}^{n}W_{ij}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)}I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)}-m\left(\Gamma_A(p)\right)\right)\left(\frac{1}{\#\Gamma_A^*(p)}\sum_{\partial_A^*=1}^{\#\Gamma_A^*(p)}I\left(\Gamma_A^{*(\partial_A^*)}\right)p_A^{*(\partial_A^*)}-m\left(\Gamma_A^*(p)\right)\right)}{\left[\sum_{j=1}^{n}W_{ij}\left(\frac{1}{\#\Gamma_A(p)}\sum_{\partial_A=1}^{\#\Gamma_A(p)}I\left(\Gamma_A^{(\partial_A)}\right)p_A^{(\partial_A)}-m\left(\Gamma_A(p)\right)\right)^2\right]^{1/2}\left[\sum_{j=1}^{n}W_{ij}\left(\frac{1}{\#\Gamma_A^*(p)}\sum_{\partial_A^*=1}^{\#\Gamma_A^*(p)}I\left(\Gamma_A^{*(\partial_A^*)}\right)p_A^{*(\partial_A^*)}-m\left(\Gamma_A^*(p)\right)\right)^2\right]^{1/2}}$$

$$\rho_{\omega 2}=\frac{\sum_{j=1}^{n}W_{ij}\left(\frac{1}{\#\mathbb{N}_A(p)}\sum_{\Im_A=1}^{\#\mathbb{N}_A(p)}I\left(\mathbb{N}_A^{(\Im_A)}\right)p_A^{(\Im_A)}-m\left(\mathbb{N}_A(p)\right)\right)\left(\frac{1}{\#\mathbb{N}_A^*(p)}\sum_{\Im_A^*=1}^{\#\mathbb{N}_A^*(p)}I\left(\mathbb{N}_A^{*(\Im_A^*)}\right)p_A^{*(\Im_A^*)}-m\left(\mathbb{N}_A^*(p)\right)\right)}{\left[\sum_{j=1}^{n}W_{ij}\left(\frac{1}{\#\mathbb{N}_A(p)}\sum_{\Im_A=1}^{\#\mathbb{N}_A(p)}I\left(\mathbb{N}_A^{(\Im_A)}\right)p_A^{(\Im_A)}-m\left(\mathbb{N}_A(p)\right)\right)^2\right]^{1/2}\left[\sum_{j=1}^{n}W_{ij}\left(\frac{1}{\#\mathbb{N}_A^*(p)}\sum_{\Im_A^*=1}^{\#\mathbb{N}_A^*(p)}I\left(\mathbb{N}_A^{*(\Im_A^*)}\right)p_A^{*(\Im_A^*)}-m\left(\mathbb{N}_A^*(p)\right)\right)^2\right]^{1/2}}$$

Step 6. Rank the alternatives in accordance with the values of weighted correlation coefficients.

Step 7. Select the best alternative according to the maximum value of weighted correlation coefficients.

Step 8. End.

4.4. The general approach for solving multi-attribute group decision-making problem

In this subsection, we propose the whole process for coping with the multi-attribute group decision-making problem under the dual probabilistic linguistic environment. The whole approach can be demonstrated by the following figure:

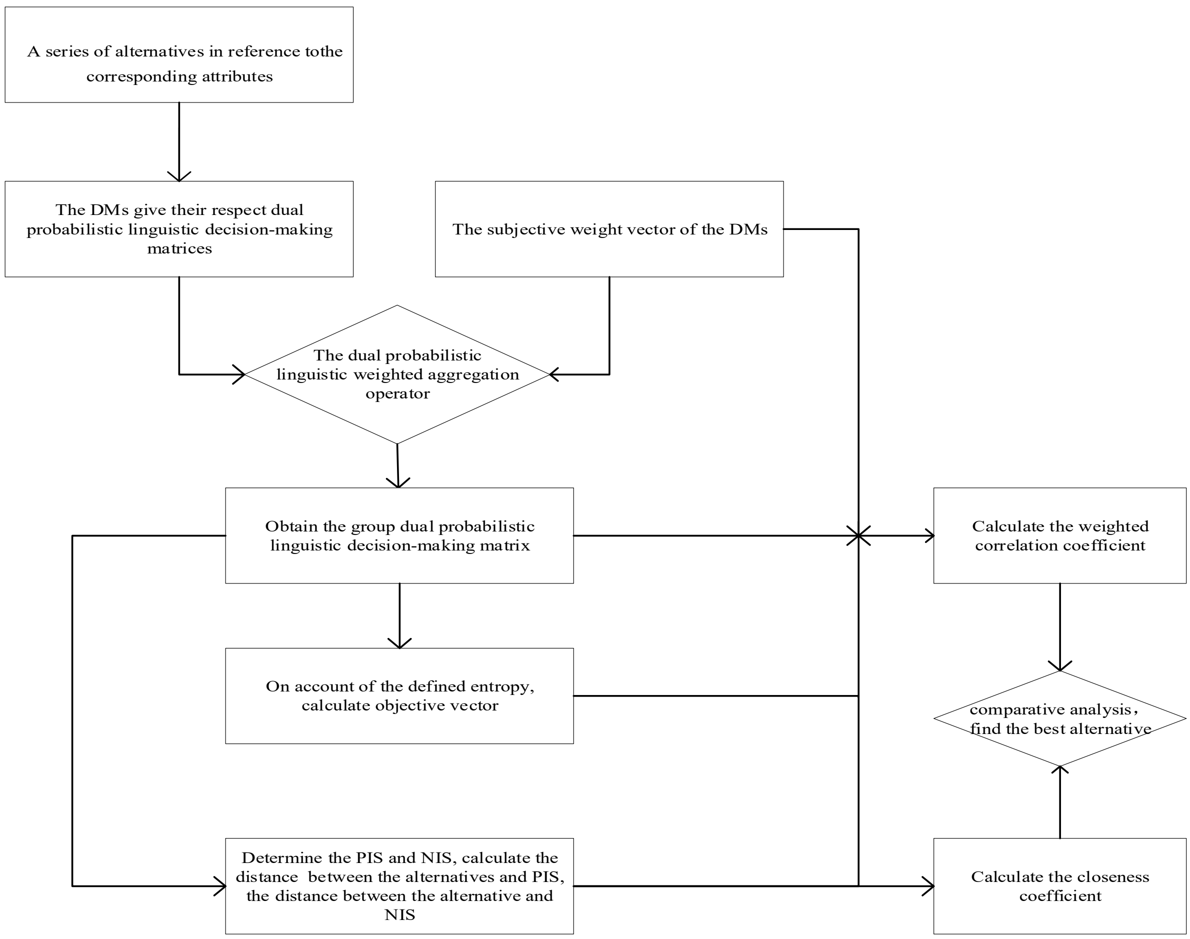

Figure 1. The whole multi-attributes group decision-making approach

5. Simulated experiment

So as to make the policy-making process more specific, this section implements a concrete simulation experiment concerned about the evaluation for the influence of AI mentioned above. Moreover, this section separates into two subsections: the first subsection is the practical experimental process to make Section 2, 3 and 4 concrete; the second subsection is the interpretation of final simulation experiment result.

5.1. Experimental process

Since March 2016, the man-machine war between AlphaGo and Li shishi has attracted people's attention to AI. The concerns about AI have always been high. To be honest, AI was first proposed by John McCarthy and other scientists in 1956. Since it was proposed, the relevant research has covered many aspects of social development, such as economic perspective [2-5], cultural aspect [6,7], education side [8-10], medical treatment perspective [11,12] that are relevant to us and so on.

In the aspect of economy, AI not only can replace the existing workforce with AI technology, and greatly increase existing labor productivity, but also can promote innovation, ameliorate existing products and services, and even create new products and services. As for the aspect of culture, AI has created many cultural wonders, made art language and ideographic system more compatible, and opened up new areas of cultural creation. For instance, image processing technology that can remodel the cultural content, and arouse people's experience of special memories and perceptual knowledge, so that cultural works are presented in more diverse ways. Besides, AI can deeply explore the connotation of culture, provide accurate information, and create a good cultural learning experience. Especially for the cultural heritage, the public does not know much about cultural heritage, AI can provide the required information to users in the shortest time, effectively make up for the lack of one-way communication and improve the efficiency of cultural communication and create the new strategic path to restore traditional culture and inherit the culture.

In terms of education, with the help of AI technology, the traditional teaching model will be broken down. Students no longer simply acquire knowledge from their teachers, and teachers are no longer just the instructors of knowledge, but the teaching service providers who meet the individual needs of students, and the growth consultants who design and implement customized learning programs. AI can not only analyze the knowledge of students in terms of knowledge linkage and population stratification, but also provide individualized and customized for each student from the aspects of brain thinking, individual character and environmental features. For the sake of medical treatment, Medical robots combined with AI can track physical health, intelligent software systems can assist doctors in the diagnosis of cancer, and surgical robots can accurately perform surgery on patients by combining multiple surgical functions.

Considering the future development potential of AI, the leaderships of one company, which includes four members, plan to invest the AI projects for the following years. Supposed that they plan to invest three concerned AI projects $a_i\left(i=1,2,3\right)$. To evaluate the three projects, they entrust one questionnaire company to investigate the impact of three AI projects under the four previously mentioned aspects. The questionnaire company regards the four mentioned-above aspects as four attributes: economic$\left(c_1\right)$, culture $\left(c_2\right)$, education $\left(c_3\right)$ medical treatment $\left(c_4\right)$. Obviously, all of the four attributes are benefit. In order to make the evaluation as objective as possible, and considering that the DPLTSs can from the two opposite aspects display the decision-making information, the questionnaire company chooses the DPLTSs as the decision-making tool for evaluation that not only reflect the membership degree of the decision-making information, but also the non-membership degree.

Assume that the dual probabilistic linguistic decision-making information that is given by four DMs for the three alternatives in relation to four attributes are as follows:

The dual probabilistic linguistic decision-making matrix D_1 given by the first DM

	c_1	c_2	c_3	c_4
x_1	$\langle\{s_{-4}(0.1),s_{-3}(0.9)\},\{s_{-2}(0.1),s_{-1}(0.5),s_0(0.4)\}\rangle$	$\langle\{s_4(0.2),s_5(0.1),s_6(0.5)\},\{s_{-1}(0.9),s_0(0.1)\}\rangle$	$\langle\{s_{-2}(0.7),s_{-1}(0.2)\},\{s_{-4}(0.2),s_{-3}(0.4)\}\rangle$	$\langle\{s_1(0.2),s_2(0.1)\},\{s_{-4}(0.1),s_{-3}(0.3),s_{-2}(0.1)\}\rangle$
x_2	$\langle\{s_0(0.2),s_1(0.7)\},\{s_{-3}(0.2),s_{-2}(0.4),s_{-1}(0.3)\}\rangle$	$\langle\{s_{-3}(0.6),s_{-2}(0.4)\},\{s_{-2}(0.1),s_{-1}(0.9)\}\rangle$	$\langle\{s_3(0.1),s_4(0.4),s_5(0.2)\},\{s_{-1}(0.2),s_0(0.8)\}\rangle$	$\langle\{s_{-3}(0.6),s_{-2}(0.4)\},\{s_2(0.3),s_3(0.4),s_4(0.1)\}\rangle$
x_3	$\langle\{s_3(0.1),s_4(0.7)\},\{s_{-2}(0.4),s_{-1}(0.6)\}\rangle$	$\langle\{s_4(0.6),s_5(0.2)\},\{s_{-3}(0.9),s_{-2}(0.1)\}\rangle$	$\langle\{s_1(0.4),s_2(0.2),s_3(0.1)\},\{s_0(0.3),s_1(0.6)\}\rangle$	$\langle\{s_1(0.6),s_2(0.3)\},\{s_{-3}(0.5),s_{-2}(0.5)\}\rangle$

The dual probabilistic linguistic decision-making matrix D_2 given by the second DM

	c_1	c_2	c_3	c_4
x_1	$\langle\{s_{-3}(0.3),s_{-2}(0.2),s_{-1}(0.4)\},\{s_{-2}(0.5),s_{-1}(0.1),s_0(0.1)\}\rangle$	$\langle\{s_1(0.3),s_2(0.2)\},\{s_{-2}(0.5)\},s_{-1}(0.3)\rangle$	$\langle\{s_{-1}(0.1),s_0(0.6)\},\{s_3(0.6),s_4(0.4)\}\rangle$	$\langle\{s_0(0.1),s_1(0.1),s_2(0.4)\},\{s_{-5}(0.7),s_{-4}(0.1)\}\rangle$
x_2	$\langle\{s_1(0.2),s_2(0.4),s_3(0.4)\},\{s_{-4}(0.6),s_{-3}(0.4)\}\rangle$	$\langle\{s_{-1}(0.2),s_0(0.2),s_1(0.4)\},\{s_{-3}(0.8),s_{-2}(0.1)\}\rangle$	$\langle\{s_2(0.1),s_3(0.5)\},\{s_0(0.1),s_1(0.3)\}\rangle$	$\langle\{s_{-6}(0.2),s_{-5}(0.4)\},\{s_1(0.2),s_2(0.6)\}\rangle$
x_3	$\langle\{s_{-2}(0.1),s_{-1}(0.1),s_0(0.5)\},\{s_5(0.3),s_6(0.5)\}\rangle$	$\langle\{s_{-1}(0.8),s_0(0.2)\},\{s_{-1}(0.4),s_0(0.4)\}\rangle$	$\langle\{s_{-1}(0.1),s_0(0.3)\},\{s_1(0.8),s_2(0.1)\}\rangle$	$\langle\{s_0(0.1),s_1(0.4),s_2(0.4)\},\{s_{-6}(0.2),s_{-5}(0.8)\}\rangle$

The dual probabilistic linguistic decision-making matrix D_3 given by the third DM

	c_1	c_2	c_3	c_4
x_1	$\langle\{s_2(0.1),s_3(0.6)\},\{s_1(0.2),s_2(0.3)\}\rangle$	$\langle\{s_{-3}(0.1),s_{-2}(0.2)\},\{s_3(0.1)\},s_4(0.2),s_5(0.5)\rangle$	$\langle\{s_{-2}(0.2),s_{-1}(0.4),s_0(0.2)\}\{s_{-1}(0.1),s_0(0.4)\}\rangle$	$\langle\{s_3(0.1),s_4(0.2),s_5(0.5)\},\{s_{-6}(0.3),s_{-5}(0.1),s_{-4}(0.4)\}\rangle$
x_2	$\langle\{s_1(0.2),s_{-2}(0.6)\},\{s_{-4}(0.1),s_{-3}(0.6),s_{-2}(0.3)\}\rangle$	$\langle\{s_{-4}(0.3),s_{-3}(0.2),s_{-2}(0.4)\}\{s_3(0.1),s_4(0.4)\}\rangle$	$\langle\{s_{-1}(0.1),s_0(0.4)\},\{s_1(0.4),s_2(0.1),s_3(0.3)\}\rangle$	$\langle\{s_{-5}(0.4),s_{-4}(0.1)\},\{s_2(0.5),s_3(0.4),s_4(0.1)\}\rangle$
x_3	$\langle\{s_5(0.3),s_6(0.5)\},\{s_{-3}(0.5),s_{-2}(0.4)\}\rangle$	$\langle\{s_{-1}(0.1),s_0(0.6),s_1(0.3)\},\{s_{-1}(0.3),s_0(0.5)\}\rangle$	$\langle\{s_2(0.2),s_3(0.5),s_4(0.2)\},\{s_{-3}(0.5),s_{-2}(0.5)\}\rangle$	$\langle\{s_2(0.4),s_3(0.2),s_4(0.4)\},\{s_{-5}(0.3),s_{-4}(0.3)\}\rangle$

The dual probabilistic linguistic decision-making matrix D_4 given by the fourth DM

	c_1	c_2	c_3	c_4
x_1	$\langle\{s_{-4}(0.4),s_{-3}(0.6)\},\{s_{-2}(0.5),s_{-1}(0.3)\}\rangle$	$\langle\{s_{-2}(0.2),s_{-1}(0.2)\},\{s_2(0.4),s_3(0.3)\},s_4(0.1)\rangle$	$\langle\{s_{-3}(0.4),s_{-2}(0.1),s_{-1}(0.4)\},\{s_{-2}(0.1),s_{-1}(0.2),s_0(0.1)\}\rangle$	$\langle\{s_4(0.4),s_5(0.1)\},\{s_{-5}(0.1),s_{-4}(0.1),s_{-3}(0.8)\}\rangle$
x_2	$\langle\{s_3(0.4),s_4(0.4)\},\{s_{-3}(0.4),s_{-2}(0.5)\}\rangle$	$\langle\{s_1(0.3),s_2(0.2)\},\{s_0(0.6),s_1(0.2),s_2(0.2)\}\rangle$	$\langle\{s_{-4}(0.5),s_{-3}(0.1),s_{-2}(0.4)\},\{s_0(0.1),s_1(0.3)\}\rangle$	$\langle\{s_{-4}(0.3),s_{-3}(0.1)\},\{s_2(0.6),s_3(0.4)\}\rangle$
x_3	$\langle\{s_1(0.3),s_2(0.6)\},\{s_{-4}(0.4),s_{-3}(0.3),s_{-2}(0.1)\}\rangle$	$\langle\{s_{-3}(0.1),s_{-2}(0.6)\},\{s_2(0.8),s_3(0.1)\}\rangle$	$\langle\{s_{-5}(0.4),s_{-4}(0.2),s_{-3}(0.4)\},\{s_2(0.1),s_3(0.2),s_4(0.5)\}\rangle$	$\langle\{s_5(0.7),s_6(0.2)\},\{s_{-3}(0.4),s_{-2}(0.6)\}\rangle$

Then in order to calculate the distance measure later, we handle those four decision-making matrices with the same cardinalities and the descending sequence as follows:

The dual probabilistic linguistic decision-making matrix $\widetilde{D}_1$ given by the first DM

	c_1	c_2	c_3	c_4
x_1	$\langle\{s_{-3}(0)s_{-4}(0.1),s_{-3}(0.9)\}$, $\{s_0(0.4),s_{-2}(0.1),s_{-1}(0.5)\}\rangle$	$\langle\{s_6(0.5),s_4(0.2),s_5(0.1)\}$, $\{s_0(0.1),s_{-1}(0),s_{-1}(0.9)\}\rangle$	$\langle\{s_{-2}(0),s_{-1}(0.2),s_{-2}(0.7)\}$, $\{s_{-3}(0),s_{-4}(0.2),s_{-3}(0.4)\}\rangle$	$\langle\{s_2(0.1),s_1(0.2),s_1(0)\}$, $\{s_{-2}(0.1),s_{-4}(0.1),s_{-3}(0.3)\}\rangle$
x_2	$\langle\{s_1(0.7),s_0(0.2),s_0(0)\}$, $\{s_{-1}(0.3),s_{-3}(0.2),s_{-2}(0.4)\}\rangle$	$\langle\{s_{-3}(0),s_{-2}(0.4),s_{-3}(0.6)\}$, $\{s_{-1}(0),s_{-2}(0.1),s_{-1}(0.9)\}\rangle$	$\langle\{s_4(0.4),s_5(0.2),s_3(0.1)\}$, $\{s_0(0.8),s_{-1}(0),s_{-1}(0.2)\}\rangle$	$\langle\{s_{-3}(0),s_{-2}(0.4),s_{-3}(0.6)\}$, $\{s_3(0.4),s_2(0.3),s_4(0.1)\}\rangle$
x_3	$\langle\{s_4(0.7),s_3(0.1),s_3(0)\}$, $\{s_{-2}(0),s_{-1}(0.6),s_{-2}(0.4)\}\rangle$	$\langle\{s_4(0.6),s_5(0.2),s_5(0)\}$, $\{s_{-3}(0),s_{-2}(0.1),s_{-3}(0.9)\}\rangle$	$\langle\{s_2(0.2),s_1(0.4),s_3(0.1)\}$, $\{s_1(0.6),s_0(0.3),s_0(0)\}\rangle$	$\langle\{s_2(0.3),s_1(0.6),s_1(0)\}$, $\{s_{-3}(0),s_{-2}(0.5),s_{-3}(0.5)\}\rangle$

The dual probabilistic linguistic decision-making matrix $\widetilde{D}_2$ given by the second DM

	c_1	c_2	c_3	c_4
x_1	$\langle\{s_{-1}(0.4),s_{-2}(0.2),s_{-3}(0.3)\}$, $\{s_0(0.1),s_{-1}(0.1),s_{-2}(0.5)\}\rangle$	$\langle\{s_2(0.2),s_1(0.3),s_1(0)\}$, $\{s_{-2}(0),s_{-1}(0.3),s_{-2}(0.5)\}\rangle$	$\langle\{s_0(0.6),s_{-1}(0),s_{-1}(0.1)\}$, $\{s_3(0.6),s_4(0.4),s_4(0)\}\rangle$	$\langle\{s_2(0.4),s_1(0.1),s_0(0.1)\}$, $\{s_{-5}(0),s_{-4}(0.1),s_{-5}(0.7)\}\rangle$
x_2	$\langle\{s_3(0.4)s_2(0.4),s_1(0.2)\}$, $\{s_{-4}(0),s_{-3}(0.4),s_{-4}(0.6),\}\rangle$	$\langle\{s_1(0.4),s_0(0.2),s_{-1}(0.2)\}$, $\{s_{-3}(0),s_{-2}(0.1),s_{-3}(0.8)\}\rangle$	$\langle\{s_3(0.5),s_2(0.1),s_2(0)\}$, $\{s_1(0.3),s_0(0.1),s_0(0)\}\rangle$	$\langle\{s_{-5}(0),s_{-6}(0.2),s_{-5}(0.4)\}$, $\{s_2(0.6),s_1(0.2),s_1(0)\}\rangle$
x_3	$\langle\{s_0(0.5),s_{-1}(0.1),s_{-2}(0.1)\}$, $\{s_6(0.5),s_5(0.3),s_5(0)\}\rangle$	$\langle\{s_0(0.2),s_{-1}(0),s_{-1}(0.8)\}$, $\{s_0(0.4),s_{-1}(0),s_{-1}(0.4)\}\rangle$	$\langle\{s_0(0.3),s_{-1}(0),s_{-1}(0.1)\}$, $\{s_1(0.8),s_2(0.1),s_2(0)\}\rangle$	$\langle\{s_2(0.4),s_1(0.4)s_0(0.1),\}$, $\{s_{-5}(0),s_{-6}(0.2),s_{-5}(0.8)\}\rangle$

The dual probabilistic linguistic decision-making matrix $\widetilde{D}_3$ given by the third DM

	c_1	c_2	c_3	c_4
x_1	$\langle\{s_3(0.6),s_2(0.1),s_2(0)\}$, $\{s_2(0.3),s_1(0.2),s_1(0)\}\rangle$	$\langle\{s_{-2}(0),s_{-3}(0.1),s_{-2}(0.2)\}$, $\{s_5(0.5),s_4(0.2),s_3(0.1)\}\rangle$	$\langle\{s_0(0.2),s_{-1}(0.4),s_{-2}(0.2)\}$ $\{s_0(0.4),s_{-1}(0),s_{-1}(0.1)\}\rangle$	$\langle\{s_5(0.5),s_4(0.2),s_3(0.1)\}$, $\{s_{-5}(0.1),s_{-4}(0.4),s_{-6}(0.3)\}\rangle$
x_2	$\langle\{s_1(0.2),s_{-2}(0),s_{-2}(0.6)\}$, $\{s_{-4}(0.1),s_{-2}(0.3),s_{-3}(0.6)\}\rangle$	$\langle\{s_{-3}(0.2),s_{-2}(0.4),s_{-4}(0.3)\}$ $\{s_4(0.4),s_3(0.1),s_3(0)\}\rangle$	$\langle\{s_0(0.4),s_{-1}(0),s_{-1}(0.1)\}$, $\{s_3(0.3),s_1(0.4),s_2(0.1)\}\rangle$	$\langle\{s_{-5}(0),s_{-4}(0.1),s_{-5}(0.4)\}$, $\{s_3(0.4),s_2(0.5),s_4(0.1)\}\rangle$
x_3	$\langle\{s_6(0.5),s_5(0.3),s_5(0)\}$, $\{s_{-2}(0),s_{-2}(0.4),s_{-3}(0.5)\}\rangle$	$\langle\{s_1(0.3),s_0(0.6),s_{-1}(0.1)\}$, $\{s_0(0.5),s_{-1}(0),s_{-1}(0.3)\}\rangle$	$\langle\{s_3(0.5),s_4(0.2),s_2(0.2)\}$, $\{s_{-2}(0),s_{-2}(0.5),s_{-3}(0.5)\}\rangle$	$\langle\{s_4(0.4),s_2(0.4),s_3(0.2)\}$, $\{s_{-5}(0),s_{-4}(0.3),s_{-5}(0.3)\}\rangle$

The dual probabilistic linguistic decision-making matrix $\widetilde{D}_4$ given by the fourth DM

	c_1	c_2	c_3	c_4
x_1	$\langle\{s_{-3}(0),s_{-4}(0.4),s_{-3}(0.6)\}$, $\{s_{-1}(0),s_{-1}(0.3),s_{-2}(0.5)\}\rangle$	$\langle\{s_{-2}(0),s_{-1}(0.2),s_{-2}(0.2)\}$, $\{s_3(0.3),s_2(0.4),s_4(0.1)\}\rangle$	$\langle\{s_{-2}(0.1),s_{-1}(0.4),s_{-3}(0.4)\}$ $\{s_0(0.1),s_{-1}(0.2),s_{-2}(0.1)\}\rangle$	$\langle\{s_4(0.4),s_5(0.1),s_5(0)\}$, $\{s_{-4}(0.1),s_{-5}(0.1),s_{-3}(0.8)\}\rangle$

x_2	$\langle\{s_4(0.4),s_3(0.4),s_3(0)\},$ $\{s_{-3}(0),s_{-2}(0.5),s_{-3}(0.4)\}\rangle$	$\langle\{s_2(0.2),s_1(0.3),s_1(0)\},$ $\{s_2(0.2),s_1(0.2),s_0(0.6)\}\rangle$	$\langle\{s_{-3}(0.1),s_{-2}(0.4),s_{-4}(0.5)\},$ $\{s_1(0.3),s_0(0.1),s_0(0)\}\rangle$	$\langle\{s_{-4}(0),s_{-3}(0.1),s_{-4}(0.3)\},$ $\{s_3(0.4),s_2(0.6),s_2(0)\}\rangle$
x_3	$\langle\{s_2(0.6),s_1(0.3),s_1(0)\},$ $\{s_{-2}(0.1),s_{-3}(0.3),s_{-4}(0.4)\}\rangle$	$\langle\{s_{-2}(0),s_{-3}(0.1),s_{-2}(0.6)\},$ $\{s_2(0.8),s_3(0.1),s_3(0)\}\rangle$	$\langle\{s_{-4}(0.2),s_{-3}(0.4),s_{-5}(0.4)\},$ $\{s_4(0.5),s_3(0.2),s_2(0.1)\}\rangle$	$\langle\{s_5(0.7),s_6(0.2),s_6(0)\},$ $\{s_{-2}(0),s_{-2}(0.6),s_{-3}(0.4)\}\rangle$

Suppose that the subject weight vector of the DMs is $\omega=(0.3,0.2,0.15,0.35)^T$, then

Step 1. According the DPLWA operator (10), we can acquire the group decision-making matrix D as follows:

The group dual probabilistic linguistic decision-making matrix D

	c_1	c_2	c_3	c_4
x_1	$\langle\{s_{-2.9}(0.0012),\ldots,s_{1.7}(0.1296)\},$ $\{s_{-0.0504}(0.0075),\ldots,s_0(0.1360)\}\rangle$	$\langle\{s_{0.25}(0.0012),\ldots,s_{1.55}(0.0040)\},$ $\{s_0(0.0512)\},s_{0.1260}(0.0225)\rangle$	$\langle\{s_{-2.15}(0.0056),\ldots,s_{-0.65}(0.0096)\}$ $\{s_{-0.1008}(0.0008),\ldots,s_0(0.102)\}\rangle$	$\langle\{s_{2.15}(0.0008),\ldots,s_{3.5}(0.002)\},$ $\{s_{0.3024}(0.0032),\ldots,s_{1.89}(0.0021)\}$
x_2	$\langle\{s_{1.4}(0.0032),s_{2.6}(0.0672)\},$ $\{s_{0.0378}(0.018),\ldots,s_{0.4536}(0.0048)\}\rangle$	$\langle\{s_{-1.35}(0.0108),s_0(0.0128)\},$ $\{s_0(0.27),s_{0.1512}(0.0064)\}\rangle$	$\langle\{s_{-0.25}(0.0005),\ldots,s_{1.4}(0.016)\},$ $\{s_{-0.0094}(0.0054),\ldots,s_0(0.1136)\}\rangle$	$\langle\{s_{-4.25}(0.0144),\ldots,s_{-3.25}(0.0016)\},$ $\{s_{0.0252}(0.018),\ldots,s_{0.3024}(0.0024)\}\rangle$
x_3	$\langle\{s_{1.6}(0.0009),\ldots,s_{2.8}(0.105)\},$ $\{s_{-0.4536}(0.04)\},\ldots,s_{-0.0630}(0.0072)$	$\langle\{s_{-0.2}(0.0048),\ldots,s_{0.95}(0.0072)\}$ $\{s_{-0.0283}(0.0108)\},\ldots,s_0(0.4680)$	$\langle\{s_{-1.35}(0.0032),\ldots,s_{0.45}(0.0024)\},$ $\{s_{-0.0756}(0.0150)\},\ldots,s_0(0.2160)\rangle$	$\langle\{s_{2.35}(0.0168),\ldots,s_{3.7}(0.0096)\},$ $\{s_{0.2520}(0.0720),\ldots,s_{0.8505}(0.0120)\}$

Remark 3. Because the size of the data D is very large, and we just use D to illustrate the aggregated result of the group dual probabilistic linguistic decision-making matrix. Therefore, the portions of the DPLTEs of D have been omitted for the sake of simplification. Moreover, the same approaches have been putted use to display the PIS and the NIS in the following section.

Let $\zeta=2$, then based on Eq. (11), we compute the entropy E of the group decision-making matrix as follows:

$$E=\begin{pmatrix} 0.9997 & 1.0000 & 1.0000 & 1.0000 \\ 0.9998 & 1.0000 & 1.0000 & 0.9999 \\ 0.9998 & 1.0000 & 1.0000 & 0.9995 \end{pmatrix}$$

Step 2. According to Eq. (12), we calculate the following objective entropy weight measure matrix C:

$$C=\begin{pmatrix}0.3849 & 0.0285 & 0.9207 & 0.0208\\ 0.3040 & 0.2770 & 0.0501 & 0.1148\\ 0.3111 & 0.6945 & 0.0292 & 0.8644\end{pmatrix}$$

Step 3. On account of Eq. (13), combined with the subject weight vector $\omega=(0.3,0.2,0.15,0.35)^T$, we calculate the final weight vector matrix W:

$$W=\begin{pmatrix}0.4332 & 0.0214 & 0.5180 & 0.0273\\ 0.4694 & 0.2851 & 0.0387 & 0.2068\\ 0.1731 & 0.2576 & 0.0081 & 0.5612\end{pmatrix}$$

Step 4. Based on Eq. (14), we determine the PIS as follows:

$$\begin{aligned}a^*=\{&\langle\{s_{1.6}(0.0009),\ldots,s_{2.8}(0.105)\},\{s_{-0.4536}(0.04),\ldots,s_{-0.0630}(0.0072)\}\rangle,\\ &\langle\{s_{-0.2}(0.0048),\ldots,s_{0.95}(0.0072)\},\{s_{-0.0283}(0.0108),\ldots,s_{0}(0.4680)\}\rangle,\\ &\langle\{s_{-0.25}(0.0005),\ldots,s_{1.4}(0.0160)\},\{s_{-0.0094}(0.0054),\ldots,s_{0}(0.1136)\}\rangle,\\ &\langle\{s_{2.35}(0.0168),\ldots,s_{3.7}(0.0096)\},\{s_{0.2520}(0.0720),\ldots,s_{0.8505}(0.0120)\}\rangle\}.\end{aligned}$$

Step 5. On the basis of Eq. (16), we figure up the matching weighted correlation coefficient between an alternative a_i and the ideal alternative a^* as follows:

$$\rho_\omega(a_1,a^*)=0.4101,\ \rho_\omega(a_2,a^*)=-0.1582,\ \rho_\omega(a_3,a^*)=0.9994$$

Step 6. Choose the best alternative:

$$\rho_\omega(a_3,a^*)>\rho_\omega(a_2,a^*)>\rho_\omega(a_1,a^*)$$

Obviously, the bigger the correlation coefficient, the better the alternative. So a_3 is the best alternative, which means the DMs should choose the third project to invest.

5.2. Result analysis with the closeness coefficient

In order to analyze the final decision-making result, in this subsection, we utilize the devised distance measure to calculate the closeness coefficient as the basis to obtain the policy-making result, and contradistinguish the differences between two methods.

Suppose that $\lambda = 0.5$, we combine the obtained final weight vector W and the distance measure d_2, we can calculate $d_{2\min}\left(a_i, a^*\right)$, where

$$d_{2\min}\left(a_i, a^*\right) = \min_{1\le i\le m} d_{2\min}\left(a_i, a^*\right) = \min_{1\le i\le m} \sum_{j=1}^{n} W_{ij} d_2\left(D_{ij}, D_j^*\right)$$

$$d_{2\min}\left(a_1, a^*\right) = 5.0686\mathrm{e}-04,\ d_{2\min}\left(a_2, a^*\right) = 0.0046,\ d_{2\min}\left(a_3, a^*\right) = 0.$$

Similar to obtain the PIS, we can also determine the NIS a^- as follows:

$$\begin{aligned} a^- = \Big\{ & \Big\langle \big\{ s_{-2.9}(0.0012), \ldots, s_{-1.7}(0.1296) \big\}, \big\{ s_{-0.0504}(0.0075), \ldots, s_0(0.1360) \big\} \Big\rangle, \\ & \Big\langle \big\{ s_{-1.35}(0.0108), \ldots, s_0(0.0128) \big\}, \big\{ s_0(0.27), \ldots, s_{0.1512}(0.0064) \big\} \Big\rangle, \\ & \Big\langle \big\{ s_{-2.15}(0.0056), \ldots, s_{-0.65}(0.0096) \big\}, \big\{ s_{-0.1008}(0.0008), \ldots, s_0(0.1020) \big\} \Big\rangle, \\ & \Big\langle \big\{ s_{-4.25}(0.0144), \ldots, s_{-3.25}(0.0016) \big\}, \big\{ s_{0.0252}(0.0180), \ldots, s_{0.3024}(0.0024) \big\} \Big\rangle \Big\} \end{aligned}$$

Moreover, we can also compute the distance between the alternatives $a_i\left(i = 1,2,3\right)$ and the NIS a^- as follows:

$$d_{2\max}\left(a_i, a^-\right) = \max_{1\le i\le m} d_2\left(a_i, a^-\right) = \max_{1\le i\le m} \sum_{j=1}^{n} W_{ij} d_2\left(D_{ij}, D_j^-\right)$$

$$d_{2\min}\left(a_1, a^-\right) = 3.16608\mathrm{e}-04,\ d_{2\min}\left(a_2, a^-\right) = 0,\ d_{2\min}\left(a_3, a^-\right) = 0.0125$$

Then we use the closeness coefficient [62] to rank the alternatives:

$$CI\left(x_i\right) = \frac{d_{2\min}\left(a_i, a^+\right)}{d_{2\max}\left(a_i, a^-\right) + d_{2\min}\left(a_i, a^+\right)}$$

Moreover, we can get $CI\left(a_1\right) = 0.625$, $CI\left(a_2\right) = 1$, $CI\left(a_3\right) = 1$, and the larger the closeness coefficient, the better the alternative. Obviously, it is easy to see the closeness coefficient $CI\left(a_2\right) = CI\left(a_3\right)$, theoretically, we can regard the alternatives a_2 and a_3 as the best alternative. However, the calculated value of the distanced is between the chose alternative a_3 and the PIS is zero, the computed value of distance measure between the selected alternative a_2 and the NIS is zero. From this perspective, the alternative a_3 is equal to the positive ideal alternative, while the alternative a_2 is equal to the negative ideal alternative. Hence, the best alternative is a_3.

