Pre-service teachers' challenges in implementing mathematical modelling: Insights into reality

Carolina Guerrero-Ortiz and Rita Borromero Ferri

Modelling has become mandatory in the school curricula in many countries across the globe, often without providing teachers the training needed to address this challenge. With a qualitative case study, we analyzed the tasks designed by secondary mathematics pre-service teachers. We recognized how participants manage their knowledge for teaching modelling in the absence of training. Elements of knowledge for teaching such as translation between languages, recognizing unknown data, covariation, and usefulness of representations for understanding and solving problems, were identified. Our results also reveal that future teachers have a tendency to create word problems when first attempting to teach modelling.

Keywords: Mathematics knowledge for teaching; Mathematics teacher training; Modelling; Word-problems

Los desafíos de los profesores en formación en la implementación de la modelación matemática. Una mirada en torno a la realidad

El modelado se ha vuelto obligatorio en los programas de estudio de muchos países, a menudo sin brindar a los profesores la capacitación necesaria para abordar este desafío. Mediante un estudio cualitativo, analizamos las tareas diseñadas por futuros profesores de matemáticas de secundaria. Identificamos cómo movilizan sus conocimientos para enseñar modelación en ausencia de formación para ello, elementos como traducción entre lenguajes, reconocimiento de datos desconocidos, covariación y uso de representaciones para la comprensión y resolución de problemas, fueron identificados. Los resultados también revelan que los participantes tienden a crear problemas verbales cuando intentan enseñar modelación por primera vez.

Términos clave: Conocimiento del contenido; Formación de profesores de matemáticas; Modelización; Problemas verbales

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The teaching of mathematical modelling is an important topic of discussion that has brought together teachers and researchers at different international conferences, such as the International Conference on the Teaching of Mathematical Modelling and Application (ICTMA) and the Congress of the European Society for Research in Mathematics Education (CERME). Modelling has also been considered one of the main elements of standardized tests such as PISA (Organisation for Economic Co-operation and Development [OECD], 2013), the focus of which is directed towards the development of students' abilities to use their mathematical knowledge and to be able to solve problems that lead to mathematical approaches in context-based situations. Following the Mathematical Literacy framework (OECD, 2013), in different countries, an attempt has been made to introduce modelling in school education. In this context, teacher education programs face a great challenge in helping future and in-service teachers build knowledge and abilities that are required to teach modelling. In many countries, however, the teaching of modelling has not yet been introduced into pre-service teacher training programs, or the universities are currently in the process of modifying their curricula to include it (Borromeo Ferri, 2021). In the case of Chile, in 2013, modelling was incorporated in the school curriculum (*Ministerio de Educación de Chile* [MINEDUC], 2018a), and the universities have advanced slowly, yet most have encountered difficulties when incorporating mathematical modelling as content for teaching in their curricula. In turn, pre-service and in-service teachers are required to teach modelling in their classrooms without possessing the necessary knowledge and training.

In the absence of modelling experience gained through teacher training, practicing teachers and pre-service teachers tend to construct pedagogical and didactical strategies that do not support effective modelling practice, as has been observed in different research and contexts (Andrews & Sayers, 2012; Paolucci & Wessels, 2017). In addition, some errors associated with modelling processes and applications of mathematical content have been observed in teachers who have not been trained in teaching modelling (Moreno et al., 2021). Consequently, effective implementation of modelling in the classrooms remains a challenge for both in-service and pre-service teachers.

In Chile, the learning objectives are defined by the national curriculum (MINEDUC, 2018a). Particularly, in the curriculum, mathematical modelling is considered as a mathematical thinking ability and is understood as a capacity for building a physical or abstract model to capture the characteristics of reality. Mathematical modelling is gradually introduced across five grades (when students are 12−17 years old) and is associated to the learning of algebra and functions. Over the course of these progressive lessons, students are expected to apply models and model situations using several functions (linear, quadratic, exponential). Activities such as making calculations, estimations, and simulations to solve problems; selecting and fitting models to solve problems considering dependencies; and evaluating the model's applicability in relation to the initial problem and its limitations are also mentioned. However, we can still find that some of the tasks suggested for implementation begin with a simplified context, in which data and clue words are given to obtain an equation and address the problem. Moreover, very few opportunities are given to teachers and students to model open, real-world problems and to validate their answers. Some examples can be found in MINEDUC (2016) and MINEDUC (2018b).

This approach to mathematical modelling differs from the approach followed in the international research and discussions of mathematical modelling education, where most researchers agree that mathematical modelling can be described as a transition process from reality to mathematics and vice versa. The importance of modelling as a process that involves several steps for solving a real-life question with the help of mathematics is often neglected in many school curricula across the globe (Borromeo Ferri, 2021). This represents a problem, as when the nature, aims, and goals of modelling in school are not explicitly defined within the curricula, many teachers do not have an idea about how to deal with modelling in the classroom (Borromeo Ferri, 2018). The lack of consensus between the notion of modelling proposed in school curricula and that addressed in mathematics research still remains latent in different parts of the world (e.g., Bolsad, 2020; Borromeo Ferri, 2021; Villa & Ruiz, 2009). Thus, as in-service and pre-service teachers interpret and implement modelling in different ways, some have difficulties in recognizing the phases of a modelling process and in working with open-ended tasks (Oropesa et al., 2018). They also have difficulties in linking curricular content with mathematical modelling problems (Paolucci & Wessels, 2017).

Currently, the topic of teacher education in mathematical modelling is actively discussed, and research in this field is still growing while its high importance is increasingly being recognized within the educational policy (Borromeo Ferri, 2021). Extant empirical and practical evidence concerning how teachers should be educated and trained to implement modelling in school has been very helpful for institutional use. However, those teachers who had the opportunity to take part in teacher training or university seminars for learning and teaching mathematical modelling are still rare, even if several projects have addressed mathematical modelling in the teacher training (some of which are cited in Borromeo Ferri, 2021; Ramos-Rodríguez et al., 2022; Wess et al., 2021). The reality is that most teachers either do not teach a single lesson of modelling or they attempt to implement modelling on the basis of what they have learned about modelling through the school curriculum (Borromeo Ferri & Blum, 2014).

In this paper, we focus on this large proportion of teachers who have not had the chance to participate in a modelling course. Our goal is to elucidate how these pre-service teachers understand the nature of mathematical modelling by analyzing their performance when teaching modelling. This paper contributes to mathematical modelling teacher education by offering information related to three questions guiding this research in the particular case that pre-service teachers have not been trained to teach modelling.

Research Questions

- How do pre-service teachers manage their knowledge to teach modelling?
- Which characteristics of modelling are present in tasks designed by mathematics pre-service teachers?
- What notion do pre-service teachers have about teaching modelling?

The first research question offers information of how pre-service teachers manage their knowledge to teach modelling when they have not been trained for it. In a modelling course a topic that should be included consider the theory and criteria to characterize modelling tasks (Borromeo Ferri, 2018), participants in this research have not had a formal opportunity to develop this knowledge, therefore we explore the characteristics of modelling tasks (Maaß, 2010) that can be evidenced in the design for teaching what they understand by modelling. With this consideration we answer the second research question. Regarding the third research question, according to Guerrero-Ortiz and Reyes-Rodríguez (2021) we interpret the participants' notions about modelling based on what is evidenced in their practice.

