Análisis de operadores y polinomios en álgebras de Banach mediante propiedades de ortogonalidad

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Un sacrificio solo vale tanto como la vida que salva.
Anillo del sacrificio, Dark Souls III.

Resumen

Esta tesis se presenta siguiendo el modelo de agrupación de publicaciones. Esto significa que cada capítulo es, en realidad, un artículo escrito por la autora de la tesis y colaboradores, con su propia introducción, desarrollo y resultados. Por tanto, cada capítulo es independiente de los anteriores, usa una notación ligeramente distinta y debe leerse de forma separada.

Nuestro objetivo es hacer avances significativos en varios problemas relacionados con la ortogonalidad en el contexto del Análisis Funcional. Para ello, estudiamos sistemáticamente aplicaciones multilineales que satisfacen algún tipo de propiedad relacionada con la ortogonalidad. Como conclusión, los resultados obtenidos muestran que algunas de las transformaciones más estudiadas en muchas de las álgebras de Banach más relevantes están determinadas por su comportamiento sobre elementos con producto cero.

El primer problema que consideramos es el de la representación lineal de los polinomios ortogonalmente aditivos. Una aplicación lineal $P\colon X \to Y$ entre espacios vectoriales es un *polinomio m-homogéneo* (para algún $m \in \mathbb{N},\ m \geq 2$) si existe una aplicación multilineal $\varphi\colon X^m \to Y$ tal que

$$P(x) = \varphi(x, \ldots, x) \quad (x \in X).$$

En el caso de que el espacio vectorial X esté equipado con un producto (por ejemplo, si X es un álgebra o un espacio de funciones), decimos que P es *ortogonalmente aditivo* si

$$x, y \in X,\ xy = yx = 0 \implies P(x + y) = P(x) + P(y).$$

Los polinomios estándar, aquellos definidos por $P(x) = \Phi(x^m)$ $(x \in X)$, donde Φ es una aplicación lineal definida en el espacio generado por $\{x^m : x \in X\}$ y con valores en Y, son ortogonalmente aditivos. El problema consiste en ver si el recíproco es cierto, es decir, si todo polinomio ortogonalmente aditivo puede expresarse mediante una aplicación lineal de esa forma.

En el Capítulo 1 ([8]), estudiamos los polinomios ortogonalmente aditivos definidos en álgebras de convolución asociadas a un grupo compacto G. En la Sección 1.2, probamos que los polinomios ortogonalmente aditivos definidos en $\mathcal{T}(G)$ admiten una representación lineal, y extendemos dicha representación a $L^1(G)$ en la Sección 1.3. La Sección 1.4 consiste en ejemplos de polinomios ortogonalmente aditivos y continuos en $L^p(\mathbb{T})$ $(1 < p \leq \infty)$ y en $C(\mathbb{T})$ que no pueden expresarse mediante aplicaciones lineales y continuas. La Sección 1.6 está dedicada a probar que los polinomios ortogonalmente aditivos definidos en una gran variedad de álgebras de convolución, que incluye a $L^p(G)$ $(1 < p < \infty)$ y a

$C(G)$, pueden representarse mediante aplicaciones lineales que son continuas respecto de una norma definida en la Sección 1.5.

El Capítulo 2 ([17]) está dedicado al estudio de este problema en el contexto de los espacios L^p no conmutativos asociados a un álgebra de von Neumann $\mathcal{M}$ equipada con una traza normal, semifinita y fiel τ. Denotamos dichos espacios por $L^p(\mathcal{M},\tau)$. En la Sección 2.2, obtenemos varios resultados positivos para polinomios ortogonalmente aditivos definidos en C^*-álgebras y con valores en cualquier espacio vectorial topológico. En la Sección 2.3, usamos estos resultados para probar la representación lineal de polinomios ortogonalmente aditivos definidos en $L^p(\mathcal{M},\tau)$ $(0 < p < \infty)$ con valores en cualquier espacio vectorial topológico.

El siguiente problema a considerar es el de la hiperreflexividad de un subespacio de operadores. Dados dos espacios de Banach X e Y, decimos que un subespacio $\mathcal{S} \subset \mathcal{B}(X,Y)$ es *reflexivo* si

$$\mathcal{S} = \big\{T \in B(X,Y) : T(x) \in \overline{\{S(x) : S \in \mathcal{S}\}}\,(x \in X)\big\}.$$

Decimos que $\mathcal{S}$ es *hiperreflexivo* si existe una constante C tal que

$$\operatorname{dist}(T,\mathcal{S}) \leq C \sup_{x\in\mathcal{X},\,\|x\|\leq 1} \inf\big\{\|T(x)-S(x)\| : S \in \mathcal{S}\big\} \quad (T \in B(X,Y)),$$

y la menor constante C se llama constante de hiperreflexividad de $\mathcal{S}$.

El Capítulo 3 ([9]) se centra en el estudio de la reflexividad e hiperreflexividad del espacio $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}),L^q(\mathcal{M}))$ de homomorfismos de $\mathcal{M}$-módulos a derecha de $L^p(\mathcal{M})$ en $L^q(\mathcal{M})$, siendo $\mathcal{M}$ un álgebra de von Neumann. Los espacios L^p que consideramos aquí son aquellos introducidos por Haagerup. Tras obtener ciertas desigualdades útiles mediante el análisis de las aplicaciones bilineales definidas en una C^*-álgebra en la Sección 3.2, probamos, en la Sección 3.3, que el subespacio $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}),L^q(\mathcal{M}))$ es reflexivo. En la Sección 3.4, probamos que dicho espacio es hiperreflexivo si $1 \leq q < p \leq \infty$, en cuyo caso la constante de hiperreflexividad es menor o igual que una constante que depende de p y q pero no de $\mathcal{M}$, y si $1 \leq p,q \leq \infty$ y $\mathcal{M}$ es inyectiva, en cuyo caso la constante de hiperreflexividad es menor o igual que 8.

Sea A un álgebra de Banach. Para cada $\varphi \in \mathcal{B}^2(A,\mathbb{C})$, consideramos las constantes

$$|\varphi|_{zp} = \sup\{|\varphi(a,b)| : a,b \in A,\ \|a\| = \|b\| = 1,\ ab = 0\},$$
$$|\varphi|_b = \sup\{|\varphi(ab,c)-\varphi(a,bc)| : a,b,c \in A,\ \|a\| = \|b\| = \|c\| = 1\}.$$

Decimos que A es *determinada por su producto cero* si

$$\varphi \in \mathcal{B}^2(A,\mathbb{C}),\ |\varphi|_{zp} = 0 \implies \varphi \in \mathcal{B}^2_\pi(a,\mathbb{C}).$$

En caso de que, además, exista una constante α tal que

$$\operatorname{dist}\big(\varphi,\mathcal{B}^2_\pi(A,\mathbb{C})\big) \leq \alpha\,|\varphi|_{zp} \quad \forall\varphi \in \mathcal{B}^2(A,\mathbb{C}),$$

decimos que A es *fuertemente determinada por su producto cero*. Por otro lado, decimos que A satisface la *propiedad* $\mathbb{B}$ si

$$\varphi \in \mathcal{B}^2(A,\mathbb{C})\ \ |\varphi|_{zp} = 0 \implies |\varphi|_b = 0,$$

y, en caso de que exista una constante β tal que

$$|\varphi|_b \leq \beta\, |\varphi|_{zp} \quad \forall \varphi \in \mathcal{B}^2(A, \mathbb{C}),$$

decimos que A satisface la *propiedad* $\mathbb{B}$ *fuerte*. Si existen, las constantes óptimas se denotan por α_A y β_A, respectivamente.

El Capítulo 4 ([10]) está dedicado al estudio de estas propiedades. En la Sección 4.2, probamos ciertas desigualdades entre las seminormas dist $\big(\cdot, \mathcal{B}^2_\pi(A, \mathbb{C})\big)$, $|\cdot|_b$, y $|\cdot|_{zp}$. Entre otras cosas, probamos que toda álgebra determinada por su producto cero satisface la propiedad $\mathbb{B}$ y toda álgebra fuertemente determinada por su producto cero satisface la propiedad $\mathbb{B}$ fuerte. Además, los recíprocos son ciertos si el álgebra admite una identidad aproximada acotada. En la Sección 4.3, damos una cota para las constantes $\alpha_{L^1(G)}$ y $\beta_{L^1(G)}$, siendo G un grupo localmente compacto, que mejora la dada en [77]. En la Sección 4.4, probamos que el álgebra $\mathcal{A}(X)$ de los operadores aproximables en un espacio de Banach X con la propiedad $(\mathbb{A})$ es fuertemente determinada por su producto cero y usamos este resultado para probar que el espacio $\mathcal{Z}^n(\mathcal{A}(X), Y^*)$ de los n-cociclos continuos de $\mathcal{A}(X)$ en Y^* es hiperreflexivo para cada $\mathcal{A}(X)$-bimódulo Y.

Finalmente, estudiamos el problema de la caracterización de los homomorfismos de Jordan con peso. Sean A y B álgebras. Una aplicación lineal $\Phi\colon A \to B$ es un *homomorfismo de Jordan* si

$$\Phi(a \circ b) = \Phi(a) \circ \Phi(b) \quad (a, b \in A),$$

donde $a \circ b$ denota el producto de Jordan $ab + ba$. Un *homomorfismo de Jordan con peso* es la composición de un homomorfismo de Jordan y un centralizador $W\colon B \to B$, es decir, una aplicación lineal que satisface

$$W(ab) = W(a)b = aW(b) \quad (a, b \in A).$$

Si $\Phi\colon A \to B$ es un homomorfismo de Jordan con peso, entonces

$$a, b \in A,\ ab = ba = 0 \implies \Phi(a) \circ \Phi(b) = 0.$$

Si el centro de B no contiene elementos nilpotentes no nulos, entonces Φ satisface una propiedad más fuerte: conserva el producto cero bilateral, es decir,

$$a, b \in A,\ ab = ba = 0 \implies \Phi(a)\Phi(b) = \Phi(b)\Phi(a) = 0.$$

El objetivo es saber si toda aplicación lineal que cumpla alguna de esas condiciones es necesariamente un homomorfismo de Jordan con peso.

En el Capítulo 5 ([27]), estudiamos la condición más débil en el contexto algebraico de los anillos unitales. En la Sección 5.2, probamos que si A es un anillo unital aditivamente generado por productos de Jordan de idempotentes y B es un anillo con $\frac{1}{2}$, entonces toda aplicación aditiva y sobreyectiva $\Phi\colon A \to B$ que cumpla $\Phi(a) \circ \Phi(b) = 0$ siempre que $ab = ba = 0$ es un homomorfismo de Jordan con peso. En la Sección 5.3, probamos que si A es un anillo primo con característica distinta de 2, 3 y 5, $\Phi\colon A \to A$ es una aplicación

aditiva y biyectiva y existe $\Psi\colon A \to A$ tal que $\Psi(a \circ b) = \Phi(a) \circ \Phi(b)$ $(a, b \in A)$, entonces Φ es un homomorfismo de Jordan con peso.

El objetivo del Capítulo 6 ([28]) es extender y unificar los resultados ya conocidos para C^*-álgebras y álgebras grupo. En la Sección 6.2, probamos que si A es un álgebra de Banach determinada por su producto cero, débilmente amenable y con identidad aproximada acotada, y B es un álgebra de Banach con identidad aproximada acotada, entonces toda aplicación de A en B lineal, continua y sobreyectiva que conserve el producto cero bilateral es un homomorfismo de Jordan con peso. Las C^*-álgebras y el álgebra $L^1(G)$ asociada a un grupo G compacto satisfacen dichas hipótesis. En la Sección 6.3, probamos que la condición más débil es suficiente si A es una C^*-álgebra o el álgebra $\mathcal{A}(X)$, donde X es un espacio de Banach tal que X^* tiene la propiedad de aproximación acotada, y B es cualquier álgebra de Banach con identidad aproximada acotada.

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Contents

Introduction

This PhD thesis follows the compendium form. This means that each chapter is, indeed, a paper written by the author and collaborators, with its own introduction, development and results. Therefore, the notation is not consistent between chapters and each one must be read as an independent entity.

The aim of this dissertation is to make significant advances in several orthogonality related problems on the field of Functional Analysis. Specifically, we study the problem or representing orthogonally additive homogeneous polynomials through linear maps, the reflexivity and hyperreflexivity of certain subspaces of operators, the strong property $\mathbb{B}$ and the property of being strongly zero product determined, and the characterization of weighted Jordan homomorphisms.

Our methodology consists on the systematic study of multilinear maps that satisfy some sort of orthogonality related property, which depends on each problem and is detailed below. As a conclusion, our results show that some of the most studied transformations on many of the most relevant Banach algebras are determined by their behaviour over elements with zero product.

Throughout all algebras and linear spaces are complex. Of course, linearity is understood to mean complex linearity. Given $m \in \mathbb{N}$ and Banach spaces X and Y, we use the notation $\mathcal{B}^m(X,Y)$ for the space of m-linear continuous maps from X^m, the m-fold Cartesian product of X, to Y. If $m = 1$ we write $\mathcal{B}(X,Y)$, and if $X = Y$ we just write $\mathcal{B}(X)$. X^* denotes the dual space of X.

Orthogonally additive polynomials

We fix $m \in \mathbb{N}$ with $m \geq 2$.

Let X and Y be linear spaces. A map $P\colon X \to Y$ is said to be an m-homogeneous polynomial if there exists an m-linear map $\varphi\colon X^m \to Y$ such that

$$P(x) = \varphi(x, \dots, x) \quad (x \in X).$$

Such a map is unique if it is required to be symmetric. This is a consequence of the so-called polarization formula, which defines φ by

$$\varphi\left(x_1, \dots, x_n\right) = \frac{1}{n!\,2^n} \sum_{\epsilon_1, \dots, \epsilon_n = \pm 1} \epsilon_1 \cdots \epsilon_n P\left(\epsilon_1 x_1 + \cdots + \epsilon_n x_n\right)$$

for all $x_1, \ldots, x_n \in X$.

Suppose that X is equipped with a multiplicative structure (for instance, X could be an algebra or a function space) and let $\mathcal{P}_m(X)$ be the linear span of the set $\{x^m : x \in X\}$. If $\Phi \colon \mathcal{P}_m(X) \to Y$ is a linear map, then the map $P \colon X \to Y$ defined by

$$P(x) = \Phi(x^m) \quad (x \in X) \tag{1}$$

is an m-homogeneous polynomial. These are the standard examples of polynomials and one might wonder if every polynomial is of this form, but the answer to this question is strongly negative. To see this, we first observe that standard polynomials are *orthogonally additive*, i.e., they satisfy the following:

$$x, y \in X,\ xy = yx = 0 \implies P(x+y) = P(x) + P(y).$$

The map $P \colon \mathbb{M}_2 \to \mathbb{C}$ defined by

$$P\big((a_{ij})\big) = a_{11}a_{22} \quad \big((a_{ij}) \in \mathbb{M}_2\big),$$

where $\mathbb{M}_2$ is the full matrix algebra of order 2, is obviously a 2-homogeneous polynomial and can't be expressed through a linear map as in (1) because

$$P(e_{11} + e_{22}) = 1 \neq 0 = P(e_{11}) + P(e_{22}),$$

where e_{ij} is the matrix whose (i,j) entry is 1 all other entries are 0. It is notably more difficult to find orthogonally additive polynomials that can't be represented by linear maps.

Our goal is to know if the necessary condition of being orthogonally additive is also sufficient to characterize polynomials that can be written as in (1). We might also wonder if orthogonal additivity on a subset of X is enough to ensure the representation and, in the case where X and Y are Banach spaces, if the representing linear map Φ is continuous provided that the polynomial P is.

Our general strategy is based on using the multilinear map φ associated to an orthogonally additive polynomial $P \colon X \to Y$ to define the representing linear map Φ, taking into account that if such a map exists and $x, e \in X$ are such that $xe = ex = x$, then, by the polarization formula,

$$\Phi(x) = \varphi(x, e, \ldots, e).$$

In [78], the author succeeded in providing a useful representation of the orthogonally additive homogeneous polynomials on the spaces $L^p([0,1])$ and ℓ^p with $1 \leq p < \infty$. In [69] (see also [29]), the authors obtained a similar representation for the space $C(K)$, for a compact Hausdorff space K. These results were generalized to Banach lattices [21] and Riesz spaces [53]. Further, the problem of representing the orthogonally additive homogeneous polynomials has been also considered in the context of Banach function algebras [15, 81] and non-commutative Banach algebras [8, 16, 67]. Notably, [67] can be thought of as the natural non-commutative analogue of the representation of orthogonally additive polynomials on $C(K)$-spaces.

Our first result on this topic was the representation of orthogonally additive polynomials on the algebra $\mathcal{A}(X)$ of approximable operators on a Banach space X such that X^* has the bounded approximation property [16].

In Chapter 1 ([8]), we study orthogonally additive polynomials on convolution algebras associated with a compact group. In Section 1.2, we prove that orthogonally additive (not necessarily continuous) polynomials on the algebra $\mathcal{T}(G)$, where G is a compact group, are standard polynomials. We extend the linear representation to the algebra $L^1(G)$ in Section 1.3. Section 1.4 shows that the representation is not always possible for the algebras $L^p(\mathbb{T})$ $(1 < p \leq \infty)$ and $C(\mathbb{T})$, where $\mathbb{T}$ denotes the circle group, if we require that the representing linear map is continuous with respect to the norm of each algebra. In Section 1.5 we define a norm in $\mathcal{P}_m(A)$ for any Banach algebra A, and, finally, in Section 1.6 we prove the representation of continuous orthogonally additive polynomials by linear maps that are continuous with respect to that norm on a large variety of convolution algebras, including $L^p(G)$ $(1 < p < \infty)$ and $C(G)$.

Chapter 2 ([17]) is devoted to the representation of orthogonally additive polynomials of non-commutative L^p-spaces associated with a von Neumann algebra $\mathcal{M}$ equipped with a normal semifinite faithful trace τ, which we denote by $L^p(\mathcal{M}, \tau)$. In Section 2.2, we prove an extension of the result of [67] for orthogonally additive polynomials on C^*-algebras in which we consider polynomials with values in any topological linear space and not necessarily a Banach space. We do so because the representing linear map of a polynomial on $L^p(\mathcal{M}, \tau)$ is supposed to be defined in $L^{p/m}(\mathcal{M}, \tau)$, which is not a Banach space if $m > p$. In Section 2.3, we use that result to prove the linear representation of orthogonally additive polynomials from $L^p(\mathcal{M}, \tau)$ for any $0 < p < \infty$ to any topological linear space.

Reflexive and hyperreflexive subspaces of operators

Let A be a closed subalgebra of the algebra $\mathcal{B}(H)$ of all continuous linear operators on the Hilbert space H. We consider the lattice of all projections onto the A-invariant subspaces of H, $\text{lat}A = \{e \in \mathcal{B}(H) \text{ projection} : e^\perp T e = 0 \ (T \in A)\}$. We can define another subalgebra of $\mathcal{B}(H)$,

$$\text{alglat}A = \big\{T \in \mathcal{B}(H) : e^\perp T e = 0 \ (e \in \text{lat}A)\big\}.$$

The subalgebra A is called *reflexive* if $A = \text{alglat}A$. Note that the inclusion $A \subset \text{alglat}A$ always holds.

If A is a $*$-subalgebra, then $\text{lat}A$ is the set of projections of the commutant of A, and $\text{alglat}A$ is the double commutant of A. Therefore, by the double commutant theorem, von Neumann algebras are precisely those reflexive subalgebras of $\mathcal{B}(H)$ that are also $*$-subalgebras. Indeed, reflexive subalgebras are also closed in the weak operator topology and contain the identity of H.

The algebra A is called *hyperreflexive* if the above condition on A is strengthened by requiring that there is a distance estimate

$$\operatorname{dist}(T,A) \le C\alpha(T,A) \quad (T \in \mathcal{B}(H))$$

for some constant C, where α is the distance defined by lat$A$,

$$\alpha(T,A) = \sup\bigl\{\|e^{\perp}Te\| : e \in \operatorname{lat}A\bigr\} \quad (T \in \mathcal{B}(H)).$$

If $T \in \mathcal{B}(H)$, $S \in A$ and $e \in \operatorname{lat}A$, then $\bigl\|e^{\perp}Te\bigr\| = \bigl\|e^{\perp}(T-S)e\bigr\| \le \|T-S\|$, so the inequality $\alpha(T,A) \le \operatorname{dist}(T,A)$ always holds. Christensen [33, 34, 35] showed that many von Neumann algebras (including injective von Neumann algebras) are hyperreflexive.

If A is a closed subalgebra of $\mathcal{B}(H)$ containing the identity of H, then

$$\alpha(T,A) = \sup_{x\in H, \|x\|\le 1} \inf\{\|T(x)-S(x)\| : S \in A\} \quad (T \in \mathcal{B}(H)).$$

This motivates the extensions of the notions of reflexivity and hyperreflexivity to subspaces of $\mathcal{B}(X,Y)$, where X and Y are Banach spaces. Following Loginov and Shulman [62], a closed linear subspace $\mathcal{S}$ of $\mathcal{B}(X,Y)$ is called reflexive if

$$\mathcal{S} = \bigl\{T \in \mathcal{B}(X,Y) : T(x) \in \overline{\{S(x) : S \in \mathcal{S}\}}\,(x \in X)\bigr\}.$$

It is worth noting that this definition makes sense for subspaces of operators between quasi-Banach spaces. In accordance with Larson [60, 61], $\mathcal{S}$ is called hyperreflexive if there exists a constant C such that

$$\operatorname{dist}(T,\mathcal{S}) \le C \sup_{x\in X,\, \|x\|\le 1} \inf\bigl\{\|T(x)-S(x)\| : S \in \mathcal{S}\bigr\} \quad (T \in \mathcal{B}(X,Y)),$$

and the optimal constant is called the hyperreflexivity constant of $\mathcal{S}$.

Chapter 3 ([9]) is devoted to studying the reflexivity and hyperreflexivity of the space $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ of all (automatically continuous) right $\mathcal{M}$-module homomorphisms from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$ for a von Neumann algebra $\mathcal{M}$. The non-commutative L^p-spaces that we consider are those introduced by Haagerup (see [46, 70, 80]). For each $0 < p \le \infty$, the space $L^p(\mathcal{M})$ is a contractive Banach $\mathcal{M}$-bimodule or a contractive p-Banach $\mathcal{M}$-bimodule according to $1 \le p$ or $p < 1$, and we will focus on the right $\mathcal{M}$-module structure of $L^p(\mathcal{M})$.

Our method relies in the analysis of a continuous bilinear map $\varphi\colon A\times A \to X$, for a C^*-algebra A and a normed space X, through the knowledge of the constant $\sup\{\|\varphi(a,b)\| : a,b \in A_+ \text{ contractions}, ab = 0\}$, alternatively, the constant $\sup\{\|\varphi(e,e^{\perp})\| : e \in A \text{ projection}\}$ in the case where A is unital and has real rank zero. This is done in Section 3.2. The original proof of Theorem 3.2.1 had a mistake that has been corrected in this dissertation.

In Section 3.3, we prove that, for each $0 < p,q \le \infty$, each right $\mathcal{M}$-module homomorphism from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$ is automatically continuous and the space $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is a reflexive subspace of $\mathcal{B}(L^p(\mathcal{M}), L^q(\mathcal{M}))$. In Section 3.4, we prove the hyperreflexivity of $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ in each of the following cases:

(i) $1 \leq q < p \leq \infty$, in which case the hyperreflexivity constant can be chosen to depend on p and q, and not on $\mathcal{M}$;

(ii) $1 \leq p, q \leq \infty$ and $\mathcal{M}$ is injective, in which case the hyperreflexivity constant can be taken to be 8.

Strongly zero product determined Banach algebras

A Banach algebra A is said to be *zero product determined* if, for every continuous bilinear functional $\varphi \in \mathcal{B}^2(A, \mathbb{C})$ satisfying $\varphi(a, b) = 0$ whenever $a, b \in A$ are such that $ab = 0$, there exists a continuous linear functional τ on A such that

$$\varphi(a, b) = \tau(ab) \quad (a, b \in A),$$

i.e., φ belongs to the space $\mathcal{B}^2_\pi(A, \mathbb{C}) = \{\xi \circ \pi : \xi \in A^*\}$ where $\pi \colon A \times A \to A$ denotes the product map. In fact, we could consider bilinear maps with values in any normed space instead of $\mathbb{C}$, in which case the continuous linear map τ is defined on the span of $\{ab : a, b \in A\}$ (see [24, Proposition 4.9]).

Let A be a zero product determined Banach algebra. Suppose that the Banach space of A is a Banach algebra under another multiplication, $*$, such that $a * b = 0$ whenever $ab = 0$. Then there is a continuous linear map τ from the span of $\{ab : a, b \in A\}$ to A such that $a * b = \tau(ab)$. Moreover, if A is unital then $\tau(a) = 1 * a$ $(a \in A)$, whence $a * b = 1 * ab$ $(a, b \in A)$. This motivates the use of the name "zero product determined".

A Banach algebra A has *property* $\mathbb{B}$ if every continuous bilinear functional $\varphi \in \mathcal{B}^2(A, \mathbb{C})$ with the property that $\varphi(a, b) = 0$ whenever $a, b \in A$ are such that $ab = 0$ satisfies

$$\varphi(ab, c) = \varphi(a, bc) \quad (a, b, c \in A).$$

Property $\mathbb{B}$ was introduced in [2] following what was called property $\mathbb{A}$. Thus, there are no deeper reasons for the name "property $\mathbb{B}$". In that same article, it was shown that this class of Banach algebras is wide enough to include a number of examples of interest: C^*-algebras, the group algebra $L^1(G)$ of any locally compact group G, and the algebra $\mathcal{A}(X)$ of approximable operators on any Banach space X.

It is evident that every zero product determined Banach algebra has property $\mathbb{B}$, and the reciprocal is true for Banach algebras having a bounded left approximate identity (Proposition 4.2.1). Therefore, C^*-algebras, group algebras and the algebra $\mathcal{A}(X)$ of any Banach space X having the bounded approximation property are zero product determined Banach algebras.

For each $\varphi \in \mathcal{B}^2(A, \mathbb{C})$ we consider the constants

$$|\varphi|_{zp} = \sup \{|\varphi(a, b)| : a, b \in A,\ \|a\| = \|b\| = 1,\ ab = 0\}$$

and

$$|\varphi|_b = \sup \{|\varphi(ab, c) - \varphi(a, bc)| : a, b, c \in A,\ \|a\| = \|b\| = \|c\| = 1\}.$$

Note that a Banach algebra A is zero product determined precisely when

$$\varphi \in \mathcal{B}^2(A,\mathbb{C}),\ |\varphi|_{zp} = 0 \implies \varphi \in \mathcal{B}^2_\pi(a,\mathbb{C}).$$

A is said to be *strongly zero product determined* if this condition is strengthened by requiring that there is a distance estimate

$$\operatorname{dist}\big(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\big) \le \alpha\,|\varphi|_{zp} \quad \forall \varphi \in \mathcal{B}^2(A,\mathbb{C})$$

for some constant α; in this case, the optimal constant α will be denoted by α_A.

Furthermore, A having property $\mathbb{B}$ is equivalent to

$$\varphi \in \mathcal{B}^2(A,\mathbb{C})\ \ |\varphi|_{zp} = 0 \implies |\varphi|_b = 0,$$

so we say that A has *strong property* $\mathbb{B}$ if

$$|\varphi|_b \le \beta\,|\varphi|_{zp} \quad \forall \varphi \in \mathcal{B}^2(A,\mathbb{C})$$

for some constant β; in this case, the optimal constant β will be denoted by β_A. The spirit of this concept first appeared in [11], and was subsequently formulated in [76] and refined in [77]. This property has proven to be useful to study the hyperreflexivity of the spaces of continuous derivations and, more generally, continuous cocycles on A (see [12, 13, 75, 76, 77]).

From Corollary 3.2.3, we obtain that if A is a C^*-algebra, then A is strongly zero product determined, has the strong property $\mathbb{B}$, and $\alpha_A, \beta_A \le 8$. It is shown in [77] that each group algebra has the strong property $\mathbb{B}$ and so (by Corollary 4.2.2) it is also strongly zero product determined.

Chapter 4 ([10]) is concerned with the study of these properties. In Section 4.2, we present some estimates that relate the seminorms $\operatorname{dist}\big(\cdot, \mathcal{B}^2_\pi(A,\mathbb{C})\big)$, $|\cdot|_b$, and $|\cdot|_{zp}$ to each other. In Section 4.3, we provide an estimate of the constants $\alpha_{L^1(G)}$ and $\beta_{L^1(G)}$ for each locally compact group G. Our estimate of $\beta_{L^1(G)}$ improves the one given in [77]. Finally, in Section 4.4 we prove that the algebra $\mathcal{A}(X)$ is strongly zero product determined for each Banach space X having property $(\mathbb{A})$ (which is a rather strong approximation property for the space X) and we use this result to show that the space $\mathcal{Z}^n(\mathcal{A}(X), Y^*)$ of continuous n-cocycles from $\mathcal{A}(X)$ into Y^* is hyperreflexive for each Banach $\mathcal{A}(X)$-bimodule Y.

Weighted Jordan homomorphisms

Let A and B be algebras. A linear map $\Phi\colon A \to B$ is said to be a *Jordan homomorphism* if it preserves Jordan products, i.e., if

$$\Phi(a \circ b) = \Phi(a) \circ \Phi(b) \quad (a, b \in A),$$

where $a \circ b$ stands for the Jordan product $ab + ba$. Obvious examples of Jordan homomorphisms are homomorphisms and antihomomorphisms. Under the mild assumption that the centre of B does not contain nonzero nilpotent elements, every Jordan homomorphism from A onto B also *preserves two-sided zero products* ([24, Lemma 7.20]), i.e.,

$$a, b \in A,\ ab = ba = 0 \implies \Phi(a)\Phi(b) = \Phi(b)\Phi(a) = 0.$$

A linear map $W\colon B \to B$ is called a *centralizer* if

$$W(ab) = W(a)b = aW(b) \quad (a, b \in A).$$

We say that Φ is a *weighted Jordan homomorphism* if there exist an invertible centralizer W of B and a Jordan homomorphism Ψ from A to B such that $\Phi = W\Psi$. Observe that Φ preserves two-sided zero products if and only if Ψ does.

Our aim is to know if every surjective linear (and continuous in the case where A and B are Banach algebras) map $\Phi\colon A \to B$ which preserves two-sided zero products is a weighted Jordan homomorphism. This problem is similar to but, as it turns out, more difficult than a more thoroughly studied question of describing zero products preserving continuous linear maps (see the most recent publications [24, 41, 42, 57, 63] for historical remarks and references). It is known that the answer is positive if either A and B are C^*-algebras [3, Theorem 3.3] or if $A = L^1(G)$ and $B = L^1(H)$ where G and H are locally compact groups with $G \in$ [SIN] (i.e., G has a base of compact neighborhoods of the identity that is invariant under all inner automorphisms) [14, Theorem 3.1 (i)]. In fact, [3, Theorem 3.3] does not require that Φ preserves two-sided zero products but only that it satisfies that

$$a, b \in A,\ ab = ba = 0 \implies \Phi(a) \circ \Phi(b) = 0.$$

This condition is also more general than the condition that Φ preserves zero Jordan products ($\Phi(a) \circ \Phi(b) = 0$ whenever $a \circ b = 0$) studied in [32].

The starting point when studying the operator Φ is to analyse the bilinear map $(a, b) \mapsto \Phi(a)\Phi(b)$ $(a, b \in A)$ if Φ preserves two-sided zero products, or $(a, b) \mapsto \Phi(a) \circ \Phi(b)$ $(a, b \in A)$ in case Φ satisfies the more general condition.

In Chapter 5 ([27]), we study this condition on the algebraic context of unital rings. If A and B are unital rings, then a centralizer $W\colon B \to B$ satisfies that $W(a) = aW(1) = W(1)a$ $(a \in A)$, so an additive map $\Phi\colon A \to B$ is a weighted Jordan homomorphism if $c = \Phi(1)$ is a invertible element of the centre of B and $\Phi(a) \circ \Phi(b) = c\Phi(a \circ b)$. In [32] it was proved that if R is a unital ring with $\frac{1}{2}$ and $A = \mathbb{M}_n(R)$ with $n \geq 4$, then surjective additive maps from A onto itself that preserve zero Jordan products are weighted Jordan homomorphisms.

In Section 5.2, we show that a surjective additive map $\Phi\colon A \to B$ satisfying $\Phi(a) \circ \Phi(b) = 0$ whenever $ab = ba = 0$ is a weighted Jordan homomorphism provided that the ring A is additively spanned by Jordan products of its idempotents and B is any ring with $\frac{1}{2}$. The condition on idempotents is fulfilled in any matrix ring $\mathbb{M}_n(R)$ with $n \geq 2$, so this theorem yields a generalization and completion of the aforementioned result of [32]. In Section 5.3, we show that if A is a prime ring with $\mathrm{char}(A) \neq 2, 3, 5$ and $\Phi : A \to A$ is a bijective additive map for which there exists an additive map $\Psi : A \to A$ such that $\Psi(a \circ b) = \Phi(a) \circ \Phi(b)$ $(a, b \in A)$, then Φ is a weighted Jordan homomorphism.

The goal of Chapter 6 ([28]) is to generalize and unify the aforementioned results from [3] and [14]. In Section 6.2, we show that a surjective continuous linear map $\Phi\colon A \to B$ between Banach algebras which preserves two-sided zero products is a weighted Jordan homomorphism provided that A is zero product determined and weakly amenable, and, additionally, both A and B have bounded approximate identities. This in particular implies that the restriction in [14, Theorem 3.1 (i)] that $G \in$ [SIN] is redundant. In Section 6.3, we show that [3, Theorem 3.3] still holds if B is any Banach algebra with a bounded approximate identity, not only a C^*-algebra, and that if X is a Banach space such that X^* has the bounded approximation property and B has a bounded approximated identity, then any continuous linear map $\Phi\colon \mathcal{A}(X) \to B$ satisfying $\Phi(a) \circ \Phi(b) = 0$ whenever $ab = ba = 0$ is a weighted Jordan homomorphism.

Chapter 1

Orthogonally additive polynomials on convolution algebras associated with a compact group

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Abstract. Let G be a compact group, let X be a Banach space, and let $P\colon L^1(G) \to X$ be an orthogonally additive, continuous n-homogeneous polynomial. Then we show that there exists a unique continuous linear map $\Phi\colon L^1(G) \to X$ such that $P(f) = \Phi\big(f * \overset{n}{\cdots} * f\big)$ for each $f \in L^1(G)$. We also seek analogues of this result about $L^1(G)$ for various other convolution algebras, including $L^p(G)$, for $1 < p \leq \infty$, and $C(G)$.

1.1 Introduction

Throughout all algebras and linear spaces are complex. Of course, linearity is understood to mean complex linearity. Moreover, we fix $n \in \mathbb{N}$ with $n \geq 2$.

Let X and Y be linear spaces. A map $P\colon X \to Y$ is said to be an *n-homogeneous polynomial* if there exists an n-linear map $\varphi\colon X^n \to Y$ such that $P(x) = \varphi(x, \ldots, x)$ $(x \in X)$. Here and subsequently, X^n stands for the n-fold Cartesian product of X. Such a map is unique if it is required to be symmetric. This is a consequence of the so-called polarization formula which defines φ by

$$\varphi(x_1, \ldots, x_n) = \frac{1}{n!\, 2^n} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \epsilon_1 \cdots \epsilon_n P(\epsilon_1 x_1 + \cdots + \epsilon_n x_n)$$

for all $x_1, \ldots, x_n \in X$. Further, in the case where X and Y are normed spaces, the polynomial P is continuous if and only if the symmetric n-linear map φ associated with P is continuous. Let A be an algebra. Then the map $P_n \colon A \to A$ defined by

$$P_n(a) = a^n \quad (a \in A)$$

is a prototypical example of n-homogeneous polynomial. The symmetric n-linear map associated with P_n is the map $S_n \colon A^n \to A$ defined by

$$S_n\left(a_1, \ldots, a_n\right) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} a_{\sigma(1)} \cdots a_{\sigma(n)} \quad \left(a_1, \ldots, a_n \in A\right),$$

where $\mathfrak{S}_n$ stands for the symmetric group of order n. Throughout, we will frequently use this map S_n. From now on, we write $\mathcal{P}_n(A)$ for the linear span of the set $\{a^n : a \in A\}$. Given a linear space Y and a linear map $\Phi \colon \mathcal{P}_n(A) \to Y$, the map $P \colon A \to Y$ defined by

$$P(a) = \Phi\left(a^n\right) \quad (a \in A) \tag{1.1}$$

yields a particularly important example of n-homogeneous polynomial, and one might wish to know an algebraic characterization of those n-homogeneous polynomials $P \colon A \to Y$ which can be expressed in the form (1.1). Further, in the case where A is a Banach algebra, Y is a Banach space, and the n-homogeneous polynomial $P \colon A \to Y$ is continuous, one should particularly like that the map Φ of (1.1) be continuous. A property that has proven valuable for this purpose is the so-called orthogonal additivity. Let A be an algebra and let Y be a linear space. A map $P \colon A \to Y$ is said to be *orthogonally additive* if

$$a, b \in A, \ ab = ba = 0 \ \Rightarrow \ P(a+b) = P(a) + P(b).$$

The polynomial defined by (1.1) is a prototypical example of orthogonally additive n-homogeneous polynomial, and the obvious questions that one can address are the following.

Q1 Let A be a specified algebra. Is it true that every orthogonally additive n-homogeneous polynomial P from A into each linear space Y can be expressed in the standard form (1.1) for some linear map $\Phi \colon \mathcal{P}_n(A) \to Y$?

Q2 Let A be a specified Banach algebra. Is it true that every orthogonally additive continuous n-homogeneous polynomial P from A into each Banach space Y can be expressed in the standard form (1.1) for some continuous linear map $\Phi \colon \mathcal{P}_n(A) \to Y$?

Q3 Let A be a specified Banach algebra. Is there any norm $|||\cdot|||$ on $\mathcal{P}_n(A)$ with the property that the orthogonally additive continuous n-homogeneous polynomials from A into each Banach space Y are exactly the polynomials of the form (1.1) for some $|||\cdot|||$-continuous linear map $\Phi \colon \mathcal{P}_n(A) \to Y$?

It seems to be convenient to remark that the demand of Q3 results precisely in the following two conditions:

- for each Banach space Y and each $|||\cdot|||$-continuous linear map $\Phi\colon \mathcal{P}_n(A) \to Y$, the prototypical polinomial $P\colon A \to Y$ defined by (1.1) is continuous, and
- every orthogonally additive continuous n-homogeneous polynomial P from A into each Banach space Y can be expressed in the standard form (1.1) for some $|||\cdot|||$-continuous linear map $\Phi\colon \mathcal{P}_n(A) \to Y$.

It is shown in [67] that the answer to Question Q2 is positive in the case where A is a C^*-algebra (see [21, 29, 69] for the case where $A = C(K)$, and see [43, 68] for the case where A is a C^*-algebra and P is a holomorphic map). The references [7, 15, 81, 82, 83] discuss Question Q2 for a variety of Banach function algebras, including the Fourier algebra $A(G)$ and the Figà-Talamanca-Herz algebra $A_p(G)$ of a locally compact group G. It is worth pointing out that the representation of orthogonally additive homogeneous polynomials has been widely discussed in the context of Banach lattices (see [21, 53] and the references therein).

This paper focuses on the questions Q1, Q2, and Q3 mentioned above for a variety of convolution algebras associated with a compact group G, such as $L^p(G)$, for $1 \le p \le \infty$, and $C(G)$. In contrast to the previous references, that are concerned with C^*-algebras and commutative Banach algebras, the algebras in this work are neither C^* nor commutative with respect to convolution.

Throughout, we are concerned with a compact group G whose Haar measure is normalized. We write $\int_G f(t)\,dt$ for the integral of $f \in L^1(G)$ with respect to the Haar measure. For $f \in L^1(G)$, we denote by f^{*n} the n-fold convolution product $f * \cdots * f$. We denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G. Let π be an irreducible unitary representation of G on a Hilbert space H_π. We set $d_\pi = \dim(H_\pi)(< \infty)$, and the character χ_π of π is the continuous function on G defined by

$$\chi_\pi(t) = \operatorname{trace}\bigl(\pi(t)\bigr) \quad (t \in G).$$

We write $\mathcal{T}_\pi(G)$ for the linear subspace of $C(G)$ generated by the set of continuous functions on G of the form $t \mapsto \langle \pi(t)u|v\rangle$ as u and v range over H_π. It should be pointed out that χ_π and $\mathcal{T}_\pi(G)$ depend only on the unitary equivalence class of π. We write $\mathcal{T}(G)$ for the linear span of the functions in $\mathcal{T}_\pi(G)$ as $[\pi]$ ranges over $\widehat{G}$. Then $\mathcal{T}(G)$ is a two-sided ideal of $L^1(G)$ whose elements are called trigonometric polynomials on G. The Fourier transform of a function $f \in L^1(G)$ at π is defined to be the operator

$$\widehat{f}(\pi) = \int_G f(t)\pi(t^{-1})\,dt$$

on H_π. Note that if π' is equivalent to π, then the operators $\widehat{f}(\pi')$ and $\widehat{f}(\pi)$ are unitarily equivalent.

In Section 1.2 we show that the answer to Question Q1 is positive for the algebra $\mathcal{T}(G)$. In Section 1.3 we show that the answer to Question Q2 is positive for the group algebra $L^1(G)$. In Section 1.4 we give a negative answer to Question Q2 for any of the convolution algebras $L^p(\mathbb{T})$, for $1 < p \le \infty$, and $C(\mathbb{T})$, where $\mathbb{T}$ denotes the circle group.

In Section 1.5 we prove that, for each Banach algebra A, there exists a largest norm topology on the linear space $\mathcal{P}_n(A)$ for which the answer to Question Q3 can be positive. Finally, in Section 1.6 we show that the answer to Question Q3 is positive for most of the significant convolution algebras associated to G, such as $L^p(G)$, for $1 < p < \infty$, and $C(G)$, when considering the norm introduced in Section 1.5.

We presume a basic knowledge of Banach algebra theory, harmonic analysis for compact groups, and polynomials on Banach spaces. For the relevant background material concerning these topics, see [36], [52], and [64], respectively.

1.2 Orthogonally additive polynomials on $\mathcal{T}(G)$

Our starting point is furnished by applying [67] to the full matrix algebra $\mathbb{M}_k$ of order k (which supplies the most elementary example of C^*-algebra).

Lemma 1.2.1. *Let $\mathcal{M}$ be an algebra isomorphic to $\mathbb{M}_k$ for some $k \in \mathbb{N}$, let X be a linear space, and let $P\colon \mathcal{M} \to X$ be an orthogonally additive n-homogeneous polynomial. Then there exists a unique linear map $\Phi\colon \mathcal{M} \to X$ such that $P(a) = \Phi(a^n)$ for each $a \in \mathcal{M}$. Further, if $\varphi\colon \mathcal{M}^n \to X$ is the symmetric n-linear map associated with P and e is the identity of $\mathcal{M}$, then $\Phi(a) = \varphi(a, e, \dots, e)$ for each $a \in \mathcal{M}$.*

Proof. Let $\Psi\colon \mathcal{M} \to \mathbb{M}_k$ be an isomorphism. Endow X with a norm, and let Y be its completion. Since $\mathbb{M}_k$ is a C^*-algebra and the map $P \circ \Psi^{-1}\colon \mathbb{M}_k \to Y$ is a continuous orthogonally additive n-homogeneous polynomial, [67, Corollary 3.1] then shows that there exists a unique linear map $\Theta\colon \mathbb{M}_k \to Y$ such that $P\big(\Psi^{-1}(M)\big) = \Theta(M^n)$ for each $M \in \mathbb{M}_k$. It is a simple matter to check that the map $\Phi = \Theta \circ \Psi$ satisfies the identity $P(a) = \Phi(a^n)$ $(a \in A)$. Now the polarization of this identity yields $\varphi(a_1, \dots, a_n) = \Phi\big(S_n(a_1, \dots, a_n)\big)$ $(a_1, \dots, a_n \in \mathcal{M})$, whence $\varphi(a, e, \dots, e) = \Phi(a)$ for each $a \in \mathcal{M}$. □

In what follows, we will require some elementary facts about the algebra $\mathcal{T}(G)$; we gather together these facts here for reference.

Lemma 1.2.2. *Let G be a compact group. Then the following results hold.*

(1) For each irreducible unitary representation π, $\mathcal{T}_\pi(G)$ is a minimal two-sided ideal of $L^1(G)$, $\mathcal{T}_\pi(G)$ is isomorphic to the full matrix algebra $\mathbb{M}_{d_\pi}$, and $d_\pi\chi_\pi$ is the identity of $\mathcal{T}_\pi(G)$.

(2) For each $f \in \mathcal{T}(G)$, the set

$$\big\{[\pi] \in \widehat{G} : f * \chi_\pi \neq 0\big\}$$

is finite and

$$f = \sum_{[\pi]\in\widehat{G}} d_\pi f * \chi_\pi.$$

Proof. (1) [52, Theorems 27.21 and 27.24(ii)].

(2) [52, Remark 27.8(a) and Theorem 27.24(ii)]. ☐

Theorem 1.2.3. *Let G be a compact group, let X be a linear space, and let $P\colon \mathcal{T}(G) \to X$ be an orthogonally additive n-homogeneous polynomial. Then there exists a unique linear map $\Phi\colon \mathcal{T}(G) \to X$ such that*

$$P(f) = \Phi\left(f^{*n}\right)$$

for each $f \in \mathcal{T}(G)$. Further, if $\varphi\colon \mathcal{T}(G)^n \to X$ is the symmetric n-linear map associated with P, then

$$\Phi(f) = \sum_{[\pi]\in\widehat{G}} \varphi\left(d_\pi f * \chi_\pi, d_\pi\chi_\pi, \dots, d_\pi\chi_\pi\right)$$

(the term $f * \chi_\pi$ being 0 for all but finitely many $[\pi] \in \widehat{G}$) for each $f \in \mathcal{T}(G)$.

Proof. We first show that

$$P(f) = \sum_{[\pi]\in\widehat{G}} P\left(d_\pi f * \chi_\pi\right) \tag{1.2}$$

for each $f \in \mathcal{T}(G)$. Set $f \in \mathcal{T}(G)$. By Lemma 1.2.2, we have

$$f = \sum_{[\pi]\in\widehat{G}} d_\pi f * \chi_\pi \tag{1.3}$$

and the set $\left\{[\pi] \in \widehat{G} : f * \chi_\pi \neq 0\right\}$ is finite, so that all save finitely many of the summands of the right-hand side of (1.3) are 0. Further, from [52, Theorem 27.24(ii)-(iii)] we see that

$$(f * \chi_\pi) * (f * \chi_{\pi'}) = f * f * \chi_\pi * \chi_{\pi'} = 0,$$

whenever $[\pi], [\pi'] \in \widehat{G}$ and $[\pi] \neq [\pi']$. The orthogonal additivity of P and (1.3) then yield(1.2).

Now set $[\pi] \in \widehat{G}$. On account of Lemma 1.2.2(1), $\mathcal{T}_\pi(G)$ is isomorphic to the full matrix algebra $\mathbb{M}_{d_\pi}$ and $d_\pi\chi_\pi$ is the identity of $\mathcal{T}_\pi(G)$. Hence, by Lemma 1.2.1, we have

$$P(f) = \varphi\left(f^{*n}, d_\pi\chi_\pi, \dots, d_\pi\chi_\pi\right)$$

for each $f \in \mathcal{T}_\pi(G)$. In particular, if $f \in \mathcal{T}(G)$, then $f * d_\pi\chi_\pi \in \mathcal{T}_\pi(G)$ and thus

$$\begin{aligned} P\left(d_\pi f * \chi_\pi\right) &= \varphi\big((f * d_\pi\chi_\pi)^{*n}, d_\pi\chi_\pi, \dots, d_\pi\chi_\pi\big) \\ &= \varphi\big(f^{*n} * d_\pi\chi_\pi, d_\pi\chi_\pi, \dots, d_\pi\chi_\pi\big). \end{aligned} \tag{1.4}$$

From (1.2) and (1.4) we conclude that

$$\begin{aligned} P(f) &= \sum_{[\pi]\in\widehat{G}} P\left(d_\pi f * \chi_\pi\right) \\ &= \sum_{[\pi]\in\widehat{G}} \varphi\left(d_\pi f^{*n} * \chi_\pi, d_\pi\chi_\pi, \dots, d_\pi\chi_\pi\right) \end{aligned}$$

for each $f \in \mathcal{T}(G)$, which completes the proof. ☐

1.3 Orthogonally additive polynomials on $L^1(G)$

A key fact in what follows is that the Banach algebra $L^1(G)$ has a central bounded approximate identity consisting of trigonometric polynomials. Indeed, by [52, Theorem 28.53], there exists a bounded approximate identity $(h_\lambda)_{\lambda\in\Lambda}$ for $L^1(G)$ such that:

- $\|h_\lambda\|_1 = 1$ for each $\lambda \in \Lambda$;
- $h_\lambda * f = f * h_\lambda$ for all $f \in L^1(G)$ and $\lambda \in \Lambda$;
- $\widehat{h_\lambda}(\pi) = \alpha_\lambda(\pi) I_\pi$ for some $\alpha_\lambda(\pi) \in \mathbb{R}_0^+$ for all $[\pi] \in \widehat{G}$ and $\lambda \in \Lambda$ (here I_π denotes the identity operator on H_π);
- $\lim_{\lambda\in\Lambda} \alpha_\lambda(\pi) = 1$ for each $[\pi] \in \widehat{G}$.

This approximate identity will be used repeatedly hereafter.

Theorem 1.3.1. *Let G be a compact group, let X be a Banach space, and let $P\colon L^1(G) \to X$ be a continuous n-homogeneous polynomial. Then the following conditions are equivalent:*

(1) the polynomial P is orthogonally additive;

(2) the polynomial P is orthogonally additive on $\mathcal{T}(G)$, i.e., $P(f+g) = P(f) + P(g)$ whenever f, $g \in \mathcal{T}(G)$ are such that $f * g = g * f = 0$;

(3) there exists a unique continuous linear map $\Phi\colon L^1(G) \to X$ such that $P(f) = \Phi\left(f^{*n}\right)$ for each $f \in L^1(G)$.

Proof. It is clear that (1) $\Rightarrow$ (2) and that (3) $\Rightarrow$ (1). We will henceforth prove that (2) $\Rightarrow$ (3).

Let $\varphi\colon L^1(G)^n \to X$ be the symmetric n-linear map associated with P, and let $\Phi_0\colon \mathcal{T}(G) \to X$ be the linear map defined by

$$\Phi_0(f) = \sum_{[\pi]\in\widehat{G}} \varphi\left(d_\pi f * \chi_\pi, d_\pi\chi_\pi, \ldots, d_\pi\chi_\pi\right)$$

for each $f \in \mathcal{T}(G)$. Since P is orthogonally additive on $\mathcal{T}(G)$, Theorem 1.2.3 yields

$$P(f) = \Phi_0\left(f^{*n}\right) \quad (f \in \mathcal{T}(G)). \tag{1.5}$$

We claim that Φ_0 is continuous. Let $(h_\lambda)_{\lambda\in\Lambda}$ be as introduced in the beginning of this section. We now note that, for $\lambda \in \Lambda$ and $[\pi] \in \widehat{G}$,

$$\begin{aligned}
\big(h_\lambda * \chi_\pi\big)(t) &= \int_G h_\lambda(s)\operatorname{trace}\big(\pi(s^{-1}t)\big)\,ds \\
&= \operatorname{trace}\left(\int_G h_\lambda(s)\pi(s^{-1}t)\,ds\right) \\
&= \operatorname{trace}\left(\int_G h_\lambda(s)\pi(s^{-1})\pi(t)\,ds\right) \\
&= \operatorname{trace}\left(\left(\int_G h_\lambda(s)\pi(s^{-1})\,ds\right)\pi(t)\right) \\
&= \operatorname{trace}\big(\widehat{h_\lambda}(\pi)\pi(t)\big) \\
&= \operatorname{trace}\big(\alpha_\lambda(\pi)\pi(t)\big) \\
&= \alpha_\lambda(\pi)\operatorname{trace}\big(\pi(t)\big) \\
&= \alpha_\lambda(\pi)\chi_\pi(t).
\end{aligned}$$

From Theorem 1.2.3 and the polarization formula we deduce that

$$\begin{aligned}
\varphi\left(f_1,\dots,f_n\right) &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\Phi_0\left(f_{\sigma(1)}*\cdots*f_{\sigma(n)}\right) \\
&= \frac{1}{n!}\sum_{[\pi]\in\widehat{G}}\sum_{\sigma\in\mathfrak{S}_n}\varphi\big(d_\pi f_{\sigma(1)}*\cdots*f_{\sigma(n)}*\chi_\pi, d_\pi\chi_\pi,\dots,d_\pi\chi_\pi\big)
\end{aligned}$$

for all $f_1,\dots,f_n \in \mathcal{T}(G)$. Pick $f \in \mathcal{T}(G)$, and set $\mathcal{F} = \big\{[\pi] : f * \chi_\pi \neq 0\big\}$. We apply the above equation in the case where $f_1 = f$ and $f_2 = \cdots = f_n = h_\lambda$ with $\lambda \in \Lambda$. Since

$$f_{\sigma(1)}*\cdots*f_{\sigma(n)}*\chi_\pi = f*h_\lambda*\cdots*h_\lambda*\chi_\pi = \alpha_\lambda(\pi)^{n-1}f*\chi_\pi$$

for each $\sigma \in \mathfrak{S}_n$, it follows that

$$\begin{aligned}
\varphi\left(f,h_\lambda,\dots,h_\lambda\right) &= \sum_{[\pi]\in\widehat{G}}\alpha_\lambda(\pi)^{n-1}\varphi\left(d_\pi f*\chi_\pi, d_\pi\chi_\pi,\dots,d_\pi\chi_\pi\right) \\
&= \sum_{[\pi]\in\mathcal{F}}\alpha_\lambda(\pi)^{n-1}\varphi\left(d_\pi f*\chi_\pi, d_\pi\chi_\pi,\dots,d_\pi\chi_\pi\right).
\end{aligned}$$

Since $\mathcal{F}$ is finite (Lemma 1.2.2(2)) and $\lim_{\lambda\in\Lambda}\alpha_\lambda(\pi) = 1$ for each $[\pi] \in \widehat{G}$, we see that the net $\big(\varphi\left(f,h_\lambda,\dots,h_\lambda\right)\big)_{\lambda\in\Lambda}$ is convergent and

$$\lim_{\lambda\in\Lambda}\varphi\left(f,h_\lambda,\dots,h_\lambda\right) = \sum_{[\pi]\in\mathcal{F}}\varphi\left(d_\pi f*\chi_\pi, d_\pi\chi_\pi,\dots,d_\pi\chi_\pi\right) = \Phi_0(f). \tag{1.6}$$

On the other hand, for each $\lambda \in \Lambda$, we have

$$\left\|\varphi\left(f,h_\lambda,\dots,h_\lambda\right)\right\| \leq \|\varphi\|\,\|h_\lambda\|^{n-1}\,\|f\| \leq \|\varphi\|\,\|f\|\,. \tag{1.7}$$

By (1.6) and (1.7),

$$\|\Phi_0(f)\| \le \|\varphi\| \, \|f\| \, ,$$

which gives the continuity of Φ_0.

Since $\mathcal{T}(G)$ is dense in $L^1(G)$ and Φ_0 is continuous, there exists a unique continuous linear map $\Phi\colon L^1(G) \to X$ which extends Φ_0. Since both P and Φ are continuous, (1.5) gives $P(f) = \Phi\left(f^{*n}\right)$ for each $f \in L^1(G)$.