Besides, no matter how to use the correlation coefficient or the closeness coefficient as the benchmark to get the ultimate decision-making consequence, the final best alternative is still a_3. In addition, compared with the closeness coefficient, the calculation of correlation coefficient does not need to adjust the number of elements for the membership degree and non-membership degree in DPLTSs and get the uniform cardinality. The decision-making information can retain the original as soon as possible. Whereas, the process for computing the correlation coefficient is relative complexity than the computation process of closeness coefficient.

5.3. Result analysis with the probabilistic linguistic decision-making matrices

For the sake of further displaying the difference of the DPLTSs, we use the same methods to the probabilistic linguistic decision-making matrices and make a comparative analysis as follows:

Suppose that the decision-making information of the DMs is presented by the PLTSs as follows:

The probabilistic linguistic decision-making matrix R_1 given by the first DM

	c_1	c_2	c_3	c_4
x_1	$\{s_{-3}(0)s_{-4}(0.1),s_{-3}(0.9)\}$	$\{s_6(0.5),s_4(0.2),s_5(0.1)\}$	$\{s_{-2}(0),s_{-1}(0.2),s_{-2}(0.7)\}$	$\{s_2(0.1),s_1(0.2),s_1(0)\}$
x_2	$\{s_1(0.7),s_0(0.2),s_0(0)\}$	$\{s_{-3}(0),s_{-2}(0.4),s_{-3}(0.6)\}$	$\{s_4(0.4),s_5(0.2),s_3(0.1)\}$	$\{s_{-3}(0),s_{-2}(0.4),s_{-3}(0.6)\}$
x_3	$\{s_4(0.7),s_3(0.1),s_3(0)\}$	$\{s_4(0.6),s_5(0.2),s_5(0)\}$	$\{s_2(0.2),s_1(0.4),s_3(0.1)\}$	$\{s_2(0.3),s_1(0.6),s_1(0)\}$

The probabilistic linguistic decision-making matrix R_2 given by the second DM

	c_1	c_2	c_3	c_4
x_1	$\{s_{-1}(0.4),s_{-2}(0.2),s_{-3}(0.3)\}$	$\{s_2(0.2),s_1(0.3),s_1(0)\}$	$\{s_0(0.6),s_{-1}(0),s_{-1}(0.1)\}$	$\{s_2(0.4),s_1(0.1),s_0(0.1)\}$

x_2	$\{s_3(0.4)s_2(0.4),s_1(0.2)\}$	$\{s_1(0.4),s_0(0.2),s_{-1}(0.2)\}$	$\{s_3(0.5),s_2(0.1),s_2(0)\}$	$\{s_{-5}(0),s_{-6}(0.2),s_{-5}(0.4)\}$
x_3	$\{s_0(0.5),s_{-1}(0.1),s_{-2}(0.1)\}$	$\{s_0(0.2),s_{-1}(0),s_{-1}(0.8)\}$	$\{s_0(0.3),s_{-1}(0),s_{-1}(0.1)\}$	$\{s_2(0.4),s_1(0.4),s_0(0.1)\}$

The probabilistic linguistic decision-making matrix R_3 given by the third DM

	c_1	c_2	c_3	c_4
x_1	$\{s_3(0.6),s_2(0.1),s_2(0)\}$	$\{s_{-2}(0),s_{-3}(0.1),s_{-2}(0.2)\}$	$\{s_0(0.2),s_{-1}(0.4),s_{-2}(0.2)\}$	$\{s_5(0.5),s_4(0.2),s_3(0.1)\}$
x_2	$\{s_1(0.2),s_{-2}(0),s_{-2}(0.6)\}$	$\{s_{-3}(0.2),s_{-2}(0.4),s_{-4}(0.3)\}$	$\{s_0(0.4),s_{-1}(0)s_{-1}(0.1)\}$	$\{s_{-5}(0),s_{-4}(0.1),s_{-5}(0.4)\}$
x_3	$\{s_6(0.5),s_5(0.3),s_5(0)\}$	$\{s_1(0.3),s_0(0.6),s_{-1}(0.1)\}$	$\{s_3(0.5),s_4(0.2),s_2(0.2)\}$	$\{s_4(0.4),s_2(0.4),s_3(0.2)\}$

The probabilistic linguistic decision-making matrix R_4 given by the fourth DM

	c_1	c_2	c_3	c_4
x_1	$\{s_{-3}(0),s_{-4}(0.4),s_{-3}(0.6)\}$	$\{s_{-2}(0),s_{-1}(0.2),s_{-2}(0.2)\}$	$\{s_{-2}(0.1),s_{-1}(0.4),s_{-3}(0.4)\}$	$\{s_4(0.4),s_5(0.1),s_5(0)\}$
x_2	$\{s_4(0.4),s_3(0.4),s_3(0)\}$	$\{s_2(0.2),s_1(0.3),s_1(0)\}$	$\{s_{-3}(0.1),s_{-2}(0.4),s_{-4}(0.5)\}$	$\{s_{-4}(0),s_{-3}(0.1),s_{-4}(0.3)\}$
x_3	$\{s_2(0.6),s_1(0.3),s_1(0)\}$	$\{s_{-2}(0),s_{-3}(0.1),s_{-2}(0.6)\}$	$\{s_{-4}(0.2),s_{-3}(0.4),s_{-5}(0.4)\}$	$\{s_5(0.7),s_6(0.2),s_6(0)\}$

Then with the same method to obtain the correlation coefficient and closeness coefficient, the entropy $\widetilde{E}$, the objective entropy weight measure matrix $\widetilde{C}$, the comprehensive weight vector matrix $\widetilde{W}$, the correlation coefficient $\widetilde{\rho_\omega}\left(a_i,a^*\right)$ and the closeness coefficient $\widetilde{CI}\left(a_i\right)$, $i=1,2,3$ can be calculated as follows:

$$\widetilde{E}=\begin{pmatrix} 0.9997 & 1.0000 & 1.0000 & 1.0000 \\ 0.9999 & 1.0000 & 1.0000 & 0.9999 \\ 0.9998 & 1.0000 & 1.0000 & 0.9995 \end{pmatrix} \qquad \widetilde{C}=\begin{pmatrix} 0.4666 & 0.0200 & 0.8868 & 0.0090 \\ 0.1598 & 0.2780 & 0.0722 & 0.1170 \\ 0.3736 & 0.7020 & 0.0410 & 0.8740 \end{pmatrix}$$

$$\widetilde{W}=\begin{pmatrix} 0.4997 & 0.0143 & 0.4749 & 0.0112 \\ 0.3086 & 0.3580 & 0.0697 & 0.2637 \\ 0.1985 & 0.2487 & 0.0109 & 0.5419 \end{pmatrix}$$

$$\widetilde{\rho_\omega}\left(a_1,a^*\right)=-0.1626,\ \widetilde{\rho_\omega}\left(a_2,a^*\right)=-0.1016,\ \widetilde{\rho_\omega}\left(a_3,a^*\right)=0.4994$$

$$\widetilde{\rho_{\omega}}\left(a_3,a^*\right)>\widetilde{\rho_{\omega}}\left(a_2,a^*\right)>\widetilde{\rho_{\omega}}\left(a_1,a^*\right)$$

$$\widetilde{CI}\left(a_1\right)=0.6418,\ \widetilde{CI}\left(a_2\right)=1,\ \widetilde{CI}\left(a_3\right)=0$$

$$\widetilde{CI}\left(a_2\right)>\widetilde{CI}\left(a_1\right)>\widetilde{CI}\left(a_3\right)$$

In order to show them clearly, here we use a table to show the final priorities of the alternatives with different methods and decision-making information as follows:

Table 1. The comparative analysis

Decision-making information	The priority of correlation coefficient	The priority of closeness coefficient
The dual probabilistic linguistic decision-making information	$\rho_{\omega}\left(a_3,a^*\right)>\rho_{\omega}\left(a_2,a^*\right)>\rho_{\omega}\left(a_1,a^*\right)$	$CI\left(a_3\right)=CI\left(a_2\right)>CI\left(a_1\right)$
The probabilistic linguistic decision-making information	$\widetilde{\rho_{\omega}}\left(a_3,a^*\right)>\widetilde{\rho_{\omega}}\left(a_2,a^*\right)>\widetilde{\rho_{\omega}}\left(a_1,a^*\right)$	$\widetilde{CI}\left(a_2\right)>\widetilde{CI}\left(a_1\right)>\widetilde{CI}\left(a_3\right)$

It is easy to see that the priorities based on the correlation coefficient are same for two kinds of decision-making information: the dual probabilistic linguistic decision-making information and the probabilistic linguistic decision-making information. However, from the perspective of the illustration of the decision-making information, the dual probabilistic linguistic decision-making information is more specific and comprehensive then the probabilistic linguistic decision-making information. Moreover, the certainty and hesitance can be expressed more complete by the dual probabilistic linguistic decision-making information. Besides, the priorities based on the closeness coefficient that are different with two different decision-making information further demonstrate the advantages of the dual probabilistic linguistic decision-making information. The more complete decision-making information that the DMs provide, the more rational priority that the alternatives can obtain. Therefore, the best project that the AI company should consider to invest project a_3.

6. Conclusions

In this paper, we have enriched the basic theory of the DPLTSs by the following directions: first defined the complement of the DPLTSs, and then defined the different distance measures for the DPLTSs with the same cardinalities. Moreover, considering the importance of the defined correlation coefficient in the policy-making area, we have proposed the correlation coefficient between the DPLTSs. Furthermore, for the sake of applying the suggested correlation coefficient to the practical policy-making problem, we have proposed the weighted correlation coefficient. In addition, in order to get the utmost out of the decision-making content, we have divided the weight vector into the subjective form and objective vector, not only considered the subjective cognition from the perspectives of the DMs, but also considered the objectivity of the attributes. On the side, we have defined the entropy for the DPLTEs to calculate the comprehensive weight. After that, we have applied the weighted correlation coefficient to the specific problem that mentioned at the beginning of the paper, and helped choose the best project for AI industry. Finally, the specific execution of the example has demonstrated the effective of the proposed theory. Besides, one comparative analysis by using the closeness coefficient based upon the distance measure has been performed to highlight the advantages and disadvantages of the correlation coefficient; the other comparative analysis has been compared with probabilistic linguistic decision-making matrices, which further demonstrates the differences of the two kinds of decision-making information.

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3 Expanding grey relational analysis with the comparable degree for dual probabilistic multiplicative linguistic term sets and its application on the cloud enterprise

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Expanding grey relational analysis with the comparable degree for dual probabilistic multiplicative linguistic term sets and its application on the cloud enterprise

WANYING XIE[1,3], ZESHUI XU[1,2], (Fellow, IEEE), ZHILIANG REN[1] and ENRIQUE HERRERA-VIEDMA[3,4]

[1]School of Economics and Management, Southeast University, Nanjing, Jiangsu 211189, China
[2]Business School, State Key Laboratory of Hydraulics and Mountain River Engineering, Sichuan University, Chengdu 610064, China
[3]Andalusian Research Institute in Data Science and Computational Intelligence, University of Granada, E-18071 Granada, Spain
[4]Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah, 21589, Saudi Arabia

Corresponding author: ZESHUI XU (e-mail: xuzeshui@263.net), ENRIQUE HERRERA-VIEDMA (e-mail: viedma@decsai.ugr.es).

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ABSTRACT Under the cloud trend of enterprises, how do traditional businesses get on the cloud becomes a worth pondering question. To help those traditional businesses that have no experience to dispel the clouds and see the sun as soon as possible, we are planning to choose one corporation with rich experience to take them into cloud market. The quintessence of dual probabilistic linguistic term sets (DPLTSs) is that it uses the combination of several linguistic terms and their proportions to reveal decision information by opposite angles. This paper proposes the dual probabilistic multiplicative linguistic preference relations (DPMLPRs) based upon the dual probabilistic multiplicative linguistic term sets (DPMLTSs). Then it defines the comparable degree between the DPMLPRs and studies the consensus of the group DPMLPR. Moreover, it probes the expanding grey relational analysis (EGRA) under the proposed comparable degree between the DPMLTSs. After that, one example of choosing the experienced cloud cooperative partner is simulated under the dual probabilistic linguistic circumstance. Besides, the comparative analysis is performed by considering the similarity among the EGRA, TODIM and VIKOR.

INDEX TERMS Dual probabilistic multiplicative linguistic preference relations, comparable degree, consensus, expanding grey relational analysis, multi-criteria decision making.

I. INTRODUCTION

Just like domino effect, since cloud computing [1] was first proposed by Eric Schmidt in 2006, the market for cloud computing is booming. Its research has been gotten a lot of attention from experts in different fields, such as internet of things [2-4], cloud storage [5, 6], cloud security [7, 8], cloud education [9, 10] and so on. The essence of cloud computing is to provide services through the network, so its architecture is centered on services, and its objective is to offer customers with faster and more convenient information services.

The currently acknowledged traits of cloud computing can be summarized as follows: (1) Supersize dimension, such as Amazon, IBM, Microsoft and Yahoo, each has hundreds of thousands of servers, "Cloud" is able to offer consumers unheard-of calculating strength; (2) Virtualization, cloud computing permits consumers to make use of application services from facultative situation utilizing all kinds of terminals. The desired resource is derived from the "Cloud" rather than an established concrete existence. The app operates someplace in the "Cloud". However, as a matter of fact, the consumers are not necessary to learn about or concern about where the app is operating. With just one laptop or one mobile phone, you are able to do everything we need through web services, even tasks like supercomputing. (3) Dynamic extendibility, the dimension of the cloud can be vibrantly scaled to fulfill the demands of adhibition and consumers scale growth. (4) High reliability, "Cloud" uses measures such as the fault tolerance for multiple copies of data and computational node isomorphism to ensure high reliability of services. Cloud computing is more responsible than utilizing local computers. (5) Commonality, cloud computing is not targeted at particular applications. With the help of "Cloud", it can structure protean applications. The identical "Cloud" can encourage diverse application operations in the mean time. (6) Service on demand, "Cloud" is a large resource pool that you are able to purchase according to the requirement, and clouds are able to be charged like water, electricity or gas. (7) Low cost and green energy saving. Because the particular fault-tolerant measures of "Cloud" can utilize rare cheap nodes to constitute a cloud, the cloud's automated centralized management eliminates the need for big business to afford cumulatively advanced data center management costs, and the versatility of "Cloud" enables the exploitation rate of resources much higher than the conventional system. Moreover, consumers are able to thoroughly enjoy the low-cost benefit of "Cloud".

Therefore, many traditional businesses begin to transform the cloud computing industry. However, majority of them do not have the relative experience, it is full of hazard for them to join in the cloud market. So it is a good choice for them to look for a good partner that with the rich experience to get twofold results with half the effort. As far as it goes, the world's four largest cloud computing companies are Amazon Web Services (AWS), Microsoft, Google and Alibaba Cloud. According to their own features, choosing one to collaborate with the four companies is the short cut for those traditional businesses that want to transform in the demand explosion period of cloud industry.

How to determine the selected company becomes the question that we will solve in this paper. The DPLTSs [11] enlarges probabilistic linguistic term sets (PLTSs)' [12] quintessence that uses the combination of several linguistic terms and their proportions to reveal decision information into the membership sentiment and non-membership sentiment. We extend it into the multiplicative linguistic scale [13] and define the dual probabilistic multiplicative linguistic term sets (DPMLTSs). Then we propose the notion of dual probabilistic multiplicative linguistic preference relations (DPMLPRs), and use the DPMLPRs as the implement to do the decision.

As most of the studies on the preference relations (PRs) [14-20], the consistency [21-26] is the common and essential condition for applying the PRs into the material decision. Different from the majority of researchers [27, 28], this paper defines the comparable degree between the DPMLPRs and utilizes it as the measure to judge the consistency of the DPMLPRs. The reason why we use the comparable degree is that the intrinsic quality between the comparable degree [29-31] and the distance measure [32, 33] is same. Moreover, because of the structure of the operator itself, the computation of the comparable degree is also separated into two angles: the membership viewpoint and the non-membership viewpoint.

After acquiring the consistent DPMLPRs, on account of the defined dual probabilistic linguistic weighted geometric aggregation operator (DPLWGA), we can obtain the group DPMLPR. Then on the foundation of the established comparable degree between the individual DPMLPRs and the group DPMLPR, the group consensus [34-38] can be checked directly. Moreover, if the consensus cannot be satisfied in the decision-making procedure, then the decision makers (DMs) need to adjust their PRs, until the consensus is satisfied in the end, and the checking is over.

The crucial intention of decision-making is to judge the sort of the alternatives. For the multi-criteria decision-making, the research for weights has been done a lot [39-42]. Most of them are divided into the following types: partially known [43-45], fully known [46, 47], total unknown [48-51]. The weights of criteria in this paper is belong to the third type that is total unknown. On the foundation of classic arithmetic averaging method [52], this paper considers the structural characteristics of DPMLTSs and designs the modified arithmetic averaging method to calculate the weights for criteria. After that, the grey relational analysis (GRA) [53] as one of the more common multi-criteria decision-making method, its superiority lies in that it does not require much of the quantity involved in the decision-making. Moreover, it does not require that the quantities to be determined conform to a typical distribution. The amount of calculation is relatively small, and the results agree well with the qualitative analysis. So the GRA has been expanded in this paper by merging with the proposed comparable degree to calculate the relational coefficient. The GRA based upon the comparable degree is named as expanding GRA (EGRA). Together with the weights of the criteria, the final priority of the alternatives is able to be procured at length.

Furthermore, we apply the proposed procedure to the case mentioned above and to help to determine the selected cooperative partner. Besides, given that the similar principle among the GRA, TODIM [54] and VIKOR [55] that studies the comparable degree between the alternative and ideal alternative, we also expand the TODIM, VIKOR into the expanding TODIM (ETODIM), expanding VIKOR (EVIKOR). Then we compare the EGRA, ETODIM and EVIKOR in the comparative analysis section, and show their several advantages and disadvantages.

In a word, the innovation points of the whole paper can be listed as follows: (1) Define the DPMLPRs; (2) Denote the comparable degree between the individual DPMLPRs; (3) Study the consistency of the individual DPMLPRs; (4) Research the consensus of group DPMLPR; (5) Propose the EGRA method based on the defined comparable degree between the DPMLTSs; (6) Expand the TODIM and VIKOR methods.

The remaining of this paper is structured as follows: Section II lists some necessary notions. Section III defines the DPMLTSs, the basic operations among the DPMLTSs, the comparable degree between the individual DPMLPRs, and study the consistency, consensus of the DPMLPRs. Section IV computes the weights of criteria, introduces the EGRA method, and the integrated multi-criteria decision-making procedure. Section V utilizes a simulation case relevant to the cloud computing industry to clarify the potential and reality of the dual probabilistic multiplicative linguistic multi-criteria group decision-making procedure. Section VI ends with some conclusions.

II. PRELIMINARIES

In this section, we will briefly recall some essential concepts, such as the linguistic terms, the dual probabilistic linguistic term set (DPLTS) and the normalized dual probabilistic linguistic term element (NDPLTE).

A. THE LINGUISTIC TERM SETS

Let $S=\left\{s_\alpha \left| \alpha\in\left[1/q,q\right]\right.\right\}$ be a continuous multiplicative linguistic label set, and q is a adequately large positive integer [13]. Moreover, if $\alpha>\beta$, then $s_\alpha>s_\beta$; if $rec\left(s_\alpha\right)=s_\beta$, then $\alpha\beta=1$; peculiarly, $rec\left(s_1\right)=s_1$. Based on the multiplicative linguistic label set S, Xu [13] introduced some basic operational laws for them as follows: $\left(s_\alpha\right)^\mu=s_{\alpha^\mu}$, $\mu\in\left[0,1\right]$; $s_\alpha\otimes s_\beta=\max\left\{s_{1/q},\min\left\{s_{\alpha\beta},s_q\right\}\right\}$; $s_\alpha\oplus s_\beta=\max\left\{s_{1/q},\min\left\{s_{\alpha+\beta},s_q\right\}\right\}$.

B. THE DPLTS

Let X be a fixed set, a DPLTS on X can be signified into the coming type [11]:

$$D=\left\{\left\langle x,\wp\left(p\right),\Upsilon\left(p\right)\right\rangle,x\in X\right\} \tag{1}$$

where

$$\wp\left(p\right)=\left\{\wp^{(i)}\left(p^{(i)}\right)\middle|\wp^{(i)}\in S_1,p^{(i)}\geq 0,\sum_{i=1}^{\#\wp(p)}p^{(i)}\leq 1\right\},$$

$$\Upsilon\left(p\right)=\left\{\Upsilon^{(j)}\left(p^{(j)}\right)\middle|\Upsilon^{(j)}\in S_1,p^{(j)}\geq 0,\sum_{j=1}^{\#\Upsilon(p)}p^{(j)}\leq 1\right\}.$$

$\wp\left(p\right)$ and $\Upsilon\left(p\right)$ stand for the conceivable membership and non-membership degrees to the element $x\in X$ for the set D with the conditions that $s_{-q}\leq\wp^{+}\oplus\Upsilon^{+}\leq s_q$, $s_{-q}\leq\wp^{-}\oplus\Upsilon^{-}\leq s_q$, $S_1=\left\{s_\alpha\left|\alpha\in\left[-q,q\right]\right.\right\}$. In addition to that, we call the pair $D=\left\langle\wp\left(p\right),\Upsilon\left(p\right)\right\rangle$ the dual probabilistic linguistic element (DPLTE).

Moreover, in the cause of reducing the trouble of the computation, Xie et al. [11] further designed the coming procedure to normalize the DPLTEs (NDPLTEs) as follows:

Assume that $D_1=\left\langle\wp_1\left(p\right),\Upsilon_1\left(p\right)\right\rangle$ and $D_2=\left\langle\wp_2\left(p\right),\Upsilon_2\left(p\right)\right\rangle$ are two unlike DPLTEs. For the first step, similar to earn the NPLTSs, there is a need to avoid the deviations in the cardinalities of the two PLTSs $\wp_1\left(p\right)$ and $\wp_2\left(p\right)$, and to score the PLTSs $\wp_1\left(p\right)$ and $\wp_2\left(p\right)$ with the identical cardinal numbers: $\#\wp_1\left(p\right)=\#\wp_2\left(p\right)$. For the second step, we need to replume the PLTSs $\wp_1\left(p\right)$ and $\wp_2\left(p\right)$ separately in the downward sort. Likewise, the PLTSs $\Upsilon_1\left(p\right)$ and $\Upsilon_2\left(p\right)$ also need to be treated with the same way. Then we can obtain two new DPLTEs $D_1'=\left\langle\wp_1'\left(p\right),\Upsilon_1'\left(p\right)\right\rangle$, $D_2'=\left\langle\wp_2'\left(p\right),\Upsilon_2'\left(p\right)\right\rangle$, where

$$\wp_l'\left(p\right)=\left\{\wp_l'^{(i)}\left(p_l'^{(i)}\right)\middle|\wp_l'^{(i)}\in S,p_l'^{(i)}\geq 0,\sum_{i=1}^{\#\wp_l'(p)}p_l'^{(i)}\leq 1\right\}$$

and

$$\Upsilon_l'\left(p\right)=\left\{\Upsilon_l'^{(j)}\left(p_l'^{(j)}\right)\middle|\Upsilon_l'^{(j)}\in S,p_l'^{(j)}\geq 0,\sum_{j=1}^{\#\Upsilon_l'(p)}p_l'^{(j)}\leq 1\right\}$$

are revealed in falling sort, $\#\wp_1'\left(p\right)=\#\wp_2'\left(p\right)$, $\#\Upsilon_1'\left(p\right)=\#\Upsilon_2'\left(p\right)$, $l=1,2$.

Moreover, we offer the definition of score function and accuracy function [11] to compare the different DPLTEs as follows:

For a DPLTE $D=\left\langle\wp\left(p\right),\Upsilon\left(p\right)\right\rangle$, it's score function is :

$$S\left(D\right)=s_{\bar{\alpha}-\bar{\beta}} \tag{2}$$

where $\bar{\alpha}=\sum_{i=1}^{\#\wp(p)}I\left(\wp^{(i)}\right)p^{(i)}/\sum_{i=1}^{\#\wp(p)}p^{(i)}$, $\bar{\beta}=\sum_{j=1}^{\#\Upsilon(p)}I\left(\Upsilon^{(j)}\right)r^{(j)}p^{(j)}/\sum_{j=1}^{\#\Upsilon(p)}p^{(j)}$ and $I\left(\bullet\right)$ is the function that can obtain the subscript of the corresponding linguistic term.

With regard to two DPLTEs $D_l\left(l=1,2\right)$, if $S\left(D_1\right)>S\left(D_2\right)$, then D_1 is superior to D_2, denoted by $D_1\succ D_2$; if $S\left(D_1\right)<S\left(D_2\right)$, then D_1 is inferior to D_2, denoted by $D_1\prec D_2$. If $S\left(D_1\right)=S\left(D_2\right)$, it is tight to tell from two DPLTEs. Thus, we state the accuracy function for the DPLTE as follows:

For a DPLTE $D=\left\langle\wp\left(p\right),\Upsilon\left(p\right)\right\rangle$, it's accuracy function can be ruled as:

$$\mathrm{A}(D)=\left(\sum_{i=1}^{\#\wp(p)}\left(p^{(i)}\left(I\left(\wp^{(i)}\right)-\bar{\alpha}\right)\right)^2\right)^{1/2}/\sum_{i=1}^{\#\wp(p)}p^{(i)}$$
$$+\left(\sum_{j=1}^{\#\Upsilon(p)}\left(p^{(j)}\left(I\left(\Upsilon^{(j)}\right)-\bar{\beta}\right)\right)^2\right)^{1/2}/\sum_{j=1}^{\#\Upsilon(p)}p^{(j)} \quad (3)$$

Hence, with regard to two DPLTEs $D_l\left(l=1,2\right)$ with $S\left(D_1\right)=S\left(D_2\right)$, if $A\left(D_1\right)<A\left(D_2\right)$, then $D_1\succ D_2$; if $A\left(D_1\right)>A\left(D_2\right)$, then $D_1\prec D_2$; if $A\left(D_1\right)=A\left(D_2\right)$, then $D_1\sim D_2$.

III. THE DUAL PROBABILISTIC MULTIPLICATIVE LINGUISTIC TERM SETS

Considering the multiplicative linguistic label set [13] and the defined DPLTS together, next we extend the DPLTS into the environment of multiplicative linguistic label set, and study the basic operations in the following section.

A. THE DPMLTS

Let X be a fixed set, a DPMLTS on X can be shown as the following style:

$$D=\left\{\left\langle x,\wp\left(p\right),\Upsilon\left(p\right)\right\rangle,x\in X\right\} \quad (4)$$

where

$$\wp\left(p\right)=\left\{\wp^{(i)}\left(p^{(i)}\right)\middle|\wp^{(i)}\in S,p^{(i)}\geq 0,\sum_{i=1}^{\#\wp(p)}p^{(i)}\leq 1\right\},$$
$$\Upsilon\left(p\right)=\left\{\Upsilon^{(j)}\left(p^{(j)}\right)\middle|\Upsilon^{(j)}\in S,p^{(j)}\geq 0,\sum_{j=1}^{\#\Upsilon(p)}p^{(j)}\leq 1\right\}.$$

$\wp\left(p\right)$ and $\Upsilon\left(p\right)$ stand for the conceivable membership and non-membership degrees to the element $x\in X$ for the set D with the situations that $s_{1/q}\leq\wp^{+}\otimes\Upsilon^{+}\leq s_q$, $s_{1/q}\leq\wp^{-}\otimes\Upsilon^{-}\leq s_q$. Additionally, we call the pair $D=\left\langle\wp\left(p\right),\Upsilon\left(p\right)\right\rangle$ the dual multiplicative probabilistic linguistic element (DPMLTE).

Then on behalf of better applying the DPMLTEs in to the practical case, we regulate the essential operation for the DPMLTEs as follows:

For two DPMLTEs $D_1=\left\langle\wp_1\left(p\right),\Upsilon_1\left(p\right)\right\rangle$ and $D_2=\left\langle\wp_2\left(p\right),\Upsilon_2\left(p\right)\right\rangle$, then the multiplicative operation is

$$D_1\otimes D_2=\left\langle\wp_1\left(p\right),\Upsilon_1\left(p\right)\right\rangle\otimes\left\langle\wp_2\left(p\right),\Upsilon_2\left(p\right)\right\rangle$$
$$=\left\langle\wp_1\left(p\right)\otimes\wp_2\left(p\right),\Upsilon_1\left(p\right)\otimes\Upsilon_2\left(p\right)\right\rangle \quad (5)$$

Based on the Ref. [3], where

$$\wp_1\left(p\right)\otimes\wp_2\left(p\right)=\bigcup_{\wp_1^{(i_1)}\in\wp_1(p),\wp_2^{(i_2)}\in\wp_2(p)}\left\{\left(\wp_1^{(i_1)}\otimes\wp_2^{(i_2)}\right)\right.$$
$$\left.\left(p_1^{(i_1)}p_2^{(i_2)}\right)\middle|i_1=1,2,\ldots,\#\wp_1\left(p\right),i_2=1,2,\ldots,\#\wp_2\left(p\right)\right\}$$

$$\Upsilon_1\left(p\right)\otimes\Upsilon_2\left(p\right)=\bigcup_{\Upsilon_1^{(j_1)}\in\Upsilon_1(p),\Upsilon_2^{(j_2)}\in\Upsilon_2(p)}\left\{\left(\Upsilon_1^{(j_1)}\otimes\Upsilon_2^{(j_2)}\right)\right.$$
$$\left.\left(p_1^{(j_1)}p_2^{(j_2)}\right)\middle|j_1=1,2,\ldots,\#\Upsilon_1\left(p\right),j_2=1,2,\ldots,\#\Upsilon_2\left(p\right)\right\}$$

The power operation is

$$\left(D_1\right)^{\lambda}=\left\langle\wp_1\left(p\right),\Upsilon_1\left(p\right)\right\rangle^{\lambda}=\left\langle\left(\wp_1\left(p\right)\right)^{\lambda},\left(\Upsilon_1\left(p\right)\right)^{\lambda}\right\rangle \quad (6)$$

where

$$\left(\wp_1\left(p\right)\right)^{\lambda}=\bigcup_{\wp_1^{(i_1)}\in\wp_1(p)}\left\{\left(\wp_1^{(i_1)}\right)^{\lambda}\left(p_1^{(i_1)}\right)^{\lambda}\middle|i_1=1,2,\ldots,\#\wp_1\left(p\right)\right\}.$$

Then let $D_1,D_2,\ldots,D_n$ be a set of DPMLTEs, then the dual probabilistic multiplicative linguistic weighted geometric aggregated (DPMLWGA) operator can be expressed as:

$$DPMLWGA\left(D_1,D_2,\ldots,D_n\right)$$
$$=\left\langle\bigotimes_{i=1}^{n}\left(\wp_i\left(p\right)\right)^{\omega_i},\bigotimes_{i=1}^{n}\left(\Upsilon_j\left(p\right)\right)^{\omega_i}\right\rangle \quad (7)$$

where $\omega=\left(\omega_1,\omega_2,\ldots,\omega_n\right)^T$ is the weight vector with respect to the DPMLTEs, and fulfills $\omega_i\in\left[0,1\right]$ and $\sum_{i=1}^{n}\omega_i=1$.

B. THE DPMLPR

In the cause of applying the DPMLTSs to the decision-making procedure, in the following, we define the dual probabilistic multiplicative linguistic preference relation (DPMLPR) as follows:

A DPMLPR on the mentioned set $S=\left\{s_\alpha\middle|\alpha\in\left[1/q,q\right]\right\}$ is defined as the matrix $D=\left(d_{ij}\right)_{n\times n}$, $d_{ij}=\left\langle\wp_{ij}\left(p\right),\Upsilon_{ij}\left(p\right)\right\rangle$, which meets

$$\wp_{ij}\left(p\right)=\Upsilon_{ji}\left(p\right),\ \Upsilon_{ij}\left(p\right)=\wp_{ji}\left(p\right) \quad (8)$$

$i\neq j$, for $i,j=1,2,\ldots,n$.

Moreover, if $i=j$, then

$$\wp_{ii}\left(p\right)=\Upsilon_{ii}\left(p\right)=\left\langle\left\{s_1\left(1\right)\right\},\left\{s_1\left(1\right)\right\}\right\rangle.$$

It is common knowledge that the consistency of PRs is the essential requirement for logical decision-making. So it is no exception to study the consistency of the defined DPMLPRs.

For a DPMLPR $D=\left\langle\wp_{ij}\left(p\right),\Upsilon_{ij}\left(p\right)\right\rangle$, if D is consistent, then it should satisfy the following conditions: for $\forall i,k,j=1,2,\ldots,n$,

$$\begin{cases}\wp_{ij}\left(p\right)=\wp_{ik}\left(p\right)\otimes\wp_{kj}\left(p\right)\\ \Upsilon_{ij}\left(p\right)=\Upsilon_{ik}\left(p\right)\otimes\Upsilon_{kj}\left(p\right)\end{cases} \quad (9)$$

which means that the DPMLPR D is consistent if and only if the membership PR $\wp=\wp_{ij}(p)$ and the non-membership $\Upsilon=\Upsilon_{ij}(p)$ are consistent at the same time.