Theoretical Background

To understand what is expected from teachers when teaching modelling, it is worth knowing the achievements that have been made in empirical research. In this section, we will emphasize three complementary elements that allow us to analyze mathematics pre-service teacher performance when teaching modelling: the competence of teachers to develop modelling problems for school purposes, the characteristics of the tasks, and teachers' mathematical knowledge for teaching.

Teachers' Competencies for Teaching Mathematical Modelling

As mentioned earlier, mathematical modelling describes the transition processes between reality and mathematics and vice versa (Borromeo Ferri, 2006). Mathematical modelling is initially a complex process for both learners and teachers, because modelling problems is open-ended and the mathematics that can be applied for solving real-life problems is often not as obvious as that presented in mathematics exercise books. Thus, mathematical modelling cannot be expected as a simple transfer, both for learners and teachers, to gain modelling competency (Blum, 2015). Therefore, teachers who possess the competency and the knowledge to offer learners modelling activities are needed. Within the research on teacher education in mathematical modelling, a well-known model for teacher competencies has been described (Borromeo Ferri, 2018; Borromeo Ferri & Blum, 2010; Wess et al., 2021). Through this model, teacher competencies for teaching modelling could be conceptualized and operationalized through the four dimensions shown in Figure 1, which have been empirically grounded (Borromeo Ferri, 2019).

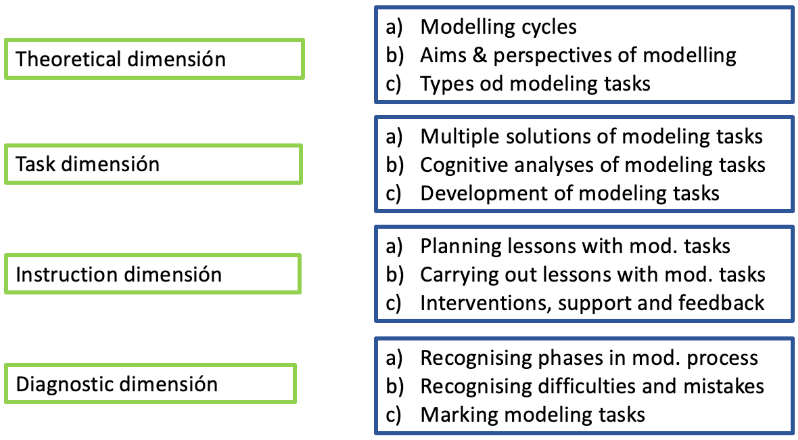

Figure 1. Model for competencies needed in teaching mathematical modelling (Borromeo Ferri, 2018; Borromeo Ferri & Blum, 2010)

The depicted model does not imply that the dimensions are hierarchical, but it is clear that a strong theoretical knowledge base concerning mathematical modelling or the modelling cycle with the phases is very helpful. It is also important for the teacher to know that one can find various types of modelling cycles (Borromeo Ferri, 2006; Doerr et al., 2017; OECD, 2013) and be able to differentiate their use for school and research purposes (Blum et al., 2007). In addition, both the Mathematical Content Knowledge (CK) and Pedagogical Content Knowledge (PCK) are necessary for teaching mathematical modelling. The above model highlights the necessary PCK (Baumert & Kunter, 2013; Wess et al., 2021). Furthermore, gaining extensive experience by solving different modelling problems helps pre-service teachers become more sensitive to how challenging the modelling will be for their learners in school (Borromeo Ferri, 2018). Most importantly, through practice, teachers can expand their knowledge concerning the nature of modelling problems compared with word problems in general. Thus, the task dimension in the above model focuses on promoting teachers' competency to create their own modelling problems.

Characteristics of the Modelling Tasks

There is also theoretical knowledge that can be related to task dimension, as indicated by Maaß (2010) who proposed criteria for a good modelling task. According to the author, modelling problems should be open, complex, authentic, and concrete, and should be solvable with all steps of the modelling cycle. In addition to these criteria, Maaß (2010) developed a classification system for modelling tasks, which is used in our study as an analytic instrument to characterize the tasks created by pre-service teachers. Maaß's classification involves the following features of a task.

Focus of Modelling Activity

In solving the problem, identify if the whole modelling process is required or only single steps. Data, consider which types of data are available (superfluous, missing, inconsistent, matching). Type of relationship to reality, distinguish the nature of the context's relationship to reality, for example if the task is authentic or is embedded in a word problem or in an artificial context. Situation, consider from which situation is the context taken, personal, occupational, public, or scientific. Type of model, distinguish between normative and descriptive models. Openness, consider solved examples, ascertaining task, reversal situation, open problem, etc. Type of representation, identify the type of representation used for present the task. Cognitive demand, consider the modelling and metacognitive modelling competences, extra-mathematical and inner-mathematical work, the use of mathematical representations, the mental objects and the level of mathematical reasoning. Mathematical content, consider the school level and the mathematical area such as arithmetic, algebra, geometry, stochastic calculus, etc. to which the modelling task is related.

The aforementioned classification, and the criteria in particular, should always be a topic in modelling courses (Borromeo Ferri & Blum, 2010) because the development of modelling problems is central to the understanding of what modelling means and how it can be taught in school. This was also confirmed by teacher educators in a recent empirical study (Borromeo Ferri, 2021) and in the results presented by Paolucci and Wessels (2017), which showed that teachers encounter challenges when developing modelling problems despite having participated in a modelling course.

Teacher's Knowledge

Creating a modelling problem is a very complex task. In addition, designing and implementing modelling tasks also requires pre-service teachers to possess mathematical knowledge for teaching (MKT) (Ball et al., 2008), particularly the three subdomains of teacher's knowledge—the Knowledge of Content and Teaching (KCT), Knowledge of Content and Students (KCS), and Specialized Content Knowledge (SCK). These domains were also used in this study as a *theoretical lens* to analyze how pre-service teachers developed modelling activities in the classroom. Mathematical modelling can be understood from two different and complementary perspectives—as content to be taught, and as a means of learning and developing mathematical skills. Therefore, in the classroom, mathematics and modelling can be assumed as content (C) to be taught (Guerrero-Ortiz, 2021). From this perspective, we interpret the MKT subdomains which are complementary with the specific approach to modelling provided by the model for teacher competencies (Borromeo Ferri, 2018). Related to the KCT domain, Ball et al. (2008) mention that tasks developed by teachers require interactions between specific mathematical understanding and understanding of pedagogical issues that affect student learning. KCT considers the mathematical knowledge involved in the instructional design, types of examples, and mathematical representations used to explain an idea. KCS involves the knowledge of students and the knowledge of mathematics; when assigning a task, teachers must be able to anticipate what students are likely to do, how they think, and what they will find confusing. Teachers also must interpret the students' reasoning through their expressions. In the SCK domain, teachers should be able to recognize what is involved in using a particular representation, linking representations to underlying ideas and to other representations, appraising and adapting the mathematical content of textbooks, modifying tasks to be either easier or harder, and asking productive mathematical questions. Sullivan et al. (2015) pointed out that KCT and SCK could be the most critical issues for task design. In addition, when teaching involves modelling tasks, some implications for the pedagogical knowledge of the teachers have been noted (Doerr, 2007; Doerr & English, 2006). Teachers must understand the approaches that students may develop to reach an answer and must listen carefully to the students' descriptions and interpretations of their models. Teachers also need to listen to student approaches that they have not expected in order to understand their meaning and to adequately support them in acquiring a better understanding. To create a rich learning environment, teachers should promote mathematical discussions and encourage students to share their interpretations, explanations, justifications, and judgments related to the qualities of their models. In their recently published work, Wess et al. (2021) organized in the modelling-specific pedagogical content knowledge the facets of knowledge about interventions, modelling processes, modelling tasks, and knowledge about aims and perspectives of mathematical modelling.