Our final task is to prove the uniqueness of the map Φ. Suppose that $\Psi\colon L^1(G) \to X$ is a continuous linear map such that $P(f) = \Psi\left(f^{*n}\right)$ for each $f \in L^1(G)$. By Theorem 1.2.3, $\Psi(f) = \Phi(f)\big(= \Phi_0(f)\big)$ for each $f \in \mathcal{T}(G)$. Since $\mathcal{T}(G)$ is dense in $L^1(G)$, and both Φ and Ψ are continuous, it follows that $\Psi(f) = \Phi(f)$ for each $f \in L^1(G)$. □

1.4 Orthogonally additive polynomials on the convolution algebras $L^p(\mathbb{T})$, $1 < p \le \infty$, and $C(\mathbb{T})$

The next examples show that, if A is any of the convolution algebras $L^p(\mathbb{T})$, for $1 < p \le \infty$, or $C(\mathbb{T})$, then there exists an orthogonally additive, continuous 2-homogeneous polynomial $P\colon A \to \mathbb{C}$ which cannot be expressed in the form $P(f) = \Phi\left(f * f\right)$ $(f \in A)$ for any continuous linear functional $\Phi\colon A \to \mathbb{C}$. Throughout this section, $\mathbb{T}$ denotes the circle group $\{z \in \mathbb{C} : |z| = 1\}$, and, for $f \in L^1(\mathbb{T})$ and $k \in \mathbb{Z}$, $\widehat{f}(k)$ denotes the kth Fourier coefficient of f. For each $k \in \mathbb{Z}$, let $\chi_k\colon \mathbb{T} \to \mathbb{C}$ be the function defined by

$$\chi_k(z) = z^k \quad (z \in \mathbb{T}).$$

Then

$$\chi_k * \chi_k = \chi_k \quad (k \in \mathbb{Z}) \tag{1.8}$$

and

$$\widehat{\chi_k}(j) = \delta_{jk} \quad (j, k \in \mathbb{Z}). \tag{1.9}$$

Example 1.4.1. Assume that $1 < p < 2$. Set $q = \frac{p}{p-1}$, $r = \frac{p}{2-p}$, and $s = \frac{q}{2}$, so that $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{r} + \frac{1}{s} = 1$. Take $h \in L^p(\mathbb{T})$ such that

$$\sum_{k=-\infty}^{+\infty} |\widehat{h}(k)|^s = +\infty. \tag{1.10}$$

Such a choice is possible because of [40, 13.5.3(1)], since $s < q$. We claim that there exists $a \in \ell^r(\mathbb{Z})$ such that the sequence $\left(\sum_{k=-m}^{m} a(k)\widehat{h}(k)\right)$ does not converge. To see this, we define the sequence (ϕ_m) in the dual of $\ell^r(\mathbb{Z})$ by

$$\phi_m(a) = \sum_{k=-m}^{m} a(k)\widehat{h}(k) \quad (a \in \ell^r(\mathbb{Z}),\ m \in \mathbb{N}).$$

It is immediate to check that

$$\|\phi_m\| = \left(\sum_{k=-m}^{m} \left|\widehat{h}(k)\right|^s \right)^{1/s} \quad (m \in \mathbb{N}).$$

From (1.10) we deduce that $(\|\phi_m\|)$ is unbounded, and the Banach-Steinhaus theorem then shows that there exists $a \in \ell^r(\mathbb{Z})$ such that $(\phi_m(a))$ does not converge, as claimed.

Let $f \in L^p(\mathbb{T})$. Then the Hausdorff-Young theorem ([40, p. 13.5.1]) yields $\|\widehat{f}\,\|_q \leq \|f\|_p$. By Hölder's inequality, we have

$$\sum_{k=-\infty}^{+\infty} \left|a(k)\widehat{f}(k)^2\right| \leq \|a\|_r \left\|\widehat{f}^{\,2}\right\|_s = \|a\|_r \left\|\widehat{f}\,\right\|_q^2 \leq \|a\|_r \|f\|_p^2 .$$

This allows us to define an orthogonally additive continuous 2-homogeneous polynomial $P\colon L^p(\mathbb{T}) \to \mathbb{C}$ by

$$P(f) = \sum_{k=-\infty}^{+\infty} a(k)\widehat{f * f}(k) = \sum_{k=-\infty}^{+\infty} a(k)\widehat{f}(k)^2 \quad (f \in L^p(\mathbb{T})).$$

Suppose that P can be expressed as $P(f) = \Phi\left(f * f\right)$ $(f \in L^p(\mathbb{T}))$ for some continuous linear functional $\Phi\colon L^p(\mathbb{T}) \to \mathbb{C}$. By (1.8) and(1.9), we have

$$\Phi(\chi_k) = \Phi(\chi_k * \chi_k) = P(\chi_k) = \sum_{j=-\infty}^{+\infty} a(j)\widehat{\chi_k}(j) = \sum_{j=-\infty}^{+\infty} a(j)\delta_{jk} = a(k)$$

for each $k \in \mathbb{Z}$. If $f \in L^p(\mathbb{T})$, then Riesz's theorem [40, p. 12.10.1] shows that the sequence $(\sum_{k=-m}^{m} \widehat{f}(k)\chi_k)$ converges to f in $L^p(\mathbb{T})$. Since Φ is continuous, it follows that the sequence

$$\left(\Phi\left(\sum_{k=-m}^{m} \widehat{f}(k)\chi_k \right) \right) = \left(\sum_{k=-m}^{m} \widehat{f}(k)\Phi(\chi_k) \right) = \left(\sum_{k=-m}^{m} \widehat{f}(k)a(k) \right)$$

converges to $\Phi(f)$. In particular, the sequence $\left(\sum_{k=-m}^{m} \widehat{h}(k)a(k)\right)$ is convergent, which contradicts the choice of h.

Example 1.4.2. Assume that $2 \leq p < \infty$. If $f \in L^p(\mathbb{T})$, then $f \in L^2(\mathbb{T})$ and therefore $\|\widehat{f}\,\|_2 = \|f\|_2 \leq \|f\|_p$. This allows us to define an orthogonally additive continuous 2-homogeneous polynomial $P\colon L^p(\mathbb{T}) \to \mathbb{C}$ by

$$P(f) = \sum_{k=-\infty}^{+\infty} \widehat{f * f}(k) = \sum_{k=-\infty}^{+\infty} \widehat{f}(k)^2 \quad (f \in L^p(\mathbb{T})).$$

Suppose that P can be represented in the form $P(f) = \Phi\left(f * f\right)$ $(f \in L^p(\mathbb{T}))$ for some continuous linear functional $\Phi\colon L^p(\mathbb{T}) \to \mathbb{C}$. Let $h \in L^q(\mathbb{T})$ be such that

$$\Phi(f) = \int_{\mathbb{T}} f(z)h(z)\,dz \quad (f \in L^p(\mathbb{T})). \tag{1.11}$$

By (1.8) and (1.9), for each $k \in \mathbb{Z}$, we have

$$\Phi(\chi_k) = \Phi(\chi_k * \chi_k) = P(\chi_k) = \sum_{j=-\infty}^{+\infty} \widehat{\chi_k}(j) = \sum_{j=-\infty}^{+\infty} \delta_{jk} = 1,$$

and (1.11) then yields

$$\widehat{h}(k) = \int_{\mathbb{T}} z^{-k} h(z)\, dz = \Phi(\chi_{-k}) = 1,$$

contrary to Riemann-Lebesgue lemma.

Example 1.4.3. If $f \in L^\infty(\mathbb{T})$, then $f \in L^2(\mathbb{T})$ and therefore $\|\widehat{f}\|_2 = \|f\|_2 \leq \|f\|_\infty$, which implies that

$$\sum_{k=0}^{+\infty} \left|\widehat{f}(-k)\right|^2 \leq \sum_{k=-\infty}^{+\infty} \left|\widehat{f}(k)\right|^2 \leq \|f\|_\infty^2.$$

Hence we can define an orthogonally additive continuous 2-homogeneous polynomial $P\colon L^\infty(\mathbb{T}) \to \mathbb{C}$ by

$$P(f) = \sum_{k=0}^{+\infty} \widehat{f * f}(-k) = \sum_{k=0}^{+\infty} \widehat{f}(-k)^2 \quad (f \in L^\infty(\mathbb{T})).$$

Suppose that P can be represented in the form $P(f) = \Phi(f * f)$ $(f \in L^\infty(\mathbb{T}))$ for some continuous linear functional $\Phi\colon L^\infty(\mathbb{T}) \to \mathbb{C}$. The restriction of Φ to $C(\mathbb{T})$ gives a continuous linear functional on $C(\mathbb{T})$ and therefore there exists a measure $\mu \in M(\mathbb{T})$ such that

$$\Phi(f) = \int_{\mathbb{T}} f(z)\, d\mu(z) \quad (f \in C(\mathbb{T})). \tag{1.12}$$

By (1.8) and (1.9), for each $k \in \mathbb{Z}$, we have

$$\Phi(\chi_k) = \Phi(\chi_k * \chi_k) = P(\chi_k) = \sum_{j=0}^{+\infty} \widehat{\chi_k}(-j) = \sum_{j=0}^{+\infty} \delta_{-jk} = \begin{cases} 1 & \text{if } k \leq 0, \\ 0 & \text{if } k > 0, \end{cases}$$

and (1.12) then yields

$$\widehat{\mu}(k) = \int_{\mathbb{T}} z^{-k}\, d\mu(z)\, dz = \Phi(\chi_{-k}) = \begin{cases} 1 & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases}$$

This contradicts the fact that the series $\sum_{k\geq 0} \chi_k$ is not a Fourier-Stieltjes series (see [40, Example 12.7.8]). It should be pointed out that we have actually shown that neither P nor the restriction of P to $C(\mathbb{T})$ can be represented in the standard form.

1.5 The largest appropriate norm topology on $\mathcal{P}_n(A)$

Since Question Q2 has been settled in the negative for the algebras $L^p(G)$, with $1 < p \leq \infty$, and $C(G)$, it is therefore reasonable to attempt to explore Question Q3 for these algebras. For this purpose, in this section, for each Banach algebra A, we make an appropriate choice of norm on $\mathcal{P}_n(A)$.

Theorem 1.5.1. *Let A be a Banach algebra. Then*

$$\mathcal{P}_n(A) = \left\{ \sum_{j=1}^{m} a_j^n : a_1, \ldots, a_m \in A,\ m \in \mathbb{N} \right\}$$

and the formula

$$\|a\|_{\mathcal{P}_n} = \inf\left\{ \sum_{j=1}^{m} \|a_j\|^n : a = \sum_{j=1}^{m} a_j^n \right\},$$

for each $a \in \mathcal{P}_n(A)$, defines a norm on $\mathcal{P}_n(A)$ such that

$$\|a^n\|_{\mathcal{P}_n} \leq \|a\|^n \quad (a \in A)$$

and

$$\|a\| \leq \|a\|_{\mathcal{P}_n} \quad (a \in \mathcal{P}_n(A)).$$

Further, the following statements hold.

(1) Suppose that $\Phi\colon \mathcal{P}_n(A) \to X$ is a $\|\cdot\|_{\mathcal{P}_n}$-continuous linear map for some Banach space X. Then the map $P\colon A \to X$ defined by $P(a) = \Phi(a^n)$ $(a \in A)$ is an orthogonally additive continuous n-homogeneous polynomial with $\|P\| = \|\Phi\|$.

(2) Suppose that $|||\cdot|||$ is a norm on $\mathcal{P}_n(A)$ for which the answer to Question Q3 is positive. Then there exist $M_1, M_2 \in \mathbb{R}^+$ such that

$$M_1 \|a\| \leq |||a||| \leq M_2 \|a\|_{\mathcal{P}_n} \quad (a \in \mathcal{P}_n(A)).$$

Proof. For $m \in \mathbb{N}$, let $a_1, \ldots, a_m \in A$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$. Take $\beta_1, \ldots, \beta_m \in \mathbb{C}$ such that $\beta_j^n = \alpha_j$ $(j \in \{1, \ldots, m\})$. Then

$$\sum_{j=1}^{m} \alpha_j a_j^n = \sum_{j=1}^{m} (\beta_j a_j)^n,$$

which establishes the first equality of the result.

Take $a \in \mathcal{P}_n(A)$, and let $a_1, \ldots, a_m \in A$ be such that $a = \sum_{j=1}^{m} a_j^n$ for some $m \in \mathbb{N}$. Then

$$\|a\| \leq \sum_{j=1}^{m} \left\|a_j^n\right\| \leq \sum_{j=1}^{m} \|a_j\|^n,$$

which proves that $\|a\| \leq \|a\|_{\mathcal{P}_n}$. In particular, if $a \in A$ is such that $\|a\|_{\mathcal{P}_n} = 0$, then we have $a = 0$.

Set $a \in \mathcal{P}_n(A)$ and $\alpha \in \mathbb{C}$. We proceed to show that $\|\alpha a\|_{\mathcal{P}_n} = |\alpha| \, \|a\|_{\mathcal{P}_n}$. Of course, we can assume that $\alpha \neq 0$. Choose $\beta \in \mathbb{C}$ such that $\beta^n = \alpha$. If $a_1, \ldots, a_m \in A$ are such that $a = \sum_{j=1}^m a_j^n$ then $\alpha a = \sum_{j=1}^m (\beta a_j)^n$ and therefore

$$\|\alpha a\|_{\mathcal{P}_n} \leq \sum_{j=1}^m \|\beta a_j\|^n = \sum_{j=1}^m |\alpha| \, \|a_j\|^n ,$$

which implies that $\|\alpha a\|_{\mathcal{P}_n} \leq |\alpha| \, \|a\|_{\mathcal{P}_n}$. On the other hand,

$$\|a\|_{\mathcal{P}_n} = \left\|\alpha^{-1}(\alpha a)\right\|_{\mathcal{P}_n} \leq |\alpha|^{-1} \|\alpha a\|_{\mathcal{P}_n} ,$$

which gives the converse inequality.

Let a, $b \in \mathcal{P}_n(A)$. Our goal is to prove that $\|a+b\|_{\mathcal{P}_n} \leq \|a\|_{\mathcal{P}_n} + \|b\|_{\mathcal{P}_n}$. To this end, set $\varepsilon \in \mathbb{R}^+$, and choose $a_1, \ldots, a_l, b_1, \ldots, b_m \in A$ such that

$$a = \sum_{j=1}^l a_j^n, \quad b = \sum_{j=1}^m b_j^n,$$

and

$$\sum_{j=1}^l \|a_j\|^n < \|a\|_{\mathcal{P}_n} + \varepsilon/2, \quad \sum_{j=1}^m \|b_j\|^n < \|b\|_{\mathcal{P}_n} + \varepsilon/2.$$

Then we have

$$a + b = \sum_{j=1}^l a_j^n + \sum_{j=1}^m b_j^n$$

and therefore

$$\|a+b\|_{\mathcal{P}_n} \leq \sum_{j=1}^l \|a_j\|^n + \sum_{j=1}^m \|b_j\|^n \leq \|a\|_{\mathcal{P}_n} + \|b\|_{\mathcal{P}_n} + \varepsilon,$$

which yields $\|a+b\|_{\mathcal{P}_n} \leq \|a\|_{\mathcal{P}_n} + \|b\|_{\mathcal{P}_n}$. Then $\|\cdot\|_{\mathcal{P}_n}$ is a norm on $\mathcal{P}_n(A)$. The space $\mathcal{P}_n(A)$ is equipped with this norm for the remainder of this proof.

It is clear that $\|a^n\|_{\mathcal{P}_n} \leq \|a\|^n$ for each $a \in A$. This property allows us to establish the statement (1). Suppose that X is a Banach space and that $\Phi \colon \mathcal{P}_n(A) \to X$ is a continuous linear map. Define $P \colon A \to X$ by $P(a) = \Phi(a^n)$ $(a \in A)$. Take $a \in A$. Then we have

$$\|P(a)\| = \|\Phi(a^n)\| \leq \|\Phi\| \, \|a^n\|_{\mathcal{P}_n} \leq \|\Phi\| \, \|a\|^n .$$

which shows that the polynomial P is continuous and that $\|P\| \leq \|\Phi\|$. On the other hand, let $a_1, \ldots, a_m \in A$ be such that $a = \sum_{j=1}^m a_j^n$. Then we have

$$\Phi(a) = \sum_{j=1}^m \Phi(a_j^n) = \sum_{j=1}^m P(a_j),$$

whence
$$\|\Phi(a)\| \leq \sum_{j=1}^{m} \|P(a_j)\| \leq \sum_{j=1}^{m} \|P\| \, \|a_j\|^n = \|P\| \sum_{j=1}^{m} \|a_j\|^n \, .$$
This shows that $\|\Phi(a)\| \leq \|P\| \, \|a\|_{\mathcal{P}_n}$, hence that $\|\Phi\| \leq \|P\|$, and finally that $\|\Phi\| = \|P\|$.

We now prove (2). Suppose that $|||\cdot|||$ is a norm on $\mathcal{P}_n(A)$ which satisfies the conditions:

- for each Banach space X and each $|||\cdot|||$-continuous linear map $\Phi\colon \mathcal{P}_n(A) \to X$, the prototypical polinomial $P\colon A \to X$ defined by $P(a) = \Phi(a^n)$ $(a \in A)$ is continuous, and
- every orthogonally additive continuous n-homogeneous polynomial P from A into each Banach space X can be expressed as $P(a) = \Phi(a^n)$ $(a \in A)$ for some $|||\cdot|||$-continuous linear map $\Phi\colon \mathcal{P}_n(A) \to X$.

The canonical polynomial $P_n\colon A \to A$ is an orthogonally additive continuous n-homogeneous polynomial. Hence there exists a $|||\cdot|||$-continuous linear map $\Phi\colon \mathcal{P}_n(A) \to A$ such that $P_n(a) = \Phi(a^n)$ $(a \in A)$. This implies that Φ is the inclusion map from $\mathcal{P}_n(A)$ into A, and the continuity of this map yields $M_1 \in \mathbb{R}^+$ such that $M_1\|a\| \leq |||a|||$ for each $a \in \mathcal{P}_n(A)$. We now take X to be the completion of the normed space $(\mathcal{P}_n(A), |||\cdot|||)$, and let $\Phi\colon \mathcal{P}_n(A) \to X$ be the inclusion map. Then, by hypothesis, the polynomial $P\colon A \to A$ defined by $P(a) = \Phi(a^n)$ $(a \in A)$ is continuous, and, in consequence, there exists $M_2 \in \mathbb{R}^+$ such that $|||a^n||| \leq M_2\|a\|^n$ for each $a \in A$. Take $a \in \mathcal{P}_n(A)$, and let $a_1, \dots, a_m \in A$ be such that $a = \sum_{j=1}^{m} a_j^n$. Then we have
$$|||a||| \leq \sum_{j=1}^{m} |||a_j^n||| \leq \sum_{j=1}^{m} M_2 \, \|a_j\|^n \, ,$$
which yields $|||a||| \leq M_2\|a\|_{\mathcal{P}_n}$, and completes the proof. □

From now on $\mathcal{P}_n(A)$ will be equipped with the norm $\|\cdot\|_{\mathcal{P}_n}$ defined in Theorem 1.5.1.

Proposition 1.5.2. *Let A be a Banach algebra. Then*
$$\mathcal{P}_n(A) = \left\{ \sum_{j=1}^{m} S_n(a_{1,j}, \dots, a_{n,j}) : a_{1,j}, \dots, a_{n,j} \in A, \ m \in \mathbb{N} \right\}$$
and the formula
$$\|a\|_{\mathcal{S}_n} = \inf\left\{ \sum_{j=1}^{m} \|a_{1,j}\| \cdots \|a_{n,j}\| : a = \sum_{j=1}^{m} S_n(a_{1,j}, \dots, a_{n,j}) \right\},$$
for each $a \in \mathcal{P}_n(A)$, defines a norm on $\mathcal{P}_n(A)$ such that
$$\|a\| \leq \|a\|_{\mathcal{S}_n} \leq \|a\|_{\mathcal{P}_n} \leq \frac{n^n}{n!} \|a\|_{\mathcal{S}_n} \quad (a \in \mathcal{P}_n(A)) \, .$$

Proof. Let $\mathcal{S}_n(A)$ denote the set of the right-hand side of the first identity in the result. Take $a \in \mathcal{P}_n(A)$. Then there exist $a_1, \ldots, a_m \in A$ such that

$$a = \sum_{j=1}^{m} a_j^n = \sum_{j=1}^{m} S_n(a_j, \ldots, a_j) \in \mathcal{S}_n(A).$$

This also implies that $\|a\|_{\mathcal{S}_n} \leq \sum_{j=1}^{m} \|a_j\| \cdots \|a_j\|$, and hence that $\|a\|_{\mathcal{S}_n} \leq \|a\|_{\mathcal{P}_n}$. We now take $a \in \mathcal{S}_n(A)$. Then there exist $a_{1,j}, \ldots, a_{n,j} \in A$ ($j \in \{1, \ldots, m\}$) such that $a = \sum_{j=1}^{m} S_n(a_{1,j}, \ldots, a_{n,j})$. The polarization formula gives

$$\begin{aligned} a &= \sum_{j=1}^{m} \frac{1}{n!\,2^n} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \epsilon_1 \cdots \epsilon_n \left(\epsilon_1 a_{1,j} + \cdots + \epsilon_n a_{n,j}\right)^n \\ &= \sum_{j=1}^{m} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \Big(\big(\tfrac{1}{n!\,2^n}\big)^{1/n} \epsilon_1^{1/n} \cdots \epsilon_n^{1/n} \left(\epsilon_1 a_{1,j} + \cdots + \epsilon_n a_{n,j}\right) \Big)^n \in \mathcal{P}_n(A). \end{aligned}$$

The proof of the fact that $\|\cdot\|_{\mathcal{S}_n}$ is a norm on $\mathcal{P}_n(A)$ is similar to Theorem 1.5.1, and it is omitted.

Our next objective is to prove the inequalities relating the three norms $\|\cdot\|$, $\|\cdot\|_{\mathcal{S}_n}$, and $\|\cdot\|_{\mathcal{P}_n}$. Take $a \in \mathcal{P}_n(A)$. We have already shown that $\|a\|_{\mathcal{S}_n} \leq \|a\|_{\mathcal{P}_n}$. Let $a_{1,j}, \ldots, a_{n,j} \in A$ ($j \in \{1, \ldots, m\}$) be such that $a = \sum_{j=1}^{m} S_n(a_{1,j}, \ldots, a_{n,j})$. Then

$$\|a\| \leq \sum_{j=1}^{m} \left\|S_n\left(a_{1,j}, \ldots, a_{n,j}\right)\right\| \leq \sum_{j=1}^{m} \|a_{1,j}\| \cdots \|a_{n,j}\|,$$

which shows that $\|a\| \leq \|a\|_{\mathcal{S}_n}$. For $l \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, we take $b_{l,j} = a_{l,j} / \|a_{l,j}\|$ if $a_{l,j} \neq 0$ and $b_{l,j} = 0$ otherwise. Then

$$a = \sum_{j=1}^{m} S_n\left(a_{1,j}, \ldots, a_{n,j}\right) = \sum_{j=1}^{m} \|a_{1,j}\| \cdots \|a_{n,j}\| \, S_n\left(b_{1,j}, \ldots, b_{n,j}\right),$$

and the polarization formula gives

$$\begin{aligned} a &= \sum_{j=1}^{m} \|a_{1,j}\| \cdots \|a_{n,j}\| \frac{1}{n!\,2^n} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \epsilon_1 \cdots \epsilon_n \left(\epsilon_1 b_{1,j} + \cdots + \epsilon_n b_{n,j}\right)^n \\ &= \sum_{j=1}^{m} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \Big(\big(\|a_{1,j}\| \cdots \|a_{n,j}\| \tfrac{1}{n!\,2^n}\big)^{1/n} \epsilon_1^{1/n} \cdots \epsilon_n^{1/n} \big(\epsilon_1 b_{1,j} + \cdots + \epsilon_n b_{n,j}\big) \Big)^n. \end{aligned}$$

We thus get

$$\|a\|_{\mathcal{P}_n} \leq \sum_{j=1}^{m} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \Big\| \big(\|a_{1,j}\| \cdots \|a_{n,j}\| \tfrac{1}{n!\,2^n}\big)^{1/n} \epsilon_1^{1/n} \cdots \epsilon_n^{1/n} \big(\epsilon_1 b_{1,j} + \cdots + \epsilon_n b_{n,j}\big) \Big\|^n$$

$$= \sum_{j=1}^{m} \sum_{\epsilon_1,\dots,\epsilon_n=\pm 1} \|a_{1,j}\| \cdots \|a_{n,j}\| \frac{1}{n!\,2^n} \big\|\epsilon_1 b_{1,j} + \cdots + \epsilon_n b_{n,j}\big\|^n$$

$$\leq \sum_{j=1}^{m} \sum_{\epsilon_1,\dots,\epsilon_n=\pm 1} \|a_{1,j}\| \cdots \|a_{n,j}\| \frac{1}{n!\,2^n} n^n = \sum_{j=1}^{m} \|a_{1,j}\| \cdots \|a_{n,j}\| \frac{n^n}{n!},$$

which shows that $\|a\|_{\mathcal{P}_n} \leq \frac{n^n}{n!} \|a\|_{\mathcal{S}_n}$. □

Proposition 1.5.3. *Let A be a Banach algebra with a central bounded approximate identity of bound M. Then $\mathcal{P}_n(A) = A$ and $\|\cdot\| \leq \|\cdot\|_{\mathcal{S}_n} \leq M^{n-1} \|\cdot\|$.*

Proof. Take $a \in A$, and let $\varepsilon \in \mathbb{R}^+$. Our objective is to show that $a \in \mathcal{P}_n(A)$ and that $\|a\|_{\mathcal{S}_n} < M^{n-1}\big(\|a\| + \varepsilon\big)$. Let $\mathcal{Z}(A)$ be the centre of A, and let $(e_\lambda)_{\lambda\in\Lambda}$ be a central bounded approximate identity for A of bound M. Then $\mathcal{Z}(A)$ is a Banach algebra and $(e_\lambda)_{\lambda\in\Lambda}$ is a bounded approximate identity for $\mathcal{Z}(A)$. Of course, A is a Banach $\mathcal{Z}(A)$-bimodule and $\mathcal{Z}(A)A$ is dense in A. We now take $a_0 = a$, and then we successively apply the factorization theorem [36, Theorem 2.9.24] to choose $a_1, \dots, a_{n-1} \in A$ and $z_1, \dots, z_{n-1} \in \mathcal{Z}(A)$ such that

$$a_{k-1} = z_k a_k, \tag{1.13}$$

$$\|a_{k-1} - a_k\| < \frac{\varepsilon}{n}, \tag{1.14}$$

and

$$\|z_k\| \leq M \tag{1.15}$$

for each $k \in \{1, \dots, n-1\}$. Then

$$a = S_n\left(z_1, \dots, z_{n-1}, a_{n-1}\right)$$

and therefore $a \in \mathcal{P}_n(A)$ with

$$\begin{aligned} \|a\|_{\mathcal{S}_n} &\leq \|z_1\| \cdots \|z_{n-1}\| \, \|a_{n-1}\| \\ &\leq M^{n-1} \|a_{n-1}\| \\ &\leq M^{n-1}\big(\|a_{n-1} - a_{n-2}\| + \cdots + \|a_1 - a\| + \|a\|\big) \\ &< M^{n-1}\big(\|a\| + \varepsilon\big), \end{aligned}$$

as claimed. Further, on account of Theorem 1.5.1, we have

$$\|a\| \leq \|a\|_{\mathcal{S}_n} \leq M^{n-1}\big(\|a\| + \varepsilon\big)$$

for each $\varepsilon \in \mathbb{R}^+$, which gives $\|a\| \leq \|a\|_{\mathcal{S}_n} \leq M^{n-1} \|a\|$. □

Corollary 1.5.4. *Let G be a compact group. Then $\mathcal{P}_n\big(L^1(G)\big) = L^1(G)$ and $\|\cdot\|_{\mathcal{S}_n} = \|\cdot\|_1$.*

Proof. The net $(h_\lambda)_{\lambda\in\Lambda}$ introduced in the beginning of Section 1.3 is a central bounded approximate identity for $L^1(G)$ of bound 1. Then Proposition 1.5.3 applies to show the result. □

1.6 Some other examples of convolution algebras

Sections 2 and 5 combine to answer Question Q3 in the positive for a variety of convolutions algebras associated with G.

Lemma 1.6.1. *Let G be a compact group, and let A be a subalgebra of $L^1(G)$ which is equipped with a norm $\|\cdot\|_A$ of its own and satisfies the following conditions:*

(a) A is a Banach algebra with respect to $\|\cdot\|_A$;

(b) $\mathcal{T}(G)$ is a dense subspace of A with respect to $\|\cdot\|_A$;

Then $\mathcal{T}(G)$ is a dense subset of $\mathcal{P}_n(A)$.

Proof. For $[\pi] \in \widehat{G}$ and $f \in \mathcal{T}_\pi(G)$, the polarization formula and Lemma 1.2.2(1) yield

$$\begin{aligned} f &= S_n\left(f, d_\pi \chi_\pi, \dots, d_\pi \chi_\pi\right) \\ &= \frac{1}{n!\,2^n} \sum_{\epsilon_1,\dots,\epsilon_n = \pm 1} \epsilon_1 \cdots \epsilon_n \left(\epsilon_1 f + \epsilon_2 d_\pi \chi_\pi + \cdots + \epsilon_n d_\pi \chi_\pi\right)^{*n}. \end{aligned}$$

This shows that $\mathcal{T}_\pi(G)$ is equal to the linear span of the set $\left\{f^{*n} : f \in \mathcal{T}_\pi(G)\right\}$, which implies that $\mathcal{T}(G) \subset \mathcal{P}_n(A)$.

On the other hand, if $f, g \in A$, then

$$\begin{aligned} f^{*n} &= P_n\big(g + (f-g)\big) \\ &= \sum_{j=0}^{n} \binom{n}{j} S_n(\underbrace{g,\dots,g}_{n-j}, \underbrace{f-g,\dots,f-g}_{j}) \\ &= g^{*n} + \sum_{j=1}^{n} \binom{n}{j} S_n(\underbrace{g,\dots,g}_{n-j}, \underbrace{f-g,\dots,f-g}_{j}) \end{aligned}$$

and therefore

$$\|f^{*n} - g^{*n}\|_{\mathcal{S}_n} \le \sum_{j=1}^{n} \binom{n}{j} \|g\|_A^{n-j} \|f-g\|_A^{j}.$$

From Proposition 1.5.2, we deduce that

$$\|f^{*n} - g^{*n}\|_{\mathcal{P}_n} \le \frac{n^n}{n!} \sum_{j=1}^{n} \binom{n}{j} \|g\|_A^{n-j} \|f-g\|_A^{j}. \tag{1.16}$$

Let $f \in A$. Then there exists a sequence (f_k) in $\mathcal{T}(G)$ such that $\|f - f_k\|_A \to 0$, and (1.16) then gives $\|f^{*n} - f_k^{*n}\|_{\mathcal{P}_n} \to 0$. This implies that $\mathcal{T}(G)$ is dense in $\mathcal{P}_n(A)$. □

Theorem 1.6.2. *Let G be a compact group, and let A be a subalgebra of $L^1(G)$ which is equipped with a norm $\|\cdot\|_A$ of its own and satisfies the following conditions:*

(a) A is a Banach algebra with respect to $\|\cdot\|_A$;

(b) $\mathcal{T}(G)$ is a dense subspace of A with respect to $\|\cdot\|_A$;

(c) A is a Banach left $L^1(G)$-module with respect to $\|\cdot\|_A$ and the convolution multiplication.

Let X be a Banach space, and let $P\colon A \to X$ be a continuous n-homogeneous polynomial. Then the following conditions are equivalent:

(1) the polynomial P is orthogonally additive;

(2) the polynomial P is orthogonally additive on $\mathcal{T}(G)$, i.e., $P(f+g) = P(f) + P(g)$ whenever $f, g \in \mathcal{T}(G)$ are such that $f * g = g * f = 0$;

(3) there exists a unique continuous linear map $\Phi\colon \mathcal{P}_n(A) \to X$ such that $P(f) = \Phi\left(f^{*n}\right)$ for each $f \in A$.

Proof. It is clear that (1) $\Rightarrow$ (2) and that (3) $\Rightarrow$ (1). We will prove that (2) $\Rightarrow$ (3).

Let $\varphi\colon A^n \to X$ be the symmetric n-linear map associated with P, and let $\Phi_0\colon \mathcal{T}(G) \to X$ be the linear map defined by

$$\Phi_0(f) = \sum_{[\pi]\in\widehat{G}} \varphi\left(d_\pi f * \chi_\pi, d_\pi\chi_\pi \dots, d_\pi\chi_\pi\right)$$

for each $f \in \mathcal{T}(G)$. Since P is orthogonally additive on $\mathcal{T}(G)$, Theorem 1.2.3 yields

$$P(f) = \Phi_0\left(f^{*n}\right) \quad \left(f \in \mathcal{T}(G)\right). \tag{1.17}$$

We claim that Φ_0 is continuous. Let $(h_\lambda)_{\lambda\in\Lambda}$ be as introduced in the beginning of Section 1.3. Set $f \in \mathcal{T}(G)$, and assume that $f = \sum_{j=1}^m f_j^{*n}$ with $f_1, \dots, f_m \in A$. For $\lambda \in \Lambda$, h_λ belongs to the centre of $L^1(G)$, and so

$$h_\lambda^{*n} * f = \sum_{j=1}^m \bigl(h_\lambda * f_j\bigr)^{*n}.$$

Since $f_j * h_\lambda \in \mathcal{T}(G)$ $(j \in \{1, \dots, m\},\ \lambda \in \Lambda)$, (1.17) yields

$$\Phi_0\left(h_\lambda^{*n} * f\right) = \sum_{j=1}^m \Phi_0\bigl((h_\lambda * f_j)^{*n}\bigr) = \sum_{j=1}^m P\left(h_\lambda * f_j\right),$$

whence

$$\begin{aligned}\left\|\Phi_0\left(h_\lambda^{*n} * f\right)\right\| = \sum_{j=1}^m \|P(h_\lambda * f_j)\| &\le \sum_{j=1}^m \|P\| \, \|h_\lambda * f_j\|_A^n \\ &\le \sum_{j=1}^m \|P\| \, \|h_\lambda\|_1^n \, \|f_j\|_A^n \le \|P\| \sum_{j=1}^m \|f_j\|_A^n .\end{aligned}$$

We thus get

$$\|\Phi_0(h_\lambda^{*n} * f)\| \leq \|P\| \, \|f\|_{\mathcal{P}_n} . \tag{1.18}$$

We now see that, for each $\lambda \in \Lambda$,

$$\begin{aligned}\|h_\lambda^{*n} * f - f\|_A &\leq \left\|h_\lambda^{*n} * f - h_\lambda^{*n-1} * f\right\|_A \\ &\quad + \cdots + \left\|h_\lambda^{*2} * f - h_\lambda * f\right\|_A + \|h_\lambda * f - f\|_A \\ &\leq \left(\|h_\lambda\|_1^{n-1} + \cdots + \|h_\lambda\|_1 + 1\right) \|h_\lambda * f - f\|_A \\ &\leq n \, \|h_\lambda * f - f\|_A .\end{aligned}$$

On account of [52, Remarks 32.33(a) and 38.6(b)], we have

$$\lim_{\lambda \in \Lambda} \|h_\lambda * f - f\|_A = 0,$$

and so

$$\lim_{\lambda \in \Lambda} \|h_\lambda^{*n} * f - f\|_A = 0.$$

Since $f \in \mathcal{T}(G)$, it follows that there exist $[\pi_1], \ldots, [\pi_l] \in \widehat{G}$ such that $f \in \mathcal{M} := \mathcal{T}_{\pi_1}(G) + \cdots + \mathcal{T}_{\pi_l}(G)$. The finite-dimensionality of $\mathcal{M}$ implies that the restriction of Φ_0 to $\mathcal{M}$ is continuous. Further, Lemma 1.2.2(1) shows that $\mathcal{M}$ is a two-sided ideal of $L^1(G)$, and so $h_\lambda \in \mathcal{M} * f$ for each $\lambda \in \Lambda$. Therefore, taking limits on both sides of equation (1.18) (and using the continuity of Φ_0 on $\mathcal{M}$), we see that

$$\|\Phi_0(f)\| \leq \|P\| \, \|f\|_{\mathcal{P}_n} ,$$

which proves our claim.

Since $\mathcal{T}(G)$ is dense in $\mathcal{P}_n(A)$ (Lemma 1.6.1), it follows that Φ_0 has a continuous extension $\Phi\colon \mathcal{P}_n(A) \to X$. Take $f \in A$. There exists a sequence (f_k) in $\mathcal{T}(G)$ with $\|f - f_k\|_A \to 0$, so that $P(f_k) \to P(f)$. Further, (1.16) gives $\|f^{*n} - f_k^{*n}\|_{\mathcal{P}_n} \to 0$, and consequently $P(f_k) = \Phi(f_k^{*n}) \to \Phi(f^{*n})$. Hence $\Phi(f^{*n}) = P(f)$.

Finally, we proceed to prove the uniqueness of the map Φ. Suppose that $\Psi\colon \mathcal{P}_n(A) \to X$ is a continuous linear map such that $P(f) = \Psi(f^{*n})$ for each $f \in A$. By Theorem 1.2.3, $\Psi(f) = \Phi(f)\big(= \Phi_0(f)\big)$ for each $f \in \mathcal{T}(G)$. Since $\mathcal{T}(G)$ is dense in $L^1(G)$ (Lemma 1.6.1), and both Φ and Ψ are continuous, it follows that $\Psi(f) = \Phi(f)$ for each $f \in A$. $\square$

Example 1.6.3. Let G be a compact group. The following convolution algebras satisfy the conditions required in Theorem 1.6.2 (see [52, Remark 38.6]).

(1) For $1 \leq p < \infty$, the algebra $L^p(G)$.

(2) The algebra $C(G)$.

(3) The algebra $A(G)$ consisting of those functions $f \in C(G)$ of the form

$$f = g * h$$

with $g, h \in L^2(G)$. The norm $\|\cdot\|_{A(G)}$ on $A(G)$ is defined by

$$\|f\|_{A(G)} = \inf\{\|g\|_2 \|h\|_2 : f = g * h,\ g, h \in L^2(G)\}$$

for each $f \in A(G)$. It is worth noting that a function $f \in L^1(G)$ is equal almost everywhere to a function in $A(G)$ if and only if

$$\sum_{[\pi]\in\widehat{G}} d_\pi \big\|\widehat{f}(\pi)\big\|_1 < \infty.$$

Further,

$$\|f\|_{A(G)} = \sum_{[\pi]\in\widehat{G}} d_\pi \big\|\widehat{f}(\pi)\big\|_1$$

for each $f \in A(G)$. Here $\|T\|_1$ denotes the trace class norm of the operator $T \in \mathcal{B}(H_\pi)$.

(4) For $1 < p < \infty$, the algebra $A_p(G)$ consisting of those functions $f \in C(G)$ of the form

$$f = \sum_{k=1}^{\infty} g_k * h_k$$

where (g_k) is a sequence in $L^p(G)$, (h_k) is a sequence in $L^q(G)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\sum_{k=1}^{\infty} \|g_k\|_p \|h_k\|_q < \infty.$$

The norm $\|\cdot\|_{A_p(G)}$ on $A_p(G)$ is defined by

$$\|f\|_{A_p(G)} = \inf\left\{\sum_{k=1}^{\infty} \|g_k\|_p \|h_k\|_q : f = \sum_{k=1}^{\infty} g_k * h_k\right\}$$

for each $f \in A_p(G)$.

(5) For $1 < p < \infty$, the algebra $S_p(G)$ consisting of the functions $f \in L^1(G)$ for which

$$\sum_{[\pi]\in\widehat{G}} d_\pi \big\|\widehat{f}(\pi)\big\|_{S^p(H_\pi)}^p < \infty.$$

Here $\|T\|_{S^p(H_\pi)}$ denotes the pth Schatten norm of the operator $T \in \mathcal{B}(H_\pi)$. The norm $\|\cdot\|_{S_p(G)}$ on $S_p(G)$ is defined by

$$\|f\|_{S_p(G)} = \|f\|_1 + \left(\sum_{[\pi]\in\widehat{G}} d_\pi \big\|\widehat{f}(\pi)\big\|_{S^p(H_\pi)}^p\right)^{1/p}$$

for each $f \in S_p(G)$.

Remark 1.6.4. Together with Corollary 1.5.4, Theorem 1.6.2 gives Theorem 1.3.1.

Remark 1.6.5. It is well-known that the Banach spaces $A(G)$ and $A_p(G)$ of Example 1.6.3 are particularly important Banach function algebras with respect to the pointwise multiplication. We emphasize that while the references [7, 15, 81, 82, 83] apply to the problem of representing the orthogonally additive homogeneous polynomials on the Banach function algebras $A(G)$ and $A_p(G)$ with respect to pointwise multiplication, Theorem 1.6.2 gives us information about that problem in the case where both $A(G)$ and $A_p(G)$ are regarded as noncommutative Banach algebras with respect to convolution.

Remark 1.6.6. We do not know whether or not the conclusion of Theorem 1.6.2 must hold for $A = L^\infty(G)$.

Chapter 2

Orthogonally additive polynomials on non-commutative L^p-spaces

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Abstract. Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ. We prove that every continuous m-homogeneous polynomial P from $L^p(\mathcal{M}, \tau)$, with $0 < p < \infty$, into each topological linear space X with the property that $P(x + y) = P(x) + P(y)$ whenever x and y are mutually orthogonal positive elements of $L^p(\mathcal{M}, \tau)$ can be represented in the form $P(x) = \Phi(x^m)$ $(x \in L^p(\mathcal{M}, \tau))$ for some continuous linear map $\Phi\colon L^{p/m}(\mathcal{M}, \tau) \to X$.

2.1 Introduction

In [78], the author succeeded in providing a useful representation of the orthogonally additive homogeneous polynomials on the spaces $L^p([0, 1])$ and ℓ^p with $1 \leq p < \infty$. In [69] (see also [29]), the authors obtained a similar representation for the space $C(K)$, for a compact Hausdorff space K. These results were generalized to Banach lattices [21] and Riesz spaces [53]. Further, the problem of representing the orthogonally additive homogeneous polynomials has been also considered in the context of Banach function algebras [15, 81] and non-commutative Banach algebras [8, 16, 67]. Notably, [67] can be thought of as the natural non-commutative analogue of the representation of orthogonally additive polynomials on $C(K)$-spaces, and the purpose to this paper is to extend the results of [78] on the representation of orthogonally additive homogeneous polynomials on L^p-spaces to the non-commutative L^p-spaces.

The non-commutative L^p-spaces that we consider are those associated with a von Neumann algebra $\mathcal{M}$ equipped with a normal semifinite faithful trace τ. From now on, $S(\mathcal{M},\tau)$ stands for the linear span of the positive elements x of $\mathcal{M}$ such that $\tau\big(\mathrm{supp}(x)\big) < \infty$; here $\mathrm{supp}(x)$ stands for the support of x. Then $S(\mathcal{M},\tau)$ is a $*$-subalgebra of $\mathcal{M}$ with the property that $|x|^p \in S(\mathcal{M},\tau)$ for each $x \in S(\mathcal{M},\tau)$ and each $0 < p < \infty$. For $0 < p < \infty$, we define $\|\cdot\|_p\colon S(\mathcal{M},\tau) \to \mathbb{R}$ by $\|x\|_p = \tau(|x|^p)^{1/p}$ ($x \in S(\mathcal{M},\tau)$). Then $\|\cdot\|_p$ is a norm or a p-norm according to $1 \le p < \infty$ or $0 < p < 1$, and the space $L^p(\mathcal{M},\tau)$ can be defined as the completion of $S(\mathcal{M},\tau)$ with respect to $\|\cdot\|_p$. Nevertheless, for our purposes here, it is important to realize the elements of $L^p(\mathcal{M},\tau)$ as measurable operators. Specifically, the set $L^0(\mathcal{M},\tau)$ of measurable closed densely defined operators affiliated to $\mathcal{M}$ is a topological $*$-algebra with respect to the strong sum, the strong product, the adjoint operation, and the topology of the convergence in measure. The algebra $\mathcal{M}$ is a dense $*$-subalgebra of $L^0(\mathcal{M},\tau)$, the trace τ extends to the positive cone of $L^0(\mathcal{M},\tau)$ in a natural way, and we can define

$$\|x\|_p = \tau\big(|x|^p\big)^{1/p} \quad (x \in L^0(\mathcal{M},\tau)),$$
$$L^p(\mathcal{M},\tau) = \big\{x \in L^0(\mathcal{M},\tau) : \|x\|_p < \infty\big\}.$$

Also we set $L^\infty(\mathcal{M},\tau) = \mathcal{M}$ (with $\|\cdot\|_\infty := \|\cdot\|$, the operator norm). Operators $x, y \in L^0(\mathcal{M},\tau)$ are mutually orthogonal, written $x \perp y$, if $xy^* = y^*x = 0$. This condition is equivalent to requiring that x and y have mutually orthogonal left, and right, supports. Further, for $x, y \in L^p(\mathcal{M},\tau)$ with $0 < p < \infty$, the condition $x \perp y$ implies that $\|x+y\|_p^p = \|x\|_p^p + \|y\|_p^p$, and conversely, if $\|x \pm y\|_p^p = \|x\|_p^p + \|y\|_p^p$ and $p \neq 2$, then $x \perp y$ (see [72, Fact 1.3]). The orthogonal additivity considered in [78] for the spaces $L^p([0,1])$ and ℓ^p can of course equally well be considered for the space $L^p(\mathcal{M},\tau)$. Let P be a map from $L^p(\mathcal{M},\tau)$ into a linear space X. Then P is:

(i) *orthogonally additive* on a subset $\mathcal{S}$ of $L^p(\mathcal{M},\tau)$ if

$$x, y \in \mathcal{S},\ x \perp y = 0 \ \Rightarrow\ P(x+y) = P(x) + P(y);$$

(ii) an *m-homogeneous polynomial* if there exists an m-linear map φ from $L^p(\mathcal{M},\tau)^m$ into X such that

$$P(x) = \varphi(x, \dots, x) \quad (x \in L^p(\mathcal{M},\tau)).$$

Here and subsequently, $m \in \mathbb{N}$ is fixed with $m \ge 2$ and the superscript m stands for the m-fold Cartesian product. Such a map is unique if it is required to be symmetric. Further, in the case where X is a topological linear space, the polynomial P is continuous if and only if the symmetric m-linear map φ associated with P is continuous.

Given a continuous linear map $\Phi\colon L^{p/m}(\mathcal{M},\tau) \to X$, where X is an arbitrary topological linear space, the map $P_\Phi\colon L^p(\mathcal{M},\tau) \to X$ defined by

$$P_\Phi(x) = \Phi(x^m) \quad (x \in L^p(\mathcal{M},\tau))$$

is a natural example of a continuous m-homogeneous polynomial which is orthogonally additive on $L^p(\mathcal{M},\tau)_{\mathrm{sa}}$ (Theorem 2.3.1), and we will prove that every continuous m-homogeneous polynomial which is orthogonally additive on $L^p(\mathcal{M},\tau)_{\mathrm{sa}}$ is actually of this special form (Theorem 2.3.2). Here and subsequently, the subscripts "sa" and + are used to denote the self-adjoint and the positive parts of a given subset of $L^0(\mathcal{M}, \tau)$, respectively.

We require a few remarks about the setting of our present work. Throughout the paper we are concerned with m-homogeneous polynomials on the space $L^p(\mathcal{M}, \tau)$ with $0 < p$, and thus one might wish to consider polynomials with values in the space $L^q(\mathcal{M}, \tau)$, especially with $q \leq p$. Further, in the case case where $p/m < 1$ and the von Neumann algebra $\mathcal{M}$ has no minimal projections, there are no non-zero continuous linear functionals on $L^{p/m}(\mathcal{M}, \tau)$; since one should like to have non-trivial "orthogonally additive" polynomials on $L^p(\mathcal{M}, \tau)$, some weakening of the normability must be allowed to the range space (see Corollary 2.3.4). For these reasons, throughout the paper, X will be a (complex and Hausdorff) topological linear space. In the case where the von Neumann algebra $\mathcal{M}$ is commutative, the prototypical polynomials P_Φ mentioned above are easily seen to be orthogonally additive on the whole domain. In contrast, we will point out in Propositions 2.2.7 and 2.3.9 that this is not the case for the von Neumann algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H whenever $\dim H \geq 2$.

We assume a basic knowledge of C^*-algebras and von Neumann algebras, tracial non-commutative L^p-spaces, and polynomials on topological linear spaces. For the relevant background material concerning these topics, see [22, 37, 79], [65, 70, 80], and [39], respectively.

2.2 C^*-algebras and von Neumann algebras

Our approach to the problem of representing the orthogonally additive homogeneous polynomials on the non-commutative L^p-spaces relies on the representation of those polynomials on the von Neumann algebras.

Recall that two elements x and y of a C^*-algebra $\mathcal{A}$ are mutually orthogonal if $xy^* = y^*x = 0$, in which case the identity $\|x + y\| = \max\{\|x\|, \|y\|\}$ holds. The reader should be aware that we have chosen the standard definition of orthogonality in the setting of non-commutative L^p-spaces. This definition is slightly different from the one used in [67], which is the standard one in the setting of Banach algebras. In [67] the orthogonality of two elements x and y is defined by the relation $xy = yx = 0$, and, further, the orthogonally additive polynomials on the self-adjoint part of a C^*-algebra are automatically orthogonally additive on the whole algebra. The important point to note here is that both the definitions of orthogonality agree on the self-adjoint part of the C^*-algebra. Thus, for a polynomial on a C^*-algebra, the property of being orthogonally additive on the self-adjoint part according to our definition is the same as being orthogonally additive according to [67]. Nevertheless, in contrast to [67], there are no non-zero orthogonally additive polynomials from the von Neumann algebra $\mathcal{B}(H)$ into any topological Banach space according to our definition (Proposition 2.2.7).

Suppose that $\mathcal{A}$ is a linear space with an involution $*$. Recall that for a linear functional $\Phi\colon \mathcal{A} \to \mathbb{C}$, the map $\Phi^*\colon \mathcal{A} \to \mathbb{C}$ defined by $\Phi^*(x) = \overline{\Phi(x^*)}$ $(x \in \mathcal{A})$ is a linear functional, and Φ is said to be *hermitian* if $\Phi^* = \Phi$. Similarly, for an m-homogeneous polynomial $P\colon \mathcal{A} \to \mathbb{C}$, the map $P^*\colon \mathcal{A} \to \mathbb{C}$ defined by $P^*(x) = \overline{P(x^*)}$ ($x \in \mathcal{A}$) is an m-homogeneous polynomial, and we call P *hermitian* if $P^* = P$.

Lemma 2.2.1. *Let X and Y be linear spaces, and let $P\colon X \to Y$ be an m-homogeneous polynomial. Suppose that P vanishes on a convex set $C \subset X$. Then P vanishes on the linear span of C.*

Proof. Set $x_1, x_2, x_3, x_4 \in C$. Let $\eta\colon Y \to \mathbb{C}$ be a linear functional, and define $f\colon \mathbb{C}^4 \to \mathbb{C}$ by

$$f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \eta\big(P(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha x_4)\big) \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}).$$

Then f is a complex polynomial function in four complex variables that vanishes on the set

$$\big\{(\rho_1, \rho_2, \rho_3, \rho_4) \in \mathbb{R}^4 : 0 \le \rho_1, \rho_2, \rho_3, \rho_4,\ \rho_1 + \rho_2 + \rho_3 + \rho_4 = 1\big\}.$$

This implies that f is identically equal to 0 on $\mathbb{C}^4$, and, in particular,

$$\eta\big(P(\rho_1 x_1 - \rho_2 x_2 + i\rho_3 x_3 - i\rho_4 x_4)\big) = f(\rho_1, -\rho_2, i\rho_3, -i\rho_4) = 0$$

for all $\rho_1, \rho_2, \rho_3, \rho_4 \ge 0$. Since this identity holds for each linear functional η, it may be concluded that $P(\rho_1 x_1 - \rho_2 x_2 + i\rho_3 x_3 - i\rho_4 x_4) = 0$ for all $\rho_1, \rho_2, \rho_3, \rho_4 \ge 0$. Thus P vanishes on the set

$$\big\{\rho_1 x_1 - \rho_2 x_2 + i\rho_3 x_3 - i\rho_4 x_4 : \rho_j \ge 0,\ x_j \in C\ (j = 1, 2, 3, 4)\big\},$$

which is exactly the linear span of the set C. □

Theorem 2.2.2. *Let $\mathcal{A}$ be a C^*-algebra, let X be a topological linear space, and let $\Phi\colon \mathcal{A} \to X$ be a continuous linear map. Then:*

(i) the map $P_\Phi\colon \mathcal{A} \to X$ defined by $P_\Phi(x) = \Phi(x^m)$ ($x \in \mathcal{A}$) is a continuous m-homogeneous polynomial which is orthogonally additive on $\mathcal{A}_{\mathrm{sa}}$;

(ii) the polynomial P_Φ is uniquely specified by the map Φ.

Suppose, further, that X is a q-normed space, $0 < q \le 1$. Then:

(iii) $2^{-1/q}\|\Phi\| \le \|P_\Phi\| \le \|\Phi\|$.

Moreover, in the case where $X = \mathbb{C}$,

(iv) the functional Φ is hermitian if and only if the polynomial P_Φ is hermitian, in which case $\|P_\Phi\| = \|\Phi\|$.

Proof. (i) It is clear that the map P_Φ is continuous and that P_Φ is the m-homogeneous polynomial associated with the symmetric m-linear map $\varphi\colon \mathcal{A}^m \to X$ defined by

$$\varphi(x_1, \dots, x_m) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \Phi\left(x_{\sigma(1)} \cdots x_{\sigma(m)}\right) \quad (x_1, \dots, x_m \in \mathcal{A});$$

here and subsequently, we write $\mathfrak{S}_m$ for the symmetric group of order m.

Suppose that $x, y \in \mathcal{A}_{\mathrm{sa}}$ are such that $x \perp y$. Then $xy = yx = 0$, and so $(x+y)^m = x^m + y^m$, which gives

$$P_\Phi(x+y) = \Phi\big((x+y)^m\big) = \Phi\big(x^m + y^m\big) = \Phi\big(x^m\big) + \Phi\big(y^m\big) = P_\Phi(x) + P_\Phi(y).$$

(ii) Assume that $\Psi\colon \mathcal{A} \to X$ is a linear map with the property that $P_\Psi = P_\Phi$. If $x \in \mathcal{A}_+$, then

$$\Phi(x) = \Phi\big((x^{1/m})^m\big) = P(x^{1/m}) = \Psi\big((x^{1/m})^m\big) = \Psi(x).$$

By linearity we also get $\Psi(x) = \Phi(x)$ for each $x \in \mathcal{A}$.

(iii) Next, assume that X is a q-normed space. For each $x \in \mathcal{A}$, we have

$$\|P_\Phi(x)\| = \|\Phi(x^m)\| \le \|\Phi\|\|x^m\| \le \|\Phi\|\|x\|^m,$$

which implies that $\|P_\Phi\| \le \|\Phi\|$. Now take $x \in \mathcal{A}$, and let $\omega \in \mathbb{C}$ with $\omega^m = -1$. Then $x = \Re x + i\Im x$, where

$$\Re x = \frac{1}{2}(x^* + x),\ \Im x = \frac{i}{2}(x^* - x) \in \mathcal{A}_{\mathrm{sa}},$$

and, further, $\|\Re x\|, \|\Im x\| \le \|x\|$. Moreover, $\Re x = x_1 - x_2$ and $\Im x = x_3 - x_4$, where $x_1, x_2, x_3, x_4 \in \mathcal{A}_+$, $x_1 \perp x_2$, and $x_3 \perp x_4$. Since $x_1 \perp x_2$ and $x_3 \perp x_4$, it follows that $x_1^{1/m} \perp x_2^{1/m}$ and $x_3^{1/m} \perp x_4^{1/m}$. Consequently,

$$\begin{aligned} \|\Re x\| &= \max\big\{\|x_1\|, \|x_2\|\big\}, \\ \|\Im x\| &= \max\big\{\|x_3\|, \|x_4\|\big\}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \big\|x_1^{1/m} + \omega x_2^{1/m}\big\| &= \max\big\{\big\|x_1^{1/m}\big\|, \big\|x_2^{1/m}\big\|\big\}, \\ \big\|x_3^{1/m} + \omega x_4^{1/m}\big\| &= \max\big\{\big\|x_3^{1/m}\big\|, \big\|x_4^{1/m}\big\|\big\}. \end{aligned} \tag{2.2}$$

Since

$$\big\|x_1^{1/m}\big\| = \|x_1\|^{1/m},\ \big\|x_2^{1/m}\big\| = \|x_2\|^{1/m},\ \big\|x_3^{1/m}\big\| = \|x_3\|^{1/m},\ \big\|x_4^{1/m}\big\| = \|x_4\|^{1/m},$$

it follows, from (2.1) and (2.2), that

$$\begin{aligned} \big\|x_1^{1/m} + \omega x_2^{1/m}\big\|^m &= \max\big\{\|x_1\|, \|x_2\|\big\} = \|\Re x\|, \\ \big\|x_3^{1/m} + \omega x_4^{1/m}\big\|^m &= \max\big\{\|x_3\|, \|x_4\|\big\} = \|\Im x\|. \end{aligned} \tag{2.3}$$

On the other hand, we have

$$\big(x_1^{1/m} + \omega x_2^{1/m}\big)^m = x_1 - x_2 = \Re x, \quad \big(x_3^{1/m} + \omega x_4^{1/m}\big)^m = x_3 - x_4 = \Im x,$$

and so

$$\begin{aligned}\Phi(x) &= \Phi(\Re x) + i\Phi(\Im x) = \Phi\big((x_1^{1/m} + \omega x_2^{1/m})^m\big) + i\Phi\big((x_3^{1/m} + \omega x_4^{1/m})^m\big) \\ &= P_\Phi\big(x_1^{1/m} + \omega x_2^{1/m}\big) + iP_\Phi\big(x_3^{1/m} + \omega x_4^{1/m}\big).\end{aligned}$$

Hence, by (2.3),

$$\begin{aligned}\|\Phi(x)\|^q &\le \big\|P_\Phi\big(x_1^{1/m} + \omega x_2^{1/m}\big)\big\|^q + \big\|P_\Phi\big(x_3^{1/m} + \omega x_4^{1/m}\big)\big\|^q \\ &\le \|P_\Phi\|^q \big\|x_1^{1/m} + \omega x_2^{1/m}\big\|^{mq} + \|P_\Phi\|^q \big\|x_3^{1/m} + \omega x_4^{1/m}\big\|^{mq} \\ &= \|P_\Phi\|^q \left(\|\Re x\|^q + \|\Im x\|^q\right) \\ &\le \|P_\Phi\|^q 2\|x\|^q.\end{aligned}$$

This clearly forces $\|\Phi\| \le 2^{1/q}\|P_\Phi\|$, as claimed.

