CONFORMAL METRICS WITH PRESCRIBED GAUSSIAN AND GEODESIC CURVATURES

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ABSTRACT. We consider the problem of prescribing the Gaussian and the geodesic curvatures of a compact surface with boundary by a conformal deformation of the metric. We derive some existence results using a variational approach, either by minimization of the Euler-Lagrange energy or via min-max methods. One of the main tools in our approach is a blow-up analysis of solutions, which in the present setting can have diverging volume. To our knowledge, this is the first time in which such an aspect is treated. Key ingredients in our arguments are: a blow-up analysis around a sequence of points different from local maxima; the use of holomorphic domain-variations; and Morse-index estimates.

1. INTRODUCTION

A classical problem in Geometry is the prescription of the Gaussian curvature on a compact Riemannian surface Σ under a conformal change of the metric, dating back to [4, 32]. Denote by $\tilde{g}$ the original metric, by g the conformal one, and by e^u the conformal factor (that is, $g = e^u \tilde{g}$). The curvature then transforms according to the law:

$$-\Delta u + 2\tilde{K}(x) = 2K(x)e^u,$$

where $\Delta = \Delta_{\tilde{g}}$ stands for the Laplace-Beltrami operator associated to the metric $\tilde{g}$, and $\tilde{K}$, K stand for the Gaussian curvatures with respect to $\tilde{g}$ and g, respectively. The solvability of this equation has been studied for several decades, and it is not possible to give here a comprehensive list of references. We refer the interested reader to Chapter 6 in the book [2].

If Σ has a boundary, other than the Gaussian curvature in Σ it is natural to prescribe also the geodesic curvature on $\partial\Sigma$. Denoting by $\tilde{h}$ and h the geodesic curvatures of the boundary with respect to $\tilde{g}$ and g respectively, we are led to the boundary value problem:

$$(1.1) \qquad \begin{cases} -\Delta u + 2\tilde{K}(x) = 2K(x)e^u, & \text{in } \Sigma, \\ \dfrac{\partial u}{\partial n} + 2\tilde{h}(x) = 2h(x)e^{u/2}, & \text{on } \partial\Sigma. \end{cases}$$

In the literature there are results on some versions of the latter problem. For example, the case $h = 0$ has been treated in [8], while the case $K = 0$ has been considered in [7, 33, 38]. There is also work on the blow-up analysis of solutions, see [3, 12], although the phenomenon is not fully understood.

The case of constant K, h has also been considered: in [5] the author used a parabolic flow to obtain solutions in the limit. Using methods from complex analysis and the structure of Liouville equations, explicit expressions for the solutions and the exact values of the constants were determined if Σ is a disk or an annulus, see [28, 30]. Some classification results for the half-plane are also available in [23, 35, 46]. However, there are almost no results for the general situation in which both curvatures are variable functions: the following ones, are the only we are aware of at the moment. In [10] partial existence results are given, but some of them include an undetermined Lagrange multiplier. A Kazdan-Warner obstruction to existence has been found in [25]. Very recently, a result for positive symmetric curvatures in the disk has appeared, see [11].

In higher dimensions the natural analogous question regards the simultaneous prescription of the scalar curvature and the boundary mean curvature. The scalar-flat case with constant mean curvature is known as the *Escobar problem*, in strong relation with the Yamabe problem. In this regard, see [1, 14, 16, 17, 21, 26, 27, 39] and the references therein.

Integrating (1.1) and applying the Gauss-Bonnet theorem, one obtains

$$(1.2) \qquad \int_\Sigma K e^u + \oint_{\partial\Sigma} h e^{u/2} = 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

In this paper we study existence and compactness of solutions of (1.1) in the negatively-curved case, namely when $K(x) < 0$. For existence, we focus on the case $\chi(\Sigma) \le 0$. The case of the disk is intrinsically more complicated due to the non-compact effect of the group of Möbius maps, and it will be studied in a future work. However one of the main goals of the paper is the blow-up analysis given in Theorem 1.4, which is very general and applies to any PDE in the form (1.1).

It is easy to see that, via a conformal change of metric, we can always prescribe the values $h = 0$, $K = sgn\chi(\Sigma)$, see Proposition 3.1. Hence, without loss of generality, we can assume that our initial metric is such that $\tilde{K}$ is constant and $\tilde{h} = 0$. The problem to study then becomes:

$$(1.3) \qquad \begin{cases} -\Delta u + 2\tilde{K} = 2K(x)e^u, & \text{in } \Sigma, \\ \dfrac{\partial u}{\partial n} = 2h(x)e^{u/2}, & \text{on } \partial\Sigma, \end{cases}$$

where $\tilde{K} = sgn\,\chi(\Sigma)$.

If $K(x) < 0$, problem (1.3) is the Euler-Lagrange equation of the energy functional $I : H^1(\Sigma) \to \mathbb{R}$,

$$(1.4) \qquad I(u) = \int_\Sigma \left(\frac{1}{2}|\nabla u|^2 + 2\tilde{K}u + 2|K(x)|e^u\right) - 4\oint_{\partial\Sigma} h(x)e^{u/2}.$$

Observe that the area and boundary terms are in competition, and a priori it is not clear whether I is bounded from below or not. For the statement of our results it will be convenient to define the function $\mathfrak{D} : \partial\Sigma \to \mathbb{R}$ as

$$(1.5) \qquad\qquad \mathfrak{D}(x) = \frac{h(x)}{\sqrt{|K(x)|}}.$$

Notice that $\mathfrak{D}$ is scale-invariant. As we shall see, the function $\mathfrak{D}$ will play a crucial role in the global behavior of the functional I, as well as in the blow-up analysis of solutions to (1.1).

Our first existence result deals with the case $\chi(\Sigma) < 0$.

Theorem 1.1. *Assume that $\tilde{K} < 0$. Let K, h be continuous functions such that $K < 0$ and $\mathfrak{D}(p) < 1$ for all $p \in \partial\Sigma$. Then I attains its infimum and hence problem (1.3) admits a solution. If moreover $h \le 0$, then the solution is unique.*

The next theorems address the case $\chi(\Sigma) = 0$.

Theorem 1.2. *Assume that $\tilde{K} = 0$. Let K, h be continuous functions such that $K < 0$ and*

(1) $\mathfrak{D}(p) < 1$ for all $p \in \partial\Sigma$;
(2) $\oint_{\partial\Sigma} h(x) > 0$.
Then I attains its infimum and hence problem (1.3) admits a solution.

Compared to Theorem 1.2, our next result is concerned with the reversed case of the inequality for $\mathfrak{D}$ and $\oint_{\partial\Sigma} h(x)$.

Theorem 1.3. *Assume that $\tilde{K} = 0$. Let K, h be C^1 functions such that $K < 0$ and*

(1) $\mathfrak{D}(p) > 1$ for some $p \in \partial\Sigma$;
(2) $\oint_{\partial\Sigma} h(x) < 0$;
(3) $\mathfrak{D}_\tau(p) \ne 0$ for any $p \in \partial\Sigma$ with $\mathfrak{D}(p) = 1$.
Then problem (1.3) admits a solution.

We point out that the integration appearing in *(2)*, for both Theorem 1.2 and Theorem 1.3, is done with respect to the (unique) conformal metric with minimal boundary obtained in Proposition 3.1. If one wishes to go back to the original formulation (1.1), these conditions should be properly rephrased.

We shall see also that the hypotheses of the above theorems are somehow natural. In the case of the annulus, for instance, the classification of solutions for constant curvatures given in [30] matches with our results. In this regard we shall also show an obstruction result for the existence of solutions, using an argument from [40]. See Section 2 for details.

We shall show that under the assumptions of Theorem 1.3 I is not bounded from below. In this case we use a min-max argument to find a saddle-type solution to the problem. However the Palais-Smale property - i.e., convergence of approximate solutions - is not known to hold for the functional I. We bypass this problem via the well-known *monotonicity trick* by Struwe; roughly speaking, we are able to prove existence if compactness of solutions to perturbed problems is guaranteed.

This motivates the study of blowing-up solutions. The question is: given a sequence u_n of solutions of problems in the form (1.1), are they bounded from above? Assuming otherwise, we may define the *singular set* as

$$(1.6) \qquad S = \{p \in \Sigma : \ \exists \, x_n \to p \text{ such that } u_n(x_n) \to +\infty\},$$

see [6]. Starting with [6, 34], the properties of such blowing-up solutions for Liouville-type equations have been much studied in the literature. In sum, near any isolated point $p \in S$ one can rescale the solution and obtain in the limit an entire solution, and then take advantage of classification results. In this framework, a finite mass condition is always assumed, as for instance:

$$\int_\Sigma |K_n(x)| e^{u_n} < +\infty.$$

In the previous literature this condition is essential in many aspects, which we enumerate below.

(1) The entire solutions of the limit problem (in the plane or the half-plane) are much more restricted in the case of finite mass. For instance, the classification of the solutions to the equation

$$-\Delta U = 2e^U \ \text{ in } \mathbb{R}^2,$$

dates back to Liouville ([37]), and form a large family of non-explicit solutions. However, under the finite mass assumption, the only solutions are given by:

$$U(x) = 2\log\left(\frac{2\lambda}{1 + \lambda^2 |x - x_0|^2}\right), \ \ x_0 \in \mathbb{R}^2, \ \lambda > 0,$$

see [9].
(2) Finite total mass implies also that the set S is finite, since e^u behaves as a finite combination of Dirac deltas with weights bounded away from zero (see [6]). The use of the Green's representation formula gives then some global information on the behavior of the solutions.
(3) Since S is finite, one has local maxima of u_n around each of the point of S, and one can pass to the limit after a suitable rescaling.