Although some empirical evidence indicates that teachers experience difficulties when creating and teaching modelling problems even when they have been trained (e.g., Borromeo Ferri, 2018; Paolucci & Wessels, 2017), there is a lack of research concerning how teachers act and adapt when they have not been trained in teaching and learning mathematical modelling and when they have to solely rely on the curriculum. Considering that teacher training programs are just being modified to include the teaching of modelling, studying pre-service teachers' practices offers insights to the teacher educators on the aspects that must be addressed. These aspects are related to pre-service teachers' understanding of modelling tasks and their development in the classroom.

The first perspective offers a guideline of the elements that teachers should master for teaching modelling and the third perspective allows us to know the mathematical knowledge for teaching involved in the teaching of modelling. These perspectives support the analysis of the results to answer research questions 1 and

3, while question 2 is addressed with support on the classification system for modelling tasks.

METHODS AND DATA ANALYSIS

This research was developed following an interpretative qualitative perspective (Cohen & Manion, 2002), particularly an exploratory case study (Yin, 2014).

Context and Case Study

The study initially involved fourteen secondary pre-service teachers of mathematics, who were all in their final year of study, all of whom were developing the final professional practicum. The teacher training program took place over four years and considered topics associated with mathematics (calculus, linear algebra, complex analysis, number theory, geometry), didactic of mathematics (e.g., didactic of numbers, functions, statistics, and geometry) and pedagogy (curricular planning and assessment, education and diversity, students' learning styles, and professional practicum). At the time of taking part in this study, the final practicum participants had completed more than 90% of the courses. Some elements of mathematical modelling can be found in the courses (closer to the applications of mathematics), but there is no specific course on modelling for the teaching. In particular, in this final professional practicum, the modelling approach utilized by PISA (OECD, 2013) was widely discussed. This had methodological implications in our research because although as researchers we adhered to the definition of modelling given by Borromeo Ferri (2006), in practice the participants only knew the definition of modelling given by the OECD, which had implications for the types of tasks they designed and in the modelling phases observed, as explained below.

The fourteen participants were encouraged to design and implement a modelling task. They freely chose the mathematical content, the learning objectives, and the characteristics of the task according to the grade level they taught. Five of the fourteen pre-service teachers created and implemented teaching tasks aimed to teach simultaneous linear equations and properties of equilateral triangles. The tasks they created involved hypothetical contexts and were closer to word problems (Greer, 1997). The other participants, due to the rules of the school in which they taught at the time[1], were unable to implement the designed tasks (some of them reported difficulties in matching mathematical content in the curriculum with modelling tasks). Thus, the work of only five participants could be considered for analysis. We chose to analyze the work of one participant, Max, in more detail based on the characteristics of the designed tasks and his agreement to participate in the research. He designed and implemented a teaching sequence

[1] In some schools, part of the year is reserved for preparing students to take national standardized tests; therefore, the courses focus on covering that content and developing algorithmic skills.

for a complete teaching unit (simultaneous linear equations and introduction to linear functions). Max's classes were aimed at teaching level eight (13- and 14-year-old students). He developed several activities for ten 90-minute sessions (for further details of the sequence of activities in the teaching unit, see Appendix B).

Prior to the implementation of the teaching activities, we observed each teacher at least once to identify the pedagogical aspects related to their practice. Their classes usually started by presenting the learning goals and reviewing the topic covered the day before, followed by presenting the new topic. Then the students solved problems individually or in groups. At the end of the class, teachers would provide a brief summary of the lesson. The planning and implementation developed by Max is representative of the work conducted by the other participants because Max considered a wide range of activities with different pedagogical methodologies, including some designs similar to those adopted by the remaining participants.

Data Collection and Analysis

The collection of data took place during a semester while Max taught a course of mathematics for secondary students. Classroom observations were the main source of information for data analysis. Ten classes, approximately 900 min in total, in which Max implemented different teaching tasks, were videotaped and transcribed (a summary of the activities developed in each class is shown in Appendix B). Written material such as lesson plans, tests given to students, and a reflective final report were also collected. Two semi-structured interviews were conducted, at the beginning and at the end of the practicum, respectively. The interviews lasted approximately 75 minutes and their aim was to verify our interpretations of Max's actions in class by questioning him about his reasons for implementing the sequence of teaching in the way he did and about his retrospective view about the class (e.g., Why did you organize the class in this way? Why this mathematical content? What are the innovations of your class in relation to what the curriculum proposes?).

The data analysis was completed in three phases, following the content analysis methodology (Bardin, 1986). We analyzed the tasks, lesson plans, and transcripts of the discussions related to sessions S1, S2 and S4. In session S1, Max aimed to introduce linear equations and functions, and to achieve this goal he designed seven tasks for students to work in groups. In S1 and S2, students worked on the tasks (T1-T7) and presented their results to the rest of the class. In session S4, the students worked on problem posing by matching a simple model with a context, and later discussed their results with their peers. These sessions were chosen because they reflect Max's understanding of modelling through his work with students, and the activities developed around the work with the tasks provided evidence of his knowledge for teaching mathematics. The final interview and lesson plan were used to confirm our interpretations from the data analysis of sessions S1, S2, and S4.

First Phase
In using Maaß's framework, we specifically considered word problems as activities that were not modelling tasks but that might support the development of modelling competencies (Maaß, 2010; Verschaffel et al., 2010). In this regard, Maaß specifies that "'reality-related task' stands for all kinds of applications of mathematics in the real world. Modelling refers to the solving of a problem from the real world […]" (p. 287). For the analysis of the tasks, questions or sentences indicating the activities that the students should develop and the characteristics of the contexts of the tasks were recognized in the text. According to the modelling task classification scheme, one of the authors characterized seven tasks for teaching linear equations and functions designed by Max (Table 2).