(iv) It is straightforward to check that $P_\Phi^* = P_{\Phi^*}$. Consequently, if Φ is hermitian, then $P_\Phi^* = P_{\Phi^*} = P_\Phi$ so that P_Φ is hermitian. Conversely, if P_Φ is hermitian, then $P_{\Phi^*} = P_\Phi^* = P_\Phi$ and (ii) implies that $\Phi^* = \Phi$. Finally, assume that Φ is a hermitian functional. For the calculation of $\|P_\Phi\|$ it suffices to check that $\|\Phi\| \le \|P_\Phi\|$. For this purpose, let $\varepsilon \in \mathbb{R}^+$, and choose $x \in \mathcal{A}$ such that $\|x\| = 1$ and $\|\Phi\| - \varepsilon < |\Phi(x)|$. We take $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $|\Phi(x)| = \alpha\Phi(x)$, so that

$$\|\Phi\| - \varepsilon < |\Phi(x)| = \Phi(\alpha x) = \overline{\Phi(\alpha x)} = \Phi\big((\alpha x)^*\big).$$

Note that $\|\Re(\alpha x)\| \le 1$ and $\|\Phi\| - \varepsilon < \Phi(\Re(\alpha x))$. Now we consider the decomposition $\Re(\alpha x) = x_1 - x_2$ with $x_1, x_2 \in \mathcal{A}_+$ and $x_1 \perp x_2$ and take $\omega \in \mathbb{C}$ with $\omega^m = -1$. As in (2.3), we see that $\big\|x_1^{1/m} + \omega x_2^{1/m}\big\| = \|\Re(\alpha x)\|^{1/m} \le 1$. Moreover, we have

$$P_\Phi\big(x_1^{1/m} + \omega x_2^{1/m}\big) = \Phi\big((x_1^{1/m} + \omega x_2^{1/m})^m\big) = \Phi(\Re(\alpha x)),$$

which gives $\|\Phi\| - \varepsilon < \|P_\Phi\|$. □

Lemma 2.2.3. *Let $\mathcal{A}$ be a C^*-algebra, let $\mathcal{R}$ be a $*$-subalgebra of $\mathcal{A}$, let X be a topological linear space, and let $\Phi\colon \mathcal{R} \to X$ be a linear map. Suppose that the polynomial $P\colon \mathcal{R} \to X$ defined by $P(x) = \Phi(x^m)$ ($x \in \mathcal{R}$) is continuous and that $\mathcal{R}$ satisfies the following conditions:*

(i) $|x| \in \mathcal{R}$ for each $x \in \mathcal{R}_{\mathrm{sa}}$;

(ii) $x^{1/m} \in \mathcal{R}$ for each $x \in \mathcal{R}_+$.

Then Φ is continuous.

Proof. Let U be a neighbourhood of 0 in X. Let V be a balanced neighbourhood of 0 in X with $V + V + V + V \subset U$. The set $P^{-1}(V)$ is a neighbourhood of 0 in $\mathcal{R}$, which implies that there exists $r \in \mathbb{R}^+$ such that $P(x) \in V$ whenever $x \in \mathcal{R}$ and $\|x\| < r$. Take $x \in \mathcal{R}$ with $\|x\| < r^m$. Since $\mathcal{R}$ is a $*$-subalgebra of $\mathcal{A}$, we see that $\Re x, \Im x \in \mathcal{R}_{\mathrm{sa}}$. We write

$\Re x = x_1 - x_2$ and $\Im x = x_3 - x_4$, as in the proof of Theorem 2.2.2, where, on account of the condition (i),

$$x_1 = \tfrac{1}{2}\big(|\Re x| + \Re x\big) \in \mathcal{R}_+, \qquad x_2 = \tfrac{1}{2}\big(|\Re x| - \Re x\big) \in \mathcal{R}_+,$$
$$x_3 = \tfrac{1}{2}\big(|\Im x| + \Im x\big) \in \mathcal{R}_+, \qquad x_4 = \tfrac{1}{2}\big(|\Im x| - \Im x\big) \in \mathcal{R}_+.$$

For each $j \in \{1,2,3,4\}$, condition (ii) gives $x_j^{1/m} \in \mathcal{R}$, and, further, we have $\|x_j^{1/m}\| = \|x_j\|^{1/m} \le \|x\|^{1/m} < r$. Hence

$$\begin{aligned}\Phi(x) &= \Phi\big((x_1^{1/m})^m - (x_2^{1/m})^m + i(x_3^{1/m})^m - i(x_4^{1/m})^m\big)\\ &= \Phi\big((x_1^{1/m})^m\big) - \Phi\big((x_2^{1/m})^m\big) + i\Phi\big((x_3^{1/m})^m\big) - i\Phi\big((x_4^{1/m})^m\big)\\ &= P\big(x_1^{1/m}\big) - P\big(x_2^{1/m}\big) + iP\big(x_3^{1/m}\big) - iP\big(x_4^{1/m}\big) \in V+V+V+V \subset U,\end{aligned}$$

which establishes the continuity of Φ. □

Theorem 2.2.4. *Let $\mathcal{A}$ be a C^*-algebra, let X be a locally convex space, and let $P\colon \mathcal{A} \to X$ be a continuous m-homogeneous polynomial. Then the following conditions are equivalent:*

(i) there exists a continuous linear map $\Phi\colon \mathcal{A} \to X$ such that $P(x) = \Phi(x^m)$ $(x \in \mathcal{A})$;

(ii) the polynomial P is orthogonally additive on $\mathcal{A}_{\mathrm{sa}}$;

(iii) the polynomial P is orthogonally additive on $\mathcal{A}_+$.

If the conditions are satisfied, then the map Φ is unique.

Proof. Theorem 2.2.2 gives (i)⇒(ii), and obviously (ii)⇒(iii). The task is now to prove that (iii)⇒(i).

Suppose that (iii) holds. For each continuous linear functional $\eta\colon X \to \mathbb{C}$, set $P_\eta = \eta \circ P$. Then P_η is a complex-valued continuous m-homogeneous polynomial. We claim that P_η is orthogonally additive on $\mathcal{A}_{\mathrm{sa}}$. Take $x, y \in \mathcal{A}_{\mathrm{sa}}$ with $x \perp y$. Then we can write $x = x_+ - x_-$ and $y = y_+ - y_-$ with $x_+, x_-, y_+, y_- \in \mathcal{A}_+$ mutually orthogonal. Define $f\colon \mathbb{C}^2 \to \mathbb{C}$ by

$$f(\alpha,\beta) = P_\eta(x_+ + \alpha x_- + y_+ + \beta y_-) - P_\eta(x_+ + \alpha x_-) - P_\eta(y_+ + \beta y_-) \quad (\alpha,\beta \in \mathbb{C}^2).$$

Then f is a complex polynomial function in two complex variables. If $\alpha, \beta \in \mathbb{R}^+$, then $x_+ + \alpha x_-, y_+ + \beta y_- \in \mathcal{A}_+$ are mutually orthogonal, and so, by hypothesis, $P(x_+ + \alpha x_- + y_+ + \beta y_-) = P(x_+ + \alpha x_-) + P(y_+ + \beta y_-)$. This shows that $f(\alpha,\beta) = 0$. Since f vanishes on $\mathbb{R}^+ \times \mathbb{R}^+$, it follows that f vanishes on $\mathbb{C}^2$, which, in particular, implies

$$P_\eta(x+y) - P_\eta(x) - P_\eta(y) = f(-1,-1) = 0.$$

Having proved that P_η is orthogonally additive on $\mathcal{A}_{\mathrm{sa}}$ we can apply [67, Theorem 2.8] to obtain a unique continuous linear functional Φ_η on $\mathcal{A}$ such that

$$\eta\big(P(x)\big) = \Phi_\eta(x^m) \quad (x \in \mathcal{A}). \tag{2.4}$$

Each $x \in \mathcal{A}$ can be written in the form $x_1^m + \cdots + x_k^m$ for suitable $x_1, \ldots, x_k \in \mathcal{A}$, and we define

$$\Phi(x) = \sum_{j=1}^{k} P(x_j).$$

Our next goal is to show that Φ is well-defined. Suppose that $x_1, \ldots, x_k \in \mathcal{A}$ are such that $x_1^m + \cdots + x_k^m = 0$. For each continuous linear functional η on X, (2.4) gives

$$\eta\left(\sum_{j=1}^{k} P(x_j)\right) = \sum_{j=1}^{k} \eta\big(P(x_j)\big) = \sum_{j=1}^{k} \Phi_\eta(x_j^m) = \Phi_\eta\left(\sum_{j=1}^{k} x_j^m\right) = 0.$$

Since X is locally convex, we conclude that $\sum_{j=1}^{k} P(x_j) = 0$.

It is a simple matter to check that Φ is linear and, by definition, $P(x) = \Phi(x^m)$ $(x \in \mathcal{A})$. The continuity of Φ then follows from Lemma 2.2.3.

The uniqueness of the map Φ follows from Theorem 2.2.2(ii). $\square$

The assumption that the space X be locally convex can be removed by requiring that the C^*-algebra $\mathcal{A}$ be sufficiently rich in projections. The real rank zero is the most important existence of projections property in the theory of C^*-algebras. We refer the reader to [22, Section V.3.2] and [37, Section V.7] for the basic properties and examples of C^*-algebras of real rank zero. This class of C^*-algebras contains the von Neumann algebras and the C^*-algebras $\mathcal{K}(H)$ of all compact operators on any Hilbert space H. Let us remark that every C^*-algebra of real rank zero has an approximate unit of projections (but not necessarily increasing).

Theorem 2.2.5. *Let $\mathcal{A}$ be a C^*-algebra of real rank zero, let X be a topological linear space, and let $P\colon \mathcal{A} \to X$ be a continuous m-homogeneous polynomial. Suppose that $\mathcal{A}$ has an increasing approximate unit of projections. Then the following conditions are equivalent:*

(i) there exists a continuous linear map $\Phi\colon \mathcal{A} \to X$ such that $P(x) = \Phi(x^m)$ $(x \in \mathcal{A})$;

(ii) the polynomial P is orthogonally additive on $\mathcal{A}_{\mathrm{sa}}$;

(iii) the polynomial P is orthogonally additive on $\mathcal{A}_+$.

If the conditions are satisfied, then the map Φ is unique.

Proof. Theorem 2.2.2 gives (i)⇒(ii), and it is clear that (ii)⇒(iii). We will henceforth prove that (iii)⇒(i).

We first note that such a map Φ is necessarily unique, because of Theorem 2.2.2(ii).

Suppose that (iii) holds and that $\mathcal{A}$ is unital. Let $\varphi\colon \mathcal{A}^m \to X$ be the symmetric m-linear map associated with P and define $\Phi\colon \mathcal{A} \to X$ by

$$\Phi(x) = \varphi(x, 1, \ldots, 1) \quad (x \in \mathcal{A}).$$

Let $Q\colon \mathcal{A} \to X$ be the m-homogeneous polynomial defined by

$$Q(x) = \Phi(x^m) \quad (x \in \mathcal{A}).$$

We will prove that $P = Q$. On account of Lemma 2.2.1, it suffices to show that $P(x) = Q(x)$ for each $x \in \mathcal{A}_{\mathrm{sa}}$.

First, consider the case where $x \in \mathcal{A}_{\mathrm{sa}}$ has finite spectrum, say $\{\rho_1, \dots, \rho_k\} \subset \mathbb{R}$. This implies that x can be written in the form

$$x = \sum_{j=1}^{k} \rho_j e_j,$$

where $e_1, \dots, e_k \in \mathcal{A}$ are mutually orthogonal projections (specifically, the projection e_j is defined by using the continuous functional calculus for x by $e_j = \chi_{\{\rho_j\}}(x)$ for each $j \in \{1, \dots, k\}$). We also set $e_0 = 1 - (e_1 + \cdots + e_k)$, so that the projections $e_0, e_1, \dots, e_k$ are mutually orthogonal, and $\rho_0 = 0$. We claim that if $j_1, \dots, j_m \in \{0, \dots, k\}$ and $j_l \neq j_{l'}$ for some $l, l' \in \{1, \dots, m\}$, then

$$\varphi(e_{j_1}, \dots, e_{j_m}) = 0. \tag{2.5}$$

Let $\Lambda_1 = \bigl\{n \in \{1, \dots, m\} : j_n = j_l\bigr\}$ and $\Lambda_2 = \bigl\{n \in \{1, \dots, m\} : j_n \neq j_l\bigr\}$. For each $\alpha_1, \dots, \alpha_m \in \mathbb{R}^+$, the elements $\sum_{n\in\Lambda_1} \alpha_n e_{j_n}$ and $\sum_{n\in\Lambda_2} \alpha_n e_{j_n}$ are positive and mutually orthogonal, so that the orthogonal additivity of P on $\mathcal{A}_+$ gives

$$P\left(\sum_{n=1}^{m} \alpha_n e_{j_n}\right) = P\left(\sum_{n\in\Lambda_1} \alpha_n e_{j_n}\right) + P\left(\sum_{n\in\Lambda_2} \alpha_n e_{j_n}\right).$$

This implies that, for each linear functional $\eta\colon X \to \mathbb{C}$, the function $f\colon \mathbb{C}^m \to \mathbb{C}$ defined by

$$f(\alpha_1, \dots, \alpha_m) = \eta\left(P\Bigl(\sum_{n=1}^{m} \alpha_n e_{j_n}\Bigr) - P\Bigl(\sum_{n\in\Lambda_1} \alpha_n e_{j_n}\Bigr) - P\Bigl(\sum_{n\in\Lambda_2} \alpha_n e_{j_n}\Bigr)\right),$$

for all $\alpha_1, \dots, \alpha_m \in \mathbb{C}$, is a complex polynomial function in m complex variables vanishing in $\bigl(\mathbb{R}^+\bigr)^m$. Therefore f vanishes on $\mathbb{C}^m$. Moreover, we observe that the coefficient of the monomial $\alpha_1 \cdots \alpha_m$ is given by $n!\eta\bigl(\varphi(e_{j_1}, \dots, e_{j_m})\bigr)$, because both Λ_1 and Λ_2 are different from $\{1, \dots, m\}$. We thus get

$$n!\eta\bigl(\varphi(e_{j_1}, \dots, e_{j_m})\bigr) = 0.$$

Since this identity holds for each linear functional η, our claim follows. Property (2.5) now leads to

$$\begin{aligned} P(x) &= \varphi\left(\sum_{j=1}^{k} \rho_j e_j, \dots, \sum_{j=1}^{k} \rho_j e_j\right) = \sum_{j_1,\dots,j_m=1}^{k} \rho_{j_1} \cdots \rho_{j_m} \varphi\left(e_{j_1}, \dots, e_{j_m}\right) \\ &= \sum_{j=1}^{k} \rho_j^m \varphi\left(e_j, \dots, e_j\right) \end{aligned}$$

and

$$Q(x) = \varphi\left(\Big(\sum_{j=0}^{k}\rho_j e_j\Big)^m, \sum_{j=0}^{k} e_j, \dots, \sum_{j=0}^{k} e_j\right) = \varphi\left(\sum_{j=0}^{k}\rho_j^m e_j, \sum_{j=0}^{k} e_j, \dots, \sum_{j=0}^{k} e_j\right)$$
$$= \sum_{j_1,\dots,j_m=0}^{k} \rho_{j_1}^m \varphi\left(e_{j_1}, \dots, e_{j_m}\right) = \sum_{j=1}^{k}\rho_j^m\varphi\left(e_j, \dots, e_j\right).$$

We thus get $P(x) = Q(x)$.

Now suppose that $x \in \mathcal{A}_{\mathrm{sa}}$ is an arbitrary element. Since $\mathcal{A}$ has real rank zero, it follows that there exists a sequence (x_n) in $\mathcal{A}_{\mathrm{sa}}$ such that each x_n has finite spectrum and $\lim x_n = x$. On account of the above case, we have $P(x_n) = Q(x_n)$ $(n \in \mathbb{N})$, and the continuity of both P and Q now yields $P(x) = \lim P(x_n) = \lim Q(x_n) = Q(x)$, as required.

We are now in a position to prove the non-unital case. By hypothesis, there exists an increasing approximate unit of projections $(e_\lambda)_{\lambda\in\Lambda}$. For each $\lambda \in \Lambda$, set $\mathcal{A}_\lambda = e_\lambda \mathcal{A} e_\lambda$. Then $\mathcal{A}_\lambda$ is a unital C^*-algebra (with identity e_λ) and has real rank zero (because $\mathcal{A}_\lambda$ is a hereditary C^*-subalgebra of $\mathcal{A}$). From what has previously been proved, it follows that there exists a unique continuous linear map $\Phi_\lambda \colon \mathcal{A}_\lambda \to X$ such that

$$P(x) = \Phi_\lambda(x^m) \quad (x \in \mathcal{A}_\lambda). \tag{2.6}$$

Define

$$\mathcal{R} = \bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda$$

and, for each $x \in \mathcal{R}$, set

$$\Phi(x) = \Phi_\lambda(x),$$

where $\lambda \in \Lambda$ is such that $x \in \mathcal{A}_\lambda$. We will show that Φ is well-defined. Suppose $\lambda, \mu \in \Lambda$ are such that $x \in \mathcal{A}_\lambda \cap \mathcal{A}_\mu$. Then there exists $\nu \in \Lambda$ with $\lambda, \mu \le \nu$. Since the net $(e_\lambda)_{\lambda\in\Lambda}$ is increasing, we see that $e_\lambda, e_\mu \le e_\nu$ and therefore $\mathcal{A}_\lambda, \mathcal{A}_\mu \subset \mathcal{A}_\nu$. The uniqueness of the representation of P on both $\mathcal{A}_\lambda$ and $\mathcal{A}_\mu$ implies that $\Phi_\nu \mid_{\mathcal{A}_\lambda} = \Phi_\lambda$ and $\Phi_\nu \mid_{\mathcal{A}_\mu} = \Phi_\mu$, which implies that $\Phi_\lambda(x) = \Phi_\nu(x) = \Phi_\mu(x)$. We now show that $\mathcal{R}$ is a $*$-subalgebra of $\mathcal{A}$ and that Φ is linear. Take $x, y \in \mathcal{R}$ and $\alpha, \beta \in \mathbb{C}$. We take $\lambda, \mu \in \Lambda$ such that $x \in \mathcal{A}_\lambda$ and $y \in \mathcal{A}_\mu$. Then $x^* \in \mathcal{A}_\lambda \subset \mathcal{R}$. Now set $\nu \in \Lambda$ with $\lambda, \mu \le \nu$. Hence $x, y \in \mathcal{A}_\nu$, so that $\alpha x + \beta y, xy \in \mathcal{A}_\nu \subset \mathcal{R}$, which shows that $\mathcal{R}$ is a subalgebra of $\mathcal{A}$. Further, we have

$$\Phi(\alpha x + \beta y) = \Phi_\nu(\alpha x + \beta y) = \alpha\Phi_\nu(x) + \beta\Phi_\nu(y) = \alpha\Phi(x) + \beta\Phi(y),$$

which shows that Φ is linear.

From (2.6) we deduce that $P(x) = \Phi(x^m)$ for each $x \in \mathcal{R}$.

Our next goal is to show that $\mathcal{R}$ satisfies the conditions of Lemma 2.2.3. If $x \in \mathcal{R}_{\mathrm{sa}}$ $(x \in \mathcal{R}_+)$, then there exists $\lambda \in \Lambda$ such that $x \in \left(\mathcal{A}_\lambda\right)_{\mathrm{sa}}$ $(x \in \left(\mathcal{A}_\lambda\right)_+$, respectively) and therefore $|x| \in \mathcal{A}_\lambda \subset \mathcal{R}$ $(x^{1/m} \in \mathcal{A}_\lambda \subset \mathcal{R}$, respectively). Since the polynomial $P\mid_{\mathcal{R}}$ is continuous, Lemma 2.2.3 shows that the map Φ is continuous.

Since $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit, it follows that $\mathcal{R}$ is dense in $\mathcal{A}$, and hence that the map Φ extends uniquely to a continuous linear map from $\mathcal{A}$ into the completion of X. By abuse of notation we continue to write Φ for this extension. Since both P and Φ are continuous, it may be concluded that $P(x) = \Phi(x^m)$ for each $x \in \mathcal{A}$. We next prove that the image of Φ is actually contained in X. Of course, it suffices to show that Φ takes $\mathcal{A}_+$ into X. If $x \in \mathcal{A}_+$, then

$$\Phi(x) = \Phi\big((x^{1/m})^m\big) = P\big(x^{1/m}\big) \in X,$$

as required. □

Since every von Neumann algebra is unital and has real rank zero, Theorem 2.2.5 applies in this setting and gives the following.

Corollary 2.2.6. *Let $\mathcal{M}$ be a von Neumann algebra, let X be a topological linear space, and let $P\colon \mathcal{M} \to X$ be a continuous m-homogeneous polynomial. Then the following conditions are equivalent:*

(i) there exists a continuous linear map $\Phi\colon \mathcal{M} \to X$ such that $P(x) = \Phi(x^m)$ $(x \in \mathcal{M})$;

(ii) the polynomial P is orthogonally additive on $\mathcal{M}_{\mathrm{sa}}$;

(iii) the polynomial P is orthogonally additive on $\mathcal{M}_+$.

If the conditions are satisfied, then the map Φ is unique.

Proposition 2.2.7. *Let H be a Hilbert space with $\dim H \geq 2$, let X be a topological linear space, and let $P\colon \mathcal{B}(H) \to X$ be a continuous m-homogeneous polynomial. Suppose that P is orthogonally additive in $\mathcal{B}(H)$. Then $P = 0$.*

Proof. For each unitary $v \in \mathcal{B}(H)$, the map $P_v\colon \mathcal{B}(H) \to X$ defined by

$$P_v(x) = P(vx) \quad (x \in \mathcal{B}(H))$$

is easily seen to be a continuous m-homogeneous polynomial that is orthogonally additive on $\mathcal{B}(H)$. In particular, P_v is orthogonally additive on $\mathcal{B}(H)_{\mathrm{sa}}$, and Corollary 2.2.6 then gives a unique continuous linear map $\Phi_v\colon \mathcal{B}(H) \to X$ such that

$$P(vx) = \Phi_v(x^m) \quad (x \in \mathcal{B}(H)).$$

We claim that, if $e, e' \in \mathcal{B}(H)$ are equivalent projections with $e \perp e'$, then $P(e) = P(e') = 0$. Let $u \in \mathcal{B}(H)$ be a partial isometry such that $u^*u = e$ and $uu^* = e'$. Then

$$\left\|u^2\right\|^4 = \left\|(u^2)^*u^2\right\|^2 = \left\|\left((u^2)^*u^2\right)^2\right\| = \left\|u^*ee'eu\right\| = 0,$$

which gives $u^2 = 0$. From this we see that $u \perp u^*$, and therefore

$$P(vu + vu^*) = P_v(u + u^*) = P_v(u) + P_v(u^*) = \Phi_v\left(u^m\right) + \Phi_v\left((u^*)^m\right) = 0. \tag{2.7}$$

We now take $\omega \in \mathbb{C}$ with $\omega^m = -1$, and define

$$v = 1 + u + u^* - e - e',$$
$$v_\omega = 1 + \omega u + u^* - e - e'.$$

It is immediately seen that both v and v_ω are unitary, and so applying (2.7) (and using the orthogonal additivity of P and that $e \perp e'$), we see that

$$0 = P(vu + vu^*) = P(e + e') = P(e) + P(e'),$$
$$0 = P(v_\omega u + v_\omega u^*) = P(e + \omega e') = P(e) + P(\omega e') = P(e) - P(e').$$

By comparing both identities, we conclude that $P(e) = P(e') = 0$, as claimed.

Our next objective is to prove that $P(e) = 0$ for each projection $e \in \mathcal{B}(H)$. Suppose that $e \in \mathcal{B}(H)$ is a rank-one projection. Since $\dim H \geq 2$, it follows that there exists an equivalent projection e' such that $e' \perp e$. Then it follows from the above claim that $P(e) = 0$. Let $e \in \mathcal{B}(H)$ be a finite projection. Then there exist mutually orthogonal projections $e_1, \ldots, e_n$ such that $e_1 + \cdots + e_n = e$. Using the preceding observation and the orthogonal additivity of P we get $P(e) = P(e_1) + \cdots + P(e_n) = 0$. We now assume that $e \in \mathcal{B}(H)$ is an infinite projection. Then there exist mutually orthogonal, equivalent projections e_1 and e_2 such that $e_1 + e_2 = e$. By the claim, we have $P(e) = P(e_1) + P(e_2) = 0$.

We finally proceed to show that $P = 0$. By Lemma 2.2.1, it suffices to show that $P(x) = 0$ for each $x \in \mathcal{B}(H)_+$. Suppose that $x \in \mathcal{B}(H)_+$ can be written in the form $x = \sum_{j=1}^k \rho_j e_j$, where $e_1, \ldots, e_k \in \mathcal{B}(H)$ are mutually orthogonal projections and $\rho_1, \ldots, \rho_k \in \mathbb{R}^+$. Then we have $P(x) = \sum_{j=1}^k {\rho_j}^m P(e_j) = 0$. Now let $x \in \mathcal{B}(H)_+$ be an arbitrary element. From the spectral decomposition we deduce that there exists a sequence (x_n) in $\mathcal{B}(H)_+$ such that each x_n is a positive linear combination of mutually orthogonal projections and $\lim x_n = x$. On account of the preceding observation, $P(x_n) = 0$ ($n \in \mathbb{N}$), and the continuity of P implies that $P(x) = \lim P(x_n) = 0$, as required. □

2.3 Non-commutative L^p-spaces

Before giving the next results we make the following preliminary remarks.

A fundamental fact for us is the behaviour of the product of $L^0(\mathcal{M}, \tau)$ when restricted to the L^p-spaces. Specifically, if $0 < p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then the Hölder inequality states that

$$x \in L^p(\mathcal{M}, \tau),\ y \in L^q(\mathcal{M}, \tau) \ \Rightarrow\ xy \in L^r(\mathcal{M}, \tau) \text{ and } \|xy\|_r \leq \|x\|_p \|y\|_q. \tag{2.8}$$

Suppose that $x, y \in L^p(\mathcal{M}, \tau)_+$, $0 < p < \infty$, are mutually orthogonal and that $\omega \in \mathbb{C}$ with $|\omega| = 1$. Then it is immediately seen that $|x + \omega y| = x + y$, and it follows, by considering the spectral resolutions of x, y, and $x + y$, that $(x + y)^p = x^p + y^p$. Hence

$$\|x + \omega y\|_p^p = \|x\|_p^p + \|y\|_p^p. \tag{2.9}$$

Each $x \in L^p(\mathcal{M}, \tau)$ can be written in the form

$$\begin{aligned} x = x_1 - x_2 + i(x_3 - x_4), \text{ with } & x_1, x_2, x_3, x_4 \in L^p(\mathcal{M}, \tau)_+, \\ & x_1 \perp x_2, \ x_3 \perp x_4, \\ & \|x_1\|_p^p + \|x_2\|_p^p = \|x_1 - x_2\|_p^p \leq \|x\|_p^p, \\ & \|x_3\|_p^p + \|x_4\|_p^p = \|x_3 - x_4\|_p^p \leq \|x\|_p^p. \end{aligned} \tag{2.10}$$

Indeed, first we write $x = \Re x + i\Im x$, where

$$\Re x = \frac{1}{2}(x^* + x), \ \Im x = \frac{i}{2}(x^* - x) \in L^p(\mathcal{M}, \tau)_{\mathrm{sa}},$$

and, since $\|x^*\|_p = \|x\|_p$, it follows that $\|\Re x\|_p, \|\Im x\|_p \leq \|x\|_p$. Further, we take the positive operators

$$x_1 = \tfrac{1}{2}\left(|\Re x| + \Re x\right), \ x_2 = \tfrac{1}{2}\left(|\Re x| - \Re x\right), \ x_3 = \tfrac{1}{2}\left(|\Im x| + \Im x\right), \ x_4 = \tfrac{1}{2}\left(|\Im x| - \Im x\right).$$

Then $x_1, x_2, x_3, x_4 \in L^p(\mathcal{M}, \tau)$, $\Re x = x_1 - x_2$ with $x_1 \perp x_2$, so that (2.9) gives

$$\|\Re x\|_p^p = \|x_1\|_p^p + \|x_2\|_p^p,$$

and $\Im x = x_3 - x_4$ with $x_3 \perp x_4$, so that (2.9) gives

$$\|\Im x\|_p^p = \|x_3\|_p^p + \|x_4\|_p^p.$$

Theorem 2.3.1. *Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ, let X be a topological linear space, and let $\Phi \colon L^{p/m}(\mathcal{M}, \tau) \to X$ be a continuous linear map with $0 < p < \infty$. Then:*

(i) the map $P_\Phi \colon L^p(\mathcal{M}, \tau) \to X$ defined by $P_\Phi(x) = \Phi(x^m)$ $(x \in L^p(\mathcal{M}, \tau))$ is a continuous m-homogeneous polynomial which is orthogonally additive on $L^p(\mathcal{M}, \tau)_{\mathrm{sa}}$;

(ii) the polynomial P_Φ is uniquely specified by the map Φ.

Suppose, further, that X is a q-normed space, $0 < q \leq 1$. Then:

(iii) $2^{-1/q}\|\Phi\| \leq \|P_\Phi\| \leq \|\Phi\|$.

Moreover, in the case where $X = \mathbb{C}$,

(iv) the functional Φ is hermitian if and only if the polynomial P_Φ is hermitian, in which case $\|P_\Phi\| = \|\Phi\|$.

Proof. The proof of this result is similar to that establishing Theorem 2.2.2.

(i) It follows immediately from (2.8) that, for each $x_1, \ldots, x_m \in L^p(\mathcal{M}, \tau)$,

$$x_1 \cdots x_m \in L^{p/m}(\mathcal{M}, \tau) \text{ and } \|x_1 \cdots x_m\|_{p/m} \leq \|x_1\|_p \cdots \|x_m\|_p. \tag{2.11}$$

On the one hand, this clearly implies that the map P_Φ is well-defined, on the other hand, the map $x \mapsto x^m$ from $L^p(\mathcal{M}, \tau)$ into $L^{p/m}(\mathcal{M}, \tau)$ is continuous, and so P_Φ is continuous.

Further, P_Φ is the m-homogeneous polynomial associated with the symmetric m-linear map $\varphi\colon L^p(\mathcal{M},\tau)^m \to X$ defined by

$$\varphi(x_1,\dots,x_m) = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} \Phi\left(x_{\sigma(1)}\cdots x_{\sigma(m)}\right) \quad (x_1,\dots,x_m \in L^p(\mathcal{M},\tau)).$$

Suppose that $x, y \in L^p(\mathcal{M},\tau)_{\mathrm{sa}}$ are such that $x \perp y$. Then $xy = yx = 0$, and so $(x+y)^m = x^m + y^m$, which gives

$$P_\Phi(x+y) = \Phi\big((x+y)^m\big) = \Phi\big(x^m + y^m\big) = \Phi\big(x^m\big) + \Phi\big(y^m\big) = P_\Phi(x) + P_\Phi(y).$$

(ii) Suppose that $\Psi\colon L^{p/m}(\mathcal{M},\tau) \to X$ is a linear map such that $P_\Psi = P_\Phi$. For each $x \in L^{p/m}(\mathcal{M},\tau)_+$, we have $x^{1/m} \in L^p(\mathcal{M},\tau)$ and

$$\Phi(x) = \Phi\big(\big(x^{1/m}\big)^m\big) = P\big(x^{1/m}\big) = \Psi\big(\big(x^{1/m}\big)^m\big) = \Psi(x).$$

By linearity we obtain $\Phi = \Psi$.

(iii) Next, assume that X is a q-normed space. For each $x \in L^p(\mathcal{M},\tau)$, by (2.11), we have

$$\|P_\Phi(x)\| = \|\Phi(x^m)\| \le \|\Phi\|\|x^m\|_{p/m} \le \|\Phi\|\|x\|_p^m,$$

which clearly implies that $\|P_\Phi\| \le \|\Phi\|$. Now take $x \in L^{p/m}(\mathcal{M},\tau)$, and take $\omega \in \mathbb{C}$ with $\omega^m = -1$. Write

$$x = \Re x + i\Im x = x_1 - x_2 + i(x_3 - x_4)$$

as in (2.10) (with p/m instead of p). Since $x_1 \perp x_2$ and $x_3 \perp x_4$, it follows that $x_1^{1/m} \perp x_2^{1/m}$ and $x_3^{1/m} \perp x_4^{1/m}$, so that (2.9) gives

$$\begin{aligned} \|\Re x\|_{p/m}^{p/m} &= \|x_1\|_{p/m}^{p/m} + \|x_2\|_{p/m}^{p/m}, \\ \|\Im x\|_{p/m}^{p/m} &= \|x_3\|_{p/m}^{p/m} + \|x_4\|_{p/m}^{p/m}, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \big\|x_1^{1/m} + \omega x_2^{1/m}\big\|_p^p &= \big\|x_1^{1/m}\big\|_p^p + \big\|x_2^{1/m}\big\|_p^p, \\ \big\|x_3^{1/m} + \omega x_4^{1/m}\big\|_p^p &= \big\|x_3^{1/m}\big\|_p^p + \big\|x_4^{1/m}\big\|_p^p. \end{aligned} \tag{2.13}$$

Further, we have $x_1^{1/m}, x_2^{1/m}, x_3^{1/m}, x_4^{1/m} \in L^p(\mathcal{M},\tau)$ and

$$\big\|x_1^{1/m}\big\|_p = \|x_1\|_{p/m}^{1/m},\ \big\|x_2^{1/m}\big\|_p = \|x_2\|_{p/m}^{1/m},\ \big\|x_3^{1/m}\big\|_p = \|x_3\|_{p/m}^{1/m},\ \big\|x_4^{1/m}\big\|_p = \|x_4\|_{p/m}^{1/m},$$

so that (2.12) and (2.13) give

$$\begin{aligned} \big\|x_1^{1/m} + \omega x_2^{1/m}\big\|_p^p &= \|\Re x\|_{p/m}^{p/m}, \\ \big\|x_3^{1/m} + \omega x_4^{1/m}\big\|_p^p &= \|\Im x\|_{p/m}^{p/m}. \end{aligned} \tag{2.14}$$

On the other hand, we have

$$\big(x_1^{1/m} + \omega x_2^{1/m}\big)^m = x_1 - x_2 = \Re x, \quad \big(x_3^{1/m} + \omega x_4^{1/m}\big)^m = x_3 - x_4 = \Im x,$$

whence

$$\begin{aligned}\Phi(x) &= \Phi(\Re x) + i\Phi(\Im x) = \Phi\big(\big(x_1^{1/m} + \omega x_2^{1/m}\big)^m\big) + i\Phi\big(\big(x_3^{1/m} + \omega x_4^{1/m}\big)^m\big) \\ &= P_\Phi\big(x_1^{1/m} + \omega x_2^{1/m}\big) + iP_\Phi\big(x_3^{1/m} + \omega x_4^{1/m}\big).\end{aligned}$$

Hence, by (2.14),

$$\begin{aligned}\|\Phi(x)\|^q &\le \big\|P_\Phi\big(x_1^{1/m} + \omega x_2^{1/m}\big)\big\|^q + \big\|P_\Phi\big(x_3^{1/m} + \omega x_4^{1/m}\big)\big\|^q \\ &\le \|P_\Phi\|^q \big\|x_1^{1/m} + \omega x_2^{1/m}\big\|_p^{mq} + \|P_\Phi\|^q \big\|x_3^{1/m} + \omega x_4^{1/m}\big\|_p^{mq} \\ &= \|P_\Phi\|^q \left(\|\Re x\|^q + \|\Im x\|^q\right) \\ &\le \|P_\Phi\|^q 2\|x\|^q.\end{aligned}$$

This clearly forces $\|\Phi\| \le 2^{1/q}\|P_\Phi\|$, as claimed.

(iv) It is straightforward to check that $P_\Phi^* = P_{\Phi^*}$. From this deduce that Φ is hermitian if and only if P_Φ is hermitian as in the proof of Theorem 2.2.2(iv). Suppose that Φ is a hermitian functional. By direct calculation, we see that P_Φ is hermitian, and it remains to prove that $\|P_\Phi\| = \|\Phi\|$. We only need to show that $\|\Phi\| \le \|P_\Phi\|$. To this end, let $\varepsilon \in \mathbb{R}^+$, and choose $x \in L^{p/m}(\mathcal{M},\tau)$ such that $\|x\|_{p/m} = 1$ and $\|\Phi\| - \varepsilon < |\Phi(x)|$. We take $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $|\Phi(x)| = \alpha\Phi(x)$, so that

$$\|\Phi\| - \varepsilon < |\Phi(x)| = \Phi(\alpha x) = \overline{\Phi(\alpha x)} = \Phi\big((\alpha x)^*\big).$$

We see that $\Re(\alpha x) \in L^{p/m}(\mathcal{M},\tau)_{\mathrm{sa}}$, $\|\Re(\alpha x)\|_{p/m} \le 1$, and $\|\Phi\| - \varepsilon < \Phi(\Re(\alpha x))$. Now we consider the decomposition $\Re(\alpha x) = x_1 - x_2$ as in (2.10) (with p/m instead of p), and take $\omega \in \mathbb{C}$ with $\omega^m = -1$. As in (2.14), we see that $\big\|x_1^{1/m} + \omega x_2^{1/m}\big\| = \|\Re(\alpha x)\|^{1/m} \le 1$. Moreover, we have

$$P_\Phi\big(x_1^{1/m} + \omega x_2^{1/m}\big) = \Phi\big(\big(x_1^{1/m} + \omega x_2^{1/m}\big)^m\big) = \Phi(\Re(\alpha x)),$$

and so $\|\Phi\| - \varepsilon < \|P_\Phi\|$. □

Theorem 2.3.2. *Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ, let X be a topological linear space, and let $P\colon L^p(\mathcal{M},\tau) \to X$ be a continuous m-homogeneous polynomial with $0 < p < \infty$. Then the following conditions are equivalent:*

(i) there exists a continuous linear map $\Phi\colon L^{p/m}(\mathcal{M},\tau) \to X$ such that $P(x) = \Phi(x^m)$ $(x \in L^p(\mathcal{M},\tau))$;

(ii) the polynomial P is orthogonally additive on $L^p(\mathcal{M},\tau)_{\mathrm{sa}}$;

(iii) the polynomial P is orthogonally additive on $S(\mathcal{M},\tau)_+$.

If the conditions are satisfied, then the map Φ is unique.

Proof. Theorem 2.3.1 shows that (i)⇒(ii), and it is obvious that (ii)⇒(iii). We proceed to prove that (iii)⇒(i).

Suppose that (iii) holds. Let $e \in \mathcal{M}$ be a projection such that $\tau(e) < \infty$, and consider the von Neumann algebra $\mathcal{M}_e = e\mathcal{M}e$. We claim that $\mathcal{M}_e \subset S(\mathcal{M}, \tau)$ and that there exists a unique continuous linear map $\Phi_e \colon \mathcal{M}_e \to X$ such that

$$P(x) = \Phi_e(x^m) \quad (x \in \mathcal{M}_e). \tag{2.15}$$

Set $x \in \mathcal{M}_e$, and write $x = (x_1 - x_2) + i(x_3 - x_4)$ with $x_1, x_2, x_3, x_4 \in \mathcal{M}_{e+}$. Then $\operatorname{supp}(x_j) \leq e$ and therefore $\tau\big(\operatorname{supp}(x_j)\big) \leq \tau(e) < \infty$ ($j \in \{1, 2, 3, 4\}$). This shows that $x_j \in S(\mathcal{M}, \tau)$ ($j \in \{1, 2, 3, 4\}$), whence $x \in S(\mathcal{M}, \tau)$. Our next goal is to show that the restriction $P\,|_{\mathcal{M}_e}$ is continuous (with respect to the norm that $\mathcal{M}_e$ inherits as a closed subspace of $\mathcal{M}$). Let $x \in \mathcal{M}_e$, and let $U \subset X$ be a neighbourhood of $P(x)$. Since P is continuous, the set $P^{-1}(U)$ is a neighbourhood of x in $L^p(\mathcal{M}, \tau)$, which implies that there exists $r \in \mathbb{R}^+$ such that $P(y) \in U$ whenever $y \in L^p(\mathcal{M}, \tau)$ and $\|y - x\|_p < r$. If $y \in \mathcal{M}_e$ is such that $\|y - x\| < r/\|e\|_p$, then, from (2.8), we obtain

$$\|y - x\|_p = \|e(y - x)\|_p \leq \|e\|_p \|y - x\| < r$$

and therefore $P(y) \in U$. Hence $P\,|_{\mathcal{M}_e}$ is continuous. Since, by hypothesis, the polynomial $P\,|_{\mathcal{M}_e}$ is orthogonally additive on $\mathcal{M}_{e+}$, Corollary 2.2.6 states that there exists a unique continuous linear map $\Phi_e \colon \mathcal{M}_e \to X$ such that (2.15) holds.

For each $x \in S(\mathcal{M}, \tau)$, define

$$\Phi(x) = \Phi_e(x),$$

where $e \in \mathcal{M}$ is any projection such that

$$ex = xe = x \quad \text{and} \quad \tau(e) < \infty. \tag{2.16}$$

We will show that Φ is well-defined. For this purpose we first check that, if $x \in S(\mathcal{M}, \tau)$, then there exists a projection e such that (2.16) holds. Indeed, we write $x = \sum_{j=1}^k \alpha_j x_j$ with $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ and $x_1, \ldots, x_k \in S(\mathcal{M}, \tau)_+$, and define $e = \operatorname{supp}(x_1) \vee \cdots \vee \operatorname{supp}(x_k)$. Then $ex = xe = x$ and $\tau(e) \leq \sum_{j=1}^k \tau\big(\operatorname{supp}(x_j)\big) < \infty$, as required. Suppose that $x \in S(\mathcal{M}, \tau)$ and that $e_1, e_2 \in \mathcal{M}$ are projections satisfying (2.16). Then the projection $e = e_1 \vee e_2$ satisfies (2.16) and $\mathcal{M}_{e_1}, \mathcal{M}_{e_2} \subset \mathcal{M}_e$. The uniqueness of the representation (2.15) on both $\mathcal{M}_{e_1}$ and $\mathcal{M}_{e_2}$ gives $\Phi_e\,|_{\mathcal{M}_{e_1}} = \Phi_{e_1}$ and $\Phi_e\,|_{\mathcal{M}_{e_2}} = \Phi_{e_2}$, which implies that $\Phi_{e_1}(x) = \Phi_e(x) = \Phi_{e_2}(x)$.

We now show that Φ is linear. Take $x_1, x_2 \in S(\mathcal{M}, \tau)$ and $\alpha, \beta \in \mathbb{C}$. Let $e_1, e_2 \in \mathcal{M}$ be projections such that $e_j x_j = x_j e_j = x_j$ and $\tau(e_j) < \infty$ ($j \in \{1, 2\}$). Then the projection $e = e_1 \vee e_2$ satisfies

$$\begin{gathered} e x_j = x_j e = x_j \quad (j \in \{1, 2\}), \\ e(\alpha x_1 + \beta x_2) = (\alpha x_1 + \beta x_2)e = \alpha x_1 + \beta x_2, \end{gathered}$$

and

$$\tau(e) \leq \tau(e_1) + \tau(e_2) < \infty.$$

Thus

$$\Phi(x_j) = \Phi_e(x_j) \quad (j \in \{1,2\})$$

and

$$\Phi(\alpha x_1 + \beta x_2) = \Phi_e(\alpha x_1 + \beta x_2) = \alpha\Phi_e(x_1) + \beta\Phi_e(x_2) = \alpha\Phi(x_1) + \beta\Phi(x_2).$$

We see from the definition of Φ that

$$P(x) = \Phi(x^m) \quad (x \in S(\mathcal{M},\tau)). \tag{2.17}$$

Our next concern will be the continuity of Φ with respect to the norm $\|\cdot\|_{p/m}$. Let U be a neighbourhood of 0 in X. Let V be a balanced neighbourhood of 0 in X with $V+V+V+V \subset U$. The set $P^{-1}(V)$ is a neighbourhood of 0 in $L^p(\mathcal{M},\tau)$, which implies that there exists $r \in \mathbb{R}^+$ such that $P(x) \in V$ whenever $x \in L^p(\mathcal{M},\tau)$ and $\|x\|_p < r$. Take $x \in S(\mathcal{M},\tau)$ with $\|x\|_{p/m} < r^m$, and write $x = (x_1 - x_2) + i(x_3 - x_4)$ as in (2.10) (with p/m instead of p). Then it is immediate to check that actually $x_1, x_2, x_3, x_4 \in S(\mathcal{M},\tau)_+$ and, further, $\|x_j\|_{p/m} \leq \|x\|_{p/m}$ $(j \in \{1,2,3,4\})$. For each $j \in \{1,2,3,4\}$, we have

$$\begin{aligned}\big\|x_j^{1/m}\big\|_p &= \tau\big(x_j^{p/m}\big)^{1/p} = \big(\tau\big(x_j^{p/m}\big)^{m/p}\big)^{1/m} = \|x_j\|_{p/m}{}^{1/m}\\ &\leq \|x\|_{p/m}^{1/m} < r,\end{aligned}$$

whence

$$\begin{aligned}\Phi(x) &= \Phi\left(\big(x_1^{1/m}\big)^m - \big(x_2^{1/m}\big)^m + i\big(x_3^{1/m}\big)^m - i\big(x_4^{1/m}\big)^m\right)\\ &= \Phi\left(\big(x_1^{1/m}\big)^m\right) - \Phi\left(\big(x_2^{1/m}\big)^m\right) + i\Phi\left(\big(x_3^{1/m}\big)^m\right) - i\Phi\left(\big(x_4^{1/m}\big)^m\right)\\ &= P\big(x_1^{1/m}\big) - P\big(x_2^{1/m}\big)\\ &\quad + iP\big(x_3^{1/m}\big) - iP\big(x_4^{1/m}\big) \in V+V+V+V \subset U,\end{aligned}$$

which establishes the continuity of Φ. Since $S(\mathcal{M},\tau)$ is dense in $L^{p/m}(\mathcal{M},\tau)$, the map Φ extends uniquely to a continuous linear map from $L^{p/m}(\mathcal{M},\tau)$ into the completion of X. By abuse of notation we continue to write Φ for this extension. Since both P and Φ are continuous, (2.17) gives $P(x) = \Phi(x^m)$ for each $x \in L^p(\mathcal{M})$. The task is now to show that the image of Φ is actually contained in X. Of course, it suffices to show that Φ takes $L^{p/m}(\mathcal{M},\tau)_+$ into X. Let $x \in L^{p/m}(\mathcal{M},\tau)_+$. Then $x^{1/m} \in L^p(\mathcal{M},\tau)_+$ and

$$\Phi(x) = \Phi\big(\big(x^{1/m}\big)^m\big) = P\big(x^{1/m}\big) \in X,$$

as required.

The uniqueness of the map Φ is given by Theorem 2.3.1(ii). $\square$

Let us note that the space of all continuous m-homogeneous polynomials from $L^p(\mathcal{M},\tau)$ into any topological linear space X which are orthogonally additive on $S(\mathcal{M},\tau)_+$ is sufficiently rich in the case where $p/m \geq 1$, because of the existence of continuous linear functionals on $L^{p/m}(\mathcal{M},\tau)$. However, some restriction on the space X must be imposed when we consider the case $p/m < 1$ and the von Neumann algebra $\mathcal{M}$ has no minimal projections, because in this case the dual of $L^{p/m}(\mathcal{M},\tau)$ is trivial ([74]). In fact, there are no non-zero continuous linear maps from $L^p(\mathcal{M},\tau)$ into any q-normed space X with $q > p$. We think that this property is probably well-known, but we have not been able to find any reference, so that we next present a proof of this result for completeness.

Proposition 2.3.3. *Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ and with no minimal projections, let X be a q-normed space, $0 < q \leq 1$, and let $\Phi\colon L^p(\mathcal{M},\tau) \to X$ be a continuous linear map with $0 < p < q$. Then $\Phi = 0$.*

Proof. The proof will be divided in a number of steps.

Our first step is to show that for each projection $e_0 \in \mathcal{M}$ with $\tau(e_0) < \infty$ and each $0 \leq \rho \leq \tau(e_0)$, there exists a projection $e \in \mathcal{M}$ such that $e \leq e_0$ and $\tau(e) = \rho$. Set

$$\mathcal{P}_1 = \bigl\{e \in \mathcal{M} : e \text{ is a projection, } e \leq e_0,\ \tau(e) \geq \rho\bigr\}.$$

Note that $e_0 \in \mathcal{P}_1$, so that $\mathcal{P}_1$ is non-empty. Let $\mathcal{C}$ be a chain in $\mathcal{P}_1$, and let $e' = \wedge_{e\in\mathcal{C}} e$. Then e' is a projection and $e' \leq e_0$. For each $e \in \mathcal{C}$, since $\tau(e_0) < \infty$, it follows that $\tau(e_0) - \tau(e) = \tau(e_0 - e)$. From the normality of τ we now deduce that

$$\begin{aligned}\tau(e_0) - \inf_{e\in\mathcal{C}} \tau(e) &= \sup_{e\in\mathcal{C}}\bigl(\tau(e_0) - \tau(e)\bigr) = \sup_{e\in\mathcal{C}} \tau(e_0 - e)\\ &= \tau\left(\vee_{e\in\mathcal{C}}(e_0 - e)\right) = \tau(e_0 - e').\end{aligned}$$

Hence $\tau(e') = \inf_{e\in\mathcal{C}} \tau(e) \geq \rho$, which shows that e' is a lower bound of $\mathcal{C}$, and so, by Zorn's lemma, $\mathcal{P}_1$ has a minimal element, say e_1. We now consider the set

$$\mathcal{P}_2 = \bigl\{e \in \mathcal{M} : e \text{ is a projection, } e \leq e_1,\ \tau(e) \leq \rho\bigr\}.$$

Note that $0 \in \mathcal{P}_2$, so that $\mathcal{P}_2$ is non-empty. Let $\mathcal{C}$ be a chain in $\mathcal{P}_2$, and let $e' = \vee_{e\in\mathcal{C}} e$. Then $e' \leq e_1$, and the normality of τ yields

$$\tau(e') = \sup_{e\in\mathcal{C}} \tau(e) \leq \rho.$$

This implies that e' is an upper bound of $\mathcal{C}$, and so, by Zorn's lemma, $\mathcal{P}_2$ has a maximal element, say e_2. Assume towards a contradiction that $e_1 \neq e_2$. Since, by hypothesis, $\mathcal{M}$ has no minimal projections, it follows that there exists a non-zero projection $e < e_1 - e_2$. Since $e \perp e_2$, we see that $e_2 + e$ is a projection. Further, we have $e_2 < e_2 + e < e_1$. The maximality of e_2 implies that $\tau(e_2 + e) > \rho$, which implies that $e_2 + e \in \mathcal{P}_1$, contradicting the minimality of e_1. Thus $e_1 = e_2$, and this clearly implies that $\tau(e_1) = \tau(e_2) = \rho$.

Our next goal is to show that $\Phi(e_0) = 0$ for each projection e_0 with $\tau(e_0) < \infty$. From the previous step, it follows that there exists a projection $e \le e_0$ with $\tau(e) = \frac{1}{2}\tau(e_0)$. Set $e' = e_0 - e$. Then $\tau(e') = \frac{1}{2}\tau(e_0)$. Further,

$$\|\Phi(e_0)\|^q = \|\Phi(e) + \Phi(e')\|^q \le \|\Phi(e)\|^q + \|\Phi(e')\|^q,$$

and therefore either $\|\Phi(e)\|^q \ge \frac{1}{2}\|\Phi(e_0)\|^q$ or $\|\Phi(e')\|^q \ge \frac{1}{2}\|\Phi(e_0)\|^q$. We define e_1 to be any of the projections e, e' for which the inequality holds. We thus get $e_1 \le e_0$, $\tau(e_1) = \frac{1}{2}\tau(e_0)$, and $\|\Phi(e_1)\| \ge 2^{-1/q}\|\Phi(e_0)\|$. By repeating the process, we get a decreasing sequence of projections (e_n) such that

$$\tau(e_n) = 2^{-n}\tau(e_0) \quad \text{and} \quad \|\Phi(e_n)\| \ge 2^{-n/q}\|\Phi(e_0)\| \quad (n \in \mathbb{N}).$$

Then

$$\left\|2^{n/q}e_n\right\|_p = 2^{n/q}\tau(e_n)^{1/p} = 2^{n(1/q-1/p)}\tau(e_0)^{1/p},$$

which converges to zero, because $p < q$. Since Φ is continuous and

$$\|\Phi(e_0)\| \le \left\|\Phi\big(2^{n/q}e_n\big)\right\|_p \quad (n \in \mathbb{N}),$$

it may be concluded that $\Phi(e_0) = 0$, as claimed.

Our next concern is to show that Φ vanishes on $S(\mathcal{M}, \tau)$. Of course, it suffices to show that Φ vanishes on $S(\mathcal{M}, \tau)_+$. Take $x \in S(\mathcal{M}, \tau)_+$, and let $e = \operatorname{supp}(x)$, so that $\tau(e) < \infty$. The spectral decomposition implies that there exists a sequence (x_n) in $\mathcal{M}_+$ such that $\lim x_n = x$ with respect to the operator norm and each x_n is of the form $x_n = \sum_{j=1}^k \rho_j e_j$, where $\rho_1, \dots, \rho_k \in \mathbb{R}^+$ and $e_1, \dots, e_k \in \mathcal{M}$ are mutually orthogonal projections with $e_j e = e e_j = e_j$ $(j \in \{1, \dots, k\})$. From the previous step, we conclude that $\Phi(x_n) = 0$ $(n \in \mathbb{N})$. Further, from (2.8) we deduce that

$$\|x - x_n\|_p = \|e(x - x_n)\|_p \le \|e\|_p\|x - x_n\| \to 0,$$

and the continuity of Φ implies that $\Phi(x) = 0$, as required.

Finally, since $S(\mathcal{M}, \tau)$ is dense in $L^p(\mathcal{M}, \tau)$ and Φ is continuous, it may be concluded that $\Phi = 0$. □

Corollary 2.3.4. *Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ and with no minimal projections, let X be a q-normed space, $0 < q \le 1$, and let $P \colon L^p(\mathcal{M}, \tau) \to X$ be a continuous m-homogeneous polynomial with $0 < p/m < q$. Suppose that P is orthogonally additive on $S(\mathcal{M}, \tau)_+$. Then $P = 0$.*

Proof. This is a straightforward consequence of Theorem 2.3.2 and Proposition 2.3.3. □

We now turn our attention to the complex-valued polynomials. In this setting the representation given in Theorem 2.3.2 has a particularly significant integral form, because of the well-known representation of the dual of the L^p-spaces. The trace gives rise to a distinguished contractive positive linear functional on $L^1(\mathcal{M},\tau)$, still denoted by τ. By (2.8), if $\frac{1}{p}+\frac{1}{q}=1$, for each $\zeta \in L^q(\mathcal{M},\tau)$, the formula

$$\Phi_\zeta(x)=\tau(\zeta x) \quad (x\in L^p(\mathcal{M},\tau)) \tag{2.18}$$

defines a continuous linear functional on $L^p(\mathcal{M},\tau)$. Further, in the case where $1\le p<\infty$, the map $\zeta\mapsto\Phi_\zeta$ is an isometric isomorphism from $L^q(\mathcal{M},\tau)$ onto the dual space of $L^p(\mathcal{M},\tau)$. It is immediate to see that $\Phi_\zeta^*=\Phi_{\zeta^*}$, so that Φ_ζ is hermitian if and only if ζ is self-adjoint.

Corollary 2.3.5. *Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ, and let $P\colon L^p(\mathcal{M},\tau)\to\mathbb{C}$ be a continuous m-homogeneous polynomial with $m\le p<\infty$. Then the following conditions are equivalent:*

(i) there exists $\zeta\in L^r(\mathcal{M},\tau)$ such that $P(x)=\tau(\zeta x^m)$ $(x\in L^p(\mathcal{M},\tau))$, where $r=p/(p-m)$ (with the convention that $p/0=\infty$);

(ii) the polynomial P is orthogonally additive on $L^p(\mathcal{M},\tau)_{\mathrm{sa}}$;

(iii) the polynomial P is orthogonally additive on $S(\mathcal{M},\tau)_+$.