For the single membership PR $\wp=\wp_{ij}(p)$, its consistent PR $C\wp=C\wp_{ij}(p)$, where

$$C\wp_{ij}(p)=\sqrt[n]{\bigotimes_{k=1}^{n}\left[\wp_{ik}(p)\otimes\wp_{kj}(p)\right]}\ .$$

By learning from the Ref. [56], its consistency index can be calculated as follows:

$$CI_{\wp}=1-\sum_{i,j=1,i<j}^{n}\frac{2}{(n-1)(n-2)}\left|\log e_{\wp_{ij}}-\log e_{C\wp_{ij}}\right| \tag{10}$$

where for a DPMLTE $D=\langle\wp(p),\Upsilon(p)\rangle$, the expected value of the DPMLTE is

$$E_D=\left\langle\sum_{i=1}^{\#\wp(p)}p^{(i)}I\left(\wp^{(i)}\right),\sum_{j=1}^{\#\Upsilon(p)}p^{(j)}I\left(\Upsilon^{(j)}\right)\right\rangle=\left\langle e_{\wp(p)},e_{\Upsilon(p)}\right\rangle \tag{11}$$

Example 1: For one DPMLTE $D=\langle\{s_{1/2}(0.4),s_3(0.6)\},\{s_2(0.3),s_1(0.5)\}\rangle$ on the certain linguistic term set $S=\{s_\alpha\,|\alpha\in[1/9,9]\}$, then expected value of the DPMLTE is

$$E_D=\left\langle\sum_{i=1}^{\#\wp(p)}p^{(i)}I\left(\wp^{(i)}\right),\sum_{j=1}^{\#\Upsilon(p)}p^{(j)}I\left(\Upsilon^{(j)}\right)\right\rangle=\langle 2.0,1.1\rangle .$$

Then for the DPMLPR D, its consistency index can be computed as below:

$$CI=1-\sum_{i,j=1,i<j}^{n}\frac{1}{(n-1)(n-2)}\left(\left|\log e_{\wp_{ij}}-\log e_{C\wp_{ij}}\right|+\left|\log e_{\Upsilon_{ij}}-\log e_{C\Upsilon_{ij}}\right|\right) \tag{12}$$

Moreover, the consistency procedure can be expressed as the Algorithm 1:

Algorithm 1. The procedure to adjust the consistency

Step 1. Set the threshold value for the consistency index Ξ, and calculate the respective consistency index $CI_i\,(i=1,2,\ldots,n)$ for the DPMLPRs;

Step 2. Judge the consistency of the DPMLPRs, if $CI_i>\Xi$, then go to **Step 4**; Otherwise, go to the next step.

Step 3. Modify the elements of the DPMLPRs according to the following method:

$$\begin{cases}\wp'_{ij}(p)=\left(\wp_{ij}(p)\right)^{\theta}\otimes\left(C\wp_{ij}(p)\right)^{1-\theta}\\ \Upsilon'_{ij}(p)=\left(\Upsilon_{ij}(p)\right)^{\theta}\otimes\left(C\Upsilon_{ij}(p)\right)^{1-\theta}\end{cases} \tag{13}$$

where $\theta\in[0,1]$ is a regulation parameter.

Step 4. Let $D'=D$, then go back to **Step 1**.

C. THE COMPATIBILITY DEGREE FOR DPMLPRS

For the obtained consistent DPMLPRs, we are devote to study the consensus of the group DPMLPRs. Usually, people like to choose the distance measure [32, 33] or the similarity measure [57-59] as the foundation to analyze the consensus of the group PR. In this paper, with an eye to the similar practical meaning among the distance measure, similarity measure and comparable degree, we utilize the comparable degree between the DPMLPRs as the foundation to research the consensus of group PR in the following subsection:

Before introducing the comparable degree between the DPMLPRs, we first give the notion of comparable degree for two DPMLTEs. For any two DPMLTEs $D_1=\langle\wp_1(p),\Upsilon_1(p)\rangle$ and $D_2=\langle\wp_2(p),\Upsilon_2(p)\rangle$, the comparable degree between two DPMLTEs can be calculated as follows:

$$C(D_1,D_2)=\frac{1}{2}\left(\left|\log e_1^{\wp}-\log e_2^{\wp}\right|+\left|\log e_1^{\Upsilon}-\log e_2^{\Upsilon}\right|\right) \tag{14}$$

Furthermore, for two different DPMLPRs $D_1=\left(d_{ij}^1\right)_{n\times n}=\left(\left\langle\wp_{ij}^1(p),\Upsilon_{ij}^1(p)\right\rangle\right)_{n\times n}$ and $D_2=\left(d_{ij}^2\right)_{n\times n}=\left(\left\langle\wp_{ij}^2(p),\Upsilon_{ij}^2(p)\right\rangle\right)_{n\times n}$, the comparable degree of D_1 and D_2 can be defined as:

$$C(D_1,D_2)=\frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\begin{array}{l}\left|\log e_{ij1}^{\wp}-\log e_{ij2}^{\wp}\right|+\\ \left|\log e_{ij1}^{\Upsilon}-\log e_{ij2}^{\Upsilon}\right|\end{array}\right)\right] \tag{15}$$

where $E_{D_1}=\left\langle e_{ij1}^{\wp},e_{ij1}^{\Upsilon}\right\rangle$ and $E_{D_2}=\left\langle e_{ij2}^{\wp},e_{ij2}^{\Upsilon}\right\rangle$ are the homologous expected value of the different DPMLPRs D_1 and D_2, respectively.

Moreover, if $e_{ij1}^{\wp}=e_{ij2}^{\wp}$, $e_{ij1}^{\Upsilon}=e_{ij2}^{\Upsilon}$, $i,j=1,2,\ldots,n$, than we call DPMLPRs D_1 and D_2 are perfectly compatible.

Theorem 1: For two DPMLPRs D_1 and D_2, then

(a) $C(D_1,D_2)\geq 0$;

(b) $C(D_1,D_2)=C(D_2,D_1)$;

(c) $C(D_1,D_2)=0$, if D_1 and D_2 are perfectly compatible.

It is easy to see that Eqs. (a-c) are apparent. Therefore, the proof is omitted.

Theorem 2: For three different DPMLPRs D_1, D_2 and D_3, we have

$$C(D_1,D_3)\leq C(D_1,D_2)+C(D_2,D_3)$$

Proof.

$$
\begin{aligned}
C(D_1,D_3) &= \frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\begin{pmatrix}\left|\log e_{ij1}^{\wp}-\log e_{ij3}^{\wp}\right|+\\ \left|\log e_{ij1}^{\Upsilon}-\log e_{ij3}^{\Upsilon}\right|\end{pmatrix}\right]\\
&= \frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\begin{pmatrix}\left|\log e_{ij1}^{\wp}-\log e_{ij2}^{\wp}+\log e_{ij2}^{\wp}-\log e_{ij3}^{\wp}\right|+\\ \left|\log e_{ij1}^{\Upsilon}-\log e_{ij2}^{\Upsilon}+\log e_{ij2}^{\Upsilon}-\log e_{ij3}^{\Upsilon}\right|\end{pmatrix}\right]\\
&\le \frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\begin{pmatrix}\left|\log e_{ij1}^{\wp}-\log e_{ij2}^{\wp}\right|+\left|\log e_{ij2}^{\wp}-\log e_{ij3}^{\wp}\right|+\\ \left|\log e_{ij1}^{\Upsilon}-\log e_{ij2}^{\Upsilon}\right|+\left|\log e_{ij2}^{\Upsilon}-\log e_{ij3}^{\Upsilon}\right|\end{pmatrix}\right]\\
&\le \frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\log e_{ij1}^{\wp}-\log e_{ij2}^{\wp}\right|+\left|\log e_{ij1}^{\Upsilon}-\log e_{ij2}^{\Upsilon}\right|\right)\right]\\
&+\frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\log e_{ij2}^{\wp}-\log e_{ij3}^{\wp}\right|+\left|\log e_{ij2}^{\Upsilon}-\log e_{ij3}^{\Upsilon}\right|\right)\right]\\
&= C(D_1,D_2)+C(D_2,D_3)
\end{aligned}
$$

Definition 1: For two DPMLPRs D_1 and D_2, if

$$C(D_1,D_2)\le\delta \tag{16}$$

then we call that D_1 and D_2 are of acceptable compatibility, where δ is the threshold value of acceptable compatibility.

For a set of DPMLPRs $D_1,D_2,\ldots,D_n$, the group DPMLPR D can be expressed as the following form:

$$
D=\begin{pmatrix}D_{11} & D_{12} & \cdots & D_{1n}\\ D_{21} & D_{22} & \cdots & D_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ D_{m1} & D_{m2} & \cdots & D_{mn}\end{pmatrix}
$$

$$
=\begin{pmatrix}
\left\langle\bigotimes_{i=1}^{n}(\wp_{11}^{i}(p))^{\omega_i},\bigotimes_{i=1}^{n}(\Upsilon_{11}^{i}(p))^{\omega_i}\right\rangle & \left\langle\bigotimes_{i=1}^{n}(\wp_{12}^{i}(p))^{\omega_i},\bigotimes_{i=1}^{n}(\Upsilon_{12}^{i}(p))^{\omega_i}\right\rangle & \cdots & \left\langle\bigotimes_{i=1}^{n}(\wp_{1n}^{i}(p))^{\omega_i},\bigotimes_{i=1}^{n}(\Upsilon_{1n}^{i}(p))^{\omega_i}\right\rangle\\
\left\langle\bigotimes_{i=1}^{n}(\wp_{21}^{i}(p))^{\omega_i},\bigotimes_{i=1}^{n}(\Upsilon_{21}^{i}(p))^{\omega_i}\right\rangle & \left\langle\bigotimes_{i=1}^{n}(\wp_{22}^{i}(p))^{\omega_i},\bigotimes_{i=1}^{n}(\Upsilon_{22}^{i}(p))^{\omega_i}\right\rangle & \cdots & \left\langle\bigotimes_{i=1}^{n}(\wp_{2n}^{i}(p))^{\omega_i},\bigotimes_{i=1}^{n}(\Upsilon_{2n}^{i}(p))^{\omega_i}\right\rangle\\
\vdots & \vdots & \ddots & \vdots\\
\left\langle\bigotimes_{i=1}^{n}(\wp_{n1}^{i}(p))^{\omega_i},\bigotimes_{i=1}^{n}(\Upsilon_{n1}^{i}(p))^{\omega_i}\right\rangle & \left\langle\bigotimes_{i=1}^{n}(\wp_{n2}^{i}(p))^{\omega_i},\bigotimes_{i=1}^{n}(\Upsilon_{n2}^{i}(p))^{\omega_i}\right\rangle & \cdots & \left\langle\bigotimes_{i=1}^{n}(\wp_{nn}^{i}(p))^{\omega_i},\bigotimes_{i=1}^{n}(\Upsilon_{nn}^{i}(p))^{\omega_i}\right\rangle
\end{pmatrix}
$$

Theorem 3: For a set of DPMLPRs $D_i=\left(d_{st}^{i}\right)_{n\times n}=\left(\left\langle\wp_{st}^{i}(p),\Upsilon_{st}^{i}(p)\right\rangle\right)_{n\times n}$, one DPMLPR $D^{*}=\left(d_{st}^{*}\right)_{n\times n}=\left(\left\langle\wp_{st}^{*}(p),\Upsilon_{st}^{*}(p)\right\rangle\right)_{n\times n}$, $s=1,2,\ldots,n$, $t=1,2,\ldots,n$ and $D=\left(d_{st}\right)_{n\times n}=\left(\left\langle\wp_{st}(p),\Upsilon_{st}(p)\right\rangle\right)_{n\times n}$ is the group DPMLPR of the set of DPMLPRs $D_i\,(i=1,2,\ldots,n)$ by utilizing the weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$, then if $C(D_i,D^{*})\le\delta$, $i=1,2,\ldots,n$, then $C(D,D^{*})\le\delta$, where δ is the threshold value of acceptable compatibility.

Proof. $D=\left(d_{st}\right)_{n\times n}=\left(\left\langle L_{st}(p),U_{st}(p)\right\rangle\right)_{n\times n}$, where

$$
\begin{aligned}
\wp_{st}(p)&=\bigotimes_{i=1}^{n}\left(\wp_{sti}(p)\right)^{\omega_i}\\
&=\bigotimes_{i=1}^{n}\left(\wp_{sti}^{\omega_i}\left(p_{sti}^{\omega_i}\right)\right)=I^{-1}\left(\prod_{i=1}^{n}\left(I\left(\wp_{sti}\right)\right)^{\omega_i}\right)\left(\prod_{i=1}^{n}p_{sti}^{\omega_i}\right),\\
\Upsilon_{st}(p)&=\bigotimes_{i=1}^{n}\left(\Upsilon_{sti}(p)\right)^{\omega_i}\\
&=\bigotimes_{i=1}^{n}\left(\Upsilon_{sti}^{\omega_i}\left(p_{sti}^{\omega_i}\right)\right)=I^{-1}\left(\prod_{i=1}^{n}\left(I\left(\Upsilon_{i}\right)\right)^{\omega_i}\right)\left(\prod_{i=1}^{n}p_{sti}^{\omega_i}\right).
\end{aligned}
$$

Then $E_{D_i}=\left(e_{sti}^{\wp},e_{sti}^{\Upsilon}\right)(i=1,2,\ldots,n)$, $E_D=\left(e_{st}^{\wp},e_{st}^{\Upsilon}\right)$, $E_{D^*}=\left(e_{st}^{\wp*},e_{st}^{\Upsilon*}\right)$, $e_{st}^{\wp}=\left(\prod_{i=1}^{n}\left(I\left(\wp_{sti}\right)\right)^{\omega_i}\right)\left(\prod_{i=1}^{n}p_{sti}^{\omega_i}\right)$,

$$e_{st}^{\Upsilon}=\left(\prod_{i=1}^{n}\left(I\left(\Upsilon_{i}\right)\right)^{\omega_i}\right)\left(\prod_{i=1}^{n}p_{sti}^{\omega_i}\right).$$

Since

$$
C(D_i,D^{*})=\frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{s=1}^{n}\sum_{t=1}^{n}\begin{pmatrix}\left|\log e_{sti}^{\wp}-\log e_{st}^{\wp*}\right|\\ +\left|\log e_{sti}^{\Upsilon}-\log e_{st}^{\Upsilon*}\right|\end{pmatrix}\right],
$$

$$\le\delta$$

then

$$
\begin{aligned}
C(D,D^{*})&=\frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{s=1}^{n}\sum_{t=1}^{n}\begin{pmatrix}\left|\log e_{st}^{\wp}-\log e_{st}^{\wp*}\right|\\ +\left|\log e_{st}^{\Upsilon}-\log e_{st}^{\Upsilon*}\right|\end{pmatrix}\right]\\
&=\frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{s=1}^{n}\sum_{t=1}^{n}\begin{pmatrix}\left|\log\left\{\left(\prod_{i=1}^{n}\left(I\left(\wp_{sti}\right)\right)^{\omega_i}\right)\left(\prod_{i=1}^{n}\left(p_{sti}\right)^{\omega_i}\right)\right\}-\log e_{st}^{\wp*}\right|\\ +\left|\log\left\{\left(\prod_{i=1}^{n}\left(I\left(\Upsilon_{sti}\right)\right)^{\omega_i}\right)\left(\prod_{i=1}^{n}\left(p_{sti}\right)^{\omega_i}\right)\right\}-\log e_{st}^{\Upsilon*}\right|\end{pmatrix}\right]\\
&=\frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{s=1}^{n}\sum_{t=1}^{n}\begin{pmatrix}\left|\log\left\{\left(\prod_{i=1}^{n}I\left(\wp_{sti}\right)\right)^{\omega_i}\left(\prod_{i=1}^{n}p_{sti}\right)^{\omega_i}\right\}-\log e_{st}^{\wp*}\right|\\ +\left|\log\left\{\left(\prod_{i=1}^{n}I\left(\Upsilon_{sti}\right)\right)^{\omega_i}\left(\prod_{i=1}^{n}p_{sti}\right)^{\omega_i}\right\}-\log e_{st}^{\Upsilon*}\right|\end{pmatrix}\right]\\
&=\frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{s=1}^{n}\sum_{t=1}^{n}\begin{pmatrix}\left|\sum_{i=1}^{n}\omega_i\left(\log\left\{I\left(\wp_{sti}\right)p_{sti}\right\}-\log e_{st}^{\wp*}\right)\right|\\ +\left|\sum_{i=1}^{n}\omega_i\left(\log\left\{I\left(\Upsilon_{sti}\right)p_{sti}\right\}-\log e_{st}^{\Upsilon*}\right)\right|\end{pmatrix}\right]\\
&\le\frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{s=1}^{n}\sum_{t=1}^{n}\begin{pmatrix}\sum_{i=1}^{n}\omega_i\left|\log\left\{I\left(\wp_{sti}\right)p_{sti}\right\}-\log e_{st}^{\wp*}\right|+\\ \sum_{i=1}^{n}\omega_i\left|\log\left\{I\left(\Upsilon_{sti}\right)p_{sti}\right\}-\log e_{st}^{\Upsilon*}\right|\end{pmatrix}\right]\\
&=\sum_{i=1}^{n}\omega_i\left\{\frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{s=1}^{n}\sum_{t=1}^{n}\begin{pmatrix}\left|\log\left\{I\left(\wp_{sti}\right)p_{sti}\right\}-\log e_{st}^{\wp*}\right|+\\ \left|\log\left\{I\left(\Upsilon_{sti}\right)p_{sti}\right\}-\log e_{st}^{\Upsilon*}\right|\end{pmatrix}\right]\right\}\\
&=\sum_{i=1}^{n}\omega_i\left\{\frac{1}{n(n-1)}\left[\frac{1}{2}\sum_{s=1}^{n}\sum_{t=1}^{n}\begin{pmatrix}\left|\log e_{sti}^{\wp}-\log e_{st}^{\wp*}\right|+\\ \left|\log e_{sti}^{\Upsilon}-\log e_{st}^{\Upsilon*}\right|\end{pmatrix}\right]\right\}\\
&\le\sum_{i=1}^{n}\omega_i\delta=\delta
\end{aligned}
$$

Thus the proof is completed.

Theorem 4: For two sets of DPMLPRs $D_i\,(i=1,2,\ldots,n)$, $\widetilde{D}_i\,(i=1,2,\ldots,n)$, D is the group DPMLPR of the set of DPMLPRs $D_i\,(i=1,2,\ldots,n)$ and $\widetilde{D}$ is the group DPMLPR of the set of DPMLPRs $\widetilde{D}_i\,(i=1,2,\ldots,n)$ by utilizing the same weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$, respectively, then if $C\left(D_i,\widetilde{D}_i\right)\le\delta$, $i=1,2,\ldots,n$, then $C\left(D,\widetilde{D}\right)\le\delta$, where δ is the threshold value of acceptable compatibility.

The proof is similar to that of *Theorem 4*, so the specific proof process is omitted.

Then the group consensus procedure can be listed below:

Algorithm 2. The procedure to adjust the consensus

Step 1: For the set of consistent DPMLPRs $D_i\left(i=1,2,\ldots,n\right)$, with the Eq. (7) and the subjective weight vector of the DMs, it is easy to obtain the group DPMLPR D.

Step 2: Let δ be the threshold value of acceptable compatibility, then calculate the compatibility degree between the individual DPMLPRs $D^{\iota}\left(\iota=1,2,\ldots,n\right)$ and the group DPMLPR $D=\left(D_{ij}\right)_{n\times n}=\left(\left\langle \wp_{ij}\left(p\right),\Upsilon_{ij}\left(p\right)\right\rangle\right)_{n\times n}$, if $C\left(D^{\iota_o},D\right)\le\sigma$, then group DMLPR is of the acceptable consensus, go to **Step 4**; Otherwise, go to the next step.

Step 3: Let $\iota_o=\iota_o+1$, $D^{\iota_o+1}=D^{\iota_o}$, where

$$\begin{cases} D_{ij}^{\iota_o+1}=\left(\left\langle \wp_{ij}^{\iota_o+1}\left(p\right),\Upsilon_{ij}^{\iota_o+1}\left(p\right)\right\rangle\right)_{n\times n}, \\ \wp_{ij}^{\iota_o+1}\left(p\right)=\eta\wp_{ij}^{\iota_o}\left(p\right)\otimes\left(1-\eta\right)\wp_{ij}\left(p\right), \\ \Upsilon_{ij}^{\iota_o+1}\left(p\right)=\eta\Upsilon_{ij}^{\iota_o}\left(p\right)\otimes\left(1-\eta\right)\Upsilon_{ij}\left(p\right). \end{cases}$$

Then go back to **Step 2** until $C\left(D^{\iota_o},D\right)\le\sigma$.

Step 4: Let $\iota_o=\iota_o+1$, $D^{\iota_o+1}=D^{\iota_o}$, then go back to **Step 1**.

IV. METHODOLOGY

In this section, the determination of weights for criteria based on the group DPMLPR and the patulous GRA are presented in detail.

A. THE WEIGHTS FOR CRITERIA

Based on the algorithm in Section III, we can get a group DPMLPR $D=\left(D_{ij}\right)_{n\times n}=\left(\left\langle \wp_{ij}\left(p\right),\Upsilon_{ij}\left(p\right)\right\rangle\right)_{n\times n}$ with the acceptable consensus degree. Then for the DPMLPR $D=\left(D_{ij}\right)_{n\times n}$, with a view to the construction features of the elements in DPMLPR, the classic arithmetic averaging method [52] cannot be used directly. So we give the following equation to calculate the weights for criteria:

$$\varpi_i=\sum_{j=1}^{n}I\left(S\left(D_{ij}\right)\right)\Big/\sum_{i=i}^{n}\sum_{j=1}^{n}I\left(S\left(D_{ij}\right)\right) \quad (17)$$

where $S\left(\bullet\right)$ is the score function of D_{ij}.

B. THE EXPANDING GREY RELATIVE ANALYSIS METHOD

With regard to the individual dual probabilistic linguistic decision-making matrices given by the DMs for the alternatives $\left(a_1,a_2,\ldots,a_m\right)$ with respect to the criteria $\left(c_1,c_2,\ldots,c_n\right)$, the group dual linguistic decision-making matrix $M=\left(M_{ij}\right)_{m\times n}$ can be acquired by Eq. (7) as follows:

$$M=\begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{array}\begin{array}{c} \begin{array}{cccc} c_1 & c_2 & \cdots & c_n \end{array} \\ \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix} \end{array} \quad (18)$$

Let $M^{+}=\left(M_1^{+},M_1^{+},\ldots,M_n^{+}\right)^T$ and $M^{-}=\left(M_1^{-},M_1^{-},\ldots,M_n^{-}\right)^T$ be the positive ideal element (PIE) and the negative ideal element (PIE) in M, respectively, where $M_j^{+}=\max_i M_{ij}$, $M_j^{-}=\min_i M_{ij}$, M_j^{+} and M_j^{-} are determined through Eq. (2) or Eq. (3).

Due to the reality that the comparable degree is similar to the distance measure in physical significance, in the light of the proposed comparable degree between the DPMLTSs, the grey relative coefficient matrices based on the PIE and the NIE are extended as follows:

$$\mu_{ij}^{+}=\frac{\min_{1\le i\le m}\min_{1\le j\le n}C\left(M_{ij},M_j^{+}\right)+\xi\max_{1\le i\le m}\max_{1\le j\le n}C\left(M_{ij},M_j^{+}\right)}{C\left(M_{ij},M_j^{+}\right)+\xi\max_{1\le i\le m}\max_{1\le j\le n}C\left(M_{ij},M_j^{+}\right)} \quad (19)$$

$$\mu_{ij}^{-}=\frac{\min_{1\le i\le m}\min_{1\le j\le n}C\left(M_{ij},M_j^{-}\right)+\xi\max_{1\le i\le m}\max_{1\le j\le n}C\left(M_{ij},M_j^{-}\right)}{C\left(M_{ij},M_j^{-}\right)+\xi\max_{1\le i\le m}\max_{1\le j\le n}C\left(M_{ij},M_j^{-}\right)} \quad (20)$$

Combined with the weights of criteria, the opposite closeness coefficient to the PIE can be determined by the coming equation:

$$OC_i=\frac{\sum_{j=1}^{n}\mu_{ij}^{+}\varpi_j}{\sum_{j=1}^{n}\mu_{ij}^{+}\varpi_j+\sum_{j=1}^{n}\mu_{ij}^{-}\varpi_j} \quad (21)$$

The bigger the opposite closeness coefficient OC_i, the better the alternative.

Then the EGRA method can be illustrated as Algorithm 3:

Step 1: Identify the PIE and the NIE of the group dual probabilistic decision-making matrix;

Step 2: Calculate the respective grey relative coefficients on the foundation of PIE and NIE;

Step 3: Obtain the opposite closeness coefficient for the alternative.

C. THE INTEGRATED PROCESS FOR SOLVING MULTI-CRITERIA GROUP DECISION-MAKING PROBLEM

On the foundation of Section III and the remaining subsection of Section IV, the integrated decision-making procedure can be concluded as follows:

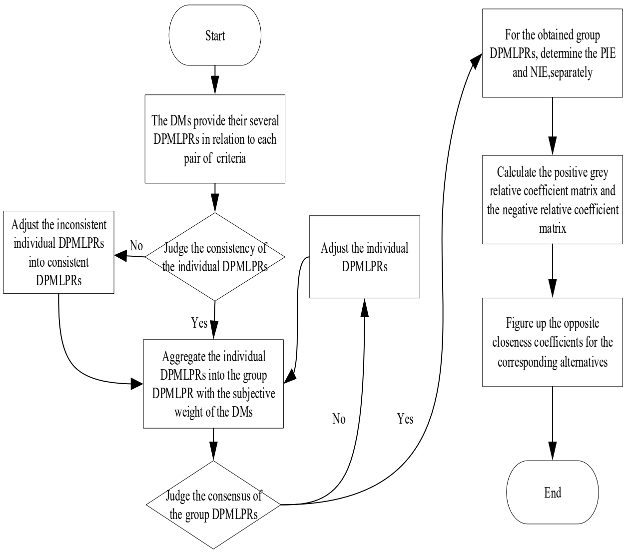

FIGURE 1. The integrated procedure to do the multi-criteria decision-making

V. SIMULATION EXPERIMENT

So as to make the decision-making procedure more detailed, this section performs a concrete simulation experiment relevant to the assessment for the manifestation of cloud enterprise mentioned above. Moreover, this section has four subsections: the first subsection is the practical experimental procedure to make Section II, III and IV particular; the second and third subsections are the comparative analysis; the four subsection is the sensitivity analysis.

A. EXPERIMENTAL PROCESS

Cloud computing [1] is a type of computing in which vibrantly scalable and always virtualized resources are supplied as a service over the internet. It was first proposed by the CEO Eric Schmidt of google at the search engine conference in 2006.

According to service types, cloud computing is able to be divided into three types: IaaS (Infrastructure-as-a-Service), consumers can get services from a complete computer infrastructure over the internet; PaaS (Platform-as-a-Service), it uses the software development environment, application environment, etc. as a service to directly provide users with the application platform required by the software; SaaS (Software-as-a- Service), it is a model for providing software over the internet. Instead of purchasing software, users rent web-based software from providers to manage business operations.

The emergence of cloud computing will reshape the IT industry landscape. There will be two clear investment opportunities: one is the new market capacity brought about by the rapid development of the cloud computing industry, and the other is to reshape the emerging industry opportunities brought about by the IT landscape. Considering the broader trend, many corporations are going to the "Cloud". For those IT corporations, they already have own IT costs and IT technology. It is much easier for them to the "Cloud". While for those traditional corporations that lack of network experience want to the "Cloud", they need to bear the cost of trial and error and the risk of failure. It is good choice for those traditional corporations to choose a good partner. Obviously, the so-called good partner shall have rich experience and enough funds to support the traditional industries in need of assistance. Globally, the four giants of the cloud industry are AWS, Microsoft, Google and Alibaba Cloud. As mentioned in Ref. [11], one good partner corporations shall equip with the four features: Corporate value, Independent research and development ability, Corporate size and Product market share. Apparently, the four features are benefit, which means the four features are positively related to the direction of growth.

Considering the future development potential of cloud computing, the enterprise who wants to get twofold results with half the effort chooses to collaborate with one of the four giants of the cloud industries: AWS, Microsoft, Google and Alibaba Cloud. Supposed that the four giants of the cloud industries are four evaluated alternatives $x_i\left(i=1,2,3,4\right)$. To evaluate the four enterprises, they entrust one questionnaire enterprise to investigate the impact of four cloud enterprises under the four previously mentioned aspects. The questionnaire enterprise regards the four mentioned-above aspects as four criteria: Corporate value $\left(c_1\right)$, Independent research and development ability $\left(c_2\right)$, Corporate size $\left(c_3\right)$ Product market share $\left(c_4\right)$. Obviously, all of the four criteria are benefit. In order to make the evaluation as objective as possible, and consider the DPMLTSs can from the two opposite aspects display the decision-making information, the questionnaire enterprise choose the DPMLTSs as the decision-making tool for evaluation. To some extent, not only reflect the membership degree of the decision-making information, but also the non-membership degree.

Assume that the DPMLPRs that are given by four DMs for the four alternatives with respect to four criteria are as follows:

$$D_1=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_6(0.1),s_7(0.7),s_8(0.2)\},\{s_{1/4}(0.4),s_{1/3}(0.4)\}\rangle & \langle\{s_4(0.3),s_5(0.4),s_6(0.3)\},\{s_1(0.2),s_2(0.2),s_3(0.2)\}\rangle & \langle\{s_4(0.3),s_5(0.2),s_6(0.2)\},\{s_{1/8}(0.2),s_{1/7}(0.2)\}\rangle \\
\langle\{s_{1/4}(0.4),s_{1/3}(0.4)\},\{s_6(0.1),s_7(0.7),s_8(0.2)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{1/3}(0.2),s_{1/2}(0.6),s_1(0.2)\},\{s_{1/4}(0.5),s_{1/3}(0.3),s_{1/2}(0.1)\}\rangle & \langle\{s_{1/5}(0.3),s_{1/4}(0.2),s_{1/3}(0.5)\},\{s_4(0.4),s_6(0.6)\}\rangle \\
\langle\{s_1(0.2),s_2(0.2),s_3(0.2)\},\{s_4(0.3),s_5(0.4),s_6(0.3)\}\rangle & \langle\{s_{1/4}(0.5),s_{1/3}(0.3),s_{1/2}(0.1)\},\{s_{1/3}(0.2),s_{1/2}(0.6),s_1(0.2)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_5(0.6),s_4(0.4)\},\{s_{1/3}(0.7),s_1(0.3)\}\rangle \\
\langle\{s_{1/8}(0.2),s_{1/7}(0.2)\},\{s_4(0.3),s_5(0.2),s_6(0.2)\}\rangle & \langle\{s_4(0.4),s_6(0.6)\},\{s_{1/5}(0.3),s_{1/4}(0.2),s_{1/3}(0.5)\}\rangle & \langle\{s_{1/3}(0.7),s_1(0.3)\},\{s_5(0.6),s_4(0.4)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

$$D_2=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_4(0.4),s_5(0.4),s_6(0.2)\},\{s_{1/5}(0.3),s_{1/3}(0.3)\}\rangle & \langle\{s_{1/8}(0.3),s_{1/6}(0.3)\},\{s_3(0.3),s_4(0.2),s_5(0.3)\}\rangle & \langle\{s_3(0.1),s_4(0.7),s_5(0.2)\},\{s_{1/6}(0.8),s_{1/4}(0.1)\}\rangle \\
\langle\{s_{1/5}(0.3),s_{1/3}(0.3)\},\{s_4(0.4),s_5(0.4),s_6(0.2)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_3(0.2),s_4(0.6)\},\{s_3(0.3),s_4(0.2)\}\rangle & \langle\{s_{1/7}(0.9),s_{1/6}(0.1)\},\{s_{1/2}(0.2),s_1(0.3),s_2(0.5)\}\rangle \\
\langle\{s_3(0.3),s_4(0.2),s_5(0.3)\},\{s_{1/8}(0.3),s_{1/6}(0.3)\}\rangle & \langle\{s_3(0.3),s_4(0.2)\},\{s_3(0.2),s_4(0.6)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{1/4}(0.5),s_{1/3}(0.3),s_{1/2}(0.1)\},\{s_6(0.4),s_7(0.1),s_8(0.4)\}\rangle \\
\langle\{s_{1/6}(0.8),s_{1/4}(0.1)\},\{s_3(0.1),s_4(0.7),s_5(0.2)\}\rangle & \langle\{s_{1/2}(0.2),s_1(0.3),s_2(0.5)\},\{s_{1/7}(0.9),s_{1/6}(0.1)\}\rangle & \langle\{s_6(0.4),s_7(0.1),s_8(0.4)\},\{s_{1/4}(0.5),s_{1/3}(0.3),s_{1/2}(0.1)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

$$D_3=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{1/2}(0.8),s_2(0.1)\},\{s_{1/6}(0.3),s_{1/5}(0.2),s_{1/4}(0.4)\}\rangle & \langle\{s_{1/9}(0.3),s_{1/7}(0.2)\},\{s_{1/5}(0.5),s_{1/4}(0.3),s_{1/3}(0.1)\}\rangle & \langle\{s_{1/4}(0.3),s_{1/3}(0.1),s_{1/2}(0.6)\},\{s_{1/7}(0.5),s_{1/6}(0.2)\}\rangle \\
\langle\{s_{1/6}(0.3),s_{1/5}(0.2),s_{1/4}(0.4)\},\{s_{1/2}(0.8),s_2(0.1)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{1/8}(0.3),s_{1/7}(0.2),s_{1/6}(0.4)\},\{s_{1/2}(0.1),s_1(0.3),s_2(0.3)\}\rangle & \langle\{s_{1/8}(0.6),s_{1/7}(0.1),s_{1/6}(0.3)\},\{s_{1/2}(0.2),s_1(0.2),s_2(0.5)\}\rangle \\
\langle\{s_{1/5}(0.5),s_{1/4}(0.3),s_{1/3}(0.1)\},\{s_{1/9}(0.3),s_{1/7}(0.2)\}\rangle & \langle\{s_{1/2}(0.1),s_1(0.3),s_2(0.3)\},\{s_{1/8}(0.3),s_{1/7}(0.2),s_{1/6}(0.4)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{1/2}(0.6),s_2(0.1)\},\{s_{1/5}(0.6),s_{1/4}(0.3),s_{1/3}(0.1)\}\rangle \\
\langle\{s_{1/7}(0.5),s_{1/6}(0.2)\},\{s_{1/4}(0.3),s_{1/3}(0.1),s_{1/2}(0.6)\}\rangle & \langle\{s_{1/2}(0.2),s_1(0.2),s_2(0.5)\},\{s_{1/8}(0.6),s_{1/7}(0.1),s_{1/6}(0.3)\}\rangle & \langle\{s_{1/5}(0.6),s_{1/4}(0.3),s_{1/3}(0.1)\},\{s_{1/2}(0.6),s_2(0.1)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

$$D_4=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_4(0.2),s_5(0.3),s_6(0.2)\},\{s_1(0.5),s_2(0.1),s_3(0.3)\}\rangle & \langle\{s_1(0.1),s_2(0.1),s_3(0.1)\},\{s_{1/3}(0.2),s_{1/2}(0.4),s_1(0.2)\}\rangle & \langle\{s_{1/5}(0.3),s_{1/4}(0.6)\},\{s_2(0.2),s_3(0.2),s_4(0.2)\}\rangle \\
\langle\{s_1(0.5),s_2(0.1),s_3(0.3)\},\{s_4(0.2),s_5(0.3),s_6(0.2)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_1(0.2),s_2(0.3),s_3(0.3)\},\{s_{1/7}(0.5),s_{1/5}(0.5)\}\rangle & \langle\{s_{1/8}(0.1),s_{1/7}(0.3),s_{1/6}(0.6)\},\{s_{1/4}(0.3),s_{1/3}(0.3),s_{1/2}(0.3)\}\rangle \\
\langle\{s_{1/3}(0.2),s_{1/2}(0.4),s_1(0.2)\},\{s_1(0.1),s_2(0.1),s_3(0.1)\}\rangle & \langle\{s_{1/7}(0.5),s_{1/5}(0.5)\},\{s_1(0.2),s_2(0.3),s_3(0.3)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_6(0.4),s_7(0.1),s_8(0.3)\},\{s_{1/2}(0.3),s_1(0.3),s_2(0.1)\}\rangle \\
\langle\{s_2(0.2),s_3(0.2),s_4(0.2)\},\{s_{1/5}(0.3),s_{1/4}(0.6)\}\rangle & \langle\{s_{1/4}(0.3),s_{1/3}(0.3),s_{1/2}(0.3)\},\{s_{1/8}(0.1),s_{1/7}(0.3),s_{1/6}(0.6)\}\rangle & \langle\{s_{1/2}(0.3),s_1(0.3),s_2(0.1)\},\{s_6(0.4),s_7(0.1),s_8(0.3)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