Second Phase
We analyzed teacher performance in class by reconstructing the teaching trajectories. For teaching trajectories, we mean the sequence of steps that a teacher follows in a class or sequences of classes to achieve the learning objectives. The formulate, employ, and interpret processes identified by the OECD (2013) were used as points of reference to divide the class into episodes. Formulate considers the identification of mathematics, which can be used to solve the problem and involves taking a given situation in a particular context and transforming it into a structure of mathematical representations through the idealization and identification of variables. Employ includes the use of mathematical reasoning, the application of concepts, procedures, and tools to obtain mathematical results, and the mathematical analysis of the information to describe results. Interpret involves reflection on solutions in terms of context and evaluation of the relevance of results in relation to the initial context (OECD, 2013). To differentiate each episode, according to the definition of the processes given by OECD, we observed the type of discussions that Max had with his students in the classroom. For example, if Max's interventions were more related to revealing the students' mathematical reasoning or the application of concepts, the episode was defined as *employing*.

Third Phase
The data were recoded to identify the ways in which Max understood the mathematical content and how he managed his knowledge to promote the use of mathematical representations or turned to different modes to explain an issue to support student learning (as done by Doerr & English, 2006).

For the analysis, the videotapes of each lesson were transcribed and examined. According to the syntactical differences, the transcripts were separated into sentences as units of analysis (the second and the third phase), then some annotations according to the episodes were added to the transcript. In each episode, the relevant mathematical content and interactions between the teacher and the students were characterized by identifying elements of mathematical knowledge for teaching (KCT, KCS, and/or SCK) in each sentence expressed by the teacher.

Finally, in each subdomain of teacher knowledge (Ball et al., 2008), we specifically recognized nuances of the teacher's knowledge for teaching with problems in context by identifying some signs of knowledge related to the work with these problems. We use the letter M to refer specifically to the translation between languages, recognizing unknown data and understanding relations between variables, recognizing the covariation in a situation, and the usefulness of representations for understanding and solving problems. These were coded as KCS_M1 and KCT_M2, KCT_M3 and KCT_M4 in the third phase, as shown in Table 1.

Table 1
Coding Examples

Mathematical knowledge for teaching	Description	Knowledge for teaching with problems in context	Description
KCT	Considers the mathematical knowledge involved in the instructional design (e.g., function, proportionality, linear relationships) and types of examples and mathematical representations used to explain an idea. The examples in this study are word problems (Appendix A)	KCT_M2	In solving problems with a context, the students have to recognize unknown data and understand relations between variables. Example line [A242]

Figure 2 summarizes the analytic process adopted in this study. All codification was performed initially by one researcher and was then discussed in a seminar with other researchers in mathematics education for member validation (Birt et al., 2016). The path of teaching was first discussed with other researchers and then confirmed in an interview with Max.

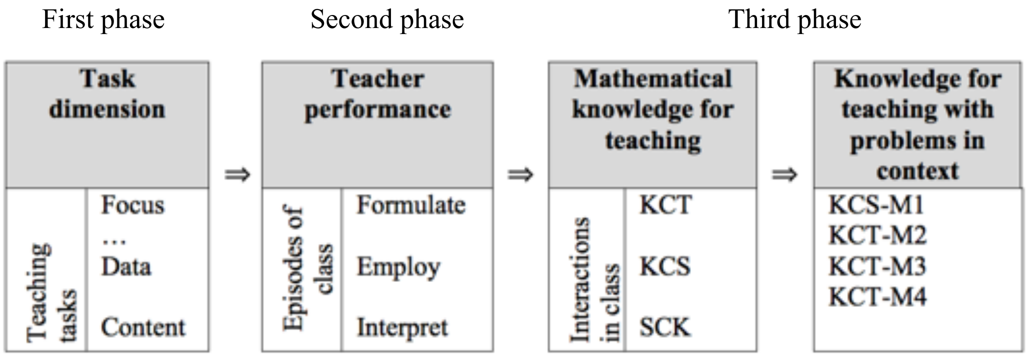

Figure 2. Protocol adopted for the analysis

The first research question was addressed in the first phase by analyzing the characteristics of the tasks according to the classification scheme (Maaß, 2010). To answer the second question, we considered information related to the knowledge pre-service teachers should possess for teaching mathematics (Ball et al., 2008), in particular the knowledge for teaching with problems in context (third phase). As Max was not trained for teaching modelling, the results concerning the task characteristics and class management provided valuable insight into his notion of modelling.

RESULTS

In this section, we discuss the relationship between the task features, the ways Max managed the class, and Max's knowledge related to supporting student learning while working with the tasks. As Max was not familiar with the modelling approach, this study will provide also information about his notion of modelling tasks.

Task Features

Below, we illustrate with the analysis of Task 2 (Figure 3) how seven tasks produced by Max were characterized. First, we describe the features of the tasks and use italics to point out the related category.

> **Task 2**
> It has been established that for a breed of foxes that inhabit the National Park called "La Campana", the relationship between their age E (months) and their weight P (kg) can be described as follows: the weight (kg) is equal 0.15 kg plus 1.2 times the age of the fox (in months). The above is valid between birth and 18 months.
> How much does a specimen weigh at 9 months of age?
> At what age does a fox weigh 7.5 kg?

Figure 3. Task 2

In designing the tasks, Max highlighted students' prior knowledge and knowledge promoted by the mathematical activity when solving the tasks. He identified unknown quantities and the notion of variable as previous knowledge of his students. As an example, Task 2 presented a situation (relationship between age and weight) as an idealized and simplified version of a part of reality. The lesson aim was to teach students about linear equations and functions. According to our conceptual framework, the task encourages only the transit through the phases understanding and mathematizing (understanding the situation and mathematizing with a linear equation). The relevant data was provided through keywords (equal, plus) in the text, where the necessary information and variables were also given (matching data). The task had the characteristics of word problems, which in the text directs the individual to choose a solution path by focusing on the keywords that evoke a mathematical operation (Greer, 1997). Thus, the intended activity for the students was embedded in the world of mathematics, and the context of the situation offered little opportunity to reflect about the reality (Maaß, 2010).

The mathematical content lies in linear equations and functions. Different representations (numerical table, graphical, algebraic, pictorial) were handled to explore and describe the original situation. The task presented a situation, and the transformation to a mathematical representation could be easily identified (ascertaining task). The cognitive demand depended on identifying the relevant information and the relationship between data (Greer, 1997), but principally relied on distinguishing the syntactic and semantic factors in the text (Andrews & Sayers, 2012). Particularly, in this case, it corresponds to the order of information from left to right. Cognitive demand also depends on the teacher encouraging mathematical reasoning by asking students to argue and justify their processes or solutions.

Following the above description, Table 2 provides a summary of the characteristics of the seven tasks analyzed, which can be found in Appendix A.

Table 2
Characterization of the seven tasks

Name of classification	Categories	Description
Focus on modelling activity	Understanding / mathematizing	Understanding to introduce the concept of variables, linear equations and functions by mathematizing
Data	Matching	The text provides the mathematical information through key words.
Nature of the relationship to reality	Embedded	The problem contexts are similar to the situations of daily life and are described using the language of daily life. Working in the world of mathematics is encouraged.
Situation	Next daily life	

Table 2
Characterization of the seven tasks

Name of classification	Categories	Description
Type of model used	Descriptive	The problem asks for descriptions of the situation in which the problem is embedded. The models are used to obtain a numerical answer.
Type of representation	Text	The situations were presented by text.
Openness of a task	Solved example/ Ascertaining task	The mathematical activity can be identified in the text.
Cognitive demand	Dealing with text containing mathematics	The level of cognitive demand depends on identifying relevant information and relations given by key words in the text.
Mathematical content	School level (Secondary school, 13-14/14-15 years old)	The mathematical content is related to dependency of variables, linear equations, systems of linear equations and linear functions.