However, in our problem, blowing-up solutions may have diverging mass. Observe that (1.2) allows some compensation between the area and the boundary terms, which can both diverge. We would like to emphasize that this is not a technical issue: indeed, those masses will diverge in some cases. This is a completely new phenomenon and one of the main goals of the paper is to give a complete description of it. Our main result in this respect is the following:

Theorem 1.4. *Let u_n be a blowing-up sequence (namely, $\sup_\Sigma u_n \to +\infty$) of solutions to*

$$(1.7) \qquad \begin{cases} -\Delta u_n + 2\tilde{K}_n(x) = 2K_n(x)e^{u_n}, & \text{in } \Sigma, \\ \frac{\partial u_n}{\partial n} + 2\tilde{h}_n(x) = 2h_n(x)e^{u_n/2}, & \text{on } \partial\Sigma, \end{cases}$$

with $\tilde{K}_n \to \tilde{K}$, $K_n \to K$, $\tilde{h}_n \to \tilde{h}$ and $h_n \to h$ in the C^1 sense, with $K < 0$. Define the singular set S as in (1.6), *and*

$$(1.8) \qquad \chi_n = \int_\Sigma \tilde{K}_n + \oint_{\partial\Sigma} \tilde{h}_n.$$

Then the following assertions hold true:

(1) $S \subset \{p \in \partial\Sigma : \mathfrak{D}(p) \geq 1\}$.

(2) *If $\int_\Sigma e^{u_n}$ is bounded, then there exists $m \in \mathbb{N}$ such that*
$$S = \{p_1, \ldots p_m\} \subset \{p \in \partial\Sigma : \mathfrak{D}(p) > 1\}.$$

   In this case $|K_n|e^{u_n} \rightharpoonup \sum_{i=1}^m \beta_i \delta_{p_i}$, $h_n e^{u_n/2} \rightharpoonup \sum_{i=1}^m (\beta_i + 2\pi)\delta_{p_i}$ for some suitable $\beta_i > 0$. In particular, $\chi_n \to 2\pi m$.

(3) *If $\int_\Sigma e^{u_n}$ is unbounded, there exists a unit positive measure σ on Σ such that:*

   a) $\dfrac{|K_n|e^{u_n}}{\int_\Sigma |K_n|e^{u_n}} \rightharpoonup \sigma$, $\dfrac{h_n e^{u_n/2}}{\oint_{\partial\Sigma} h_n e^{u_n/2}} \rightharpoonup \sigma|_{\partial\Sigma}$;

   b) $supp\, \sigma \subset \{p \in \partial\Sigma : \mathfrak{D}(p) \geq 1, \mathfrak{D}_\tau(p) = 0\}$.

(4) *If there exists $m \in \mathbb{N}$ such that $ind(u_n) \leq m$ for all n, then $S = S_0 \cup S_1$, where:*
$$S_0 \subset \{p \in \partial\Sigma : \mathfrak{D}(p) = 1, \mathfrak{D}_\tau(p) = 0\},$$
$$S_1 = \{p_1, \ldots p_k\} \subset \{p \in \partial\Sigma : \mathfrak{D}(p) > 1\}, \quad k \leq m.$$
   If moreover $\chi_n \leq 0$, then S_1 is empty.

In the above statement, $\mathfrak{D}_\tau(p)$ refers to the derivative of $\mathfrak{D}$ with respect to the tangential direction τ along $\partial\Sigma$. Moreover, $ind(u_n)$ stands for the Morse index of the function u_n, namely the maximal dimension of a subspace $E \subset H^1(\Sigma)$ such that the quadratic form $Q_n : H^1(\Sigma) \to \mathbb{R}$

$$(1.9) \qquad Q_n(\psi) = \int_\Sigma [|\nabla\psi|^2 + 2|K_n(x)|e^{u_n}\psi^2] - \oint_{\partial\Sigma} h_n e^{u_n/2}\psi^2,$$

is negative definite when restricted to E.

We will also give some description on the asymptotic profiles of u_n around the singular points, see Proposition 7.2 a) and b).

Some comments are in order:

(1) In general we cannot ensure that the singular set S is finite. Indeed we will show explicit examples, given by [30], in which it is not, see Subsection 2.2. This is an entirely new feature of this kind of blow-up.

(2) We are able to pass to a limit problem around any singular point $p \in S$. This is tricky since p need not be isolated and we cannot guarantee the existence of local maxima around such a point. We do this by choosing carefully a sequence $x_n \to p$ with $u_n(x_n) \to +\infty$, even if $u_n(x_n)$ are not local maxima. Our main tool for that is the *Ekeland variational principle* (in a finite-dimensional fashion).

   We point out that this is a technical novelty even for the finite mass case and allows one to pass to a limit problem, as in [34], without knowing a priori that S is finite.

(3) As we can see, the terms $\int_\Sigma |K_n| e^{u_n}$, $\oint_{\partial\Sigma} h_n e^{u_n/2}$ can become unbounded, but when normalized they converge to the same measure. Hence they are in strong competition, and this cancellation implies that a normalization technique is not of use here. Despite, we are able to use a Pohozaev-type identity to show that $\operatorname{supp}\sigma \subset \{p \in \partial\Sigma : \mathfrak{D}_\tau(p) = 0\}$. The main obstacle here is that we do not have any control on the Dirichlet energy of the solutions. We are able to bypass this problem by using *holomorphic maps* as test fields in the Pohozaev-type identity, see Proposition 6.2 for more details.

(4) The limit problem here is posed in a half-plane:

$$(1.10) \qquad \begin{cases} -\Delta v = -2e^v, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial v}{\partial n} = 2\mathfrak{D}(p)e^{v/2}, & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

In the spirit of [9], solutions of the above problem with finite mass have been classified in [46], and they exist only if $\mathfrak{D}(p) > 1$.

However we shall need a classification of *all solutions*, with bounded or unbounded mass, that has been given in [23] (in the spirit of Liouville, [37]). Those solutions exist only for $\mathfrak{D}(p) \geq 1$ and are linked to the family of holomorphic maps from the half-plane to disks on the hyperbolic space, so they form a much broader family of solutions. However in Section 4 we are able to completely characterize the stability of all such solutions, and this allows us to exclude many blow-up profiles by means of the Morse index restriction. As a consequence the unique possible limit profile with infinite mass is the solution of (1.10) with $\mathfrak{D}(p) = 1$, namely the 1-D function:

$$v(s,t) = 2\log\left(\frac{\lambda}{1 + \lambda t}\right), \quad \lambda > 0.$$

Observe that there could also be finite-mass blow-up on a finite number of points, if $\chi_n \geq 0$. It is an interesting open question whether those two types of blowing-up behavior can coexist in general.

(5) Assumptions on the boundedness of the Morse index of solutions are relatively natural in the framework of minimal surfaces (we refer the reader e.g. to the papers [22], [41], [42], and also to [19] in a PDE setting), but are completely new in this kind of problems. Moreover, we can show an explicit family of blowing-up solutions in the annulus, given in [30], with diverging Morse index and with a different limit profile, see Subsection 2.2.

The rest of the paper is organized as follows. In Section 2 we discuss the naturality of our assumptions, showing that they are in some cases sharp. Section 3 is devoted to the variational study of the functional I, completing the proof of Theorem 1.1 and 1.2. We also combine the monotonicity trick of Struwe (in a version by Jeanjean) with index bounds by Ghoussoub and Fang. In this way we find solutions of perturbed problems with Morse

index 1, under the setting of Theorem 1.3. In Section 4 we recall the classification results for entire solutions in half spaces and determine their Morse index. The blow-up analysis of this kind of sequences is started in Section 5. Section 6 is devoted to the unbounded mass case, whereas the bounded mass or bounded Morse index case is studied in Section 7. We conclude the paper with an Appendix devoted to evaluate certain test functions on the energy functional I.

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Notation. An open ball of center p and radius r will be denoted by $B_p(r)$, while $A(p; r, R)$ stands for an annulus of center p and radii $0 < r < R$. We also use the notation:

$$B_p^+(r) = \{(s,t) \in B_p(r) \subset \mathbb{R}^2 : t \geq 0\};$$
$$\Gamma_p^+(r) = \{(s,t) \in \partial B_p^+(r) : t = 0\}, \quad \partial^+ B_p(r) = \partial B_p^+(r) \setminus \Gamma_p^+(r).$$

We shall use the symbols $o(1)$, $O(1)$ in a standard way to denote quantities that converge to 0 or are bounded, respectively. Analogously, we will write $o(\rho)$, $O(\rho)$ to denote quantities that, divided by ρ, converge to 0 or are bounded, respectively.

Given any function f, we denote by $f^- = \min\{f, 0\}$ and $f^+ = \max\{f, 0\}$, so that $f = f^+ + f^-$. Moreover, many times in the paper we drop the element of area or length, that is we shall only write $\int_\Sigma K e^u$ or $\int_{\partial\Sigma} h e^{u/2}$; these expressions refer to the standard measure given by the metric. We also use the symbol $\fint f$ to denote the mean value of f, that is

$$\fint_\Sigma f = \frac{1}{|\Sigma|} \int_\Sigma f.$$

2. OBSTRUCTIONS TO EXISTENCE AND EXAMPLES OF BLOW-UP

In this section we include two types of comments related to our problem. First we show some obstructions for the existence of solutions to problem (1.3), which somehow complement the existence results given in Theorems 1.1, 1.2, 1.3. Secondly, we will provide some explicit examples of blowing-up solutions for problem (1.7) with diverging mass, in order to highlight the differences of this phenomenon with respect to the more standard finite-mass blow-up.