Step 1. Let $\Xi=0.9$, then we check and improve the consistency of individual DPMLPRs $\tilde{D}_i\left(i=1,2,3,4\right)$ by Algorithm 1 as follows:

TABLE 1
THE CONSISTENT DEGREE OF INDIVIDUAL DPMLPRS

DPMLPRs	D_1	D_2	D_3	D_4
CI	0.6636	0.6161	0.5610	0.6851

Obviously, based on Table 1, all of the four individual DPMLPRs are not consistent. On the foundation of Algorithm 1, they can be adjusted as follows:

$$\tilde{D}_1=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{5.1537}(0.1117),\ldots,s_{8.8247}(0.1273)\},\{s_{0.1713}(0.1820),\ldots,s_{0.3494}(0.2003)\}\rangle & \langle\{s_{2.8586}(0.1689),\ldots,s_{6.3325}(0.1521)\},\{s_{0.4947}(0.1817),\ldots,s_{1.2786}(0.1246)\}\rangle & \langle\{s_{3.9218}(0.1770),\ldots,s_{6.6835}(0.1596)\},\{s_{0.2219}(0.1794),\ldots,s_{0.4147}(0.1651)\}\rangle \\
\langle\{s_{0.1713}(0.1820),\ldots,s_{0.3494}(0.2003)\},\{s_{5.1537}(0.1117),\ldots,s_{8.8247}(0.1273)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.3029}(0.1834),\ldots,s_{0.9268}(0.1722)\},\{s_{0.9375}(0.2383),\ldots,s_{2.1352}(0.0992)\}\rangle & \langle\{s_{0.3657}(0.2128),\ldots,s_{0.7432}(0.2515)\},\{s_{1.4142}(0.2225),\ldots,s_{2.7591}(0.2059)\}\rangle \\
\langle\{s_{0.4947}(0.1817),\ldots,s_{1.2786}(0.1246)\},\{s_{2.8586}(0.1689),\ldots,s_{6.3325}(0.1521)\}\rangle & \langle\{s_{0.9375}(0.2383),\ldots,s_{2.1352}(0.0992)\},\{s_{0.3029}(0.1834),\ldots,s_{0.9268}(0.1722)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{1.4694}(0.3004),\ldots,s_{2.9225}(0.1759)\},\{s_{0.4664}(0.2940),\ldots,s_{1.3594}(0.1868)\}\rangle \\
\langle\{s_{0.2219}(0.1794),\ldots,s_{0.4147}(0.1651)\},\{s_{3.9218}(0.1770),\ldots,s_{6.6835}(0.1596)\}\rangle & \langle\{s_{1.4142}(0.2225),\ldots,s_{2.7591}(0.2059)\},\{s_{0.3657}(0.2128),\ldots,s_{0.7432}(0.2515)\}\rangle & \langle\{s_{0.4664}(0.2940),\ldots,s_{1.3594}(0.1868)\},\{s_{1.4694}(0.3004),\ldots,s_{2.9225}(0.1759)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

$$\tilde{D}_2=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{2.1352}(0.1724),\ldots,s_{4.3735}(0.1401)\},\{s_{0.2740}(0.2614),\ldots,s_{0.4864}(0.1441)\}\rangle & \langle\{s_{0.7469}(0.1605),\ldots,s_{1.1826}(0.1972)\},\{s_{0.9950}(0.2526),\ldots,s_{1.9541}(0.1172)\}\rangle & \langle\{s_{0.9341}(0.1372),\ldots,s_{1.7160}(0.0910)\},\{s_{0.3644}(0.3449),\ldots,s_{0.7782}(0.1117)\}\rangle \\
\langle\{s_{0.2740}(0.2614),\ldots,s_{0.4864}(0.1441)\},\{s_{2.1352}(0.1724),\ldots,s_{4.3735}(0.1401)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.9666}(0.1923),\ldots,s_{1.4600}(0.2530)\},\{s_{2.1440}(0.2056),\ldots,s_{4.5004}(0.1231)\}\rangle & \langle\{s_{0.2551}(0.3150),\ldots,s_{0.4090}(0.0826)\},\{s_{1.0332}(0.1985),\ldots,s_{3.1870}(0.1939)\}\rangle \\
\langle\{s_{0.9950}(0.2526),\ldots,s_{1.9541}(0.1172)\},\{s_{0.7469}(0.1605),\ldots,s_{1.1826}(0.1972)\}\rangle & \langle\{s_{2.1440}(0.2056),\ldots,s_{4.5004}(0.1231)\},\{s_{0.9666}(0.1923),\ldots,s_{1.4600}(0.2530)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.5415}(0.2628),\ldots,s_{1.0989}(0.0672)\},\{s_{1.6000}(0.2360),\ldots,s_{2.9852}(0.2332)\}\rangle \\
\langle\{s_{0.3644}(0.3449),\ldots,s_{0.7782}(0.1117)\},\{s_{0.9341}(0.1372),\ldots,s_{1.7160}(0.0910)\}\rangle & \langle\{s_{1.0332}(0.1985),\ldots,s_{3.1870}(0.1939)\},\{s_{0.2551}(0.3150),\ldots,s_{0.4090}(0.0826)\}\rangle & \langle\{s_{1.6000}(0.2360),\ldots,s_{2.9852}(0.2332)\},\{s_{0.5415}(0.2628),\ldots,s_{1.0989}(0.0672)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

$$\tilde{D}_3=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.2554}(0.2659),\ldots,s_{1.2194}(0.1117)\},\{s_{0.0768}(0.2634),\ldots,s_{0.1249}(0.1820)\}\rangle & \langle\{s_{0.0859}(0.2614),\ldots,s_{0.1724}(0.1180)\},\{s_{0.1399}(0.2681),\ldots,s_{0.3597}(0.0765)\}\rangle & \langle\{s_{0.1454}(0.2977),\ldots,s_{0.4172}(0.1808)\},\{s_{0.1017}(0.3053),\ldots,s_{0.1898}(0.1140)\}\rangle \\
\langle\{s_{0.0768}(0.2634),\ldots,s_{0.1249}(0.1820)\},\{s_{0.2554}(0.2659),\ldots,s_{1.2194}(0.1117)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.0646}(0.2477),\ldots,s_{0.1016}(0.1820)\},\{s_{0.3247}(0.1342),\ldots,s_{1.8536}(0.1133)\}\rangle & \langle\{s_{0.0893}(0.3355),\ldots,s_{0.1798}(0.1972)\},\{s_{0.2567}(0.1817),\ldots,s_{1.1632}(0.1613)\}\rangle \\
\langle\{s_{0.1399}(0.2681),\ldots,s_{0.3597}(0.0765)\},\{s_{0.0859}(0.2614),\ldots,s_{0.1724}(0.1180)\}\rangle & \langle\{s_{0.3247}(0.1342),\ldots,s_{1.8536}(0.1133)\},\{s_{0.0646}(0.2477),\ldots,s_{0.1016}(0.1820)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.2199}(0.3004),\ldots,s_{0.8970}(0.0891)\},\{s_{0.1000}(0.3004),\ldots,s_{0.2032}(0.0959)\}\rangle \\
\langle\{s_{0.1017}(0.3053),\ldots,s_{0.1898}(0.1140)\},\{s_{0.1454}(0.2977),\ldots,s_{0.4172}(0.1808)\}\rangle & \langle\{s_{0.2567}(0.1817),\ldots,s_{1.1632}(0.1613)\},\{s_{0.0893}(0.3355),\ldots,s_{0.1798}(0.1972)\}\rangle & \langle\{s_{0.1000}(0.3004),\ldots,s_{0.2032}(0.0959)\},\{s_{0.2199}(0.3004),\ldots,s_{0.8970}(0.0891)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

$$\tilde{D}_4=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.9416}(0.1328),\ldots,s_{1.9900}(0.1512)\},\{s_{0.6276}(0.1703),\ldots,s_{2.2628}(0.1868)\}\rangle & \langle\{s_{0.8421}(0.0826),\ldots,s_{3.1663}(0.0826)\},\{s_{0.5568}(0.1756),\ldots,s_{1.7403}(0.1512)\}\rangle & \langle\{s_{0.4494}(0.1237),\ldots,s_{0.9132}(0.2530)\},\{s_{0.8499}(0.1512),\ldots,s_{2.9225}(0.1118)\}\rangle \\
\langle\{s_{0.6276}(0.1703),\ldots,s_{2.2628}(0.1868)\},\{s_{0.9416}(0.1328),\ldots,s_{1.9900}(0.1512)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.5946}(0.1081),\ldots,s_{2.4415}(0.1441)\},\{s_{0.3375}(0.2383),\ldots,s_{0.6636}(0.2258)\}\rangle & \langle\{s_{0.2821}(0.1035),\ldots,s_{0.5922}(0.3355)\},\{s_{0.3786}(0.1806),\ldots,s_{0.9909}(0.1469)\}\rangle \\
\langle\{s_{0.5568}(0.1756),\ldots,s_{1.7403}(0.1512)\},\{s_{0.8421}(0.0826),\ldots,s_{3.1663}(0.0826)\}\rangle & \langle\{s_{0.3375}(0.2383),\ldots,s_{0.6636}(0.2258)\},\{s_{0.5946}(0.1081),\ldots,s_{2.4415}(0.1441)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.8670}(0.1898),\ldots,s_{1.5777}(0.2526)\},\{s_{0.5694}(0.1335),\ldots,s_{2.6516}(0.0725)\}\rangle \\
\langle\{s_{0.8499}(0.1512),\ldots,s_{2.9225}(0.1118)\},\{s_{0.4494}(0.1237),\ldots,s_{0.9132}(0.2530)\}\rangle & \langle\{s_{0.3786}(0.1806),\ldots,s_{0.9909}(0.1469)\},\{s_{0.2821}(0.1035),\ldots,s_{0.5922}(0.3355)\}\rangle & \langle\{s_{0.5694}(0.1335),\ldots,s_{2.6516}(0.0725)\},\{s_{0.8670}(0.1898),\ldots,s_{1.5777}(0.2526)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

The consistent degree of four adjusted individual DPMLPRs are listed as follows:

TABLE 2
THE CONSISTENT DEGREE OF INDIVIDUAL DPMLPRS

DPMLPRs	$\tilde{D}_1$	$\tilde{D}_2$	$\tilde{D}_3$	$\tilde{D}_4$
CI	0.9210	0.9055	0.9105	0.9247

Step 2. Let the subjective weight of the DMs $\hat{\omega}=\left(0.3,0.2,0.15,0.35\right)^T$, then we utilize the aggregation operator (7) to figure out the group DPMLPR D as follows:

$$D=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{1.2557}(0.1831),\ldots,s_{3.0679}(0.1272)\},\{s_{0.2334}(0.1994),\ldots,s_{0.3447}(0.1795)\}\rangle & \langle\{s_{0.5619}(0.1829),\ldots,s_{1.1558}(0.1337)\},\{s_{0.4043}(0.2069),\ldots,s_{0.9351}(0.1068)\}\rangle & \langle\{s_{0.7433}(0.1750),\ldots,s_{1.4310}(0.1597)\},\{s_{0.3057}(0.1841),\ldots,s_{0.4795}(0.1265)\}\rangle \\
\langle\{s_{1.2557}(0.1831),\ldots,s_{3.0679}(0.1272)\},\{s_{0.2334}(0.1994),\ldots,s_{0.3447}(0.1795)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.3590}(0.1846),\ldots,s_{0.5414}(0.1846)\},\{s_{0.7109}(0.2005),\ldots,s_{1.9796}(0.1228)\}\rangle & \langle\{s_{0.2515}(0.2198),\ldots,s_{0.3879}(0.1930)\},\{s_{0.8494}(0.2004),\ldots,s_{1.8001}(0.1776)\}\rangle \\
\langle\{s_{0.4043}(0.2069),\ldots,s_{0.9351}(0.1068)\},\{s_{0.5619}(0.1829),\ldots,s_{1.1558}(0.1337)\}\rangle & \langle\{s_{0.7109}(0.2005),\ldots,s_{1.9796}(0.1228)\},\{s_{0.3590}(0.1846),\ldots,s_{0.5414}(0.1846)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.5836}(0.2527),\ldots,s_{1.4491}(0.1207)\},\{s_{0.4509}(0.1952),\ldots,s_{0.9043}(0.1342)\}\rangle \\
\langle\{s_{0.3057}(0.1841),\ldots,s_{0.4795}(0.1265)\},\{s_{0.7433}(0.1750),\ldots,s_{1.4310}(0.1597)\}\rangle & \langle\{s_{0.8494}(0.2004),\ldots,s_{1.8001}(0.1776)\},\{s_{0.2515}(0.2198),\ldots,s_{0.3879}(0.1930)\}\rangle & \langle\{s_{0.5836}(0.2527),\ldots,s_{1.4491}(0.1207)\},\{s_{0.4509}(0.1952),\ldots,s_{0.9043}(0.1342)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

Step 3. Let $\sigma=0.7$, then we figure out the comparable degree between the individual DPMLPRs with the group DPMLPR by Eq. (15) as follows:

TABLE 3
THE COMPARABLE DEGREE BETWEEN INDIVIDUAL DPMLPRS AND GROUP DPMLPR

DPMLPRs	D_1	D_2	D_3	D_4
Comparable degree	0.7652	0.6092	1.1478	0.6028

This table shows that the individual DPMLPRs D_1 and D_3 are not of the acceptable comparable degrees, so we adjust the individual DPMLPRs by Algorithm 2 as follows:

$$\tilde{D}'_1=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{2.7168}(0.1681),\ldots,s_{5.2032}(0.1273)\},\{s_{0.2357}(0.1881),\ldots,s_{0.3470}(0.1896)\}\rangle & \langle\{s_{1.3816}(0.1892),\ldots,s_{2.7054}(0.1426)\},\{s_{0.4472}(0.1939),\ldots,s_{1.0934}(0.1154)\}\rangle & \langle\{s_{1.8506}(0.1682),\ldots,s_{3.0925}(0.1596)\},\{s_{0.2604}(0.1817),\ldots,s_{0.4459}(0.1445)\}\rangle \\
\langle\{s_{0.2357}(0.1881),\ldots,s_{0.3470}(0.1896)\},\{s_{2.7168}(0.1681),\ldots,s_{5.2032}(0.1273)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.4596}(0.1785),\ldots,s_{0.7084}(0.1783)\},\{s_{0.8462}(0.2051),\ldots,s_{2.0559}(0.1104)\}\rangle & \langle\{s_{0.3454}(0.2219),\ldots,s_{0.5370}(0.2203)\},\{s_{1.0960}(0.2111),\ldots,s_{2.2286}(0.1912)\}\rangle \\
\langle\{s_{0.4472}(0.1939),\ldots,s_{1.0934}(0.1154)\},\{s_{1.3816}(0.1892),\ldots,s_{2.7054}(0.1426)\}\rangle & \langle\{s_{0.8462}(0.2051),\ldots,s_{2.0559}(0.1104)\},\{s_{0.4596}(0.1785),\ldots,s_{0.7084}(0.1783)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.9260}(0.2755),\ldots,s_{2.0579}(0.1457)\},\{s_{0.4586}(0.2396),\ldots,s_{1.1088}(0.1583)\}\rangle \\
\langle\{s_{0.2604}(0.1817),\ldots,s_{0.4459}(0.1445)\},\{s_{1.8506}(0.1682),\ldots,s_{3.0925}(0.1596)\}\rangle & \langle\{s_{1.0960}(0.2111),\ldots,s_{2.2286}(0.1912)\},\{s_{0.3454}(0.2219),\ldots,s_{0.5370}(0.2203)\}\rangle & \langle\{s_{0.4586}(0.2396),\ldots,s_{1.1088}(0.1583)\},\{s_{0.9260}(0.2755),\ldots,s_{2.0579}(0.1457)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

$$\tilde{D}'_3=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.5663}(0.2207),\ldots,s_{1.9341}(0.1192)\},\{s_{0.1519}(0.2028),\ldots,s_{0.2075}(0.1807)\}\rangle & \langle\{s_{0.2197}(0.2187),\ldots,s_{0.4463}(0.1256)\},\{s_{0.2485}(0.2101),\ldots,s_{0.5799}(0.0904)\}\rangle & \langle\{s_{0.3408}(0.1990),\ldots,s_{0.7727}(0.1699)\},\{s_{0.2127}(0.1999),\ldots,s_{0.3017}(0.1201)\}\rangle \\
\langle\{s_{0.1519}(0.2028),\ldots,s_{0.2075}(0.1807)\},\{s_{0.5663}(0.2207),\ldots,s_{1.9341}(0.1192)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.1679}(0.1960),\ldots,s_{0.2346}(0.1833)\},\{s_{0.5239}(0.1882),\ldots,s_{1.9156}(0.1179)\}\rangle & \langle\{s_{0.1715}(0.2090),\ldots,s_{0.2641}(0.1951)\},\{s_{0.6153}(0.1896),\ldots,s_{1.4470}(0.1693)\}\rangle \\
\langle\{s_{0.2485}(0.2101),\ldots,s_{0.5799}(0.0904)\},\{s_{0.2197}(0.2187),\ldots,s_{0.4463}(0.1256)\}\rangle & \langle\{s_{0.5239}(0.1882),\ldots,s_{1.9156}(0.1179)\},\{s_{0.1679}(0.1960),\ldots,s_{0.2346}(0.1833)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.3582}(0.2755),\ldots,s_{1.1401}(0.1037)\},\{s_{0.2344}(0.1746),\ldots,s_{0.4287}(0.1134)\}\rangle \\
\langle\{s_{0.2127}(0.1999),\ldots,s_{0.3017}(0.1201)\},\{s_{0.3408}(0.1990),\ldots,s_{0.7727}(0.1699)\}\rangle & \langle\{s_{0.6153}(0.1896),\ldots,s_{1.4470}(0.1693)\},\{s_{0.1715}(0.2090),\ldots,s_{0.2641}(0.1951)\}\rangle & \langle\{s_{0.2344}(0.1746),\ldots,s_{0.4287}(0.1134)\},\{s_{0.3582}(0.2755),\ldots,s_{1.1401}(0.1037)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

Until all the individual DPMLPRs fulfill with the $C\left(\widetilde{D_i}, D\right) \le 0.7\left(i = 1,2,3,4\right)$, by using Eq. (7), we can get a group DPMLPR as follows:

$$D=\begin{pmatrix}
\langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{1.3164}(0.1759),\ldots,s_{3.0771}(0.1301)\},\{s_{0.2705}(0.2017),\ldots,s_{0.4108}(0.1761)\}\rangle & \langle\{s_{0.5959}(0.1701),\ldots,s_{1.2494}(0.1341)\},\{s_{0.4651}(0.2099),\ldots,s_{1.0546}(0.1107)\}\rangle & \langle\{s_{0.7427}(0.1693),\ldots,s_{1.4089}(0.1562)\},\{s_{0.3641}(0.1796),\ldots,s_{0.5764}(0.1238)\}\rangle \\
\langle\{s_{0.2705}(0.2017),\ldots,s_{0.4108}(0.1761)\},\{s_{1.3164}(0.1759),\ldots,s_{3.0771}(0.1301)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.4349}(0.1825),\ldots,s_{0.6694}(0.1870)\},\{s_{0.7507}(0.2036),\ldots,s_{1.9799}(0.1285)\}\rangle & \langle\{s_{0.2614}(0.2231),\ldots,s_{0.4025}(0.1848)\},\{s_{0.8809}(0.2011),\ldots,s_{1.8225}(0.1766)\}\rangle \\
\langle\{s_{0.4651}(0.2099),\ldots,s_{1.0546}(0.1107)\},\{s_{0.5959}(0.1701),\ldots,s_{1.2494}(0.1341)\}\rangle & \langle\{s_{0.7507}(0.2036),\ldots,s_{1.9799}(0.1285)\},\{s_{0.4349}(0.1825),\ldots,s_{0.6694}(0.1870)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle & \langle\{s_{0.6027}(0.2388),\ldots,s_{1.4186}(0.1204)\},\{s_{0.5640}(0.1909),\ldots,s_{1.1046}(0.1354)\}\rangle \\
\langle\{s_{0.3641}(0.1796),\ldots,s_{0.5764}(0.1238)\},\{s_{0.7427}(0.1693),\ldots,s_{1.4089}(0.1562)\}\rangle & \langle\{s_{0.8809}(0.2011),\ldots,s_{1.8225}(0.1766)\},\{s_{0.2614}(0.2231),\ldots,s_{0.4025}(0.1848)\}\rangle & \langle\{s_{0.5640}(0.1909),\ldots,s_{1.1046}(0.1354)\},\{s_{0.6027}(0.2388),\ldots,s_{1.4186}(0.1204)\}\rangle & \langle\{s_1(1)\},\{s_1(1)\}\rangle
\end{pmatrix}$$

Then, we use the Eq. (17) can get the final weight for the criteria as:

$$\varpi = \left(0.2488, 0.2608, 0.2456, 0.2448\right)$$

Moreover, through the combination of the subjective weights of DMs and the four individual dual probabilistic linguistic decision-making matrices $M_i\left(i = 1,2,3,4\right)$

$$\mathrm{M}_1=\begin{bmatrix}
\langle\{s_{1/8}(0.5),s_{1/7}(0.3),s_{1/6}(0.1)\},\{s_{1/7}(0.2),s_{1/6}(0.8)\}\rangle & \langle\{s_5(0.3),s_6(0.2),s_7(0.4)\},\{s_{1/3}(0.1),s_{1/2}(0.3),s_1(0.5)\}\rangle & \langle\{s_4(0.1),s_6(0.9)\},\{s_{1/9}(0.5),s_{1/8}(0.3),s_{1/7}(0.2)\}\rangle & \langle\{s_{1/5}(0.3),s_{1/4}(0.4)\},\{s_2(0.2),s_3(0.2),s_4(0.5)\}\rangle \\
\langle\{s_{1/7}(0.1),s_{1/6}(0.4),s_{1/5}(0.2)\},\{s_{1/2}(0.6),s_2(0.1)\}\rangle & \langle\{s_2(0.3),s_4(0.6)\},\{s_{1/2}(0.3),s_1(0.5)\}\rangle & \langle\{s_{1/3}(0.1),s_1(0.4)\},\{s_5(0.2),s_6(0.4)\}\rangle & \langle\{s_{1/8}(0.3),s_{1/6}(0.5)\},\{s_{1/8}(0.3),s_{1/7}(0.2),s_{1/6}(0.2)\}\rangle \\
\langle\{s_7(0.3),s_9(0.3)\},\{s_4(0.7),s_6(0.3)\}\rangle & \langle\{s_7(0.1),s_8(0.4),s_9(0.3)\},\{s_3(0.2),s_4(0.3)\}\rangle & \langle\{s_{1/5}(0.6),s_{1/3}(0.4)\},\{s_6(0.5),s_7(0.2),s_8(0.2)\}\rangle & \langle\{s_{1/3}(0.5),s_{1/2}(0.4)\},\{s_{1/6}(0.4),s_{1/5}(0.4),s_{1/4}(0.2)\}\rangle \\
\langle\{s_3(0.4),s_4(0.2),s_5(0.3)\},\{s_{1/5}(0.2),s_{1/4}(0.2)\}\rangle & \langle\{s_{1/7}(0.1),s_{1/5}(0.8)\},\{s_{1/8}(0.1),s_{1/7}(0.1),s_{1/6}(0.6)\}\rangle & \langle\{s_7(0.2),s_8(0.3),s_9(0.1)\},\{s_3(0.5),s_4(0.1),s_5(0.3)\}\rangle & \langle\{s_4(0.6),s_5(0.2),s_6(0.2)\},\{s_{1/2}(0.5),s_1(0.1)\}\rangle
\end{bmatrix}$$

$$\mathrm{M}_2=\begin{bmatrix}
\langle\{s_{1/8}(0.5),s_{1/7}(0.3)\},\{s_8(0.2),s_9(0.1)\}\rangle & \langle\{s_5(0.1),s_7(0.2)\},\{s_8(0.2),s_9(0.2)\}\rangle & \langle\{s_{1/6}(0.6),s_{1/5}(0.2)\},\{s_2(0.4),s_3(0.1),s_4(0.3)\}\rangle & \langle\{s_3(0.1),s_4(0.1),s_5(0.8)\},\{s_{1/4}(0.1),s_{1/2}(0.8)\}\rangle \\
\langle\{s_{1/2}(0.3),s_2(0.1)\},\{s_4(0.5),s_5(0.3)\}\rangle & \langle\{s_{1/3}(0.1),s_{1/2}(0.7)\},\{s_{1/5}(0.5),s_{1/4}(0.2),s_{1/3}(0.2)\}\rangle & \langle\{s_6(0.2),s_8(0.4)\},\{s_{1/8}(0.2),s_{1/7}(0.3),s_{1/6}(0.1)\}\rangle & \langle\{s_3(0.2),s_4(0.4),s_5(0.4)\},\{s_{1/8}(0.4),s_{1/7}(0.1),s_{1/6}(0.1)\}\rangle \\
\langle\{s_1(0.1),s_2(0.5),s_3(0.1)\},\{s_1(0.2),s_3(0.3)\}\rangle & \langle\{s_5(0.8),s_6(0.2)\},\{s_4(0.3),s_6(0.7)\}\rangle & \langle\{s_{1/2}(0.3),s_1(0.3),s_2(0.3)\},\{s_3(0.3),s_4(0.2),s_5(0.5)\}\rangle & \langle\{s_{1/5}(0.2),s_{1/4}(0.3),s_{1/3}(0.3)\},\{s_{1/7}(0.2),s_{1/6}(0.8)\}\rangle \\
\langle\{s_3(0.5),s_4(0.2),s_5(0.2)\},\{s_5(0.4),s_6(0.2),s_7(0.4)\}\rangle & \langle\{s_7(0.8),s_8(0.2)\},\{s_{1/9}(0.4),s_{1/8}(0.4)\}\rangle & \langle\{s_{1/5}(0.3),s_{1/3}(0.7)\},\{s_{1/4}(0.5),s_{1/3}(0.1)\}\rangle & \langle\{s_6(0.1),s_7(0.2),s_8(0.6)\},\{s_{1/3}(0.4),s_{1/2}(0.6)\}\rangle
\end{bmatrix}$$

$$\mathrm{M}_3=\begin{bmatrix}
\langle\{s_6(0.4),s_7(0.1),s_8(0.5)\},\{s_1(0.3),s_2(0.2),s_3(0.5)\}\rangle & \langle\{s_2(0.3),s_3(0.2)\},\{s_{1/7}(0.3),s_{1/6}(0.2),s_{1/5}(0.3)\}\rangle & \langle\{s_3(0.1),s_4(0.6),s_5(0.2)\},\{s_{1/3}(0.2),s_{1/2}(0.4)\}\rangle & \langle\{s_6(0.2),s_7(0.4),s_8(0.3)\},\{s_{1/5}(0.1),s_{1/4}(0.4),s_{1/3}(0.3)\}\rangle \\
\langle\{s_4(0.5),s_5(0.3),s_6(0.1)\},\{s_5(0.3),s_6(0.1),s_7(0.3)\}\rangle & \langle\{s_{1/8}(0.1),s_{1/7}(0.1)\},\{s_7(0.1),s_8(0.2),s_9(0.4)\}\rangle & \langle\{s_{1/5}(0.5),s_{1/3}(0.1)\},\{s_5(0.4),s_6(0.5)\}\rangle & \langle\{s_{1/4}(0.2),s_{1/3}(0.5),s_{1/2}(0.3)\},\{s_1(0.5),s_2(0.1),s_3(0.3)\}\rangle \\
\langle\{s_{1/4}(0.4),s_{1/3}(0.1),s_{1/2}(0.4)\},\{s_{1/3}(0.3),s_1(0.2)\}\rangle & \langle\{s_5(0.4),s_6(0.1),s_7(0.2)\},\{s_3(0.1),s_4(0.4),s_5(0.2)\}\rangle & \langle\{s_6(0.4),s_7(0.3),s_8(0.1)\},\{s_4(0.2),s_5(0.1),s_6(0.7)\}\rangle & \langle\{s_3(0.5),s_4(0.1),s_5(0.3)\},\{s_4(0.1),s_5(0.2),s_6(0.3)\}\rangle \\
\langle\{s_1(0.1),s_2(0.5),s_3(0.3)\},\{s_{1/8}(0.2),s_{1/7}(0.6),s_{1/6}(0.2)\}\rangle & \langle\{s_{1/6}(0.2),s_{1/5}(0.6)\},\{s_2(0.4),s_3(0.3)\}\rangle & \langle\{s_{1/6}(0.5),s_{1/5}(0.2),s_{1/4}(0.3)\},\{s_{1/6}(0.3),s_{1/5}(0.4),s_{1/4}(0.3)\}\rangle & \langle\{s_3(0.7),s_5(0.2)\},\{s_{1/3}(0.3),s_1(0.4)\}\rangle
\end{bmatrix}$$

$$\mathrm{M}_4=\begin{bmatrix}
\langle\{s_6(0.4),s_7(0.1),s_8(0.3)\},\{s_{1/2}(0.3),s_1(0.3),s_2(0.1)\}\rangle & \langle\{s_{1/6}(0.1),s_{1/5}(0.2),s_{1/4}(0.3)\},\{s_{1/2}(0.3),s_1(0.2),s_2(0.3)\}\rangle & \langle\{s_4(0.1),s_5(0.3),s_6(0.2)\},\{s_5(0.2),s_6(0.7),s_7(0.1)\}\rangle & \langle\{s_{1/7}(0.3),s_{1/6}(0.6),s_{1/5}(0.1)\},\{s_5(0.6),s_6(0.2)\}\rangle \\
\langle\{s_{1/5}(0.2),s_{1/4}(0.4),s_{1/3}(0.3)\},\{s_{1/7}(0.2),s_{1/6}(0.1),s_{1/5}(0.6)\}\rangle & \langle\{s_{1/8}(0.3),s_{1/7}(0.3),s_{1/6}(0.3)\},\{s_{1/8}(0.4),s_{1/7}(0.3),s_{1/6}(0.3)\}\rangle & \langle\{s_5(0.5),s_6(0.2),s_7(0.2)\},\{s_{1/5}(0.2),s_{1/4}(0.3),s_{1/3}(0.5)\}\rangle & \langle\{s_{1/2}(0.2),s_1(0.5)\},\{s_{1/2}(0.1),s_2(0.6)\}\rangle \\
\langle\{s_{1/3}(0.5),s_{1/2}(0.1),s_1(0.4)\},\{s_3(0.3),s_5(0.7)\}\rangle & \langle\{s_{1/4}(0.5),s_{1/3}(0.3),s_{1/2}(0.2)\},\{s_{1/5}(0.2),s_{1/4}(0.8)\}\rangle & \langle\{s_4(0.2),s_5(0.1),s_6(0.3)\},\{s_{1/9}(0.1),s_{1/8}(0.5),s_{1/7}(0.2)\}\rangle & \langle\{s_4(0.4),s_5(0.1),s_6(0.5)\},\{s_4(0.3),s_6(0.7)\}\rangle \\
\langle\{s_{1/8}(0.5),s_{1/7}(0.2)\},\{s_4(0.4),s_5(0.4),s_6(0.2)\}\rangle & \langle\{s_8(0.7),s_9(0.2)\},\{s_5(0.2),s_6(0.6),s_7(0.2)\}\rangle & \langle\{s_{1/3}(0.2),s_{1/2}(0.4),s_1(0.2)\},\{s_{1/4}(0.6),s_{1/3}(0.1),s_{1/2}(0.2)\}\rangle & \langle\{s_5(0.6),s_6(0.4)\},\{s_{1/5}(0.3),s_{1/4}(0.3),s_{1/3}(0.2)\}\rangle
\end{bmatrix}$$

the group dual probabilistic multiplicative linguistic decision-making matrix M can be figured up as follows:

$$\mathrm{M}=\begin{bmatrix}
\langle\{s_{0.8660}(0.4472),\ldots,s_{1.1196}(0.2580)\},\{s_{0.7620}(0.2449),\ldots,s_{1.4776}(0.3278)\}\rangle & \langle\{s_{2.1783}(0.2042),\ldots,s_{3.1567}(0.2617)\},\{s_{0.4972}(0.1990),\ldots,s_{0.9803}(0.3224)\}\rangle & \langle\{s_{1.9155}(0.1431),\ldots,s_{2.8511}(0.3140)\},\{s_{0.5150}(0.3024),\ldots,s_{0.7732}(0.2491)\}\rangle & \langle\{s_{1.0748}(0.2090),\ldots,s_{1.4805}(0.3375)\},\{s_{0.6762}(0.1611),\ldots,s_{1.1753}(0.4004)\}\rangle \\
\langle\{s_{0.6196}(0.2428),\ldots,s_{1.1254}(0.1452)\},\{s_{1.4060}(0.3849),\ldots,s_{2.6366}(0.2394)\}\rangle & \langle\{s_{0.3494}(0.1639),\ldots,s_{0.5104}(0.2979)\},\{s_{0.8516}(0.2362),\ldots,s_{1.3238}(0.3566)\}\rangle & \langle\{s_{0.7459}(0.2569),\ldots,s_{1.3816}(0.2219)\},\{s_{1.4753}(0.2549),\ldots,s_{1.8993}(0.3389)\}\rangle & \langle\{s_{0.3704}(0.2259),\ldots,s_{0.6324}(0.3999)\},\{s_{0.3186}(0.3222),\ldots,s_{0.6654}(0.2366)\}\rangle \\
\langle\{s_{0.9359}(0.2875),\ldots,s_{1.8895}(0.2781)\},\{s_{1.2167}(0.3567),\ldots,s_{2.7147}(0.2956)\}\rangle & \langle\{s_{3.5290}(0.3135),\ldots,s_{4.9264}(0.2259)\},\{s_{2.1169}(0.1702),\ldots,s_{3.0944}(0.3573)\}\rangle & \langle\{s_{1.2381}(0.3844),\ldots,s_{2.2381}(0.2226)\},\{s_{2.4915}(0.2573),\ldots,s_{3.6001}(0.3724)\}\rangle & \langle\{s_{0.9427}(0.4026),\ldots,s_{1.4984}(0.3531)\},\{s_{0.7917}(0.2053),\ldots,s_{1.1293}(0.3670)\}\rangle \\
\langle\{s_{1.2679}(0.2662),\ldots,s_{2.4531}(0.2603)\},\{s_{0.5062}(0.2549),\ldots,s_{0.6804}(0.2297)\}\rangle & \langle\{s_{0.6006}(0.2587),\ldots,s_{0.7403}(0.4453)\},\{s_{0.5603}(0.2378),\ldots,s_{0.7580}(0.3681)\}\rangle & \langle\{s_{0.5886}(0.2989),\ldots,s_{0.9553}(0.2405)\},\{s_{0.4572}(0.4297),\ldots,s_{0.7218}(0.2266)\}\rangle & \langle\{s_{4.0559}(0.4425),\ldots,s_{5.9625}(0.2764)\},\{s_{0.3487}(0.3704),\ldots,s_{0.7383}(0.2579)\}\rangle
\end{bmatrix}$$

Step 4. Determine the PIE M_j^+ and the NIE M_j^- for the DPMLPR M and

$$M_j^+ = \left(\left\langle\{s_{0.9359}(0.2875),\ldots,s_{1.8895}(0.2781)\},\{s_{1.2167}(0.3567),\ldots,s_{2.7147}(0.2956)\}\right\rangle,\right.$$
$$\left\langle\{s_{3.5290}(0.3135),\ldots,s_{4.9264}(0.2259)\},\{s_{2.1169}(0.1702),\ldots,s_{3.0944}(0.3573)\}\right\rangle,$$
$$\left\langle\{s_{1.2381}(0.3844),\ldots,s_{2.2381}(0.2226)\},\{s_{2.4915}(0.2573),\ldots,s_{3.6001}(0.3724)\}\right\rangle,$$
$$\left.\left\langle\{s_{4.0559}(0.4425),\ldots,s_{5.9625}(0.2764)\},\{s_{0.3487}(0.3704),\ldots,s_{0.7383}(0.2579)\}\right\rangle\right)$$

$$M_j^- = \left(\left\langle\{s_{0.8660}(0.4472),\ldots,s_{1.1196}(0.2580)\},\{s_{0.7620}(0.2449),\ldots,s_{1.4776}(0.3278)\}\right\rangle,\right.$$
$$\left\langle\{s_{0.3494}(0.1639),\ldots,s_{0.5104}(0.2979)\},\{s_{0.8516}(0.2362),\ldots,s_{1.3238}(0.3566)\}\right\rangle,$$
$$\left\langle\{s_{0.5886}(0.2989),\ldots,s_{0.9553}(0.2405)\},\{s_{0.4572}(0.4297),\ldots,s_{0.7218}(0.2266)\}\right\rangle,$$
$$\left.\left\langle\{s_{0.3704}(0.2259),\ldots,s_{0.6324}(0.3999)\},\{s_{0.3186}(0.3222),\ldots,s_{0.6654}(0.2366)\}\right\rangle\right)$$

Step 5. Let $\xi = 0.5$, by using Eqs. (19) and (20), we figure out the grey relative coefficient matrices μ_{ij}^+ and μ_{ij}^-:

$$\mu^+ = \begin{pmatrix} 0.6551 & 0.4437 & 0.4704 & 0.4570 \\ 0.6744 & 0.3333 & 0.5966 & 0.3797 \\ 1 & 1 & 1 & 0.4482 \\ 0.4873 & 0.3685 & 0.3954 & 1 \end{pmatrix}$$

and

$$\mu^- = \begin{pmatrix} 1 & 0.4141 & 0.5568 & 0.4982 \\ 0.6540 & 1 & 0.5396 & 1 \\ 0.6551 & 0.3333 & 0.3954 & 0.4850 \\ 0.5660 & 0.5420 & 1 & 0.3797 \end{pmatrix}$$

Step 6. Using Eq. (21) to figure out the closeness coefficient:

$$OC = \left(0.4512, 0.3816, 0.6500, 0.4739\right)$$

Therefore, the priority of the alternatives is $a_3 \succ a_4 \succ a_1 \succ a_2$.