The seven tasks provided an interesting and familiar context for the students, where the data and, in some cases, the corresponding variables were given. However, there was not a real translation from the reality to mathematics, and the students were not challenged to make assumptions, simplifications, or idealizations. As Max pointed out in the learning objectives and in the interview, understanding variables and their interdependency was the main focus for his students: “It was the first time that I [taught] this unit. … I think that by this way we can address the teaching of dependency … for this reason more than idealize, it has a relation with the dependency of variables because is always difficult for them…” This explained why students were required to work in the mathematical world by giving some meaning to the mathematical elements related to the context. This point was confirmed when we asked Max what aspects of dependency he considered difficult for students: “It has to be seen by them [students], it has to be an exercise that they can understand through a picture, it has to be expressed in another way… with words.” However, Max searched for contexts that could be close to the daily lives of students and be easy and interesting to them (for example, the growing of animals and payment of services), but students were not encouraged to reflect about the context of the tasks. In Task 2, questions such as “How many species of foxes are there in the area?”, “do they all have the same weight?” or “is it true that they grow in that relation?” would help students reflect on the context.

As demonstrated before, the designed tasks do not have the modelling task characteristics, but nonetheless served as a useful mathematical activity for the students. The teacher's management of the class could contribute to increasing or decreasing the cognitive demand by transforming the tasks to promote students' active engagement with mathematics (Henningsen & Stein, 1997). We discuss this point in the next section.

The Tasks and Teacher Performance in Class

The direction that Max gives to the class is reflected in his interactions with the students, in his management of the class, and in the questions he asked the students. As mentioned before, although Max created several tasks for ten teaching sessions, we focus on how Max used his knowledge to support student learning in sessions S1, S2 and S4. We identify in the first class the episodes of Max's teaching where we extract some indications of his knowledge. Below, we analyze and summarize relevant elements of the three sessions. A summary of all activities developed by Max in ten sessions is provided in Appendix B.

Max's Class

At the beginning of the first class, Max placed the students in seven teams and gave each group a different task to analyze and solve (see Appendix A). Students were encouraged to choose several ways to represent the problem and their solutions, and were also asked to explain their understanding to the rest of the class. From the students' explanations, we identified that they all first worked on some arithmetic operations, and then some of them constructed algebraic expressions representing unknown values (denoted by letters) and their relationship. They also attempted to represent the situation and solution to the problems graphically and pictorially. In the discussion below, students are represented by E1, E2, …, En, and the teacher by T.

In the next episode, one group explained to the rest of the class how they solved task T1 (Appendix A). The students answered two questions, the first of which required an arithmetic substitution, and the second required algebraic operations.

When formulating, Max emphasized the identification of the mathematical elements in the problem, such as the representation of significant data using variables, and the relationship between them. In the next fragment of transcription, a group of students explained how they determined the solution, and Max commented and asked questions about the solution process.

[A16] E1: In raising the Chilean Coypu, it is considered that the weight P in kilograms of the specimens varies with age E in months, according to the following relationship: weight is equal to 4 kilograms plus 2 times the age in months. [Coypu is also known as nutria]

[A17] E1: To answer the following 2 questions [At what age does the Coypu weigh 24 kg? How much does a 7-month-old specimen weigh?], you must

follow the data for weight and age. First, the weight would be added to the age multiplied by 2 plus 4, which would be ten months. In the second exercise, we answered the question "How much does a 7-month-old weigh?", which would be 7 times 2 equals 14 plus 4 equals 18.

[A18] T: […] And what did you write below the picture? Explain that part.

The student's explanations (line A17) show how the translation of information from the natural language to mathematical language was done, as Max had planned. The seven teams worked with similar problems, and were encouraged to explain their solution processes. Besides, in other episodes, Max insisted that students realized the process of translation. This finding shows Max's concern related to choosing a type of problem to help students in making translations from natural language to symbolic language, a difficulty that is frequently reported in the literature (Greer, 1997). In this way, his knowledge of content and students allowed Max to focus this part of the class on a problematic aspect for students. Particularly, this sign (KCS_M1) denotes Max's awareness of difficulties students encounter in the translation between languages when solving problems with a context. Additionally, in line A18, Max highlighted the expression written by students (*Weight = Age × 2 + 4, Age = Weight – 4/2*). This group was the first to explain the process, so that through questioning them, Max was able to make the process understandable for other students, which reflected his knowledge of content and teaching (KCT). In addition, this question helped students transition to the next phase of the solution process.

The focus in this episode was on the translation from natural language to mathematics. Because of the types of tasks, students lost the opportunity to formulate a problem or reflect about the characteristics of the context. The selection of relevant information was restricted to the information given in the text. This represents an example of Max's incomplete knowledge about modelling tasks (Borromeo Ferri, 2018). However, although we know that in word problems all data are given and students are only required to develop algorithms, this process was not easy for students in this class.

When employing mathematical concepts and procedures, students were encouraged to use different mathematical representations to understand and solve problems. In this episode, students were challenged to explain the relationships between different mathematical representations and to prove coherence between them. The development of mathematical operations, the identification of variables, and their relationships were discussed.

[A20] T: […] How do you get the first expression [Weight = Aged × 2 + 4]? […]

[A21] E1: Because here it comes out.

[A24] T: And the second expression [Age = Weight – 4/2]? […]

[A25] E1: We did it the other way around. Instead of adding it, we subtracted it, and instead of multiplying it, we divided it.

Max's first question was intended to help the rest of the class understand the translation from the information in a word problem to a mathematical expression, specifically to the linear equation [A20]. However, the students did not provide enough information to help Max discern how they built the expression. Related to the second point, Max focused on the processes developed to determine age. When students explained the process [A25], they uncovered that their operations were supported by the notion of the arithmetic equation, where the result was considered a consequence of the equality and the equation was solved by undoing (Filloy & Rojano, 1989). Max did not make annotations related to the notion of equality, potentially because he was more interested in students evidencing their understanding of inverse algebraic operations. This finding could evidence part of his specialized content knowledge for teaching.

When interpreting, Max and his students discussed the coherence between the context of the problem and the data represented in their solution processes. Max's knowledge related to content and students was evident when he prompted students to recognize in the problem the presence of something unknown that could be unveiled by considering the conditions of the context. He also explained that the unknown was called *variable*. Max insisted the students understand the meaning of the variable, the dependency of mathematical variables, and the connection with the context. Processes of interpreting and re-interpreting were further motivated by discussing the contexts of different problems.