2.1. **Obstructions to existence.** In [30] the problem of prescribing constant curvatures K, h_1, h_2 on an annulus is treated. In this case all solutions are known explicitly; in particular, the following result is proved (see [30], [Corollary 1, Section 3]):

Theorem 2.1. *Let Σ be an annulus with $K \equiv -1$ and constant geodesic curvatures h_1, h_2 at the two boundary components. Then:*

i) either $h_1 + h_2 > 0$ and both $h_i < 1$,
ii) or $h_1 + h_2 < 0$ and some $h_i > 1$,
iii) or $h_1 = 1$, $h_2 = -1$ or vice-versa.

As it can be seen, Theorem 1.2 is the counterpart of *i)*, whereas Theorem 1.3 is related to *ii)*. Case *iii)* is somehow borderline.

The assumptions of Theorems 1.1, 1.2, 1.3 are natural also in view of the following result:

Theorem 2.2. *Let Σ be a compact surface with boundary $\partial\Sigma$, and denote by h its geodesic curvature. Assume that $h(x) > \sqrt{|K^-(p)|}$ for all $x \in \partial\Sigma$, $p \in \Sigma$. Then Σ is homeomorphic to a disc.*

Proof. The proof follows closely an argument by Rosenberg in [40] (see also [45, Appendix A]), hence we will be sketchy. Assume by contradiction that Σ is not a topological disk, and consider two cases:

Case 1: $\partial\Sigma$ is not connected.
Let Λ_0 be a connected component of $\partial\Sigma$, and γ be a curve minimizing the distance between Λ_0 and $\partial\Sigma \setminus \Lambda_0$. Namely, γ is a solution of the problem:

$$Inf\{Length(\gamma): \ \gamma: [0, L] \to \Sigma, \ \gamma(0) \in \Lambda_0, \ \gamma(L) \in \partial\Sigma \setminus \Lambda_0\}.$$

We can assume that γ is parametrized by arc-length, and set $p = \gamma(0)$, $q = \gamma(L)$. Take a neighbourhood $O \subset \Sigma$ of $\gamma[0, L]$ and define $\Lambda_l \subset O$ to be a parallel curve to Λ_0 at distance $l \in [0, L]$, that is:

$$\Lambda_l = \{x \in O: \ d(x, \Lambda_0) = l\},$$

where d denotes the geodesic distance in Σ. Using the above curvature condition, it was shown in [45, Appendix A] that Λ_l is a regular curve if O is a sufficiently small neighbourhood. Define $\hat{h}(t)$ to be the geodesic curvature of the curve Λ_l at the point $\gamma(t)$. It can be shown (see again [45, Appendix A]) that $\hat{h}(t)$ is an increasing function of t, so $\hat{h}(L) > \hat{h}(0) = h(p)$. Observe now that Λ_l is in contact with $\partial\Sigma \setminus \Lambda_0$, so $h(q) \leq -\hat{h}(L)$, a contradiction.

Case 2: $\partial\Sigma$ is connected.
Let us denote by $\tilde{\Sigma}$ the compact surface obtained by identifying $\partial\Sigma$ to a single point. If Σ is not a topological disk, then $\tilde{\Sigma}$ is not homeomorphic to a sphere, hence it is not simply connected. Consider the following minimization problem:

$$Inf\left\{Length(\gamma): \ \gamma: [0, L] \to \Sigma, \ \{\gamma(0), \ \gamma(L)\} \subset \partial\Sigma, \ [\tilde{\gamma}] \text{ not trivial}\right\}.$$

Above, we are denoting $\tilde{\gamma}: [0, L] \to \tilde{\Sigma}$ to be the curve obtained by applying the identification to γ, and $[\tilde{\gamma}]$ to be its homotopy class. Clearly this infimum is attained at a certain γ (as in Figure 1), which is a local minimizer of the distance. We can now follow the arguments of Case 1 and still arrive to a contradiction.

$\square$

Let us point out that in case of a disk we can have $h(x) > \sqrt{|K^-(p)|}$ for all $x \in \partial\Sigma$, $p \in \Sigma$. For instance, in the Poincaré disk there are balls with constant geodesic curvature $h > 1$.
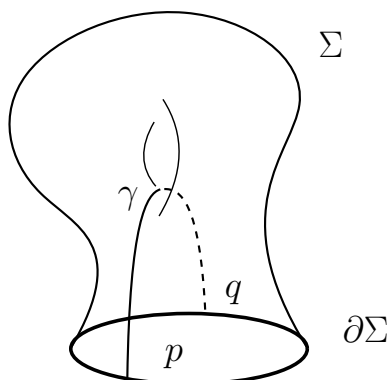
FIGURE 1. A curve minimizing the distance in Σ with non-trivial homotopy class in $\tilde{\Sigma}$

2.2. **Explicit examples of infinite mass blow-up.** Consider the problem:

(2.1)
$$
\begin{cases}
-\Delta u = -2e^u, & \text{in } A(0; r, 1), \\
\dfrac{\partial u}{\partial n} + 2 = 2h_1 e^{u/2}, & \text{on } |x| = 1, \\
\dfrac{\partial u}{\partial n} - 2/r = 2h_2 e^{u/2}, & \text{on } |x| = r,
\end{cases}
$$

where h_1, h_2 are two real constants. All explicit solutions of this problem have been classified by [30]; here we exhibit some of them observing that they form families of blowing-up solutions with infinite mass.

As a first example (see [30, Equation (3) and Lemma 2]), the functions

$$
u(x) = \log\left(\frac{4}{|x|^2(\lambda + 2\log|x|)^2}\right), \quad \text{for any } \lambda \notin [0, -2\log r],
$$

are solutions of (2.1); if $\lambda < 0$ then $h_1 = 1$ and $h_2 = -1$, whereas if $\lambda > -2\log r$, $h_1 = -1$ and $h_2 = 1$. Observe that if λ tends to 0 or $2\pi r$, then the functions u blow up at a whole component of the boundary, which is indeed the one with curvature equal to 1. In particular, the blowing-up set S is infinite here.

Let us give a second example. Given any $h_1 > 1$, for any $\gamma \in \mathbb{N}$, there exists an explicit solution in the form (we use complex variable notation):

(2.2)
$$
u_\gamma(z) = 2\log\left(\frac{\gamma|z|^{\gamma-1}}{h_1 + Re(z^\gamma)}\right),
$$

where $h_2 = -h_1 r^{-\gamma}$. See [30, equation (5) and Lemma 2].

Observe that the solutions in (2.2) form a blowing-up sequence as $\gamma \to +\infty$, keeping $h_1 > 1$ fixed. Indeed,

$$
u_\gamma(z) \to -\infty \text{ if } |z| < 1, \quad u_\gamma(z) \to +\infty \text{ if } |z| = 1.
$$

Hence in this case $S = \{|z| = 1\}$ and $h_1 > 1$; it is also easy to check that $\sigma = \delta_{\{|z|=1\}}$. Let us compute the asymptotic profile of these solutions. Fix a point of blow-up $z = -1$, and define the rescaling:

$$v_\gamma(z) = u_\gamma \left(1 - \frac{z}{\gamma} \right) - 2 \log \gamma.$$

Hence v can be written in the form:

$$v_\gamma(z) = 2 \log \left(\frac{|1 - \frac{z}{\gamma}|^{\gamma-1}}{h_1 + Re((1 - \frac{z}{\gamma})^\gamma)} \right).$$

Clearly,

$$\lim_{\gamma \to +\infty} \left(1 - \frac{z}{\gamma} \right)^\gamma = e^{-z},$$

so v_γ converges, at least point-wise, to the function:

$$v(z) = 2 \log \left(\frac{|e^{-z}|}{h_1 + Re(e^{-z})} \right) = 2 \log \left(\frac{e^{-t}}{h_1 + e^{-t} \cos s} \right),$$

which is defined in the half-plane $\{(s,t) \in \mathbb{R}^2 : t \geq 0\}$. This is indeed a solution to the limit problem in the half-space with $K = -1$, $h_1 > 1$, infinite mass and infinite Morse index, according to our results in Section 4.

This example shows that infinite Morse index blowing up solutions exist, even in this geometric context. Moreover, the blow-up limits can be entire solutions in the half-plane with infinite Morse index. Let us point out, though, that this example is not under the conditions of Theorem 1.4 since $h_2 = -h_1 r^{-\gamma}$ diverges negatively. Whether the infinite Morse index assumption is necessary or not in Theorem 1.4 *(4)* remains as an interesting open problem.

3. STUDY OF THE ENERGY FUNCTIONAL

In this section we will perform a variational study of the energy functional I defined in (1.4). From this we will derive the proof of Theorems 1.1, 1.2, and also 1.3, provided that Theorem 1.4 holds. This last result, which addresses the blow-up analysis, will be proved later.

As a first preliminary result, let us show that we can always prescribe zero geodesic curvature and constant Gaussian curvature.

Proposition 3.1. *Equation* (1.1) *is always solvable if $h = 0$ and*

$$K = 0 \text{ if } \chi(\Sigma) = 0, \quad K = -1 \text{ if } \chi(\Sigma) < 0, \quad K = 1 \text{ if } \chi(\Sigma) = 1.$$

The solution is unique up to a constant in the first case, unique in the second case, and unique up to Möbius transformations in the third case.

Proof. If $\chi(\Sigma) < 0$, we need to study the (strictly convex) energy functional:

$$J(u) = \int_\Sigma \left(\frac{1}{2} |\nabla u|^2 + 2\tilde{K}u + 2e^u \right) + 2 \oint_{\partial\Sigma} \tilde{h}u.$$

Taking into account that $\int_\Sigma \tilde{K} + \oint_{\partial\Sigma} \tilde{h} = 2\pi\chi(\Sigma) < 0$, it is easy to show the existence of a unique minimizer for J.