B. RESULT ANALYSIS WITH EXPANDING TODIM

In this subsection, based on the proposed comparable degree, we propose the ETODIM.

As the conventional introduction for the TODIM, it usually concludes the following procedures:(1) Obtain the group decision-making information; (2) Divide the index value into two classifications: the benefit type and the cost type and normalize the group decision-making information; (3) Figure up the relative weight between the fixed indexes; (4) Count the comparative dominance between the selected alternatives; (5) Compute the prospect value on account of the acquired dominance and receive the ranking of the picked alternatives.

In this subsection, different from the traditional TODIM that uses the distance measure to measure the deviation between the alternatives, we use the comparable degree to calculate the comparative dominance between the selected alternatives. Concretely, the ETODIM can be stated below:

Step 1. Acquire the group dual probabilistic linguistic decision-making matrix $M = \left(M_{ij}\right)_{n\times n}$;

Step 2. Normalize the dual probabilistic linguistic decision-making matrix $\bar{M} = \left(\bar{M}_{ij}\right)_{m\times n}$, if $\bar{M}_{ij} = \begin{cases} \bar{M}_{ij}, & if \quad c_j \quad is \quad benefit \\ M_{ij}, & if \quad c_j \quad is \quad cost \end{cases}$;

Step 3. Figure up the weights for criteria $\breve{\varpi}=\left(\breve{\varpi}_1,\breve{\varpi}_2,\ldots,\breve{\varpi}_n\right)$ by the Eq. (17), then we count the comparative weight $\breve{\varpi}_{jr} = \breve{\varpi}_j / \breve{\varpi}_r$, where $\breve{\varpi}_r = \max\left(\breve{\varpi}_1,\breve{\varpi}_2,\ldots,\breve{\varpi}_n\right)$;

Step 4. Count the comparative dominance $\mathbb{Q}\left(a_i,a_k\right)=\sum_{j=1}^{n}\psi_j\left(a_i,a_k\right)$ for $\forall\left(i,k\right)$, where

$$\psi_j\left(a_i,a_k\right)=\begin{cases} \sqrt{\breve{\varpi}_{jr}C\left(\bar{M}_{ij},\bar{M}_{kj}\right)/\sum_{j=1}^{n}\breve{\varpi}_{jr}}, & if \quad S\left(\bar{M}_{ij}\right)>S\left(\bar{M}_{kj}\right); \\ 0, & if \quad S\left(\bar{M}_{ij}\right)=S\left(\bar{M}_{kj}\right); \\ -\frac{1}{\varsigma}\sqrt{C\left(\bar{M}_{kj},\bar{M}_{ij}\right)\left(\sum_{j=1}^{n}\breve{\varpi}_{jr}\right)\Big/\breve{\varpi}_{jr}}, & if \quad S\left(\bar{M}_{ij}\right)<S\left(\bar{M}_{kj}\right); \end{cases} \tag{22}$$

Moreover, the parameter ς is the attenuation factor of the losses, here we take $\varsigma=1$ which manifests that the losses will make contribution to their real value to the global value and $\left(a_i,a_k\right)$ is any pair of alternatives.

Step 5. Compute the prospect value of the picked alternatives as:

$$\aleph\left(a_i\right)=\frac{\sum_{k=1}^{m}\mathbb{Q}\left(a_i,a_k\right)-\min_i\left\{\sum_{k=1}^{m}\mathbb{Q}\left(a_i,a_k\right)\right\}}{\max_i\left\{\sum_{k=1}^{m}\mathbb{Q}\left(a_i,a_k\right)\right\}-\min_i\left\{\sum_{k=1}^{m}\mathbb{Q}\left(a_i,a_k\right)\right\}} \tag{23}$$

Then with the ETODIM method, we can get the comparative weight $\breve{\varpi}=\left(0.9540,1,0.9415,0.9384\right)$, the comparative dominance matrix $\mathbb{Q}=\begin{pmatrix} 0 & 1.8378 & -1.0826 & -2.8501 \\ -7.3405 & 0 & -5.3131 & -7.7640 \\ -3.1812 & -0.0735 & 0 & -5.5764 \\ -2.0592 & 1.9365 & -0.8096 & 0 \end{pmatrix}$, the prospect values of the picked alternatives are as follows: $\aleph\left(a_1\right)=0.9421$, $\aleph\left(a_2\right)=0$, $\aleph\left(a_3\right)=0.5957$, $\aleph\left(a_4\right)=1$. Then we can get the priority of the alternatives as $a_4 \succ a_1 \succ a_3 \succ a_2$.

C. RESULT ANALYSIS WITH EXPANDING VIKOR

Owing to that the TODIM and the GRA are with the same principle that use the distance as the basis to compute, so we consider to use the other relative classic VIKOR for comparative analysis in this subsection. First, we state simply the classic VIKOR as follows: (1) Seek out the PIE and the NIE for the benefit and cost criteria, respectively; (2) Determine the weight of the criteria; (3) Figure up the ordering value; (4) Count the compromise solution of the chosen alternatives and confirm the priority for the alternatives. Similarly, on the foundation of the suggested comparable degree, we present the following EVIKOR below:

Step 1. For the obtained group dual probabilistic linguistic decision-making matrix $M=\left(M_{ij}\right)_{n\times n}$, seek out the PIE:

$$M_j^*=\begin{cases} \max_i M_{ij}, & if \quad c_j \quad is \quad benefit; \\ \min_i M_{ij}, & if \quad c_j \quad is \quad cost. \end{cases} \tag{24}$$

and the NIE:

$$M_j^-=\begin{cases} \min_i M_{ij}, & if \quad c_j \quad is \quad benefit; \\ \max_i M_{ij}, & if \quad c_j \quad is \quad cost. \end{cases} \tag{25}$$

Step 2. Determine the weights for criteria $\bar{\varpi}=\left(\bar{\varpi}_1,\bar{\varpi}_2,\ldots,\bar{\varpi}_n\right)$ by Eq. (17).

Step 3. Figure up the ordering value Υ_j and Z_j as follows:

$$\begin{cases} \Upsilon_i=\sum_{j=1}^{n}\bar{\varpi}_j\frac{C\left(M_j^*,M_{ij}\right)}{C\left(M_j^*,M_j^-\right)} \\ Z_i=\max_i\left(\bar{\varpi}_j\frac{C\left(M_j^*,M_{ij}\right)}{C\left(M_j^*,M_j^-\right)}\right) \end{cases}, \quad i=1,2,\ldots,m. \tag{26}$$

Step 4. Count the compromise solution of the chosen alternatives as:

$$\begin{cases} \Lambda_i=\rho\frac{\Upsilon_i-\Upsilon^*}{\Upsilon^--\Upsilon^*}+\left(1-\rho\right)\frac{Z_i-Z^*}{Z^--Z^*} \\ i=1,2,\ldots,m \end{cases} \tag{27}$$

where the parameter $\rho\in\left(0,1\right)$ shows the weight of Υ_i and the decision-making tactic of the DMs. The ultimate sort outcome is steady with a decision-making tactic, which accords with the majority rule if $\rho>0.5$, or the consensus rule if $\rho=0.5$, or the veto rule if $\rho<0.5$. $\Upsilon^*=\min(\Upsilon_i)$, $\Upsilon^-=\max(\Upsilon_i)$, $Z^*=\min\left(Z_i\right)$, $Z^-=\max\left(Z_i\right)$.

Then by using EVIKOR method, the ordering value $\Upsilon=\left(0.7711,0.8423,0.1844,0.9663\right)$, $Z=\left(0.2488,0.2608,0.1844,0.4973\right)$ and the compromise solution of the chosen alternatives $\Lambda_1=0.4781$, $\Lambda_2=0.5428$, $\Lambda_3=0$, $\Lambda_4=1$. Therefore, we can get the priority of the alternatives as $a_3 \succ a_1 \succ a_2 \succ a_4$.

So as to present the results clearly, we give the following table:

TABLE 4
THE PRIORITY OF ALTERNATIVES WITH DIFFERENT METHODS

Method	The priority of alternatives

EGRA	$a_3 \succ a_4 \succ a_1 \succ a_2$
ETODIM	$a_4 \succ a_1 \succ a_3 \succ a_2$
EVIKOR	$a_3 \succ a_1 \succ a_2 \succ a_4$

Apparently, the obtained optimal decisions by three different methods are different. For the EGRA method, the optimal alternative is a_3. Usually, it is based on the degree of similarity or dissimilarity between the development trends of factors, that is, the "grey correlation degree", as a method to measure the degree of association between factors. It considers the relative comparable degree between the ideal solution and the alternative. It has the advantage of being simple to calculate. For the ETODIM method, the optimal alternative is a_4. It is a typical decision-making method considering the mental behavior of DMs based on the prospect theory. It sorts and optimizes the solution by calculating the dominance of the alternatives over other scenarios. The salient features of it are that it not only accelerates the risk factor in the system, but also enriches the range of decision-making procedure. Moreover, it provides a chance for us to check gains and losses for any two alternatives with regard to any criteria. While for the EVIKOR method, the optimal alternative is a_3. If there is a conflict between the indicators, it sorts the scheme according to a certain method, so as to obtain an optimal solution. Because it maximizes group benefits and minimizes individual losses, it leads to a compromise solution that can be acknowledged by DMs. Moreover, the compromise solution is the optimal solution in the solution space.

D. SENSITIVITY ANALYSIS WITH THE PARAMETER ξ

Let ξ vary from 0 to 1, then we distinguish the variation of three different final priorities by the following figure:

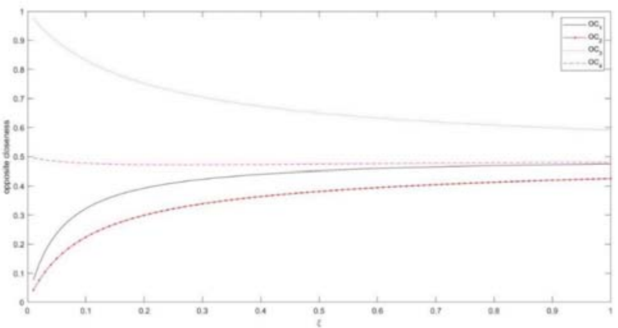

FIGURE 2. The priority of alternatives with the variation of parameter ξ

From the figure 2, it is to see when the parameter ξ increases, there are fewer and fewer differences between the schemes of the alternatives. The purpose of decision-making is to choose the preferred alternative among the selected alternatives. In this paper, let $\xi = 0.5$, then we not only can obtain the priority of alternatives, but also in the risk neutral status. To some extent, the choice of the parameter is rational.

VI. CONCLUSIONS

In this paper, we have enriched the basic theory of the DPLTSs by putting forward the DPMLTSs and the DPMLPRs, separately. Moreover, we have considered the importance of the consistency of the PRs in the procedure of obtaining the logical decision result, and probed the consistency of the DPMLPRs. Furthermore, on the foundation of the proposed comparable degree between the DPMLPRs, we have researched the consensus of the group DPMLPR. In addition, in order to obtain the final decision result, we have proposed the EGRA method. On the side, we have also developed the ETODIM method and the EVIKOR method based upon the comparable degrees. After that, we have applied the proposed method to settle the problem that mentioned at the beginning of the paper, and helped choose the best cooperative enterprise for cloud enterprise. Finally, the specific execution of the example has demonstrated the effective of the proposed theory. Besides, two comparative analyses have been utilized to highlight the advantages and disadvantages of the proposed method.

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A new multi-criteria decision model based on incomplete dual probabilistic linguistic preference relations

Wanying Xie [1,3], Zeshui Xu [1,2,*] , Zhiliang Ren [1], Enrique Herrera-Viedma [3,4,*]

[1] School of Economics and Management, Southeast University, Nanjing, Jiangsu 211189, China

[2] Business School, Sichuan University, Chengdu, Sichuan, 610064, China

[3] Andalusian Research Institute in Data Science and Computational Intelligence, University of Granada, E-18071 Granada, Spain

[4] Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah, 21589, Saudi Arabia

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Abstract

The use of dual probabilistic linguistic term sets (DPLTSs) to represent the use's preferences in decision making can reflect the decision maker's cognitive certainty and uncertainty. Additionally, the appearance of incomplete preferences is a recurring phenomenon that must be taken into account if you want to make a successful decision. This paper presents a new multi-criteria decision model based on the incomplete dual probabilistic linguistic preference relations (IDPLPRs). We first propose a step-by-step repairing method to repair the linguistic section and probabilistic section of IDPLPRs separately. The superiority is that this step-by-step method conforms to the principle of element generation. After that, the consistency index based on the distance measure between the dual probabilistic linguistic preference relations (DPLPRs) is defined to check and improve the consistency of DPLPRs. Then the weights of criteria can be obtained by information fusion. Moreover, we construct optimistic and pessimistic data envelopment analysis models under the dual probabilistic linguistic environment to do the sorting process. Optimistic and pessimistic data envelopment analysis models can demonstrate the efficiency of each decision-making unit (DMU) from the perspective of the most and least favorable. Finally, we simulate a cased of 5G industry market to help enterprises choose appropriate 5G partners by using proposed methods.

Keywords: 5G; Incomplete dual probabilistic linguistic preference relations; Repairing; Consistency; Dual probabilistic linguistic envelopment analysis.

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[*] Corresponding author. E-mail addresses: xie_wanyinglai@163.com (W.Y. Xie), xuzeshui@263.net (Z.S. Xu), ren_zhi_liang@126.com (Z.L. Ren), viedma@decsai.ugr.es (E. Herrera-Viedma).

1. Introduction

The fifth generation mobile phone mobile communication standard, also known as the fifth generation mobile communication technology, the abbreviation is 5G. At one time, the emergence of mobile phones and text messages was very exciting. Playing games on the mobile phone with colored screen in the subway or bus was enviable. Although 4G networks [1] are becoming more and more popular, using traffic to watch video is still a "local tyrant". In the 5G era, when the eyes are closed, an ultra-high-definition movie has been downloaded. In addition to the fast speed, it can also bring "magic power". 5G will not only enhance greatly the high-bandwidth business experience of mobile internet users [2], but also lead to profound changes in production modes and lifestyles. At the same time, it can also meet the needs of massive applications with large connections and wide coverage. It plays an important role in accelerating the digitalization, networking and intelligent development of production activities and promoting the transformation and development of the real economy.

Those superiorities lead to such a result that 5G has become a technological highland that global communication equipment giants have seized. Presently, Huawei, ZTE, Qualcomm and Ericsson can be regarded as the four magnates in the field of communication equipment. For those manufacturers who have no experience but want to share the highland, the way to get twice the result with half the effort is to follow a magnate. How to determine the magnate in the selected magnates is the practical decision-making problem that we want to settle in this paper.

For the decision-making problem, one of the most commonly used decision-making tools is the preference relations (PRs) [3-9]. It takes a significant role in the opposite measurement issues [10, 11]. Dual probabilistic linguistic term set (DPLTS) [12], as one of the comparatively new type of decision-making sets, can reveal the decision-making information through the association of the membership part and non-membership part. Then the DPLPRs based on the DPLTSs also have the same features. Moreover, because of various subjective and objective reasons, similar to the absence of other preference information [13-20], the DPLPRs cannot be determined completely and directly. How to repair efficaciously the IDPLPRs is the second problem that we want to settle in the paper.

In this paper, we study multi-criteria decision making with incomplete preference information. Generally, this incomplete information often leads to the inability to determine fully the weight of the criterion. The weights of criteria for multi-criteria decision-making problem that we want to solve are set to unknown. We choose to determine the weight of the criterion based on the complete preference information. Obviously, the elements of DPLPRs are constituted by DPLTSs, while the DPLTSs are constituted by two probabilistic linguistic term sets (PLTSs) [21]: One is the membership, the other is the non-membership. Both are able to be regarded as the stochastic variables because of their structures. By drawing from Ref. [22], firstly, let's ignore the probability part of the PLTSs for the membership. Through the construction of the linear programming model, we can get the complete linguistic membership. Secondly, similar to other methods [13-20] of repairing the incomplete PRs, the consistency [23-29] is the foundational qualification for the PRs to fulfill with. Zhang et al. [30] defined the additive consistency for the probabilistic linguistic preference relations (PLPRs). Through using the feature of additive consistency for PLPRs, the remaining incomplete probability part can be added completely. Then we can get the complete membership. The same method is also applied to the remaining non-membership. Then the incomplete DPLPRs can be repaired completely and the second problem of this paper is solved.

However, because that the repairing of the incomplete PRs under the dual probabilistic linguistic situation is divided into two steps: the linguistic section and the probabilistic section, for the obtained complete DPLPRs, the overall consistency of DPLPRs cannot be guaranteed although all the repairing steps are based on consistency. Hence, in order to avoid unreasonable decision-making results as much as possible, the study of overall consistency for the obtained complete DPLPRs is essential. In this paper, for the sake of making the consistency process conveniently, in the context of thinking about the relationship between the membership and the non-membership, we provide the consistency checking and improving method in light of Ref. [30]. Considering that the PRs are acquired by the paired comparison of decision-making criteria. Consistent PRs can give birth to the weight of the criterion.

Moreover, data envelopment analysis (DEA) is an efficient evaluation method for multiple decision units with the ratio of multiple inputs to multiple outputs. Since it was proposed in 1978 by Charnes et al. [31], it has drawn people's extensive concerns. The reasons can be summarized as three aspects: Firstly, it is visualized and direct to construct the non-parameter model. Secondly, generally, the non-parameter model is simple to deal with or can be converted into the relative sample model to solve. Thirdly, the model can help to improve the alternative or do the efficiency analysis. Moreover, because of those multiple inputs and multiple outputs usually involve in many factors in the decision-making procedure, many scholars who commit to the research for the uncertain decision-making choose it as the research tool. Generally, in practical application, the input and output data in the traditional DEA is specific number. To some extent, this situation may not always be effective in practical applications. That is to say, the situations that inputs and outputs are imprecise often exist. The imprecision in the input/output data can be presented in the form of fuzzy numbers, interval numbers, intuitional fuzzy numbers, hesitant fuzzy numbers, and so on. For instance, the fuzzy data envelopment analysis (FDEA): Pambudi and Nananukul [32] proposed the hierarchical fuzzy DEA approach with multivariate factors to determine the integrated efficiency scores of DMUs from both district and province levels and determine wind power plant sites in Indonesia. The superiority is that the proposed hierarchical fuzzy DEA method is expected to be used for location selection decisions. Rashidi and Cullinane [33] combined the fuzzy DEA method with fuzzy TOPSIS to solve the problem of sustainable supply chain selection. Nastis et al. [34] applied the fuzzy DEA method to agricultural systems and estimation of efficiency scores for a sample of organic farms by incorporating uncertainty in measurement. Liu and Lee [35] proposed a novel method that considers simultaneously all possible weights of all the DMUs to calculate directly the fuzzy cross-efficiency, and the choice of weights is not required. The superiority is that fuzzy cross efficiency in data envelopment analysis has discriminative power in ranking the DMUs when the data are fuzzy numbers. The interval DEA: An et al. [36] combined the DEA and analytic hierarchy process (AHP) to rank fully the DMUs that considers all possible cross efficiencies of a DMU with respect to all the other DMUs. This method of merging not only avoids overestimation of DMUs' efficiency by only self-evaluation, but also eliminates the subjectivity of pairwise comparison between DMUs in AHP. The interval DEA and goal programming model proposed by Torres-Ruiz and Ravindran [37] can be applied to do the dynamic eco-efficiency assessment for sustainable supplier management, as well as supplier evaluation, selection and monitoring. The intuitionistic FDEA: Liu et al. [38] investigated a novel approach for group decision making based on DEA cross-efficiency with intuitionistic fuzzy preference relations, which can avoid information distortion and obtain more credible decision-making results. Arya and Yadav [39] developed intuitionistic fuzzy data envelopment analysis (IFDEA) and dual IFDEA (DIFDEA) models based on α- and β-cuts, and proposed an index ranking approach to rank the DMUs in the application of health sector. Zhou and Xu [40] proposed a novel intuitionistic fuzzy decision-making approach from the perspective of envelopment analysis, which can help make a decision by calculating the intuitionistic fuzzy efficiencies of all the alternatives. Otay et al. [41] used an integrated intuitionistic fuzzy AHP&DEA methodology to do multi-expert performance evaluation of healthcare institutions. Zhou et al. [42] first developed the hesitant fuzzy envelopment analysis (HFEA) model based on the defined score per unit. Then they developed the deviation-oriented hesitant fuzzy envelopment analysis (DHFEA) model and the score-oriented hesitant fuzzy envelopment analysis (SHFEA) model in terms of score and deviation values. Moreover, they constructed the hesitant fuzzy preference envelopment analysis (HFPEA) model by integrating the attributes' preferences. Combined with the HFEA model, the sort value of the scheme and the improvement of the non-optimal ones in the bidding evaluation problem can be obtained by the envelopment values and the obtained parameters, and so on.

The inputs and outputs in these examples [32-42] are deterministic in the decision-making procedure. If either the input or the output is random, how do we solve this problem? In 2010, Azadeh and Alem [43] proposed the stochastic DEA method to deal with the situation that these inputs are deterministic and these outputs are random. In this paper, by considering the feature of the dual probabilistic linguistic term element (DPLTE), we regard the DPLTEs as the stochastic variable, and expand the DEA into the dual probabilistic linguistic environment. Moreover, we assume that these inputs and outputs are all DPLTEs, which means that these inputs and outputs are both stochastic variables. Based on the idea of Ref. [44], we build the model to measure the efficiency. Moreover, because that optimistic and pessimistic efficiencies can reflect the efficiency of each DMU from the most and least favorable situations, respectively. By virtue of Ref. [45], we divide the model into two categories: one is the optimistic situation, and the other is the pessimistic situation.

Similar to majority of stochastic DEAs [46-49], for those two kinds of models, we solve it by studying respective distributions of these inputs and outputs. The difference is that these inputs and outputs in this dual probabilistic linguistic DEA model are two dimensional discrete random variables. In order to obtain the final decision-making consequence, we use the score function of these discrete random variables as the inputs, the accuracy function of these dimensional discrete random variables as the outputs to obtain the ultima decision-making result. The advantage of this method is that the model with the stochastic variable can be converted into the model which does not contain the stochastic variable but the specific value.

Broadly speaking, this paper aims to study the dual probabilistic linguistic multi-criteria decision-making problem with the unknown weight of the criterion and use dual probabilistic linguistic DEA to do the sorting process. The research contributions of the paper can be summarized as follows: (1) Because the DPLPRs can reveal the decision-making information through the association of the membership part and the non-membership part, we choose the DPLPRs to reflect the preference information in the procedure of dealing with the uncertain decision-making problem. (2) In view of various subjective and objective reasons, the DPLPRs can't always be obtained fully. Then, we study the case of incomplete preference information that is the IDPLPRs. (3) Complete preference information is the precondition to make a decision. Therefore, how to repair the incomplete preference information is the third contribution we make in this paper. (4) For the sake of obtaining the relatively meaningful decision-making results, we check and improve the consistency of completed preference information. (5) All work on PRs is to determine the criteria for the multi-criteria decision-making problem under study. Based on those PRs that already satisfy consistency, we can obtain the criteria for the decision issue to be addressed. (6) The final aim of making a decision is to get the final decision-making result by looking for the applicable method. For the features of dual probabilistic linguistic preference information, we construct optimistic and pessimistic dual probabilistic DEA model to make final decision.

The remainder of the paper is formed as follows: Section 2 describes some fundamental notions involving the DPLTSs. Section 3 first defines the DPLPRs and the incomplete DPLPRs. Moreover, it can be separated into two steps: one is to propose repairing methods for incomplete DPLPRs. The other is the discussion of consistency for the obtained complete DPLPRs. Section 4 constructs two dual probabilistic linguistic DEA models to get the final decision-making result. Section 5 applies the proposed decision-making procedure to the precise 5G case, compares and analyzes the diversities. Section 6 concludes the paper with some conclusions.

2. Preliminaries

For this section, we are going to list momently some inevitable notions with reference to the linguistic terms and DPLTSs.

2.1. The linguistic terms

This subsection aims to introduce the linguistic terms and provide some basic operational laws that will be used later.

For a linguistic term set [50] $S=\left\{s_\alpha \middle| \alpha\in[0,2q]\right\}$, q is a positive integer large enough, s_α and s_β are any two linguistic terms on the mentioned set S, then these two linguistic terms fulfill these coming foundational operations:

1) $s_\alpha \oplus s_\beta = \min\left\{s_{\alpha+\beta}, s_{2q}\right\}$;

2) $s_\alpha \Theta s_\beta = \max\left\{s_{\alpha-\beta}, s_0\right\}$;

3) $s_\alpha \otimes s_\beta = \min\left\{s_{\alpha\beta}, s_{2q}\right\}$;

4) $neg\left(s_\alpha\right) = s_{2q-\alpha}$;

5) $\lambda s_\alpha = s_{\lambda\alpha}$, $\lambda \geq 0$.

2.2. The dual probabilistic linguistic term element

The purpose of this subsection is to introduce the DPLTSs, their basic operational laws, the aggregation operator, score function and accuracy function.

The DPLTSs were first proposed by Xie et al. [12], its general formula could be exhibited as follows:

$$D = \left\{\left\langle x, \mathbb{C}(p), \mathbb{Q}(p)\right\rangle, x \in X\right\} \tag{1}$$

where X is a settled reference set, $\mathbb{C}(p) = \left\{\mathbb{C}^{(i)}\left(p^{(i)}\right) \middle| \mathbb{C}^{(i)} \in S, p^{(i)} \geq 0, \sum_{i=1}^{\#\mathbb{C}(p)} p^{(i)} \leq 1\right\}$, $\mathbb{Q}(p) = \left\{\mathbb{Q}^{(j)}\left(p^{(j)}\right) \middle| \mathbb{Q}^{(j)} \in S, p^{(j)} \geq 0, \sum_{j=1}^{\#\mathbb{Q}(p)} p^{(j)} \leq 1\right\}$, $s_0 \leq \mathbb{C}^+ \oplus \mathbb{Q}^+ \leq s_{2q}$, $s_0 \leq \mathbb{C}^- \oplus \mathbb{Q}^- \leq s_{2q}$, $\mathbb{C}^+$ and $\mathbb{C}^-$ are the linguistic terms of the max and min elements [12] of the PLTS $\mathbb{C}(p)$, $\mathbb{Q}^+$ and $\mathbb{Q}^-$ are the linguistic terms of the max and min elements of the PLTS $\mathbb{Q}(p)$, $S = \left\{s_\alpha \middle| \alpha \in [0, 2q]\right\}$, q is a large enough positive integer. For the sake of applying it into the real problem, in the following section, we set $q = 3$. Moreover, Xie et al. [12] named $D = \left\langle \mathbb{C}(p), \mathbb{Q}(p)\right\rangle$ as dual probabilistic linguistic term element (DPLTE).

Additionally, these DPLTEs $D_1 = \left\langle \mathbb{C}_1(p), \mathbb{Q}_1(p)\right\rangle$ and $D_2 = \left\langle \mathbb{C}_2(p), \mathbb{Q}_2(p)\right\rangle$ fulfill these coming foundational operational laws:

$$D_1 \oplus D_2 = \left\langle \mathbb{C}_1(p) \oplus \mathbb{C}_2(p), \mathbb{Q}_1(p) \otimes \mathbb{Q}_2(p)\right\rangle \tag{2}$$

$$\lambda D_1 = \left\langle \lambda\mathbb{C}_1(p), \lambda\mathbb{Q}_1(p)\right\rangle \tag{3}$$

$$D_1 \Theta D_2 = D_1 \oplus D_2^c = \left\langle \mathbb{C}_1(p) \oplus \mathbb{Q}_2(p), \mathbb{Q}_1(p) \otimes \mathbb{C}_2(p)\right\rangle \tag{4}$$

where $\mathbb{C}_1(p) \oplus \mathbb{C}_2(p) = \left\{\mathbb{C}_1^{(i_1)} \oplus \mathbb{C}_2^{(i_2)}\left(p_1^{(i_1)} p_2^{(i_2)}\right)\right\}$, $\mathbb{Q}_1(p) \otimes \mathbb{Q}_2(p) = \left\{\mathbb{Q}_1^{(j_1)} \otimes \mathbb{Q}_2^{(j_2)}\left(p_1^{(j_1)} p_2^{(j_2)}\right)\right\}$, $\lambda\mathbb{C}_1(p) =$

$\left\{\lambda\mathbb{C}_1^{(i)}\left(p_1^{(i)}\right)\right\}$, $\lambda\mathbb{Q}_1(p)=\left\{\lambda\mathbb{Q}_1^{(j)}\left(p_1^{(j)}\right)\right\}$, $D_2^c=\langle\mathbb{Q}_2(p),\mathbb{C}_2(p)\rangle$ and $i_l\,(l=1,2)$ are the $i_l th$ elements in $\mathbb{C}_l(p)$, j_l are the $j_l th$ element in $\mathbb{Q}_l(p)$, separately.

With regard to a series of DPLTEs $D_i=\langle\mathbb{C}_i(p),\mathbb{Q}_i(p)\rangle$, $i=1,2,\ldots,n$, and the respective weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$, the dual probabilistic linguistic weighted aggregation operator (DPLWA) is given as follows:

$$DPLWA(D_1,D_2,\ldots,D_n)=\omega_1 D_1\oplus\omega_1 D_2\oplus\ldots\oplus\omega_n D_n \tag{5}$$

Moreover, we introduce the score function and accuracy function of DPLTEs as follows:

For a DPLTE $D=\langle\mathbb{C}(p),\mathbb{Q}(p)\rangle$, its score function can be expressed as follows:

$$S(D)=s_{\bar{\alpha}-\bar{\beta}} \tag{6}$$

where $\bar{\alpha}=\sum_{i=1}^{\#\mathbb{C}(p)} f\left(\mathbb{C}^{(i)}\right)p^{(i)}/\sum_{i=1}^{\#\mathbb{C}(p)}p^{(i)}$, $\bar{\beta}=\sum_{j=1}^{\#\mathbb{Q}(p)} f\left(\mathbb{Q}^{(j)}\right)p^{(j)}/\sum_{j=1}^{\#\mathbb{Q}(p)}p^{(j)}$, $f(\bullet)$ means the subscript of linguistic term in parenthesis.

For a DPLTE $D=\langle\mathbb{C}(p),\mathbb{Q}(p)\rangle$, its accuracy function can be expressed as follows:

$$\begin{aligned}A(D)=&\left(\sum_{i=1}^{\#\mathbb{C}(p)}\left(p^{(i)}\left(f\left(\mathbb{C}^{(i)}\right)-\bar{\alpha}\right)\right)^2\right)^{1/2}/\sum_{i=1}^{\#\mathbb{C}(p)}p^{(i)}\\&+\left(\sum_{j=1}^{\#\mathbb{Q}(p)}\left(p^{(j)}\left(f\left(\mathbb{Q}^{(j)}\right)-\bar{\beta}\right)\right)^2\right)^{1/2}/\sum_{j=1}^{\#\mathbb{Q}(p)}p^{(j)}\end{aligned} \tag{7}$$

3. The dual probabilistic linguistic preference relations

For this section, we first introduce the DPLPRs, then for the reason that the complex and changeful policy-making circumstance, the complete DPLPRs is hard to earn indeed. So in the remaining section, we define the IDPLPRs and look for the suitable mean to restore the IDLPRs as follows:

3.1. The dual probabilistic linguistic preference relations and incomplete dual probabilistic linguistic preference relations

Provided that the matrix $D=\left(d_{ij}(p)\right)_{n\times n}=\left(\langle\mathbb{C}_{ij}(p),\mathbb{Q}_{ij}(p)\rangle\right)_{n\times n}$ fulfills these coming qualifications:

$$\mathbb{C}_{ij}(p)=\mathbb{Q}_{ji}(p),\quad \mathbb{Q}_{ij}(p)=\mathbb{C}_{ji}(p),\quad \mathbb{C}_{ii}(p)=\mathbb{Q}_{ii}(p)=\{s_q\}$$

Then we call the matrix $D=\left(d_{ij}(p)\right)_{n\times n}$ a DPLPR.