[A26] T:	[…] What would be the variables of the problem? […]
[A27] E2:	Values that could vary…
[A28] T:	[…] What things vary in the problem you have?
[A29] S2:	The questions, but these formulas are for any age or any month.
[A30] T:	[...] So what varies there?
[A31] E3:	Does the formula vary?
[A32] E1:	The formula between weight and age.
[A33] T:	[…] Weight and age is what varies! Why do you say that is a formula?
[A34] E2:	Because if that formula is followed using any data they give you, the result will be obtained.

When Max asked about the variables, students' ambiguous answers revealed that this concept was unclear to them. However, from the students' answers, it appeared that they appreciated the existence of a relationship between quantities and that the expression reflected the coordination of two changing things [A32, A34]. The concept of covariation was implicitly present in the discussion as students were aware that changes in one variable are coordinated with changes in another variable (Thompson & Carlson, 2017). This assertion is confirmed in line A34 of the transcription. Making sense of the relations involved in variation is particularly important for modelling dynamic events (Carlson et al., 2002) and is a topic that frequently causes difficulties for students, and therefore necessary to be addressed in the teaching. The fact that Max insisted in students' grasping the relationship between variables is a sign of his knowledge of content and teaching (KCT), particularly in solving problems with a context, where students have to recognize unknown data and understand the relations between variables (KCT_M2).

After all teams finished exposing the processes they followed to arrive at their solutions, to restart the discussion about the dependency of variables, Max resumed the work done by the teams focusing on the covariation of variables, as we can observe in the next fragment of the transcription related to the solution of task T2.

[A164] T: […] They placed kilograms on the horizontal line and months on the vertical axis [Figure 4]. The problem was about a fox that was growing with age, as time went by, its weight increased. If you look at what the group did, they made the fox a little bigger each time the kilograms increased, [...] and they were marking months, here 1 month, 2 months, 3 months [...].

[A165] T: […] What relationship can we see between the variables in this problem […]?

[A166] E27: Whenever the fox grows, the weight also increases.

[A167] T: Perfect!

[A168] E24: Direct proportionality.

[A169] T: Ok, why?

[A170] E24: Because if one increases the other does too.

[A171] T: If what increases, what increases?

[A172] E24: If months increase, the size increases.

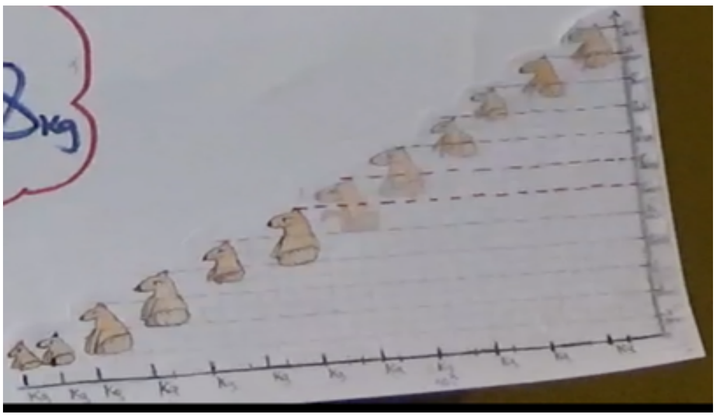

Figure 4. Students' production when solving Task 2

In the previous discussion, Max's attention was focused on students' understanding of the relationship between two groups, i.e., the weight as a function of the age. We relate this finding to his knowledge of content and teaching as, according to Thompson and Carlson (2017), in solving problems with a context, students must recognize the covariation in a situation (KCT_M3). However, even though Max's students attempted to identify relations of dependency and covariation in their problems, this was not completely clear for some of them. Thus, Max dedicated a new round of whole class discussion to students' solutions to task T1.

[A221] E2:	The weight depends on age because when months increase, 5 months, 6 months, weight increases.
[A222] T:	So, what depends on what?
[A223] E2:	Weight depends on age.
[A225] T:	Why can't it be the other way around?
[A226] E2:	Because age can continue advancing without depending on weight. Age controls weight.

In the previous paragraph, students explained the relation of dependency. However, the discussion about the rationality of the context was missing, particularly that related to non-linear increase in Coypu's weight and its limited lifespan. This reflects a common difficulty in sense-making when working with word problems (Verschaffel et al., 2010), but also indicates Max's lack of knowledge about what modelling tasks mean, particularly related herein to the realistic and authentic context (Borromeo Ferri, 2018). Thus, owing to his inexperience in working with modelling problems, Max could not anticipate these elements in the task design, representing a source of weakness in his KCT.

Finally, Max closed the class by asking the students to express their understanding related to dependency of variables.

[A242] T:	[…] There are two types of variables—dependent variables and independent variables. In the exercises, we could find both types. Does anyone know the meaning of dependent and independent variables? […]
[A243] E15:	The dependent variable is the one that has to be accompanied by something, and the independent variable is something that goes by itself.
[A245] E18:	It is something that is dependent on another.
[A249] E19:	A variable that depends on other data.
[A250] E14:	The dependent variable needs something to change.
[A254] T:	[…] The independent variables do not depend on any factor to be modified, but instead dependent ones depend on phenomena so that they can change […]

In the first session, three moments were identified where KCS and KCT were evidenced when Max prompted students to make sense of relevant elements in the problem. He encouraged students to explain their mathematical reasoning by asking questions about the arithmetic operations and mathematical elements involved in their solution processes. He also motivated the students to create different and useful representations of the problem. Then, Max asked students to explain how the algebraic expressions or graphs were built and the information represented therein, which helped him understand the ways students think about the quantities and relations between data. Max used the examples of students' solutions to show to the rest of the group that problems can be solved using different strategies.

Max supported the students in transiting from situations presented as word problems to a mathematical solution by enhancing their problem-solving strategies and sharing their own solution processes. This result demonstrates Max's knowledge about solving problems with a context, where the students have to learn using different strategies to obtain the solution. He was also aware of the usefulness of representations for understanding and solving problems, which is a sign of his KCT_M4. In solving mathematical problems, it is desirable that students use different representations and strategies, but from the modelling perspective, the activity given to the students should go beyond the mathematical world.

In the second session, Max initiated the class by summarizing the content of the previous lesson. Then, by referring to the solution process used for task T1, he asked the whole class about their understanding of the algebraic model (term introduced by him).

[A42] T: […] What do you understand by an algebraic model??

[A43] E1: A kind of example or sample to understand the problem you have.

[A44] T: Ok, another group, what do you think algebraic model means?

[A47] E2: A different way to represent something [...]

[A56] T: […] What would be a different way that this group used to represent the problem?

[A57] E1: The Coypus [Student means the picture of coypus].

[A58] T: Ok, the Coypus. We are going to mark it from 1; we may find more than one way. Is there any other ... some other way?

[A63] E3: The formula.

[A64] T: Ok, correct, then this is the model ... why is it an algebraic model? Because, if you look, the two variables are expressed therein.