If the conditions are satisfied, then ζ is unique and $\|P\|\le\|\zeta\|_r\le 2\|P\|$; moreover, if P is hermitian, then ζ is self-adjoint and $\|\zeta\|_r=\|P\|$.

Proof. This follows from Theorems 2.3.1 and 2.3.2. □

Let H be a Hilbert space. We denote by Tr the usual trace on the von Neumann algebra $\mathcal{B}(H)$. Then $L^p(\mathcal{B}(H),\mathrm{Tr})$, with $0<p<\infty$, is the Schatten class $S^p(H)$. In the case where $0<p<q$, we have $S^p(H)\subset S^q(H)\subset\mathcal{K}(H)$ and $\|x\|\le\|x\|_q\le\|x\|_p$ $(x\in S^p(H))$. It is clear that $S(\mathcal{B}(H),\mathrm{Tr})=\mathcal{F}(H)$, the two-sided ideal of $\mathcal{B}(H)$ consisting of the finite-rank operators. Thus, the following result is an immediate consequence of Corollary 2.3.5.

Corollary 2.3.6. *Let H be a Hilbert space, and let $P\colon S^p(H)\to\mathbb{C}$ be a continuous m-homogeneous polynomial with $m<p<\infty$. Then the following conditions are equivalent:*

(i) there exists $\zeta\in S^r(H)$ such that $P(x)=\mathrm{Tr}(\zeta x^m)$ $(x\in S^p(H))$, where $r=p/(p-m)$;

(ii) the polynomial P is orthogonally additive on $S^p(H)_{\mathrm{sa}}$;

(iii) the polynomial P is orthogonally additive on $\mathcal{F}(H)_+$.

If the conditions are satisfied, then ζ is unique and $\|P\|\le\|\zeta\|_r\le 2\|P\|$; moreover, if P is hermitian, then ζ is self-adjoint and $\|\zeta\|_r=\|P\|$.

Corollary 2.3.7. *Let H be a Hilbert space, and let $P\colon\mathcal{K}(H)\to\mathbb{C}$ be a continuous m-homogeneous polynomial. Then the following conditions are equivalent:*

(i) there exists $\zeta\in S^1(H)$ such that $P(x)=\mathrm{Tr}(\zeta x^m)$ $(x\in\mathcal{K}(H))$;

(ii) the polynomial P is orthogonally additive on $\mathcal{K}(H)_{\mathrm{sa}}$;

(iii) the polynomial P is orthogonally additive on $\mathcal{F}(H)_+$.

If the conditions are satisfied, then ζ is unique and $\|P\| \le \|\zeta\|_1 \le 2\|P\|$; moreover, if P is hermitian, then ζ is self-adjoint and $\|\zeta\|_1 = \|P\|$.

Proof. In order to prove the equivalence of the conditions we are reduced to prove that (iii)⇒(i). Suppose that (iii) holds. Let $x, y \in \mathcal{K}(H)_+$ such that $x \perp y$. From the spectral decomposition of both x and y we deduce that there exist sequences (x_n) and (y_n) in $\mathcal{F}(H)_+$ such that $\lim x_n = x$, $\lim y_n = y$, and $x_m \perp y_n$ $(m, n \in \mathbb{N})$. Then

$$P(x+y) = \lim P(x_n + y_n) = \lim\bigl(P(x_n) + P(y_n)\bigr) = P(x) + P(y).$$

This shows that P is orthogonally additive on $\mathcal{K}(H)_+$. Since the C^*-algebra $\mathcal{K}(H)$ has real rank zero and the net consisting of all finite-rank projections is an increasing approximate unit, Theorem 2.2.5 applies and gives a continuous linear functional Φ on $\mathcal{K}(H)$ such that $P(x) = \Phi(x^m)$ $(x \in \mathcal{K}(H))$. It is well-known that the map $\zeta \mapsto \Phi_\zeta$, as defined in (2.18), gives an isometric isomorphism from $S^1(H)$ onto the dual of $\mathcal{K}(H)$, so that there exists $\zeta \in S^1(H)$ such that $\Phi(x) = \mathrm{Tr}(\zeta x)$ $(x \in \mathcal{K}(H))$ and $\|\zeta\|_1 = \|\Phi\|$. Thus we obtain (i). The additional properties of the result follow from Theorem 2.2.2. □

Corollary 2.3.8. *Let H be a Hilbert space, and let $P\colon S^p(H) \to \mathbb{C}$ be a continuous m-homogeneous polynomial with $0 < p \le m$. Then the following conditions are equivalent:*

(i) there exists $\zeta \in \mathcal{B}(H)$ such that $P(x) = \mathrm{Tr}(\zeta x^m)$ $(x \in S^p(H))$;

(ii) the polynomial P is orthogonally additive on $S^p(H)_{\mathrm{sa}}$;

(iii) the polynomial P is orthogonally additive on $\mathcal{F}(H)_+$.

If the conditions are satisfied, then ζ is unique and $\|P\| \le \|\zeta\| \le 2\|P\|$; moreover, if P is hermitian, then ζ is self-adjoint and $\|\zeta\| = \|P\|$.

Proof. By Theorems 2.3.1 and 2.3.2, it suffices to show that the map $\zeta \mapsto \Phi_\zeta$, as defined in (2.18), gives isometric isomorphism from $\mathcal{B}(H)$ onto the dual of $S^{p/m}(H)$. This is probably well-known, but we are not aware of any reference. Consequently, it may be helpful to include a proof of this fact. If $\zeta \in \mathcal{B}(H)$ and $x \in S^{p/m}(H)$, then, by (2.8), $\zeta x \in S^{p/m}(H)$, so that $\zeta x \in S^1(H)$ and

$$\bigl|\mathrm{Tr}(\zeta x)\bigr| \le \|\zeta x\|_1 \le \|\zeta\|\|x\|_1 \le \|\zeta\|\|x\|_{p/m},$$

which shows that Φ_ζ is a continuous linear functional on $S^{p/m}(H)$ with $\|\Phi_\zeta\| \le \|\zeta\|$. Conversely, assume that Φ is a continuous linear functional on $S^{p/m}(H)$. For each $\xi, \eta \in H$, let $\xi \otimes \eta \in \mathcal{F}(H)$ defined by

$$\bigl(\xi \otimes \eta\bigr)(\psi) = \langle \psi | \eta \rangle \xi \quad (\psi \in H),$$

and define $\varphi\colon H \times H \to \mathbb{C}$ by

$$\varphi(\xi, \eta) = \Phi(\xi \otimes \eta) \quad (\xi, \eta \in H).$$

It is easily checked that φ is a continuous sesquilinear functional with $\|\varphi\| \leq \|\Phi\|$. Therefore there exists $\zeta \in \mathcal{B}(H)$ such that $\langle \zeta(\xi)|\eta\rangle = \varphi(\xi, \eta)$ for all $\xi, \eta \in H$ and $\|\zeta\| \leq \|\Phi\|$. The former condition implies that

$$\Phi_\zeta(\xi \otimes \eta) = \mathrm{Tr}(\zeta\xi \otimes \eta) = \langle \zeta(\xi)|\eta\rangle = \varphi(\xi, \eta) = \Phi(\xi \otimes \eta)$$

for all $\xi, \eta \in H$, which gives $\Phi_\zeta(x) = \Phi(x)$ for each $x \in \mathcal{F}(H)$. Since $\mathcal{F}(H)$ is dense in $S^{p/m}(H)$, it follows that $\Phi_\zeta = \Phi$. Further, we have $\|\zeta\| \leq \|\Phi\| = \|\Phi_\zeta\| \leq \|\zeta\|$. Finally, it is immediate to see that $\Phi_\zeta^* = \Phi_{\zeta^*}$, so that Φ_ζ is hermitian if and only if ζ is self-adjoint. □

Proposition 2.3.9. *Let H be a Hilbert space with $\dim H \geq 2$, let X be a topological linear space, and let $P\colon S^p(H) \to X$ be a continuous m-homogeneous polynomial with $0 < p < \infty$. Suppose that P is orthogonally additive on $S^p(H)$. Then $P = 0$.*

Proof. Since $\mathcal{F}(H)$ is dense in $S^p(H)$ and P is continuous, it suffices to prove that P vanishes on $\mathcal{F}(H)$. On account of Lemma 2.2.1, we are also reduced to prove that P vanishes on $\mathcal{F}(H)_{\mathrm{sa}}$. We continue to use the notation $\xi \otimes \eta$ which was introduced in the proof of Corollary 2.3.8.

Let $x \in \mathcal{F}(H)_{\mathrm{sa}}$. Then $x = \sum_{j=1}^k \alpha_j \xi_j \otimes \xi_j$, where $k \geq 2$, $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, and $\{\xi_1, \ldots, \xi_k\}$ is an orthonormal subset of H. It is clear that the subalgebra $\mathcal{M}$ of $\mathcal{B}(H)$ generated by $\big\{\xi_i \otimes \xi_j : i, j \in \{1, \ldots, k\}\big\}$ is contained in $\mathcal{F}(H)$ and it is $*$-isomorphic to the von Neumann algebra $\mathcal{B}(K)$, where K is the linear span of the set $\{\xi_1, \ldots, \xi_k\}$. By Proposition 2.2.7, $P\,|_{\mathcal{M}} = 0$, and therefore $P(x) = 0$. We thus get $P\,|_{\mathcal{F}(H)_{\mathrm{sa}}} = 0$, as required. □

Chapter 3

Hyperreflexivity of the space of module homomorphisms between non-commutative L^p-spaces

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Abstract. Let $\mathcal{M}$ be a von Neumann algebra, and let $0 < p, q \leq \infty$. Then the space $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ of all right $\mathcal{M}$-module homomorphisms from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$ is a reflexive subspace of the space of all continuous linear maps from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$. Further, the space $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive in each of the following cases: (i) $1 \leq q < p \leq \infty$; (ii) $1 \leq p, q \leq \infty$ and $\mathcal{M}$ is injective, in which case the hyperreflexivity constant is at most 8.

3.1 Introduction

Let $\mathcal{A}$ be a closed subalgebra of the algebra $B(\mathcal{H})$ of all continuous linear operators on the Hilbert space $\mathcal{H}$. Then $\mathcal{A}$ is called reflexive if

$$\mathcal{A} = \big\{T \in B(\mathcal{H}) : e^{\perp}Te = 0 \ (e \in \mathrm{lat}\mathcal{A})\big\},$$

where $\mathrm{lat}\mathcal{A} = \{e \in B(\mathcal{H}) \text{ projection} : e^{\perp}Te = 0 \ (T \in \mathcal{A})\}$ is the set of all projections onto the $\mathcal{A}$-invariant subspaces of $\mathcal{H}$. The double commutant theorem shows that each von Neumann algebra on $\mathcal{H}$ is certainly reflexive. The algebra $\mathcal{A}$ is called hyperreflexive if the above condition on $\mathcal{A}$ is strengthened by requiring that there is a distance estimate

$$\mathrm{dist}(T, \mathcal{A}) \leq C \sup\big\{\|e^{\perp}Te\| : e \in \mathrm{lat}\mathcal{A}\big\} \quad (T \in B(\mathcal{H}))$$

for some constant C. The inequality

$$\sup\{\|e^{\perp}Te\| : e \in \mathrm{lat}\mathcal{A}\} \leq \mathrm{dist}(T, \mathcal{A}) \quad (T \in B(\mathcal{H}))$$

is always true and elementary. This quantitative version of reflexivity was introduced by Arveson [19] and has proven to be a powerful tool when it is available. Christensen [33, 34, 35] showed that many von Neumann algebras are hyperreflexive by relating the hyperreflexivity to the vanishing of certain cohomology group. Notably each injective von Neumann algebra $\mathcal{M}$ on the Hilbert space $\mathcal{H}$ is hyperreflexive and

$$\mathrm{dist}(T, \mathcal{M}) \leq 4 \sup\{\|e^{\perp}Te\| : e \in \mathcal{M} \text{ projection}\} \quad (T \in B(\mathcal{H}))$$

(see [33, Theorem 2.3] and [38, p. 340]).

Both notions, reflexivity and hyperreflexivity, were extended to subspaces of $B(\mathcal{X}, \mathcal{Y})$, the Banach space of all continuous linear maps from the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$. Following Loginov and Shulman [62], a closed linear subspace $\mathcal{S}$ of $B(\mathcal{X}, \mathcal{Y})$ is called reflexive if

$$\mathcal{S} = \{T \in B(\mathcal{X}, \mathcal{Y}) : T(x) \in \overline{\{S(x) : S \in \mathcal{S}\}}\, (x \in \mathcal{X})\}.$$

In accordance with Larson [60, 61], $\mathcal{S}$ is called hyperreflexive if there exists a constant C such that

$$\mathrm{dist}(T, \mathcal{S}) \leq C \sup_{x \in \mathcal{X},\, \|x\| \leq 1} \inf\{\|T(x) - S(x)\| : S \in \mathcal{S}\} \quad (T \in B(\mathcal{X}, \mathcal{Y})),$$

and the optimal constant is called the hyperreflexivity constant of $\mathcal{S}$. The inequality

$$\sup_{x \in \mathcal{X},\, \|x\| \leq 1} \inf\{\|T(x) - S(x)\| : S \in \mathcal{S}\} \leq \mathrm{dist}(T, \mathcal{S}) \quad (T \in B(\mathcal{X}, \mathcal{Y})).$$

is always true.

The ultimate objective of this paper is to study the hyperreflexivity of the space $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ of all (automatically continuous) right $\mathcal{M}$-module homomorphisms from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$ for a von Neumann algebra $\mathcal{M}$. The non-commutative L^p-spaces that we consider throughout are those introduced by Haagerup (see [46, 70, 80]). For each $0 < p \leq \infty$, the space $L^p(\mathcal{M})$ is a contractive Banach $\mathcal{M}$-bimodule or a contractive p-Banach $\mathcal{M}$-bimodule according to $1 \leq p$ or $p < 1$, and we will focus on the right $\mathcal{M}$-module structure of $L^p(\mathcal{M})$.

Our method relies in the analysis of a continuous bilinear map $\varphi \colon \mathcal{A} \times \mathcal{A} \to \mathcal{X}$, for a C^*-algebra $\mathcal{A}$ and a normed space $\mathcal{X}$, through the knowledge of the constant $\sup\{\|\varphi(a, b)\| : a, b \in \mathcal{A}_+ \text{ contractions}, ab = 0\}$, alternatively, the constant $\sup\{\|\varphi(e, e^{\perp})\| : e \in \mathcal{A} \text{ projection}\}$ in the case where $\mathcal{A}$ is unital and has real rank zero. This is done in Section 3.2. Here and subsequently, $\mathcal{A}_+$ stands for the cone of positive elements in $\mathcal{A}$.

In Section 3.3 we prove that, for each $0 < p, q \leq \infty$, each right $\mathcal{M}$-module homomorphism from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$ is automatically continuous and that the space

$\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ of all right $\mathcal{M}$-module homomorphisms is a reflexive subspace of $B(L^p(\mathcal{M}), L^q(\mathcal{M}))$ (the notion of reflexivity makes perfect sense for subspaces of operators between quasi-Banach spaces).

Section 3.4 is devoted to study the hyperreflexivity of $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ for $1 \le p, q \le \infty$. The space $B(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is a Banach $\mathcal{M}$-bimodule for the operations specified by

$$(aT)(x) = T(xa), \quad (Ta)(x) = T(x)a$$

for all $T \in B(L^p(\mathcal{M}), L^q(\mathcal{M}))$, $a \in \mathcal{M}$, and $x \in L^p(\mathcal{M})$ (note that the left $\mathcal{M}$-module structure of both $L^p(\mathcal{M})$ and $L^q(\mathcal{M})$ is disregarded), and we will prove that there is a distance estimate

$$\mathrm{dist}\big(T, \mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))\big) \le C \sup\big\{\|e^{\perp}Te\| : e \in \mathcal{M} \text{ projection}\big\}$$

for each $T \in B(L^p(\mathcal{M}), L^q(\mathcal{M}))$ in each of the following cases:

(i) $1 \le q < p \le \infty$, in which case the constant C can be chosen to depend on p and q, and not on $\mathcal{M}$;

(ii) $1 \le p, q \le \infty$ and $\mathcal{M}$ is injective, in which case the constant C can be taken to be 8.

Further,

$$\begin{aligned} &\sup\big\{\|e^{\perp}Te\| : e \in \mathcal{M} \text{ projection}\big\} \\ \le &\sup_{x \in L^p(\mathcal{M}),\, \|x\|_p \le 1} \inf\big\{\|T(x) - \Phi(x)\|_q : \Phi \in \mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))\big\}, \end{aligned}$$

and thus, in both cases, it turns out that the space $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive.

It is perhaps worth remarking that most of the discussion of reflexivity and hyperreflexivity is accomplished for continuous homomorphisms between modules over a C^*-algebra.

Throughout this paper we write $\mathcal{X}^*$ for the dual of a Banach space $\mathcal{X}$ and $\langle \cdot, \cdot \rangle$ for the duality between $\mathcal{X}$ and $\mathcal{X}^*$.

3.2 Analysing bilinear maps through orthogonality

Goldstein proved in [44] (albeit with sesquilinear functionals) that, for each C^*-algebra $\mathcal{A}$, every continuous bilinear functional $\varphi\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ with the property that $\varphi(a, b) = 0$ whenever $a, b \in \mathcal{A}_{sa}$ satisfy $ab = 0$ can be represented in the form $\varphi(a, b) = \omega_1(ab) + \omega_2(ba)$ $(a, b \in \mathcal{A})$ for some $\omega_1, \omega_2 \in \mathcal{A}^*$. Independently, it was shown in [2] that if $\mathcal{A}$ is a C^*-algebra or the group algebra $L^1(G)$ of a locally compact group G, then every continuous bilinear functional $\varphi\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ with the property that $\varphi(a, b) = 0$ whenever $a, b \in \mathcal{A}$ are such that $ab = 0$ necessarily satisfies the condition $\varphi(ab, c) = \varphi(a, bc)$ $(a, b, c \in \mathcal{A})$, which in turn implies the existence of $\omega \in \mathcal{A}^*$ such that $\varphi(a, b) = \omega(ab)$ $(a, b \in \mathcal{A})$.

Actually, [11] gives more, namely, the norms $\|\varphi(ab,c)-\varphi(a,bc)\|$ with $a,b,c\in\mathcal{A}$ can be estimated through the constant $\sup\{\|\varphi(a,b)\| : a,b\in\mathcal{A},\ \|a\|=\|b\|=1,\ ab=0\}$. This property has proven to be useful to study the hyperreflexivity of the spaces of derivations and continuous cocycles on $\mathcal{A}$ (see [12, 13, 75, 76, 77]). This section provides an improvement of the above mentioned property in the case of C^*-algebras, and this will be used later to study the hyperreflexivity of the space $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},\mathcal{Y})$ of all continuous module homomorphisms between the Banach right $\mathcal{A}$-modules $\mathcal{X}$ and $\mathcal{Y}$.

Theorem 3.2.1. *Let $\mathcal{A}$ be a C^*-algebra, let $\mathcal{Z}$ be a normed space, let $\varphi\colon \mathcal{A}\times\mathcal{A}\to\mathcal{Z}$ be a continuous bilinear map, and let the constant $\varepsilon\geq 0$ be such that*

$$a,b\in\mathcal{A}_+,\ ab=0 \implies \|\varphi(a,b)\|\leq\varepsilon\|a\|\|b\|.$$

Suppose that $(e_j)_{j\in J}$ is a net in $\mathcal{A}$ such that $(e_j)_{j\in J}$ converges to $1_{\mathcal{A}^{**}}$ in $\mathcal{A}^{**}$ with respect to the weak topology. Then, for each $a\in\mathcal{A}$, the nets $(\varphi(a,e_j))_{j\in J}$ and $(\varphi(e_j,a))_{j\in J}$ converge in $\mathcal{Z}^{**}$ with respect to the weak* topology and*

$$\Big\|\lim_{j\in J}\varphi(a,e_j)-\lim_{j\in J}\varphi(e_j,a)\Big\|\leq 8\varepsilon\|a\|.$$

In particular, if $\mathcal{A}$ is unital, then

$$\|\varphi(a,1_{\mathcal{A}})-\varphi(1_{\mathcal{A}},a)\|\leq 8\varepsilon\|a\| \quad (a\in\mathcal{A}).$$

Proof. First, we regard φ as a continuous bilinear map with values in $\mathcal{Z}^{**}$. By applying [55, Theorem 2.3] to $\mathcal{A}$ acting on the Hilbert space of its universal representation, we obtain that φ extends uniquely, without change of norm, to a continuous bilinear map $\psi\colon\mathcal{A}^{**}\times\mathcal{A}^{**}\to\mathcal{Z}^{**}$ which is separately weak* continuous.

Now, since $(e_j)_{j\in J}\to 1_{\mathcal{A}^{**}}$ with respect to the weak* topology and ψ is separately weak* continuous, we see that, for each $a\in\mathcal{A}$, the nets $(\varphi(a,e_j))_{j\in J}$ and $(\varphi(e_j,a))_{j\in J}$ converge to $\psi(a,1_{\mathcal{A}^{**}})$ and $\psi(1_{\mathcal{A}^{**}},a)$, respectively, with respect to the weak* topology of $\mathcal{Z}^{**}$. Consequently, the proof of the theorem is completed by showing that

$$\big\|\psi(a,1_{\mathcal{A}^{**}})-\psi(1_{\mathcal{A}^{**}},a)\big\|\leq 8\varepsilon\|a\| \quad (a\in\mathcal{A}). \tag{3.1}$$

For this purpose we will show in the next lines that

$$p,q\in\mathcal{A}^{**}\text{ projections, } pq=0 \implies \|\psi(p,q)\|\leq\varepsilon. \tag{3.2}$$

Take projections $p,q\in\mathcal{A}^{**}$ with $pq=0$. For each $\zeta\in\mathcal{Z}^*$ with $\|\zeta\|=1$, we define $\psi_\zeta\colon\mathcal{A}^{**}\times\mathcal{A}^{**}\to\mathbb{C}$ by

$$\psi_\zeta(x,y)=\langle\zeta,\psi(x,y)\rangle \quad \forall x,y\in\mathcal{A}^{**},$$

and we observe that ψ_ζ is a separately weak* continuous bilinear functional so that, by [48, Proposition 2.3], ψ_ζ is jointly σ-strong* continuous. Set $x=p-q\in\mathcal{A}^{**}$. Then x is a self-adjoint element in the unit ball of $\mathcal{A}^{**}$ and hence Kaplansky's density theorem shows that there exists a net $(a_i)_{i\in I}$ in the unit ball of $\mathcal{A}_{\mathrm{sa}}$ converging to x with respect to the strong topology. Consequently, $({a_i}^+)_{i\in I} \to x^+ = p$ and $({a_i}^-)_{i\in I} \to x^- = q$ with respect to the strong topology, where $(\cdot)^+$ and $(\cdot)^-$ stand for the continuous functional calculus for the functions $t \mapsto \max\{t, 0\}$ and $t \mapsto \{-t, 0\}$, respectively. We observe that

$${a_i}^+, {a_i}^- \in \mathcal{A}_+, \quad {a_i}^+ {a_i}^- = 0 \quad \forall i \in I,$$

so that

$$|\psi_\zeta({a_i}^+, {a_i}^-)| = |\langle \zeta, \varphi({a_i}^+, {a_i}^-)\rangle| \le \|\varphi({a_i}^+, {a_i}^-)\| \le \varepsilon,$$

and further $({a_i}^+)_{i\in I} \to x^+ = p$ and $({a_i}^-)_{i\in I} \to x^- = q$ with respect to the σ-strong* topology (because we are working on the self-adjoint part of the unit ball of $\mathcal{A}^{**}$). Since ψ_ζ is jointly σ-strong* continuous, it follows that $|\psi_\zeta(p, q)| \le \varepsilon$. Since this holds for each $\zeta \in \mathcal{Z}^*$ with $\|\zeta\| = 1$, we conclude that (3.2) holds.

Our next objective is to prove (3.1).

We begin with the case $a \in \mathcal{A}_+$, and we can assume that $\|a\| \le 1$. For each $n \in \mathbb{N}$, let $h_n \colon [0, 1] \to \mathbb{R}$ be the bounded Borel function defined by

$$h_n = \frac{1}{n+1} \sum_{k=1}^{n} \chi_{]k/(n+1),1]}$$

(we use the notation χ_Δ for the characteristic function of a subset Δ of $[0, 1]$). We also consider the sequence in $\mathcal{A}^{**}$ given by $\bigl(h_n(a)\bigr)$. Since the sequence (h_n) converges uniformly on $[0, 1]$ to the identity map on $[0, 1]$, it follows that $\bigl(h_n(a)\bigr) \to a$ with respect to the norm topology. Thus

$$\begin{aligned}\psi(a, 1_{\mathcal{A}^{**}}) - \psi(1_{\mathcal{A}^{**}}, a) &= \lim_{n\to\infty} \Bigl(\psi\bigl(h_n(a), 1_{\mathcal{A}^{**}}\bigr) - \psi\bigl(1_{\mathcal{A}^{**}}, h_n(a)\bigr)\Bigr) \\ &= \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=1}^{n} \Bigl(\psi\bigl(\chi_{]\frac{k}{n+1},1]}(a), 1_{\mathcal{A}^{**}}\bigr) - \psi\bigl(1_{\mathcal{A}^{**}}, \chi_{]\frac{k}{n+1},1]}(a)\bigr)\Bigr).\end{aligned} \tag{3.3}$$

Further, for each $k \in \{1, \dots, n\}$, we have

$$1_{\mathcal{A}^{**}} = \chi_{[0,k/(n+1)]}(a) + \chi_{]k/(n+1),1]}(a),$$

so that

$$\begin{aligned}&\psi\bigl(\chi_{]k/(n+1),1]}(a), 1_{\mathcal{A}^{**}}\bigr) - \psi\bigl(1_{\mathcal{A}^{**}}, \chi_{]k/(n+1),1]}(a)\bigr) \\ &\qquad = \psi\bigl(\chi_{]k/(n+1),1]}(a), \chi_{[0,k/(n+1)]}(a)\bigr) - \psi\bigl(\chi_{[0,k/(n+1)]}(a), \chi_{]k/(n+1),1]}(a)\bigr),\end{aligned}$$

and (3.2) then give

$$\bigl\|\psi\bigl(\chi_{]k/(n+1),1]}(a), 1_{\mathcal{A}^{**}}\bigr) - \psi\bigl(1_{\mathcal{A}^{**}}, \chi_{]k/(n+1),1]}(a)\bigr)\bigr\| \le 2\varepsilon. \tag{3.4}$$

From (3.3) and (3.4) it may be concluded that

$$\bigl\|\psi(a, 1_{\mathcal{A}^{**}}) - \psi(1_{\mathcal{A}^{**}}, a)\bigr\| \le \lim_{n\to\infty} \frac{n2\varepsilon}{n+1} = 2\varepsilon. \tag{3.5}$$

Now suppose that $a \in \mathcal{A}_{\rm sa}$. Then we can write $a = a_+ - a_-$, where $a_+, a_- \in \mathcal{A}_+$ are mutually orthogonal, and (3.5) gives

$$\begin{aligned}\big\|\psi(a, 1_{\mathcal{A}^{**}}) - \psi(1_{\mathcal{A}^{**}}, a)\big\| &\leq \big\|\psi(a_+, 1_{\mathcal{A}^{**}}) - \psi(1_{\mathcal{A}^{**}}, a_+)\big\| \\ &\quad + \big\|\psi(a_-, 1_{\mathcal{A}^{**}}) - \psi(1_{\mathcal{A}^{**}}, a_-)\big\| \\ &\leq 2\varepsilon\|a_+\| + 2\varepsilon\|a_-\| \\ &\leq 4\varepsilon \max\{\|a_+\|, \|a_-\|\} = 4\varepsilon\|a\|.\end{aligned}$$

Finally take $a \in \mathcal{A}$, and write

$$a = \Re a + i\Im a, \tag{3.6}$$

where

$$\Re a = \frac{1}{2}(a^* + a),\ \Im a = \frac{i}{2}(a^* - a) \in \mathcal{A}_{\rm sa},$$

and, further, $\|\Re a\|, \|\Im a\| \leq \|a\|$. From what has previously been proved, it follows that

$$\begin{aligned}\big\|\psi(a, 1_{\mathcal{A}^{**}}) - \psi(1_{\mathcal{A}^{**}}, a)\big\| &\leq \big\|\psi(\Re a, 1_{\mathcal{A}^{**}}) - \psi(1_{\mathcal{A}^{**}}, \Re a)\big\| \\ &\quad + \big\|\psi(\Im a, 1_{\mathcal{A}^{**}}) - \psi(1_{\mathcal{A}^{**}}, \Im a)\big\| \\ &\leq 4\varepsilon\|\Re a\| + 4\varepsilon\|\Im a\| \\ &\leq 8\varepsilon\|a\|.\end{aligned}$$

This gives (3.1) and completes the proof of the theorem. □

Theorem 3.2.2. *Let $\mathcal{A}$ be a unital C^*-algebra of real rank zero, let $\mathcal{Z}$ be a topological vector space, and let $\varphi\colon \mathcal{A} \times \mathcal{A} \to \mathcal{Z}$ be a continuous bilinear map.*

(i) Suppose that

$$e \in \mathcal{A} \text{ projection } \implies \varphi(e, e^\perp) = 0.$$

Then

$$\varphi(a, 1_{\mathcal{A}}) = \varphi(1_{\mathcal{A}}, a) \quad (a \in \mathcal{A}).$$

(ii) Suppose that $\mathcal{Z}$ is a normed space and let the constant $\varepsilon \geq 0$ be such that

$$e \in \mathcal{A} \text{ projection } \implies \|\varphi(e, e^\perp)\| \leq \varepsilon.$$

Then

$$\|\varphi(a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a)\| \leq 8\varepsilon\|a\| \quad (a \in \mathcal{A}).$$

Proof. Let $a \in \mathcal{A}_+$, and assume that a has finite spectrum, say $\{\alpha_1, \ldots, \alpha_n\}$. Of course, we can suppose that $0 \leq \alpha_1 < \cdots < \alpha_n = \|a\|$. This implies that a can be written in the form

$$a = \sum_{k=1}^{n} \alpha_k p_k,$$

where $p_1, \dots, p_n \in \mathcal{A}$ are mutually orthogonal projections (specifically, the projection p_k is defined by using the continuous functional calculus for a by $p_k = \chi_{\{\alpha_k\}}(a)$ for each $k \in \{1, \dots, n\}$ because $\chi_{\{\alpha_k\}}$ is a continuous function on the spectrum of a, being this set finite). In case (i), we have

$$\begin{aligned}
\varphi(a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a) &= \sum_{k=1}^{n} \alpha_k \big(\varphi(p_k, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, p_k)\big) \\
&= \sum_{k=1}^{n} \alpha_k \big(\varphi(p_k, p_k^{\perp} + p_k) - \varphi(p_k^{\perp} + p_k, p_k)\big) \\
&= \sum_{k=1}^{n} \alpha_k \big(\varphi(p_k, p_k^{\perp}) - \varphi(p_k^{\perp}, p_k)\big) = 0.
\end{aligned} \tag{3.7}$$

In case (ii), we define real numbers $\lambda_1, \dots, \lambda_n \in [0, \infty[$ and projections $e_1, \dots, e_n \in \mathcal{A}$ by

$$\begin{aligned}
&\lambda_1 = \alpha_1, \\
&\lambda_k = \alpha_k - \alpha_{k-1} \quad (1 < k \leq n),
\end{aligned}$$

and

$$e_k = p_k + \cdots + p_n \quad (1 \leq k \leq n).$$

It is a simple matter to check that

$$a = \sum_{k=1}^{n} \lambda_k e_k.$$

For each $k \in \{1, \dots, n\}$, we have

$$\begin{aligned}
\|\varphi(e_k, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, e_k)\| &= \left\|\varphi(e_k, e_k^{\perp} + e_k) - \varphi({e_k}^{\perp} + e_k, e_k)\right\| \\
&= \left\|\varphi(e_k, e_k^{\perp}) - \varphi(e_k^{\perp}, e_k)\right\| \\
&\leq \left\|\varphi(e_k, e_k^{\perp})\right\| + \left\|\varphi(e_k^{\perp}, e_k)\right\| \leq 2\varepsilon.
\end{aligned}$$

We thus get

$$\begin{aligned}
\|\varphi(a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a)\| &= \left\|\sum_{k=1}^{n} \lambda_k \big(\varphi(e_k, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, e_k)\big)\right\| \\
&\leq \sum_{k=1}^{n} \lambda_k \big\|\varphi(e_k, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, e_k)\big\| \\
&\leq \sum_{k=1}^{n} \lambda_k 2\varepsilon = 2\varepsilon\alpha_n = 2\varepsilon\|a\|.
\end{aligned} \tag{3.8}$$

Now suppose that $a \in \mathcal{A}_{\mathrm{sa}}$ and that a has finite spectrum. Then we can write $a = a_+ - a_-$, where a_+, $a_- \in \mathcal{A}_+$ are mutually orthogonal. Since $a_+ = f(a)$ and $a_- = g(a)$, where f, $g \colon \mathbb{R} \to \mathbb{R}$ are the continuous functions defined by

$$f(t) = \max\{t, 0\}, \quad g(t) = \max\{-t, 0\}, \quad (t \in \mathbb{R}),$$

it follows that both a_+ and a_- have finite spectra. In case (i), (3.7) gives

$$\varphi(a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a) = \big(\varphi(a_+, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a_+)\big) - \big(\varphi(a_-, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a_-)\big) = 0. \tag{3.9}$$

In case (ii), on account of (3.8), we have

$$\begin{aligned} \|\varphi(a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a)\| &\leq \|\varphi(a_+, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a_+)\| \\ &\quad + \|\varphi(a_-, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a_-)\| \\ &\leq 2\varepsilon\|a_+\| + 2\varepsilon\|a_-\| \\ &\leq 4\varepsilon \max\{\|a_+\|, \|a_-\|\} = 4\varepsilon\|a\|. \end{aligned} \tag{3.10}$$

Let a be an arbitrary element of $\mathcal{A}_{\mathrm{sa}}$. Since $\mathcal{A}$ has real rank zero, it follows that there exists a sequence (a_n) in $\mathcal{A}_{\mathrm{sa}}$ such that each a_n has finite spectrum and $(a_n) \to a$ with respect to the norm topology. In case (i), (3.9) gives

$$\varphi(a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a) = \lim_{n\to\infty} \big(\varphi(a_n, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a_n)\big) = 0.$$

In case (ii), from (3.10) it follows that

$$\|\varphi(a_n, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a_n)\| \leq 4\varepsilon\|a_n\| \quad (n \in \mathbb{N}),$$

and the continuity of φ now yields

$$\begin{aligned} \|\varphi(a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a)\| &= \lim_{n\to\infty} \|\varphi(a_n, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a_n)\| \\ &\leq 4\varepsilon \lim_{n\to\infty} \|a_n\| = 4\varepsilon\|a\|. \end{aligned}$$

Finally set $a \in \mathcal{A}$, and write $a = \Re a + i\Im a$ as in (3.6). From the previous step we see that, in case (i),

$$\varphi(a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a) = \varphi(\Re a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, \Re a) + \varphi(\Im a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, \Im a) = 0,$$

and, in case (ii),

$$\begin{aligned} \|\varphi(a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a)\| &\leq \|\varphi(\Re a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, \Re a)\| \\ &\quad + \|\varphi(\Im a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, \Im a)\| \\ &\leq 4\varepsilon\|\Re a\| + 4\varepsilon\|\Im a\| \leq 8\varepsilon\|a\|, \end{aligned}$$

giving the result. □

Corollary 3.2.3. *Let $\mathcal{A}$ be a C^*-algebra, let $\mathcal{Z}$ be a normed space, let $\varphi\colon \mathcal{A}\times\mathcal{A}\to\mathcal{Z}$ be a continuous bilinear map, and let the constant $\varepsilon\geq 0$ be such that*

$$a,b\in\mathcal{A},\ ab=0 \implies \|\varphi(a,b)\|\leq\varepsilon\|a\|\|b\|.$$

Then

$$\|\varphi(ab,c)-\varphi(a,bc)\|\leq 8\varepsilon\|a\|\|b\|\|c\| \quad (a,b,c\in\mathcal{A}).$$

Further, if either $\mathcal{A}$ is unital or $\mathcal{Z}$ is a dual Banach space, then there exists a continuous linear map $\Phi\colon\mathcal{A}\to\mathcal{Z}$ such that

$$\|\varphi(a,b)-\Phi(ab)\|\leq 8\varepsilon\|a\|\|b\| \quad (a,b\in\mathcal{A})$$

and $\|\Phi\|\leq\|\varphi\|$.

Proof. Set $a,c\in\mathcal{A}$, and consider the continuous bilinear map $\psi\colon\mathcal{A}\times\mathcal{A}\to\mathcal{Z}$ defined by

$$\psi(u,v)=\varphi(au,vc) \quad (u,v\in\mathcal{A}).$$

If $u,v\in A$ are such that $uv=0$, then $au,vc\in\mathcal{A}$ and $(au)(vc)=0$, which gives

$$\|\psi(u,v)\|=\|\varphi(au,vc)\|\leq\varepsilon\|au\|\|vc\|\leq\varepsilon\|a\|\|c\|\|u\|\|v\|$$

by hypothesis.

Let $(e_i)_{i\in I}$ be an approximate identity for $\mathcal{A}$ of bound one. Since the net $(e_i)_{i\in I}$ is bounded, it has a subnet $(e_j)_{j\in J}$ which converges to an element $E\in\mathcal{A}^{**}$ with respect to the weak* topology. We claim that $E=1_{\mathcal{A}^{**}}$. Indeed, let $a\in\mathcal{A}$. Then $(ae_j)_{j\in J}\to a$ with respect to the norm topology and, further, $(ae_j)_{j\in J}\to aE$ with respect to the weak* topology. Thus $aE=a$. From the weak* density of $\mathcal{A}$ in $\mathcal{A}^{**}$ and the separate weak* continuity of the product of $\mathcal{A}^{**}$, we conclude that $AE=A$ for each $A\in\mathcal{A}^{**}$, hence that $E=1_{\mathcal{A}^{**}}$, as claimed.

From Theorem 3.2.1 we deduce that, for each $b\in\mathcal{A}$, the nets $(\psi(b,e_j))_{j\in J}$ and $(\psi(e_j,b))_{j\in J}$ converge in $\mathcal{Z}^{**}$ with respect to the weak* topology and

$$\big\|\lim_{j\in J}\psi(b,e_j)-\lim_{j\in J}\psi(e_j,b)\big\|\leq 8(\varepsilon\|a\|\|c\|)\|b\|. \tag{3.11}$$

Since $(ae_j)_{j\in J}$ converges to a and $(e_jc)_{j\in J}$ converges to c in norm and φ is continuous, it follows that

$$\lim_{j\in J}\psi(b,e_j)=\lim_{j\in J}\varphi(ab,e_jc)=\varphi(ab,c)$$

and

$$\lim_{j\in J}\psi(e_j,b)=\lim_{j\in J}\varphi(ae_j,bc)=\varphi(a,bc)$$

in norm for each $b\in\mathcal{A}$, which establishes the required inequality when combined with (3.11).

Now suppose that $\mathcal{A}$ is unital, and define $\Phi\colon\mathcal{A}\to\mathcal{Z}$ by

$$\Phi(a)=\varphi(a,1_{\mathcal{A}}) \quad (a\in\mathcal{A}).$$

Then Φ is a continuous linear map, and clearly $\|\Phi\| \leq \|\varphi\|$. For each a, $b \in \mathcal{A}$, we have

$$\|\varphi(a,b) - \Phi(ab)\| = \|\varphi(a, b1_{\mathcal{A}}) - \varphi(ab, 1_{\mathcal{A}})\| \leq 8\varepsilon\|a\|\|b\|,$$

as claimed.

Finally suppose that $\mathcal{Z}$ is the dual of a Banach space $\mathcal{Z}_*$. Let $\mathcal{U}$ be an ultrafilter on J containing the order filter on J. It follows from the Banach-Alaoglu theorem that each bounded subset of $\mathcal{Z}$ is relatively compact with respect to the weak* topology on $\mathcal{Z}$. Consequently, each bounded net $(z_j)_{j\in J}$ in $\mathcal{Z}$ has a unique limit with respect to the weak* topology along the ultrafilter $\mathcal{U}$, and we write $\lim_{\mathcal{U}} z_j$ for this limit. Further, it is worth noting that

$$\left\|\lim_{\mathcal{U}} z_j\right\| \leq \lim_{\mathcal{U}} \|z_j\|. \tag{3.12}$$

Indeed, for each $\zeta \in \mathcal{Z}_*$ such that $\|\zeta\| = 1$, we have $|\langle z_j, \zeta\rangle| \leq \|z_j\|$ $(j \in J)$, and hence

$$\left|\left\langle \lim_{\mathcal{U}} z_j, \zeta \right\rangle\right| = \lim_{\mathcal{U}} |\langle z_j, \zeta\rangle| \leq \lim_{\mathcal{U}} \|z_j\|,$$

which establishes (3.12).

For each $a \in \mathcal{Z}$, we have

$$\|\varphi(a, e_j)\| \leq \|\varphi\|\|a\| \quad (j \in J), \tag{3.13}$$

and hence the net $(\varphi(a, e_j))_{j\in J}$ is bounded. Consequently, we can define the map $\Phi\colon \mathcal{A} \to \mathcal{Z}$ by

$$\Phi(a) = \lim_{\mathcal{U}} \varphi(a, e_j) \quad (a \in \mathcal{A}).$$

The linearity of the limit along an ultrafilter on a topological linear space gives the linearity of Φ. Take $a \in \mathcal{A}$. From (3.12) and (3.13) we deduce that $\|\Phi(a)\| \leq \|\varphi\|\|a\|$, which gives the continuity of Φ and $\|\Phi\| \leq \|\varphi\|$. Now take $a, b \in \mathcal{A}$. We have

$$\|\varphi(ab, e_j) - \varphi(a, be_j)\| \leq 8\varepsilon\|a\|\|b\| \quad (j \in J). \tag{3.14}$$

Since $(be_j)_{j\in J} \to b$ in norm, the continuity of φ gives $(\varphi(a, be_j))_{j\in J} \to \varphi(a, b)$ in norm. Since $\mathcal{U}$ refines the order filter on J we see that $\lim_{\mathcal{U}} \varphi(a, be_j) = \varphi(a, b)$. Taking the limit in (3.14) along the ultrafilter $\mathcal{U}$, and using (3.12), we obtain $\|\Phi(ab) - \varphi(a, b)\| \leq 8\varepsilon\|a\|\|b\|$, as required. □

It is worth noting that Corollary 3.2.3 gives a sharper estimate for the constant of the strong property $\mathbb{B}$ of a C^*-algebra to the one given in [77, Theorem 3.4], where our constant 8 is replaced by $384\pi^2(1 + \sqrt{2})$. The hyperreflexivity constant given in [77, Theorem 4.4] for C^*-algebras can be sharpened as well accordingly.

3.3 Primary estimates and reflexivity

3.3.1 Homomorphisms between modules over a C^*-algebra

Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{X}$ and $\mathcal{Y}$ be quasi-Banach right $\mathcal{A}$-modules. For a linear map $T\colon \mathcal{X} \to \mathcal{Y}$ and $a \in \mathcal{A}$, define linear maps aT, $Ta\colon \mathcal{X} \to \mathcal{Y}$ by

$$(aT)(x) = T(xa), \quad (Ta)(x) = T(x)a \quad (x \in \mathcal{X}). \tag{3.15}$$

Then the space $B(\mathcal{X}, \mathcal{Y})$ of all continuous linear maps from $\mathcal{X}$ to $\mathcal{Y}$ is a quasi-Banach $\mathcal{A}$-bimodule for the operations specified by (3.15). For $T \in B(\mathcal{X}, \mathcal{Y})$, let $\operatorname{ad}(T)\colon \mathcal{A} \to B(\mathcal{X}, \mathcal{Y})$ denote the inner derivation implemented by T, so that

$$\operatorname{ad}(T)(a) = aT - Ta \quad (a \in \mathcal{A}).$$

It is clear that T is a right $\mathcal{A}$-module homomorphism if and only if $\operatorname{ad}(T) = 0$, and, in the case where $\mathcal{X}$ and $\mathcal{Y}$ are Banach $\mathcal{A}$-modules, the constant

$$\| \operatorname{ad}(T)\| \tag{3.16}$$

is intended to estimate the distance from T to the space $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ of all continuous module homomorphisms from $\mathcal{X}$ to $\mathcal{Y}$. This is actually very much in the spirit of [33, 34, 35]. We will use several additional alternative ways to estimate the distance $\operatorname{dist}(T, \operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}))$ that equally make sense, namely

$$\alpha(T) = \sup\bigl\{\|e^{\perp}Te\| : e \in \mathcal{A} \text{ projection}\bigr\}, \tag{3.17}$$

$$\beta(T) = \sup\bigl\{\|eTf\| : e, f \in \mathcal{A} \text{ projections, } ef = 0\bigr\}, \tag{3.18}$$

$$\gamma(T) = \sup\bigl\{\|aTb\| : a, b \in \mathcal{A}_+ \text{ contractions, } \ ab = 0\bigr\}, \tag{3.19}$$

$$\delta(T) = \sup_{x \in \mathcal{X},\, \|x\| \le 1} \inf\bigl\{\|T(x) - \Phi(x)\| : \Phi \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})\bigr\}. \tag{3.20}$$

For (3.17), the algebra is supposed to be unital, and this constant is actually very much in the spirit of the celebrated Arveson's distance formula [20]. The significance of the constants (3.17) and (3.18) for our purposes requires that the C^*-algebra $\mathcal{A}$ to be sufficiently rich in projections (as does a C^*-algebra of real rank zero) and they are in the spirit of [58, 59]. We take the constant (3.20) in accordance with [60, 61].

Throughout we suppose that the Banach $\mathcal{A}$-modules are contractive. The statements of our results can be easily adapted to non-contractive Banach modules.

Proposition 3.3.1. *Let $\mathcal{A}$ be a C^*-algebra, let $\mathcal{X}$ and $\mathcal{Y}$ be Banach right $\mathcal{A}$-modules, and let $T\colon \mathcal{X} \to \mathcal{Y}$ be a continuous linear map.*

(i) $\beta(T) \le \gamma(T) \le \delta(T) \le \operatorname{dist}(T, \operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}))$.

(ii) $\gamma(T) \le \| \operatorname{ad}(T)\| \le 2 \operatorname{dist}(T, \operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}))$.

(iii) If $\mathcal{A}$ is unital, then $\alpha(T) = \beta(T)$.

(iv) If $\mathcal{A}$ is unital and has real rank zero, then $\beta(T) = \gamma(T)$.

Proof. (i) This is immediate.

(ii) Let $a, b \in \mathcal{A}_+$ contractions with $ab = 0$. Then $aTb = (aT - Ta)b$ and therefore $\|aTb\| \leq \|aT - Ta\| \leq \|\operatorname{ad}(T)\|$. We thus get $\gamma(T) \leq \|\operatorname{ad}(T)\|$.

Now take a sequence (Φ_n) in $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ such that

$$\operatorname{dist}\bigl(T, \operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})\bigr) = \lim_{n\to\infty} \|T - \Phi_n\|.$$

Let $x \in \mathcal{X}$ and $a \in \mathcal{A}$ with $\|x\| = \|a\| = 1$. Then

$$\begin{aligned}\|T(xa) - T(x)a\| &= \|T(xa) - T(x)a - (\Phi_n(xa) - \Phi_n(x)a)\| \\ &\leq \|T(xa) - \Phi_n(xa)\| + \|(T(x) - \Phi_n(x))a\| \\ &\leq 2\|T - \Phi_n\|,\end{aligned}$$

which gives $\|(aT - Ta)(x)\| \leq 2 \operatorname{dist}\bigl(T, \operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})\bigr)$, and the second inequality is proved.

(iii) It suffices to prove that $\beta(T) \leq \alpha(T)$. Let $e, f \in \mathcal{A}$ mutually orthogonal projections. Then $e \leq f^\perp$, so that $eTf = e(f^\perp T f)$ and therefore

$$\|eTf\| = \|e(f^\perp Tf)\| \leq \|f^\perp Tf\| \leq \alpha(T),$$

which implies that $\beta(T) \leq \alpha(T)$, as required.

(iv) It suffices to show that $\gamma(T) \leq \beta(T)$. Take $a, b \in \mathcal{A}_+$ mutually orthogonal contractions, and the task is to show that $\|aTb\| \leq \beta(T)$.

First, assume that both a and b have finite spectrum, say $\{\alpha_1, \ldots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_n\}$ with $0 \leq \alpha_1 < \cdots < \alpha_m = \|a\| \leq 1$ and $0 \leq \beta_1 < \cdots < \beta_n = \|b\| \leq 1$, and write

$$a = \sum_{j=1}^m \alpha_j p_j, \quad b = \sum_{k=1}^n \beta_k q_k,$$

where $p_1, \ldots, p_m, q_1, \ldots, q_n \in \mathcal{A}$ are mutually orthogonal projections, and, further,

$$a = \sum_{j=1}^m \lambda_j e_j, \quad b = \sum_{k=1}^n \mu_k f_k,$$

where $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n \in [0, \infty[$ and $e_1, \ldots, e_m, f_1, \ldots, f_n \in \mathcal{A}$ are defined by

$$\begin{aligned}&\lambda_1 = \alpha_1, && \mu_1 = \beta_1, \\ &\lambda_j = \alpha_j - \alpha_{j-1} \ (1 < j \leq m), && \mu_k = \beta_k - \beta_{k-1} \ (1 < k \leq n), \\ &e_j = p_j + \cdots + p_m \ (1 \leq j \leq m), && f_k = q_k + \cdots + q_n \ (1 \leq k \leq n),\end{aligned}$$

as in the proof of Theorem 3.2.2. Since $e_j f_k = 0$ for all $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$, it follows that

$$\begin{aligned}\|aTb\| &= \left\|\sum_{j=1}^m \sum_{k=1}^n \lambda_j \mu_k e_j T f_k\right\| \leq \sum_{j=1}^m \sum_{k=1}^n \lambda_j \mu_k \|e_j T f_k\| \\ &\leq \sum_{j=1}^m \sum_{k=1}^n \lambda_j \mu_k \beta(T) = \|a\|\|b\|\beta(T) \leq \beta(T),\end{aligned}$$

as required.

Now consider the general case. Since $\mathcal{A}$ has real rank zero, it follows that there exists a sequence (c_n) in $\mathcal{A}_{\mathrm{sa}}$ such that each c_n has finite spectrum and (c_n) converges to $a-b$ in norm. For each $n \in \mathbb{N}$, set

$$a_n = \tfrac{1}{2}\left(|c_n| + c_n\right), \quad b_n = \tfrac{1}{2}\left(|c_n| - c_n\right).$$

Since $(|c_n|) \to a+b$, we see that $(a_n) \to a$ and $(b_n) \to b$. Further, for each $n \in \mathbb{N}$, a_n, $b_n \in \mathcal{A}_+$, have finite spectra, and $a_n b_n = 0$. From the previous step we deduce that

$$\|a_n T b_n\| \leq \beta(T)\|a_n\|\|b_n\| \quad (n \in \mathbb{N}),$$

and so, taking limits on both sides of the above inequality, we obtain $\|aTb\| \leq \beta(T)$, which completes the proof. $\square$

Now let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{X}$ a Banach right $\mathcal{A}$-module. Then $\mathcal{X}^*$ is a Banach left $\mathcal{A}$-module with respect to the module operation specified by

$$\langle x, a\phi\rangle = \langle xa, \phi\rangle \quad (\phi \in \mathcal{X}^*,\ a \in \mathcal{A},\ x \in \mathcal{X}).$$

This module has the property that the map $\phi \mapsto a\phi$ from $\mathcal{X}^*$ to $\mathcal{X}^*$ is weak* continuous for each $a \in \mathcal{A}$. Similarly, if $\mathcal{X}$ is a Banach left $\mathcal{A}$-module, then $\mathcal{X}^*$ is a Banach right $\mathcal{A}$-module with respect to the module operation specified by

$$\langle x, \phi a\rangle = \langle ax, \phi\rangle \quad (\phi \in \mathcal{X}^*,\ a \in \mathcal{A},\ x \in \mathcal{X}),$$

and the map $\phi \mapsto \phi a$ from $\mathcal{X}^*$ to $\mathcal{X}^*$ is weak* continuous for each $a \in \mathcal{A}$.

Theorem 3.3.2. *Let $\mathcal{A}$ be a C^*-algebra, let $\mathcal{X}$ and $\mathcal{Y}$ be Banach right $\mathcal{A}$-modules with $\mathcal{X}$ essential, and let $T\colon \mathcal{X} \to \mathcal{Y}$ be a continuous linear map.*

(i) Suppose that $\{y \in \mathcal{Y} : y\mathcal{A} = 0\} = \{0\}$ and that

$$a, b \in \mathcal{A}_+,\ ab = 0 \implies aTb = 0.$$

Then T is a right $\mathcal{A}$-module homomorphism.

(ii) Suppose that $\mathcal{Y}$ satisfies the condition

$$\|y\| = \sup\{\|ya\| : a \in \mathcal{A},\ \|a\| = 1\}$$

for each $y \in \mathcal{Y}$. Then

$$\|\operatorname{ad}(T)\| \leq 8 \sup\big\{\|aTb\| : a, b \in \mathcal{A}_+ \text{ contractions, } ab = 0\big\}.$$

Further, the above condition holds in each of the following cases:

(a) $\mathcal{Y}$ is essential;

(b) $\mathcal{Y}$ is the dual of an essential Banach left $\mathcal{A}$-module.

Proof. Define the continuous bilinear map $\varphi\colon \mathcal{A}\times\mathcal{A}\to B(\mathcal{X},\mathcal{Y})$ by

$$\varphi(a,b) = aTb \quad (a,b\in\mathcal{A}),$$

and set $\varepsilon = \sup\bigl\{\|aTb\| : a,b\in\mathcal{A}_+ \text{ contractions}, ab=0\bigr\}$. Then $\|\varphi(a,b)\|\le\varepsilon\|a\|\|b\|$ whenever $a,b\in\mathcal{A}_+$ are such that $ab=0$.

Let $(e_i)_{i\in I}$ be an approximate identity for $\mathcal{A}$ of bound one. As in the proof of Corollary 3.2.3, we see that $(e_i)_{i\in I}$ has a subnet $(e_j)_{j\in J}$ which converges to $1_{\mathcal{A}^{**}}$ in $\mathcal{A}^{**}$ with respect to the weak* topology.

From Theorem 3.2.1 it follows that, for each $a\in\mathcal{A}$, the nets $(\varphi(a,e_j))_{j\in J}$ and $(\varphi(e_j,a))_{j\in J}$ converge in $B(\mathcal{X},\mathcal{Y})^{**}$ with respect to the weak* topology and

$$\Bigl\|\lim_{j\in J}\varphi(a,e_j)-\lim_{j\in J}\varphi(e_j,a)\Bigr\|\le 8\varepsilon\|a\|,$$

whence

$$\Bigl\|\lim_{j\in J} aTe_j-\lim_{j\in J} e_jTa\Bigr\|\le 8\varepsilon\|a\|.$$

In particular, for each $x\in\mathcal{X}$, $a,b\in\mathcal{A}$, and $\phi\in\mathcal{Y}^*$, the net

$$\bigl(\langle b\phi,(aTe_j-e_jTa)(x)\rangle\bigr)_{j\in J}=\bigl(\langle\phi,T(xa)e_jb-T(xe_j)ab\rangle\bigr)_{j\in J}$$

converges and

$$\Bigl|\lim_{j\in J}\langle\phi,T(xa)e_jb-T(xe_j)ab\rangle\Bigr|\le 8\varepsilon\|a\|\|b\|\|x\|\|\phi\|. \tag{3.21}$$

Since $\mathcal{X}$ is essential, it follows that $(xe_j)_{j\in J}\to x$ in norm for each $x\in\mathcal{X}$. Thus (3.21) gives

$$\bigl|\langle\phi,T(xa)b-T(x)ab\rangle\bigr|\le 8\varepsilon\|a\|\|b\|\|x\|\|\phi\|$$

for all $x\in\mathcal{X}$, $a,b\in\mathcal{A}$, and $\phi\in\mathcal{Y}^*$, and hence

$$\|(T(xa)-T(x)a)b\|\le 8\varepsilon\|a\|\|b\|\|x\| \tag{3.22}$$

for all $x\in\mathcal{X}$, and $a,b\in\mathcal{A}$.

(i) In this case, we have $\varepsilon=0$ and, for each $x\in\mathcal{X}$ and $a\in\mathcal{A}$, (3.22) gives

$$\bigl(T(xa)-T(x)a\bigr)b=0 \quad (b\in\mathcal{A}),$$

which yields $T(xa)=T(x)a$. Hence T is a right $\mathcal{A}$-module homomorphism.