Example 1. On the linguistic term set $S=\{s_\alpha|\alpha\in[0,6]\}$, the DPLPR can be displayed as:

$$D=\begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.1),s_2(0.7)\},\{s_0(0.4),s_1(0.4)\}\rangle & \langle\{s_3(0.3),s_4(0.4)\},\{s_1(0.2),s_2(0.2)\}\rangle & \langle\{s_4(0.3),s_5(0.2)\},\{s_0(0.2),s_1(0.2)\}\rangle \\ \langle\{s_0(0.4),s_1(0.4)\},\{s_1(0.1),s_2(0.7)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_2(0.3),s_3(0.4)\},\{s_1(0.2),s_2(0.2)\}\rangle & \langle\{s_1(0.2),s_2(0.5)\},\{s_0(0.4),s_1(0.6)\}\rangle \\ \langle\{s_1(0.2),s_2(0.2)\},\{s_3(0.3),s_4(0.4)\}\rangle & \langle\{s_1(0.2),s_2(0.2)\},\{s_2(0.3),s_3(0.4)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_3(0.6),s_4(0.4)\},\{s_0(0.7),s_1(0.3)\}\rangle \\ \langle\{s_0(0.2),s_1(0.2)\},\{s_4(0.3),s_5(0.2)\}\rangle & \langle\{s_0(0.4),s_1(0.6)\},\{s_1(0.2),s_2(0.5)\}\rangle & \langle\{s_0(0.7),s_1(0.3)\},\{s_3(0.6),s_4(0.4)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

Moreover, due to some reasons that either the boundedness of the knowledge of the DMs or the intricacies of the factors involved in policy making, these constructed elements in DPLPRs are not always given completely, which leads to the produce of the IDPLPR. They can be shown as follows:

If some of the elements of the matrix $D=\left(d_{ij}(p)\right)_{n\times n}=\left(\left\langle \mathbb{C}_{ij}(p),\mathbb{Q}_{ij}(p)\right\rangle\right)_{n\times n}$ are missing, then the matrix is named as IDPLPR, where $\mathbb{C}_{ij}(p)=\mathbb{Q}_{ji}(p)$, $\mathbb{Q}_{ij}(p)=\mathbb{C}_{ji}(p)$, $s_0\le\mathbb{C}_{ij}(p)\oplus\mathbb{Q}_{ij}(p)\le s_{2q}$, $\mathbb{C}_{ij}(p)=\left\{\mathbb{C}_{ij}^{(l)}\left(p_{ij}^{(l)}\right)\middle|l=1,2,\ldots,\neq\mathbb{C}_{ij}(p)\right\}$, $\mathbb{C}_{ij}^{(l)}$ is the lth linguistic term in $\mathbb{C}_{ij}(p)$, $p_{ij}^{(l)}$ is the possibility of the linguistic term $\mathbb{C}_{ij}^{(l)}$. $\mathbb{Q}_{ij}(p)=\left\{\mathbb{Q}_{ij}^{(\ell)}\left(p_{ij}^{(\ell)}\right)\middle|\ell=1,2,\ldots,\neq\mathbb{Q}_{ij}(p)\right\}$, $\mathbb{Q}_{ij}^{(\ell)}$ is the ℓth linguistic term in $\mathbb{Q}_{ij}(p)$, $p_{ij}^{(\ell)}$ is the possibility of the linguistic term $\mathbb{Q}_{ij}^{(\ell)}$. $d_{ij}(p)\in\Omega_D$, Ω_D is the set of all the known elements. To facilitate the application, in all of the following sections, we set $l=\ell$, which means that the membership part and the non-membership part have the same number of elements.

Furthermore, if each unknown element of the IDPLPR can be acquired by its adjacent known elements, then the IDPLPR is called acceptable, where the adjacent known elements mean that for the two elements d_{ij} and d_{st} in the IDPLPR $D=\left(d_{ij}(p)\right)_{n\times n}$, if $(i,j)\cap(s,t)\neq\varnothing$, the elements d_{ij} and d_{st} are adjacent. For convenience, all the IDPLPRs in the remaining paper are acceptable.

Example 2. On the linguistic term set $S=\{s_\alpha|\alpha\in[0,6]\}$, the IDPLPR can be displayed as:

$$ID=\begin{cases}\langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.1),s_2(0.7)\},\{s_0(0.4),s_1(0.4)\}\rangle \\ \langle\{s_0(0.4),s_1(0.4)\},\{s_1(0.1),s_2(0.7)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \\ d_{31} & \langle\{s_1(0.2),s_2(0.2)\},\{s_2(0.3),s_3(0.4)\}\rangle \\ \langle\{s_0(0.2),s_1(0.2)\},\{s_4(0.3),s_5(0.2)\}\rangle & \langle\{s_0(0.4),s_1(0.6)\},\{s_1(0.2),s_2(0.5)\}\rangle \end{cases}$$

$$\left.\begin{matrix} d_{13} & \langle\{s_4(0.3),s_5(0.2)\},\{s_0(0.2),s_1(0.2)\}\rangle \\ \langle\{s_2(0.3),s_3(0.4)\},\{s_1(0.2),s_2(0.2)\}\rangle & \langle\{s_1(0.2),s_2(0.5)\},\{s_0(0.4),s_1(0.6)\}\rangle \\ \langle\{s_3(1)\},\{s_3(1)\}\rangle & d_{34} \\ d_{43} & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{matrix}\right\}$$

3.2. The repairing for incomplete dual probabilistic linguistic preference relations

From the construction of the DPLPRs, it is clear to know that it is constructed by the membership part and non-membership part. Moreover, whether is the membership part or the non-membership part, generally speaking, there are several elements in the membership part or the non-membership part.

In this paper, we consider the lack of preference information from three perspectives: consider simply the absence of membership part, consider simply the absence of non-membership part and both membership part and non-membership part are missing. Please see the following example for details.

Example 3. On the linguistic term set $S=\{s_\alpha|\alpha\in[0,6]\}$, the IDPLPR can be displayed as:

$$ID=\begin{cases}\langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.1),s_2(0.7)\},\{s_0(0.4),s_1(0.4)\}\rangle \\ \langle\{s_0(0.4),s_1(0.4)\},\{s_1(0.1),s_2(0.7)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \\ x & \langle\{s_1(0.2),s_2(0.2)\},\{s_4(0.3),s_5(0.4)\}\rangle \\ \langle\{s_0(0.2),s_1(0.2)\},x\rangle & \langle\{s_0(0.4),s_1(0.6)\},\{s_1(0.2),s_2(0.5)\}\rangle \end{cases}$$

$$\left.\begin{matrix} x & \langle x,\{s_0(0.2),s_1(0.2)\}\rangle \\ \langle\{s_4(0.3),s_5(0.4)\},\{s_1(0.2),s_2(0.2)\}\rangle & \langle\{s_1(0.2),s_2(0.5)\},\{s_0(0.4),s_1(0.6)\}\rangle \\ \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_3(0.6),s_4(0.4)\},x\rangle \\ \langle x,\{s_3(0.6),s_4(0.4)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{matrix}\right\}$$

Since complete preference information is a prerequisite for effective decision-making. Hence, in the following section, we start to restore the incomplete preference information to the complete preference information.

Through the study of Ref. [22], Ref. [22] can only repair the individual linguistic intuitionistic fuzzy preference relations that the constituent elements contain only one membership and one non-membership. For those constituent elements in the incomplete DPLPRs are several memberships and several non-memberships, the repairing method of Ref. [22] cannot satisfy the demand. Therefore, based on the unique feature of DPLTSs, we make the following improvement in the paper to the method of Ref. [22]. For the first step, we use the coming model to repair the lacking linguistic section:

$$\Phi = \min \sum_{i=1}^{j-1} \sum_{j=i+1}^{n} \left(u_{ij}^{+} + u_{ij}^{-} + v_{ij}^{+} + v_{ij}^{-} \right)$$

$$s.t. \begin{cases} \theta_{ij}^{(l)} \begin{bmatrix} (n-2)\left(\eta_{ij}^{(l)} f\left(\mathbb{C}_{ij}^{(l)}\right) + \left(1-\eta_{ij}^{(l)}\right)\left(2q - f\left(\mathbb{Q}_{ij}^{(l)}\right)\right) + q \right) - \\ \sum_{k=1,k\neq i,j}^{n} \left(\eta_{ik}^{(l)} f\left(\mathbb{C}_{ik}^{(l)}\right) + \left(1-\eta_{ik}^{(l)}\right)\left(2q - f\left(\mathbb{Q}_{ik}^{(l)}\right)\right) \right) \\ + \eta_{kj}^{(l)} f\left(\mathbb{C}_{kj}^{(l)}\right) + \left(1-\eta_{kj}^{(l)}\right)\left(2q - f\left(\mathbb{Q}_{kj}^{(l)}\right)\right) - u_{ij}^{(l)+} + u_{ij}^{(l)-} \end{bmatrix} = 0 \\ \theta_{ij}^{(l)} \begin{bmatrix} (n-2)\left(\eta_{ij}^{(l)} f\left(\mathbb{Q}_{ij}^{(l)}\right) + \left(1-\eta_{ij}^{(l)}\right)\left(2q - f\left(\mathbb{C}_{ij}^{(l)}\right)\right) + q \right) - \\ \sum_{k=1,k\neq i,j}^{n} \left(\eta_{ik}^{(l)} f\left(\mathbb{Q}_{ik}^{(l)}\right) + \left(1-\eta_{ik}^{(l)}\right)\left(2q - f\left(\mathbb{C}_{ik}^{(l)}\right)\right) \right) \\ + \eta_{kj}^{(l)} f\left(\mathbb{Q}_{kj}^{(l)}\right) + \left(1-\eta_{kj}^{(l)}\right)\left(2q - f\left(\mathbb{C}_{kj}^{(l)}\right)\right) - v_{ij}^{(l)+} + v_{ij}^{(l)-} \end{bmatrix} = 0 \\ 0 \le f\left(\mathbb{C}_{ij}^{(l)}\right) \le 2q - f\left(\mathbb{Q}_{ij}^{(l)}\right),\ f\left(\mathbb{C}_{ij}^{(l)}\right) \in \mho_{\mathbb{C}} \wedge f\left(\mathbb{Q}_{ij}^{(l)}\right) \notin \mho_{\mathbb{Q}} \\ 0 \le f\left(\mathbb{Q}_{ij}^{(l)}\right) \le 2q - f\left(\mathbb{C}_{ij}^{(l)}\right),\ f\left(\mathbb{C}_{ij}^{(l)}\right) \notin \mho_{\mathbb{C}} \wedge f\left(\mathbb{Q}_{ij}^{(l)}\right) \in \mho_{\mathbb{Q}} \\ 0 \le f\left(\mathbb{C}_{ij}^{(l)}\right), f\left(\mathbb{Q}_{ij}^{(l)}\right), f\left(\mathbb{C}_{ij}^{(l)}\right) + f\left(\mathbb{Q}_{ij}^{(l)}\right) \le 2q,\ f\left(\mathbb{C}_{ij}^{(l)}\right) \in \mho_{\mathbb{C}} \wedge f\left(\mathbb{Q}_{ij}^{(l)}\right) \in \mho_{\mathbb{Q}} \\ \eta_{ij}^{(l)} = 0 \vee 1, i, j = 1,2,\ldots,n \\ u_{ij}^{(l)+}, u_{ij}^{(l)-}, v_{ij}^{(l)+}, v_{ij}^{(l)-} \ge 0, i, j = 1,2,\ldots,n, i<j \end{cases} \tag{8}$$

where

$$\mho_{\mathbb{C}} = \left\{ f\left(\mathbb{C}_{ij}^{(l)}\right) \middle| \mathbb{C}_{ij}^{(l)} \text{ is missing for all } i, j = 1,2,\ldots,n \text{ with } i<j \right\},$$

$$\mho_{\mathbb{Q}} = \left\{ f\left(\mathbb{Q}_{ij}^{(l)}\right) \middle| \mathbb{Q}_{ij}^{(l)} \text{ is missing for all } i, j = 1,2,\ldots,n \text{ with } i<j \right\}$$

and for each pair (i, j) with $i<j$,

$$\theta_{ij}^{(l)} = \begin{cases} 0, & d_{ik} \text{ and } d_{kj} \text{ are both known for all } k = 1,2,\ldots,n \\ 1, & \text{otherwise} \end{cases}$$

Next step, based on the principle of additive consistency in PLPRs, we construct the following model to repair these lacking probabilities for the incomplete PLPRs $\mathbb{C}' = \left(\mathbb{C}'_{ij}(p)\right)_{n\times n}$:

$$\text{Min}\ \ \varepsilon_{ij}=\frac{1}{n}\sum_{k=1}^{n}\left|S\left(\mathbb{C}'_{ij}\left(p\right)\right)-S\left(\mathbb{C}'_{ik}\left(p\right)\oplus\mathbb{C}'_{kj}\left(p\right)\right)\right|$$

$$\text{s.t.}\ \begin{cases}\sum_{\eta=1}^{\neq\mathbb{C}'_{ij}(p)} p^{(\eta)}=1\\ 0\le p^{(\eta)}\le 1\end{cases} \tag{9}$$

where $\mathbb{C}'_{ij}\left(p\right)=\left\{\mathbb{C}'^{(\eta)}_{ij}\left(p^{(\eta)}\right)\middle|\mathbb{C}'^{(\eta)}_{ij}\in S, p^{(\eta)}\ge 0, \sum_{\eta=1}^{\#\mathbb{C}'_{ij}(p)} p^{(\eta)}\le 1\right\}$. So far, if $\varepsilon_{ij}=0$, then the lacking of membership part for the IDPLPRs can be repaired completely by utilizing Eq. (9). Then we apply Eqs. (8) and (9) to the lacking non-membership part for the IDPLPRs, the IDPLPRs can be repaired completely, the whole repairing procedure is over.

Obviously, the repairing process is divided into two parts: the repairing for linguistic terms and the repairing for corresponding probabilities. Theoretically speaking, those missing elements in the incomplete DPLPRs should be repaired as a whole. But owing to the complexity of the element in the incomplete DPLPRs, it is hard to realize. The DPLTSs evolved from the PLTSs. It is well known that those elements in the PLTSs are obtained by group decision-making rather than given directly. General speaking, the probabilistic section is determined by the linguistic section. From this perspective, the linguistic section and the probabilistic section in the PLTSs do not seem to be a whole. Therefore, the step-by-step repairing process in the paper is reasonable.

3.3. Consistency

After obtaining the complete DPLPRs, for the sake of obtaining the meaningful result, the checking for consistency of DPLPRs is necessary. In the majority of references [13-20], the researches first defined the consistency index as the measure to check the consistency. This method is direct and convenient. Without loss of generality, according the distance measure between the DPLTSs in Ref. [51], the distance measure between the two DPLTEs $D_1=\left\langle\mathbb{C}_1\left(p\right),\mathbb{Q}_1\left(p\right)\right\rangle$ and $D_2=\left\langle\mathbb{C}_2\left(p\right),\mathbb{Q}_2\left(p\right)\right\rangle$ should be as follows:

$$dm\left(D_1,D_2\right)=\left[\frac{1}{2}\left(\frac{1}{\#\mathbb{C}_1(p)}\sum_{l_1=1}^{\#\mathbb{C}_1(p)}\left|\frac{I\left(\mathbb{C}_1^{(l_1)}\right)}{q}\times p_1^{(l_1)}-\frac{I\left(\mathbb{C}_2^{(l_2)}\right)}{q}\times p_2^{(l_2)}\right|^{\lambda}+\right.\right.$$

$$\left.\left.\frac{1}{\#\mathbb{Q}_1(p)}\sum_{\ell_1=1}^{\#\mathbb{Q}_1(p)}\left|\frac{I\left(\mathbb{Q}_1^{(\ell_1)}\right)}{q}\times p_1^{(\ell_1)}-\frac{I\left(\mathbb{Q}_2^{(\ell_2)}\right)}{q}\times p_2^{(\ell_2)}\right|^{\lambda}\right)\right]^{1/\lambda} \tag{10}$$

Moreover, the distance measure between the two DPLPRs $D_1=\left(d_{ij1}(p)\right)_{n\times n}$ and $D_2=\left(d_{ij2}(p)\right)_{n\times n}$ can be defined as follows:

$$dm\left(D_1,D_2\right)=\sqrt{\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n}\left[dm\left(d_{ij1}(p),d_{ij2}(p)\right)\right]^2} \tag{11}$$

As Zhang et al. [30] mentioned, if the PLPRs fulfill the following qualification:

$$\mathbb{C}_{ij}(p)\cong\mathbb{C}_{ik}(p)\oplus\mathbb{C}_{kj}(p) \tag{12}$$

then the PLPRs is consistent.

Remark 1. The symbol "$\cong$" means that the score function [30] of $\mathbb{C}_{ij}(p)$ is the same as $\mathbb{C}_{ik}(p)\oplus\mathbb{C}_{kj}(p)$, where for the PLTS $\mathbb{C}(p)=\left\{\mathbb{C}^{(i)}\left(p^{(i)}\right)\middle|\mathbb{C}^{(i)}\in S,p^{(i)}\geq 0,\sum_{i=1}^{\#\mathbb{C}(p)}p^{(i)}\leq 1\right\}$, its score function is

$$\hat{S}\left(\mathbb{C}(p)\right)=\left(\sum_{i=1}^{\#\mathbb{C}(p)}f\left(\mathbb{C}^{(i)}\right)p^{(i)}\right)\Big/\sum_{i=1}^{\#\mathbb{C}(p)}p^{(i)} \tag{13}$$

and

$$\hat{S}\left(\mathbb{C}_{ij}(p)\right)=\hat{S}\left(\frac{1}{n}\left(\oplus_{k=1}^{n}\left(\mathbb{C}_{ik}(p)\oplus\mathbb{C}_{kj}(p)\right)\right)\right)=\frac{1}{n}\hat{S}\left(\oplus_{k=1}^{n}\left(\mathbb{C}_{ik}(p)\oplus\mathbb{C}_{kj}(p)\right)\right) \tag{14}$$

Based on this, for a DPLPR $D=\left(d_{ij}(p)\right)_{n\times n}$, $d_{ij}(p)=\left\langle\mathbb{C}_{ij}(p),\mathbb{Q}_{ij}(p)\right\rangle$, D is consistent if and only if it fulfills the following qualifications:

$$\begin{cases}\mathbb{C}_{ij}(p)\cong\mathbb{C}_{ik}(p)\oplus\mathbb{C}_{kj}(p)\\ \mathbb{Q}_{ij}(p)\cong\mathbb{Q}_{ik}(p)\oplus\mathbb{Q}_{kj}(p)\end{cases} \tag{15}$$

Therefore, its consistent DPLPR $D'=\left(d'_{ij}(p)\right)_{n\times n}=\left(\left\langle\mathbb{C}'_{ij}(p),\mathbb{Q}'_{ij}(p)\right\rangle\right)_{n\times n}$ can be obtained as follows:

$$\begin{cases} \mathbb{C}'_{ij}(p) = \begin{cases} \frac{1}{n}\left(\oplus_{k=1}^{n}\left(\mathbb{C}_{ik}(p)\oplus\mathbb{C}_{kj}(p)\right)\right), i,j=1,2,\ldots,n & i\neq j \\ \left\{s_q(1)\right\}, & otherwise \end{cases} \\ \mathbb{Q}'_{ij}(p) = \begin{cases} \frac{1}{n}\left(\oplus_{k=1}^{n}\left(\mathbb{Q}_{ik}(p)\oplus\mathbb{Q}_{kj}(p)\right)\right), i,j=1,2,\ldots,n & i\neq j \\ \left\{s_q(1)\right\}, & otherwise \end{cases} \end{cases} \tag{16}$$

For a DPLPR $D=\left(d_{ij}(p)\right)_{n\times n}=\left(\left\langle\mathbb{C}_{ij}(p),\mathbb{Q}_{ij}(p)\right\rangle\right)_{n\times n}$ and its consistent DPLPR $D'=\left(d'_{ij}(p)\right)_{n\times n}$ $=\left(\left\langle\mathbb{C}'_{ij}(p),\mathbb{Q}'_{ij}(p)\right\rangle\right)_{n\times n}$, the consistency index of D can be measured by the following equation:

$$\begin{aligned} CI &= 1-dm\left(D,D'\right) \\ &= 1-\sqrt{\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n}\left[dm\left(d_{ij}(p),d'_{ij}(p)\right)\right]^2} \end{aligned} \tag{17}$$

Obviously, the bigger the consistency index CI, the more consistent the DPLPR D.

Furthermore, given a threshold Δ ahead of time, if the consistency index $CI<\Delta$, which means the DPLPRs with the unacceptable consistency, then the DMs need to modify their DPLPRs for the sake of obtaining the higher level of consistency. Next, we introduce the method to adjust the DPLPRs as follows:

For a DPLPR $D=\left(d_{ij}(p)\right)_{n\times n}=\left(\left\langle\mathbb{C}_{ij}(p),\mathbb{Q}_{ij}(p)\right\rangle\right)_{n\times n}$ with the unacceptable consistency, the modified DPLPR $\breve{D}=\left(\breve{d}_{ij}(p)\right)_{n\times n}=\left(\left\langle\breve{\mathbb{C}}_{ij}(p),\breve{\mathbb{Q}}_{ij}(p)\right\rangle\right)_{n\times n}$ with the acceptable consistency can be acquired as:

$$\begin{cases} \breve{\mathbb{C}}_{ij}(p)=(1-\tau)\mathbb{C}_{ij}(p)\oplus\tau\mathbb{C}'_{ij}(p) \\ \breve{\mathbb{Q}}_{ij}(p)=(1-\tau)\mathbb{Q}_{ij}(p)\oplus\tau\mathbb{Q}'_{ij}(p) \end{cases} \tag{18}$$

where τ is the adjusted parameter and fulfills $\tau\in[0,1]$.

Then the consistency of modified complete DPLPRs can be checked and improved. Moreover, based on Eq. (5) and the principle of information fusion, the weights of criteria can be determined. Next, we are dedicated to doing the sorting process for determining the ranking order of the multi-criteria decision-making problem.

4. The dual probabilistic linguistic efficiency analysis model

As one of the most popular technique, the superiority of DEA is that it can evaluate the set of DMs by measuring the relative efficiency without assuming prior weights on the inputs and outputs. Generally, in practical application, both input data and output data in the traditional DEA are concrete values. To some extent, this situation may not always be effective in practical applications. That is to say, these situations that inputs and outputs are imprecise often occurs. The imprecision in the input/output data can be presented in the form of fuzzy numbers or other fuzzy forms of preference information. In this section, considering the advantage that optimistic and pessimistic efficiencies are able to demonstrate the efficiency of each DMU from the perspective of the most and least favorable, respectively. We first introduce simply these traditional optimistic and pessimistic efficiency models [45]. The DPLTEs can reveal the decision-making information by considering cognitive certainty and cognitive uncertainty. Then we regard DPLTEs as inputs and outputs, and propose these dual probabilistic linguistic optimistic and pessimistic efficiency models to further enrich DEA's ability to deal with decision-making problems. Please see the following content for details.

4.1. The representation of the basic efficiency model

For n DMUs $DMU_j\left(j=1,2,\ldots,n\right)$ that will be measured, $DMU_j\left(j=1,2,\ldots,n\right)$ utilize these m inputs $\chi_{ij}\left(i=1,2,\ldots,m\right)$ to produce t outputs $y_{kj}\left(k=1,2,\ldots,t\right)$, ϑ_{ij} and ϖ_{kj} is the matching weight for the input χ_{ij} and the output y_{kj}, separately, then these optimistic and pessimistic efficiency models [45] can be displayed separately as follows:

$$\text{Max} \quad E_j^O = \frac{\sum_{k=1}^{t} \varpi_{kj} y_{kj}}{\sum_{i=1}^{m} \vartheta_{ij} \chi_{ij}}$$

$$\text{s.t.} \begin{cases} \dfrac{\sum_{k=1}^{t} \varpi_{kj} y_{kj}}{\sum_{i=1}^{m} \vartheta_{ij} \chi_{ij}} \leq 1, \quad \forall j = 1,2,\ldots,n \\ \varpi_{kj} \geq \varepsilon, \quad \forall k = 1,2,\ldots,t \\ \vartheta_{ij} \geq \varepsilon, \quad \forall i = 1,2,\ldots,m \\ \varepsilon > 0 \end{cases} \tag{19}$$

$$\text{Min}\quad E_j^P = \frac{\sum_{k=1}^{t} \varpi_{kj} y_{kj}}{\sum_{i=1}^{m} \vartheta_{ij} \chi_{ij}}$$

$$\text{s.t.}\begin{cases} \dfrac{\sum_{k=1}^{t} \varpi_{kj} y_{kj}}{\sum_{i=1}^{m} \vartheta_{ij} \chi_{ij}} \geq 1, \quad \forall j = 1,2,\ldots,n \\ \varpi_{kj} \geq \varepsilon, \quad \forall k = 1,2,\ldots,t \\ \vartheta_{ij} \geq \varepsilon, \quad \forall i = 1,2,\ldots,m \\ \varepsilon > 0 \end{cases} \tag{20}$$

4.2. Dual probabilistic linguistic optimistic and pessimistic efficiency models

In the policy-making procedure, suppose that inputs and outputs are all the DPLTEs. From the structural feature of the DPLTEs, it is easy to know that it is constructed by several linguistic terms and these matching probabilities. Whatever is Eq. (19) or Eq. (20), owing to its structure of DPLTEs, these two models cannot be utilized directly when these DMUs are DPLTEs. That is to say, it is impossible to apply the DPLTEs to the model (19) or the model (20) as the inputs or outputs. Thus, in this section, we propose new efficiency model that can deal with the dual probabilistic linguistic DMUs. In the view of the principle of information fusion, we want to use the score function as the inputs and the accuracy function as the outputs. On the one hand, whether the score function or the accuracy function, they are the form of straight values. They can be applied directly to the model (19) and the model (20). On the other hand, because of the principle of information fusion, by using the score function instead of the DPLTEs as the inputs and the accuracy function instead of the DPLTEs as the outputs, the data information will not be lost. Please see the following for details.

Assume that the variables $X_{ij}=D_{ij}(p)(i=1,2,\ldots,m)$ and $Y_{kj}=O_{kj}(p)(k=1,2,\ldots,t)$ are these inputs and outputs of the $jth\ DMU_j(j=1,2,\ldots,n)$, where $D_{ij}(p)=\langle \mathbb{C}_{ij}(p),\ \mathbb{Q}_{ij}(p)\rangle$, $O_{kj}(p)=\langle \mathbb{C}_{kj}(p),\ \mathbb{Q}_{kj}(p)\rangle$, ϖ_k and ϑ_i are these matching weights of these variables X_{ij} and Y_{kj}, respectively, then these dual probabilistic linguistic optimistic and pessimistic efficiency models can be displayed separately as follows:

$$\text{Max} \quad SE_{\varepsilon}^{O}=\frac{\sum_{k=1}^{t}\varpi_{k} f\left[S\left(X_{k\varepsilon}\right)\right]}{\sum_{i=1}^{m}\vartheta_{i} A\left(X_{i\varepsilon}\right)}$$

$$\text{s.t.}\begin{cases}\dfrac{\sum_{k=1}^{t}\varpi_{k} f\left[S\left(X_{k\varepsilon}\right)\right]}{\sum_{i=1}^{m}\vartheta_{i} A\left(X_{i\varepsilon}\right)} \leq 1, \quad \forall j=1,2,\ldots,n \\ \varpi_{k} \geq 0, \quad \forall k=1,2,\ldots,t \\ \vartheta_{i} \geq 0, \quad \forall i=1,2,\ldots,m \\ \varepsilon \in\{1,2,\ldots,n\}\end{cases} \tag{21}$$

$$\text{Min} \quad SE_{\varepsilon}^{P}=\frac{\sum_{k=1}^{t}\varpi_{k} f\left[S\left(X_{k\varepsilon}\right)\right]}{\sum_{i=1}^{m}\vartheta_{i} A\left(X_{i\varepsilon}\right)}$$

$$\text{s.t.}\begin{cases}\dfrac{\sum_{k=1}^{t}\varpi_{k} f\left[S\left(X_{k\varepsilon}\right)\right]}{\sum_{i=1}^{m}\vartheta_{i} A\left(X_{i\varepsilon}\right)} \geq 1, \quad \forall j=1,2,\ldots,n \\ \varpi_{k} \geq 0, \quad \forall k=1,2,\ldots,t \\ \vartheta_{i} \geq 0, \quad \forall i=1,2,\ldots,m \\ \varepsilon \in\{1,2,\ldots,n\}\end{cases} \tag{22}$$

where $A(\bullet)$ is the accuracy function of the corresponding dual probabilistic linguistic term in parentheses.

Because these models (21) and (22) are nonlinear programming, it is not simple to solve them. Hence, in terms of solving the model (21), we transform it into the following linear programming through Ref. [42] as follows:

$$\text{Max} \quad SE_{\varepsilon}^{O}=z\sum_{k=1}^{t}\varpi_{k} f\left[S\left(X_{k\varepsilon}\right)\right]=\sum_{k=1}^{t}z\varpi_{k} f\left[S\left(X_{k\varepsilon}\right)\right]=\sum_{k=1}^{t}\xi_{k} f\left[S\left(X_{k\varepsilon}\right)\right]$$

$$\text{s.t.}\begin{cases}\sum_{k=1}^{t}\xi_k f\left[S\left(X_{kj}\right)\right]-\sum_{i=1}^{m}\psi_i A\left(X_{ij}\right)\le 0, & \forall j=1,2,\ldots,n\\ \sum_{i=1}^{m}\psi_i A\left(X_{ij}\right)=1 & \\ \xi_k\ge 0, \quad \forall k=1,2,\ldots,t & \\ \psi_i\ge 0, \quad \forall i=1,2,\ldots,m & \\ \varepsilon\in\{1,2,\ldots,n\} & \end{cases} \tag{23}$$

where $z=\left[\sum_{i=1}^{m}\vartheta_i A\left(X_{ij}\right)\right]^{-1}$, $\xi_k=z\varpi_k$ and $\psi_i=z\vartheta_i$.

Moreover, its dual modality can be expressed as follows:

$$\text{Min } DE_{\varepsilon}^{O}$$

$$\text{s.t.}\begin{cases}\sum_{j=1}^{n}\phi_j A\left(X_{ij}\right)\le DE_{\varepsilon}^{O} A\left(X_{i\varepsilon}\right), & \forall j=1,2,\ldots,n\\ \sum_{j=1}^{n}\phi_j f\left[S\left(X_{kj}\right)\right]\ge f\left[S\left(X_{k\varepsilon}\right)\right] & \\ \phi_j\ge 0, \quad \forall j=1,2,\ldots,n & \\ \varepsilon\in\{1,2,\ldots,n\} & \\ \forall k=1,2,\ldots,m & \\ \forall i=1,2,\ldots,m & \end{cases} \tag{24}$$

For the pessimistic efficiency model, its dual form can be expressed as follows:

$$\text{Max } DE_{\varepsilon}^{P}$$

$$\text{s.t.}\begin{cases}\sum_{j=1}^{n}\phi_j A\left(X_{ij}\right)\ge DE_{\varepsilon}^{P} A\left(X_{i\varepsilon}\right), & \forall j=1,2,\ldots,n\\ \sum_{j=1}^{n}\phi_j f\left[S\left(X_{kj}\right)\right]\le f\left[S\left(X_{k\varepsilon}\right)\right] & \\ \phi_j\ge 0, \quad \forall j=1,2,\ldots,n & \\ \varepsilon\in\{1,2,\ldots,n\} & \\ \forall k=1,2,\ldots,m & \\ \forall i=1,2,\ldots,m & \end{cases} \tag{25}$$

Then the envelopment analysis value [52] $E_{\varepsilon}^{geometric}=\sqrt{DE_{\varepsilon}^{O}\times DE_{\varepsilon}^{P}}$ can be calculated directly, the more the envelopment analysis value, the better the DMUs.

Based on two models (24) and (25), the final sorting process can be realized. The final ranking of alternatives of the multi-criteria decision-making problem can be obtained, too.

4.3. The general procedure to do the multi-criteria decision based on dual probabilistic linguistic term sets

In general, the implementation of the method can be summarized as follows:

Step1. Categorizing the PRs: the complete DPLPRs and the incomplete DPLPRs;

Step 2. Repairing the incomplete DPLPRs: the repairing for linguistic section and the repairing for probabilistic section;

Step 3. Checking and improving the consistency of the complete DPLPRs;

Step 4. Aggregating all the consistent DPLPRs into the group DPLPR and determine the weight vector of the criterion;

Step 5. To fuse all the decision-making information with these weights of criteria and get the group decision-making matrix;

Step 6. Building respectively optimistic and pessimistic efficiency models based on the group decision-making matrix and complete the sorting process.

Moreover, we use a snapshot of the prototype of the implemented method to show the general procedure to do the multi-criteria decision-making for DPLTSs.