Following this discussion, Max asked the students to identify different representations in the problems on which they worked so far (i.e., algebraic, numerical, graphical, pictorial, and natural language). In the remaining sessions, the first three were called models [A 67-68]. From here, Max's notion related to models was associated with the mathematical representations used to solve the problems (he also considered pictorial representation as a part of a model). Although mathematical representations are frequently associated with mathematical models, in modelling perspectives, a model is not defined according to its representations, as its definition is associated with its role in representing a part of simplified reality with a specific normative or descriptive intention (Maaß, 2010). It could be that the limited attention given to modelling in academic training encourages to a greater extent the association of mathematical models to the mathematical representations than to their utility in representing a part of simplified reality. It could be also associated with the theories of didactics that pre-services teachers learn in their training. This is an aspect that can also be associated with Max's specialized content knowledge. Regardless, Max's notion about models is close to having a tool to communicate and represent something, as we can see in the next transcription. This point had some implications for student learning because some of Max's students viewed that having an algebraic model is the same as performing arithmetic operations.

[67] T: […] because through the representation of a drawing, in this case, they attempt to communicate something about the problem [...].

[A 68] T: If you look at the other group, which also has a table, all those models are going to be used to communicate things that are immersed in the problem and may be done because some people find it easier to understand in one way or another […]. There are others who find it easier to work with numbers and associate numbers and price as in the exercise on parking. That is the idea, each person represents how he or she interprets and understands the problem in the way that is most comfortable for him or her […].

Our interpretation of Max's notion of modelling related to mathematical representations could be contrasted in the conversation when he reflected on the progress of the class. The excerpt from the conversation also shows that Max is aware of his lack of experience in teaching modelling.

It is the first time that I am working with modelling and I am a bit intuitive. I read a couple of things about modelling, but I do not know everything which is required to say that we are modelling [...]. I based myself on what they [students] are able to do from ingenuity because they do not have formalized knowledge about what we are going to see in the class. They should be able from intuition, from previous knowledge, to solve an idealized real-life problem to get a mathematical conjecture [...], that's like my idea of mathematical modelling. The idea later is to extrapolate it to functions and to tables. And, from functions and tables they are able to invent a problem [...].

In session four, teams were given different linear equations and were asked to create a problem matching each expression, which all teams found very challenging. First, all teams who solved the task made some mistakes. Then, the teams explained to the rest of the class their solution processes, and were given feedback to improve their creations. This part of the discussion represented an enrichment for all students because they had the opportunity to reformulate the initial situation where some data did not match the algebraic expression or their questions were unclear. The discussion presented revolves around a problem [A149] created by one team to match the expression d = 450 + 2.5t

[A149] E1: Some explorers traveled to complete their research on the hottest volcano in the world. On one of their trips, they found a volcano and discovered the following: the maximum temperature of the volcano is 500 °C. The question we have is how many days would it take to reach 500 °C?

[A150] E1: The model you gave us is 450 + 2.5t; 450 would be the days and 2.5 would be the thickness of the volcano [...], I don't know how to explain it […].

[A162] E1: I was wrong, I was wrong! It would be multiplied by the temperature and every 450 days they add 2.5° […]

In discussing the problem, the dependency between time and temperature [A200−203, A208], the association of relevant information in the text with letters in the expression [A206], notion of covariation [A208], and minimum temperature [A202, A227] were key elements in rectifying the issues with the way students posed the problem.

[A200] T: […] they are mixing two variables on one side. As E4 said, we have two variables, the temperature and the days. Which one depends on which one? [...] What depends on what? [...]

[A201] E2: The one that depends would be the degrees.

[A202] T: What do the grades depend on?

[A203] E1: On the days.

[A222] T: According to what we were saying, we are reformulating the problem, what does the 450 represent? […]

[A227] E3: It should be the base temperature, the minimum temperature.

Finally, the students and the teacher reformulated the text and the question to correctly match the problem to the mathematical expression. In reformulating the text, students were given the opportunity to analyze the data and make sense of the terms in the algebraic expressions. Here, problem posing was a medium to support students to match a situation with a mathematical expression, which is somewhat similar to the use of known mathematical models to study a new situation. When modelling is necessary, students reflect on the coherence between the problem and the mathematical model, so problem posing could in some sense support the transit across the phases of the modelling cycle. Studying the implications of problem posing in developing modelling competences is a research topic that needs be further investigated. Problem posing has been characterized as potentially effective for developing creativity, as well as for generalizing and providing students with opportunities to think as mathematicians, and should be a component of teacher knowledge (Crespo & Sinclair, 2008).

CONCLUSIONS

The findings yielded by this study highlight aspects that need to be considered for inclusion in teacher training. The first important issue is the lack of experience and knowledge that future teachers often have to integrate modelling in their lessons. Although, the findings reported here were related to the Chilean context, the teaching of modelling is a topic that concerns researchers around the world (Bolsad, 2020; Villa & Ruiz, 2009). Therefore, the results we presented here contribute to different fields of research. The training of future teachers is a complex process where diverse knowledge must be developed, as several researchers have pointed out (Borromeo Ferri & Blum, 2010; Wess et al., 2021). Regarding the question Q1 related to the pre-service teachers' management of their knowledge to teach modelling, Max's teaching shows that he possesses rich Pedagogical Content Knowledge (Ball et al., 2008), as he was aware of the difficulties students will have in recognizing the relationships between contexts and mathematical expressions. He also promoted students' understanding of mathematical concepts and included problem-posing activities into his lessons. The analyses presented here also highlight Max's ability to manage the classroom, promote interactions between students, and facilitate collaborative work, which helped increase student motivation and improve their attitudes toward class activities (see the summary in Table B1). On the other hand, Max's knowledge for teaching modelling was weak, as evidenced by the tasks he designed where the context was far from having realistic considerations, and the focus for the students was on obtaining the right answers and not on reasoning how close to real-life situations problem contexts were. Consequently, the students did not have the opportunity to transition through the modelling cycle. Based on these findings, it is clear that for teaching modelling, teachers need to have knowledge related to the four dimensions indicated by Borromeo Ferri (2018).

Linked with the question Q2 about the characteristics of modelling task designed by pre-service teachers, Max's tasks reflected a notion about modelling associated to word problems (Greer, 1997) but realistic contexts were not evident. Related to the teaching with contextual problems, we identified four elements of the knowledge of content and students and knowledge of content and teaching (KCS_M1, KCT_M2, KCT_M3, KCT_4) that allowed us to recognize Max's concern about important difficulties in working with word problems, the meaning of variables and operations involving them, variation and covariation, dependency between variables, and the use of different representations and strategies to solve problems. Although these issues are not the focus of the study of modelling, in the practice they have an impact on students' proficiency. Students that took part in Max's classes were given the opportunity to develop problem-posing skills by reformulating real-world textual problems they had created. Although this activity was initially embedded in a word problem when students created a context matching an expression, the complexity of the task increased, and consequently, cognitive activity also increased. Tasks that require students to pose problems from given mathematical equations require understanding of the meaning of the operations, and students usually follow a process with a focus on the operational and not the semantic structure of the problem (Christou et al., 2005). The skills developed through these activities could later support students in modelling activities, but this issue must be further explored. The answer to question Q3 is a consequence of analyzing the tasks and their implementation. Max's notion, associated with modelling, was related to working with tasks that are precursors to modelling (Verschaffel et al., 2010), while the models were related to the different mathematical representations. The association of mathematical models to mathematical representations has been identified in the students' productions and offers information of the knowledge that students use to create a model (Montejo-Gámez et al., 2021), in addition the results here shown a natural tendency to include different representations in the teaching of modelling. Being the first time Max implemented a modelling-based teaching sequence, his notion about modelling and models is nuanced by the experience in his academic training (Guerrero-Ortiz & Reyes-Rodríguez, 2021). We emphasize that the variety of interpretations about modelling in the curriculum influences the type of notions that teachers and future teachers develop, such as has also been shown in other research involving teachers in training without experience in modelling, where similar conceptions have been evidenced (Guerrero-Ortiz & Reyes-Rodríguez, 2021). Teachers' notions are also permeated by the pedagogical content knowledge and by their teaching and learning models. In addition, we must take into account that these notions are changing and can evolve as teachers have an approach to the knowledge of modelling theory, tasks, instruction and diagnosis (Borromeo Ferri, 2018).