If $\chi(\Sigma) = 0$, we just need to find a solution to:

$$\begin{cases} -\Delta u + 2\tilde{K} = 0, & \text{in } \Sigma, \\ \dfrac{\partial u}{\partial n} + 2\tilde{h} = 0, & \text{on } \partial\Sigma. \end{cases}$$

This problem is linear and has a unique solution, up to addition of constants.

Finally, if $\chi(\Sigma) = 1$ then Σ is topologically a disk. By the Uniformization Theorem, Σ is conformally equivalent to the hemisphere so the statement follows. $\qquad\square$

With this result at hand, we can assume that our initial metric has constant Gaussian curvature and that $\partial\Sigma$ is geodesic. Hence our problem reduces to (1.3). As commented in the introduction, (1.3) is the Euler-Lagrange equation of the energy functional I defined in (1.4). The following estimate will be crucial in the variational study of I.

Lemma 3.2. *Define* $\bar{\mathfrak{D}} = \max\{\mathfrak{D}^+(x),\ x \in \partial\Sigma\}$. *For any* $\varepsilon > 0$ *there exists* $C > 0$ *such that:*

$$4 \oint_{\partial\Sigma} h e^{u/2} \leq (\bar{\mathfrak{D}} + \varepsilon) \left[\int_\Sigma \frac{1}{2}|\nabla u|^2 + 2|K|e^u \right] + C.$$

Proof. Take a finite partition of unity $\{\phi_j\}_{j=1}^M$ of $\partial\Sigma$ such that for all j one has $diam(supp\,\phi_j) < \delta$ for some fixed $\delta > 0$. We denote by Σ_j the support of ϕ_j. Take a smooth vector field N in Σ such that $N(x) = \nu(x)$ on the boundary, $|N(x)| \leq 1$. Then, we use the divergence theorem to obtain:

$$\oint_{\partial\Sigma} \phi_j e^{u/2} = \oint_{\partial\Sigma} \phi_j e^{u/2} N(x) \cdot \nu(x)$$

$$= \int_\Sigma e^{u/2} \left[\nabla \phi_j \cdot N + \phi_j div\,N + \frac{1}{2}\phi_j \nabla u \cdot N \right] \leq C \int_\Sigma e^{u/2} + \frac{1}{2}\int_\Sigma e^{u/2}|\nabla u|\phi_j.$$

Let us set $\bar{h}_j = \sup_{x\in\Sigma_j}\{h^+(x)\}$. If the diameter of Σ_j is small enough, one has that $\bar{h}_j \leq (\bar{\mathfrak{D}}+\varepsilon)\sqrt{|K(x)|}$ for any $x \in \Sigma_j$. Then, we apply Schwartz's inequality:

$$4 \oint_{\partial\Sigma} h e^{u/2} = 4\sum_{j=1}^M \oint_{\partial\Sigma} \phi_j h e^{u/2} \leq 4\sum_{j=1}^M \bar{h}_j \oint_{\partial\Sigma} \phi_j e^{u/2}$$

$$\leq C\int_\Sigma e^{u/2} + (\bar{\mathfrak{D}} + \varepsilon)\sum_{j=1}^M 2\int_\Sigma \sqrt{|K|}\phi_j e^{u/2}|\nabla u|$$

$$\leq C\int_\Sigma e^{u/2} + (\bar{\mathfrak{D}} + \varepsilon)\left(\sum_{j=1}^M 2\int_\Sigma |K|\phi_j e^u + \frac{1}{2}\int_\Sigma \phi_j|\nabla u|^2 \right)$$

$$= C\int_\Sigma e^{u/2} + (\bar{\mathfrak{D}} + \varepsilon)\left(2\int_\Sigma |K|e^u + \frac{1}{2}\int_\Sigma |\nabla u|^2 \right).$$

We conclude by observing that $\int_\Sigma e^{u/2} \leq \varepsilon \int_\Sigma e^u + C$ and renaming ε appropriately.

$\square$

Proof of Theorem 1.1. Under the assumptions of Theorem 1.1, recalling the notation from Lemma 3.2 we have $\bar{\mathfrak{D}} < 1$, so we can choose $\varepsilon > 0$ with $\bar{\mathfrak{D}} + \varepsilon < 1 - \varepsilon$. By inserting the inequality in Lemma 3.2 into the definition of I, we obtain:

$$I(u) \geq \int_\Sigma \left(\frac{\varepsilon}{2}|\nabla u|^2 + 2\varepsilon|K|e^u + 2\tilde{K}u \right) - C.$$

Since $\tilde{K} < 0$,

$$\lim_{u \to \pm\infty} 2\varepsilon|K|e^u + 2\tilde{K}u = +\infty.$$

Then I is coercive. It is also standard to check that I is weakly lower semicontinuous, so that it attains its infimum.

If moreover $h \leq 0$, then it is easy to check that the functional I is strictly convex. As a consequence its minimum corresponds to its unique critical point.

$\square$

Proof of Theorem 1.2. Here $\tilde{K} = 0$ and by Lemma 3.2 we just obtain:

$$(3.1) \qquad I(u) \geq \int_\Sigma \left(\frac{\varepsilon}{2}|\nabla u|^2 + 2\varepsilon|K|e^u \right) - C.$$

This implies that I is bounded from below but we do not have coercivity in this case. First, let us show that $\inf I < 0$. Indeed, take $u = -c$, with $c > 0$. Then:

$$I(-c) = e^{-c} \int_\Sigma 2|K| - 4e^{-c/2} \int_{\partial\Sigma} h,$$

which is negative for large c, since $\int_{\partial\Sigma} h > 0$.

Take now a minimizing sequence u_n for I. By (3.1), $\int_\Sigma |\nabla u_n|^2$ is bounded. We conclude if we show that $\fint_\Sigma u_n$ is bounded.

Up to a subsequence, we can assume that

$$u_n - \fint_\Sigma u_n \rightharpoonup u_0 \quad \text{in } H^1(\Sigma).$$

As a consequence,

$$e^{u_n - \fint_\Sigma u_n} \to e^{u_0} \text{ in } L^1(\Sigma), \ e^{(u_n - \fint_\Sigma u_n)/2} \to e^{u_0/2} \text{ in } L^1(\partial\Sigma).$$

If now $\fint_\Sigma u_n \to +\infty$, then

$$\int_\Sigma |K|e^{u_n} = e^{\fint_\Sigma u_n} \int_\Sigma |K|e^{u_n - \fint_\Sigma u_n} \to +\infty,$$

contradicting (3.1).

So, we now consider the remaining case $\fint_\Sigma u_n \to -\infty$. In such a case,

$$\oint_{\partial\Sigma} h\, e^{u_n/2} = e^{\frac{\fint_\Sigma u_n}{2}} \oint_{\partial\Sigma} h\, e^{(u_n - \fint_\Sigma u_n)/2} \to 0.$$

By the definition of I we conclude that

$$\liminf_{n \to +\infty} I(u_n) \geq 0,$$

which is a contradiction.

$\square$

In what follows we show that under the assumptions of Theorem 1.3, the functional I has a min-max structure.

Proposition 3.3. *Assume that $\tilde{K} = 0$, $K < 0$, $\oint_{\partial \Sigma} h < 0$ and $\mathfrak{D}(p) > 1$ for some $p \in \partial \Sigma$. Then there exist u_0, $u_1 \in H^1(\Sigma)$ such that:*

$$c = \inf_{\gamma} \max_{t} \{I(\gamma(t)) : \ t \in [0,1], \ \gamma \in \Gamma\} > \max\{I(u_0), I(u_1)\} > 0,$$

where $\Gamma = \{\gamma : [0,1] \to H^1(\Sigma) \text{ continuous} : \gamma(0) = u_0, \ \gamma(1) = u_1\}$.

The proof of Proposition 3.3 requires two preliminary results:

Lemma 3.4. *If $K < 0$ and $\mathfrak{D}(p) > 1$ for some $p \in \partial\Sigma$, then there exists a sequence u_n such that $I(u_n) \to -\infty$ and $\oint_{\partial\Sigma} e^{u_n/2} \to +\infty$.*

Proof. The proof of this result is postponed to the Appendix. $\square$

Lemma 3.5. *If $\tilde{K} = 0$ and $\oint_{\partial\Sigma} h < 0$, then there exists $\varepsilon > 0$, $\delta > 0$ such that*

$$I(u) > \varepsilon \quad \forall \, u \in H^1(\Sigma) \quad \text{with} \quad \oint_{\partial\Sigma} e^{u/2} = \delta.$$

Proof. First we claim that for any $\delta > 0$, the infimum:

$$\alpha_\delta = \inf \left\{ I(u) : u \in H^1(\Sigma) \text{ with } \oint_{\partial\Sigma} e^{u/2} = \delta \right\}$$

is attained. Recall the definition of I given in (1.4); this implies that $\alpha_\delta \neq -\infty$. Moreover, for a minimizing sequence u_n the integral $\int_\Sigma |\nabla u_n|^2$ must be bounded, otherwise $I(u_n) \to +\infty$. Up to a subsequence,

$$u_n - \fint_\Sigma u_n \rightharpoonup u_0 \text{ in } H^1(\Sigma),$$

which implies that

$$e^{u_n - \fint_\Sigma u_n} \to e^{u_0} \text{ in } L^1(\Sigma), \ e^{(u_n - \fint_\Sigma u_n)/2} \to e^{u_0/2} \text{ in } L^1(\partial\Sigma).$$

Moreover,

$$0 < \delta = \oint_{\partial\Sigma} e^{u/2} = e^{\frac{\fint_\Sigma u_n}{2}} \oint_{\partial\Sigma} h e^{(u_n - \fint_\Sigma u_n)/2},$$

which implies that $\fint_\Sigma u_n$ is bounded. Hence u_n is bounded in $H^1(\Sigma)$ and the conclusion follows from standard arguments.