(ii) In this case, for each $x\in\mathcal{X}$ and $a\in\mathcal{A}$, (3.22) gives

$$\|T(xa)-T(x)a\|\le 8\varepsilon\|a\|\|x\|,$$

so that $\|\operatorname{ad}(T)\|\le 8\varepsilon$, as claimed.

Now suppose that $\mathcal{Y}$ satisfies either of the additional assumptions (a) or (b); we will prove that $\|y\|=\sup\{\|ya\| : a\in\mathcal{A},\ \|a\|=1\}$ for each $y\in\mathcal{Y}$. Take $y\in\mathcal{Y}$, and set $\alpha=\sup\{\|ya\| : a\in\mathcal{A},\ \|a\|=1\}$. It is clear that $\alpha\le\|y\|$.

In case (a), since $(ye_j)_{j\in J} \to y$, it follows that $(\|ye_j\|)_{j\in J} \to \|y\|$, and consequently $\|y\| \le \alpha$.

In case (b), $\mathcal{Y}$ is the dual of an essential Banach left $\mathcal{A}$-module $\mathcal{Y}_*$. Take $\varepsilon > 0$, and let $\phi \in \mathcal{Y}_*$ with $\|\phi\| = 1$ and $\|y\| - \varepsilon < |\langle \phi, y\rangle|$. Then $(e_j\phi)_{j\in J} \to \phi$ and the continuity of ϕ gives $(\langle \phi, ye_j\rangle)_{j\in J} = (\langle e_j\phi, y\rangle)_{j\in J} \to \langle \phi, y\rangle$, which implies that there exists $j \in J$ such that $\|y\| - \varepsilon < |\langle \phi, ye_j\rangle| \le \alpha$. We thus get $\|y\| \le \alpha + \varepsilon$ for each $\varepsilon > 0$, and hence $\|y\| \le \alpha$, which completes the proof. □

Theorem 3.3.3. *Let $\mathcal{A}$ be a unital C^*-algebra of real rank zero, let $\mathcal{X}$ and $\mathcal{Y}$ be unital quasi-Banach right $\mathcal{A}$-modules, and let $T\colon \mathcal{X} \to \mathcal{Y}$ be a continuous linear map.*

(i) Suppose that

$$e \in \mathcal{A} \text{ projection} \implies e^\perp T e = 0.$$

Then T is a right $\mathcal{A}$-module homomorphism.

(ii) Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach modules. Then

$$\|\operatorname{ad}(T)\| \le 8 \sup\bigl\{\|e^\perp T e\| : e \in \mathcal{A} \text{ projection}\bigr\}.$$

Proof. Define the continuous bilinear map $\varphi\colon \mathcal{A} \times \mathcal{A} \to B(\mathcal{X}, \mathcal{Y})$ by

$$\varphi(a, b) = aTb \quad (a, b \in \mathcal{A}).$$

(i) In this case, $\varphi(e, e^\perp) = 0$ for each projection $e \in \mathcal{A}$. Consequently, Theorem 3.2.2(i) shows that $\varphi(a, 1_{\mathcal{A}}) = \varphi(1_{\mathcal{A}}, a)$ for each $a \in \mathcal{A}$, which gives $aT = Ta$, and this is precisely the assertion of (i).

(ii) Set $\varepsilon = \sup\bigl\{\|e^\perp T e\| : e \in \mathcal{A} \text{ projection}\bigr\}$. Then $\|\varphi(e, e^\perp)\| \le \varepsilon$ for each projection $e \in \mathcal{A}$, and Theorem 3.2.2(ii) now shows that

$$\|aT - Ta\| = \|\varphi(a, 1_{\mathcal{A}}) - \varphi(1_{\mathcal{A}}, a)\| \le 8\varepsilon\|a\|$$

for each $a \in \mathcal{A}$. We thus get $\|\operatorname{ad}(T)\| \le 8\varepsilon$, as claimed. □

Theorem 3.3.4. *Let $\mathcal{A}$ be a C^*-algebra of real rank zero, let $\mathcal{X}$ and $\mathcal{Y}$ be Banach right $\mathcal{A}$-modules with $\mathcal{X}$ essential, and let $T\colon \mathcal{X} \to \mathcal{Y}$ be a continuous linear map.*

(i) Suppose that $\{y \in \mathcal{Y} : y\mathcal{A} = 0\} = \{0\}$ and that

$$e, f \in \mathcal{A} \text{ projections, } ef = 0 \implies eTf = 0.$$

Then T is a right $\mathcal{A}$-module homomorphism.

(ii) Suppose that $\mathcal{Y}$ satisfies the condition

$$\|y\| = \sup\{\|ya\| : a \in \mathcal{A},\ \|a\| = 1\}$$

for each $y \in \mathcal{Y}$. Then

$$\|\operatorname{ad}(T)\| \le 8 \sup\bigl\{\|eTf\| : e, f \in \mathcal{A} \text{ projections, } ef = 0\bigr\}.$$

Further, the above condition holds in each of the following cases:

(a) $\mathcal{Y}$ *is essential;*

(b) $\mathcal{Y}$ *is the dual of an essential Banach left* $\mathcal{A}$*-module.*

Proof. Take an approximate identity $(e_j)_{j\in J}$ for $\mathcal{A}$ consisting of projections. Fix $j \in J$, define the continuous bilinear map $\varphi_j : e_j\mathcal{A}e_j \times e_j\mathcal{A}e_j \to B(\mathcal{X}, \mathcal{Y})$ by

$$\varphi_j(a, b) = aTb \quad (a, b \in e_j\mathcal{A}e_j),$$

and set $\varepsilon = \sup\big\{\|eTf\| : e, f \in \mathcal{A} \text{ projections}, ef = 0\big\}$. Then $e_j\mathcal{A}e_j$ is a unital C^*-algebra (with unit e_j) and has real rank zero. Further, $\|\varphi(e, e_j - e)\| \le \varepsilon$ for each projection $e \in e_j\mathcal{A}e_j$. From Theorem 3.2.2(ii) it follows that

$$\|aTe_j - e_jTa\| = \|\varphi(a, e_j) - \varphi(e_j, a)\| \le 8\varepsilon\|a\|$$

for each $a \in e_j\mathcal{A}e_j$. Hence

$$\|T(xe_jae_j)e_jb - T(xe_j)e_jae_jb\| \le 8\varepsilon\|x\|\|a\|\|b\| \quad (j \in J,\ x \in \mathcal{X},\ a, b \in \mathcal{A}). \tag{3.23}$$

For each $x \in \mathcal{X}$ and $a, b \in \mathcal{A}$, we have

- $(e_jae_j)_{j\in J} \to a$ and $(e_jb)_{j\in J} \to b$ in norm, so that (using the continuity of T) $(T(xe_jae_j)e_jb)_{j\in J} \to T(xa)b$ in norm;
- $(xe_j)_{j\in J} \to x$ in norm, because $\mathcal{X}$ is essential, and $(e_jae_jb)_{j\in J} \to ab$ in norm, and hence (using the continuity of T) $(T(xe_j)e_jae_jb)_{j\in J} \to T(x)ab$ in norm.

Thus, taking the limit in (3.23) we see that

$$\|T(xa)b - T(x)ab\| \le 8\varepsilon\|x\|\|a\|\|b\| \quad (x \in \mathcal{X},\ a, b \in \mathcal{A}).$$

The rest of the proof goes through as for Theorem 3.3.2 (from inequality (3.22) on). □

Corollary 3.3.5. *Let* $\mathcal{A}$ *be a* C^**-algebra, and let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be Banach right* $\mathcal{A}$*-modules. Suppose that* $\mathcal{X}$ *is essential and that* $\{y \in \mathcal{Y} : y\mathcal{A} = 0\} = \{0\}$*. Then the space* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ *is reflexive.*

Proof. Take $T \in B(\mathcal{X}, \mathcal{Y})$ such that

$$T(x) \in \overline{\{\Phi(x) : \Phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})\}} \quad (x \in \mathcal{X}).$$

Let $a, b \in \mathcal{A}$ be such that $ab = 0$. We claim that $aTb = 0$. For each $x \in \mathcal{X}$, there exists a sequence (Φ_n) in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ such that $(\Phi_n(xa)) \to T(xa)$ in norm, and hence

$$(aTb)(x) = T(xa)b = \lim_{n\to\infty} \Phi_n(xa)b = \lim_{n\to\infty} \Phi_n(xab) = 0,$$

which proves our claim.

From Theorem 3.3.2(i), it follows that $T \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$. □

Corollary 3.3.6. *Let* $\mathcal{A}$ *be a unital* C^**-algebra of real rank zero, and let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be unital quasi-Banach right* $\mathcal{A}$*-modules. Then the space* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ *is reflexive.*

Proof. This follows by the same method as in Corollary 3.3.5, with Theorem 3.3.2(i) replaced by Theorem 3.3.3(i). □

3.3.2 Homomorphisms between non-commutative L^p-spaces

Let $\mathcal{M}$ be a von Neumann algebra. For each $0 < p \leq \infty$, the space $L^p(\mathcal{M})$ is a contractive Banach $\mathcal{M}$-bimodule or a contractive p-Banach $\mathcal{M}$-bimodule according to $p \geq 1$ or $p < 1$. More generally, if $0 < p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ (we adopt throughout the convention that $\frac{1}{\infty} = 0$), then

$$x \in L^p(\mathcal{M}),\ y \in L^q(\mathcal{M}) \implies xy \in L^r(\mathcal{M}) \text{ and } \|xy\|_r \leq \|x\|_p \|y\|_q.$$

This is the non-commutative Hölder's inequality. Recall that the spaces $L^p(\mathcal{M})$ are constructed as spaces of measurable operators affiliated with a certain semi-finite von Neumann algebra containing $\mathcal{M}$, namely, the crossed product of $\mathcal{M}$ by one of its modular automorphism groups, so that in Hölder's inequality xy stands for the product of the unbounded operators x and y. From now on we confine attention to the right $\mathcal{M}$-module structure of $L^p(\mathcal{M})$.

Theorem 3.3.7. *Let $\mathcal{M}$ be a von Neumann algebra, let $0 < p, q \leq \infty$, and let $T\colon L^p(\mathcal{M}) \to L^q(\mathcal{M})$ be a linear map. Suppose that the map $e^\perp Te\colon L^p(\mathcal{M}) \to L^q(\mathcal{M})$ is continuous for each projection $e \in \mathcal{M}$. Then T is continuous.*

Proof. We first observe that $eT - Te = eTe^\perp - e^\perp Te$ is continuous for each projection $e \in \mathcal{M}$.

Now we consider the separating space of T, which is defined by

$$\mathcal{S}(T) = \{y \in L^q(\mathcal{M}) : \text{there exists } (x_n) \to 0 \text{ in } L^p(\mathcal{M}) \text{ with } (T(x_n)) \to y\}.$$

It is an immediate restatement of the closed graph theorem that T is continuous if and only if $\mathcal{S}(T) = 0$.

We claim that $\mathcal{S}(T)$ is a closed right submodule of $L^q(\mathcal{M})$. By [36, Proposition 5.1.2], $\mathcal{S}(T)$ is a closed subspace of $L^q(\mathcal{M})$. Let $y \in \mathcal{S}(T)$, and let e be a projection in $\mathcal{M}$. Take a sequence (x_n) in $L^p(\mathcal{M})$ with $\lim x_n = 0$ and $\lim T(x_n) = y$. Then $\lim x_n e = 0$ and, using the first observation,

$$T(x_n e) = (eT - Te)(x_n) + T(x_n)e \to ye.$$

Thus $ye \in \mathcal{S}(T)$. Now let $a \in \mathcal{M}$ be an arbitrary element. Then there exists a sequence (a_n) in $\mathcal{M}$ such that each a_n is a linear combination of projections and $\lim a_n = a$. Since $\mathcal{S}(T)$ is a closed subspace of $L^q(\mathcal{M})$, it follows that $ya_n \in \mathcal{S}(T)$ $(n \in \mathbb{N})$ and that $ya = \lim ya_n \in \mathcal{S}(T)$. Hence $\mathcal{S}(T)$ is a right submodule of $L^q(\mathcal{M})$, as claimed.

We now consider the subspace $\mathcal{I}(T)$ defined by

$$\mathcal{I}(T) = \{a \in \mathcal{M} : \mathcal{S}(T)a = 0\}.$$

It is clear that $\mathcal{I}(T)$ is a closed right ideal of $\mathcal{M}$. Further, since $\mathcal{S}(T)$ is a right submodule of $L^q(\mathcal{M})$, it follows immediately that $\mathcal{I}(T)$ is also a left ideal of $\mathcal{M}$. Our next goal is to prove that $\mathcal{I}(T)$ is weak* closed in $\mathcal{M}$. Take $y \in L^q(\mathcal{M})$, and let $s_r(y)$ be the right support projection of y. Then

$$\{a \in \mathcal{M} : ya = 0\} = \{a \in \mathcal{M} : s_r(y)a = 0\}$$

(see [72, Fact 1.2(ii)]) and, since $s_r(y) \in \mathcal{M}$, this latter set is clearly weak* closed in $\mathcal{M}$. Since

$$\mathcal{I}(T) = \bigcap_{y \in \mathcal{S}(T)} \{a \in \mathcal{M} : ya = 0\},$$

we conclude that $\mathcal{I}(T)$ is weak* closed.

Since $\mathcal{I}(T)$ is a weak* closed two-sided ideal of $\mathcal{M}$, it follows that there exists a central projection $e \in \mathcal{M}$ such that

$$\mathcal{I}(T) = e\mathcal{M}.$$

We now claim that $\dim e^\perp \mathcal{M} < \infty$. Assume towards a contradiction that $\dim e^\perp \mathcal{M} = \infty$. Then we can take a sequence (e_n) of non-zero mutually orthogonal projections in $e^\perp \mathcal{M}$. For $n \in \mathbb{N}$, we define the projection $f_n \in \mathcal{M}$ by

$$f_n = \bigvee_{k \geq n} e_k,$$

and consider the maps $R_n \in B(L^p(\mathcal{M}), L^p(\mathcal{M}))$ and $S_n \in B(L^q(\mathcal{M}), L^q(\mathcal{M}))$ defined by

$$R_n(x) = xf_n, \quad S_n(y) = yf_n \quad (x \in L^p(\mathcal{M}),\ y \in L^q(\mathcal{M})).$$

Our next objective is to apply a fundamental result about the separating space: the so-called stability lemma. By hypothesis, $TR_n - S_nT$ is continuous for each $n \in \mathbb{N}$, and hence, by [36, Corollary 5.2.7], $\big(\overline{S_1 \cdots S_n(\mathcal{S}(T))}\big)$ is a nest in $L^q(\mathcal{M})$ which stabilizes. Since $f_{n+1} \leq f_n$ for each $n \in \mathbb{N}$, it follows that $S_1 \cdots S_n = S_n$, and hence that

$$\overline{S_1 \cdots S_n(\mathcal{S}(T))} = \overline{\mathcal{S}(T)f_n} = \mathcal{S}(T)f_n$$

for each $n \in \mathbb{N}$. Thus there exists $N \in \mathbb{N}$ such that

$$\mathcal{S}(T)f_N = \mathcal{S}(T)f_n \quad (N \leq n).$$

In particular, since $f_N e_N = e_N$ and $f_{N+1}e_N = 0$, we have

$$\mathcal{S}(T)e_N = \big(\mathcal{S}(T)f_N\big)e_N = \big(\mathcal{S}(T)f_{N+1}\big)e_N = 0.$$

Hence $e_N \in \mathcal{I}(T) = e\mathcal{M}$. But this is a contradiction of the facts that $e_N \in e^\perp\mathcal{M}$ and $e_N \neq 0$.

Our next claim is that the map $Te^\perp \colon L^p(\mathcal{M}) \to L^q(\mathcal{M})$ is continuous. Since the projection $e^\perp$ is central, we see that $e^\perp x = xe^\perp$ for each $x \in L^p(\mathcal{M})$, and hence $e^\perp L^p(\mathcal{M}) = e^\perp L^p(\mathcal{M})e^\perp$. Moreover, [72, Fact 1.4] shows that the subspace $e^\perp L^p(\mathcal{M})e^\perp$ is isometrically isomorphic to $L^p(e^\perp \mathcal{M} e^\perp)$. Since $\dim e^\perp \mathcal{M} < \infty$, it follows that

$\dim L^p(e^\perp \mathcal{M} e^\perp) < \infty$, so that $\dim e^\perp L^p(\mathcal{M}) < \infty$. Thus the restriction of T to the subspace $e^\perp L^p(\mathcal{M})$ is continuous, and hence the map

$$e^\perp T \colon L^p(\mathcal{M}) \to L^q(\mathcal{M}),\ x \mapsto T(xe^\perp)$$

is continuous. On the other hand,

$$Te^\perp = e^\perp T - (e^\perp T - Te^\perp),$$

which implies that $Te^\perp$ is continuous, as claimed.

Finally, we are in a position to prove the continuity of T. From the above claim we deduce that $\mathcal{S}(T)e^\perp = 0$, and hence that $e^\perp \in \mathcal{I}(T) = e\mathcal{M}$. This implies that $e^\perp = 0$, whence $1_\mathcal{M} = e \in \mathcal{I}(T)$, which gives $\mathcal{S}(T) = \mathcal{S}(T)1_\mathcal{M} = 0$ and T is continuous. □

Corollary 3.3.8. *Let $\mathcal{M}$ be a von Neumann algebra, let $0 < p, q \leq \infty$, and let $T \colon L^p(\mathcal{M}) \to L^q(\mathcal{M})$ be a right $\mathcal{M}$-module homomorphism. Then T is continuous.*

Proof. It is clear that T satisfies the requirement in Theorem 3.3.7 and hence T is continuous. □

Suppose that $0 < p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. By Hölder's inequality, for each $\xi \in L^r(\mathcal{M})$, we can define the left composition map $L_\xi \colon L^p(\mathcal{M}) \to L^q(\mathcal{M})$ by

$$L_\xi(x) = \xi x \quad (x \in L^p(\mathcal{M})).$$

Further L_ξ is continuous with $\|L_\xi\| \leq \|\xi\|_r$, and it is obvious that L_ξ is a right $\mathcal{M}$-module homomorphism.

Theorem 3.3.9. *Let $\mathcal{M}$ be a von Neumann algebra, and let $0 < p, q \leq \infty$.*

(i) Suppose that $p \geq q$, and let $0 < r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Then the map

$$\xi \mapsto L_\xi,\ \ L^r(\mathcal{M}) \to \mathrm{Hom}_\mathcal{M}\big(L^p(\mathcal{M}), L^q(\mathcal{M})\big)$$

is an isometric linear bijection.

(ii) Suppose that $p < q$ and that $\mathcal{M}$ has no minimal projection. Then

$$\mathrm{Hom}_\mathcal{M}\big(L^p(\mathcal{M}), L^q(\mathcal{M})\big) = \{0\}\,.$$

Proof. (i) By [56, Theorem 2.5], this map is a surjection. We proceed to show that it is an isometry. Let $\xi \in L^r(\mathcal{M}) \setminus \{0\}$. We have already seen that $\|L_\xi\| \leq \|\xi\|_r$. We now establish the reverse inequality by considering three cases.

Assume that $p = \infty$. Then $r = q$ and

$$\|\xi\|_r = \|L_\xi(1_\mathcal{M})\|_q \leq \|L_\xi\|\|1_\mathcal{M}\| = \|L_\xi\|,$$

as required.

Now assume that $p < \infty$ and that $r = \infty$. Then $p = q$, and, for each $x \in L^p(\mathcal{M})$, we have

$$\begin{aligned}\|L_\xi(x)\|_p = \|\xi x\|_p = \big\|(\xi x)^*(\xi x)\big\|_{p/2}^{1/2} = \big\|x^*\xi^*\xi x\big\|_{p/2}^{1/2} \\ = \big\|x^*|\xi|^2 x\big\|_{p/2}^{1/2} = \big\|(|\xi|x)^*(|\xi|x)\big\|_{p/2}^{1/2} = \big\||\xi|x\big\|_p = \big\|L_{|\xi|}(x)\big\|_p.\end{aligned}$$

Thus $\|L_\xi\| = \|L_{|\xi|}\|$, and [56, Lemma 2.1] shows that $\|L_{|\xi|}\| = \||\xi|\| = \|\xi\|$.

Finally, assume that $p, r < \infty$. Then $|\xi|^{r/p} \in L^p(\mathcal{M})$, and we have

$$\begin{aligned}\big\|L_\xi\big(|\xi|^{r/p}\big)\big\|_q = \big\|\xi|\xi|^{r/p}\big\|_q = \big\||\xi|^{r/p}\xi^*\xi|\xi|^{r/p}\big\|_{q/2}^{1/2} \\ = \big\||\xi|^{2(1+r/p)}\big\|_{q/2}^{1/2} = \big\||\xi|^{2r/q}\big\|_{q/2}^{1/2} = \|\xi\|_r^{r/q}.\end{aligned}$$

On the other hand, we have

$$\big\|L_\xi\big(|\xi|^{r/p}\big)\big\|_q \le \|L_\xi\|\big\||\xi|^{r/p}\big\|_p = \|L_\xi\|\|\xi\|_r^{r/p},$$

and hence

$$\|\xi\|_r = \|\xi\|_r^{r/q-r/p} \le \|L_\xi\|,$$

as required.

(ii) [56, Corollary 2.7]. □

Corollary 3.3.10. *Let $\mathcal{M}$ be a von Neumann algebra, let $0 < p, q \le \infty$, and let $T\colon L^p(\mathcal{M}) \to L^q(\mathcal{M})$ be a continuous linear map.*

(i) Suppose that

$$e \in \mathcal{M} \text{ projection} \implies e^\perp T e = 0.$$

Then T is a right $\mathcal{M}$-module homomorphism.

(ii) Suppose that $1 \le p, q \le \infty$. Then

$$\|\operatorname{ad}(T)\| \le 8 \sup\big\{\|e^\perp T e\| : e \in \mathcal{M} \text{ projection}\big\}.$$

Proof. Since every von Neumann algebra has real rank zero, this result follows from Theorem 3.3.3. □

By Corollary 3.3.6, the space $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is reflexive. However we next show that this space is not merely reflexive.

Corollary 3.3.11. *Let $\mathcal{M}$ be a von Neumann algebra, let $0 < p, q \le \infty$, and let $T\colon L^p(\mathcal{M}) \to L^q(\mathcal{M})$ be a linear map such that*

$$T(x) \in \overline{\{\Phi(x) : \Phi \in \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))\}} \quad (x \in L^p(\mathcal{M})).$$

Then $T \in \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$. In particular, the space $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is reflexive.

Proof. Take a projection $e \in \mathcal{M}$. For each $x \in L^p(\mathcal{M})$, there exists a sequence (Φ_n) in $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ such that $T(xe^\perp) = \lim \Phi_n(xe^\perp)$ in norm, and hence

$$(e^\perp Te)(x) = T(xe^\perp)e = \lim_{n\to\infty} \Phi_n(xe^\perp)e = \lim_{n\to\infty} \Phi_n(x)e^\perp e = 0.$$

This shows that $e^\perp Te = 0$.

From Theorem 3.3.7, it follows that T is continuous, and Corollary 3.3.10 then gives $T \in \mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$. □

3.4 Distance estimates

Besides the investigation of the hyperreflexivity of $\mathrm{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ for a von Neumann algebra $\mathcal{M}$ and $1 \le p, q \le \infty$, the motivation behind this section comes from the analysis of almost annihilator preservers offered in [66, Section 2].

3.4.1 Homomorphisms between modules over a C^*-algebra

Let $\mathcal{M}$ be a von Neumann algebra, and let $\mathcal{X}$ a Banach right $\mathcal{M}$-module. Then the Banach left $\mathcal{M}$-module $\mathcal{X}^*$ is called normal if the map $a \mapsto a\phi$ from $\mathcal{M}$ to $\mathcal{X}^*$ is weak* continuous for each $\phi \in \mathcal{X}^*$. Similarly, if $\mathcal{X}$ is a Banach left $\mathcal{M}$-module, then the Banach right $\mathcal{M}$-module $\mathcal{X}^*$ is called normal if the map $a \mapsto \phi a$ from $\mathcal{M}$ to $\mathcal{X}^*$ is weak* continuous for each $\phi \in \mathcal{X}^*$.

Theorem 3.4.1. *Let $\mathcal{M}$ be an injective von Neumann algebra, let $\mathcal{X}$ and $\mathcal{Y}$ be Banach right and left $\mathcal{M}$-modules, respectively, and let $T\colon \mathcal{X} \to \mathcal{Y}^*$ be a continuous linear map.*

(i) The subset $\mathcal{W}(T)$ of $\mathcal{X}$ consisting of the elements $x \in \mathcal{X}$ with the property that the bilinear map $(a, b) \mapsto T(xa)b$ from $\mathcal{M} \times \mathcal{M}$ to $\mathcal{Y}^*$ is separately weak continuous is a closed submodule of $\mathcal{X}$. Further, if the modules $\mathcal{X}^*$ and $\mathcal{Y}^*$ are normal, then $\mathcal{W}(T) = \mathcal{X}$.*

(ii) There exists a continuous linear map $\Phi\colon \mathcal{X} \to \mathcal{Y}^*$ such that:

(a) $\|\Phi\| \le \|T\|$;

(b) $\Phi(xa) = \Phi(x)a$ for all $x \in \mathcal{W}(T)$ and $a \in \mathcal{M}$; in particular, if both $\mathcal{X}^*$ and $\mathcal{Y}^*$ are normal, then Φ is a right $\mathcal{M}$-module homomorphism;

(c) $\|1_{\mathcal{M}}T - \Phi\| \le \|\operatorname{ad}(T)\|$; in particular, if the module $\mathcal{X}$ is unital, then $\|T - \Phi\| \le \|\operatorname{ad}(T)\|$;

(d) $\|T1_{\mathcal{M}} - \Phi\| \le \|\operatorname{ad}(T)\|$; in particular, if the module $\mathcal{Y}$ is unital, then $\|T - \Phi\| \le \|\operatorname{ad}(T)\|$.

Proof. (i) Routine verifications show that $\mathcal{W}(T)$ is a submodule of $\mathcal{X}$. To show that $\mathcal{W}(T)$ is closed, take a sequence (x_n) in $\mathcal{W}(T)$ and $x \in \mathcal{X}$ such that $(x_n) \to x$ in norm. We define continuous bilinear maps $\tau, \tau_n\colon \mathcal{M} \times \mathcal{M} \to \mathcal{Y}^*$ by

$$\tau(a, b) = T(xa)b,\ \ \tau_n(a, b) = T(x_n a)b \quad (a, b \in \mathcal{M},\ n \in \mathbb{N}).$$

Then $(\tau_n) \to \tau$ in norm, and each τ_n is separately weak* continuous. This implies that τ is separately weak* continuous, which shows that $x \in \mathcal{W}(T)$.

Suppose that both $\mathcal{X}^*$ and $\mathcal{Y}^*$ are normal, and take $x \in \mathcal{X}$. For each $a \in \mathcal{M}$, $T(xa) \in \mathcal{Y}^*$, so that, by definition, the map $b \mapsto T(xa)b$ from $\mathcal{M}$ to $\mathcal{Y}^*$ is weak* continuous. For each $b \in \mathcal{M}$ and each $y \in \mathcal{Y}$, define $\phi_{b,y} \in \mathcal{X}^*$ by

$$\langle x, \phi_{b,y} \rangle = \langle y, T(x)b \rangle \quad (x \in \mathcal{X}).$$

Then

$$\langle y, T(xa)b \rangle = \langle x, a\phi_{b,y} \rangle \quad (x \in \mathcal{X},\, a \in \mathcal{M}),$$

and, since the map $a \mapsto a\phi_{b,y}$ is weak* continuous, it follows that the functional $a \mapsto \langle y, T(xa)b \rangle$ is weak* continuous for each $x \in \mathcal{X}$.

(ii) Let G be the discrete semigroup of the isometries of $\mathcal{M}$. A mean on G is a state μ on $\ell^\infty(G)$ and, for a given mean μ, we use the formal notation

$$\int_G \phi(u)\, d\mu(u) := \langle \phi, \mu \rangle \quad (\phi \in \ell^\infty(G)).$$

By [47, Theorem 2.1], there exists a mean μ on G with the property that

$$\int_G \tau(au^*, u)\, d\mu(u) = \int_G \tau(u^*, ua)\, d\mu(u) \tag{3.24}$$

for each separately weak* continuous bilinear functional $\tau\colon \mathcal{M} \times \mathcal{M} \to \mathbb{C}$ and each $a \in \mathcal{M}$.

Define $\Phi\colon \mathcal{X} \to \mathcal{Y}^*$ by

$$\langle y, \Phi(x) \rangle = \int_G \langle y, T(xu^*)u \rangle\, d\mu(u) \quad (x \in \mathcal{X},\, y \in \mathcal{Y}).$$

Then Φ is well-defined and linear. Further, for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and $u \in G$, we have

$$\begin{aligned} |\langle y, T(xu^*)u \rangle| &\le \|y\|\|T(xu^*)u\| \le \|y\|\|T(xu^*)\| \\ &\le \|y\|\|T\|\|xu^*\| \le \|y\|\|T\|\|x\|, \end{aligned}$$

which implies that

$$\left| \int_G \langle y, T(xu^*)u \rangle\, d\mu(u) \right| \le \|T\|\|x\|\|y\|$$

and hence that $\|\Phi(x)\| \le \|T\|\|x\|$. Thus Φ is continuous and (a) holds.

Let $x \in \mathcal{W}(T)$, $a \in \mathcal{M}$, and $y \in \mathcal{Y}$. Since, by definition, the bilinear functional $(u, v) \mapsto \langle y, T(xu)v \rangle$ is separately weak* continuous, it follows from (3.24) that

$$\begin{aligned} \langle y, \Phi(xa) \rangle &= \int_G \langle y, T(xau^*)u \rangle\, d\mu(u) = \int_G \langle y, T(xu^*)ua \rangle\, d\mu(u) \\ &= \int_G \langle ay, T(xu^*)u \rangle\, d\mu(u) = \langle ay, \Phi(x) \rangle = \langle y, \Phi(x)a \rangle. \end{aligned}$$

This establishes (b).

Now let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ with $\|x\| = \|y\| = 1$. Then

$$\begin{aligned}
|\langle y, T(x1_{\mathcal{M}}) - \Phi(x)\rangle| &= \left|\int_G \langle y, T(x1_{\mathcal{M}}) - T(xu^*)u\rangle \, d\mu(u)\right| \\
&= \left|\int_G \langle y, T(xu^*u) - T(xu^*)u\rangle \, d\mu(u)\right| \\
&\leq \int_G \left|\langle y, T(xu^*u) - T(xu^*)u)\rangle\right| d\mu(u) \\
&\leq \int_G \|T(xu^*u) - T(xu^*)u\| \; d\mu(u) \\
&\leq \int_G \|\operatorname{ad}(T)\|\|xu^*\|\|u\| \, d\mu(u) \\
&\leq \int_G \|\operatorname{ad}(T)\|\|x\|\|u^*\|\|u\| \, d\mu(u) = \|\operatorname{ad}(T)\|.
\end{aligned}$$

This gives (c). We also have

$$\begin{aligned}
|\langle y, T(x)1_{\mathcal{M}} - \Phi(x)\rangle| &= \left|\int_G \langle y, T(x)1_{\mathcal{M}} - T(xu^*)u\rangle \, d\mu(u)\right| \\
&= \left|\int_G \langle y, T(x)u^*u - T(xu^*)u\rangle \, d\mu(u)\right| \\
&\leq \int_G \left|\langle y, T(x)u^*u - T(xu^*)u\rangle\right| d\mu(u) \\
&\leq \int_G \|T(x)u^*u - T(xu^*)u\| \, d\mu(u) \\
&\leq \int_G \|T(x)u^* - T(xu^*)\| \, d\mu(u) \\
&\leq \int_G \|\operatorname{ad}(T)\|\|x\|\|u^*\| \, d\mu(u) = \|\operatorname{ad}(T)\|,
\end{aligned}$$

and this gives (d). □

Theorem 3.4.2. *Let $\mathcal{A}$ be a nuclear C^*-algebra, let $\mathcal{X}$ and $\mathcal{Y}$ be Banach right and left $\mathcal{A}$-modules, respectively, and let $T\colon \mathcal{X} \to \mathcal{Y}^*$ be a continuous linear map. Then there exists a continuous right $\mathcal{A}$-module homomorphism $\Phi\colon \mathcal{X} \to \mathcal{Y}^*$ such that:*

(a) $\|\Phi\| \leq \|T\|$;

(b) $\|aT - a\Phi\| \leq \|\operatorname{ad}(T)\|\|a\|$ $(a \in \mathcal{A})$; *moreover, if the module $\mathcal{X}$ is essential, then* $\|T - \Phi\| \leq \|\operatorname{ad}(T)\|$;

(c) $\|Ta - \Phi a\| \leq \|\operatorname{ad}(T)\|\|a\|$ $(a \in \mathcal{A})$; *moreover, if the module $\mathcal{Y}$ is essential, then* $\|T - \Phi\| \leq \|\operatorname{ad}(T)\|$.

Proof. Consider the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$, and let $\pi\colon \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ be the continuous linear map defined through

$$\pi(a \otimes b) = ab \quad (a, b \in \mathcal{A}).$$

The Banach space $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a contractive Banach $\mathcal{A}$-bimodule with respect to the operations defined through

$$(a \otimes b)c = a \otimes bc,\ \ c(a \otimes b) = ca \otimes b \quad (a, b, c \in \mathcal{A}).$$

By [47, Theorem 3.1], there exists a virtual diagonal for $\mathcal{A}$ of norm one. This is an element $\mathrm{M} \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ with $\|\mathrm{M}\| = 1$ such that, for each $a \in \mathcal{A}$, we have

$$a\mathrm{M} = \mathrm{M}a \quad \text{and} \quad \pi^{**}(\mathrm{M})a = a.$$

Here, both $(\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ and $\mathcal{A}^{**}$ are considered as dual $\mathcal{A}$-bimodules in the usual way. For each continuous bilinear functional $\tau\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ there exists a unique element $\widehat{\tau} \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^*$ such that

$$\widehat{\tau}(a \otimes b) = \tau(a, b) \quad (a, b \in \mathcal{A}),$$

and we use the formal notation

$$\int_{\mathcal{A}\times\mathcal{A}} \tau(u, v)\, d\mathrm{M}(u, v) := \langle \widehat{\tau}, \mathrm{M}\rangle.$$

Using this notation, the defining properties of M can be written as

$$\int_{\mathcal{A}\times\mathcal{A}} \tau(au, v)\, d\mathrm{M}(u, v) = \int_{\mathcal{A}\times\mathcal{A}} \tau(u, va)\, d\mathrm{M}(u, v) \tag{3.25}$$

and

$$\int_{\mathcal{A}\times\mathcal{A}} \langle auv, \phi\rangle\, d\mathrm{M}(u, v) = \langle a, \phi\rangle \tag{3.26}$$

for each continuous bilinear functional $\tau\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$, each $a \in \mathcal{A}$, and each $\phi \in \mathcal{A}^*$; further, it will be helpful noting that

$$\left|\int_{\mathcal{A}\times\mathcal{A}} \tau(u, v)\, d\mathrm{M}(u, v)\right| \leq \|\mathrm{M}\|\|\widehat{\tau}\| = \|\tau\|. \tag{3.27}$$

Define $\Phi\colon \mathcal{X} \to \mathcal{Y}^*$ by

$$\langle y, \Phi(x)\rangle = \int_{\mathcal{A}\times\mathcal{A}} \langle y, T(xu)v\rangle\, d\mathrm{M}(u, v) \quad (x \in \mathcal{X},\ y \in \mathcal{Y}).$$

Then Φ is well-defined and linear. For each $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and $u, v \in \mathcal{A}$, we have

$$\begin{aligned} |\langle y, T(xu)v\rangle| &\leq \|T(xu)v\|\|y\| \leq \|T(xu)\|\|v\|\|y\| \\ &\leq \|T\|\|xu\|\|v\|\|y\| \leq \|T\|\|x\|\|u\|\|v\|\|y\|. \end{aligned}$$

Then, using (3.27), we have

$$\left| \int_{\mathcal{A}\times\mathcal{A}} \langle y, T(xu)v\rangle \, d\mathrm{M}(u,v) \right| \le \|T\|\|x\|\|y\|,$$

which implies that $\|\Phi(x)\| \le \|T\|\|x\|$. Thus Φ is continuous and (a) holds.

We claim that Φ is a right $\mathcal{A}$-module homomorphism. Indeed, for $x \in \mathcal{X}$, $a \in \mathcal{A}$, and each $y \in \mathcal{Y}$, (3.25) gives

$$\begin{aligned}\langle y, \Phi(xa)\rangle &= \int_{\mathcal{A}\times\mathcal{A}} \langle y, T(xau)v\rangle \, d\mathrm{M}(u,v) = \int_{\mathcal{A}\times\mathcal{A}} \langle y, T(xu)va\rangle \, d\mathrm{M}(u,v) \\ &= \int_{\mathcal{A}\times\mathcal{A}} \langle ay, T(xu)v\rangle \, d\mathrm{M}(u,v) = \langle ay, \Phi(x)\rangle = \langle y, \Phi(x)a\rangle.\end{aligned}$$

Our next objective is to prove (b). Take $x \in \mathcal{X}$, $a \in \mathcal{A}$, and $y \in \mathcal{Y}$ with $\|x\| = \|a\| = \|y\| = 1$, and define $\phi \in \mathcal{A}^*$ and $\tau \colon \mathcal{A}\times\mathcal{A} \to \mathbb{C}$ by

$$\begin{aligned}\langle u, \phi\rangle &= \langle y, T(xu)\rangle \quad (u \in \mathcal{A}), \\ \tau(u,v) &= \langle y, T(xauv) - T(xau)v\rangle \quad (u, v \in \mathcal{A}).\end{aligned}$$

For each $u, v \in \mathcal{A}$, we have

$$\begin{aligned}|\tau(u,v)| &\le \|T(xauv) - T(xau)v\| \le \|\operatorname{ad}(T)\|\|xau\|\|v\| \\ &\le \|\operatorname{ad}(T)\|\|x\|\|au\|\|v\| \le \|\operatorname{ad}(T)\|\|u\|\|v\|,\end{aligned}$$

so that $\|\tau\| \le \|\operatorname{ad}(T)\|$. By (3.26),

$$\langle y, T(xa)\rangle = \langle a, \phi\rangle = \int_{\mathcal{A}\times\mathcal{A}} \langle auv, \phi\rangle \, d\mathrm{M}(u,v) = \int_{\mathcal{A}\times\mathcal{A}} \langle y, T(xauv)\rangle \, d\mathrm{M}(u,v),$$

and, using the definition of Φ, we obtain

$$\langle y, T(xa) - \Phi(xa)\rangle = \int_{\mathcal{A}\times\mathcal{A}} \tau(u,v) \, d\mathrm{M}(u,v).$$

By (3.27), $|\langle y, T(xa) - \Phi(xa)\rangle| \le \|\operatorname{ad}(T)\|$. Since this inequality holds for each $y \in \mathcal{Y}$ with $\|y\| = 1$, it follows that

$$\|T(xa) - \Phi(xa)\| \le \|\operatorname{ad}(T)\|.$$

Now assume that $\mathcal{X}$ is essential. Take an approximate identity $(e_j)_{j\in J}$ for $\mathcal{A}$ of bound 1. Then $(e_j)_{j\in J}$ is a right approximate identity for $\mathcal{X}$ and, for each $x \in \mathcal{X}$ with $\|x\| = 1$,

$$\|T(xe_j) - \Phi(xe_j)\| \le \|\operatorname{ad}(T)\| \quad (j \in J),$$

so that, using the continuity of T and Φ, we see that $\|T(x) - \Phi(x)\| \le \|\operatorname{ad}(T)\|$. Thus $\|T - \Phi\| \le \|\operatorname{ad}(T)\|$.

Finally, we proceed to prove (c). Take $x \in \mathcal{X}$, $a \in \mathcal{A}$, and $y \in \mathcal{Y}$ with $\|x\| = \|a\| = \|y\| = 1$, and define $\phi \in \mathcal{A}^*$ and $\tau\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ by

$$\langle u, \phi \rangle = \langle y, T(x)u \rangle \quad (u \in \mathcal{A}),$$
$$\tau(u, v) = \langle y, T(x)auv - T(xau)v \rangle \quad (u, v \in \mathcal{A}).$$

For each $u, v \in \mathcal{A}$, we have

$$\begin{aligned} |\tau(u, v))| &\leq \|T(x)auv - T(xau)v\| \leq \|T(x)au - T(xau)\|\|v\| \\ &\leq \|\operatorname{ad}(T)\|\|x\|\|au\|\|v\| \leq \|\operatorname{ad}(T)\|\|u\|\|v\|, \end{aligned}$$

so that $\|\tau\| \leq \|\operatorname{ad}(T)\|$. By (3.26),

$$\langle y, T(x)a \rangle = \langle a, \phi \rangle = \int_{\mathcal{A}\times\mathcal{A}} \langle auv, \phi \rangle \, d\mathrm{M}(u, v) = \int_{\mathcal{A}\times\mathcal{A}} \langle y, T(x)auv \rangle \, d\mathrm{M}(u, v),$$

and, using the definition of Φ, we obtain

$$\langle y, T(x)a - \Phi(x)a \rangle = \langle y, T(x)a - \Phi(xa) \rangle = \int_{\mathcal{A}\times\mathcal{A}} \tau(u, v) \, d\mathrm{M}(u, v).$$

From (3.27) we see that $|\langle y, T(x)a - \Phi(x)a \rangle| \leq \|\operatorname{ad}(T)\|$. Thus

$$\|T(x)a - \Phi(x)a\| \leq \|\operatorname{ad}(T)\|.$$

Assume that $\mathcal{Y}$ is essential, and take an approximate identity $(e_j)_{j\in J}$ for $\mathcal{A}$ of bound 1. For each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ with $\|x\| = \|y\| = 1$, we have

$$|\langle e_j y, T(x) - \Phi(x) \rangle| = |\langle y, T(x)e_j - \Phi(x)e_j \rangle| \leq \|\operatorname{ad}(T)\| \quad (j \in J).$$

Since $\mathcal{Y}$ is essential, it follows that $(e_j)_{j\in J}$ is a right approximate identity for $\mathcal{Y}$ and hence, taking limit, we see that $|\langle y, T(x) - \Phi(x) \rangle| \leq \|\operatorname{ad}(T)\|$. Therefore $\|T(x) - \Phi(x)\| \leq \|\operatorname{ad}(T)\|$, and the proof is complete. □

Corollary 3.4.3. *Let $\mathcal{M}$ be an injective von Neumann algebra, and let $\mathcal{X}$ and $\mathcal{Y}$ be unital Banach right and left $\mathcal{M}$-modules, respectively, with both $\mathcal{X}^*$ and $\mathcal{Y}^*$ normal. Then*

$$\operatorname{dist}\big(T, \operatorname{Hom}_{\mathcal{M}}(\mathcal{X}, \mathcal{Y}^*)\big) \leq 8 \sup\big\{\|e^{\perp} T e\| : e \in \mathcal{M} \textit{ projection}\big\}$$

for each $T \in B(\mathcal{X}, \mathcal{Y}^*)$. In particular, the space $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}^*)$ *is hyperreflexive.*

Proof. Take $T \in B(\mathcal{X}, \mathcal{Y}^*)$. Then Theorem 3.4.1 gives $\Phi \in \operatorname{Hom}_{\mathcal{M}}(\mathcal{X}, \mathcal{Y}^*)$ such that $\|\Phi\| \leq \|T\|$ and $\|T - \Phi\| \leq \|\operatorname{ad}(T)\|$. Theorem 3.3.3(ii) now shows that

$$\|T - \Phi\| \leq 8 \sup\big\{\|e^{\perp} T e\| : e \in \mathcal{M} \text{ projection}\big\},$$

which establishes our estimate of the distance to $\operatorname{Hom}_{\mathcal{M}}(\mathcal{X}, \mathcal{Y}^*)$.

The hyperreflexivity follows from the estimates in Proposition 3.3.1. □

Corollary 3.4.4. *Let $\mathcal{A}$ be a nuclear C^*-algebra, and let $\mathcal{X}$ and $\mathcal{Y}$ be essential Banach right and left $\mathcal{A}$-modules, respectively. Then*

$$\operatorname{dist}\bigl(T, \operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}^*)\bigr) \le 8 \sup\bigl\{\|aTb\| : a, b \in \mathcal{A}_+ \text{ contractions, } ab = 0\bigr\}$$

for each $T \in B(\mathcal{X}, \mathcal{Y}^*)$. In particular, the space $\operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}^*)$ is hyperreflexive.

Proof. The estimate follows from Theorem 3.3.2(ii) and Theorem 3.4.2, as in Corollary 3.4.3. The hyperreflexivity follows from the estimates in Proposition 3.3.1. □

Corollary 3.4.5. *Let $\mathcal{A}$ be a unital nuclear C^*-algebra of real rank zero, and let $\mathcal{X}$ and $\mathcal{Y}$ be unital Banach right and left $\mathcal{A}$-modules, respectively. Then*

$$\operatorname{dist}\bigl(T, \operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}^*)\bigr) \le 8 \sup\bigl\{\|e^{\perp}Te\| : e \in \mathcal{A} \text{ projection}\bigr\}$$

for each $T \in B(\mathcal{X}, \mathcal{Y}^*)$.

Proof. The estimate follows from Theorem 3.3.3(ii) and Theorem 3.4.2, as in Corollary 3.4.3. □

Corollary 3.4.6. *Let $\mathcal{A}$ be a nuclear C^*-algebra of real rank zero, and let $\mathcal{X}$ and $\mathcal{Y}$ be essential Banach right and left $\mathcal{A}$-modules, respectively. Then*

$$\operatorname{dist}\bigl(T, \operatorname{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}^*)\bigr) \le 8 \sup\bigl\{\|eTf\| : e, f \in \mathcal{A} \text{ projections, } ef = 0\bigr\}$$

for each $T \in B(\mathcal{X}, \mathcal{Y}^*)$.

Proof. The estimate follows from Theorem 3.3.4(ii) and Theorem 3.4.2, as in Corollary 3.4.3. □

3.4.2 Homomorphisms between non-commutative L^p-spaces

Let $\mathcal{M}$ be a von Neumann algebra. For each $1 \le p \le \infty$, define $1 \le p^* \le \infty$ by the requirement that $\frac{1}{p} + \frac{1}{p^*} = 1$. There exists a natural isomorphism $\omega \mapsto x_\omega$ from $\mathcal{M}_*$ onto $L^1(\mathcal{M})$ (this isomorphism preserves the adjoint operation, positivity, and polar decomposition), and hence the space $L^1(\mathcal{M})$ is equipped with a distinguished contractive positive linear functional Tr defined by $\operatorname{Tr}(x_\omega) = \omega(1_{\mathcal{M}})$ $(\omega \in \mathcal{M}_*)$. This functional implements, for each $1 \le p \le \infty$, the duality $\langle\cdot,\cdot\rangle\colon L^p(\mathcal{M}) \times L^{p^*}(\mathcal{M}) \to \mathbb{C}$ defined by

$$\langle x, y\rangle = \operatorname{Tr}(xy) = \operatorname{Tr}(yx) \quad (x \in L^p(\mathcal{M}),\ y \in L^{p^*}(\mathcal{M})).$$

In the case where $p \ne \infty$, the above duality gives an isometric isomorphism from $L^{p^*}(\mathcal{M})$ onto $L^p(\mathcal{M})^*$. Moreover the duality satisfies the following properties:

$$\langle ax, y\rangle = \langle x, ya\rangle, \quad \langle xa, y\rangle = \langle x, ay\rangle, \tag{3.28}$$

$$\langle ax, y\rangle = \langle xy, a\rangle, \quad \langle xa, y\rangle = \langle yx, a\rangle \tag{3.29}$$

for all $x \in L^p(\mathcal{M})$, $y \in L^{p^*}(\mathcal{M})$, and $a \in \mathcal{M}$. Condition (3.28) shows that, for $p \ne \infty$, the identification of $L^{p^*}(\mathcal{M})$ with $L^p(\mathcal{M})^*$ is an isomorphism of $\mathcal{M}$-bimodules, and, further, condition (3.29) shows that $L^p(\mathcal{M})^*$ is a normal $\mathcal{M}$-bimodule.

Theorem 3.4.7. *Let $\mathcal{M}$ be a von Neumann algebra, and let $1 \le p, q \le \infty$. Then*

$$\operatorname{dist}\big(T, \operatorname{Hom}_{\mathcal{M}}(L^\infty(\mathcal{M}), L^q(\mathcal{M}))\big) \le 8 \sup\big\{\|e^\perp T e\| : e \in \mathcal{M} \text{ projection}\big\}$$

for each $T \in B(L^\infty(\mathcal{M}), L^q(\mathcal{M}))$, and

$$\operatorname{dist}\big(T, \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^1(\mathcal{M}))\big) \le 8 \sup\big\{\|e^\perp T e\| : e \in \mathcal{M} \text{ projection}\big\}$$

for each $T \in B(L^p(\mathcal{M}), L^1(\mathcal{M}))$. In particular, the spaces $\operatorname{Hom}_{\mathcal{M}}(L^\infty(\mathcal{M}), L^q(\mathcal{M}))$ *and* $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^1(\mathcal{M}))$ *are hyperreflexive.*

Proof. Suppose that $T \in B(L^\infty(\mathcal{M}), L^q(\mathcal{M}))$. Define $\xi = T(1_{\mathcal{M}}) \in L^q(\mathcal{M})$. Then, for each $x \in \mathcal{M}$, we have

$$\|(T - L_\xi)(x)\|_q = \|T(1_{\mathcal{M}} x) - T(1_{\mathcal{M}})x\|_q \le \|\operatorname{ad}(T)\| \|1_{\mathcal{M}}\| \|x\|,$$

so that $\|T - L_\xi\| \le \|\operatorname{ad}(T)\|$. Corollary 3.3.10 now gives

$$\|T - L_\xi\| \le 8 \sup\big\{\|e^\perp T e\| : e \in \mathcal{M} \text{ projection}\big\},$$

which establishes the required inequality.

Now suppose that $T \in B(L^p(\mathcal{M}), L^1(\mathcal{M}))$. In order to get the desired inequality, we are reduced to consider the case $p \neq \infty$. Consider the continuous linear functional ϕ on $L^p(\mathcal{M})$ defined by

$$\langle x, \phi \rangle = \operatorname{Tr}(T(x)) \quad (x \in L^p(\mathcal{M})).$$

Then there exists $\xi \in L^{p^*}(\mathcal{M})$ such that $\|\xi\| = \|\phi\| \le \|T\|$ and

$$\operatorname{Tr}(\xi x) = \langle x, \phi \rangle = \operatorname{Tr}\big(T(x)\big) \quad \big(x \in L^p(\mathcal{M})\big).$$

For each $x \in L^p(\mathcal{M})$ and $a \in \mathcal{M}$, we see that

$$\begin{aligned} \big|\operatorname{Tr}\big((T - L_\xi)(x)a\big)\big| &= \big|\operatorname{Tr}\big(T(x)a - \xi x a)\big)\big| = \big|\operatorname{Tr}\big(T(x)a - T(xa)\big)\big| \\ &\le \|T(x)a - T(xa)\|_1 \le \|\operatorname{ad}(T)\| \|x\|_p \|a\|. \end{aligned}$$

This implies that $\|(T - L_\xi)(x)\|_1 \le \|\operatorname{ad}(T)\| \|x\|_p$, whence $\|T - L_\xi\| \le \|\operatorname{ad}(T)\|$, and Corollary 3.3.10 shows that

$$\|T - L_\xi\| \le 8 \sup\big\{\|e^\perp T e\| : e \in \mathcal{M} \text{ projection}\big\}.$$

The hyperreflexivity follows from the estimates in Proposition 3.3.1. □

Theorem 3.4.8. *Let $\mathcal{M}$ be an injective von Neumann algebra, and let $1 \le p, q \le \infty$. Then*

$$\operatorname{dist}\big(T, \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M})\big) \le 8 \sup\big\{\|e^\perp T e\| : e \in \mathcal{M} \text{ projection}\big\}$$

for each $T \in B(L^p(\mathcal{M}), L^q(\mathcal{M}))$. In particular, the space $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ *is hyperreflexive and the hyperreflexivity constant is at most 8.*

Proof. By Theorem 3.4.7, we need only to consider the case where $p \neq \infty$ and $q \neq 1$, and then the result follows from Corollary 3.4.3, because both modules $L^p(\mathcal{M})^*$ and $L^q(\mathcal{M})\big(= L^{q^*}(\mathcal{M})^*\big)$ are normal. □

At the expense of replacing the condition $1 \le p, q \le \infty$ by $1 \le q < p \le \infty$ and losing the bound 8 on the distance estimate, we may remove the injectivity of the von Neumann algebra $\mathcal{M}$ in Theorem 3.4.8. To this end, we will be involved with the ultraproduct of non-commutative L^p-spaces. We summarize some of its main properties.

Let $(\mathcal{X}_n)$ be a sequence of Banach spaces and let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. Let $\prod \mathcal{X}_n$ be the ℓ^∞-sum of the sequence $(\mathcal{X}_n)$ and take

$$N_{\mathcal{U}} = \big\{(x_n) \in \textstyle\prod \mathcal{X}_n : \lim_{\mathcal{U}} \|x_n\| = 0\big\}.$$

Then the ultraproduct $\prod_{\mathcal{U}} \mathcal{X}_n$ of the sequence $(\mathcal{X}_n)$ along $\mathcal{U}$ is the quotient Banach space $\prod \mathcal{X}_n / N_{\mathcal{U}}$. Given $(x_n) \in \prod \mathcal{X}_n$, we write $(x_n)_{\mathcal{U}}$ for its corresponding equivalence class in $\prod_{\mathcal{U}} \mathcal{X}_n$. The norm on $\prod_{\mathcal{U}} \mathcal{X}_n$ is given by

$$\big\|(x_n)_{\mathcal{U}}\big\| = \lim_{\mathcal{U}} \|x_n\|$$

for each $(x_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} \mathcal{X}_n$. Let $(\mathcal{Y}_n)$ be another sequence of Banach spaces and let $(T_n) \in \prod B(\mathcal{X}_n, \mathcal{Y}_n)$. Then we define $\prod_{\mathcal{U}} T_n \colon \prod_{\mathcal{U}} \mathcal{X}_n \to \prod_{\mathcal{U}} \mathcal{Y}_n$ by

$$\textstyle\prod_{\mathcal{U}} T_n\big((x_n)_{\mathcal{U}}\big) = \big(T_n(x_n)\big)_{\mathcal{U}}$$

for each $(x_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} \mathcal{X}_n$. Of course, it can be checked that the definition we make is independent of the choice of the representative of the equivalence class. Moreover, $\prod_{\mathcal{U}} T_n$ is continuous and

$$\big\|\textstyle\prod_{\mathcal{U}} T_n\big\| = \lim_{\mathcal{U}} \|T_n\|. \tag{3.30}$$

All the above statements are also valid for quasi-Banach spaces. We refer the reader to [49] for the basics of ultraproducts.

If $(\mathcal{A}_n)$ is a sequence of C^*-algebras, then $\prod_{\mathcal{U}} \mathcal{A}_n$ is again a C^*-algebra. The ultraproduct of a sequence $(\mathcal{M}_n)$ of von Neumann algebras is not as straightforward as the C^*-algebra case. According to [45, 71], it is known that $\prod_{\mathcal{U}} L^1(\mathcal{M}_n)$ is isometrically isomorphic to the predual of a von Neumann algebra $\mathcal{M}_{\mathcal{U}}$. Further, it is shown in [71] that $\mathcal{M}_{\mathcal{U}}$ has such a nice behaviour as $\prod_{\mathcal{U}} L^p(\mathcal{M}_n)$ is isometrically isomorphic to $L^p(\mathcal{M}_{\mathcal{U}})$ for each $p < \infty$. Specifically,

- there exists an isometric $*$-homomorphism

  $$\iota \colon \textstyle\prod_{\mathcal{U}} \mathcal{M}_n \to \mathcal{M}_{\mathcal{U}}$$

  from the C^*-algebra $\prod_{\mathcal{U}} \mathcal{M}_n$ into the von Neumann algebra $\mathcal{M}_{\mathcal{U}}$ such that $\iota\big(\prod_{\mathcal{U}} \mathcal{M}_n\big)$ is weak* dense in $\mathcal{M}_{\mathcal{U}}$, and,

- for each $p < \infty$, there exists an isometric isomorphism
$$\Lambda_p\colon \textstyle\prod_{\mathcal{U}} L^p(\mathcal{M}_n) \to L^p(\mathcal{M}_{\mathcal{U}})$$
such that
$$\Lambda_p\big((a_n)_{\mathcal{U}}(x_n)_{\mathcal{U}}(b_n)_{\mathcal{U}}\big) = \iota\big((a_n)_{\mathcal{U}}\big)\Lambda_p\big((x_n)_{\mathcal{U}}\big)\iota\big((b_n)_{\mathcal{U}}\big)$$
and, for $0 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$,
$$\Lambda_r\big((x_n)_{\mathcal{U}}(y_n)_{\mathcal{U}}\big) = \Lambda_p\big((x_n)_{\mathcal{U}}\big)\Lambda_q\big((y_n)_{\mathcal{U}}\big)$$
for all $(a_n)_{\mathcal{U}}, (b_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} \mathcal{M}_n$, $(x_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} L^p(\mathcal{M}_n)$, and $(y_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} L^q(\mathcal{M}_n)$.