The second issue is related to the several interpretations of modelling in the scholar curriculum and textbooks (Borromeo Ferri, 2021), and the concern of schools to accomplish the curriculum. Although here we only present the case of Max, the tasks created by the other participants were similar, and corresponded in general to the tasks referring to modelling proposed in the textbooks. This could be the reason why teachers in this study designed tasks with word problem characteristics and aligned them to specific mathematical content in the curriculum, but not to modelling tasks. These findings highlight the need to help future and in-service teachers to develop knowledge about the aims and perspectives of modelling and the range of references to reality (Wess et al., 2021) so they can be able to select, evaluate, and improve the activities proposed in the textbooks and the curriculum.

Finally, from this study, we learn that when the teacher changes the traditional way of exploring word problems, the activity becomes more meaningful for students (here, we refer to the change introduced when Max turned from solving a problem to encourage the students to create a problem). Therefore, teachers and students should be able to judge the meaningfulness of a problem and its mathematical answers. Requiring students to formulate problems by themselves, as Max did, could be a significant step in encouraging students to reflect about the coherence between the mathematical representation, the solutions, and the information given in the problem. In this paper, we showed how pre-service teachers without prior training in modelling start experimenting with tasks and implementing changes in the activities proposed by the curriculum, gradually learning from their experience and their mistakes. Here, we identify an attempt to change the manner in which the content is traditionally taught in mathematics courses in schools. Unfortunately, tensions emerged between the curriculum-imposed content teaching strategies and the activities that pre-service teachers want to develop.

When interpreting these findings, several limitations to this study should be noted. First, as the study participants (all of whom were pre-service teachers in their final year of the course) needed to accomplish specific mathematical content for teaching, the tasks they implemented were primarily dependent to the content and did not specifically focus on teaching students the specifics of modelling. Second, none of the participants had prior modelling training or experience. Thus, it is likely that different results would be obtained if participants had some prior skills for teaching modelling. Finally, in the research on the teaching of modelling, more progress has been made in the exploration of PCK (Doerr 2007; Wess et al., 2021). While research on the characteristics of the other elements of teacher knowledge (KCS, KCT, SCK) and their relationship with modelling is slowly progressing, therefore this study yields results that can be discussed depending of the theoretical perspective of researchers.

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Carolina Guerrero Ortiz
Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Chile
c_cguerrero@yahoo.com.mx

Rita Borromeo Ferri
Institute of Mathematics, University of Kassel, Germany
borromeo@mathematik.uni-kassel.de

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APPENDIX A

T1

In the raising Chilean Coypu, the weight P, in kilograms of specimens, varies with the age E, in months, according to the following relationship: the weight is equal to 4 kilograms plus 2 times the age in months of the coypus.
At what age does the coypu weigh 24 kg?
How much does a 7-month-old specimen weight?

T2

It has been established that for a breed of foxes that inhabit the National Park called "La Campana", the relationship between their age E (months) and their weight P (kg) can be described as follows: the weight (kg) is equal 0.15kg plus 1.2 times the age of the fox (in months). The above is valid between birth and 18 months.
How much does a specimen weight at 9 months of age?
At what age does a fox weight 7.5 kg?

T3

A parking lot charges $2,000 pesos for the first hour of parking and $1,000 for each additional hour.
How much should a person who parked 3 hours and a half pay?
How long parked a person who pays $7,000 pesos?

T4

Juan owes money to the bank and to pay off his debt he pays $150,000 each month, after ten months his remaining debt was $6,900,000.
-What was Juan's original debt?
-How long did it take to pay off his debt?

T5

A telephone company has a plan according to the minutes that the user speaks. This plan advertised the follow: the monthly cost will be $1000 pesos, plus $50 pesos for each minute spoken during the month.
How much will a person who spoke 3 hours and 50 minutes during the month have to pay?
If Pablo has a debt of $15 000 pesos, how many hours does he speak?

T6

From a study of the Chilean Chinchilla, his weight (gr) and age are related according to its sex in the following way: the weight of the male Chinchilla is equal to 172 gr plus 0.96 times the Chinchilla's age. The weight of the female Chinchilla is equal to 162 grams plus 1.2 times the age of the Chinchilla.
How much does a male Chinchilla weigh at 16 days?
After how many days does a female Chinchilla weigh 460 grams?

T7

Within its first year of life, the weight P of a certain kind of hedgehog varies with age E as follows, the weight (kg) is equal to 25 gr plus 3 gr for each day the pet hedgehog is kept. How much does a hedgehog weigh at 30 days old?

APPENDIX B

Summary of the activities developed by Max in the ten class sessions.

Table B1
Summary of activities developed in Max's classes.

Session[a]	Type of tasks[b]	Learning objective[c]	Classroom actions[d]	Mathematical concepts[e]
S1	Word problem	Identifying variables by using word problems	Students work in groups Students solve a word problem Students present their work to the whole class Teacher's explanations	Different ways of representing the situation emerged Variables Dependency between variables
S2, S3	Word problem	Identifying variables	Teacher reviews and comments on the student's work in the whole class Students identify variables and dependency in simple situations	Graphical, pictorial, and tabular representations Dependency between variables
S4, S5	Problem posing	Modelling problems using a known expression	Students work in groups Students create a problem matching the algebraic expression Students present their work to the whole class	Relations between variables Dependency Coherence between context and mathematical representations Domain and range

Table B1

Summary of activities developed in Max's classes.

Session[a]	Type of tasks[b]	Learning objective[c]	Classroom actions[d]	Mathematical concepts[e]
S6	Word problem	Relation between mathematical representations and the context of problem	Students work in groups Students match data in graphs, tables and sagittal diagrams with the context of the problems Teacher reviews and comments on the students' work in the whole class	Relations between representations Continuity
S7, S8	Analyzing mathematical representations	Matching different representations	Students work in groups Teacher explains mathematical concepts	Relations between graphical and algebraic representations Definition of linear equations Image y and pre-image Graphing methods
S9, S10	Whole class test		Students explain their processes of solution to the whole course Students answer specific questions from the teachers and other students	

Note. [a] number of session, [b] type of tasks worked in each session, [c] learning objective in the session, [d] teacher and students' actions, [e] mathematical concepts present in each session