We now turn our attention to the proof of Lemma 3.5. By contradiction, assume that there exists $u_n \in H^1(\Sigma)$ such that:

$$\liminf_n I(u_n) \leq 0, \qquad \oint_{\partial\Sigma} e^{u_n/2} \to 0.$$

By the definition of I it readily follows that $\int_\Sigma |\nabla u_n|^2 \to 0$, hence $u_n - \fint_\Sigma u_n \to 0$ in $H^1(\Sigma)$. Therefore,

$$I(u_n) = \int_\Sigma \frac{1}{2}|\nabla u_n|^2 + 2|K|e^{u_n} - 4\int_{\partial\Sigma} he^{u_n/2} \geq -4e^{\fint_\Sigma u_n/2}\int_{\partial\Sigma} he^{(u_n/-\fint_\Sigma u_n)/2},$$

but $\int_{\partial\Sigma} he^{(u_n/-\fint_\Sigma u_n)/2} \to \int h < 0$, hence $I(u_n) > 0$ for sufficiently large n. $\qquad\square$

Proof of Proposition 3.3. Fix $\varepsilon > 0$, $\delta > 0$ as given by Lemma 3.5. Take a constant $c > 0$ so large that

$$I(-c) = 2\int_\Sigma |K|e^{-c} - 4\oint_{\partial\Sigma} he^{-c/2} \in \left(0, \frac{\varepsilon}{2}\right),$$

and moreover $\oint_{\partial\Sigma} e^{-c/2} < \delta$. We define $u_0 = -c$.

By Lemma 3.4, there exists also u_1 with $\oint_{\partial\Sigma} e^{u_1/2} > \delta$ satisfying $I(u_1) < 0$.

Observe that for any $\gamma \in \Gamma$ there exists $t \in (0,1)$ such that $\oint_{\partial\Sigma} e^{\gamma(t)/2} = \delta$. As a consequence, $c \geq \varepsilon > \max\{I(u_0),\ I(u_1)\} > 0$. $\qquad\square$

The main issue in order to prove Theorem 1.3 is the fact that we do not know whether the Palais-Smale condition holds or not for I. The strategy consists in producing a sequence of solutions u_n to a perturbed version of (1.3), namely (1.7) (where $\tilde{K}_n$ and $\tilde{h}_n$ are as in Theorem 1.4), with Morse index not exceeding 1. The main tool is a monotonicity argument originally attributed to Struwe (see [43]), combined with a deformation argument from [18].

For ε close to zero, we consider the following family of functionals

$$I_\varepsilon(u) = I(u) + \varepsilon J(u),$$

where I is as in (1.4), and

$$J(u) = \int_\Sigma \left(|\nabla u|^2 + e^u - u\right).$$

Notice that, since

$$e^u - u \to +\infty \qquad \text{as } u \to \pm\infty,$$

$J(u)$ is coercive on $H^1(\Sigma)$.

For a small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$ we can reason as in the proof of Proposition 3.3, to find two elements $u_0, u_1 \in H^1(\Sigma)$ for which

$$(3.2) \qquad c_\varepsilon := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\varepsilon(\gamma(t)) > \max\{I_\varepsilon(u_0), I_\varepsilon(u_1)\} + \delta > \delta,$$

where $\delta > 0$ is a fixed positive number and where Γ, again, stands for the class of admissible curves

$$\Gamma = \left\{\gamma \in C([0,1]; H^1(\Sigma)) \ : \ \gamma(0) = u_0, \gamma(1) = u_1\right\}.$$

Under these assumptions, the function $\varepsilon \mapsto c_\varepsilon$ is monotone non-decreasing and therefore c_ε is a.e. differentiable in ε.

Consider a value ε where c_ε is differentiable, and let $\varepsilon_n \nearrow \varepsilon$. It is shown in the proof of Proposition 2.1 in [29] that at such a value, if $\gamma_n \in \Gamma$ satisfies

$$\max_{t \in [0,1]} I_\varepsilon(\gamma_n(t)) \le c_\lambda + (2 + c'_\lambda)(\lambda - \lambda_n),$$

and if t is such that

$$I_\varepsilon(\gamma_n(t)) \ge c_\varepsilon - (\varepsilon - \varepsilon_n),$$

then $\|\gamma_n(t)\| \le M$, with M depending on c'_ε.

Given $\alpha > 0$, define

$$(3.3) \qquad F_\alpha = \left\{ u \in H^1(\Sigma) \ : \ \|u\| \le M + 1 \text{ and } |I_\varepsilon - c_\varepsilon| \le \alpha \right\}.$$

The classical deformation lemma is used in [29] to prove that I_ε has a critical point in F_α.

On the other hand, in [18] the following result is shown. Assume that a functional $\mathcal{I}$ on a Hilbert space H is of class C^2 has a mountain-pass structure with mountain-pass level c, and assume there exists $\rho > 0$ such that $\mathcal{I}'$ and $\mathcal{I}''$ are uniformly Hölder continuous on the set $\{c - \rho \le \mathcal{I} \le c + \rho\}$. Then a Palais-Smale sequence x_n at level c is found, satisfying also the following second-order property (see Corollary 1 in [18]).

(i) If $\mathcal{I}''(x_n)[u, u] < -\frac{1}{n}\|u\|^2$ for all u belonging to a subspace $E \subseteq H$, then $\dim(E) \le 1$.

Such a sequence is found starting from curves $f(t)$, $t \in [0, 1]$, for which $\sup_{t \in [0,1]} \mathcal{I}(f(t))$ is close to c, and then deforming them properly at points for which $\mathcal{I}(f(t))$ is large enough. Such a deformation displaces the path $f(t)$ near its highest level only by a small amount, see formula (21) in the proof of Theorem 1.bis in [18].

In our case, level sub-supersets of the form $\{c_\varepsilon - \rho \le I_\varepsilon \le c_\varepsilon + \rho\}$ are unbounded, and therefore the above uniform Hölder continuity property is not guaranteed. However, by the above monotonicity argument, the deformation in [18] can be localized to the set F_α in (3.3), where such uniform Hölder continuity holds true. As a consequence, we obtain a bounded Palais-Smale sequence for I_ε at level c_ε also satisfying the property **(i)**. Recall that the maps $u \mapsto e^u$ and $u \mapsto e^{u/2}|_{\partial\Sigma}$ are compact from $H^1(\Sigma)$ into $L^1(\Sigma)$ and $L^1(\partial\Sigma)$ respectively. Therefore, passing to a proper subsequence, the limit of the latter Palais-Smale sequence exists and is a critical point of I_ε with Morse index less or equal to 1.

We can summarize the above discussion in the next proposition.

Proposition 3.6. *Suppose that we are under the assumptions of Theorem 1.3. Then there exists a sequence $\varepsilon_n \searrow 0$ and solutions u_n to problem* (1.7), *with*

$$\tilde{K}_n = \frac{-\varepsilon_n/2}{1 + 2\varepsilon_n}; \quad K_n(x) = \frac{K(x) - \varepsilon_n/2}{1 + 2\varepsilon_n}; \quad h_n(x) = \frac{h(x)}{1 + 2\varepsilon_n}; \quad \tilde{h}_n = 0.$$

Moreover $ind(u_n) \le 1$.

Proof of Theorem 1.3 We consider the sequence u_n given by Proposition 3.6, with $\varepsilon_n \searrow 0$. Observe in particular that $\chi_n \le 0$. By Theorem 1.4 *(4)* and our

assumptions on the function $\mathfrak{D}$, which has no critical points in $\{\mathfrak{D} = 1\}$, u_n must be uniformly bounded from above.

By these upper bounds on u_n, the right-hand sides of (1.7) are uniformly bounded. By standard elliptic regularity theory, a subsequence of $u_n - \fint_\Sigma u_n$ converges in $C^1(\Sigma)$ to some function w.

If we assume by contradiction that $\inf_\Sigma u_n \to -\infty$, then $\fint_\Sigma u_n \to -\infty$. However this would imply

$$e^{u_n} = e^{u_n - \fint_\Sigma u_n} e^{\fint_\Sigma u_n} \to 0,$$

uniformly on Σ. As a consequence, w is harmonic in Σ and satisfies homogeneous Neumann boundary conditions. Since $\int_\Sigma w = 0$, we must have $w \equiv 0$ and therefore, recalling the notation from the beginning of this section, $I_{\varepsilon_n}(u_n) \to 0$. This is however in contradiction to the fact that u_n is a min-max sequence, with min-max value satisfying (3.2). ∎

4. SOLUTIONS OF THE LIMIT PROBLEM IN THE HALF-SPACE AND THEIR MORSE INDEX

In this section we study the Morse index of solutions to the limit problem:

$$(4.1) \qquad \begin{cases} -\Delta v = 2K_0 e^v, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial v}{\partial n} = 2h_0 e^{v/2}, & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

where $K_0 < 0$, h_0 are constants, and $\mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 : t \geq 0\}$. The solutions of this problem have been classified in [23, 46], and they are as follows:

Theorem 4.1. *Define:* $\mathfrak{D}_\circ = \dfrac{h_0}{\sqrt{|K_0|}}$. *The following assertions hold true:*

(1) *If* $\mathfrak{D}_\circ < 1$ *then* (4.1) *does not admit any solution.*
(2) *If* $\mathfrak{D}_\circ = 1$ *the only solutions of* (4.1) *are given by:*

$$(4.2) \qquad v_\lambda(s,t) = 2\log\left(\frac{\lambda}{1 + \lambda t}\right) - \log|K_0|, \ \lambda > 0.$$

(3) *If* $\mathfrak{D}_\circ > 1$ *there exists a locally univalent holomorphic map g from* $\mathbb{R}^2_+$ *to a disk of geodesic curvature* $\mathfrak{D}_\circ$ *in the Poincaré disk* $\mathbb{H}^2$ *such that*

$$v(z) = 2\log\left(\frac{2|g'(z)|}{1 - |g(z)|^2}\right) - \log|K_0|.$$

We recall that $\mathbb{H}^2$ *is the unit disk in* $\mathbb{C}$ *equipped with the metric* $\frac{4}{(1-|z|^2)^2} \, dz$.