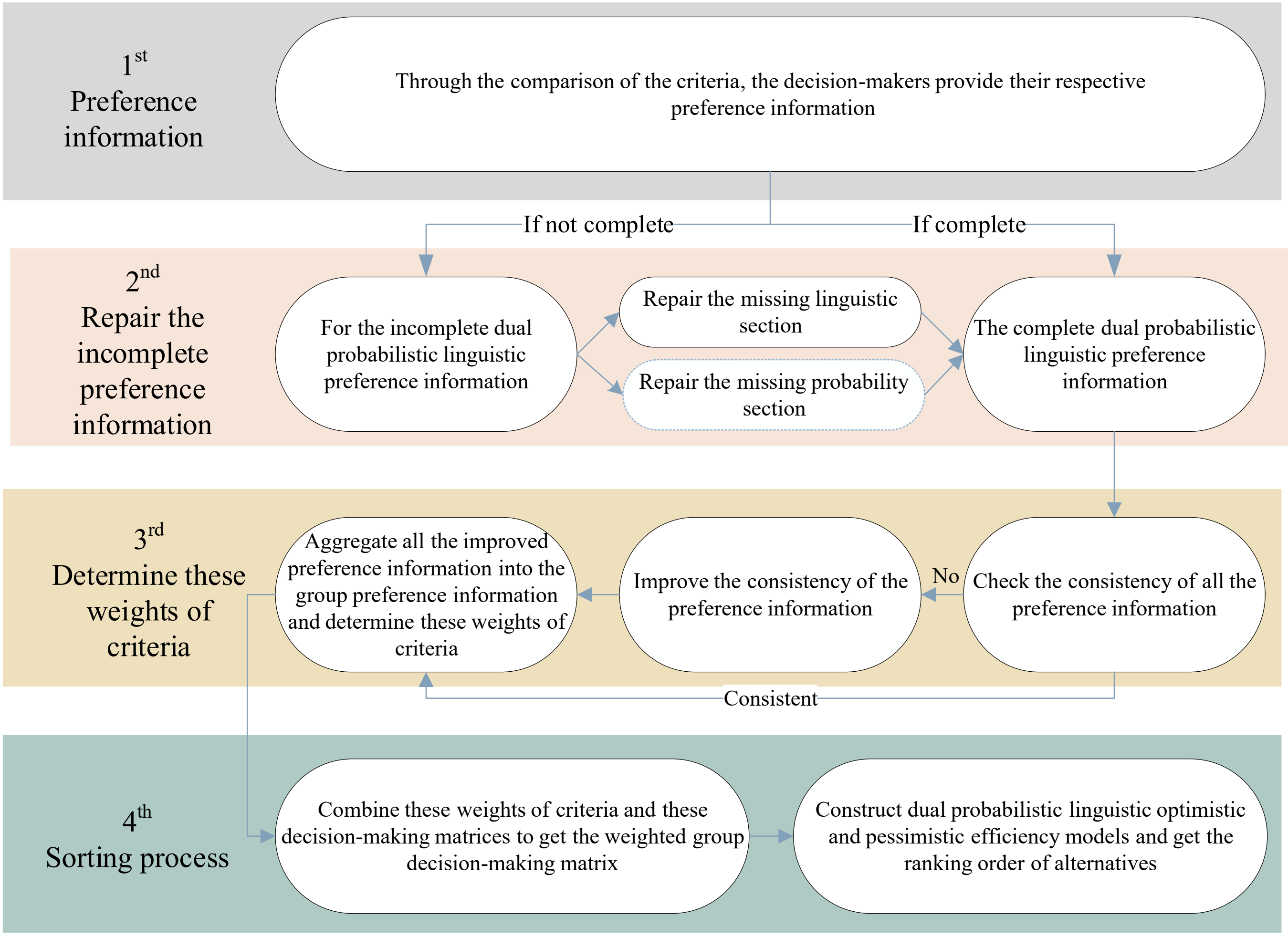

Fig. 1. Procedure of the multi-criteria decision-making for DPLTEs

Besides, we make the following table to discuss and analyze all the relevant work.

Table 1. The discussion and analysis to all relevant work

All relevant work	Advantages	Shortcomings
Divide the PRs into two categories	Clear thinking and clear structure	---
Repair the linguistic section and the probability section in steps	Conform to the principle of generating original preference information	The repair process is relatively complex
Check and improve the consistency	It is the premise to obtain a reasonable decision	---
Determine the weight of the criterion	Relatively objective and reasonable	---
Data envelopment analysis	It can be analyzed from the perspective of optimism and	The data analyzed is processed data, not original

	pessimism	data

5. Simulation test

For this section, we apply the proposed theory and method to the specific case relevant to the 5G and to check the validity of both. Please see these following contents for details.

5.1. The application background

This subsection introduces briefly the development status of 5G industry. Based on this, we propose a multi-criteria decision-making problem.

For more than 30 years, mobile communications [53] have changed fundamentally the world. Since 1980, analog communications have been around. In 1990, second generation (2G) was put into commercial use. At the beginning of this century, 3G was put into commercial use. Until 4G was put into commercial use in 2010, wireless communication technology will make a leap every ten years. What are the personal feelings of these technological leaps? For the 1G, it could only make calls. For the 2G, it can view the .txt text online. For the 3G, it allows to see the .jpg picture online. For the 4G, it can watch .avi video online. By 2020, what kind of revolutionary innovation will the new mobile communication 5G bring to human society?

Compared with 4G, the new mobile communication 5G can bring roughly these three changes [54]: (1) The first is to promote a new round of industrial development. From the moment when 5G appeared, the industry has high hopes for its future commercial use. When 5G is commercialized, it will promote the development of a new round of industry, and will make a great contribution to the long-term sustainable growth of global GDP and become a large-scale economy at the new level. For the industry competition, the emergence of 5G will break the monopoly situation, and the relatively equal competition opportunities will provide opportunities for more innovative enterprises. (2) The second is the "Internet of Everything." The development of each generation of mobile communication technology will bring great changes to the society. 4G technology makes people's connection with merchants and friends more smooth through mobile phones, and makes the “new four inventions (high-speed rail, mobile payment, shared bicycle, online shopping) that are flourishing on the basis of mobile planning is widely known. The industry believes that once the faster 5G technology is ready for commercial use, it will point to the “Internet of Everything”, which will drive the development of applications such as Internet of Vehicles, Internet of Things, drones, and cloud computing. 5G will not only become a new round of development opportunities for the global communications industry, but also bring revolutionary changes to emerging information technologies. (3) The third is the smart life experience. In the 5G era of "Internet of Everything", people will get closer to the exquisite life of their dreams. All members of the Internet of Things will become a point on the terminal network, enabling efficient connectivity and intelligent links for all types of life. Perhaps, when you wake up in the morning, your curtains will automatically open slowly, and the warm sunlight will spill into the room; when you walk into the room, these soft lights will be scattered in every corner of the room after being automatically opened; When you walk into the bathroom and prepare to brush your teeth, the cup has been automatically filled with warm water.... In short, 5G will make the “smart life” of the Internet of Things become a reality. A new technology commercialization will surely drive the corresponding economic output value. In the face of 5G's tempting cake that breaks the output value of 10 trillion US dollars, all related companies are naturally gearing up.

Moreover, compared with the 4G, the 5G has following preponderances [54]: (1) continuous wide area coverage; (2) hotspot high capacity; (3) low power consumption and large connection; (4) low latency and high reliability. Owing to these four preponderances, future 5G applications are mainly concentrated in four scenarios [2] recently: continuous wide-area coverage scenarios such as high-speed rail and subway; hot-spot high-capacity scenarios such as residential areas, office areas, and open-air gatherings; and low-power large-connection scenarios such as smart cities, environmental monitoring, and intelligent agriculture; low latency and high reliability scenarios such as Internet of Vehicles, industrial control, virtual reality, and wearable devices. At present, the world's major manufacturers are focused on these four scenarios to develop 5G technology and product development.

The time point for 5G commercialization in 2020 is getting closer, the industry has started related research and has made many achievements. In order to present these specific research results and progresses of various manufacturers, the "List of 5G Industrial Heroes" was established [55] in the whole world of the communication. In the process of 5G, Huawei stood bravely at the head of the tide and took on the mission of 5G. Huawei has been investing in 5G research and development since 2009. It has been working extensively with industry-leading partners. It has signed more than 30 cooperation memos on 5G and has created multiple records in the field of ultra-large broadband technology, low-latency high-reliability connection technology and ultra-large connection technology. These records far exceed the ITU requirements for 5G networks and are ahead of entire industry. In the 5G test that is in full swing in various countries, ZTE has demonstrated its leading position and refreshed many industry records such as cell throughput, mass connection and low latency in the 5G national test two-stage test. Its high-low frequency commercial test base station and commercial scene test progress have attracted much attention. Qualcomm has been engaged in 5G forward-looking research since many years ago, and has done a lot of work in 5G basic technology, standardization, prototype testing and other aspects. It is worth mentioning that Qualcomm has launched the world's first 5G modulator-demodulator. Since the launch of the 5G project by the International Organization for Standardization (ISO) in 2013, Ericsson has been actively involved in technology standardization and the research has become a global 5G leader. Moreover, Ericsson is committed to building and promoting global 5G cooperation, and has signed cooperation agreements with many first-line operators and repeatedly refreshing the 5G transmission rate record in fast moving state.

On the background of the current 5G industry development, the multi-criteria decision-making problem can be constructed as follows: the four selected alternatives: Ericsson (a_1), ZTE (a_2), Qualcomm (a_3), Huawei (a_4), four DMs (e_1, e_2, e_3 and e_4) are invited to do the assessment. By learning from Ref. [56], these criteria are considered as Corporate value(c_1), Independent research and development ability (c_2), Corporate size (c_3) and Product market share (c_4). The weights of the criteria are set to be unknown.

5.2. Simulation process

This section aims to present the specific implementation procedure of the proposed method. By studying and analyzing the derived characteristics of the data in DPLTEs, all the data of the multi-criteria decision-making problem to be solved is randomly generated by Matlab software. Although the data is simulated in the paper, but the background of application is real. Moreover, the key of this section is to show the application process of the proposed method. Our focus is to do the uncertain decision-making research or the fuzzy decision-making research. However, if the real data can be obtained in the real decision-making process, the proposed method is also applicable. For the real data, all the involved operations and models in the proposed method need to be modified based on the requirement of the real-world data.

Then owing to these limitations of the DMs and these ranges of decision-making issues are enormous, by comparing these alternatives in pairs, these DMs offer three IDPLPRs $ID_i\left(i=1,2,3\right)$ and one complete DPLPR D_4 on the linguistic term set $S=\left\{s_\alpha \middle| \alpha\in\left[0,6\right]\right\}$ as follows:

$$ID_1=\begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.4),s_2(0.4)\},\{s_1(0.2),s_3(0.6)\}\rangle & \langle\ \rangle & \langle\{s_0(0.2),s_1(0.5)\},\{s_0(0.5),s_1(0.2)\}\rangle \\ \langle\{s_1(0.2),s_3(0.6)\},\{s_1(0.4),s_2(0.4)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_3(0.4),s_4(0.2)\},\{s_1(0.4),s_2(0.3)\}\rangle & \langle\{s_0(0.7),s_2(0.2)\},\{s_2(0.2),s_4(0.3)\}\rangle \\ \langle\ \rangle & \langle\{s_1(0.4),s_2(0.3)\},\{s_3(0.4),s_4(0.2)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.7),s_2(0.3)\},\{s_2(0.5),s_3(0.3)\}\rangle \\ \langle\{s_0(0.5),s_1(0.2)\},\{s_0(0.2),s_1(0.5)\}\rangle & \langle\{s_2(0.2),s_4(0.3)\},\{s_0(0.7),s_2(0.2)\}\rangle & \langle\{s_2(0.5),s_3(0.3)\},\{s_1(0.7),s_2(0.3)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

$$ID_2=\begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_3(0.5),s_4(0.2)\},\{s_1(0.3),s_2(0.3)\}\rangle & \langle\{s_0(0.2),s_2(0.7)\},\{s_2(0.2),s_3(0.6)\}\rangle & \langle\{s_2(0.5),s_4(0.2)\},\{s_1(0.3),s_2(0.7)\}\rangle \\ \langle\{s_1(0.3),s_2(0.3)\},\{s_3(0.5),s_4(0.2)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{\ \},\{s_0(0.5),s_1(0.3)\}\rangle & \langle\{s_1(0.3),s_3(0.2)\},\{s_1(0.4),s_3(0.2)\}\rangle \\ \langle\{s_2(0.2),s_3(0.6)\},\{s_0(0.2),s_2(0.7)\}\rangle & \langle\{s_0(0.5),s_1(0.3)\},\{\ \}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.6),s_3(0.4)\},\{s_1(0.4),s_2(0.6)\}\rangle \\ \langle\{s_1(0.3),s_2(0.7)\},\{s_2(0.5),s_4(0.2)\}\rangle & \langle\{s_1(0.4),s_3(0.2)\},\{s_1(0.3),s_3(0.2)\}\rangle & \langle\{s_1(0.4),s_2(0.6)\},\{s_1(0.6),s_3(0.4)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

$$ID_3=\begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.3),s_2(0.1)\},\{s_3(0.1),s_4(0.4)\}\rangle & \langle\{s_4(0.2),s_5(0.2)\},\{s_0(0.4),s_1(0.6)\}\rangle & \langle\{s_3(0.2),s_4(0.6)\},\{s_1(0.5),s_2(0.2)\}\rangle \\ \langle\{s_3(0.1),s_4(0.4)\},\{s_1(0.3),s_2(0.1)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_0(0.5),s_1(0.4)\},\{s_2(0.2),s_4(0.4)\}\rangle & \langle\{s_3(0.4),s_4(0.4)\},\{\ \}\rangle \\ \langle\{s_0(0.4),s_1(0.6)\},\{s_4(0.2),s_5(0.2)\}\rangle & \langle\{s_2(0.2),s_4(0.4)\},\{s_0(0.5),s_1(0.4)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_0(0.6),s_1(0.4)\},\{s_4(0.4),s_5(0.6)\}\rangle \\ \langle\{s_1(0.5),s_2(0.2)\},\{s_3(0.2),s_4(0.6)\}\rangle & \langle\{\ \},\{s_3(0.4),s_4(0.4)\}\rangle & \langle\{s_4(0.4),s_5(0.6)\},\{s_0(0.6),s_1(0.4)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

$$D_4=\begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.2),s_2(0.6)\},\{s_3(0.4),s_4(0.6)\}\rangle & \langle\{s_0(0.4),s_1(0.5)\},\{s_4(0.3),s_5(0.2)\}\rangle & \langle\{s_3(0.3),s_4(0.5)\},\{s_0(0.1),s_1(0.8)\}\rangle \\ \langle\{s_3(0.4),s_4(0.6)\},\{s_1(0.2),s_2(0.6)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_0(0.5),s_2(0.3)\},\{s_2(0.5),s_3(0.5)\}\rangle & \langle\{s_0(0.3),s_1(0.4)\},\{s_3(0.1),s_4(0.8)\}\rangle \\ \langle\{s_4(0.3),s_5(0.2)\},\{s_0(0.4),s_1(0.5)\}\rangle & \langle\{s_2(0.5),s_3(0.5)\},\{s_0(0.5),s_2(0.3)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_3(0.4),s_5(0.5)\},\{s_0(0.6),s_1(0.4)\}\rangle \\ \langle\{s_0(0.1),s_1(0.8)\},\{s_3(0.3),s_4(0.5)\}\rangle & \langle\{s_3(0.1),s_4(0.8)\},\{s_0(0.3),s_1(0.4)\}\rangle & \langle\{s_0(0.6),s_1(0.4)\},\{s_3(0.4),s_5(0.5)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

Then by Eqs. (8) and (9), three incomplete DPLPRs $D_i\left(i=1,2,3\right)$ can be repaired as follows:

$$D_1=\begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.4),s_2(0.4)\},\{s_1(0.2),s_3(0.6)\}\rangle & \langle\{s_6(0.5),s_2(0.5)\},\{s_0(0.1),s_1(0.9)\}\rangle & \langle\{s_0(0.2),s_1(0.5)\},\{s_0(0.5),s_1(0.2)\}\rangle \\ \langle\{s_1(0.2),s_3(0.6)\},\{s_1(0.4),s_2(0.4)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_3(0.4),s_4(0.2)\},\{s_1(0.4),s_2(0.3)\}\rangle & \langle\{s_0(0.7),s_2(0.2)\},\{s_2(0.2),s_4(0.3)\}\rangle \\ \langle\{s_0(0.1),s_1(0.9)\},\{s_6(0.5),s_2(0.5)\}\rangle & \langle\{s_1(0.4),s_2(0.3)\},\{s_3(0.4),s_4(0.2)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.7),s_2(0.3)\},\{s_2(0.5),s_3(0.3)\}\rangle \\ \langle\{s_0(0.5),s_1(0.2)\},\{s_0(0.2),s_1(0.5)\}\rangle & \langle\{s_2(0.2),s_4(0.3)\},\{s_0(0.7),s_2(0.2)\}\rangle & \langle\{s_2(0.5),s_3(0.3)\},\{s_1(0.7),s_2(0.3)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

$$D_2=\begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_3(0.5),s_4(0.2)\},\{s_1(0.3),s_2(0.3)\}\rangle & \langle\{s_0(0.2),s_2(0.7)\},\{s_2(0.2),s_3(0.6)\}\rangle & \langle\{s_2(0.5),s_4(0.2)\},\{s_1(0.3),s_2(0.7)\}\rangle \\ \langle\{s_1(0.3),s_2(0.3)\},\{s_3(0.5),s_4(0.2)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{3.2027}(0.9),s_{1.5}(0.1)\},\{s_0(0.5),s_1(0.3)\}\rangle & \langle\{s_1(0.3),s_3(0.2)\},\{s_1(0.4),s_3(0.2)\}\rangle \\ \langle\{s_2(0.2),s_3(0.6)\},\{s_0(0.2),s_2(0.7)\}\rangle & \langle\{s_0(0.5),s_1(0.3)\},\{s_{3.2027}(0.9),s_{1.5}(0.1)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.6),s_3(0.4)\},\{s_1(0.4),s_2(0.6)\}\rangle \\ \langle\{s_1(0.3),s_2(0.7)\},\{s_2(0.5),s_4(0.2)\}\rangle & \langle\{s_1(0.4),s_3(0.2)\},\{s_1(0.3),s_3(0.2)\}\rangle & \langle\{s_1(0.4),s_2(0.6)\},\{s_1(0.6),s_3(0.4)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

$$D_3=\begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_1(0.3),s_2(0.1)\},\{s_3(0.1),s_4(0.4)\}\rangle & \langle\{s_4(0.2),s_5(0.2)\},\{s_0(0.4),s_1(0.6)\}\rangle & \langle\{s_3(0.2),s_4(0.6)\},\{s_1(0.5),s_2(0.2)\}\rangle \\ \langle\{s_3(0.1),s_4(0.4)\},\{s_1(0.3),s_2(0.1)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_0(0.5),s_1(0.4)\},\{s_2(0.2),s_4(0.4)\}\rangle & \langle\{s_3(0.4),s_4(0.4)\},\{s_1(0.1),s_2(0.9)\}\rangle \\ \langle\{s_0(0.4),s_1(0.6)\},\{s_4(0.2),s_5(0.2)\}\rangle & \langle\{s_2(0.2),s_4(0.4)\},\{s_0(0.5),s_1(0.4)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_0(0.6),s_1(0.4)\},\{s_4(0.4),s_5(0.6)\}\rangle \\ \langle\{s_1(0.5),s_2(0.2)\},\{s_3(0.2),s_4(0.6)\}\rangle & \langle\{s_1(0.1),s_2(0.9)\},\{s_3(0.4),s_4(0.4)\}\rangle & \langle\{s_4(0.4),s_5(0.6)\},\{s_0(0.6),s_1(0.4)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

After that, based on Eq. (15), we calculate the corresponding consistent DPLPRs $D_i'\left(i=1,2,3,4\right)$ as follows:

$$D_1'=\begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(0.0013),\ldots,s_{1.5}(0.0036)\},\{s_{1.5}(5.602\text{e-}04),\ldots,s_{1.5}(0.0026)\}\rangle & \langle\{s_{1.5}(0.004),\ldots,s_{1.5}(0.003)\},\{s_{1.5}(2.801\text{e-}04),\ldots,s_{1.5}(0.0087)\}\rangle & \langle\{s_{1.5}(0.0039),\ldots,s_{1.5}(0.003)\},\{s_{1.5}(5.01\text{e-}04),\ldots,s_{1.5}(0.0019)\}\rangle \\ \langle\{s_{1.5}(5.602\text{e-}04),\ldots,s_{1.5}(0.0026)\},\{s_{1.5}(0.0013),\ldots,s_{1.5}(0.0036)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(0.0056),\ldots,s_{1.5}(0.0007)\},\{s_{1.5}(0.0009),\ldots,s_{1.5}(0.0029)\}\rangle & \langle\{s_{1.5}(0.0055),\ldots,s_{1.5}(0.0007)\},\{s_{1.5}(0.0016),\ldots,s_{1.5}(0.0006)\}\rangle \\ \langle\{s_{1.5}(2.801\text{e-}04),\ldots,s_{1.5}(0.0087)\},\{s_{1.5}(0.004),\ldots,s_{1.5}(0.003)\}\rangle & \langle\{s_{1.5}(0.0009),\ldots,s_{1.5}(0.0029)\},\{s_{1.5}(0.0056),\ldots,s_{1.5}(0.0007)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(0.0027),\ldots,s_{1.5}(0.0024)\},\{s_{1.5}(0.005),\ldots,s_{1.5}(0.0005)\}\rangle \\ \langle\{s_{1.5}(5.01\text{e-}04),\ldots,s_{1.5}(0.0019)\},\{s_{1.5}(0.0039),\ldots,s_{1.5}(0.003)\}\rangle & \langle\{s_{1.5}(0.0016),\ldots,s_{1.5}(0.0006)\},\{s_{1.5}(0.0055),\ldots,s_{1.5}(0.0007)\}\rangle & \langle\{s_{1.5}(0.005),\ldots,s_{1.5}(0.0005)\},\{s_{1.5}(0.0027),\ldots,s_{1.5}(0.0024)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

Remark 2. All the sizes of these consistent DPLPRs are very large. For the sake of simplicity, we present only a portion of the data to show preference information. Here just take DPLPR D_1' as an example, please refer to the appendix for additional preference information. Similar to the consistent DPLPR D_1', the following modified DPLPRs $\breve{D}_i\left(i=1,2,3,4\right)$ have been treated with the same way.

Then we check the consistency of the four complete DPLPRs $D_i\left(i=1,2,3,4\right)$ based on Eq. (16) as follows:

Table 2. The consistency indices CI of the DPLPRs $D_i\left(i=1,2,3,4\right)$

DPLPRs	D_1	D_2	D_3	D_4
CI	0.8924	0.8886	0.8808	0.8800

Let the threshold value be $\Delta=0.9$, then we judge the relationship between the threshold value and the consistency index. According to Table 2, for all the DPLPRs $D_i\left(i=1,2,3,4\right)$, $CI_i<\Delta\left(i=1,2,3,4\right)$. Therefore, we need to adjust all the DPLPRs $D_i\left(i=1,2,3,4\right)$. Here we set the adjusted parameter $\tau=0.5$, based on Eq. (17), we obtain the modified DPLPRs with the acceptable consistency as follows:

$$\breve{D}_1=\begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.25}(0.0005),\ldots,s_{1.75}(0.0014)\},\{s_{1.25}(0.0001),\ldots,s_{2.25}(0.0016)\}\rangle & \langle\{s_{3.75}(0.002),\ldots,s_{1.75}(0.0015)\},\{s_{0.75}(2.80\text{e-}05),\ldots,s_{1.25}(0.0079)\}\rangle & \langle\{s_{0.75}(0.0008),\ldots,s_{1.25}(0.0015)\},\{s_{0.75}(0.0003),\ldots,s_{1.25}(0.0004)\}\rangle \\ \langle\{s_{1.25}(0.0001),\ldots,s_{2.25}(0.0016)\},\{s_{1.25}(0.0005),\ldots,s_{1.75}(0.0014)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{2.25}(0.0022),\ldots,s_{2.75}(0.0001)\},\{s_{1.25}(0.0004),\ldots,s_{1.75}(0.0009)\}\rangle & \langle\{s_{0.75}(0.0038),\ldots,s_{1.75}(0.0001)\},\{s_{1.75}(0.0003),\ldots,s_{2.75}(0.0002)\}\rangle \\ \langle\{s_{0.75}(2.80\text{e-}05),\ldots,s_{1.25}(0.0079)\},\{s_{3.75}(0.002),\ldots,s_{1.75}(0.0015)\}\rangle & \langle\{s_{1.25}(0.0004),\ldots,s_{1.75}(0.0009)\},\{s_{2.25}(0.0022),\ldots,s_{2.75}(0.0001)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.25}(0.0019),\ldots,s_{1.75}(0.0007)\},\{s_{1.75}(0.0025),\ldots,s_{2.25}(0.0002)\}\rangle \\ \langle\{s_{0.75}(0.0003),\ldots,s_{1.25}(0.0004)\},\{s_{0.75}(0.0008),\ldots,s_{1.25}(0.0015)\}\rangle & \langle\{s_{1.75}(0.0003),\ldots,s_{2.75}(0.0002)\},\{s_{0.75}(0.0038),\ldots,s_{1.75}(0.0001)\}\rangle & \langle\{s_{1.75}(0.0025),\ldots,s_{2.25}(0.0002)\},\{s_{1.25}(0.0019),\ldots,s_{1.75}(0.0007)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

Then, we calculate the consistency index of the modified DPLPRs based on Eq. (17) as the following table:

Table 3. The consistency index CI of these modified DPLPRs $\breve{D}_i\left(i=1,2,3,4\right)$

Modified DPLPRs	$\breve{D}_1$	$\breve{D}_2$	$\breve{D}_3$	$\breve{D}_4$
CI	0.9994	0.9993	0.9995	0.9991

Suppose that these weights of the DMs are $\omega=\left(0.15,0.2,0.3,0.35\right)$, based on Eq. (5), we integrate four modified DPLPRs $\breve{D}_i\left(i=1,2,3,4\right)$ to the group DPLPR as:

$$D=\begin{pmatrix}
\langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.45}(1.3271\text{e-}15),\ldots,s_{1.95}(4.7029\text{e-}14)\},\{s_{0.0249}(5.6435\text{e-}16),\ldots,s_{0.0938}(7.4895\text{e-}14)\}\rangle & \langle\{s_{1.8}(1.5925\text{e-}13),\ldots,s_{2.025}(2.1947\text{e-}12)\},\{s_{0.0085}(5.0165\text{e-}16),\ldots,s_{0.0360}(8.4257\text{e-}12)\}\rangle & \langle\{s_{1.925}(5.2673\text{e-}14),\ldots,s_{2.525}(3.5832\text{e-}14)\},\{s_{0.0028}(9.3312\text{e-}17),\ldots,s_{0.0151}(2.4575\text{e-}12)\}\rangle \\
\langle\{s_{0.0249}(5.6435\text{e-}16),\ldots,s_{0.0938}(7.4895\text{e-}14)\},\{s_{1.45}(1.3271\text{e-}15),\ldots,s_{1.95}(4.7029\text{e-}14)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.2953}(1.6931\text{e-}11),\ldots,s_{1.70}(8.8061\text{e-}16)\},\{s_{0.009}(1.8579\text{e-}14),\ldots,s_{0.0426}(6.9657\text{e-}14)\}\rangle & \langle\{s_{1.3}(2.0906\text{e-}12),\ldots,s_{1.975}(3.9138\text{e-}14)\},\{s_{0.0194}(4.4237\text{e-}17),\ldots,s_{0.0938}(3.1346\text{e-}12)\}\rangle \\
\langle\{s_{0.0085}(5.0165\text{e-}16),\ldots,s_{0.0360}(8.4257\text{e-}12)\},\{s_{1.8}(1.5925\text{e-}13),\ldots,s_{2.025}(2.1947\text{e-}12)\}\rangle & \langle\{s_{0.009}(1.8579\text{e-}14),\ldots,s_{0.0426}(6.9657\text{e-}14)\},\{s_{1.2953}(1.6931\text{e-}11),\ldots,s_{1.70}(8.8061\text{e-}16)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.45}(7.4332\text{e-}12),\ldots,s_{2.225}(3.7150\text{e-}13)\},\{s_{0.0142}(4.7776\text{e-}13),\ldots,s_{0.0504}(5.0153\text{e-}13)\}\rangle \\
\langle\{s_{0.0028}(9.3312\text{e-}17),\ldots,s_{0.0151}(2.4575\text{e-}12)\},\{s_{1.925}(5.2673\text{e-}14),\ldots,s_{2.525}(3.5832\text{e-}14)\}\rangle & \langle\{s_{0.0194}(4.4237\text{e-}17),\ldots,s_{0.0938}(3.1346\text{e-}12)\},\{s_{1.3}(2.0906\text{e-}12),\ldots,s_{1.975}(3.9138\text{e-}14)\}\rangle & \langle\{s_{0.0142}(4.7776\text{e-}13),\ldots,s_{0.0504}(5.0153\text{e-}13)\},\{s_{1.45}(7.4332\text{e-}12),\ldots,s_{2.225}(3.7150\text{e-}13)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle
\end{pmatrix}$$

Then by using the following equation [57], the weights of criteria are able to be figured up:

$$\hat{\omega}=\sum_{j=1}^{n}A\left(D_{ij}\right)\Big/\sum_{i=i}^{n}\sum_{j=1}^{n}A\left(D_{ij}\right)$$

$$\hat{\omega}=\left(0.1857,0.3139,0.2564,0.2440\right)$$

Moreover, these decision-making matrices offered by the DMs for the alternatives $a_i\left(i=1,2,3,4\right)$ with respect to the corresponding criteria $c_j\left(j=1,2,3,4\right)$ are shown as follows:

$$M_1=\begin{array}{cc} & \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{array}{c} a_1\\ a_2\\ a_3\\ a_4 \end{array} & \begin{pmatrix}
\langle\{s_0(0.2),s_2(0.7)\},\{s_2(0.2),s_3(0.6)\}\rangle & \langle\{s_2(0.5),s_4(0.2)\},\{s_1(0.3),s_2(0.7)\}\rangle & \langle\{s_1(0.8),s_2(0.1)\},\{s_0(0.5),s_1(0.3)\}\rangle & \langle\{s_1(0.3),s_3(0.2)\},\{s_1(0.4),s_3(0.2)\}\rangle \\
\langle\{s_1(0.6),s_3(0.4)\},\{s_1(0.4),s_2(0.6)\}\rangle & \langle\{s_1(0.5),s_3(0.5)\},\{s_1(0.7),s_3(0.2)\}\rangle & \langle\{s_3(0.2),s_4(0.2)\},\{s_1(0.1),s_2(0.4)\}\rangle & \langle\{s_2(0.1),s_3(0.9)\},\{s_1(0.5),s_2(0.4)\}\rangle \\
\langle\{s_1(0.6),s_3(0.3)\},\{s_2(0.3),s_3(0.3)\}\rangle & \langle\{s_1(0.6),s_3(0.1)\},\{s_0(0.5),s_2(0.4)\}\rangle & \langle\{s_1(0.6),s_2(0.1)\},\{s_2(0.8),s_4(0.1)\}\rangle & \langle\{s_1(0.2),s_2(0.6)\},\{s_3(0.4),s_4(0.6)\}\rangle \\
\langle\{s_0(0.4),s_1(0.5)\},\{s_4(0.3),s_5(0.2)\}\rangle & \langle\{s_3(0.3),s_4(0.5)\},\{s_0(0.1),s_1(0.8)\}\rangle & \langle\{s_0(0.5),s_2(0.3)\},\{s_2(0.5),s_3(0.5)\}\rangle & \langle\{s_0(0.3),s_1(0.4)\},\{s_3(0.1),s_4(0.8)\}\rangle
\end{pmatrix}\end{array}$$

$$M_2=\begin{array}{cc} & \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{array}{c} a_1\\ a_2\\ a_3\\ a_4 \end{array} & \begin{pmatrix}
\langle\{s_3(0.4),s_5(0.5)\},\{s_0(0.6),s_1(0.4)\}\rangle & \langle\{s_1(0.1),s_2(0.5)\},\{s_1(0.8),s_2(0.2)\}\rangle & \langle\{s_1(0.6),s_3(0.3)\},\{s_0(0.5),s_1(0.1)\}\rangle & \langle\{s_1(0.5),s_2(0.2)\},\{s_0(0.3),s_2(0.1)\}\rangle \\
\langle\{s_0(0.6),s_1(0.2)\},\{s_2(0.7),s_4(0.3)\}\rangle & \langle\{s_1(0.7),s_3(0.2)\},\{s_1(0.5),s_2(0.2)\}\rangle & \langle\{s_1(0.2),s_2(0.7)\},\{s_2(0.4),s_4(0.6)\}\rangle & \langle\{s_1(0.3),s_2(0.1)\},\{s_3(0.1),s_4(0.4)\}\rangle \\
\langle\{s_4(0.2),s_5(0.2)\},\{s_0(0.4),s_1(0.6)\}\rangle & \langle\{s_3(0.2),s_4(0.6)\},\{s_1(0.5),s_2(0.2)\}\rangle & \langle\{s_0(0.5),s_1(0.4)\},\{s_2(0.2),s_4(0.4)\}\rangle & \langle\{s_3(0.4),s_4(0.4)\},\{s_0(0.3),s_1(0.4)\}\rangle \\
\langle\{s_0(0.6),s_1(0.4)\},\{s_4(0.4),s_5(0.6)\}\rangle & \langle\{s_4(0.2),s_5(0.3)\},\{s_0(0.2),s_1(0.6)\}\rangle & \langle\{s_1(0.6),s_2(0.2)\},\{s_0(0.5),s_2(0.4)\}\rangle & \langle\{s_2(0.7),s_4(0.1)\},\{s_1(0.3),s_2(0.3)\}\rangle
\end{pmatrix}\end{array}$$

$$M_3=\begin{array}{cc} & \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{array}{c} a_1\\ a_2\\ a_3\\ a_4 \end{array} & \begin{pmatrix}
\langle\{s_1(0.4),s_2(0.2)\},\{s_0(0.8),s_2(0.1)\}\rangle & \langle\{s_1(0.4),s_2(0.4)\},\{s_3(0.2),s_4(0.1)\}\rangle & \langle\{s_0(0.1),s_1(0.4)\},\{s_4(0.2),s_5(0.8)\}\rangle & \langle\{s_1(0.2),s_2(0.4)\},\{s_1(0.3),s_3(0.7)\}\rangle \\
\langle\{s_0(0.1),s_2(0.3)\},\{s_0(0.2),s_1(0.3)\}\rangle & \langle\{s_0(0.1),s_1(0.6)\},\{s_1(0.7),s_3(0.3)\}\rangle & \langle\{s_3(0.2),s_4(0.4)\},\{s_1(0.4),s_2(0.5)\}\rangle & \langle\{s_0(0.5),s_1(0.3)\},\{s_3(0.2),s_5(0.5)\}\rangle \\
\langle\{s_1(0.2),s_2(0.3)\},\{s_3(0.7),s_4(0.3)\}\rangle & \langle\{s_2(0.3),s_4(0.4)\},\{s_1(0.4),s_2(0.4)\}\rangle & \langle\{s_3(0.4),s_4(0.5)\},\{s_1(0.7),s_2(0.3)\}\rangle & \langle\{s_2(0.4),s_3(0.6)\},\{s_1(0.3),s_2(0.3)\}\rangle \\
\langle\{s_4(0.5),s_5(0.3)\},\{s_0(0.2),s_1(0.5)\}\rangle & \langle\{s_1(0.4),s_2(0.3)\},\{s_1(0.8),s_3(0.1)\}\rangle & \langle\{s_3(0.1),s_4(0.8)\},\{s_0(0.7),s_2(0.2)\}\rangle & \langle\{s_2(0.2),s_4(0.3)\},\{s_1(0.7),s_2(0.3)\}\rangle
\end{pmatrix}\end{array}$$

$$M_4=\begin{array}{cc} & \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{array}{c} a_1\\ a_2\\ a_3\\ a_4 \end{array} & \begin{pmatrix}
\langle\{s_2(0.5),s_3(0.3)\},\{s_2(0.3),s_3(0.5)\}\rangle & \langle\{s_2(0.3),s_4(0.2)\},\{s_1(0.1),s_2(0.6)\}\rangle & \langle\{s_1(0.5),s_3(0.4)\},\{s_1(0.4),s_3(0.2)\}\rangle & \langle\{s_1(0.8),s_2(0.2)\},\{s_3(0.1),s_4(0.5)\}\rangle \\
\langle\{s_1(0.9),s_3(0.1)\},\{s_0(0.7),s_1(0.2)\}\rangle & \langle\{s_1(0.4),s_2(0.4)\},\{s_2(0.2),s_4(0.4)\}\rangle & \langle\{s_3(0.4),s_4(0.5)\},\{s_1(0.3),s_2(0.3)\}\rangle & \langle\{s_1(0.1),s_2(0.4)\},\{s_2(0.4),s_3(0.2)\}\rangle \\
\langle\{s_1(0.1),s_3(0.6)\},\{s_0(0.8),s_1(0.2)\}\rangle & \langle\{s_0(0.5),s_1(0.3)\},\{s_1(0.7),s_2(0.3)\}\rangle & \langle\{s_0(0.5),s_1(0.4)\},\{s_1(0.2),s_3(0.8)\}\rangle & \langle\{s_2(0.2),s_3(0.2)\},\{s_0(0.2),s_2(0.5)\}\rangle \\
\langle\{s_2(0.5),s_4(0.2)\},\{s_0(0.4),s_1(0.5)\}\rangle & \langle\{s_0(0.1),s_1(0.5)\},\{s_2(0.6),s_4(0.3)\}\rangle & \langle\{s_1(0.6),s_3(0.2)\},\{s_2(0.3),s_3(0.6)\}\rangle & \langle\{s_0(0.4),s_2(0.4)\},\{s_2(0.8),s_3(0.1)\}\rangle
\end{pmatrix}\end{array}$$