Actually, [71] is concerned with the ultrapower of $L^p(\mathcal{M})$ for a given von Neumann algebra, but it is also emphasized there that the results are equally valid for the above situation.

Theorem 3.4.9. *Let $1 \leq q < p \leq \infty$. Then there exists a constant $C_{p,q} \in \mathbb{R}^+$ with the property that, for each von Neumann algebra $\mathcal{M}$ and each continuous linear map $T\colon L^p(\mathcal{M}) \to L^q(\mathcal{M})$, we have*
$$\operatorname{dist}\big(T, \operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))\big) \leq C_{p,q} \sup\big\{\|e^{\perp}Te\| : e \in \mathcal{M} \textit{ projection}\big\}.$$
In particular, the space $\operatorname{Hom}_{\mathcal{M}}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ is hyperreflexive.

Proof. In the case where either $p = \infty$ or $q = 1$, we apply Theorem 3.4.7 to obtain the result.

Suppose that $1 < q < p < \infty$, and take $1 < r < \infty$ such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Our objective is to prove that there exists a constant $c_{p,q} \in \mathbb{R}^+$ with the property that for each von Neumann algebra $\mathcal{M}$ and each $T \in B(L^p(\mathcal{M}), L^q(\mathcal{M}))$, we have
$$\operatorname{dist}\big(T, \operatorname{Hom}_{\mathcal{M}}\big(L^p(\mathcal{M}), L^q(\mathcal{M})\big)\big) \leq c_{p,q}\, \|\operatorname{ad}(T)\|. \tag{3.31}$$
Assume towards a contradiction that the clause is false, and there is no such constant $c_{p,q}$. Then, for each $n \in \mathbb{N}$, there exists a von Neumann algebra $\mathcal{M}_n$ and a continuous linear map $R_n\colon L^p(\mathcal{M}_n) \to L^q(\mathcal{M}_n)$ such that
$$\delta_n := \operatorname{dist}\big(R_n, \operatorname{Hom}_{\mathcal{M}_n}\big(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n)\big)\big) > n\|\operatorname{ad}(R_n)\|.$$
For each $n \in \mathbb{N}$, set $S_n = \delta_n^{-1} R_n$. Then
$$\|\operatorname{ad}(S_n)\| < 1/n \quad (n \in \mathbb{N}) \tag{3.32}$$
and
$$\operatorname{dist}\big(S_n, \operatorname{Hom}_{\mathcal{M}_n}\big(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n)\big)\big) = 1 \quad (n \in \mathbb{N}). \tag{3.33}$$
Since the sequence (S_n) need not to be bounded, we replace it with a bounded one that still satisfies both (3.32) and (3.33). For this purpose, for each $n \in \mathbb{N}$, we take

$\Psi_n \in \operatorname{Hom}_{\mathcal{M}_n}\big(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n)\big)$ such that $\|S_n - \Psi_n\| < 1 + 1/n$ and consider the map $T_n = S_n - \Psi_n$. Then (T_n) is bounded. It is clear that $\|\operatorname{ad}(T_n)\| = \|\operatorname{ad}(S_n)\|$ and that

$$\operatorname{dist}\big(T_n, \operatorname{Hom}_{\mathcal{M}_n}\big(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n)\big)\big) = \operatorname{dist}\big(S_n, \operatorname{Hom}_{\mathcal{M}_n}\big(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n)\big)\big)$$

for each $n \in \mathbb{N}$, so that (3.32) and (3.33) give

$$\|\operatorname{ad}(T_n)\| < 1/n \quad (n \in \mathbb{N}), \tag{3.34}$$

$$\operatorname{dist}\big(T_n, \operatorname{Hom}_{\mathcal{M}_n}\big(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n)\big)\big) = 1 \quad (n \in \mathbb{N}). \tag{3.35}$$

Take a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Consider the ultraproduct von Neumann algebra

$$\mathcal{M}_{\mathcal{U}} = \big(\textstyle\prod_{\mathcal{U}} L^1(\mathcal{M}_n)\big)^*$$

and the maps

$$\begin{aligned} \iota\colon \textstyle\prod_{\mathcal{U}}\mathcal{M}_n &\to \mathcal{M}_{\mathcal{U}}, \\ \Lambda_p\colon \textstyle\prod_{\mathcal{U}} L^p(\mathcal{M}_n) &\to L^p(\mathcal{M}_{\mathcal{U}}), \\ \Lambda_q\colon \textstyle\prod_{\mathcal{U}} L^q(\mathcal{M}_n) &\to L^q(\mathcal{M}_{\mathcal{U}}), \\ \Lambda_r\colon \textstyle\prod_{\mathcal{U}} L^r(\mathcal{M}_n) &\to L^r(\mathcal{M}_{\mathcal{U}}) \end{aligned}$$

introduced in the preliminary remark. Further, take the ultraproduct map

$$\textstyle\prod_{\mathcal{U}} T_n\colon \prod_{\mathcal{U}} L^p(\mathcal{M}_n) \to \prod_{\mathcal{U}} L^q(\mathcal{M}_n).$$

We claim that $\prod_{\mathcal{U}} T_n$ is a right $\prod_{\mathcal{U}}\mathcal{M}_n$-module homomorphism. Take elements $(x_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} L^p(\mathcal{M}_n)$ and $(a_n)_{\mathcal{U}} \in \prod_{\mathcal{U}}\mathcal{M}_n$. Then (3.34) gives

$$\begin{aligned} \Big\|\textstyle\prod_{\mathcal{U}} T_n\big((x_n)_{\mathcal{U}}(a_n)_{\mathcal{U}}\big) - \prod_{\mathcal{U}} T_n\big((x_n)_{\mathcal{U}}\big)(a_n)_{\mathcal{U}}\Big\| &= \lim_{\mathcal{U}} \|T_n(x_n a_n) - T_n(x_n)a_n\| \\ &\le \lim_{\mathcal{U}} \big(\|\operatorname{ad}(T_n)\|\|x_n\|\|a_n\|\big) \\ &\le \lim_{\mathcal{U}} \big(\tfrac{1}{n}\|x_n\|\|a_n\|\big) = 0. \end{aligned}$$

Define $\mathbf{T}\colon L^p(\mathcal{M}_{\mathcal{U}}) \to L^q(\mathcal{M}_{\mathcal{U}})$ by

$$\mathbf{T} = \Lambda_q \circ \textstyle\prod_{\mathcal{U}} T_n \circ {\Lambda_p}^{-1}.$$

Then $\mathbf{T}$ is a right $\iota\big(\prod_{\mathcal{U}}\mathcal{M}_n\big)$-module homomorphism. We now note that:

- $\iota\big(\prod_{\mathcal{U}}\mathcal{M}_n\big)$ is weak* dense in $\mathcal{M}_{\mathcal{U}}$;
- the module maps $\mathbf{a} \mapsto \mathbf{xa}$ and $\mathbf{a} \mapsto \mathbf{ya}$ are weak*-weak* continuous for all $\mathbf{x} \in L^p(\mathcal{M}_{\mathcal{U}})$ and $\mathbf{y} \in L^q(\mathcal{M}_{\mathcal{U}})$;
- the map $\mathbf{T}$ is weak*-weak* continuous, since both $L^p(\mathcal{M}_{\mathcal{U}})$ and $L^q(\mathcal{M}_{\mathcal{U}})$ are reflexive (being $1 < p, q < \infty$).

The above conditions imply that $\mathbf{T}$ is a right $\mathcal{M}_{\mathcal{U}}$-module homomorphism. By Theorem 3.3.9, there exists $\Xi \in L^r(\mathcal{M}_{\mathcal{U}})$ such that

$$\mathbf{T}(\mathbf{x}) = \Xi\mathbf{x} \quad (\mathbf{x} \in L^p(\mathcal{M}_{\mathcal{U}})).$$

Set $(\xi_n)_{\mathcal{U}} = {\Lambda_r}^{-1}(\Xi) \in \prod_{\mathcal{U}} L^r(\mathcal{M}_n)$, and, for each $n \in \mathbb{N}$, take the left composition map $L_{\xi_n} \colon L^p(\mathcal{M}_n) \to L^q(\mathcal{M}_n)$. Then, for each $\mathbf{x} \in L^p(\mathcal{M}_{\mathcal{U}})$, we have

$$\begin{aligned}\mathbf{T}(\mathbf{x}) &= \Xi\mathbf{x} = \Lambda_r\big((\xi_n)_{\mathcal{U}}\big)\Lambda_p\big({\Lambda_p}^{-1}(\mathbf{x})\big) = \Lambda_q\big((\xi_n)_{\mathcal{U}}{\Lambda_p}^{-1}(\mathbf{x})\big)\\ &= \big(\Lambda_q \circ \textstyle\prod_{\mathcal{U}} L_{\xi_n} \circ {\Lambda_p}^{-1}\big)(\mathbf{x}),\end{aligned}$$

whence $\prod_{\mathcal{U}} T_n = \prod_{\mathcal{U}} L_{\xi_n}$, so that (3.30) gives $\lim_{\mathcal{U}} \|T_n - L_{\xi_n}\| = 0$ and hence

$$\lim_{\mathcal{U}} \operatorname{dist}\big(T_n, \operatorname{Hom}_{\mathcal{M}_n}\big(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n)\big)\big) \leq \lim_{\mathcal{U}} \|T_n - L_{\xi_n}\| = 0,$$

contrary to (3.35).

Finally, (3.31) and Corollary 3.3.10 give the desired inequality with $C_{p,q} = 8c_{p,q}$.

The hyperreflexivity follows from the estimates in Proposition 3.3.1. □

Chapter 4

Strongly zero product determined Banach algebras

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Abstract. C^*-algebras, group algebras, and the algebra $\mathcal{A}(X)$ of approximable operators on a Banach space X having the bounded approximation property are known to be zero product determined. In this paper we give a quantitative estimate of this property by showing that, for the Banach algebra A, there exists a constant α with the property that for every continuous bilinear functional $\varphi\colon A\times A\to\mathbb{C}$ there exists a continuous linear functional ξ on A such that

$$\sup_{\|a\|=\|b\|=1}|\varphi(a,b)-\xi(ab)|\le\alpha\sup_{\substack{\|a\|=\|b\|=1,\\ ab=0}}|\varphi(a,b)|$$

in each of the following cases: (i) A is a C^*-algebra, in which case $\alpha=8$; (ii) $A=L^1(G)$ for a locally compact group G, in which case $\alpha=60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}$; (iii) $A=\mathcal{A}(X)$ for a Banach space X having property $(\mathbb{A})$ (which is a rather strong approximation property for X), in which case $\alpha=60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}C^2$, where C is a constant associated with the property $(\mathbb{A})$ that we require for X.

4.1 Introduction

Let A be a Banach algebra. Then $\pi\colon A\times A\to A$ denotes the product map, we write A^* for the dual of A, and $\mathcal{B}^2(A,\mathbb{C})$ for the space of continuous bilinear functionals on A.

The Banach algebra A is said to be *zero product determined* if every $\varphi \in \mathcal{B}^2(A,\mathbb{C})$ with the property

$$a, b \in A,\ ab = 0 \ \Rightarrow\ \varphi(a,b) = 0 \tag{4.1}$$

belongs to the space

$$\mathcal{B}^2_\pi(A,\mathbb{C}) = \{\xi \circ \pi : \xi \in A^*\}.$$

This concept implicitly appeared in [2] as an additional outcome of the so-called property $\mathbb{B}$ which was introduced in that paper, and was the basis of subsequent Jordan and Lie versions (see [4, 5, 6]). For a comprehensive survey of the theory of the zero product determined Banach algebras we refer the reader to [24]. The algebra A is said to have *property* $\mathbb{B}$ if every $\varphi \in \mathcal{B}^2(A,\mathbb{C})$ satisfying (4.1) belongs to the closed subspace $\mathcal{B}^2_b(A,\mathbb{C})$ of $\mathcal{B}^2(A,\mathbb{C})$ defined by

$$\mathcal{B}^2_b(A,\mathbb{C}) = \{\psi \in \mathcal{B}^2(A,\mathbb{C}) : \psi(ab,c) = \psi(a,bc)\ \forall a,b,c \in A\}.$$

In [2] it was shown that this class of Banach algebras is wide enough to include a number of examples of interest: C^*-algebras, the group algebra $L^1(G)$ of any locally compact group G, and the algebra $\mathcal{A}(X)$ of approximable operators on any Banach space X.

Throughout, we confine ourselves to Banach algebras having a bounded left approximate identity. Then $\mathcal{B}^2_\pi(A,\mathbb{C}) = \mathcal{B}^2_b(A,\mathbb{C})$ (Proposition 4.2.1), and hence A is a zero product determined Banach algebra if and only if A has property $\mathbb{B}$. For example, this applies to C^*-algebras, group algebras and the algebra $\mathcal{A}(X)$ on any Banach space X having the bounded approximation property, so that all of them are zero product determined Banach algebras.

For each $\varphi \in \mathcal{B}^2(A,\mathbb{C})$, the distance from φ to $\mathcal{B}^2_\pi(A,\mathbb{C})$ is

$$\operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right) = \inf\left\{\|\varphi - \psi\| : \psi \in \mathcal{B}^2_\pi(A,\mathbb{C})\right\},$$

which can be easily estimated through the constant

$$|\varphi|_b = \sup\{|\varphi(ab,c) - \varphi(a,bc)| : a,b,c \in A,\ \|a\| = \|b\| = \|c\| = 1\}$$

(Proposition 4.2.1 below). Our purpose is to estimate $\operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right)$ through the constant

$$|\varphi|_{zp} = \sup\{|\varphi(a,b)| : a,b \in A,\ \|a\| = \|b\| = 1,\ ab = 0\}.$$

Note that A is zero product determined precisely when

$$\varphi \in \mathcal{B}^2(A,\mathbb{C}),\ |\varphi|_{zp} = 0 \ \Rightarrow\ \varphi \in \mathcal{B}^2_\pi(A,\mathbb{C}). \tag{4.2}$$

We call the Banach algebra A *strongly zero product determined* if condition (4.2) is strengthened by requiring that there is a distance estimate

$$\operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right) \le \alpha\, |\varphi|_{zp} \quad \forall \varphi \in \mathcal{B}^2(A,\mathbb{C}) \tag{4.3}$$

for some constant α; in this case, the optimal constant α for which (4.3) holds will be denoted by α_A. The inequality $|\varphi|_{zp} \leq \operatorname{dist}\big(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\big)$ is always true (Proposition 4.2.1 below). We also note that A has property $\mathbb{B}$ exactly in the case when

$$\varphi \in \mathcal{B}^2(A,\mathbb{C}),\ |\varphi|_{zp} = 0 \ \Rightarrow\ |\varphi|_b = 0,$$

and the algebra A is said to have the *strong property* $\mathbb{B}$ if there is an estimate

$$|\varphi|_b \leq \beta\, |\varphi|_{zp} \quad \forall \varphi \in \mathcal{B}^2(A,\mathbb{C}) \tag{4.4}$$

for some constant β; in this case, the optimal constant β for which (4.4) holds will be denoted by β_A. The inequality $|\varphi|_{zp} \leq M\, |\varphi|_b$ is always true for some constant M (Proposition 4.2.1 below). The spirit of this concept first appeared in [11], and was subsequently formulated in [76] and refined in [77]. This property has proven to be useful to study the hyperreflexivity of the spaces of continuous derivations and, more generally, continuous cocycles on A (see [12, 13, 75, 76, 77]).

From [9, Corollary 1.3], we obtain the following result.

Theorem 4.1.1. *Let A be a C^*-algebra. Then A is strongly zero product determined, has the strong property $\mathbb{B}$, and $\alpha_A, \beta_A \leq 8$.*

It is shown in [77] that each group algebra has the strong property $\mathbb{B}$ and so (by Corollary 4.2.2 below) it is also strongly zero product determined. In Theorem 4.3.3 we prove that, for each group G,

$$\alpha_{L^1(G)} \leq \beta_{L^1(G)} \leq 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}.$$

This gives a sharper estimate for the constant of the strong property $\mathbb{B}$ of $L^1(G)$ to the one given in [77, Theorem 3.4]. The estimates given in Theorems 4.1.1 and 4.3.3 can be used to sharp the upper bound given in [77, Theorem 4.4] for the hyperreflexivity constant of $\mathcal{Z}^n(A,X)$, the space of continuous n-cocycles from A into X, where A is a C^*-algebra or the group algebra of a group with an open subgroup of polynomial growth and X is a Banach A-bimodule for which the n^{th} Hochschild cohomology group $\mathcal{H}^{n+1}(A,X)$ is a Banach space.

Finally, in Theorem 4.4.1 we prove that the algebra $\mathcal{A}(X)$ is strongly zero product determined for each Banach space X having property $(\mathbb{A})$ (which is a rather strong approximation property for the space X). Further, we will use this result to show that the space $\mathcal{Z}^n(\mathcal{A}(X),Y^*)$ is hyperreflexive for each Banach $\mathcal{A}(X)$-bimodule Y.

There is no reason for an arbitrary zero product Banach algebra to be strongly zero product determined. However, as yet, we do not know an example of a zero product determined Banach algebra which is not strongly zero product determined.

Throughout, our reference for Banach algebras, and particularly for group algebras, is the monograph [36].

4.2 Elementary estimates

In the following result we gather together some estimates that relate the seminorms $\operatorname{dist}\left(\cdot, \mathcal{B}^2_\pi(A,\mathbb{C})\right)$, $|\cdot|_b$, and $|\cdot|_{zp}$ on $\mathcal{B}^2_\pi(A,\mathbb{C})$ to each other.

Proposition 4.2.1. *Let A be a Banach algebra with a left approximate identity of bound M. Then $\mathcal{B}^2_\pi(A,\mathbb{C}) = \mathcal{B}^2_b(A,\mathbb{C})$ and, for each $\varphi \in \mathcal{B}^2(A,\mathbb{C})$, the following properties hold:*

(i) *The distance* $\operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right)$ *is attained;*

(ii) $\frac{1}{2}|\varphi|_b \le \operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right) \le M|\varphi|_b$*;*

(iii) $|\varphi|_{zp} \le \operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right)$.

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a left approximate identity of bound M.

(i) Let (ξ_n) be a sequence in A^* such that

$$\operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right) = \lim_{n\to\infty} \|\varphi - \xi_n \circ \pi\|.$$

For each $n \in \mathbb{N}$ and $a \in A$, we have

$$|\xi_n(e_\lambda a)| = |(\xi_n \circ \pi)(e_\lambda, a)| \le M \|\xi_n \circ \pi\| \|a\| \quad \forall \lambda \in \Lambda$$

and hence, taking limit in the above inequality and using that $\lim_{\lambda\in\Lambda} e_\lambda a = a$, we see that $|\xi_n(a)| \le M\|\xi_n \circ \pi\| \|a\|$, which shows that $\|\xi_n\| \le M\|\xi_n \circ \pi\|$. Further, since

$$\|\xi_n \circ \pi\| \le \|\varphi - \xi_n \circ \pi\| + \|\varphi\| \quad \forall n \in \mathbb{N},$$

it follows that the sequence $(\|\xi_n\|)$ is bounded. By the Banach–Alaoglu theorem, the sequence (ξ_n) has a weak*-accumulation point, say ξ, in A^*. Let $(\xi_\nu)_{\nu\in N}$ be a subnet of (ξ_n) such that $\text{w}^*\text{-}\lim_{\nu\in N} \xi_\nu = \xi$. The task is now to show that

$$\|\varphi - \xi \circ \pi\| = \operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right).$$

For each $a, b \in A$ with $\|a\| = \|b\| = 1$, we have

$$|\varphi(a,b) - \xi_\nu(ab)| \le \|\varphi - \xi_\nu \circ \pi\| \quad \forall \nu \in N,$$

and so, taking limits on both sides of the above inequality and using that

$$\lim_{\nu\in N} \xi_\nu(ab) = \xi(ab)$$

and that $(\|\varphi - \xi_\nu \circ \pi\|)_{\nu\in N}$ is a subnet of the convergent sequence $(\|\varphi - \xi_n \circ \pi\|)$, we obtain

$$|\varphi(a,b) - \xi(ab)| \le \operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right).$$

This implies that $\|\varphi - \xi \circ \pi\| \le \operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right)$, and the converse inequality $\operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right) \le \|\varphi - \xi \circ \pi\|$ trivially holds.

(ii) For each $\lambda \in \Lambda$ define $\xi_\lambda \in A^*$ by

$$\xi_\lambda(a) = \varphi(e_\lambda, a) \quad \forall a \in A.$$

Then $\|\xi_\lambda\| \le M \|\varphi\|$ for each $\lambda \in \Lambda$, so that $(\xi_\lambda)_{\lambda\in\Lambda}$ is a bounded net in A^* and hence the Banach–Alaoglu theorem shows that it has a weak*-accumulation point, say ξ, in A^*. Let $(\xi_\nu)_{\nu\in N}$ be a subnet of $(\xi_\lambda)_{\lambda\in\Lambda}$ such that w*-$\lim_{\nu\in N}\xi_\nu = \xi$. For each $a, b \in A$ with $\|a\| = \|b\| = 1$, we have

$$|\varphi(e_\nu a, b) - \varphi(e_\nu, ab)| \le M\, |\varphi|_b \quad \forall \nu \in N$$

and hence, taking limit and using that $(e_\nu a)_{\nu\in N}$ is a subnet of the convergent net $(e_\lambda a)_{\lambda\in\Lambda}$ and that $\lim_{\nu\in N}\varphi(e_\lambda, ab) = \xi(ab)$, we see that

$$|\varphi(a, b) - \xi(ab)| \le M\, |\varphi|_b\,.$$

This gives $\|\varphi - \xi\circ\pi\| \le M\,|\varphi|_b$, whence

$$\operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right) \le M|\varphi|_b.$$

Set $\xi \in A^*$. For each $a, b, c \in A$ with $\|a\| = \|b\| = \|c\| = 1$, we have

$$\begin{aligned}
|\varphi(ab, c) - \varphi(a, bc)| &= |\varphi(ab, c) - (\xi\circ\pi)(ab, c) + (\xi\circ\pi)(a, bc) - \varphi(a, bc)| \\
&\le |\varphi(ab, c) - (\xi\circ\pi)(ab, c)| + |(\xi\circ\pi)(a, bc) - \varphi(a, bc)| \\
&\le \|\varphi - \xi\circ\pi\|\,\|ab\|\,\|c\| + \|\varphi - \xi\circ\pi\|\,\|a\|\,\|bc\| \\
&\le 2\,\|\varphi - \xi\circ\pi\|
\end{aligned}$$

and therefore $|\varphi|_b \le 2\,\|\varphi - \xi\circ\pi\|$. Since this inequality holds for each $\xi \in A^*$, it follows that

$$|\varphi|_b \le 2\operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right).$$

(iii) Let $a, b \in A$ with $\|a\| = \|b\| = 1$ and $ab = 0$. For each $\xi \in A^*$, we see that

$$|\varphi(a, b)| = |\varphi(a, b) - (\xi\circ\pi)(a, b)| \le \|\varphi - \xi\circ\pi\|\,,$$

and consequently $|\varphi|_{zp} \le \|\varphi - \xi\circ\pi\|$. Since the above inequality holds for each $\xi \in A^*$, we conclude that

$$|\varphi|_{zp} \le \operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right).$$

Finally, it is clear that $\mathcal{B}^2_\pi(A,\mathbb{C}) \subset \mathcal{B}^2_b(A,\mathbb{C})$. To prove the reverse inclusion take $\varphi \in \mathcal{B}^2_b(A,\mathbb{C})$. Then $|\varphi|_b = 0$, hence (ii) shows that $\operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\right) = 0$, and (i) gives $\psi \in \mathcal{B}^2_\pi(A,\mathbb{C})$ such that $\|\varphi - \psi\| = 0$, which implies that $\varphi = \psi \in \mathcal{B}^2_\pi(A,\mathbb{C})$. □

The following result is an immediate consequence of assertion (ii) in Proposition 4.2.1.

Corollary 4.2.2. *Let A be a Banach algebra with a left approximate identity of bound M. Then A is a strongly zero product determined Banach algebra if and only if has the strong property $\mathbb{B}$, in which case*

$$\tfrac{1}{2}\beta_A \le \alpha_A \le M\beta_A.$$

Let X and Y be Banach spaces, and let $n \in \mathbb{N}$. We write $\mathcal{B}^n(X,Y)$ for the Banach space of all continuous n-linear maps from $X \times \overset{n}{\cdots} \times X$ to Y. As usual, we abbreviate $\mathcal{B}^1(X,Y)$ to $\mathcal{B}(X,Y)$, $\mathcal{B}(X,X)$ to $\mathcal{B}(X)$, and $\mathcal{B}(X,\mathbb{C})$ to X^*. The identity operator on X is denoted by I_X. Further, we write $\langle \cdot, \cdot \rangle$ for the duality between X and X^*. For each subspace E of X, $E^\perp$ denotes the annihilator of E in X^*.

For a Banach algebra A and a Banach space X, and for each $\varphi \in \mathcal{B}^2(A,X)$, we continue to use the notations

$$|\varphi|_b = \sup\left\{|\varphi(ab,c) - \varphi(a,bc)| : a,b,c \in A,\ \|a\| = \|b\| = \|c\| = 1\right\},$$
$$|\varphi|_{zp} = \sup\left\{|\varphi(a,b)| : a,b \in A,\ \|a\| = \|b\| = 1,\ ab = 0\right\}.$$

Proposition 4.2.3. *Let A be a Banach algebra with a left approximate identity of bound M and having the strong property $\mathbb{B}$. Let X be a Banach space, and let $\varphi \in \mathcal{B}^2(A,X)$. Then the following properties hold:*

(i) $|\varphi|_b \le \beta_A |\varphi|_{zp}$;

(ii) *If X is a dual Banach space, then there exists $\Phi \in \mathcal{B}(A,X)$ such that $\|\varphi - \Phi \circ \pi\| \le M\beta_A$.*

Proof. (i) For each $\xi \in X^*$, we have

$$|\xi \circ \varphi|_b \le \beta_A |\xi \circ \varphi|_{zp}.$$

It follows from the Hahn-Banach theorem that

$$|\varphi|_b = \sup\big\{|\xi \circ \varphi|_b : \xi \in X^*,\ \|\xi\| = 1\big\},$$
$$|\varphi|_{zp} = \sup\big\{|\xi \circ \varphi|_{zp} : \xi \in X^*,\ \|\xi\| = 1\big\}.$$

In this way we obtain (i).

(ii) Suppose that X is the dual of a Banach space X_*. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a left approximate identity for A of bound M, and define a net $(\Phi_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{B}(A,X)$ by setting

$$\Phi_\lambda(a) = \varphi(e_\lambda, a) \quad \forall a \in A,\ \forall \lambda \in \Lambda.$$

Since each bounded subset of $\mathcal{B}(A,X)$ is relatively compact with respect to the weak* operator topology on $\mathcal{B}(A,X)$ and the net $(\Phi_\lambda)_{\lambda\in\Lambda}$ is bounded, it follows that there exist $\Phi \in \mathcal{B}(A,X)$ and a subnet $(\Phi_\nu)_{\nu\in N}$ of $(\Phi_\lambda)_{\lambda\in\Lambda}$ such that wo*-$\lim_{\nu\in N}\Phi_\nu = \Phi$. For each $a,b \in A$ with $\|a\| = \|b\| = 1$, and $x_* \in X_*$ with $\|x_*\| = 1$, we have

$$\big|\langle x_*, \varphi(e_\nu a, b)\rangle - \langle x_*, \varphi(e_\nu, ab)\rangle\big| \le \|\varphi(e_\nu a, b) - \varphi(e_\nu, ab)\| \le M\beta_A \quad \forall \nu \in N$$

and hence, taking limit and using that $(e_\nu a)_{\nu\in N}$ is a subnet of the net $(e_\lambda a)_{\lambda\in\Lambda}$ (which converges to a with respect to the norm topology) and that $\lim_{\nu\in N}\langle x_*, \varphi(e_\nu, ab)\rangle = \langle x_*, \Phi(ab)\rangle$ (by definition of Φ), we see that

$$\big|\langle x_*, \varphi(a,b) - \Phi(ab)\rangle\big| = M\beta_A.$$

This gives $\|\varphi - \Phi \circ \pi\| \le M\beta_A$. □

4.3 Group algebras

In this section we prove that the group algebra $L^1(G)$ of each locally compact group G is a strongly zero product determined Banach algebra and we provide an estimate of the constants $\alpha_{L^1(G)}$ and $\beta_{L^1(G)}$. Our estimate of $\beta_{L^1(G)}$ improves the one given in [77]. For the basic properties of this important class of Banach algebras we refer the reader to [36, Section 3.3].

Throughout this section, $\mathbb{T}$ denotes the circle group, and we consider the normalized Haar measure on $\mathbb{T}$. We write $A(\mathbb{T})$ and $A(\mathbb{T}^2)$ for the Fourier algebras of $\mathbb{T}$ and $\mathbb{T}^2$, respectively. For each $f \in A(\mathbb{T})$, $F \in A(\mathbb{T}^2)$, and $j, k \in \mathbb{Z}$, we write $\widehat{f}(j)$ and $\widehat{F}(j,k)$ for the Fourier coefficients of f and F, respectively. Let $\mathbf{1}, \zeta \in A(\mathbb{T})$ denote the functions defined by

$$\mathbf{1}(z) = 1, \quad \zeta(z) = z \quad \forall z \in \mathbb{T}.$$

Let $\Delta\colon A(\mathbb{T}^2) \to A(\mathbb{T})$ be the bounded linear map defined by

$$\Delta(F)(z) = F(z,z) \quad \forall z \in \mathbb{T},\ \forall F \in A(\mathbb{T}^2).$$

For $f, g \in A(\mathbb{T})$, let $f \otimes g\colon \mathbb{T}^2 \to \mathbb{C}$ denote the function defined by

$$(f \otimes g)(z,w) = f(z)g(w) \quad \forall z, w \in \mathbb{T},$$

which is an element of $A(\mathbb{T}^2)$ with $\|f \otimes g\| = \|f\|\,\|g\|$.

Lemma 4.3.1. *Let $\Phi\colon A(\mathbb{T}^2) \to \mathbb{C}$ be a continuous linear functional, and let the constant $\varepsilon \geq 0$ be such that*

$$f, g \in A(\mathbb{T}),\ fg = 0 \ \Rightarrow\ |\Phi(f \otimes g)| \leq \varepsilon \, \|f\| \, \|g\| \, .$$

Then

$$|\Phi(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta)| \leq \|\Phi\,|_{\ker \Delta}\| \, 2 \sin \tfrac{\pi}{10} + 60\sqrt{27}\left(1 + \sin \tfrac{\pi}{10}\right) \varepsilon.$$

Proof. Set

$$E = \left\{ e^{\theta i} : -\tfrac{1}{5}\pi \leq \theta \leq \tfrac{1}{5}\pi \right\},$$
$$W = \left\{ (z,w) \in \mathbb{T}^2 : zw^{-1} \in E \right\},$$

and let $F \in A(\mathbb{T}^2)$ be such that

$$F(z,w) = 0 \quad \forall (z,w) \in W. \tag{4.5}$$

Our objective is to prove that

$$|\Phi(F)| \leq 30\sqrt{27}\, \|F\| \, \varepsilon. \tag{4.6}$$

For this purpose, we take

$$\begin{aligned}
a &= e^{\frac{1}{15}\pi i},\\
A &= \left\{e^{\theta i} : 0 < \theta \le \tfrac{1}{15}\pi\right\},\\
B &= \left\{e^{\theta i} : \tfrac{2}{15}\pi < \theta \le \tfrac{29}{15}\pi\right\},\\
U &= \left\{e^{\theta i} : -\tfrac{1}{30}\pi < \theta < \tfrac{1}{30}\pi\right\},
\end{aligned}$$

and we define functions $\omega, \upsilon \in A(\mathbb{T})$ by

$$\omega = 30\,\chi_A * \chi_U, \quad \upsilon = 30\,\chi_B * \chi_U.$$

We note that

$$\begin{aligned}
\{z \in \mathbb{T} : \omega(z) \neq 0\} = AU = \left\{e^{\theta i} : -\tfrac{1}{30}\pi < \theta < \tfrac{1}{10}\pi\right\},\\
\{z \in \mathbb{T} : \upsilon(z) \neq 0\} = BU = \left\{e^{\theta i} : \tfrac{1}{10}\pi < \theta < \tfrac{59}{30}\pi\right\},
\end{aligned}$$

and, with $\|\cdot\|_2$ denoting the norm of $L^2(\mathbb{T})$,

$$\begin{aligned}
\|\omega\| \le 30\,\|\chi_A\|_2\,\|\chi_U\|_2 = 30\,\frac{1}{\sqrt{30}}\,\frac{1}{\sqrt{30}} = 1,\\
\|\upsilon\| \le 30\,\|\chi_B\|_2\,\|\chi_U\|_2 = 30\,\frac{\sqrt{27}}{\sqrt{30}}\,\frac{1}{\sqrt{30}} = \sqrt{27}.
\end{aligned}$$

Since

$$\bigcup_{k=0}^{29} a^k A = \mathbb{T}, \quad \bigcup_{k=2}^{28} a^k A = B,$$

it follows that

$$\sum_{k=0}^{29} \delta_{a^k} * \chi_A = \sum_{k=0}^{29} \chi_{a^k A} = \mathbf{1}, \quad \sum_{k=2}^{28} \delta_{a^k} * \chi_A = \sum_{k=2}^{28} \chi_{a^k A} = \chi_B,$$

and thus, for each $j \in \mathbb{Z}$, we have

$$\sum_{k=j}^{j+29} \delta_{a^k} * \omega = 30\delta_{a^j} * \sum_{k=0}^{29} \delta_{a^k} * \chi_A * \chi_U = 30\delta_{a^j} * \mathbf{1} * \chi_U = \mathbf{1}, \tag{4.7}$$

$$\sum_{k=j+2}^{j+28} \delta_{a^k} * \omega = 30\delta_{a^j} * \sum_{k=2}^{28} \delta_{a^k} * \chi_A * \chi_U = 30\delta_{a^j} * \chi_B * \chi_U = \delta_{a^j} * \upsilon. \tag{4.8}$$

If $j \in \mathbb{Z}$, $k \in \{j-1, j, j+1\}$, and $z, w \in \mathbb{T}$ are such that $(\delta_{a^j} * \omega)(z)(\delta_{a^k} * \omega)(w) \neq 0$, then

$$zw^{-1} \in a^j AU\big(a^k AU\big)^{-1} \subset a^{j-k}\left\{e^{\theta i} : -\tfrac{2}{15}\pi < \theta < \tfrac{2}{15}\pi\right\} \subset E,$$

whence $\{(z,w) \in \mathbb{T}^2 : (\delta_{a^j} * \omega) \otimes (\delta_{a^k} * \omega)(z,w) \neq 0\} \subset W$ and (4.5) gives

$$F(\delta_{a^j} * \omega) \otimes (\delta_{a^k} * \omega) = 0. \tag{4.9}$$

Since $AU \cap BU = \varnothing$, it follows that $\omega\upsilon = 0$, and therefore

$$(\delta_{a^k} * \omega)(\delta_{a^k} * \upsilon) = 0 \quad \forall k \in \mathbb{Z}. \tag{4.10}$$

From (4.7), (4.8), and (4.9) we deduce that

$$\begin{aligned}
F &= F \sum_{j=0}^{29} \sum_{k=j-1}^{j+28} (\delta_{a^j} * \omega) \otimes (\delta_{a^k} * \omega) \\
&= \sum_{j=0}^{29} \sum_{k=j-1}^{j+1} F(\delta_{a^j} * \omega) \otimes (\delta_{a^k} * \omega) + \sum_{j=0}^{29} \sum_{k=j+2}^{j+28} F(\delta_{a^j} * \omega) \otimes (\delta_{a^k} * \omega) \\
&= \sum_{j=0}^{29} \sum_{k=j+2}^{j+28} F(\delta_{a^j} * \omega) \otimes (\delta_{a^k} * \omega) = \sum_{j=0}^{29} F(\delta_{a^j} * \omega) \otimes (\delta_{a^j} * \upsilon).
\end{aligned}$$

As

$$F = \sum_{j,k=-\infty}^{\infty} \widehat{F}(j,k)\zeta^j \otimes \zeta^k$$

we have

$$F = \sum_{j,k=-\infty}^{\infty} \sum_{l=0}^{29} \widehat{F}(j,k)\big(\zeta^j(\delta_{a^l} * \omega)\big) \otimes \big(\zeta^k(\delta_{a^l} * \upsilon)\big),$$

so that

$$\Phi(F) = \sum_{j,k=-\infty}^{\infty} \sum_{l=0}^{29} \widehat{F}(j,k)\Phi\Big(\big(\zeta^j(\delta_{a^l} * \omega)\big) \otimes \big(\zeta^k(\delta_{a^l} * \upsilon)\big)\Big).$$

By (4.10), for each $j,k,l \in \mathbb{Z}$,

$$\big(\zeta^j(\delta_{a^l} * \omega)\big)\big(\zeta^k(\delta_{a^l} * \upsilon)\big) = 0$$

and therefore

$$\begin{aligned}
\left|\Phi\left(\big(\zeta^j(\delta_{a^l} * \omega)\big) \otimes \big(\zeta^k(\delta_{a^l} * \upsilon)\big)\right)\right| &\le \varepsilon \left\|\zeta^j(\delta_{a^l} * \omega)\right\| \left\|\zeta^k(\delta_{a^l} * \upsilon)\right\| \\
&= \varepsilon \|\omega\| \|\upsilon\| \le \sqrt{27}\,\varepsilon.
\end{aligned}$$

We thus get

$$\begin{aligned}
|\Phi(F)| &= \sum_{j,k=-\infty}^{\infty} \sum_{l=0}^{29} \left|\widehat{F}(j,k)\right| \left|\Phi\Big(\big(\zeta^j(\delta_{a^l} * \omega)\big) \otimes \big(\zeta^k(\delta_{a^l} * \upsilon)\big)\Big)\right| \\
&\le \sum_{j,k=-\infty}^{\infty} \sum_{l=0}^{29} \left|\widehat{F}(j,k)\right| \sqrt{27}\,\varepsilon = 30\sqrt{27}\,\|F\|\,\varepsilon,
\end{aligned}$$

and (4.6) is proved.

Let $f \in A(\mathbb{T})$ be such that $f(z) = 0$ for each $z \in E$, and define the function $F\colon \mathbb{T}^2 \to \mathbb{C}$ by

$$F(z, w) = f(zw^{-1})w = \sum_{k=-\infty}^{\infty} \widehat{f}(k) z^k w^{-k+1} \quad \forall z, w \in \mathbb{T}.$$

Then $F \in A(\mathbb{T}^2)$, $\|F\| = \|f\|$, $\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta - F \in \ker \Delta$, and

$$(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta - F)(z, w) = \Big(1 - \widehat{f}(1)\Big) z + \Big(-1 - \widehat{f}(0)\Big) w - \sum_{k \neq 0,1} \widehat{f}(k) z^k w^{-k+1},$$

which certainly implies that

$$\|\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta - F\| = \Big|1 - \widehat{f}(1)\Big| + \Big|-1 - \widehat{f}(0)\Big| + \sum_{k \neq 0,1} \Big|\widehat{f}(k)\Big| = \|\zeta - \mathbf{1} - f\|.$$

According to (4.6), we have

$$\begin{aligned}
|\Phi(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta)| &\leq |\Phi(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta - F)| + |\Phi(F)| \\
&\leq \|\Phi|_{\ker \Delta}\| \, \|\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta - F\| + 30\sqrt{27}\, \|F\|\, \varepsilon \\
&= \|\Phi|_{\ker \Delta}\| \, \|\zeta - \mathbf{1} - f\| + 30\sqrt{27}\, \|f\|\, \varepsilon \\
&\leq \|\Phi|_{\ker \Delta}\| \, \|\zeta - \mathbf{1} - f\| + 30\sqrt{27}\big(\|\zeta - \mathbf{1} - f\| + 2\big)\varepsilon
\end{aligned}$$

(as $\|f\| \leq \|\zeta - \mathbf{1} - f\| + \|\zeta - \mathbf{1}\|$). Further, this inequality holds for each function from the set $\mathcal{I}$ consisting of all functions $f \in A(\mathbb{T})$ such that $f(z) = 0$ for each $z \in E$. Consequently,

$$|\Phi(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta)| \leq \|\Phi|_{\ker \Delta}\| \operatorname{dist}(\zeta - \mathbf{1}, \mathcal{I}) + 30\sqrt{27}\,(\operatorname{dist}(\zeta - \mathbf{1}, \mathcal{I}) + 2)\,\varepsilon.$$

On the other hand, it is shown at the beginning of the proof of [18, Corollary 3.3] that

$$\operatorname{dist}(\zeta - \mathbf{1}, \mathcal{I}) \leq 2 \sin \tfrac{\pi}{10},$$

and we thus get

$$|\Phi(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta)| \leq \|\Phi|_{\ker \Delta}\| \, 2 \sin \tfrac{\pi}{10} + 30\sqrt{27}\,\big(2 \sin \tfrac{\pi}{10} + 2\big)\,\varepsilon,$$

which completes the proof. □

Lemma 4.3.2. *Let $\Phi\colon A(\mathbb{T}^2) \to \mathbb{C}$ be a continuous linear functional, and let the constant $\varepsilon \geq 0$ be such that*

$$f, g \in A(\mathbb{T}),\ fg = 0 \ \Rightarrow\ |\Phi(f \otimes g)| \leq \varepsilon\, \|f\|\, \|g\|.$$

Then

$$\big|\Phi\big(F - \mathbf{1} \otimes \Delta F\big)\big| \leq 60\sqrt{27}\, \frac{1 + \sin \frac{\pi}{10}}{1 - 2 \sin \frac{\pi}{10}}\, \varepsilon\, \|F\|$$

for each $F \in A(\mathbb{T}^2)$.

Proof. Fix $j, k \in \mathbb{Z}$. We claim that

$$\left|\Phi(\zeta^j \otimes \zeta^k - \mathbf{1} \otimes \zeta^{j+k})\right| \le \|\Phi\,|_{\ker\Delta}\|\, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon. \tag{4.11}$$

Of course, we are reduced to proving (4.11) for $j \neq 0$. We define $d_j\colon A(\mathbb{T}) \to A(\mathbb{T})$, and $D_j, L_k\colon A(\mathbb{T}^2) \to A(\mathbb{T}^2)$ by

$$d_j f(z) = f(z^j) \quad \forall f \in A(\mathbb{T}),\ \forall z \in \mathbb{T}$$

and

$$D_j F(z,w) = F(z^j, w^j), \quad L_k F(z,w) = F(z,w)w^k \quad \forall F \in A(\mathbb{T}^2),\ \forall z,w \in \mathbb{T},$$

respectively. Further, we consider the continuous linear functional $\Phi \circ L_k \circ D_j$. If $f, g \in A(\mathbb{T})$ are such that $fg = 0$, then $(d_j f)(\zeta^k d_j g) = \zeta^k d_j(fg) = 0$, and so, by hypothesis,

$$|\Phi \circ L_k \circ D_j(f \otimes g)| = \left|\Phi(d_j f \otimes \zeta^k d_j g)\right| \le \varepsilon\,\|d_j f\|\left\|\zeta^k d_j g\right\| = \varepsilon\,\|f\|\,\|g\|\,.$$

By applying Lemma 4.3.1, we obtain

$$\begin{aligned}\left|\Phi(\zeta^j \otimes \zeta^k - \mathbf{1} \otimes \zeta^{j+k})\right| &= |\Phi \circ L_k \circ D_j(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta)| \\ &\le \|\Phi \circ L_k \circ D_j\,|_{\ker\Delta}\|\, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon.\end{aligned}$$

We check at once that $(L_k \circ D_j)(\ker\Delta) \subset \ker\Delta$, which gives

$$\|\Phi \circ L_k \circ D_j\,|_{\ker\Delta}\| \le \|\Phi\,|_{\ker\Delta}\|\,,$$

and therefore (4.11) is proved.

Take $F \in A(\mathbb{T}^2)$. Then

$$F = \sum_{j,k=-\infty}^{\infty} \widehat{F}(j,k)\zeta^j \otimes \zeta^k$$

and

$$\Delta F = \sum_{j,k=-\infty}^{\infty} \widehat{F}(j,k)\zeta^{j+k}.$$

Consequently,

$$\Phi(F - \mathbf{1} \otimes \Delta F) = \sum_{j,k=-\infty}^{\infty} \widehat{F}(j,k)\Phi(\zeta^j \otimes \zeta^k - \mathbf{1} \otimes \zeta^{j+k}),$$

and (4.11) gives

$$\begin{aligned}|\Phi(F - \mathbf{1}\otimes\Delta F)| &\le \sum_{j,k=-\infty}^{\infty} \left|\widehat{F}(j,k)\right| \left|\Phi(\zeta^j\otimes\zeta^k - \mathbf{1}\otimes\zeta^{j+k})\right| \\ &\le \sum_{j,k=-\infty}^{\infty} \left|\widehat{F}(j,k)\right| \left[\|\Phi\,|_{\ker\Delta}\|\, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon\right] \\ &= \|F\| \left[\|\Phi\,|_{\ker\Delta}\|\, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon\right].\end{aligned} \tag{4.12}$$

In particular, for each $F \in \ker\Delta$, we have

$$\|\Phi(F)\| \le \|F\| \left[\|\Phi\,|_{\ker\Delta}\|\, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon\right].$$

Thus

$$\|\Phi\,|_{\ker\Delta}\| \le \|\Phi\,|_{\ker\Delta}\|\, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon,$$

so that

$$\|\Phi\,|_{\ker\Delta}\| \le 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,\varepsilon.$$

Using this estimate in (4.12), we obtain

$$\begin{aligned}|\Phi(F - \mathbf{1}\otimes\Delta F)| &\le \|F\| \left[60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,\varepsilon 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon\right] \\ &= \|F\|\, 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,\varepsilon\end{aligned}$$

for each $F \in A(\mathbb{T}^2)$, which completes the proof. □

Theorem 4.3.3. *Let G be a locally compact group. Then the Banach algebra $L^1(G)$ is strongly zero product determined and*

$$\alpha_{L^1(G)} \le \beta_{L^1(G)} \le 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}.$$

Proof. On account of Corollary 4.2.2, it suffices to prove that $L^1(G)$ has the strong property $\mathbb{B}$ with

$$\beta_{L^1(G)} \le 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}, \tag{4.13}$$

because $L^1(G)$ has an approximate identity of bound 1. For this purpose set $\varphi \in \mathcal{B}^2(L^1(G),\mathbb{C})$.

Let $t \in G$, and let δ_t be the point mass measure at t on G. We define a contractive homomorphism $T\colon A(\mathbb{T}) \to M(G)$ by

$$T(u) = \sum_{k=-\infty}^{\infty} \widehat{u}(k)\delta_{t^k} \quad \forall u \in A(\mathbb{T}).$$

Take $f, h \in L^1(G)$ with $\|f\| = \|h\| = 1$, and define a continuous linear functional $\Phi\colon A(\mathbb{T}^2) \to \mathbb{C}$ by

$$\Phi(F) = \sum_{(j,k)\in\mathbb{Z}^2} \widehat{F}(j,k)\varphi(f * \delta_{t^j}, \delta_{t^k} * h) \quad \forall F \in A(\mathbb{T}^2).$$

Further, if $u, v \in A(\mathbb{T})$, then

$$\Phi(u \otimes v) = \sum_{(j,k)\in\mathbb{Z}^2} \widehat{u}(j)\widehat{v}(k)\varphi(f * \delta_{t^j}, \delta_{t^k} * h) = \varphi(f * T(u), T(v) * h);$$

in particular, if $uv = 0$, then $(f * T(u)) * (T(v) * h) = f * T(uv) * h = 0$, and so

$$\begin{aligned}|\Phi(u \otimes v)| = |\varphi(f * T(u), T(v) * h)| &\le |\varphi|_{zp} \|f * T(u)\| \|T(v) * h\| \\ &\le |\varphi|_{zp} \|u\| \|v\| .\end{aligned}$$

By applying Lemma 4.3.2 with $F = \zeta \otimes \mathbf{1}$, we see that

$$|\varphi(f * \delta_t, h) - \varphi(f, \delta_t * h)| = |\Phi(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta)| \le 60\sqrt{27}\,\frac{1 + \sin\frac{\pi}{10}}{1 - 2\sin\frac{\pi}{10}}\,|\varphi|_{zp} .$$

We now take $g \in L^1(G)$ with $\|g\| = 1$. By multiplying the above inequality by $|g(t)|$, we arrive at

$$|\varphi(g(t)f * \delta_t, h) - \varphi(f, g(t)\delta_t * h)| \le 60\sqrt{27}\,\frac{1 + \sin\frac{\pi}{10}}{1 - 2\sin\frac{\pi}{10}}\,|\varphi|_{zp}\,|g(t)| . \tag{4.14}$$

Since the convolutions $f * g$ and $g * h$ can be expressed as

$$\begin{aligned} f * g &= \int_G g(t) f * \delta_t \, dt, \\ g * h &= \int_G g(t)\delta_t * h \, dt, \end{aligned}$$

where the expressions on the right-hand side are considered as Bochner integrals of $L^1(G)$-valued functions of t, it follows that

$$\varphi(f * g, h) - \varphi(f, g * h) = \int_G \left[\varphi(g(t)f * \delta_t, h) - \varphi(f, g(t)\delta_t * h)\right] dt.$$

From (4.14) we now deduce that

$$\begin{aligned} |\varphi(f * g, h) - \varphi(f, g * h)| &\le \int_G |\varphi(g(t)f * \delta_t, h) - \varphi(f, g(t)\delta_t * h)| \, dt \\ &\le 60\sqrt{27}\,\frac{1 + \sin\frac{\pi}{10}}{1 - 2\sin\frac{\pi}{10}}\,|\varphi|_{zp} \int_G |g(t)| \, dt \\ &= 60\sqrt{27}\,\frac{1 + \sin\frac{\pi}{10}}{1 - 2\sin\frac{\pi}{10}}\,|\varphi|_{zp} . \end{aligned}$$

We thus get

$$|\varphi|_b \le 60\sqrt{27}\,\frac{1 + \sin\frac{\pi}{10}}{1 - 2\sin\frac{\pi}{10}}\,|\varphi|_{zp} ,$$

and (4.13) is proved. □

4.4 Algebras of approximable aperators

Let X be a Banach space. Then we write $\mathcal{F}(X)$ for the two-sided ideal of $\mathcal{B}(X)$ consisting of finite-rank operators, and $\mathcal{A}(X)$ for the closure of $\mathcal{F}(X)$ in $\mathcal{B}(X)$ with respect to the operator norm. For each $x \in X$ and $\phi \in X^*$, we define $x \otimes \phi \in \mathcal{F}(X)$ by $(x \otimes \phi)(y) = \langle y, \phi \rangle x$ for each $y \in X$. A *finite, biorthogonal system* for X is a set

$$\big\{(x_j, \phi_k) : j, k = 1, \dots, n\big\}$$

with $x_1, \dots, x_n \in X$ and $\phi_1, \dots, \phi_n \in X^*$ such that

$$\langle x_j, \phi_k \rangle = \delta_{j,k} \quad \forall j, k \in \{1, \dots, n\}.$$

Each such system defines an algebra homomorphism

$$\theta \colon \mathbb{M}_n \to \mathcal{F}(X), \quad (a_{j,k}) \mapsto \sum_{j,k=1}^{n} a_{j,k} x_j \otimes \phi_k,$$

where $\mathbb{M}_n$ is the full matrix algebra of order n over $\mathbb{C}$. The identity matrix is denoted by I_n.

The Banach space X is said to have *property* $(\mathbb{A})$ if there is a directed set Λ such that, for each $\lambda \in \Lambda$, there exists a finite, biorthogonal system

$$\big\{(x_j^\lambda, \phi_k^\lambda) : j, k = 1, \dots, n_\lambda\big\}$$

for X with corresponding algebra homomorphism $\theta_\lambda \colon \mathbb{M}_{n_\lambda} \to \mathcal{F}(X)$ such that:

(i) $\lim_{\lambda \in \Lambda} \theta_\lambda(I_{n_\lambda}) = I_X$ uniformly on the compact subsets of X;

(ii) $\lim_{\lambda \in \Lambda} \theta_\lambda(I_{n_\lambda})^* = I_{X^*}$ uniformly on the compact subsets of X^*;

(iii) for each index $\lambda \in \Lambda$, there is a finite subgroup G_λ of the group of all invertible $n_\lambda \times n_\lambda$ matrices over $\mathbb{C}$ whose linear span is all of $\mathbb{M}_{n_\lambda}$, such that

$$\sup_{\lambda \in \Lambda} \sup_{t \in G_\lambda} \|\theta_\lambda(t)\| < \infty. \tag{4.15}$$

Property $(\mathbb{A})$ forces the Banach algebra $\mathcal{A}(X)$ to be amenable. For an exhaustive treatment of this topic (including a variety of interesting examples of spaces with property $(\mathbb{A})$) we refer to [73, Section 3.3].

The notation of the above definition will be standard for the remainder of this section. Furthermore, our basic reference for this section is the monograph [73].

Theorem 4.4.1. *Let X be a Banach space with property $(\mathbb{A})$. Then the Banach algebra $\mathcal{A}(X)$ is strongly zero product determined. Specifically, if C denotes the supremum in (4.15), then*

$$\tfrac{1}{2}\beta_{\mathcal{A}(X)} \le \alpha_{\mathcal{A}(X)} \le 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,C^2.$$

Proof. For each $\lambda \in \Lambda$ we define $\Phi_\lambda \colon \ell^1(G_\lambda) \to \mathcal{F}(X)$ by

$$\Phi_\lambda(f) = \sum_{t \in G_\lambda} f(t)\theta_\lambda(t) \quad \forall f \in \ell^1(G_\lambda).$$

We claim that Φ_λ is an algebra homomorphism. It is clear the Φ_λ is a linear map and, for each $f, g \in \ell^1(G_\lambda)$, we have

$$\begin{aligned}
\Phi_\lambda(f * g) &= \sum_{t \in G_\lambda} (f * g)(t)\theta_\lambda(t) = \sum_{t \in G_\lambda} \sum_{s \in G_\lambda} f(s)g(s^{-1}t)\theta_\lambda(t) \\
&= \theta_\lambda\left(\sum_{t \in G_\lambda} \sum_{s \in G_\lambda} f(s)g(s^{-1}t)t \right) = \theta_\lambda\left(\sum_{s \in G_\lambda} f(s)s \sum_{t \in G_\lambda} g(s^{-1}t)s^{-1}t \right) \\
&= \theta_\lambda\left(\sum_{s \in G_\lambda} f(s)s \sum_{r \in G_\lambda} g(r)r \right) = \theta_\lambda\left(\sum_{s \in G_\lambda} f(s)s \right) \theta_\lambda\left(\sum_{r \in G_\lambda} g(r)r \right) \\
&= \Phi_\lambda(f)\Phi_\lambda(g).
\end{aligned}$$

Of course, Φ_λ is continuous because $\ell^1(G_\lambda)$ is finite-dimensional, and, further, for each $f \in \ell^1(G_\lambda)$, we have

$$\|\Phi_\lambda(f)\| \le \sum_{t \in G_\lambda} |f(t)| \, \|\theta_\lambda(t)\| \le \sum_{t \in G_\lambda} |f(t)| \, C = C \, \|f\|_1 .$$

Hence $\|\Phi_\lambda\| \le C$.