Moreover, g is a Möbius map if and only if

$$(4.3) \qquad \text{either } \int_{\mathbb{R}^2_+} e^v < +\infty \text{ or } \oint_{\partial\mathbb{R}^2_+} e^{v/2} < +\infty.$$

In such case v can be written as:

$$(4.4) \qquad v_\lambda(s,t) = 2\log\left(\frac{2\lambda}{(s-s_0)^2 + (t+t_0)^2 - \lambda^2}\right) - \log|K_0|,$$

where $\lambda > 0$, $s_0 \in \mathbb{R}$, $t_0 = \mathfrak{D}_\circ \lambda$. Moreover,

$$\int_{\mathbb{R}^2} |K_0| e^{v_\lambda} = \beta, \quad \oint_{\partial \mathbb{R}^2_+} h_0 e^{v_\lambda/2} = \beta + 2\pi,$$

with

$$(4.5) \qquad \beta = 2\pi\left(\frac{h_0}{\sqrt{h_0^2 + K_0}} - 1\right).$$

We will next compute the Morse index of the above solutions. By the change of variable $v = v + \log(|K_0|)$, we can assume $K_0 = -1$ and pass to the equivalent problem:

$$(4.6) \qquad \begin{cases} -\Delta v = -2e^v, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial v}{\partial n} = 2\mathfrak{D}_\circ e^{v/2}, & \text{on } \partial \mathbb{R}^2_+. \end{cases}$$

We will be concerned with the study of the quadratic form:

$$(4.7) \qquad Q(\psi) = \int_{\mathbb{R}^2_+} |\nabla\psi|^2 dV_g + 2\int_{\mathbb{R}^2_+} e^v \psi^2 \, dV_g - \mathfrak{D}_\circ \oint_{\partial \mathbb{R}^2_+} e^{v/2} \psi^2 \, dy_g,$$

where $\psi \in C_0^\infty(\mathbb{R}^2_+)$, the set of test functions with compact support (not necessarily zero on the boundary). We define the Morse index of a solution v of (4.6) as:

$$ind(v) = \sup\{dim(E): E \subset C_0^\infty(\mathbb{R}^2_+) \text{ vector space}, Q(\psi) < 0 \; \forall \psi \in E\}.$$

We understand that $ind(v) = +\infty$ if the above set is not bounded from above.

Theorem 4.2. *Let v be a solution of problem (4.6). Then:*
   a) If $\mathfrak{D}_\circ = 1$, then $ind(v) = 0$, namely v is stable.
   b) If $\mathfrak{D}_\circ > 1$, then:
       If (4.3) is satisfied, then $ind(v) = 1$. Otherwise, $ind(v) = +\infty$.

Proof. In case a) the solution v is given by (4.2). Let us consider the linearized problem:

$$\begin{cases} -\Delta\psi + 2\dfrac{1}{(1+t)^2}\psi = 0, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial\psi}{\partial n} = \psi, & \text{on } \partial \mathbb{R}^2_+. \end{cases}$$

An explicit solution is $\psi(s,t) = \psi(t) = \frac{1}{t+1}$, which is a positive function. This implies stability (see for instance [15], Section 1.2) .

In case b), let g be given by the above classification. By composing with a symmetry of $\mathbb{H}^2$ we can assume, without loss of generality, that $g(\mathbb{R}^2_+)$ is

contained in the disk $D_R \subset \mathbb{H}^2$ centred at 0. Here R denotes its euclidean radius, that satisfies $R = \mathfrak{D}_\mathfrak{o} - \sqrt{\mathfrak{D}_\mathfrak{o}^2 - 1} < 1$. In such a case, the Gaussian and geodesic curvatures of D_R in $\mathbb{H}^2$ translate to $\mathbb{R}_+^2$ via g.

In what follows we shall write $\rho(s) = \frac{4}{(1-s^2)^2}$. In order to study stability we pass to the disk D_R and study the following quadratic form:

$$Q_R(\psi) = \int_{D_R} |\nabla\psi|^2 dz + 2\int_{D_R} \psi^2 \rho(|z|)dz - \mathfrak{D}_\mathfrak{o} \oint_{\partial D_R} \psi^2 \sqrt{\rho(R)}dz,$$

for $\psi \in C^\infty(\overline{D_R})$. We claim that Q_R has Morse index 1. Its associated linear operator can be written as

(4.8)
$$\begin{cases} -\Delta\psi + 2\rho(|z|)\psi = 0, & \text{on } D_R, \\ \dfrac{\partial\psi}{\partial n} = \mathfrak{D}_\mathfrak{o}\sqrt{\rho(R)}\gamma, & \text{on } \partial D_R. \end{cases}$$

One can easily check that the functions:

$$\frac{x_1}{1-|x|^2}, \qquad \frac{x_2}{1-|x|^2},$$

satisfy (4.8). These elements in the kernel are of course related to the invariances of our problem. It is easy to observe, by using Fourier decomposition, that those functions correspond to the second mode expansion. As a consequence, there is a unique negative eigenvalue with radially symmetric eigenfunction. This eigenfunction is indeed explicit:

$$\gamma(x) = \frac{1+|x|^2}{1-|x|^2}.$$

Observe that this function is bounded in D_R, for $R \in (0,1)$. Moreover, γ solves the boundary value problem:

$$\begin{cases} -\Delta\psi + 2\rho(|z|)\psi = 0, & \text{on } D_R, \\ \dfrac{\partial\psi}{\partial n} = \dfrac{1}{\mathfrak{D}_\mathfrak{o}}\sqrt{\rho(R)}\gamma, & \text{on } \partial D_R. \end{cases}$$

Observe that this equation is very similar to (4.8), but with $\mathfrak{D}_\mathfrak{o}$ replaced by $\frac{1}{\mathfrak{D}_\mathfrak{o}}$. Since $\mathfrak{D}_\mathfrak{o} > 1$, we have that $Q_R(\gamma) < 0$. This finishes the proof of the claim.

Define now $\psi = \gamma \circ g$. Clearly, ψ solves:

(4.9)
$$\begin{cases} -\Delta\psi + 2e^v\psi = 0, & \text{in } \mathbb{R}_+^2, \\ \dfrac{\partial\psi}{\partial n} = \dfrac{1}{\mathfrak{D}_\mathfrak{o}}e^{v/2}\psi, & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

Let us first consider the case of finite mass, that is, assume that (4.3) holds. Using the invertibility of the Möbius map g, we can relate the second variation in D_R to that in $\mathbb{R}_+^2$, which implies that $Q(\psi) < 0$.

We observe now that, by definition, ψ is uniformly bounded in $\mathbb{R}_+^2$; since $e^v \in L^1(\mathbb{R}_+^2)$ and $e^{v/2} \in L^1(\partial\mathbb{R}_+^2)$ and since ψ has a limit at infinity, we

can find a compactly-supported function $\tilde{\psi}$ such that, still $Q(\tilde{\psi}) < 0$. As a consequence, $ind(v) \geq 1$.

If by contradiction we had the strict inequality, we would be able to find a two-dimensional space E_2 spanned by two (smooth) compactly-supported functions such that the above quadratic form would be negative-definite on E_2. Considering then the two-dimensional space $\hat{E}_2$ of functions on the disk D_R defined by

$$\hat{E}_2 = \left\{ \phi \circ g^{-1} \; : \; \phi \in E_2 \right\},$$

we would have that $Q_R(\hat{\phi}) < 0$ for all $\hat{\phi} \in \hat{E}_2$. This would contradict the fact that the index of Q_R is precisely 1. Therefore, we proved that if (4.3) holds then $ind(v) = 1$.

Assume now that $\oint_{\partial \mathbb{R}^2_+} e^{v/2} = +\infty$: we will show that the solution v is not stable outside any compact set. This implies in particular that the Morse index is infinite.

We multiply (4.9) by $\phi^2 \psi$ and integrate, where ϕ is a conveniently chosen cut-off function:

$$0 = \int_{\mathbb{R}^2_+} (-\Delta \psi + 2e^v \psi)\phi^2 \psi = \int_{\mathbb{R}^2_+} \nabla \psi \cdot \nabla(\phi^2 \psi) + 2e^v \psi^2 \phi^2 - \oint_{\partial \mathbb{R}^2_+} \frac{1}{\mathfrak{D}_\circ} e^{v/2} \psi^2 \phi^2.$$

Taking into account that $|\nabla(\phi\psi)|^2 = \nabla \psi \cdot \nabla(\phi^2 \psi) + \psi^2 |\nabla \phi|^2$, we have:

$$(4.10) \qquad \int_{\mathbb{R}^2_+} \psi^2 |\nabla \phi|^2 = \int_{\mathbb{R}^2_+} |\nabla(\phi\psi)|^2 + 2e^v(\psi\phi)^2 - \oint_{\partial \mathbb{R}^2_+} \frac{1}{\mathfrak{D}_\circ} e^{v/2}(\psi\phi)^2.$$

Observe that the above right-hand side is similar to the expression of $Q(\phi\psi)$ given in (4.7), but with $\frac{1}{\mathfrak{D}_\circ}$ instead of $\mathfrak{D}_\circ$.