Together with the weights of criteria $\hat{\omega}$, the weighted group dual probabilistic linguistic decision-making matrix can be computed as:

$$M=\left(\begin{array}{llll}\left\langle\left\{s_{0.297}(0.016),\ldots,s_{0.5477}(0.021)\right\},\left\{s_0(0.0288),\ldots,s_{0.0105}(0.012)\right\}\right\rangle & \left\langle\left\{s_{0.4708}(0.006),\ldots,s_{0.9416}(0.008)\right\},\left\{s_{0.003}(0.0048),\ldots,s_{0.0316}(0.0084)\right\}\right\rangle & \left\langle\left\{s_{0.1795}(0.024),\ldots,s_{0.577}(0.0048)\right\},\left\{s_0(0.02),\ldots,s_{0.0121}(0.0048)\right\}\right\rangle & \left\langle\left\{s_{0.244}(0.024),\ldots,s_{0.5247}(0.0032)\right\},\left\{s_0(0.0036),\ldots,s_{0.0553}(0.007)\right\}\right\rangle \\ \left\langle\left\{s_{0.0928}(0.0324),\ldots,s_{0.427}(0.0024)\right\},\left\{s_0(0.0392),\ldots,s_{0.0047}(0.0108)\right\}\right\rangle & \left\langle\left\{s_{0.2197}(0.014),\ldots,s_{0.6435}(0.024)\right\},\left\{s_{0.002}(0.049),\ldots,s_{0.0712}(0.0048)\right\}\right\rangle & \left\langle\left\{s_{0.6667}(0.0032),\ldots,s_{0.9231}(0.028)\right\},\left\{s_{0.0016}(0.0048),\ldots,s_{0.0258}(0.036)\right\}\right\rangle & \left\langle\left\{s_{0.2074}(0.0015),\ldots,s_{0.4515}(0.0108)\right\},\left\{s_{0.0138}(0.004),\ldots,s_{0.0922}(0.016)\right\}\right\rangle \\ \left\langle\left\{s_{0.297}(0.0024),\ldots,s_{0.5755}(0.0108)\right\},\left\{s_0(0.0672),\ldots,s_{0.007}(0.0108)\right\}\right\rangle & \left\langle\left\{s_{0.4237}(0.018),\ldots,s_{0.8789}(0.0072)\right\},\left\{s_0(0.07),\ldots,s_{0.0158}(0.0096)\right\}\right\rangle & \left\langle\left\{s_{0.2692}(0.06),\ldots,s_{0.5257}(0.008)\right\},\left\{s_{0.0032}(0.0224),\ldots,s_{0.0775}(0.0096)\right\}\right\rangle & \left\langle\left\{s_{0.5003}(0.0064),\ldots,s_{0.7443}(0.0288)\right\},\left\{s_0(0.0072),\ldots,s_{0.0123}(0.036)\right\}\right\rangle \\ \left\langle\left\{s_{0.3527}(0.006),\ldots,s_{0.6034}(0.012)\right\},\left\{s_0(0.0096),\ldots,s_{0.0146}(0.03)\right\}\right\rangle & \left\langle\left\{s_{0.4865}(0.0024),\ldots,s_{0.8004}(0.0225)\right\},\left\{s_0(0.0096),\ldots,s_{0.0119}(0.0144)\right\}\right\rangle & \left\langle\left\{s_{0.3718}(0.018),\ldots,s_{0.7565}(0.0096)\right\},\left\{s_0(0.0525),\ldots,s_{0.0291}(0.024)\right\}\right\rangle & \left\langle\left\{s_{0.244}(0.0168),\ldots,s_{0.6955}(0.0048)\right\},\left\{s_{0.0046}(0.0168),\ldots,s_{0.0369}(0.0072)\right\}\right\rangle\end{array}\right)$$

Then by utilizing these models (24) and (25), the dual probabilistic linguistic optimistic envelopment analysis value DE_{ε}^{O} and the dual probabilistic linguistic pessimistic envelopment analysis value DE_{ε}^{P} can be calculated separately. Moreover, the geometric envelopment analysis value $E_{\varepsilon}^{geometric}=\sqrt{DE_{\varepsilon}^{O}\times DE_{\varepsilon}^{P}}$ can be calculated easily. Then based on the geometric envelopment analysis value, the priorities of alternatives can be obtained as follows:

Table 4. The priorities and envelopment analysis value of alternatives

Alternatives	$E_{\varepsilon}^{geometric}$	DE_{ε}^{O}	DE_{ε}^{P}	**Priority**
a_1	1.2043	1.5855	0.9147	**2**
a_2	0.8291	1.1168	0.6155	**4**
a_3	0.8693	1.2291	0.6148	**3**
a_4	1.0343	1.0740	0.9960	**1**

Based on Table 4, it is easy to see that the proposed method can solve the problem mentioned at the beginning of the paper that is to choose the applicable cooperative corporation. Our work is mainly to solve the uncertain multi-criteria decision-making problem. The two keys to solving this problem are to choose the right decision-making tools and to determine the weights of criteria. In this paper, in light of the complexity of decision-making problems in real-world applications and the limitations of the available knowledge to DMs, the preference information is not always fully available. Therefore, it is reasonable to consider an uncertain multi-criteria decision-making problem with incomplete preference information in this paper. To some extent, this consideration is close to real-world application. Moreover, now that the preference information is not complete, we think about to deal with the decision-making problem with tools that can give the decision maker as much information as possible. The DPLTSs can reveal the decision-making information through the association of the membership part and the non-membership part. Moreover, whether it is the membership element or the non-membership element, both are composed by several linguistic terms and the corresponding probabilities. From this point of view, the DPLTSs can reflect the decision information as fully as possible. Hence, we choose to study the incomplete dual probabilistic linguistic multi-criteria decision-making problem. Based on the proposed method, the weights of criteria can be determined by a series of steps. Moreover, combining the consideration of incomplete decision and the choice of decision tools, our proposed method meets the needs of most practical problems. The proposed method is of great importance and practical applications for solving uncertain multi-criteria decision-making problem.

Besides, the proposed method has some limitations. As for the repairing process for incomplete DPLPRs, it is complex. In view of the data envelopment analysis, because of the complex structure of the DPLTEs, those inputs and outputs are not the original DPLTEs, but the score functions and accuracy functions. Although from the perspective of information fusion, there is no lack of information. After all, the form is changed. The best method is to do envelope analysis on the original data as far as possible, which is our future research direction.

5.3. Comparative analysis with probabilistic linguistic decision-making matrices

In this subsection, for the sake of comparative analysis, we apply a series of proposed theories and methods to the probabilistic linguistic environment. In terms of preference information, because of the relationship between DPLPRs and PLPRs, there is no need to generate data by Matlab software. The PLPRs can be obtained by the membership part of DPLPRs. Similarly, based on the same principle, the probabilistic linguistic decision-making matrices can be obtained as follows:

$$M_1 = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad c_1 \quad & \quad c_2 \quad & \quad c_3 \quad & \quad c_4 \quad \end{array} \\ \begin{pmatrix} \langle\{s_0(0.2), s_2(0.7)\}\rangle & \langle\{s_2(0.5), s_4(0.2)\}\rangle & \langle\{s_1(0.8), s_2(0.1)\}\rangle & \langle\{s_1(0.3), s_3(0.2)\}\rangle \\ \langle\{s_1(0.6), s_3(0.4)\}\rangle & \langle\{s_1(0.5), s_3(0.5)\}\rangle & \langle\{s_3(0.2), s_4(0.2)\}\rangle & \langle\{s_2(0.1), s_3(0.9)\}\rangle \\ \langle\{s_1(0.6), s_3(0.3)\}\rangle & \langle\{s_1(0.6), s_3(0.1)\}\rangle & \langle\{s_1(0.6), s_2(0.1)\}\rangle & \langle\{s_1(0.2), s_2(0.6)\}\rangle \\ \langle\{s_0(0.4), s_1(0.5)\}\rangle & \langle\{s_3(0.3), s_4(0.5)\}\rangle & \langle\{s_0(0.5), s_2(0.3)\}\rangle & \langle\{s_0(0.3), s_1(0.4)\}\rangle \end{pmatrix} \end{array}$$

$$
M_2 = \begin{array}{cc} & \begin{array}{cccc} c_1 & \quad\quad\quad c_2 & \quad\quad\quad c_3 & \quad\quad\quad c_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \begin{pmatrix} \langle\{s_3(0.4),s_5(0.5)\}\rangle & \langle\{s_1(0.1),s_2(0.5)\}\rangle & \langle\{s_1(0.6),s_3(0.3)\}\rangle & \langle\{s_1(0.5),s_2(0.2)\}\rangle \\ \langle\{s_0(0.6),s_1(0.2)\}\rangle & \langle\{s_1(0.7),s_3(0.2)\}\rangle & \langle\{s_1(0.2),s_2(0.7)\}\rangle & \langle\{s_1(0.3),s_2(0.1)\}\rangle \\ \langle\{s_4(0.2),s_5(0.2)\}\rangle & \langle\{s_3(0.2),s_4(0.6)\}\rangle & \langle\{s_0(0.5),s_1(0.4)\}\rangle & \langle\{s_3(0.4),s_4(0.4)\}\rangle \\ \langle\{s_0(0.6),s_1(0.4)\}\rangle & \langle\{s_4(0.2),s_5(0.3)\}\rangle & \langle\{s_1(0.6),s_2(0.2)\}\rangle & \langle\{s_2(0.7),s_4(0.1)\}\rangle \end{pmatrix} \end{array}
$$

$$
M_3 = \begin{array}{cc} & \begin{array}{cccc} c_1 & \quad\quad\quad c_2 & \quad\quad\quad c_3 & \quad\quad\quad c_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \begin{pmatrix} \langle\{s_1(0.4),s_2(0.2)\}\rangle & \langle\{s_1(0.4),s_2(0.4)\}\rangle & \langle\{s_0(0.1),s_1(0.4)\}\rangle & \langle\{s_1(0.2),s_2(0.4)\}\rangle \\ \langle\{s_0(0.1),s_2(0.3)\}\rangle & \langle\{s_0(0.1),s_1(0.6)\}\rangle & \langle\{s_3(0.2),s_4(0.4)\}\rangle & \langle\{s_0(0.5),s_1(0.3)\}\rangle \\ \langle\{s_1(0.2),s_2(0.3)\}\rangle & \langle\{s_2(0.3),s_4(0.4)\}\rangle & \langle\{s_3(0.4),s_4(0.5)\}\rangle & \langle\{s_2(0.4),s_3(0.6)\}\rangle \\ \langle\{s_4(0.5),s_5(0.3)\}\rangle & \langle\{s_1(0.4),s_2(0.3)\}\rangle & \langle\{s_3(0.1),s_4(0.8)\}\rangle & \langle\{s_2(0.2),s_4(0.3)\}\rangle \end{pmatrix} \end{array}
$$

$$
M_4 = \begin{array}{cc} & \begin{array}{cccc} c_1 & \quad\quad\quad c_2 & \quad\quad\quad c_3 & \quad\quad\quad c_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \begin{pmatrix} \langle\{s_2(0.5),s_3(0.3)\}\rangle & \langle\{s_2(0.3),s_4(0.2)\}\rangle & \langle\{s_1(0.5),s_3(0.4)\}\rangle & \langle\{s_1(0.8),s_2(0.2)\}\rangle \\ \langle\{s_1(0.9),s_3(0.1)\}\rangle & \langle\{s_1(0.4),s_2(0.4)\}\rangle & \langle\{s_3(0.4),s_4(0.5)\}\rangle & \langle\{s_1(0.1),s_2(0.4)\}\rangle \\ \langle\{s_1(0.1),s_3(0.6)\}\rangle & \langle\{s_0(0.5),s_1(0.3)\}\rangle & \langle\{s_0(0.5),s_1(0.4)\}\rangle & \langle\{s_2(0.2),s_3(0.2)\}\rangle \\ \langle\{s_2(0.5),s_4(0.2)\}\rangle & \langle\{s_0(0.1),s_1(0.5)\}\rangle & \langle\{s_1(0.6),s_3(0.2)\}\rangle & \langle\{s_0(0.4),s_2(0.4)\}\rangle \end{pmatrix} \end{array}
$$

Then with the same method to do the group decision-making with dual probabilistic linguistic envelopment analysis, the probabilistic linguistic optimistic envelopment analysis value E_ε^O and the probabilistic linguistic pessimistic envelopment analysis value E_ε^P can be calculated separately. Moreover, the geometric envelopment analysis value $E_\varepsilon^{geometric} = \sqrt{E_\varepsilon^O \times E_\varepsilon^P}$ can be calculated easily. Then based on the geometric envelopment analysis value, the priorities of alternatives can be obtained as follows:

Table 5. The priorities and envelopment analysis value of alternatives

Alternatives	$E_\varepsilon^{geometric}$	E_ε^O	E_ε^P	**Priority**
a_1	0.9031	1.2655	0.6445	**2**
a_2	0.8454	1.7711	0.4035	**3**
a_3	0.7814	1.0234	0.5966	**4**
a_4	0.9781	1.0919	0.8761	**1**

Compared with the results in Table 4, the ranking orders of alternatives in Table 5 are a little different. The main differences are between the alternatives a_2 and a_3. The reasons for the difference can be attributed to the following two aspects: one is that the scale of the experimental data is not large enough. The other is that the decision-making information takes different forms. In relation to probabilistic linguistic decision-making information, the dual probabilistic linguistic decision-making information is able to show the evaluator's attitude towards both sides of the subject, namely, it can show the evaluator's degree of likeness and dislike of the things being evaluated. The key to a good decision is whether the decision maker can make full use of all the available preference information to choose the appropriate method to make a decision. Therefore, here in this paper, there is a better choice to choose dual probabilistic linguistic term sets to collect preference information than probabilistic linguistic term sets.

5.4. Comparative analysis with previous methods

Owing to the fact that the dual probabilistic linguistic term sets was first defined by us, there is very little related research for now. Previous methods in this paper can be divided into two parts: (1) The repairing method for incomplete DPLPRs. For the case of incomplete DPLPRs, it is first defined in this paper. Owing to the structural features of the DPLTSs, the existing methods are not suitable for repairing incomplete DPLPRs. Therefore, we first propose the repairing method to deal with the incomplete dual probabilistic linguistic situation. For this repairing method, its applicability is mainly reflected in the solution of case problem. Based on the simulation result on the subsection 5.2, for those incomplete DPLPRs, the proposed repairing method can fix them indeed.

(2) The decision-making method for dual probabilistic linguistic uncertain decision-making problem. Currently, the DPLTSs were first proposed by us in Ref. [12]. There has been little previous research on decision-making methods. In Ref. [12], the decision-making method is multi-criteria decision-making method. The weight vector of the criterion is given in advance. Then the decision is made based on the most traditional operator integration method. To some extent, it is relatively unreasonable and arbitrary to give criterion weight subjectively. This is the shortcoming of the current existing dual probabilistic linguistic decision-making method. However, in this paper, we let the weights of criteria be unknown. The weight vector of the criterion is determined objectively by the preference information of the DMs. After that, considering the advantage of DEA for multiple decision units with the ratio of multiple inputs to multiple

outputs. We regard the DPLTEs as the stochastic variable, and expand the DEA into the dual probabilistic linguistic environment. Moreover, we assume that these inputs and outputs are all DPLTEs, which means that these inputs and outputs are both stochastic variables. Based on the idea of Ref. [44], we build the model to measure the efficiency. Similarly, the simulation result on the subsection 5.2 demonstrates the validity of the dual probabilistic linguistic data envelopment analysis method.

5.5. Discussion

In this subsection, the research contents are classified and discussed from the following aspects: theoretical aspect, methodological aspect and applicable aspect.

Theoretical aspect: Considering the fact that practical decision-making problems often cover many aspects of things. The DMs who are invited to make decisions are not familiar with all the decision issues. Hence, in reality, most of the decisions are made under uncertainties. In this paper, our aim is devoted to doing the research of incomplete DPLPRs. The elements of DPLPRs can not only show the cognitive certainty and uncertainty of DMs, but also further show the hesitation degree of DMs between several linguistic terms. To some extent, the consideration of incomplete DPLPRs is close to the actual situation of most uncertain decisions. Moreover, from the theoretical point of view, complete preference information is the presupposition for making a decision. Therefore, we do the theoretical research for incomplete DPLPRs. In general, in this paper, the basic theory of the DPLTSs has been enriched by defining the DPLPRs and the IDPLPRs.

Methodological aspect: For the repairing for incomplete DPLPRs, we choose a step-by-step repairing method. We first repair the linguistic section of the incomplete DPLPRs, then repair the possibilistic section. This step-by step repairing method conforms to the principle of element generation in the DPLPRs. This is also the advantage of the step-by-step repairing method. Moreover, after obtaining the complete DPLPRs, consistency is the basis to get a logical decision result. Based on the distance between the DPLPRs, we define the consistency index to check and improve the consistency of DPLPRs. After that, we choose the DEA method to the sorting process. The advantage of DEA is that it can evaluate the set of DMs by measuring the relative efficiency without assuming prior weights on the inputs and outputs.

Applicable aspect: The DPLTSs can reveal the decision-making information through the association of the membership part and non-membership part. Moreover, whether it is the membership element or the non-membership element, both are composed by several linguistic terms and corresponding probabilities.

From this point of view, the DPLTSs can reflect the decision information as fully as possible. Hence, we choose to study the incomplete dual probabilistic linguistic multi-criteria decision-making problem. Based on a series of proposed method, the weights of criteria can be determined by a series of steps. Moreover, combining the consideration of incomplete decision and the choice of decision tools, our proposed methods meet the needs of most practical problems. The proposed method is of great importance and practical applications for solving uncertain multi-criteria decision-making problem. Especially, the proposed method in this paper can also be used to do the uncertain assessment for sustainable supplier management, as well as supplier evaluation, selection and monitoring.

6. Conclusions

In this paper, we have enriched the basic theory of the DPLTSs by these following directions: firstly, we have defined the DPLPRs, and then defined the IDPLPRs. Moreover, for the sake of obtaining the logical decision-making result, we have constructed different linear programming models to repair the missing linguistic portion and the probabilistic portion, and then, we have probed the consistency of the DPLPRs. Furthermore, we have built the dual probabilistic linguistic DEA model to make decisions. After that, we have applied the proposed method to solve the problem mentioned at the beginning of the paper, and helped to choose the best project for 5G enterprise. The research result shows that the enterprise should choose Huawei as a partner to develop the 5G industry. To some extent, the decision-making result is in line with the current status of 5G industry development.

The contributions of the paper can be highlighted as follows: (1) We first define the incomplete DPLPRs and develop the step-by-step method to repair incomplete DPLPRs. The step-by-step method repairs innovatively the linguistic section and the probabilistic section separately, which is in line with the principle that preference information is derived in the first place. (2) In order to prepare for getting logical decision results, we use the distance measure between the DLPRs as the basis to study the consistency of preference relationships. (3) We extend innovatively the DPLTSs to the DEA environment. Considering the structural features of DPLTSs, it is not realistic to calculate directly if we use the DPLTSs as inputs or outputs. We use score functions and accuracy functions as inputs and outputs based on the principle of information fusion. The combination of dual probabilistic linguistic information and DEA conforms to the characters of most uncertain decisions. Because most of uncertain decision-making problem cannot always provide specific input and output. Generally, the inputs and outputs may be vague or uncertain. The DPLTSs just reflect the fuzziness of uncertain decisions.

Although the repairing method for incomplete DPLPRs is reasonable, but the procedure is relatively complex. Moreover, the dual probabilistic linguistic data envelopment analysis is performed on score functions and accuracy functions, not the original DPLTSs. Therefore, in the future, we can make further research from the following two perspectives: (1) Looking for simpler and more straightforward repairing methods. (2) Using the original data for data envelopment analysis as soon as possible.

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Appendix

These corresponding consistent DPLPRs $D_i'\left(i=2,3,4\right)$ are as follows:

$$D_2'=\begin{pmatrix}
\langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(0.005),\ldots,s_{1.5}(0.0003)\},\{s_{1.5}(0.0015),\ldots,s_{1.5}(0.0008)\}\rangle & \langle\{s_{1.5}(0.0036),\ldots,s_{1.5}(0.0012)\},\{s_{1.5}(0.0011),\ldots,s_{1.5}(0.0091)\}\rangle & \langle\{s_{1.5}(0.0045),\ldots,s_{1.5}(0.0004)\},\{s_{1.5}(0.0009),\ldots,s_{1.5}(0.0106)\}\rangle \\
\langle\{s_{1.5}(0.0015),\ldots,s_{1.5}(0.0008)\},\{s_{1.5}(0.005),\ldots,s_{1.5}(0.0003)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(0.0058),\ldots,s_{1.5}(0.0003)\},\{s_{1.5}(0.006),\ldots,s_{1.5}(0.0009)\}\rangle & \langle\{s_{1.5}(0.0073),\ldots,s_{1.5}(9.600\text{e-}05)\},\{s_{1.5}(0.0048),\ldots,s_{1.5}(0.0010)\}\rangle \\
\langle\{s_{1.5}(0.0011),\ldots,s_{1.5}(0.0091)\},\{s_{1.5}(0.0036),\ldots,s_{1.5}(0.0012)\}\rangle & \langle\{s_{1.5}(0.006),\ldots,s_{1.5}(0.0009)\},\{s_{1.5}(0.0058),\ldots,s_{1.5}(0.0003)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(0.0054),\ldots,s_{1.5}(0.0012)\},\{s_{1.5}(0.0035),\ldots,s_{1.5}(0.0035)\}\rangle \\
\langle\{s_{1.5}(0.0009),\ldots,s_{1.5}(0.0106)\},\{s_{1.5}(0.0045),\ldots,s_{1.5}(0.0004)\}\rangle & \langle\{s_{1.5}(0.0048),\ldots,s_{1.5}(0.0010)\},\{s_{1.5}(0.0073),\ldots,s_{1.5}(9.600\text{e-}05)\}\rangle & \langle\{s_{1.5}(0.0035),\ldots,s_{1.5}(0.0035)\},\{s_{1.5}(0.0054),\ldots,s_{1.5}(0.0012)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle
\end{pmatrix}$$

$$D_3'=\begin{pmatrix}
\langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(7.200\text{e-}05),\ldots,s_{1.5}\},\{s_{1.5}(0.0004),\ldots,s_{1.5}(0.0031)\}\rangle & \langle\{s_{1.5}(0.0005),\ldots,s_{1.5}(0.0006)\},\{s_{1.5}(0.001),\ldots,s_{1.5}(0.0046)\}\rangle & \langle\{s_{1.5}(0.0006),\ldots,s_{1.5}(0.0012)\},\{s_{1.5}(0.0004),\ldots,s_{1.5}(0.0052)\}\rangle \\
\langle\{s_{1.5}(0.0004),\ldots,s_{1.5}(0.0031)\},\{s_{1.5}(7.200\text{e-}05),\ldots,s_{1.5}\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(0.0008),\ldots,s_{1.5}(0.0031)\},\{s_{1.5}(0.0003),\ldots,s_{1.5}(0.0035)\}\rangle & \langle\{s_{1.5}(0.0010),\ldots,s_{1.5}(0.0061)\},\{s_{1.5}(0.0001),\ldots,s_{1.5}(0.0039)\}\rangle \\
\langle\{s_{1.5}(0.001),\ldots,s_{1.5}(0.0046)\},\{s_{1.5}(0.0005),\ldots,s_{1.5}(0.0006)\}\rangle & \langle\{s_{1.5}(0.0003),\ldots,s_{1.5}(0.0035)\},\{s_{1.5}(0.0008),\ldots,s_{1.5}(0.0031)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(0.0023),\ldots,s_{1.5}(0.0092)\},\{s_{1.5}(0.0008),\ldots,s_{1.5}(0.0052)\}\rangle \\
\langle\{s_{1.5}(0.0004),\ldots,s_{1.5}(0.0052)\},\{s_{1.5}(0.0006),\ldots,s_{1.5}(0.0012)\}\rangle & \langle\{s_{1.5}(0.0001),\ldots,s_{1.5}(0.0039)\},\{s_{1.5}(0.0010),\ldots,s_{1.5}(0.0061)\}\rangle & \langle\{s_{1.5}(0.0008),\ldots,s_{1.5}(0.0052)\},\{s_{1.5}(0.0023),\ldots,s_{1.5}(0.0092)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle
\end{pmatrix}$$

$$D_4'=\begin{pmatrix}
\langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(0.0002),\ldots,s_{1.5}(0.0360)\},\{s_{1.5}(0.0007),\ldots,s_{1.5}(0.0069)\}\rangle & \langle\{s_{1.5}(0.0029),\ldots,s_{1.5}(0.0090)\},\{s_{1.5}(0.0007),\ldots,s_{1.5}(0.0048)\}\rangle & \langle\{s_{1.5}(0.00086),\ldots,s_{1.5}(0.0150)\},\{s_{1.5}(7.200\text{e-}05),\ldots,s_{1.5}(0.0246)\}\rangle \\
\langle\{s_{1.5}(0.0007),\ldots,s_{1.5}(0.0069)\},\{s_{1.5}(0.0002),\ldots,s_{1.5}(0.0360)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(0.0072),\ldots,s_{1.5}(0.0043)\},\{s_{1.5}(0.0006),\ldots,s_{1.5}(0.0120)\}\rangle & \langle\{s_{1.5}(0.0022),\ldots,s_{1.5}(0.0072)\},\{s_{1.5}(6.0000\text{e-}05),\ldots,s_{1.5}(0.0614)\}\rangle \\
\langle\{s_{1.5}(0.0007),\ldots,s_{1.5}(0.0048)\},\{s_{1.5}(0.0029),\ldots,s_{1.5}(0.0090)\}\rangle & \langle\{s_{1.5}(0.0006),\ldots,s_{1.5}(0.0120)\},\{s_{1.5}(0.0072),\ldots,s_{1.5}(0.0043)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.5}(0.0022),\ldots,s_{1.5}(0.005)\},\{s_{1.5}(0.0007),\ldots,s_{1.5}(0.0154)\}\rangle \\
\langle\{s_{1.5}(7.200\text{e-}05),\ldots,s_{1.5}(0.0246)\},\{s_{1.5}(0.00086),\ldots,s_{1.5}(0.0150)\}\rangle & \langle\{s_{1.5}(6.0000\text{e-}05),\ldots,s_{1.5}(0.0614)\},\{s_{1.5}(0.0022),\ldots,s_{1.5}(0.0072)\}\rangle & \langle\{s_{1.5}(0.0007),\ldots,s_{1.5}(0.0154)\},\{s_{1.5}(0.0022),\ldots,s_{1.5}(0.005)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle
\end{pmatrix}$$

These modified DPLPRs $\breve{D}_i\left(i=2,3,4\right)$ with the acceptable consistency are as follows:

$$\breve{D}_2 = \begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{2.25}(0.0025),\ldots,s_{2.75}(6.7200\text{e-}05)\},\{s_{1.25}(0.0004),\ldots,s_{1.75}(0.0002)\}\rangle & \langle\{s_{0.75}(0.0007),\ldots,s_{1.75}(0.0008)\},\{s_{1.75}(0.0002),\ldots,s_{2.25}(0.0054)\}\rangle & \langle\{s_{1.75}(0.0023),\ldots,s_{2.75}(8.9600\text{e-}05)\},\{s_{1.25}(0.0003),\ldots,s_{1.75}(0.0074)\}\rangle \\ \langle\{s_{1.25}(0.0004),\ldots,s_{1.75}(0.0002)\},\{s_{2.25}(0.0025),\ldots,s_{2.75}(6.7200\text{e-}05)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{2.3514}(0.0052),\ldots,s_{1.5}(2.5200\text{e-}05)\},\{s_{0.75}(0.0030),\ldots,s_{1.25}(0.0003)\}\rangle & \langle\{s_{1.25}(0.0022),\ldots,s_{2.25}(1.9200\text{e-}05)\},\{s_{1.25}(0.0019),\ldots,s_{2.25}(0.0002)\}\rangle \\ \langle\{s_{1.75}(0.0002),\ldots,s_{2.25}(0.0054)\},\{s_{0.75}(0.0007),\ldots,s_{1.75}(0.0008)\}\rangle & \langle\{s_{0.75}(0.0030),\ldots,s_{1.25}(0.0003)\},\{s_{2.3514}(0.0052),\ldots,s_{1.5}(2.5200\text{e-}05)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.25}(0.0032),\ldots,s_{2.25}(0.0005)\},\{s_{1.25}(0.0014),\ldots,s_{1.75}(0.0021)\}\rangle \\ \langle\{s_{1.25}(0.0003),\ldots,s_{1.75}(0.0074)\},\{s_{1.75}(0.0023),\ldots,s_{2.75}(8.9600\text{e-}05)\}\rangle & \langle\{s_{1.25}(0.0019),\ldots,s_{2.25}(0.0002)\},\{s_{1.25}(0.0022),\ldots,s_{2.25}(1.9200\text{e-}05)\}\rangle & \langle\{s_{1.25}(0.0014),\ldots,s_{1.75}(0.0021)\},\{s_{1.25}(0.0032),\ldots,s_{2.25}(0.0005)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

$$\breve{D}_3 = \begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.25}(2.1600\text{e-}05),\ldots,s_{1.75}(4.320\text{e-}05)\},\{s_{2.25}(4.0000\text{e-}05),\ldots,s_{2.75}(0.0012)\}\rangle & \langle\{s_{2.75}(9.600\text{e-}05),\ldots,s_{3.25}(0.0001)\},\{s_{0.75}(0.0004),\ldots,s_{1.25}(0.0028)\}\rangle & \langle\{s_{2.25}(0.0001),\ldots,s_{2.75}(0.0007)\},\{s_{1.25}(0.0002),\ldots,s_{1.75}(0.0010)\}\rangle \\ \langle\{s_{2.25}(4.0000\text{e-}05),\ldots,s_{2.75}(0.0012)\},\{s_{1.25}(2.1600\text{e-}05),\ldots,s_{1.75}(4.320\text{e-}05)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{0.75}(0.0004),\ldots,s_{1.25}(0.0012)\},\{s_{1.75}(5.7600\text{e-}05),\ldots,s_{2.75}(0.0014)\}\rangle & \langle\{s_{2.25}(0.0004),\ldots,s_{2.75}(0.0025)\},\{s_{1.25}(1.200\text{e-}05),\ldots,s_{1.75}(0.0035)\}\rangle \\ \langle\{s_{0.75}(0.0004),\ldots,s_{1.25}(0.0028)\},\{s_{2.75}(9.600\text{e-}05),\ldots,s_{3.25}(0.0001)\}\rangle & \langle\{s_{1.75}(5.7600\text{e-}05),\ldots,s_{2.75}(0.0014)\},\{s_{0.75}(0.0004),\ldots,s_{1.25}(0.0012)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{0.75}(0.0014),\ldots,s_{1.25}(0.0037)\},\{s_{2.75}(0.0003),\ldots,s_{3.25}(0.0031)\}\rangle \\ \langle\{s_{1.25}(0.0002),\ldots,s_{1.75}(0.0010)\},\{s_{2.25}(0.0001),\ldots,s_{2.75}(0.0007)\}\rangle & \langle\{s_{1.25}(1.200\text{e-}05),\ldots,s_{1.75}(0.0035)\},\{s_{2.25}(0.0004),\ldots,s_{2.75}(0.0025)\}\rangle & \langle\{s_{2.75}(0.0003),\ldots,s_{3.25}(0.0031)\},\{s_{0.75}(0.0014),\ldots,s_{1.25}(0.0037)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

$$\breve{D}_4 = \begin{pmatrix} \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{1.25}(4.800\text{e-}05),\ldots,s_{1.75}(0.0216)\},\{s_{2.25}(0.0003),\ldots,s_{2.75}(0.0041)\}\rangle & \langle\{s_{0.75}(0.0012),\ldots,s_{1.25}(0.0045)\},\{s_{2.75}(0.0002),\ldots,s_{3.25}(0.0010)\}\rangle & \langle\{s_{2.25}(0.0003),\ldots,s_{2.75}(0.0075)\},\{s_{0.75}(7.200\text{e-}06),\ldots,s_{1.25}(0.0197)\}\rangle \\ \langle\{s_{2.25}(0.0003),\ldots,s_{2.75}(0.0041)\},\{s_{1.25}(4.800\text{e-}05),\ldots,s_{1.75}(0.0216)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{0.75}(0.0036),\ldots,s_{1.75}(0.0013)\},\{s_{1.75}(0.0003),\ldots,s_{2.25}(0.0060)\}\rangle & \langle\{s_{0.75}(0.0006),\ldots,s_{1.25}(0.0029)\},\{s_{2.25}(6.0000\text{e-}06),\ldots,s_{2.75}(0.0492)\}\rangle \\ \langle\{s_{2.75}(0.0002),\ldots,s_{3.25}(0.0010)\},\{s_{0.75}(0.0012),\ldots,s_{1.25}(0.0045)\}\rangle & \langle\{s_{1.75}(0.0003),\ldots,s_{2.25}(0.0060)\},\{s_{0.75}(0.0036),\ldots,s_{1.75}(0.0013)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle & \langle\{s_{2.25}(0.0009),\ldots,s_{3.25}(0.0025)\},\{s_{0.75}(0.0004),\ldots,s_{1.25}(0.0061)\}\rangle \\ \langle\{s_{0.75}(7.200\text{e-}06),\ldots,s_{1.25}(0.0197)\},\{s_{2.25}(0.0003),\ldots,s_{2.75}(0.0075)\}\rangle & \langle\{s_{2.25}(6.0000\text{e-}06),\ldots,s_{2.75}(0.0492)\},\{s_{0.75}(0.0006),\ldots,s_{1.25}(0.0029)\}\rangle & \langle\{s_{0.75}(0.0004),\ldots,s_{1.25}(0.0061)\},\{s_{2.25}(0.0009),\ldots,s_{3.25}(0.0025)\}\rangle & \langle\{s_3(1)\},\{s_3(1)\}\rangle \end{pmatrix}$$

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