Let $\varphi \in \mathcal{B}^2(\mathcal{A}(X), \mathbb{C})$. Let us prove that

$$\left| \varphi(S\theta_\lambda(t), \theta_\lambda(t^{-1})T) - \varphi(S\theta_\lambda(I_{n_\lambda}), \theta_\lambda(I_{n_\lambda})T) \right| \le \beta_{\ell^1(G_\lambda)} C^2 \, \|S\| \, \|T\| \, |\varphi|_{zp} \tag{4.16}$$

for all $\lambda \in \Lambda$, $S, T \in \mathcal{A}(X)$, and $t \in G_\lambda$. For this purpose, take $\lambda \in \Lambda$ and $S, T \in \mathcal{A}(X)$, and define $\varphi_\lambda \colon \ell^1(G_\lambda) \times \ell^1(G_\lambda) \to \mathbb{C}$ by

$$\varphi_\lambda(f, g) = \varphi(S\Phi_\lambda(f), \Phi_\lambda(g)T) \quad \forall f, g \in \ell^1(G_\lambda).$$

Then φ_λ is continuous and, for each $f, g \in \ell^1(G_\lambda)$ such that $f * g = 0$, we have $(S\Phi_\lambda(f))(\Phi_\lambda(g)T) = S(\Phi_\lambda(f * g))T = 0$ and therefore

$$|\varphi_\lambda(f, g)| \le |\varphi|_{zp} \, \|S\Phi_\lambda(f)\| \, \|\Phi_\lambda(g)T\| \le |\varphi|_{zp} \, C^2 \, \|S\| \, \|T\| \, \|f\|_1 \, \|g\|_1 ,$$

whence

$$|\varphi_\lambda|_{zp} \le C^2 \, \|S\| \, \|T\| \, |\varphi|_{zp} .$$

For each $t \in G_\lambda$, we have

$$\left| \varphi_\lambda(\delta_t, \delta_{t^{-1}}) - \varphi_\lambda(\delta_{I_{n_\lambda}}, \delta_{I_{n_\lambda}}) \right| = \left| \varphi_\lambda(\delta_{I_{n_\lambda}} * \delta_t, \delta_{t^{-1}}) - \varphi_\lambda(\delta_{I_{n_\lambda}}, \delta_t * \delta_{t^{-1}}) \right| \le$$

$$|\varphi_\lambda|_b \le \beta_{\ell^1(G_\lambda)} \, |\varphi_\lambda|_{zp} \le \beta_{\ell^1(G_\lambda)} C^2 \, \|S\| \, \|T\| \, |\varphi|_{zp} ,$$

which gives (4.16).

The projective tensor product $\mathcal{A}(X)\widehat{\otimes}\mathcal{A}(X)$ becomes a Banach $\mathcal{A}(X)$-bimodule for the products defined by

$$R\cdot(S\otimes T)=(RS)\otimes T,\quad (S\otimes T)\cdot R=S\otimes(TR)\quad \forall R,S,T\in\mathcal{A}(X).$$

We define a continuous linear functional $\widehat{\varphi}\in\big(\mathcal{A}(X)\widehat{\otimes}\mathcal{A}(X)\big)^*$ through

$$\langle S\otimes T,\widehat{\varphi}\rangle=\varphi(S,T)\quad \forall S,T\in\mathcal{A}(X).$$

For each $\lambda\in\Lambda$, set $P_\lambda=\theta_\lambda(I_{n_\lambda})$ and

$$D_\lambda=\frac{1}{|G_\lambda|}\sum_{t\in G_\lambda}\theta_\lambda(t)\otimes\theta_\lambda(t^{-1}).$$

Then $(P_\lambda)_{\lambda\in\Lambda}$ is a bounded approximate identity for $\mathcal{A}(X)$ and $(D_\lambda)_{\lambda\in\Lambda}$ is an approximate diagonal for $\mathcal{A}(X)$ (see [73, Theorem 3.3.9]), so that $(\|S\cdot D_\lambda-D_\lambda\cdot S\|)_{\lambda\in\Lambda}\to 0$ for each $S\in\mathcal{A}(X)$.

For each $\lambda\in\Lambda$ and $S,T\in\mathcal{A}(X)$, (4.16) shows that

$$\begin{aligned}
&|\langle S\cdot D_\lambda\cdot T,\widehat{\varphi}\rangle-\varphi(SP_\lambda,P_\lambda T)|\\
&=\left|\frac{1}{|G_\lambda|}\sum_{t\in G_\lambda}\big[\varphi(S\theta_\lambda(t),\theta_\lambda(t^{-1})T)-\varphi(S\theta_\lambda(I_{n_\lambda}),\theta_\lambda(I_{n_\lambda})T)\big]\right|\\
&\leq \beta_{\ell^1(G_\lambda)}C^2\,\|S\|\,\|T\|\,|\varphi|_{zp}
\end{aligned}$$

and Theorem 4.3.3 then gives

$$|\langle S\cdot D_\lambda\cdot T,\widehat{\varphi}\rangle-\varphi(SP_\lambda,P_\lambda T)|\leq 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,C^2\,\|S\|\,\|T\|\,|\varphi|_{zp}\,. \tag{4.17}$$

For each $\lambda\in\Lambda$, define $\xi_\lambda\in\mathcal{A}(X)^*$ by

$$\langle T,\xi_\lambda\rangle=\langle D_\lambda\cdot T,\widehat{\varphi}\rangle\quad \forall T\in\mathcal{A}(X).$$

Note that

$$\|\xi_\lambda\|\leq\|\widehat{\varphi}\|\,\|D_\lambda\|\leq\|\varphi\|\,C^2\quad \forall\lambda\in\Lambda$$

and therefore $(\xi_\lambda)_{\lambda\in\Lambda}$ is a bounded net in $\mathcal{A}(X)^*$. By the Banach–Alaoglu theorem the net $(\xi_\lambda)_{\lambda\in\Lambda}$ has a weak*-accumulation point, say ξ, in $\mathcal{A}(X)^*$. Take a subnet $(\xi_\nu)_{\nu\in N}$ of $(\xi_\lambda)_{\lambda\in\Lambda}$ such that w*-$\lim_{\nu\in N}\xi_\nu=\xi$. Take $S,T\in\mathcal{A}(X)$. For each $\nu\in N$, we have

$$\begin{gathered}
\varphi(SP_\nu,P_\nu T)-\xi_\lambda(ST)=\\
\varphi(SP_\nu,P_\nu T)-\langle S\cdot D_\nu\cdot T,\widehat{\varphi}\rangle+\langle(S\cdot D_\nu-D_\nu\cdot S)\cdot T,\widehat{\varphi}\rangle
\end{gathered}$$

so that (4.17) gives

$$|\varphi(SP_\nu,P_\nu T)-\langle ST,\xi_\lambda\rangle|\leq$$

$$60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,C^2\,\|S\|\,\|T\|\,|\varphi|_{zp}+\|\varphi\|\,\|S\cdot D_\nu-D_\nu\cdot S\|\,\|T\|\,.$$

Taking limits on both sides of the above inequality, and using that $(SP_\nu)_{\nu\in N}\to S$, $(P_\nu T)_{\nu\in N}\to T$, and $(\|S\cdot D_\nu-D_\nu\cdot S\|)_{\nu\in N}\to 0$, we see that

$$|\varphi(S,T)-\langle ST,\xi\rangle|\le 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,C^2\,\|S\|\,\|T\|\,|\varphi|_{zp}\,.$$

We thus get

$$\operatorname{dist}\big(\varphi,\mathcal{B}^2_\pi(\mathcal{A}(X),\mathbb{C})\big)\le 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,C^2\,|\varphi|_{zp}\,,$$

which proves the theorem. □

The hyperreflexivity of the space $\mathcal{Z}^n(A,X)$ of continuous n-cocycles from A into X, where A is a C^*-algebra or a group algebra and X is a Banach A-bimodule has been already studied in [77, Theorem 4.4]. We conclude this section with a look at the hyperreflexivity of the space $\mathcal{Z}^n(\mathcal{A}(X),Y^*)$. For this purpose we introduce some terminology.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Set

$$L_X=\sup\big\{\|a\cdot x\|:x\in X,\ a\in A,\ \|x\|=\|a\|=1\big\}$$

and

$$R_X=\sup\big\{\|x\cdot a\|:x\in X,\ a\in A,\ \|x\|=\|a\|=1\big\}.$$

For each $n\in\mathbb{N}$, let $\delta^n\colon\mathcal{B}^n(A,X)\to\mathcal{B}^{n+1}(A,X)$ be the n-coboundary operator defined by

$$\begin{aligned}(\delta^nT)(a_1,\dots,a_{n+1})=a_1\cdot T(a_2,\dots,a_{n+1})&\\ +\sum_{k=1}^{n}(-1)^kT(a_1,\dots,a_ka_{k+1},\dots,a_{n+1})&\\ +(-1)^{n+1}T(a_1,\dots,a_n)\cdot a_{n+1}&\end{aligned}$$

for all $T\in\mathcal{B}^n(A,X)$ and $a_1,\dots,a_{n+1}\in A$. Further, $\delta^0\colon X\to\mathcal{B}(A,X)$ is defined by

$$(\delta^0x)(a)=a\cdot x-x\cdot a\quad\forall x\in X,\ \forall a\in A.$$

The space of continuous n-cocycles, $\mathcal{Z}^n(A,X)$, is defined as $\ker\delta^n$. The space of continuous n-coboundaries, $\mathcal{N}^n(A,X)$, is the range of δ^{n-1}. Then $\mathcal{N}^n(A,X)\subset\mathcal{Z}^n(A,X)$, and the quotient $\mathcal{H}^n(A,X)=\mathcal{Z}^n(A,X)/\mathcal{N}^n(A,X)$ is the n^{th} Hochschild cohomology group. For each $T\in\mathcal{B}^n(A,X)$, the constant

$$\operatorname{dist}_r\big(T,\mathcal{Z}^n(A,X)\big):=$$

$$\sup_{\|a_1\|=\cdots=\|a_n\|=1}\inf\big\{\|T(a_1,\dots,a_n)-S(a_1,\dots,a_n)\|:S\in\mathcal{Z}^n(A,X)\big\}$$

is intended to estimate the usual distance from T to $\mathcal{Z}^n(A,X)$, and, in accordance with [76, 77], the space $\mathcal{Z}^n(A,X)$ is called hyperreflexive if there exists a constant K such that

$$\operatorname{dist}\big(T,\mathcal{Z}^n(A,X)\big)\le K\operatorname{dist}_r\big(T,\mathcal{Z}^n(A,X)\big)\quad\forall T\in\mathcal{B}^n(A,X).$$

The inequality $\operatorname{dist}_r\big(T,\mathcal{Z}^n(A,X)\big)\le\operatorname{dist}\big(T,\mathcal{Z}^n(A,X)\big)$ is always true.

Proposition 4.4.2. *Let A be a C-amenable Banach algebra, and let X be a Banach A-bimodule. Then there exist projections $P,Q\in\mathcal{B}(X^*)$ onto $(X\cdot A)^\perp$ and $(A\cdot X)^\perp$, respectively, with $\|P\|\le 1+R_XC$, $\|Q\|\le 1+L_XC$, and such that*

$$\operatorname{dist}\big(T,\mathcal{Z}^1(A,X^*)\big)\le C\big(R_X+L_X\|P\|+\|P\|\|Q\|\big)\|\delta^1T\|$$

for all $T\in\mathcal{B}(A,X^*)$. In particular, if the module X is essential, then

$$\operatorname{dist}\big(T,\mathcal{Z}^1(A,X^*)\big)\le R_XC\|\delta^1T\|$$

for all $T\in\mathcal{B}(A,X^*)$.

Proof. The Banach algebra A has a virtual diagonal D with $\|\mathrm{D}\|\le C$. This is an element $\mathrm{D}\in(A\widehat{\otimes}A)^{**}$ such that, for each $a\in A$, we have

$$a\cdot\mathrm{D}=\mathrm{D}\cdot a\quad\text{and}\quad a\cdot\widehat{\pi}^{**}(\mathrm{D})=a. \tag{4.18}$$

Here, the Banach space $A\widehat{\otimes}A$ turns into a contractive Banach A-bimodule with respect to the operations defined through

$$(a\otimes b)c=a\otimes bc,\ c(a\otimes b)=ca\otimes b\quad\forall a,b,c\in A,$$

and both $(A\widehat{\otimes}A)^{**}$ and A^{**} are considered as dual A-bimodules in the usual way. The map $\widehat{\pi}\colon A\widehat{\otimes}A\to A$ is the projective induced product map defined through

$$\widehat{\pi}(a\otimes b)=ab\quad\forall a,b\in A.$$

For each $\varphi\in\mathcal{B}^2(A,\mathbb{C})$ there exists a unique element $\widehat{\varphi}\in(A\widehat{\otimes}A)^*$ such that

$$\widehat{\varphi}(a\otimes b)=\varphi(a,b)\quad\forall a,b\in A,$$

and we use the formal notation

$$\int_{A\times A}\varphi(u,v)\,d\mathrm{D}(u,v):=\langle\widehat{\varphi},\mathrm{D}\rangle.$$

Using this notation, the properties (4.18) can be written as

$$\int_{A\times A}\varphi(au,v)\,d\mathrm{D}(u,v)=\int_{A\times A}\varphi(u,va)\,d\mathrm{D}(u,v) \tag{4.19}$$

and

$$\int_{A\times A} \langle auv, \xi\rangle \, d\mathrm{D}(u,v) = \langle a, \xi\rangle \tag{4.20}$$

for all $\varphi \in \mathcal{B}^2(A, \mathbb{C})$, $a \in A$, and $\xi \in A^*$; further, it will be helpful noting that

$$\left| \int_{A\times A} \varphi(u,v) \, d\mathrm{D}(u,v) \right| \leq \|\mathrm{D}\| \|\widehat{\varphi}\| \leq C\|\varphi\|. \tag{4.21}$$

We proceed to define the projections P and Q. For this purpose we first define $P_0, Q_0 \in \mathcal{B}(X^*)$ by

$$\begin{aligned} \langle x, P_0\xi\rangle &= \int_{A\times A} \langle x\cdot(uv), \xi\rangle \, d\mathrm{D}(u,v), \\ \langle x, Q_0\xi\rangle &= \int_{A\times A} \langle (uv)\cdot x, \xi\rangle \, d\mathrm{D}(u,v) \end{aligned}$$

for all $x \in X$ and $\xi \in X^*$, and set

$$P = I_{X^*} - P_0, \quad Q = I_{X^*} - Q_0.$$

From (4.21) we obtain $\|P_0\| \leq R_X C$ and $\|Q_0\| \leq L_X C$, so that $\|P\| \leq 1 + R_X C$ and $\|Q\| \leq 1 + L_X C$.

We claim that

$$a \cdot P_0\xi = P_0(a\cdot\xi) = a\cdot\xi, \tag{4.22}$$

$$P_0\xi\cdot a = P_0(\xi\cdot a) \tag{4.23}$$

for all $a \in A$ and $\xi \in X^*$. Indeed, for $a \in A$, $\xi \in X^*$, and each $x \in X$, (4.19) and (4.20) gives

$$\begin{aligned} \langle x, a\cdot P_0\xi\rangle = \langle x\cdot a, P_0\xi\rangle &= \int_{A\times A} \langle x\cdot(auv), \xi\rangle \, d\mathrm{D}(u,v) \\ &= \langle x\cdot a, \xi\rangle = \langle x, a\cdot\xi\rangle, \\ \langle x, P_0(a\cdot\xi)\rangle &= \int_{A\times A} \langle x\cdot(uv), a\cdot\xi\rangle \, d\mathrm{D}(u,v) \\ &= \int_{A\times A} \langle x\cdot(uva), \xi\rangle \, d\mathrm{D}(u,v) \\ &= \int_{A\times A} \langle x\cdot(auv), \xi\rangle \, d\mathrm{D}(u,v) = \langle x, a\cdot\xi\rangle, \end{aligned}$$

and

$$\begin{aligned} \langle x, P_0\xi\cdot a\rangle = \langle a\cdot x, P_0\xi\rangle &= \int_{A\times A} \langle (a\cdot x)\cdot(uv), \xi\rangle \, d\mathrm{D}(u,v) \\ &= \int_{A\times A} \langle x\cdot(uv), \xi\cdot a\rangle \, d\mathrm{D}(u,v) = \langle x, P_0(\xi\cdot a)\rangle, \end{aligned}$$

which proves (4.22) and (4.23). From (4.22) we deduce that

$$\langle x \cdot a, P\xi \rangle = \langle x, a \cdot \xi - a \cdot P_0\xi \rangle = 0,$$

and so $P\xi \in (X \cdot A)^\perp$. Further, if $\xi \in (X \cdot A)^\perp$, then

$$\langle x, P_0\xi \rangle = \int_{A \times A} \langle \underbrace{x \cdot (uv)}_{\in X \cdot A}, \xi \rangle \, d\mathrm{D}(u, v) = 0,$$

and so $P\xi = \xi$. The operator P is a projection onto $(X \cdot A)^\perp$. From (4.22) we deduce immediately that

$$P(A \cdot X^*) = \{0\}. \tag{4.24}$$

The operator Q can be handled in much the same way as P, and we obtain

$$\begin{aligned} &Q_0\xi \cdot a = Q_0(\xi \cdot a) = \xi \cdot a, \\ &a \cdot Q_0\xi = Q_0(a \cdot \xi) \end{aligned}$$

for all $a \in A$ and $\xi \in X^*$, the operator Q is a projection onto $(A \cdot X)^\perp$, and

$$Q(X^* \cdot A) = \{0\}. \tag{4.25}$$

Set $T \in \mathcal{B}(A, X^*)$, and define $\phi \in X^*$ by

$$\langle x, \phi \rangle = \int_{A \times A} \langle x, u \cdot T(v) \rangle \, d\mathrm{D}(u, v) \quad \forall x \in X.$$

For each $x \in X$ and $a \in A$ we have

$$\langle x, P_0 T(a) \rangle = \int_{A \times A} \langle x \cdot (uv), T(a) \rangle \, d\mathrm{D}(u, v) = \int_{A \times A} \langle x, (uv) \cdot T(a) \rangle \, d\mathrm{D}(u, v)$$

and

$$\begin{aligned} \langle x, (\delta^0\phi)(a) \rangle &= \langle x, a \cdot \phi - \phi \cdot a \rangle = \langle x \cdot a - a \cdot x, \phi \rangle \\ &= \int_{A \times A} \langle x \cdot a - a \cdot x, u \cdot T(v) \rangle \, d\mathrm{D}(u, v) \\ &= \int_{A \times A} \langle x, (au) \cdot T(v) - u \cdot T(v) \cdot a \rangle \, d\mathrm{D}(u, v) \\ &= \int_{A \times A} \langle x, u \cdot T(va) - u \cdot T(v) \cdot a \rangle \, d\mathrm{D}(u, v), \end{aligned}$$

so that

$$\begin{aligned} \langle x, (P_0 T - \delta^0\phi)(a) \rangle &= \int_{A \times A} \langle x, u \cdot (\delta^1 T)(v, a) \rangle \, d\mathrm{D}(u, v) \\ &= \int_{A \times A} \langle x \cdot u, (\delta^1 T)(v, a) \rangle \, d\mathrm{D}(u, v). \end{aligned}$$

From the latter identity and (4.21) we conclude that

$$|\langle x, (P_0T - \delta^0\phi)(a)\rangle| \leq CR_X\|\delta^1 T\|\|a\|\|x\|,$$

whence

$$\|P_0T - \delta^0\phi\| \leq CR_X\|\delta^1 T\|. \tag{4.26}$$

Write $S = PT$. From (4.22) and (4.23) it follows that $\delta^1 S(a,b) = P\delta^1 T(a,b)$, and so

$$\|\delta^1 S\| \leq \|P\|\|\delta^1 T\|. \tag{4.27}$$

We now define $\psi \in X^*$ by

$$\langle x, \psi\rangle = \int_{A\times A} \langle x, S(u)\cdot v\rangle \, d\mathrm{D}(u,v) \quad \forall x \in X.$$

For each $x \in X$ and $a \in A$ we have

$$\langle x, Q_0S(a)\rangle = \int_{A\times A} \langle (uv)\cdot x, S(a)\rangle \, d\mathrm{D}(u,v) = \int_{A\times A} \langle x, S(a)\cdot(uv)\rangle \, d\mathrm{D}(u,v)$$

and

$$\begin{aligned}
\langle x, (\delta^0\psi)(a)\rangle &= \langle x, a\cdot\psi - \psi\cdot a\rangle = \langle x\cdot a - a\cdot x, \psi\rangle \\
&= \int_{A\times A} \langle x\cdot a - a\cdot x, S(u)\cdot v\rangle \, d\mathrm{D}(u,v) \\
&= \int_{A\times A} \langle x, a\cdot S(u)\cdot v - S(u)\cdot(va)\rangle \, d\mathrm{D}(u,v) \\
&= \int_{A\times A} \langle x, a\cdot S(u)\cdot v - S(au)\cdot v\rangle \, d\mathrm{D}(u,v),
\end{aligned}$$

and hence

$$\begin{aligned}
\langle x, (Q_0S + \delta^0\psi)(a)\rangle &= \int_{A\times A} \langle x, (\delta^1 S)(a,u)\cdot v\rangle \, d\mathrm{D}(u,v) \\
&= \int_{A\times A} \langle v\cdot x, (\delta^1 S)(a,u)\rangle \, d\mathrm{D}(u,v).
\end{aligned}$$

From the latter identity and (4.21) we conclude that

$$|\langle x, (Q_0S + \delta^0\psi)(a)\rangle| \leq CL_X\|\delta^1 S\|\|a\|\|x\|.$$

Thus $\|Q_0S + \delta^0\psi\| \leq CL_X\|\delta^1 S\|$ and (4.27) then gives

$$\|Q_0S + \delta^0\psi\| \leq CL_X\|P\|\|\delta^1 T\|. \tag{4.28}$$

Our next goal is to estimate $\|QPT\|$. For each $u, v, a \in A$, we have

$$\delta^1 T(a, uv) = a\cdot T(uv) - T(auv) + T(a)\cdot(uv),$$

(4.23) and (4.24) gives

$$P(\delta^1 T(a, uv)) = \underbrace{P(a \cdot T(uv))}_{=0} - PT(auv) + PT(a) \cdot (uv),$$

and finally (4.25) yields

$$QP(\delta^1 T(a, uv)) = -QPT(auv) + \underbrace{Q(PT(a) \cdot (uv))}_{=0} = -QPT(auv).$$

We thus get

$$\begin{aligned} \langle x, QPT(a) \rangle &= \int_{A \times A} \langle x, QPT(auv) \rangle \, d\mathrm{D}(u, v) \\ &= \int_{A \times A} \langle x, -QP(\delta^1 T)(a, uv) \rangle \, d\mathrm{D}(u, v) \end{aligned}$$

and (4.21) implies

$$|\langle x, QPT(a) \rangle| \le C \|QP(\delta^1 T)\| \|x\| \|a\| \le C \|Q\| \|P\| \|\delta^1 T\| \|x\| \|a\|.$$

Hence

$$\|QPT\| \le C \|Q\| \|P\| \|\delta^1 T\|. \tag{4.29}$$

Finally, since

$$T - \delta^0 \phi + \delta^0 \psi = QPT + (P_0 T - \delta^0 \phi) + (Q_0 PT + \delta^0 \psi),$$

(4.26), (4.28), and (4.29) show that

$$\begin{aligned} \|T - \delta^0 \phi + \delta^0 \psi\| &\le \|P_0 T - \delta^0 \phi\| + \|Q_0 PT + \delta^0 \psi\| + \|QPT\| \\ &\le C R_X \|\delta^1 T\| + C L_X \|P\| \|\delta^1 T\| + C \|Q\| \|P\| \|\delta^1 T\|. \end{aligned}$$

Since $-\delta^0 \phi + \delta^0 \psi \in \mathcal{Z}^1(A, X^*)$, it follows that

$$\operatorname{dist}\big(T, \mathcal{Z}^1(A, X^*)\big) \le C R_X \|\delta^1 T\| + C L_X \|P\| \|\delta^1 T\| + C \|Q\| \|P\| \|\delta^1 T\|$$

as required. □

Corollary 4.4.3. *Let A be a C-amenable Banach algebra, let X be a Banach A-bimodule, and let $n \in \mathbb{N}$. Then*

$$\operatorname{dist}\big(T, \mathcal{Z}^n(A, X^*)\big) \le 2(n + L_X)(1 + R_X) C^3 \|\delta^n T\|$$

for each $T \in \mathcal{B}^n(A, X^*)$.

Proof. Of course, we need only consider the case where A is a non-zero Banach algebra, which implies that $C \geq 1$.

Suppose that $n = 1$, and $T \in \mathcal{B}(A, X^*)$. By Proposition 4.4.2,

$$\begin{aligned}\operatorname{dist}\bigl(T, \mathcal{Z}^1(A, X^*)\bigr) &\leq C\bigl(R_X + L_X(1 + R_X C) + (1 + L_X C)(1 + R_X C)\bigr)\|\delta^1 T\| \\ &\leq 2(1 + L_X)(1 + R_X)C^3\|\delta^1 T\|,\end{aligned}$$

as $C \geq 1$.

The Banach space $\mathcal{B}^n(A, X^*)$ is a Banach A-bimodule with respect to the operations

$$(a \cdot T)(a_1, \ldots, a_n) = a \cdot T(a_1, \ldots, a_n)$$

and

$$\begin{aligned}(T \cdot a)(a_1, \ldots, a_n) = T(aa_1, \ldots, a_n)& \\ &+ \sum_{k=1}^{n-1}(-1)^k T(a, a_1, \ldots, a_k a_{k+1}, \ldots, a_n) \\ &+ (-1)^n T(a, a_1, \ldots, a_{n-1}) \cdot a_n\end{aligned}$$

for all $T \in \mathcal{B}^n(A, X^*)$, and $a, a_1, \ldots, a_n \in A$. Let

$$\Delta^1 \colon \mathcal{B}(A, \mathcal{B}^n(A, X^*)) \to \mathcal{B}^2(A, \mathcal{B}^n(A, X^*))$$

be the 1-coboundary operator. We also consider the maps

$$\begin{aligned}\tau_1^n &\colon \mathcal{B}^{1+n}(A, X^*) \to \mathcal{B}(A, \mathcal{B}^n(A, X^*)), \\ \tau_2^n &\colon \mathcal{B}^{2+n}(A, X^*) \to \mathcal{B}^2(A, \mathcal{B}^n(A, X^*))\end{aligned}$$

defined by

$$\begin{aligned}(\tau_1^n T)(a)(a_1, \ldots, a_n) &= T(a, a_1, \ldots, a_n), \\ (\tau_2^n T)(a, b)(a_1, \ldots, a_n) &= T(a, b, a_1, \ldots, a_n).\end{aligned}$$

Then:

- τ_1^n and τ_2^n are isometric isomorphisms;
- $\Delta^1 \circ \tau_1^n = \tau_2^n \circ \delta^{n+1}$;
- $\tau_1^n \mathcal{Z}^{n+1}(A, X^*) = \mathcal{Z}^1(A, \mathcal{B}^n(A, X^*))$.

For each $T \in \mathcal{B}^{1+n}(A, X^*)$ we have

$$\begin{aligned}\operatorname{dist}\bigl(T, \mathcal{Z}^{n+1}(A, X^*)\bigr) &= \operatorname{dist}\bigl(\tau_1^n T, \tau_1^n \mathcal{Z}^{n+1}(A, X^*)\bigr) \\ &= \operatorname{dist}\bigl(\tau_1^n T, \mathcal{Z}^1(A, \mathcal{B}^n(A, X^*))\bigr).\end{aligned} \tag{4.30}$$

Our next objective is to apply Proposition 4.4.2 to estimate the distance of the last term in (4.30). To this end, we realize that $\mathcal{B}^n(A, X^*)$ is a dual Banach A-bimodule by setting

$$Y = \underbrace{A\widehat{\otimes} \cdots \widehat{\otimes} A}_{n\text{-times}} \widehat{\otimes} X.$$

Then:

- Y is a Banach A-bimodule with respect to the operations

$$(a_1 \otimes \cdots \otimes a_n \otimes x) \cdot a = a_1 \otimes \cdots \otimes a_n \otimes (x \cdot a)$$

and

$$\begin{aligned} a \cdot (a_1 \otimes \cdots \otimes a_n \otimes x) &= (aa_1) \otimes \cdots \otimes a_n \otimes x \\ &\quad + \sum_{k=1}^{n-1} (-1)^k a \otimes a_1 \otimes \cdots \otimes (a_k a_{k+1}) \otimes \cdots \otimes a_n \otimes x \\ &\qquad + (-1)^n a \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes (a_n \cdot x) \end{aligned}$$

for all $a, a_1, \ldots, a_n \in A$, and $x \in X$;

- we have the estimates

$$L_Y \leq n + L_X, \quad R_Y \leq R_X;$$

- the Banach A-bimodule $\mathcal{B}^n(A, X^*)$ is isometrically isomorphic to the Banach A-bimodule Y^* through the duality

$$\langle a_1 \otimes \cdots \otimes a_n \otimes x, T \rangle = \langle x, T(a_1, \ldots, a_n) \rangle$$

for all $T \in \mathcal{B}^n(A, X^*)$, $a_1, \ldots, a_n \in A$, and $x \in X$.

Proposition 4.4.2 now leads to

$$\begin{aligned} \operatorname{dist}\big(\tau_1^n T, \mathcal{Z}^1(A, \mathcal{B}^n(A, X^*))\big) &= \operatorname{dist}\big(\tau_1^n T, \mathcal{Z}^1(A, Y^*)\big) \\ &\leq 2(1 + L_Y)(1 + R_Y) C^3 \left\| \Delta^1 \tau_1^n T \right\| \\ &\leq 2(1 + n + L_X)(1 + R_X) C^3 \left\| \Delta^1 \tau_1^n T \right\| \\ &= 2(1 + n + L_X)(1 + R_X) C^3 \left\| \tau_2^n \delta^{n+1} T \right\| \\ &= 2(1 + n + L_X)(1 + R_X) C^3 \left\| \delta^{n+1} T \right\|. \end{aligned}$$

Combining (4.30) with the inequality above, we obtain precisely the estimate of the corollary. □

Theorem 4.4.4. *Let X be a Banach space with property $(\mathbb{A})$, let Y be a Banach $\mathcal{A}(X)$-bimodule, and let $n \in \mathbb{N}$. Then the space $\mathcal{Z}^n(\mathcal{A}(X), Y^*)$ is hyperreflexive. Specifically, if C denotes the supremum in (4.15), then*

$$\operatorname{dist}\big(T, \mathcal{Z}^n(\mathcal{A}(X), Y^*)\big) \leq$$
$$(n + L_Y)(1 + R_Y) C^6 2^n \big(C^2 \beta_{\mathcal{A}(X)} + (C+1)^2\big)^{n+1} \operatorname{dist}_r\big(T, \mathcal{Z}^n(\mathcal{A}(X), Y^*)\big)$$

for each $T \in \mathcal{B}^n(\mathcal{A}(X), Y^*)$, where

$$\beta_{\mathcal{A}(X)} \leq 120\sqrt{27}\, \frac{1 + \sin\frac{\pi}{10}}{1 - 2\sin\frac{\pi}{10}}\, C^2.$$

Proof. From Theorem 4.4.1 we see that $\mathcal{A}(X)$ has the strong property $\mathbb{B}$ and the estimate for $\beta_{\mathcal{A}(X)}$ holds.

The Banach algebra $\mathcal{A}(X)$ has an approximate identity of bound C. Further, for each $T \in \mathcal{F}(X)$ there exists $S \in \mathcal{F}(X)$ such that $ST = TS = T$, and [76, Proposition 5.4] then shows that $\mathcal{A}(X)$ has bounded local units.

By [73, Theorem 3.3.9], $\mathcal{A}(X)$ is C^2-amenable, and Corollary 4.4.3 now gives

$$\operatorname{dist}\big(T, \mathcal{Z}^n(\mathcal{A}(X), Y^*)\big) \leq 2(n + L_Y)(1 + R_Y)C^6 \|\delta^n T\|$$

for each $T \in \mathcal{B}^n(\mathcal{A}(X), Y^*)$. This estimate shows that the map

$$\begin{aligned} \mathcal{B}^n(\mathcal{A}(X), Y^*)/\mathcal{Z}^n(\mathcal{A}(X), Y^*) &\to \mathcal{N}^{n+1}(\mathcal{A}(X), Y^*) \\ T + \mathcal{Z}^n(\mathcal{A}(X), Y^*) &\mapsto \delta^n T \end{aligned}$$

is an isomorphism, hence $\mathcal{N}^{n+1}(\mathcal{A}(X), Y^*)$ is closed in $\mathcal{B}^{n+1}(\mathcal{A}(X), Y^*)$ and this implies that the n^{th} Hochschild cohomology group $\mathcal{H}^{n+1}(\mathcal{A}(X), Y^*)$ is a Banach space. By applying [77, Theorem 4.3] we obtain the hyperreflexivity of the space $\mathcal{Z}^n(\mathcal{A}(X), Y^*)$ as well as the statement about the estimate of $\operatorname{dist}\big(T, \mathcal{Z}^n(\mathcal{A}(X), Y^*)\big)$. □

Chapter 5

Weighted Jordan homomorphisms

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Abstract. Let A and B be unital rings. An additive map $T : A \to B$ is called a weighted Jordan homomorphism if $c = T(1)$ is an invertible central element and $cT(x^2) = T(x)^2$ for all $x \in A$. We provide assumptions, which are in particular fulfilled when $A = B = M_n(R)$ with $n \geq 2$ and R any unital ring with $\frac{1}{2}$, under which every surjective additive map $T : A \to B$ with the property that $T(x)T(y) + T(y)T(x) = 0$ whenever $xy = yx = 0$ is a weighted Jordan homomorphism. Further, we show that if A is a prime ring with $\text{char}(A) \neq 2, 3, 5$, then a bijective additive map $T : A \to A$ is a weighted Jordan homomorphism provided that there exists an additive map $S : A \to A$ such that $S(x^2) = T(x)^2$ for all $x \in A$.

5.1 Introduction

Let A and B be unital rings. Recall that an additive map $\Phi : A \to B$ is called a *Jordan homomorphism* if $\Phi(x \circ y) = \Phi(x) \circ \Phi(y)$ for all $x, y \in A$, where $x \circ y$ stands for the Jordan product $xy + yx$ of x and y. We say that an additive map $T : A \to B$ is a *weighted Jordan homomorphism* if $c = T(1)$ is an invertible element lying in the center of B and $x \mapsto c^{-1}T(x)$ is a Jordan homomorphism, that is,

$$cT(x \circ y) = T(x) \circ T(y) \quad (x, y \in A).$$

Weighted Jordan homomorphisms can be also defined for rings without unity, see [24, p. 121]. However, we will work only with unital rings in this paper.

Weighted Jordan homomorphisms naturally appear in some preserver problems. In [32], Chebotar, Ke, Lee, and Zhang used functional identities to prove that if R is a unital ring with $\frac{1}{2}$ and $A = M_n(R)$ with $n \geq 4$ (i.e., A is the ring of $n \times n$ matrices over R), then a surjective additive map $T : A \to A$ which preserves zero Jordan products (i.e., $T(x) \circ T(y) = 0$ whenever $x \circ y = 0$) is a weighted Jordan homomorphism. We also mention more recent papers [30, 31] which are close to [32] and also involve weighted Jordan homomorphisms. Further, Alaminos, Brešar, Extremera, and Villena [3] proved that if A and B are C^*-algebras and $T : A \to B$ is a continuous linear map with the property that for all $x, y \in A$,

$$xy = yx = 0 \implies T(x) \circ T(y) = 0, \tag{5.1}$$

then T is a weighted Jordan homomorphism. The proof was based on the theory of zero product determined algebras which is surveyed in the recent book [24].

In Section 5.2, we show that a surjective additive map $T \colon A \to B$ satisfying (5.1) is a weighted Jordan homomorphism provided that the ring A is additively spanned by Jordan products of its idempotents and B is any ring with $\frac{1}{2}$ (Theorem 5.2.3). The condition on idempotents is fulfilled in any matrix ring $M_n(R)$ with $n \geq 2$, so this theorem yields a generalization and completion of the aforementioned result of [32] (Corollary 5.2.4). In the first step of the proof of Theorem 5.2.3, which is similar to that in [3], we reduce the problem to the situation where there exists an additive map $S : A \to B$ such that

$$S(x \circ y) = T(x) \circ T(y) \quad (x, y \in A). \tag{5.2}$$

In the second step, which is based on elementary but tricky calculations, we show that (5.2) implies that T is a weighted Jordan homomorphism.

The condition (5.2) is a simple, natural generalization of the condition that a map is a (weighted) Jordan homomorphism, and we find it interesting in its own right. Our interest also stems from the recent paper [25] in which this condition unexpectedly occurred when studying problems that are rather unrelated to those in this paper. We therefore believe that (5.2) deserves a systematic treatment. In Section 5.3, we show that if A is a prime ring with $\mathrm{char}(A) \neq 2, 3, 5$ and $T : A \to A$ is a bijective additive map for which there exists an additive map $S : A \to A$ such that (5.2) holds, then T is a weighted Jordan homomorphism (Theorem 5.3.8). The proof is more complex than the proof in Section 5.2. It combines the results from the theory of functional identities, the theory of polynomial identities, the classical structure theory or rings, and linear algebra.

5.2 Maps satisfying $xy = yx = 0 \implies T(x) \circ T(y) = 0$

The proof of the main theorem of this section depends on some ideas presented in the book [24]. However, we cannot refer directly to the results in this book since it is (mostly) written in the context of algebras over fields while we wish to work in the context of rings. The following is the ring version of Theorem 2.15 (in conjunction with Proposition 1.3) and Theorem 3.23 (in conjunction with Remark 3.24) from [24].

Proposition 5.2.1. *Let A be a unital ring, let B be an additive group, and let $\varphi : A \times A \to B$ be a biadditive map. If A is generated as a ring by idempotents, then:*

(a) *If $\varphi(x, y) = 0$ whenever $x, y \in A$ are such that $xy = 0$, then $\varphi(x, y) = \varphi(xy, 1)$ for all $x, y \in A$.*

(b) *If φ is symmetric and $\varphi(x, y) = 0$ whenever $x, y \in A$ are such that $xy = yx = 0$, then $2\varphi(x, y) = \varphi(x \circ y, 1)$ for all $x, y \in A$.*

The proof of (a) is literally the same as the proof of Lemma 2.2 and Theorem 2.3 from [24]. Using (a), one can prove (b) by simply following the proof of Theorem 3.23 from [24]. (We will actually need only (b), but we stated also (a) to explain the proof of (b)).

We continue with a simple lemma which will be also needed in the next section.

Lemma 5.2.2. *Let A and B be unital rings and let $T : A \to B$ be a surjective additive map satisfying*

$$2T(x) \circ T(y) = T(x \circ y) \circ c \quad (x, y \in A), \tag{5.3}$$

where $c = T(1)$. Assume that B is 2-torsion free (i.e., $2b = 0$ with $b \in B$ implies $b = 0$) and denote the center of B by Z. The following conditions are equivalent:

(i) T is a weighted Jordan homomorphism.

(ii) $c^2 \in Z$.

(iii) $c \in Z$.

Proof. (i) $\implies$ (ii). This is a consequence of the definition of a weighted Jordan homomorphism.

(ii) $\implies$ (iii). Let $b \in A$ be such that $T(b) = 1$. Using (5.3) we see that $c^2 \in Z$ implies

$$4[T(x), c] = [T(x \circ b) \circ c, c] = [T(x \circ b), c^2] = 0 \quad (x \in A)$$

(here, as usual, $[x, y]$ stands for $xy - yx$). Since T is surjective and B is 2-torsion free, $c \in Z$ follows.

(iii) $\implies$ (i). Asssuming that $c \in Z$ it follows from (5.3) that $4 = 4T(b^2)c$. As B is 2-torsion free this shows that c is invertible with $c^{-1} = T(b^2)$. Since (5.3) implies that $T(x) \circ T(y) = cT(x \circ y)$, T is a weighted Jordan homomorphism. $\square$

We will say that a ring A is *additively spanned by Jordan products of its idempotents* if A is equal to its additive subgroup generated by elements of the form $e \circ f$ where e and f are idempotents. By saying that B is a ring with $\frac{1}{2}$ we mean that $1 + 1$ is an invertible element in B; such a ring is of course 2-torsion free. (We remark that under the assumption that $\frac{1}{2} \in B$, the condition (5.3) is equivalent to the condition (5.2) pointed out in Section 5.1, see the beginning of Section 5.3).

We are now ready to state the main result of this section.

Theorem 5.2.3. *Let A and B be unital rings. Assume that A is additively spanned by Jordan products of its idempotents and assume that $\frac{1}{2} \in B$. If $T : A \to B$ is a surjective additive map such that for all $x, y \in A$,*

$$xy = yx = 0 \implies T(x) \circ T(y) = 0, \tag{5.4}$$

then T is a weighted Jordan homomorphism.

Proof. Define $\varphi : A \times A \to B$ by
$$\varphi(x, y) = T(x) \circ T(y).$$
Note that φ is symmetric and that (5.4) shows that $\varphi(x, y) = 0$ whenever $xy = yx = 0$. Since our assumption on A in particular implies that A is generated by idempotents, it follows from Proposition 5.2.1 (b) that $2\varphi(x, y) = \varphi(x \circ y, 1)$ for all $x, y \in A$. That is, (5.3) holds (where $c = T(1)$). In other words, we have
$$W(T(x \circ y)) = T(x) \circ T(y) \quad (x, y \in A), \tag{5.5}$$
where $W\colon B \to B$ is defined by
$$W(x) = \frac{1}{2} x \circ c.$$
Setting $x = y$ in (5.5) we obtain
$$W(T(x^2)) = T(x)^2 \quad (x \in A). \tag{5.6}$$
From $y = \frac{1}{2}\big((y + 1)^2 - y^2 - 1^2\big)$ we see that B is additively spanned by squares of its elements. Therefore, W is surjective since T is surjective. Further, (5.5) shows that
$$W(W(T(x \circ y))) = W(T(x) \circ T(y)) = \frac{1}{2}(T(x) \circ T(y)) \circ c$$
and hence
$$[W(W(T(x \circ y))), c] = \frac{1}{2}[T(x) \circ T(y), c^2] \quad (x, y \in A). \tag{5.7}$$

Let $e \in A$ be an idempotent. By (5.6),
$$\frac{1}{2} T(e) \circ c = W(T(e)) = W(T(e^2)) = T(e)^2. \tag{5.8}$$
This implies that $[T(e), T(e) \circ c] = 0$, i.e., $[T(e)^2, c] = 0$. Hence, (5.8) shows that $[T(e) \circ c, c] = 0$, i.e.,
$$[T(e), c^2] = 0.$$
Together with (5.7), this yields
$$[W(W(T(e \circ f))), c] = 0$$
for all idempotents e and f. Since W and T are surjective, our assumption on A implies that c belongs to the center of B. The desired conclusion that T is a weighted Jordan homomorphism now follows from Lemma 5.2.2. □

The following corollary generalizes [32, Theorem 1.1]; in particular, it shows that the assumption that $n \geq 4$ in this theorem is redundant.

Corollary 5.2.4. *Let R be a unital ring with $\frac{1}{2}$ and let $A = M_n(R)$, $n \geq 2$. If a surjective additive map $T : A \to A$ satisfies (5.4) (in particular, if T preserves zero Jordan products), then T is a weighted Jordan homomorphism.*

Proof. By e_{ij} we denote the standard matrix units and by xe_{ij} the matrix whose (i,j) entry is $x \in R$ all other entries are 0. Of course, each e_{ii} is an idempotent. Let $i \neq j$. Note that

$$xe_{ij} + e_{ii} \quad \text{and} \quad x(e_{ii} + e_{ji}) + (1-x)(e_{ij} + e_{jj})$$

are idempotents and

$$\begin{aligned} xe_{ij} &= (xe_{ij} + e_{ii}) - e_{ii}, \\ xe_{ii} &= \frac{1}{2}\big(\big(x(e_{ii} + e_{ji}) + (1-x)(e_{ij} + e_{jj})\big) \circ e_{ii} - xe_{ji} - (1-x)e_{ij}\big). \end{aligned}$$

Since $2e = e \circ e$ for every idempotent e and since $\frac{1}{2} \in R$ (and so $xe_{ij} = 2(\frac{1}{2}xe_{ij})$) it follows that A is additively spanned by Jordan products of idempotents. Thus, Theorem 5.2.3 applies. $\square$

Remark 5.2.5. The assumption that $\frac{1}{2} \in R$ cannot be removed. Indeed, if R is a ring with $\mathrm{char}(R) = 2$, then any map $T : A \to A$ of the form $T(x) = x + \lambda(x)1$, where $\lambda : A \to Z$ is an additive map, satisfies (5.4). It is easy to find examples where such a map is surjective but is not a weighted Jordan homomorphism.

Remark 5.2.6. Every Jordan homomorphism on a matrix ring $M_n(R)$ is the sum of a homomorphism and an antihomomorphism [54]. The result of Corollary 5.2.4 can thus be stated as that $T(x) = c\big(\Phi_1(x) + \Phi_2(x)\big)$ where Φ_1 is a homomorphism and Φ_2 is an antihomomorphism.

The matrix ring $M_n(R)$ is our basic and motivating example of a ring satisfying the condition of Theorem 5.2.3 regarding idempotents. However, there are other examples.

Example 5.2.7. Let A and B be unital rings with $\frac{1}{2}$ that are additively spanned by Jordan products of idempotents. It is an easy exercise to show that the triangular ring $\mathrm{Tri}(A, M, B)$, where M is a unital (A, B)-bimodule, is also additively spanned by Jordan products of idempotents (compare [24, Corollary 2.5]).

5.3 Pairs of maps satisfying $S(x^2) = T(x)^2$

Until further notice, we assume that A is a *unital prime ring* with $\mathrm{char}(A) \neq 2$ and $S, T : A \to A$ are additive maps satisfying

$$S(x^2) = T(x)^2 \quad (x \in A). \tag{5.9}$$

The standard linearization trick shows that (5.9) is equivalent to

$$S(x \circ y) = T(x) \circ T(y) \quad (x, y \in A). \tag{5.10}$$

We assume that T *is bijective*. Our goal is to prove that, under some additional restrictions on $\mathrm{char}(A)$ which will be imposed later, T is a weighted Jordan homomorphism.

The center of A will be denoted by Z. Further, we denote $c = T(1)$ and $b = T^{-1}(1)$. Note that (5.10) shows that

$$2T(x) = S(x \circ b) \quad (x \in A).$$

and

$$2S(x) = T(x) \circ c \quad (x \in A), \tag{5.11}$$

and that (5.10) and (5.11) yield

$$T(x \circ y) \circ c = 2T(x) \circ T(y) \quad (x, y \in A). \tag{5.12}$$

Thus, (5.9) is just a small variation of the condition (5.12) that was already studied in the preceding section. Under the presence of the element $\frac{1}{2}$, the two conditions are equivalent.

We need some more notation. By C we denote the *extended centroid* C of A. Recall that C is a field containing the center Z (see [23, Section 7.5] for details). Let $x \in A$. We write $\deg(x) = n$ if x is algebraic of degree n over C, and $\deg(x) = \infty$ if x is not algebraic over C. Set $\deg(A) = \sup\{\deg(x) \mid x \in A\}$. It is well known that the condition that $\deg(A) < \infty$ is equivalent to the condition that A is a PI-ring.

Our first lemma was essentially proved in [32]. More precisely, noticing that (5.10) implies

$$T(y) \circ T(xyx) = T(x) \circ T(yxy)$$

we see that this lemma is evident from the proof of [32, Theorem 2.4] along with a basic result on functional identities which states that a prime ring A is a d-free subset of $Q = Q_{ml}(A)$, the maximal left ring of quotients of A, if and only if $\deg(A) \geq d$ [26, Corollary 5.12]. Therefore, we state it without proof.

Lemma 5.3.1. *If* $\deg(A) \geq 4$*, then* T *is a weighted Jordan homomorphism.*

It should be emphasized that Lemma 5.3.1 covers the case where $\deg(A) = \infty$. We thus only need to consider the case where A is a PI-ring with $\deg(A) < 4$. The $\deg(A) = 1$ case is trivial.

Lemma 5.3.2. *If* $\deg(A) = 1$*, then* T *is a weighted Jordan homomorphism.*

Proof. The condition that $\deg(A) = 1$ means that A is commutative, so $c \in Z$ automatically holds and we may apply Lemma 5.2.2. □

Hence, there are only two cases left: $\deg(A) = 2$ and $\deg(A) = 3$. The rings that remain to be considered are thus very specific. However, for problems that can be solved by means of functional identities, the low degree situations are usually the more difficult ones.

In our next lemma we will not yet need the degree restriction. Its proof is also based on functional identities. The reader is referred to [26] for the explanation of some notions that will be used.

Lemma 5.3.3. *There exists a ring monomorphism $\mu : Z \to Z$ such that*

$$T(zx) = \mu(z)T(x) \quad (z \in Z, x \in A),$$

$$S(zx) = \mu(z)S(x) \quad (z \in Z, x \in A).$$

Proof. Fix $z \in Z$. Since $zx \circ y = x \circ zy$ for all $x, y \in A$ it follows from (5.10) that $T(zx) \circ T(y) = T(x) \circ T(zy)$, that is,

$$T(zx)T(y) - T(zy)T(x) + T(y)T(zx) - T(x)T(zy) = 0 \quad (x, y \in A). \tag{5.13}$$

In view of Lemma 5.3.2, the lemma is trivial if A is commutative. We may thus assume that A is not commutative, which implies that it is a 2-free subset of Q [26, Corollary 5.12]. Hence, applying [26, Theorem 4.3] to (5.13) we see that there exist uniquely determined $p_1, p_2 \in Q$ and maps $\lambda_1, \lambda_2 : A \to C$ such that

$$T(zx) = T(x)p_1 + \lambda_1(x) \quad (x \in A), \tag{5.14}$$
$$-T(zy) = T(y)p_2 + \lambda_2(y) \quad (y \in A), \tag{5.15}$$
$$T(zx) = -p_2T(x) - \lambda_1(x) \quad (x \in A), \tag{5.16}$$
$$-T(zy) = -p_1T(y) - \lambda_2(y) \quad (y \in A). \tag{5.17}$$

Comparing (5.14) and (5.17) we obtain

$$T(x)p_1 - p_1T(x) = \lambda_2(x) - \lambda_1(x) \in C \quad (x \in A).$$

Again using the 2-freeness of A along with [26, Theorem 4.3] it follows that $\lambda_1 = \lambda_2$ and $p_1 \in C$ (similarly (5.15) and (5.16) show that $p_2 \in C$). Next, comparing (5.14) and (5.15) we obtain

$$T(x)(p_1 + p_2) = -\big(\lambda_1(x) + \lambda_2(x)\big) \in C \quad (x \in A),$$

and so, again by the 2-freeness and [26, Theorem 4.3], $\lambda_1 + \lambda_2 = 0$ (and $p_1 = -p_2$). Since we have shown above that $\lambda_1 = \lambda_2$ and since $\operatorname{char}(A) \neq 2$ by assumption, it follows that $\lambda_1 = 0$. Thus, $T(zx) = T(x)p_1$ for all $x \in A$. As above, let $b \in A$ be such that $T(b) = 1$. From $p_1 = T(b)p_1 = T(zb)$ we see that $p_1 \in C \cap A = Z$. Defining $\mu(z) = p_1 \in Z$ we thus have $T(zx) = \mu(z)T(x)$ for all $x \in A$. From (5.11) we see that this immediately implies that $S(zx) = \mu(z)S(x)$ holds too.

For any $z_1, z_2 \in Z$, we have

$$\mu(z_1 + z_2) = T\big((z_1 + z_2)b\big) = T(z_1b) + T(z_2b) = \mu(z_1) + \mu(z_2)$$

and

$$\mu(z_1z_2) = T(z_1z_2b) = \mu(z_1)T(z_2b) = \mu(z_1)\mu(z_2),$$

so μ is a ring endomorphism. Since T is injective we see from $\mu(z) = T(zb)$ that μ is injective too. □

From now on we assume that $\deg(A)$ equals 2 or 3. In particular, $\deg(A) < \infty$, which implies that $Q_Z(A)$, the ring of central quotients of A, is a finite-dimensional central simple algebra over the field of quotients of Z. This is the content of Posner's Theorem, see [23, Theorem 7.58]. The elements of the ring $Q_Z(A)$ can be written as $z^{-1}x$ where $z \in Z \setminus \{0\}$ and $x \in A$.

Lemma 5.3.4. *There exist additive maps $\mathcal{S}, \mathcal{T} \colon Q_Z(A) \to Q_Z(A)$ such that $\mathcal{S}|_A = S$, $\mathcal{T}|_A = T$, and $\mathcal{S}(q^2) = \mathcal{T}(q)^2$ for every $q \in Q_Z(A)$.*

Proof. For any $z \in Z \setminus \{0\}$ and $x \in A$, define

$$\mathcal{S}(z^{-1}x) = \mu(z)^{-1}S(x).$$

Assume that $z, z' \in Z \setminus \{0\}$ and $x, x' \in A$ are such that $z^{-1}x = z'^{-1}x'$. Then $z'x = zx'$ and hence $\mu(z')S(x) = \mu(z)S(x')$, that is, $\mu(z)^{-1}S(x) = \mu(z')^{-1}S(x')$. This shows that $\mathcal{S}$ is well-defined. It is clear that $\mathcal{S}|_A = S$. Let $z, w \in Z \setminus \{0\}$ and $x, y \in A$. Then

$$\begin{aligned} \mathcal{S}(z^{-1}x + w^{-1}y) &= \mathcal{S}\big((zw)^{-1}(wx + zy)\big) = \mu(zw)^{-1}S(wx + zy) \\ =&\mu(z)^{-1}\mu(w)^{-1}\big(\mu(w)S(x) + \mu(z)S(y)\big) = \mathcal{S}(z^{-1}x) + \mathcal{S}(w^{-1}y), \end{aligned}$$

so $\mathcal{S}$ is additive.

Similarly we see that

$$\mathcal{T}(z^{-1}x) = \mu(z)^{-1}T(x)$$

is a well-defined additive map which extends T. Finally,

$$\mathcal{S}\big((z^{-1}x)^2\big) = \mathcal{S}(z^{-2}x^2) = \mu(z^2)^{-1}S(x^2) = \mu(z)^{-2}T(x)^2 = \mathcal{T}(z^{-1}x)^2,$$

which proves that $\mathcal{S}(q^2) = \mathcal{T}(q)^2$ for every $q \in Q_Z(A)$. □

Lemma 5.3.4 shows that there is no loss of generality in assuming that $A = Q_Z(A)$ is a central simple algebra such that $\deg(A) = 2$ or $\deg(A) = 3$, or equivalently, $\dim_Z(A) = 4$ or $\dim_Z(A) = 9$ (see [26, Theorem C.2]). Furthermore, in light of Corollary 5.2.4 we may assume that A is not a ring of $n \times n$ matrices, $n \geq 2$, over some ring, and hence, by the classical Wedderburn's structure theorem, we may assume that *A is a division ring.*

Lemma 5.3.5. *If $\deg(A) = 2$, then T is a weighted Jordan homomorphism.*

Proof. Since $\deg(A) = 2$, there exist an additive map $\tau : A \to Z$ and a biadditive map $\delta : A^2 \to Z$ such that

$$x^2 = \tau(x)x + \delta(x, x) \tag{5.18}$$

for every $x \in A$ (see [26, Corollary C.3]). We may assume that δ is symmetric, since otherwise we replace it by $(x, y) \mapsto \frac{1}{2}\big(\delta(x, y) + \delta(y, x)\big)$. Linearizing (5.18) we obtain that for all $x, y \in A$,

$$x \circ y = \tau(x)y + \tau(y)x + 2\delta(x, y).$$

From (5.11) we thus see that

$$2S(x) = \tau(T(x))c + \tau(c)T(x) + 2\delta(T(x), c) \tag{5.19}$$

for all $x \in A$. Next, applying S to (5.18) and using (5.9) we obtain

$$T(x)^2 = \mu(\tau(x))S(x) + \mu\big(\delta(x, x)\big)c^2.$$

Applying (5.18) and (5.19) we see that this can be rewritten as

$$\begin{aligned} &\tau(T(x))T(x) + \delta(T(x), T(x)) \\ =&\frac{1}{2}\mu(\tau(x))\tau(T(x))c + \frac{1}{2}\mu(\tau(x))\tau(c)T(x) \\ &+ \mu(\tau(x))\delta(T(x), c) + \mu\big(\delta(x, x)\big)c^2. \end{aligned} \tag{5.20}$$

Commuting this identity with c we obtain

$$\Big(\tau(T(x)) - \frac{1}{2}\mu(\tau(x))\tau(c)\Big)[T(x), c] = 0$$

for all $x \in A$. Accordingly, for each $x \in A$ we have either

$$\tau(T(x)) = \frac{1}{2}\mu(\tau(x))\tau(c) \tag{5.21}$$

or $[T(x), c] = 0$. The set of all $x \in A$ that satisfy one of these two conditions is an additive subgroup of A. Since a group cannot be the union of two proper subgroups, one of the two conditions must hold for every $x \in A$. If $[T(x), c] = 0$ for every $x \in A$, then $c \in Z$ and so T is a weighted Jordan homomorphism by Lemma 5.2.2. We may thus assume that (5.21) holds for every $x \in A$.