Indeed, given a fixed $M_0 > 0$ and M large enough, take $\phi = \phi_M$ a non-negative cut-off such function such that

$$\phi = 0 \text{ in } B_0(M_0), \phi = 1 \text{ in } A(0; 2M_0, M), \; \phi = 0 \text{ in } B_0(2M)^c,$$

$$\int_{\mathbb{R}^2} |\nabla \phi_M|^2 \leq C, \quad \text{independently of } M.$$

Recall now (4.10), and let us estimate:

$$\int_{\mathbb{R}^2_+} \psi^2 |\nabla \phi|^2 \leq C,$$

However $\psi \geq 1$, and so

$$\oint_{\partial \mathbb{R}^2_+} e^{v/2}(\psi\phi)^2 \geq \oint_{\partial \mathbb{R}^2_+ \cap A(0; 2M_0, M)} e^{v/2}.$$

Since $\oint_{\partial \mathbb{R}^2_+} e^{v/2} = +\infty$ and M_0 is fixed, the above term diverges as $M \to +\infty$. Hence we can choose M so that $Q(\phi\psi) < 0$. Since M_0 is arbitrary we obtain instability outside any compact set, as claimed. $\qquad \square$

5. BLOW-UP ANALYSIS. GENERAL PROPERTIES

The goal of the rest of the paper is to prove Theorem 1.4. In this section we focus on some general properties of blowing-up sequences of solutions to (1.7). In particular we will derive the proof of Theorem 1.4, *(1)*.

Proposition 5.1. *The singular set S defined in* (1.6) *satisfies*

$$S \subset \{p \in \partial\Sigma \ : \ \mathfrak{D}(p) \geq 1\} .$$

The spirit of the proof is simple: if $p \in S$, then one can rescale around p to obtain a solution on the half-plane, and this is possible only if $\mathfrak{D}(p) \geq 1$, as recalled in the previous section. The non-trivial point here is to be able to rescale and pass to a limit problem even if p is not isolated in S. We show that this is possible by choosing carefully a sequence $x_n \in \Sigma$ (not necessarily local maxima) such that $x_n \to p$ and $u_n(x_n) \to +\infty$. Let us point out that this is a technical novelty even for the classical problem considered in [6, 34]: one can pass to the limit around a singular point *without knowing if the singular set is finite.*

For this purpose we shall use Ekeland's variational principle, which we recall below:

Theorem 5.2 (see Chapter 5 in [44]). *Let (X, d) be a complete metric space and consider a function $\varphi : X \to (-\infty, +\infty]$ that is lower semi-continuous, bounded from below and not identical to $+\infty$. Let $\varepsilon > 0$ and $\lambda > 0$ be given and let $x \in X$ be such that $\varphi(x) \leq \inf_X \varphi + \varepsilon$. Then there exists $x_\varepsilon \in X$ such that*

(1) $\varphi(x_\varepsilon) \leq \varphi(x)$,
(2) $d(x_\varepsilon, x) \leq \lambda$,
(3) $\varphi(x_\varepsilon) < \varphi(z) + \varepsilon \dfrac{1}{\lambda} d(x_\varepsilon, z)$ for every $z \neq x_\varepsilon$.

Proof. Let p be a point in S. By conformal invariance, we can pass from a neighborhood of p to a domain $B \subset \mathbb{R}^2$, where B denotes $B_0(r) \subset \mathbb{R}^2$, if $p \in int(\Sigma)$, or $B_0^+(r)$, if $p \in \partial\Sigma$. We can choose this conformal map so that p is mapped to 0.

Let us take a sequence y_n in B such that $y_n \to 0$ and $u_n(y_n) \to +\infty$, and define

$$\varepsilon_n = e^{-\frac{u_n(y_n)}{2}}.$$

We apply Theorem 5.2 taking $\varphi = e^{-\frac{u_n}{2}}$ and $\lambda = \sqrt{\varepsilon_n}$: then there exists a sequence $x_n \in B$ such that

a) $u_n(y_n) \leq u_n(x_n)$,
b) $d(x_n, y_n) \leq \sqrt{\varepsilon_n}$,
c) $e^{-\frac{u_n(x_n)}{2}} < e^{-\frac{u_n(z)}{2}} + \sqrt{\varepsilon_n} d(x_n, z)$ for every $z \neq x_n$.

As a consequence of a) and b) above, $x_n \to 0$ and $u_n(x_n) \to +\infty$. The idea is that the new sequence x_n is convenient for rescaling and passing to a limit problem.

Now, set:

$$(5.1) \qquad \delta_n = e^{\frac{-u_n(x_n)}{2}} \to 0, \ B_n = B_{x_n}(r/2) \cap B,$$

and

(5.2) $$v_n(x) = u_n(\delta_n x + x_n) + 2 \log \delta_n,$$

which is defined in $\tilde{B}_n = \frac{1}{\delta_n} B_n$. Clearly, $v_n(0) = 0$. We claim that, given any $R > 0$, $\varepsilon > 0$,

(5.3) $$v_n(x) \leq \varepsilon \ \ \forall x \in \tilde{B}_n, \ |x| < R,$$

for sufficiently large n. Indeed, we use c) in the choice of the sequence x_n and recall the definition of δ_n; we conclude that if $|z - x_n| < R\delta_n$, then

$$(1 - \sqrt{\varepsilon_n} R) e^{-\frac{u_n(x_n)}{2}} < e^{-\frac{u_n(z)}{2}}.$$

From this we get:

$$u_n(z) < u_n(x_n) - 2 \log(1 - \sqrt{\varepsilon_n} R), \ \text{if } |z - x_n| < R\delta_n.$$

And this implies (5.3).

In what follows we distinguish two cases:

Case 1 $p \in \partial \Sigma$ and, up to a subsequence:

$$d(x_n, \Gamma_0^+(r)) = O(\delta_n) \quad \text{as } n \to +\infty.$$

Passing to a subsequence we can assume that $\frac{d(x_n, \Gamma_0^+(r))}{\delta_n} \to t_0 \geq 0$. Then, the function v_n solves

(5.4) $$\begin{cases} -\Delta v_n + 2\delta_n^2 \tilde{K}_n(\delta_n x + x_n) = 2K_n(\delta_n x + x_n) e^{v_n}, & \text{in } \tilde{B}_n, \\ \dfrac{\partial v_n}{\partial n} + 2\delta_n \tilde{h}_n(\delta_n x + x_n) = 2h_n(\delta_n x + x_n) e^{v_n/2}, & \text{on } \tilde{\Gamma}_n, \end{cases}$$

where $\tilde{\Gamma}_n$ is the straight portion of $\partial \tilde{B}_n$.

Taking (5.3) into account, by Harnack type inequalities (see Lemma A.2 in [31]), v_n is uniformly bounded in $L_{loc}^\infty(\mathbb{R} \times (-t_0, +\infty))$. Therefore, up to subsequence,

(5.5) $$v_n \to v \quad \text{in } C_{loc}^2(\mathbb{R} \times (-t_0, +\infty)),$$

which is a solution of the equation

(5.6) $$\begin{cases} -\Delta v = 2K(0) e^v, & \text{in } \mathbb{R} \times (-t_0, +\infty), \\ \dfrac{\partial v}{\partial n} = 2h(0) e^{v/2}, & \text{if } t = -t_0. \end{cases}$$

If $K(0) < 0$ and $h(0) > 0$, the latter problem admits solutions only if $\mathfrak{D}(0) \geq 1$, see Section 4.

Case 2

$$\frac{d(x_n, \Gamma_0^+(r))}{\delta_n} \to +\infty, \quad \text{as } n \to +\infty.$$

In this situation the rescaled domains $\tilde{B}_n$ invade all of $\mathbb{R}^2$. Hence, reasoning as before, up to subsequence we have

$$v_n \to v \quad \text{uniformly in } C^2_{loc}(\mathbb{R}^2),$$

which is a solution of the equation

(5.7) $$-\Delta v = 2K(0)e^v, \qquad \text{in } \mathbb{R}^2.$$

If $K(0) < 0$, is it well known, via Liouville's formula, that problem (5.7) does not admit any solution.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 5.3. *In the above proof, in Case 1, we arrive to a solution of* (5.6). *Moreover* $v(0) = 0$ *and, by* (5.3), $v(x) \leq 0$ *for all* $x \in \mathbb{R} \times [-t_0, +\infty)]$. *By the maximum principle this implies that actually* $t_0 = 0$. *That is, one obtains the more precise conclusion* $d(x_n, \Gamma_0^+(r)) = o(\delta_n)$.

We now state and prove a couple of lemmas for later use.

Lemma 5.4. *There holds:*

  a) For any $q \in (1, 2)$, $\int_\Sigma |\nabla u_n^-|^q < O(1)$.
  b) If, moreover, $\chi_n \leq 0$, *then* $\int_\Sigma |\nabla u_n^-|^2 < O(1)$.
  c) For any $K \subset \Sigma$ *compact set with* $K \cap S = \emptyset$, *we have that:*

$$\sup_K u_n - \inf_K u_n = O(1), \quad \int_K |\nabla u_n|^2 = O(1).$$

Estimates like *c)* are commonly used in Liouville type problems, and are usually proved by using a Green's representation argument. However, in our case this is not possible unless the total mass is finite. Here we give a different proof, which is based on estimates for u_n^- and on local regularity arguments.

The estimates of u_n^- follow the idea of [13], where the Kato inequality is used. The problem is that here it is not so clear which boundary condition is satisfied by u_n^-, hence we use a smooth approximation of the function $u \mapsto u^-$.