Note that (5.20) along with $c^2 = \tau(c)c + \delta(c, c)$ now shows that

$$\Big(\frac{1}{2}\mu(\tau(x))\tau(T(x)) + \mu\big(\delta(x, x)\big)\tau(c)\Big)c \in Z.$$

Therefore, either $c \in Z$ or

$$\frac{1}{2}\mu(\tau(x))\tau(T(x)) + \mu\big(\delta(x, x)\big)\tau(c) = 0$$

for every $x \in A$. Assume that the latter holds. By (5.21), we can rewrite this identity as

$$\Big(\frac{1}{4}\mu(\tau(x)^2) + \mu\big(\delta(x, x)\big)\Big)\tau(c) = 0. \tag{5.22}$$

If $\tau(c) = 0$ then it follows from (5.21) that $\tau(T(x)) = 0$ and hence $T(x)^2 \in Z$ for every $x \in A$. Since T is surjective, this means that $y^2 \in Z$ for every $y \in A$, which leads to a contradiction that $y = \frac{1}{2}\big((y + 1)^2 - y^2 - 1\big) \in Z$ for every $y \in A$. Thus, $\tau(c) \neq 0$ and so (5.22) implies that

$$\frac{1}{4}\mu(\tau(x)^2) + \mu\big(\delta(x, x)\big) = 0$$

for every $x \in A$. Since μ is injective it follows that

$$\frac{1}{4}\tau(x)^2 = -\delta(x,x).$$

Together with (5.18) this yields

$$\big(x - \frac{1}{2}\tau(x)\big)^2 = x^2 - \tau(x)x + \frac{1}{4}\tau(x)^2 = x^2 - \tau(x)x - \delta(x,x) = 0.$$

Since, as mentioned before the statement of the lemma, we may assume that A is a division ring, this implies that $x = \frac{1}{2}\tau(x) \in Z$ for every $x \in A$, a contradiction. Therefore, $c \in Z$ and so T is a weighted Jordan homomorphism by Lemma 5.2.2. $\square$

The case where $\deg(A) = 3$ is more involved. To handle it, we need the following linear algebra lemma.

Lemma 5.3.6. *Let K be an algebraically closed field with* $\mathrm{char}(K) \neq 2, 3$. *Let $t \in M_3(K)$. Then the set*

$$S_y = \{t, t \circ y, (t \circ y) \circ y, ((t \circ y) \circ y) \circ y\}$$

is linearly dependent for every $y \in M_3(K)$ if and only if t is a scalar matrix.

Proof. The "if" part follows from the Cayley-Hamilton Theorem. To prove the "only if" part, assume that t is not a scalar matrix. Our goal is to find a $y \in A$ such that the set S_y is linearly independent. Since K is algebraically closed, we may assume that t is in the Jordan normal form.

We divide the proof into four cases.

1. Assume that

$$t = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

where $\lambda \in K$. If

$$y = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

then $t \circ y$, $(t \circ y) \circ y$, $(t \circ y) \circ y) \circ y$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 2\lambda & 2 & 0 \\ 0 & 2\lambda & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 4\lambda & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6 & 0 & 0 \end{pmatrix},$$

respectively. It is easy to check that S_y is linearly independent.

2. Assume that
$$t = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$
where $\lambda_1, \lambda_2 \in K$. If
$$y = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
then $t \circ y$, $(t \circ y) \circ y$, $(t \circ y) \circ y) \circ y$ are
$$\begin{pmatrix} 1 & 0 & 0 \\ 2\lambda_1 & 1 & 0 \\ 0 & \lambda_1 + \lambda_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3\lambda_1 + \lambda_2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix},$$
respectively. Again, it is easy to see that S_y is linearly independent.

3. Assume that
$$t = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$
where $\lambda_1, \lambda_2, \lambda_3 \in K$ are not all equal and $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$. If
$$y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
then $t \circ y$, $(t \circ y) \circ y$, $(t \circ y) \circ y) \circ y$ are
$$\begin{pmatrix} 0 & 0 & \lambda_1 + \lambda_3 \\ \lambda_1 + \lambda_2 & 0 & 0 \\ 0 & \lambda_2 + \lambda_3 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & \lambda_1 + \lambda_2 + 2\lambda_3 & 0 \\ 0 & 0 & 2\lambda_1 + \lambda_2 + \lambda_3 \\ \lambda_1 + 2\lambda_2 + \lambda_3 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 2\lambda_1 + 3\lambda_2 + 3\lambda_3 & 0 & 0 \\ 0 & 3\lambda_1 + 2\lambda_2 + 3\lambda_3 & 0 \\ 0 & 0 & 3\lambda_1 + 3\lambda_2 + 2\lambda_3 \end{pmatrix},$$
respectively. A slightly more tedious but still elementary argument shows that S_y is linearly independent in this case too.

4. We now consider the last remaining case where
$$t = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix}$$

with $\lambda_1, \lambda_2 \in K$ and $\lambda_1 \neq 0$. If

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

then $t \circ y$, $(t \circ y) \circ y$, $(t \circ y) \circ y) \circ y$ are

$$\begin{pmatrix} 2\lambda_1 & 0 & 0 \\ \lambda_1 + \lambda_2 & 0 & 0 \\ 0 & -\lambda_1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 4\lambda_1 & 0 & 0 \\ 3\lambda_1 + \lambda_2 & 0 & 0 \\ \lambda_2 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 8\lambda_1 & 0 & 0 \\ 7\lambda_1 + \lambda_2 & 0 & 0 \\ 3\lambda_1 + 2\lambda_2 & 0 & 0 \end{pmatrix},$$

respectively. One easily checks that S_y is linearly independent.

We have thus proved that for each t that is not a scalar matrix there is a matrix y such that S_y is linearly independent. □

We are now ready to tackle the $\deg(A) = 3$ case.

Lemma 5.3.7. *If* $\deg(A) = 3$ *and* $\operatorname{char}(A)$ *is also different from* 3 *and* 5*, then* T *is a weighted Jordan homomorphism.*

Proof. We start similarly as in the proof of Lemma 5.3.5. As $\deg(A) = 3$, there exist symmetric multiadditive maps $\tau\colon A \to Z$, $\alpha\colon A^2 \to Z$ and $\delta\colon A^3 \to Z$ such that

$$x^3 = \tau(x)x^2 + \alpha(x,x)x + \delta(x,x,x)$$

for every $x \in A$, with $\delta(x,x,x)$ being the reduced norm of x. Since, as pointed out above, Corollary 5.2.4 enables us to assume that A is a division ring, $\delta(x,x,x) \neq 0$ for every nonzero $x \in A$. In particular,

$$\gamma = \mu\big(\delta(b,b,b)\big) \neq 0$$

(here, as always, $b = T^{-1}(1)$).

Let $x \in A$. From (5.12) we see that $T(x^2) \circ T(x) = T(x^3) \circ c$. Hence,

$$\begin{aligned} T(x^2) \circ T(x) =& T\big(\tau(x)x^2 + \alpha(x,x)x + \delta(x,x,x)\big) \circ c \\ =& \mu(\tau(x))T(x^2) \circ c + \mu(\alpha(x,x))T(x) \circ c \\ &+ \mu(\delta(x,x,x))c \circ c. \end{aligned}$$

Since, again by (5.12), $T(x^2) \circ c = T(x) \circ T(x)$, it follows that

$$\big(T(x^2) - \mu(\tau(x))T(x) - \mu(\alpha(x,x))c\big) \circ T(x) = 2\mu(\delta(x,x,x))c^2. \tag{5.23}$$

Define $f\colon A^2 \to A$ by

$$f(x,y) = \frac{1}{2}T(x \circ y) - \frac{1}{2}\mu(\tau(x))T(y) - \frac{1}{2}\mu(\tau(y))T(x) - \mu(\alpha(x,y))c.$$

Observe that f is a symmetric biadditive map which, by (5.23), satisfies

$$f(x,x) \circ T(x) = 2\mu(\delta(x,x,x))c^2. \tag{5.24}$$

Linearizing this identity we obtain

$$f(x,y) \circ T(z) + f(z,x) \circ T(y) + f(y,z) \circ T(x) = 6\mu(\delta(x,y,z))c^2. \tag{5.25}$$

By (5.24),

$$f(b,b) = \frac{1}{2}f(b,b) \circ T(b) = \mu(\delta(b,b,b))c^2 = \gamma c^2.$$

Putting $y = z = b$ in (5.25) we obtain

$$4f(x,b) + \gamma c^2 \circ T(x) = 6\mu(\delta(x,b,b))c^2.$$

Next, applying (5.25) with $y = x$ and $z = b$ we arrive at

$$f(x,x) + f(x,b) \circ T(x) = 3\mu(\delta(x,x,b))c^2.$$

The last two identities yield

$$f(x,x) = \Big(\frac{\gamma}{4}c^2 \circ T(x) - \frac{3}{2}\mu(\delta(x,b,b))c^2\Big) \circ T(x) + 3\mu(\delta(x,x,b))c^2.$$

Returning to (5.24), we now have

$$\begin{aligned}\frac{\gamma}{4}\big((c^2 \circ T(x)) \circ T(x)\big) \circ T(x) - \frac{3}{2}\mu(\delta(x,b,b))(c^2 \circ T(x)) \circ T(x)& \\ +3\mu(\delta(x,x,b))c^2 \circ T(x) = 2\mu(\delta(x,x,x))c^2.&\end{aligned}$$

Since $\gamma \neq 0$ and T is surjective, this shows that for each $y \in A$, the set

$$\{c^2, c^2 \circ y, (c^2 \circ y) \circ y, ((c^2 \circ y) \circ y) \circ y\}$$

is linearly dependent. We will now use the fact known from the theory of polynomial identities that the linear dependence can be characterized through a special identity, see [23, Theorem 7.45]. Denoting by c_4 the 4th *Capelli polynomial*, this theorem implies that

$$c_4\big(c^2, c^2 \circ y, (c^2 \circ y) \circ y, ((c^2 \circ y) \circ y) \circ y, x_1, x_2, x_3\big) = 0$$

for all $y, x_1, x_2, x_3 \in A$. Since c_4 is multilinear, the linearization of this identity gives

$$\sum_{\sigma \in S_6} c_4\big(c^2, c^2 \circ y_{\sigma(1)}, (c^2 \circ y_{\sigma(2)}) \circ y_{\sigma(3)}, ((c^2 \circ y_{\sigma(4)}) \circ y_{\sigma(5)}) \circ y_{\sigma(6)}, x_1, x_2, x_3\big) = 0$$

for all $x_i, y_j \in A$, $i = 1, 2, 3$, $j = 1, \dots, 6$. Let K be the algebraic closure of Z and let $\overline{A} = K \otimes_Z A$. Since each x_i and each y_j occurs linearly in the last identity, it follows that $t = 1 \otimes c^2 \in \overline{A}$ satisfies

$$\sum_{\sigma \in S_6} c_4\big(t, t \circ y_{\sigma(1)}, (t \circ y_{\sigma(2)}) \circ y_{\sigma(3)}, ((t \circ y_{\sigma(4)}) \circ y_{\sigma(5)}) \circ y_{\sigma(6)}, x_1, x_2, x_3\big) = 0$$

for all $x_i, y_j \in \overline{A}$, $i = 1, 2, 3$, $j = 1, \dots, 6$. Take each y_i to be equal to y. Our characteristic assumption implies that $6!u = 0$ with $u \in \overline{A}$ implies $u = 0$, so we have

$$c_4\big(t, t \circ y, (t \circ y) \circ y, ((t \circ y) \circ y) \circ y, x_1, x_2, x_3\big) = 0$$

for all $x_i, y \in A$, $i = 1, 2, 3$. We may now again use [23, Theorem 7.45], this time in the opposite direction, to conclude that the set

$$\{t, t \circ y, (t \circ y) \circ y, ((t \circ y) \circ y) \circ y\}$$

is linearly dependent for every $y \in \overline{A}$. Since $\overline{A} \cong M_3(K)$ [23, Theorem 4.39], Lemma 5.3.6 shows that t lies in the center of $\overline{A}$. Consequently, $c^2 \in Z$ and so Lemma 5.2.2 tells us that T is a weighted Jordan homomorphism. □

We can now state the main result of this section.

Theorem 5.3.8. *Let A be a prime ring with* $\mathrm{char}(A) \neq 2, 3, 5$ *and let $S, T : A \to A$ be additive maps such that $S(x^2) = T(x)^2$ for every $x \in A$. If T is bijective, then it is a weighted Jordan homomorphism.*

Proof. Apply Lemmas 5.3.1, 5.3.2, 5.3.5, and 5.3.7. □

Remark 5.3.9. The conclusion of Theorem 5.3.8 is that $T(x) = c\Phi(x)$ where Φ is a Jordan automorphism of A. It is well known that, since A is prime, Φ is either an automorphism or an antiautomorphism [50].

Remark 5.3.10. The injectivity of T was used only once in the proof of Theorem 5.3.8, that is, when showing that μ is injective. If Z is a field, in particular if A is simple, then μ is automatically injective and so we may weaken the assumption that T is bijective to T being only surjective.

In our final result we return to the condition studied in Section 5.2.

Corollary 5.3.11. *Let A be a unital simple ring with* $\mathrm{char}(A) \neq 2, 3, 5$. *If A contains a nontrivial idempotent, then every surjective additive map $T : A \to A$ such that for all $x, y \in A$,*

$$xy = yx = 0 \implies T(x) \circ T(y) = 0,$$

is a weighted Jordan homomorphism.

Proof. It is well known that the existence of one nontrivial idempotent in a simple ring A implies that A is generated by idempotents [51, Corollaries on p. 9 and p. 18]. We can therefore repeat the argument from the beginning of the proof of Theorem 5.2.3 and conclude that T satisfies condition (5.12), which is of course an equivalent version of the condition $S(x^2) = T(x)^2$ studied in Theorem 5.3.8. As pointed out in Remark 5.3.10, in this setting the injectivity of T is not needed for reaching the conclusion that T is a weighted Jordan homomorphism. $\square$

Theorems 5.2.3 and 5.3.8 show that in quite general rings, weighted Jordan homomorphisms are the only bijective additive maps T with the property that $S(x^2) = T(x)^2$ for some additive map S. We conclude the paper with an example showing that there exist rings in which this does not hold.

Example 5.3.12. Let A be the Grassmann algebra in two generators over a field F with $\mathrm{char}(F) \neq 2$. That is, A is the 4-dimensional algebra with basis $1, u, v, uv$ where $u^2 = v^2 = uv + vu = 0$. For each $x \in A$, let $\lambda(x)$ be the element in F satisfying $x - \lambda(x)1 \in \mathrm{span}\{u, v, uv\}$. Note that $x \mapsto \lambda(x)$ is an algebra homomorphism from A to F and that $x \circ u = 2\lambda(x)u$ for every $x \in A$. Define $S, T : A \to A$ by

$$T(x) = x + \lambda(x)u, \quad S(x) = x + 2\lambda(x)u.$$

Then S and T are linear maps, T is bijective, and

$$\begin{aligned} S(x^2) =& x^2 + 2\lambda(x^2)u = x^2 + 2\lambda(x)^2 u \\ =& x^2 + \lambda(x)(x \circ u) = (x + \lambda(x)u)^2 = T(x)^2 \end{aligned}$$

for every $x \in A$. However, $T(1) = 1 + u$ is not a central element, so T is not a weighted Jordan homomorphism.

Chapter 6

Maps preserving two-sided zero products on Banach algebras

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Abstract. Let A and B be Banach algebras with bounded approximate identities and let $\Phi\colon A \to B$ be a surjective continuous linear map which preserves two-sided zero products (i.e., $\Phi(a)\Phi(b) = \Phi(b)\Phi(a) = 0$ whenever $ab = ba = 0$). We show that Φ is a weighted Jordan homomorphism provided that A is zero product determined and weakly amenable. These conditions are in particular fulfilled when A is the group algebra $L^1(G)$ with G any locally compact group. We also study a more general type of continuous linear maps $\Phi\colon A \to B$ that satisfy $\Phi(a)\Phi(b) + \Phi(b)\Phi(a) = 0$ whenever $ab = ba = 0$. We show in particular that if Φ is surjective and A is a C^*-algebra, then Φ is a weighted Jordan homomorphism.

6.1 Introduction

Let A and B be Banach algebras. We will say that a linear map $\Phi\colon A \to B$ *preserves two-sided zero products* if for all $a, b \in A$,

$$ab = ba = 0 \implies \Phi(a)\Phi(b) = \Phi(b)\Phi(a) = 0. \tag{6.1}$$

Obvious examples of such maps are homomorphisms and antihomomorphisms. Their common generalizations are *Jordan homomorphisms*, i.e., linear maps $\Psi\colon A \to B$ satisfying

$$\Psi(a \circ b) = \Psi(a) \circ \Psi(b) \quad \forall a, b \in A,$$

where $a \circ b$ stands for the Jordan product $ab + ba$. Under the mild assumption that the centre of B does not contain nonzero nilpotent elements, every Jordan homomorphism from A onto B also preserves two-sided zero products [24, Lemma 7.20]. Next we recall that a linear map $W\colon B \to B$ is called a *centralizer* if

$$W(ab) = W(a)b = aW(b) \quad \forall a, b \in B.$$

We say that Φ is a *weighted Jordan homomorphism* if there exist an invertible centralizer W of B and a Jordan homomorphism Ψ from A to B such that $\Phi = W\Psi$. Observe that Φ preserves two-sided zero products if and only if Ψ does. We also remark that, by the closed graph theorem, every centralizer W is automatically continuous if B is a faithful algebra (i.e., $bB = Bb = \{0\}$ implies $b = 0$), and so, in this case, Φ is continuous if and only if Ψ is.

Is every surjective continuous linear map $\Phi\colon A \to B$ which preserves two-sided zero products a weighted Jordan homomorphism? This question is similar to but, as it turns out, more difficult than a more thoroughly studied question of describing zero products preserving continuous linear maps (see the most recent publications [24, 41, 42, 57, 63] for historical remarks and references). It is known that the answer is positive if either A and B are C^*-algebras [3, Theorem 3.3] or if $A = L^1(G)$ and $B = L^1(H)$ where G and H are locally compact groups with $G \in$ [SIN] (i.e., G has a base of compact neighborhoods of the identity that is invariant under all inner automorphisms) [14, Theorem 3.1 (i)]. In fact, [3, Theorem 3.3] does not require that Φ satisfies (6.1) but only that for all $a, b \in A$,

$$ab = ba = 0 \implies \Phi(a) \circ \Phi(b) = 0. \tag{6.2}$$

This condition was also considered in the recent algebraic paper [27]. Observe also that it is more general than the condition that Φ preserves zero Jordan products ($a \circ b = 0$ implies $\Phi(a) \circ \Phi(b) = 0$) studied in [32].

The goal of this paper is to generalize and unify the aforementioned results from [3] and [14]. Our approach is based on the concept of a *zero product determined* Banach algebra. These are Banach algebras A with the property that every continuous bilinear functional $\varphi\colon A \times A \to \mathbb{C}$ satisfying $\varphi(a, b) = 0$ whenever $ab = 0$ is of the form $\varphi(a, b) = \tau(ab)$ for some continuous linear functional τ on A. We refer to the recent book [24] for a survey of these algebras. Let us for now only mention that they form a fairly large class of Banach algebras whose main representatives, C^*-algebras and group algebras of locally compact groups, are also weakly amenable Banach algebras having bounded approximate identities.

In Section 6.2, we show that the answer to our question is positive, i.e., a surjective continuous linear map $\Phi\colon A \to B$ which preserves two-sided zero products is a weighted Jordan homomorphism, provided that A is zero product determined and weakly amenable, and, additionally, both A and B have bounded approximate identities (Theorem 6.2.5). This in particular implies that that the restriction in [14, Theorem 3.1 (i)] that $G \in$ [SIN] is redundant (Corollary 6.2.7).

Section 6.3 is devoted to condition (6.2). We show that [3, Theorem 3.3] still holds if B is any Banach algebra with a bounded approximate identity, not only a C^*-algebra (Theorem 6.3.3). Our second main result regarding (6.2) considers the case where $A = \mathcal{A}(X)$ is the algebra of approximable operators (Theorem 6.3.4).

6.2 Maps preserving two-sided zero products

Throughout, for a Banach space X, we write X^* for the dual of X and $\langle \cdot , \cdot \rangle$ for the duality between X and X^*. Let A be a Banach algebra. We turn A^* into a Banach A-bimodule by letting

$$\langle b, a \cdot \omega \rangle = \langle ba, \omega \rangle, \quad \langle b, \omega \cdot a \rangle = \langle ab, \omega \rangle \quad \forall a, b \in A, \ \forall \omega \in A^*.$$

The space of continuous derivations from A into A^* is denoted by $\mathcal{Z}^1(A, A^*)$. The main representatives of $\mathcal{Z}^1(A, A^*)$ are the so-called inner derivations. The inner derivation implemented by $\omega \in A^*$ is the map $\delta_\omega \colon A \to A^*$ defined by

$$\delta_\omega(a) = a \cdot \omega - \omega \cdot a \quad \forall a \in A.$$

The Banach algebra is called *weakly amenable* if every element of $\mathcal{Z}^1(A, A^*)$ is inner. For a thorough treatment of this property and an account of many interesting examples of weakly amenable Banach algebras we refer the reader to [36]. We should remark that the group algebra $L^1(G)$ of each locally compact group G and each C^*-algebra are weakly amenable [36, Theorems 5.6.48 and 5.6.77].

Our first result is a sharpening of [5, Theorem 2.7] (see also [24, Theorem 6.6]).

Theorem 6.2.1. *Let A be a Banach algebra, and suppose that:*

(a) A is zero product determined;

(b) A has a bounded approximate identity;

(c) A is weakly amenable.

Then there exists a constant $C \in \mathbb{R}^+$ such that for each continuous bilinear functional $\varphi \colon A \times A \to \mathbb{C}$ with the property that for all $a, b \in A$,

$$ab = ba = 0 \implies \varphi(a, b) = 0 \tag{6.3}$$

there exist $\sigma, \tau \in A^*$ such that

$$\|\sigma\| \le C\|\varphi\|, \quad \|\tau\| \le C\|\varphi\|,$$

and

$$\varphi(a, b) = \sigma(ab) + \tau(ba) \quad \forall a, b \in A.$$

Proof. Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for A of bound M.

Since A is weakly amenable, the map $\omega \mapsto \delta_\omega$ from A^* to the Banach space $\mathcal{Z}^1(A, A^*)$ is a continuous linear surjection, and so, by the open mapping theorem, there exists a constant $N \in \mathbb{R}^+$ such that, for each $D \in \mathcal{Z}^1(A, A^*)$, there exists an $\omega \in A^*$ with

$$\|\omega\| \le N\|D\|$$

and

$$a \cdot \omega - \omega \cdot a = D(a) \quad \forall a \in A.$$

Towards the proof of the theorem, we proceed through a detailed inspection of the proof of [5, Theorem 2.7].

Define $\varphi_1, \varphi_2 \colon A \times A \to \mathbb{C}$ by

$$\varphi_1(a,b) = \tfrac{1}{2}\big[\varphi(a,b) + \varphi(b,a)\big],$$
$$\varphi_2(a,b) = \tfrac{1}{2}\big[\varphi(a,b) - \varphi(b,a)\big] \quad \forall a,b \in A.$$

It is clear that both φ_1 and φ_2 satisfy condition (6.3) and that

$$\|\varphi_1\| \leq \|\varphi\|, \quad \|\varphi_2\| \leq \|\varphi\|.$$

Since φ_1 is symmetric it follows from the last assertion of [5, Lemma 2.6] (or [24, Theorem 6.1]) that there exists a $\xi \in A^*$ such that

$$2\varphi_1(a,b) = \langle a \circ b, \xi\rangle \quad \forall a,b \in A. \tag{6.4}$$

For each $a \in A$, we observe that

$$\langle a \circ e_\lambda, \xi\rangle = 2\varphi_1(a, e_\lambda) \quad \forall \lambda \in \Lambda$$

and hence that

$$|\langle a \circ e_\lambda, \xi\rangle| \leq 2M\|\varphi\|\|a\| \quad \forall \lambda \in \Lambda.$$

Taking limit and using $\lim_{\lambda\in\Lambda} a \circ e_\lambda = 2a$ we see that

$$2\langle a, \xi\rangle \leq 2M\|\varphi\|\|a\|.$$

This shows that

$$\|\xi\| \leq M\|\varphi\|. \tag{6.5}$$

Our next concern will be the behaviour of the skew-symmetric functional φ_2. By [4, Lemma 4.1] (or [24, Theorem 6.1]), there exists a $\psi \in A^*$ such that

$$\varphi_2(ab,c) - \varphi_2(b,ca) + \varphi_2(bc,a) = \langle abc, \psi\rangle \quad \forall a,b,c \in A.$$

The proof reveals that the functional ψ is defined by

$$\langle a, \psi\rangle = \lim_{\lambda\in\Lambda} \varphi_2(a, e_\lambda) \quad \forall a \in A,$$

so that

$$\|\psi\| \leq M\|\varphi\|. \tag{6.6}$$

By [5, Lemma 2.6] (or the proof of [24, Theorem 6.5]) , the map

$$D\colon A \to A^*, \quad \langle b, D(a)\rangle = \varphi_2(a,b) + \frac{1}{2}\langle a \circ b, \psi\rangle$$

is a continuous derivation, and clearly (using (6.6))

$$\|D\| \leq \|\varphi_2\| + \|\psi\| \leq (1+M)\|\varphi\|.$$

Consequently, there exists an $\omega \in A^*$ such that

$$\|\omega\| \leq N\|D\| \leq N(1+M)\|\varphi\| \tag{6.7}$$

and

$$a \cdot \omega - \omega \cdot a = D(a) \quad \forall a \in A,$$

and hence

$$\langle ba - ab, \omega\rangle - \varphi_2(a,b) = \tfrac{1}{2}\langle a \circ b, \psi\rangle \quad \forall a, b \in A.$$

Viewing this expression as a bilinear functional on $A \times A$, we see that the left-hand side is skew-symmetric and the right-hand side is symmetric. Therefore, both sides are zero. Thus

$$\varphi_2(a,b) = \langle ba - ab, \omega\rangle \quad \forall a, b \in A. \tag{6.8}$$

We then define

$$\sigma = \tfrac{1}{2}\xi - \omega, \quad \tau = \tfrac{1}{2}\xi + \omega.$$

From (6.5) and (6.7) we see that

$$\|\sigma\| \leq (\tfrac{1}{2}M + N + NM)\|\varphi\|,$$

$$\|\tau\| \leq (\tfrac{1}{2}M + N + NM)\|\varphi\|,$$

and from (6.4) and (6.8) we deduce that

$$\varphi(a,b) = \sigma(ab) + \tau(ba) \quad \forall a, b \in A. \qquad \square$$

We can now start our consideration of maps preserving two-sided zero products.

Lemma 6.2.2. *Let A be a Banach algebra, and suppose that:*

(a) A is zero product determined;

(b) A has a bounded approximate identity;

(c) A is weakly amenable.

Let B be a Banach algebra and let $\Phi\colon A \to B$ be a continuous linear map which preserves two-sided zero products. Then there exist:

- *a closed left ideal L of B containing $\Phi(A)$ and a continuous linear map $U\colon L \to B$,*
- *a closed right ideal R of B containing $\Phi(A)$ and a continuous linear map $V\colon R \to B$*

such that

$$U(xy) = xU(y), \quad V(zx) = V(z)x \quad \forall x \in B,\ \forall y \in L,\ \forall z \in R,$$

$$U(\Phi(a)) = V(\Phi(a)) \quad \forall a \in A,$$

and

$$U(\Phi(a \circ b)) = V(\Phi(a \circ b)) = \Phi(a) \circ \Phi(b) \quad \forall a, b \in A.$$

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for A of bound M, and let C be the constant given in Theorem 6.2.1.

We define

$$L = \Big\{x \in B\colon \text{ the net } \big(x\Phi(e_\lambda)\big)_{\lambda\in\Lambda} \text{ is convergent}\Big\},$$

$$U\colon L \to B, \quad U(x) = \lim_{\lambda\in\Lambda} x\Phi(e_\lambda) \quad \forall x \in L,$$

and

$$R = \Big\{x \in B\colon \text{ the net } \big(\Phi(e_\lambda)x\big)_{\lambda\in\Lambda} \text{ is convergent}\Big\},$$

$$V\colon R \to B, \quad V(x) = \lim_{\lambda\in\Lambda} \Phi(e_\lambda)x \quad \forall x \in R.$$

It is clear that L is a left ideal of B, R is a right ideal of B, and routine verifications, using that the net $(\Phi(e_\lambda))_{\lambda\in\Lambda}$ is bounded, show that both L and R are closed subspaces of B. It is also obvious that both U and V are continuous linear maps with $\|U\| \le M\|\Phi\|$ and $\|V\| \le M\|\Phi\|$ and that

$$U(xy) = xU(y), \quad V(zx) = V(z)x \quad \forall x \in B,\ \forall y \in L,\ \forall z \in R.$$

In the remainder of this proof we will verify that

$$\Phi(A) \subset L \cap R, \tag{6.9}$$

$$U(\Phi(a)) = V(\Phi(a)) \quad \forall a \in A, \tag{6.10}$$

and

$$U(\Phi(a \circ b)) = V(\Phi(a \circ b)) = \Phi(a) \circ \Phi(b) \quad \forall a, b \in A. \tag{6.11}$$

For this purpose, we proceed to show that

$$\big(\Phi(a^2)\Phi(e_\lambda)\big)_{\lambda\in\Lambda} \to \Phi(a)^2 \quad \forall a \in A. \tag{6.12}$$

Fix an $a \in A$. By the Hahn-Banach theorem, for each $\lambda \in \Lambda$ there exists a $\xi_\lambda \in B^*$ with $\|\xi_\lambda\| = 1$ and

$$\big\langle \Phi(a^2)\Phi(e_\lambda) - \Phi(a)^2, \xi_\lambda \big\rangle = \big\|\Phi(a^2)\Phi(e_\lambda) - \Phi(a)^2\big\|. \tag{6.13}$$

For each $\lambda \in \Lambda$, we consider the continuous bilinear functional

$$\varphi_\lambda\colon A \times A \to \mathbb{C}, \quad \varphi_\lambda(u, v) = \langle \Phi(u)\Phi(v), \xi_\lambda\rangle \quad \forall u, v \in A,$$

which clearly satisfies (6.3) and

$$\|\varphi_\lambda\| \leq \|\Phi\|^2.$$

Hence Theorem 6.2.1 yields the existence of $\sigma_\lambda, \tau_\lambda \in A^*$ such that

$$\|\sigma_\lambda\| \leq C\|\Phi\|^2, \quad \|\tau_\lambda\| \leq C\|\Phi\|^2,$$

and

$$\langle \Phi(u)\Phi(v), \xi_\lambda \rangle = \sigma_\lambda(uv) + \tau_\lambda(vu) \quad \forall u, v \in A. \tag{6.14}$$

From (6.13) and (6.14) we deduce that

$$\begin{aligned}\|\Phi(a^2)\Phi(e_\lambda) - \Phi(a)^2\| &= \langle \Phi(a^2)\Phi(e_\lambda), \xi_\lambda \rangle - \langle \Phi(a)^2, \xi_\lambda \rangle \\ &= \sigma_\lambda(a^2 e_\lambda) + \tau_\lambda(e_\lambda a^2) - \sigma_\lambda(a^2) - \tau_\lambda(a^2) \\ &= \sigma_\lambda(a^2 e_\lambda - a^2) + \tau_\lambda(e_\lambda a^2 - a^2) \quad \forall \lambda \in \Lambda.\end{aligned} \tag{6.15}$$

We now observe that

$$\begin{aligned}|\sigma_\lambda(a^2 e_\lambda - a^2)| &\leq C\|\Phi\|^2 \|a^2 e_\lambda - a^2\|, \\ |\tau_\lambda(e_\lambda a^2 - a^2)| &\leq C\|\Phi\|^2 \|e_\lambda a^2 - a^2\| \quad \forall \lambda \in \Lambda,\end{aligned}$$

and so, taking limits and using that

$$\lim_{\lambda \in \Lambda} \|a^2 e_\lambda - a^2\| = \lim_{\lambda \in \Lambda} \|e_\lambda a^2 - a^2\| = 0,$$

we see that

$$\lim_{\lambda \in \Lambda} \sigma_\lambda(a^2 e_\lambda - a^2) = \lim_{\lambda \in \Lambda} \tau_\lambda(e_\lambda a^2 - a^2) = 0.$$

Taking limit in (6.15) we now deduce that

$$\lim_{\lambda \in \Lambda} \|\Phi(a^2)\Phi(e_\lambda) - \Phi(a)^2\| = 0,$$

which gives (6.12).

Of course, (6.12) gives

$$\Phi(a^2) \in L, \quad U(\Phi(a^2)) = \Phi(a)^2 \quad \forall a \in A. \tag{6.16}$$

In the same way as (6.12) one proves that

$$\big(\Phi(e_\lambda)\Phi(a^2)\big)_{\lambda \in \Lambda} \to \Phi(a)^2 \quad \forall a \in A,$$

which clearly yields

$$\Phi(a^2) \in R, \quad V(\Phi(a^2)) = \Phi(a)^2 \quad \forall a \in A. \tag{6.17}$$

From (6.16) and (6.17) we deduce immediately that

$$\Phi(a \circ b) \in L \cap R \quad \forall a, b \in A \tag{6.18}$$

and that (6.11) holds. Finally we are in a position to verify (6.9) and (6.10). Take $a \in A$. By [1, Theorem II.16], there exist $b, c \in A$ such that $a = bcb$, and hence

$$a = \tfrac{1}{2} b \circ (b \circ c) - \tfrac{1}{2} b^2 \circ c.$$

From (6.18) we deduce that

$$\Phi(a) = \tfrac{1}{2}\Phi(b \circ (b \circ c)) - \tfrac{1}{2}\Phi(b^2 \circ c) \in L \cap R,$$

and (6.9) is proved. Further, from (6.11) we see that

$$U(\Phi(a)) = \tfrac{1}{2} U(\Phi(b \circ (b \circ c))) - \tfrac{1}{2} U(\Phi(b^2 \circ c)) =$$

$$\tfrac{1}{2} V(\Phi(b \circ (b \circ c))) - \tfrac{1}{2} V(\Phi(b^2 \circ c)) = V(\Phi(a)),$$

which gives (6.10). □

Lemma 6.2.3. *Let A and B be Banach algebras, and suppose that:*

(a) A is zero product determined;

(b) A has a bounded approximate identity;

(c) A is weakly amenable;

(d) B is faithful.

Let $\Phi\colon A \to B$ be a continuous linear map having dense range and preserving two-sided zero products. Then there exists an injective continuous centralizer $W\colon B \to B$ such that

$$W(\Phi(a \circ b)) = \Phi(a) \circ \Phi(b) \quad \forall a, b \in A.$$

Proof. We apply Lemma 6.2.2. Since Φ has dense range, it follows that $L = R = B$ and that $U = V$. Set $W = U\,(= V)$. Then W is a centralizer on B and

$$W(\Phi(a \circ b)) = \Phi(a) \circ \Phi(b) \quad \forall a, b \in A.$$

The only point remaining concerns the injectivity of W. We claim that

$$\ker W B^3 = B^3 \ker W = \{0\}. \tag{6.19}$$

Let $x \in \ker W$. For each $a \in A$, we have

$$0 = W(x)\Phi(a^2) = W(x\Phi(a^2)) = xW(\Phi(a^2)) = x\Phi(a)^2,$$

and, since the range of Φ is dense, we arrive at

$$xy^2 = 0 \quad \forall y \in B.$$

We thus get

$$x(yz + zy) = 0 \quad \forall x \in \ker W,\ \forall y, z \in B.$$

For all $x \in \ker W$ and $y, z, w \in B$ we have (using that $xz \in \ker W$)

$$(xyz)w = (-xzy)w = -(xz)yw = (xz)wy = x(zw)y = -xy(zw),$$

whence $xyzw = 0$, and so $\ker W B^3 = \{0\}$. Similarly we see that $B^3 \ker W = \{0\}$. Thus, (6.19) holds.

It is an elementary exercise to show that an element b in a faithful algebra B satisfying $bB^3 = B^3 b = \{0\}$ must be 0. Indeed, one first observes that every $c \in B^2 b B^2$ satisfies $cB = Bc = \{0\}$, which yields $B^2 b B^2 = \{0\}$. Similarly we see that this implies $B^2 b B = B b B^2 = \{0\}$, hence $BbB = B^2 b = bB^2 = \{0\}$, and finally $bB = Bb = \{0\}$. Therefore, $b = 0$.

Thus, (6.19) shows that $\ker W = \{0\}$. $\square$

Lemma 6.2.4. *Let A and B be Banach algebras, and suppose that B has a bounded approximate identity. Let $\Phi\colon A \to B$ be a surjective linear map, and let $W\colon B \to B$ be a linear map such that*

$$W(\Phi(a \circ b)) = \Phi(a) \circ \Phi(b) \quad \forall a, b \in A.$$

Then W is surjective.

Proof. Set $x \in B$. By [1, Theorem II.16], there exist $y, z \in B$ such that $x = yzy$, so that

$$x = \tfrac{1}{2} y \circ (y \circ z) - \tfrac{1}{2} y^2 \circ z.$$

Since Φ is surjective, we can choose $a, b, c, d \in A$ with

$$\Phi(a) = y, \quad \Phi(b) = y \circ z, \quad \Phi(c) = y^2, \quad \Phi(d) = z.$$

The condition on W now gives

$$\begin{aligned} W\big(\Phi(\tfrac{1}{2} a \circ b - \tfrac{1}{2} c \circ d)\big) &= \tfrac{1}{2}\Phi(a) \circ \Phi(b) - \tfrac{1}{2}\Phi(c) \circ \Phi(d) \\ &= \tfrac{1}{2} y \circ (y \circ z) - \tfrac{1}{2} y^2 \circ z = x. \end{aligned}$$

$\square$

We are now ready to establish our main result.

Theorem 6.2.5. *Let A and B be Banach algebras, and suppose that:*

(a) A is zero product determined;

(b) A has a bounded approximate identity;

(c) A is weakly amenable;

(d) B has a bounded approximate identity.

Let $\Phi\colon A \to B$ be a surjective continuous linear map which preserves two-sided zero products. Then Φ is a weighted Jordan homomorphism.

Proof. Since B has a bounded approximate identity, it follows that B is faithful. We conclude from Lemma 6.2.3 that there exists an injective continuous centralizer $W\colon B \to B$ such that

$$W(\Phi(a \circ b)) = \Phi(a) \circ \Phi(b) \quad \forall a, b \in A. \tag{6.20}$$

Lemma 6.2.4 now shows that W is surjective.

Having proved that W is an invertible centralizer, we can define $\Psi = W^{-1}\Phi$ which is a surjective continuous linear map and, further, we deduce from (6.20) that Ψ is a Jordan homomorphism. Of course, $\Phi = W\Psi$. □

The crucial examples of zero product determined Banach algebras are the group algebras $L^1(G)$ for each locally compact group G and C^*-algebras [24, Theorems 5.19 and 5.21]. Furthermore, these Banach algebras are also weakly amenable and have bounded approximate identities. Therefore, it is legitimate to apply Theorem 6.2.5 in the case where A is a group algebra or a C^*-algebra.

Corollary 6.2.6. *Let G be a locally compact group, let B be a Banach algebra having a bounded approximate identity, and let $\Phi\colon L^1(G) \to B$ be a surjective continuous linear map which preserves two-sided zero products. Then Φ is a weighted Jordan homomorphism.*

Our final corollary generalizes [14, Theorem 3.1 (i)].

Corollary 6.2.7. *Let G and H be locally compact groups, and let $\Phi\colon L^1(G) \to L^1(H)$ be a surjective continuous linear map which preserves two-sided zero products. Then there exist a surjective continuous Jordan homomorphism $\Psi\colon L^1(G) \to L^1(H)$ and an invertible central measure $\mu \in M(H)$ such that $\Phi(f) = \mu * \Psi(f)$ for each $f \in L^1(G)$.*

Proof. By Corollary 6.2.6, there exist an invertible centralizer W of $L^1(H)$ and a surjective continuous Jordan homomorphism $\Psi\colon L^1(G) \to L^1(H)$ such that $\Phi = W\Psi$. The centralizer W can be thought of as an element of the centre of the multiplier algebra of $L^1(H)$ which is, by Wendel's Theorem (see [36, Theorem 3.3.40]), isomorphic to the measure algebra $M(H)$. This gives a measure $\mu \in M(H)$ as required. □

6.3 Maps satisfying $ab = ba = 0 \implies \Phi(a) \circ \Phi(b) = 0$

We will not discuss Theorem 6.2.5 in the case where A is a C^*-algebra, because in this case condition (6.1) can be weakened to condition (6.2). Showing this is the main purpose of this section.

Lemma 6.3.1. *Let A be a Banach algebra, and suppose that:*

(a) A is zero product determined;

(b) A has a bounded approximate identity.

Let B be a Banach algebra and let $\Phi\colon A \to B$ be a continuous linear map satisfying condition (6.2). Then there exist a closed linear subspace J of B containing $\Phi(A)$ and a continuous linear map $W\colon J \to B$ such that

$$W(\Phi(a \circ b)) = \Phi(a) \circ \Phi(b) \quad \forall a, b \in A.$$

Moreover, if B has a bounded approximate identity and Φ is surjective, then W is a surjective map from B onto itself.

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for A of bound M.

We define

$$J = \Big\{x \in B\colon \text{ the net } \big(\Phi(e_\lambda) \circ x\big)_{\lambda\in\Lambda} \text{ is convergent}\Big\}$$

and

$$W\colon J \to B, \quad W(x) = \lim_{\lambda\in\Lambda} \tfrac{1}{2}\Phi(e_\lambda) \circ x \quad \forall x \in J.$$

It is clear that J is a linear subspace of B and routine verifications, using that the net $(\Phi(e_\lambda))_{\lambda\in\Lambda}$ is bounded, show that J is a closed linear subspace of B. It is also obvious that W is a continuous linear map with $\|W\| \leq M\|\Phi\|$.

Applying [24, Theorem 6.1 and Remark 6.2] to the continuous bilinear map $\varphi\colon A\times A \to B$ defined by

$$\varphi(a, b) = \Phi(a) \circ \Phi(b) \quad \forall a, b \in A$$

we see that there exists a continuous linear map $S\colon A \to B$ such that

$$\Phi(a) \circ \Phi(b) = S(a \circ b) \quad \forall a, b \in A. \tag{6.21}$$

For each $a \in A$, we thus have

$$\Phi(e_\lambda) \circ \Phi(a) = S(e_\lambda \circ a) \quad \forall \lambda \in \Lambda.$$

Using $\lim_{\lambda\in\Lambda} e_\lambda \circ a = 2a$ and the continuity of S, we see by taking limit that

$$\lim_{\lambda\in\Lambda} \Phi(e_\lambda) \circ \Phi(a) = 2S(a).$$

This shows that $\Phi(a) \in J$ and that $W(\Phi(a)) = S(a)$. On the other hand, using (6.21), we see that

$$W(\Phi(a \circ b)) = S(a \circ b) = \Phi(a) \circ \Phi(b) \quad \forall a, b \in A. \tag{6.22}$$

Of course, if Φ is surjective, then $J = B$. Now suppose that, in addition, B has a bounded approximate identity. Then, on account of (6.22), Lemma 6.2.4 shows that W is surjective. ☐

Lemma 6.3.2. *Let A and B be a Banach algebras, let $\Phi\colon A \to B$ be a continuous linear map, and let $\omega \in B$. Suppose that:*

(a) A is the closed linear span of its idempotents.

(b) $\Phi(a^2) \circ \omega = 2\Phi(a)^2$ *for each* $a \in A$.

Then $\omega^2\Phi(a) = \Phi(a)\omega^2$ *for each* $a \in A$.

Proof. Let $e \in A$ be an idempotent. From (b) we see that

$$\omega\Phi(e) + \Phi(e)\omega = \Phi(e) \circ \omega = \Phi(e^2) \circ \omega = 2\Phi(e)^2. \tag{6.23}$$

By multiplying (6.23) by $\Phi(e)$ on the left we obtain

$$\Phi(e)\omega\Phi(e) + \Phi(e)^2\omega = 2\Phi(e)^3 \tag{6.24}$$

and multiplying by $\Phi(e)$ on the right we get

$$\omega\Phi(e)^2 + \Phi(e)\omega\Phi(e) = 2\Phi(e)^3. \tag{6.25}$$

From (6.24) and (6.25) we arrive at $\omega\Phi(e)^2 = \Phi(e)^2\omega$, which, on account of (6.23), yields

$$\omega^2\Phi(e) = \Phi(e)\omega^2.$$

Since A is the closed linear span of its idempotents, it follows that

$$\omega^2\Phi(a) = \Phi(a)\omega^2 \quad \forall a \in A.$$ □

In the proof of the next results we will use the first Arens product on the second dual A^{**} of a Banach algebra A. We will denote this product by juxtaposition. Furthermore, we will use the following basic facts about the weak* continuity of the first Arens product which the reader can find in [36].

(A1) For each $a \in A$, the map $\xi \mapsto a\xi$ from A^{**} to itself is weak* continuous.

(A2) For each $\xi \in A^{**}$, the map $\zeta \mapsto \zeta\xi$ from A^{**} to itself is weak* continuous.

(A3) If A is a C^*-algebra, then the product in A^{**} is separately weak* continuous.

Theorem 6.3.3. *Let* A *be a* C^**-algebra, let* B *be a Banach algebra having a bounded approximate identity, and let* $\Phi\colon A \to B$ *be a surjective continuous linear map such that for all* $a, b \in A$,

$$ab = ba = 0 \implies \Phi(a) \circ \Phi(b) = 0.$$

Then Φ *is a weighted Jordan homomorphism.*

Proof. By Lemma 6.3.1 there exists a surjective continuous linear map $W\colon B \to B$ such that

$$W(\Phi(a \circ b)) = \Phi(a) \circ \Phi(b) \quad \forall a, b \in A. \tag{6.26}$$

We write $\Phi^{**}\colon A^{**} \to B^{**}$ and $W^{**}\colon B^{**} \to B^{**}$ for the second duals of the continuous linear maps $\Phi\colon A \to B$ and $W\colon B \to B$, respectively. We claim that

$$W^{**}(\Phi^{**}(x \circ y)) = \Phi^{**}(x) \circ \Phi^{**}(y) \quad \forall x, y \in A^{**}. \tag{6.27}$$

Set $x, y \in A^{**}$, and take nets $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ in A such that

$$\begin{aligned} &(a_i)_{i\in I} \to x, \\ &(b_j)_{j\in J} \to y \quad \text{in } (A^{**}, \sigma(A^{**}, A^*)). \end{aligned}$$

On account of (6.26), we have

$$W(\Phi(a_i b_j + b_j a_i)) = \Phi(a_i)\Phi(b_j) + \Phi(b_j)\Phi(a_i) \quad \forall i \in I,\ \forall j \in J, \tag{6.28}$$

and the task is now to take the iterated limit $\lim_{j\in J} \lim_{i\in I}$ on both sides of the above equation. Throughout the proof, the limits $\lim_{i\in I}$ and $\lim_{j\in J}$ are taken with respect to the weak* topology. From (A3) and the weak* continuity of both Φ^{**} and W^{**} we deduce that

$$\begin{aligned} \lim_{j\in J} \lim_{i\in I} W^{**}\big(\Phi^{**}(a_i b_j + b_j a_i)\big) &= \lim_{j\in J} W^{**}\big(\Phi^{**}(x b_j + b_j x)\big) \\ &= W^{**}\big(\Phi^{**}(xy + yx)\big). \end{aligned} \tag{6.29}$$

From (A1)-(A2) (applied to the Arens product of B^{**}) and the weak* continuity of Φ^{**} we deduce that

$$\begin{aligned} \lim_{j\in J} \lim_{i\in I} \Phi(b_j)\Phi(a_i) &= \lim_{j\in J} \Phi(b_j)\Phi^{**}(x) \\ &= \Phi^{**}(y)\Phi^{**}(x). \end{aligned} \tag{6.30}$$

The remaining iterated limits

$$\lim_{j\in J} \lim_{i\in I} \Phi(a_i)\Phi(b_j)$$

must be treated with much more care than the previous ones. We regard A as a C^*-algebra acting on the Hilbert space of its universal representation, and we regard the continuous bilinear map

$$A \times A \to B, \quad (a, b) \to \Phi(a)\Phi(b)$$

as a continuous bilinear map with values in the Banach space B^{**} which is separately ultraweak-weak* continuous. By applying [55, Theorem 2.3], we obtain that the bilinear map above extends uniquely, without change of norm, to a continuous bilinear map $\phi\colon A^{**} \times A^{**} \to B^{**}$ which is separately weak* continuous. From this, and using (A1)-(A2) and the weak* continuity of Φ^{**}, we obtain

$$\begin{aligned} \lim_{j\in J} \lim_{i\in I} \Phi(a_i)\Phi(b_j) &= \lim_{j\in J} \lim_{i\in I} \phi(a_i, b_j) \\ &= \phi(x, y) \\ &= \lim_{i\in I} \lim_{j\in J} \phi(a_i, b_j) \\ &= \lim_{i\in I} \lim_{j\in J} \Phi(a_i)\Phi(b_j) \\ &= \lim_{i\in I} \Phi(a_i)\Phi^{**}(y) \\ &= \Phi^{**}(x)\Phi^{**}(y). \end{aligned} \tag{6.31}$$

From (6.28), (6.29), (6.30), and (6.31), it may be concluded that

$$\begin{aligned} W^{**}(\Phi^{**}(x \circ y)) =& \lim_{j\in J} \lim_{i\in I} W(\Phi(a_i \circ b_j)) \\ =& \lim_{j\in J} \lim_{i\in I} \Phi(a_i)\Phi(b_j) + \lim_{j\in J} \lim_{i\in I} \Phi(b_j)\Phi(a_i) \\ =& \Phi^{**}(x) \circ \Phi^{**}(y), \end{aligned}$$

and (6.27) is proved.

Define $\omega = \Phi^{**}(1) \in B^{**}$, where 1 is the unit of the von Neumann algebra A^{**}. Setting $x = y$ in (6.27) we conclude that

$$W^{**}(\Phi^{**}(x^2)) = \Phi^{**}(x)^2 \quad \forall x \in A^{**}, \tag{6.32}$$

and setting $y = 1$ in (6.27) we see that

$$2W^{**}(\Phi^{**}(x)) = \omega\Phi^{**}(x) + \Phi^{**}(x)\omega \quad \forall x \in A^{**}. \tag{6.33}$$

From (6.32) and (6.33) we deduce that

$$\Phi^{**}(x^2) \circ \omega = 2\Phi^{**}(x)^2 \quad \forall x \in A^{**}.$$

Since A^{**} is a von Neumann algebra, it is the closed linear span of its projections and we are in a position to apply Lemma 6.3.2, which gives

$$\omega^2\Phi^{**}(x)) = \Phi^{**}(x)\omega^2 \quad \forall x \in A^{**}.$$

In particular,

$$\omega^2\Phi(a) = \Phi(a)\omega^2 \quad \forall a \in A.$$

Since Φ is surjective, it may be concluded that

$$\omega^2 u = u\omega^2 \quad \forall u \in B.$$

From (6.33) we see that, for each $a \in A$,

$$\begin{aligned} \omega W(\Phi(a)) =& \omega\tfrac{1}{2}\big(\omega\Phi(a) + \Phi(a)\omega\big) \\ =& \tfrac{1}{2}\big(\omega^2\Phi(a) + \omega\Phi(a)\omega\big) \\ =& \tfrac{1}{2}\big(\Phi(a)\omega^2 + \omega\Phi(a)\omega\big) \\ =& \tfrac{1}{2}\big(\Phi(a)\omega + \omega\Phi(a)\big)\omega \\ =& W(\Phi(a))\omega. \end{aligned}$$

Since both Φ and W are surjective, it may be concluded that

$$\omega u = u\omega \quad \forall u \in B. \tag{6.34}$$

From (6.33) we now deduce that

$$W(\Phi(a)) = \tfrac{1}{2}\big(\omega\Phi(a) + \Phi(a)\omega\big) = \omega\Phi(a) \quad \forall a \in A,$$

and hence that

$$W(u) = \omega u \quad \forall u \in B.$$

Furthermore, for all $a, b \in B$, using (6.34) we obtain

$$W(ab) = \omega ab = W(a)b,$$

$$W(ab) = (\omega a)b = (a\omega)b = aW(b),$$

whence W is a centralizer on B. In order to prove that W is an invertible centralizer, it remains to show that W is injective. If $a \in \ker W$, then

$$aB = aW(B) = W(a)B = \{0\}$$

and therefore $a = 0$.

Since W is an invertible centralizer on B, (6.26) shows that $W^{-1}\Phi$ is a Jordan homomorphism, and hence Φ is a weighted Jordan homomorphism. □

Our final concern will be the algebra $\mathcal{A}(X)$ of approximable operators on a Banach space X. It is shown in [2] that $\mathcal{A}(X)$ has the so-called property $\mathbb{B}$ for each Banach space X (see also [24, Example 5.15]). Further, it is known that $\mathcal{A}(X)$ has a bounded left approximate identity if and only if the Banach space X has the bounded approximation property (see [36, Theorem 2.9.37]). In this case, $\mathcal{A}(X)$ is actually a zero product determined Banach algebra (see [2, Lemma 2.3] or, alternatively, [24, Proposition 5.5]). Another remarkable feature of $\mathcal{A}(X)$ is that it has a bounded approximate identity if and only if X^* has the bounded approximation property (see [36, Theorem 2.9.37]).

Theorem 6.3.4. *Let X be a Banach space such that X^* has the bounded approximation property, let B be a Banach algebra having a bounded approximate identity, and let $\Phi\colon \mathcal{A}(X) \to B$ be a surjective continuous linear map such that for all $S, T \in \mathcal{A}(X)$,*

$$ST = TS = 0 \implies \Phi(S) \circ \Phi(T) = 0.$$

Then Φ is a weighted Jordan homomorphism.

Proof. We begin by applying Lemma 6.3.1 to obtain a surjective continuous linear map $W\colon B \to B$ such that

$$W(\Phi(S \circ T)) = \Phi(S) \circ \Phi(T) \quad \forall S, T \in \mathcal{A}(X). \tag{6.35}$$

Let $(E_\lambda)_{\lambda\in\Lambda}$ be a bounded approximate identity for $\mathcal{A}(X)$. Then we regard $\big(\Phi(E_\lambda)\big)_{\lambda\in\Lambda}$ as a bounded net in the second dual B^{**} of B. It follows from the Banach-Alaoglu theorem that this net has a $\sigma(B^{**}, B^*)$-convergent subnet. Hence, by passing to a subnet, we may suppose that $(E_\lambda)_{\lambda\in\Lambda}$ is a bounded approximate identity for $\mathcal{A}(X)$ such that

$$\lim_{\lambda\in\Lambda} \Phi(E_\lambda) = \omega \quad \text{in } (B^{**}, \sigma(B^{**}, B^*))$$

for some $\omega \in B^{**}$.

Set $T \in \mathcal{A}(X)$. Writting E_λ for S in (6.35), we obtain

$$W(\Phi(E_\lambda T + TE_\lambda)) = \Phi(E_\lambda)\Phi(T) + \Phi(T)\Phi(E_\lambda) \quad \forall \lambda \in \Lambda, \tag{6.36}$$

and our next goal is to take limits on both sides of (6.36). Since

$$\lim_{\lambda \in \Lambda}(E_\lambda T + TE_\lambda) = 2T \quad \text{in } (\mathcal{A}(X), \|\cdot\|),$$

the continuity of $W\Phi$ gives

$$\lim_{\lambda \in \Lambda} W(\Phi(E_\lambda T + TE_\lambda)) = 2W(\Phi(T)) \quad \text{in } (B, \|\cdot\|). \tag{6.37}$$

On the other hand, since

$$\lim_{\lambda \in \Lambda} \Phi(E_\lambda) = \omega \quad \text{in } (B^{**}, \sigma(B^{**}, B^*))$$

and $\Phi(T) \in B$, we can appeal to (A1)-(A2) to deduce that

$$\begin{aligned} &\lim_{\lambda \in \Lambda} \Phi(E_\lambda)\Phi(T) = \omega\Phi(T), \\ &\lim_{\lambda \in \Lambda} \Phi(T)\Phi(E_\lambda) = \Phi(T)\omega \quad \text{in } (B^{**}, \sigma(B^{**}, B^*)). \end{aligned} \tag{6.38}$$

Hence, taking limits in (6.36) and using (6.37) and (6.38), we obtain

$$2W(\Phi(T)) = \omega\Phi(T) + \Phi(T)\omega. \tag{6.39}$$

Having (6.35) and (6.39) and using that $\mathcal{A}(X)$ is the closed linear span of its idempotents, we can now apply Lemma 6.3.2 to obtain

$$\omega^2\Phi(T) = \Phi(T)\omega^2 \quad \forall T \in \mathcal{A}(X).$$

From the surjectivity of Φ we deduce that

$$\omega^2 a = a\omega^2 \quad \forall a \in B.$$

By using the same method as in the proof of Theorem 6.3.3 we verify that $\omega a = a\omega$ for each $a \in B$, that $W(a) = \omega a$ for each $a \in A$, and that W is an invertible centralizer on B such that $W^{-1}\Phi$ is a Jordan homomorphism. $\square$

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