Proof. Define $w : \mathbb{R} \to \mathbb{R}$ a C^2 function such that:

$$\begin{cases} w(u) = u & u \leq 0, \\ w(u) = 1 & u \geq 1, \\ w'(u) \geq 0 & u \in \mathbb{R}, \\ w''(u) \leq 0 & u \in \mathbb{R}. \end{cases}$$

Then the function $w_n = w(u_n)$ satisfies

(5.8) $$\begin{cases} -\Delta w_n = -w''(u_n)|\nabla u_n|^2 - 2w'(u_n)(\tilde{K}_n + |K_n|e^{u_n}), & x \in \Sigma, \\ \dfrac{\partial w_n}{\partial \nu} + 2w'(u_n)\tilde{h}_n = 2w'(u_n)h_n e^{u_n/2}, & x \in \partial\Sigma. \end{cases}$$

Just by integrating we find that

$$0 \leq \int_\Sigma -w''(u_n)|\nabla u_n|^2 \leq O(1).$$

As a consequence, the right-hand side of the first equation in (5.8) is bounded in L^1, whereas the boundary data are bounded in L^∞. By elliptic regularity estimates, $\int_\Sigma |\nabla w_n|^q < O(1)$ for any $q \in (1,2)$. This implies *a)*.

We now prove *b)*. We multiply (1.7) by u_n^- and integrate, to get

$$\int_\Sigma |\nabla u_n^-|^2 + 2u_n^-(\tilde{K}_n + |K_n|e^{u_n})] = 2\oint_{\partial\Sigma} u_n^-(h_n e^{u_n/2} - \tilde{h}_n).$$

Clearly the functions $u_n^- e^{u_n}$, $u_n^- e^{u_n/2}$ are bounded in L^∞. Hence,

$$\int_\Sigma |\nabla u_n^-|^2 = -2\left[\int_\Sigma \tilde{K}_n u_n^- + \oint_{\partial\Sigma} \tilde{h}_n u_n^-\right] + O(1).$$

We are able to estimate the right hand side by the assumption on χ_n. We split $u_n^- = (u_n^- - \fint_\Sigma u_n^-) + \fint_\Sigma u_n^-$, and, making use of *a)*, we obtain:

$$\left|\int_\Sigma \tilde{K}_n(u_n^- - \fint_\Sigma u_n^-)\right| \le O(1), \quad \left|\oint_{\partial\Sigma} \tilde{h}_n(u_n^- - \fint_\Sigma u_n^-)\right| \le O(1).$$

Moreover,

$$-2\left[\int_\Sigma \tilde{K}_n \fint_\Sigma u_n^- + \oint_{\partial\Sigma} \tilde{h}_n \fint_\Sigma u_n^-\right] = -2\chi_n \fint_\Sigma u_n^- \le 0,$$

yielding the assertion.

Finally, let us show *c)*. We can assume that $K = \overline{B}_p(r)$ where $B_p(4r) \cap S = \emptyset$ with $p \in \Sigma$ and:

(1) either $B_p(4r) \subset int(\Sigma)$,
(2) or $p \in \partial\Sigma$.

Via a conformal map we can pass to a problem in $B_0(4r) \subset \mathbb{R}^2$ or $B_0^+(4r) \subset \mathbb{R}_+^2$.

Observe that in any case $u_n \le C$ in $B_p(2r)$ for some $C > 0$. Recall that as in a), we have that the function $v_n = \min\{u_n, C\}$ satisfies $\int_\Sigma |\nabla v_n|^q = O(1)$ for all $q \in [1,2)$. As a consequence, the function $\tilde{u}_n = u_n - \fint_{B_p(2r)} u_n$ is bounded in $L^{q'}(B_p(2r))$ for any $q' > 1$.

In case (1) we have that $\tilde{u}_n$ solves:

$$-\Delta\tilde{u}_n = f_n, \qquad \text{in } B_0(2r),$$

with f_n bounded in L^∞. We conclude then by local regularity estimates (see for instance [24, Theorem 8.17 and Theorem 8.32]).

In case 2 we are led to the problem:

$$\begin{cases} -\Delta\tilde{u}_n = f_n, & \text{in } B_0^+(2r), \\ \dfrac{\partial\tilde{u}_n}{\partial n} = g_n, & \text{on } \Gamma_0^+(2r), \end{cases}$$

with f_n, g_n bounded in L^∞. We conclude by local regularity estimates for the Neumann problem (see for instance [36, Theorem 5.36 and Lemma 5.51]).

$\square$

The following lemma gives a Pohozaev type identity, depending on an arbitrary field F, which we state and prove for the sake of completeness.

Lemma 5.5. *Let u be a solution of*

$$-\Delta u + 2\tilde{K} = 2K(x)e^u, \qquad in\ \Sigma.$$

Then, given any vector field $F : \Sigma \to T\Sigma$, there holds:

$$\oint_{\partial\Sigma}[4K(x)e^u(F \cdot \nu) + 2(\nabla u \cdot \nu)(\nabla u \cdot F) - |\nabla u|^2 F \cdot \nu]$$

$$= \int_{\Sigma}[4\tilde{K}\nabla u \cdot F + 4e^u(\nabla K \cdot F + K\,\nabla \cdot F) + 2DF(\nabla u, \nabla u) - \nabla \cdot F|\nabla u|^2].$$

Proof. We will make use of the following basic identity:

$$2\Delta u(\nabla u \cdot F) = \nabla \cdot (2(\nabla u \cdot F)\nabla u - |\nabla u|^2 F) - 2DF(\nabla u, \nabla u) + |\nabla u|^2 \nabla \cdot F.$$

With this identity at hand, we just multiply the equation by $2\nabla u \cdot F$ and integrate, using the divergence theorem. Take into account also that, again by the divergence theorem,

$$4\int_{\Sigma} Ke^u\nabla u \cdot F = 4\left[\oint_{\partial\Sigma} Ke^u F \cdot \nu - \int_{\Sigma} e^u(\nabla K \cdot F + K\,\nabla \cdot F)\right].$$

$\square$

6. BLOW-UP WITH UNBOUNDED MASS

In this section we address the question of infinite-mass blow-up. As we shall see later, this is the only possible blow-up scenario if $\chi_n \leq 0$, and it is a completely new phenomenon in this type of problems. Let us set:

$$\rho_n = \oint_{\partial\Sigma} h_n e^{u_n/2} = \int_{\Sigma} |K_n|e^{u_n} + O(1) \to +\infty.$$

Hence, up to a subsequence, we obtain that

(1) $\rho_n^{-1}he^{u_n/2} \rightharpoonup \sigma$,
(2) $\rho_n^{-1}|K|e^{u_n} \rightharpoonup \xi$,

where σ, ξ are unit positive measures defined on $\partial\Sigma$ and Σ, respectively, and the above is weak convergence of measures.

The next proposition implies *(3) a)* of Theorem 1.4.

Proposition 6.1. $\xi|_{int(\Sigma)} = 0$ and $\xi|_{\partial\Sigma} = \sigma$.

Proof. Fix $\phi \in C^2(\Sigma)$, multiply equation (1.3) by ϕ and use Green's formula, to obtain:

$$2\oint_{\partial\Sigma}[h_n e^{u_n/2} - \tilde{h}_n]\phi - 2\int_{\Sigma}[\hat{K}_n\phi + |K_n|e^{u_n}\phi] = \int_{\Sigma} u_n\Delta\phi + \oint_{\partial\Sigma} \frac{\partial\phi}{\partial\nu}u_n.$$

We now estimate the right hand side taking into account the positive and negative parts of u_n, where we recall that in our notation $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$:

$$\left|\int_{\Sigma} u_n^+\Delta\phi + \oint_{\partial\Sigma} \frac{\partial\phi}{\partial\nu}u_n^+\right| \leq C\int_{\Sigma} u_n^+ + \oint_{\partial\Sigma} u_n^+ = o(\rho_n).$$

Moreover,

$$\int_\Sigma u_n^- \Delta\phi + \oint_{\partial\Sigma} \frac{\partial\phi}{\partial\nu} u_n^- = \int_\Sigma (u_n^- - \fint_\Sigma u_n^-)\Delta\phi + \oint_{\partial\Sigma} \frac{\partial\phi}{\partial\nu}(u_n^- - \fint_\Sigma u_n^-),$$

and

$$\int_\Sigma |u_n^- - \fint_\Sigma u_n^-| + \oint_{\partial\Sigma} |u_n^- - \fint_\Sigma u_n^-| \le C \int_\Sigma |\nabla u_n^-|^q,$$

for any $q \in (1,2)$, and this last quantity is bounded by Lemma 5.4, *a)*.

Since ϕ is arbitrary, we can conclude. $\qquad\square$

Observe that $supp\,\sigma \subset S \subset \{p \in \partial\Sigma : \mathfrak{D}(p) \ge 1\}$ by Proposition 5.1. In what follows we will show that $\mathrm{supp}\,\sigma \subset \{p \in \partial\Sigma : \mathfrak{D}_\tau(p) = 0\}$. This will be accomplished by making use of the Pohozaev-type identity given in Lemma 5.5 on fields that are tangential to $\partial\Sigma$. However, here the question is delicate since the support of σ need not be finite. Moreover, and more importantly, we do not have any control of the asymptotic behavior of the Dirichlet energy of the solutions.

The idea is to apply Lemma 5.5 to holomorphic fields F. In this way, the Cauchy-Riemann equations imply that the terms involving the Dirichlet energy vanish. For this, we will need to first pass to an analytic setting via a conformal map.

Proposition 6.2. $\mathrm{supp}\,\sigma \subset \{p \in \partial\Sigma : \mathfrak{D}_\tau(p) = 0\}$.

Proof. Let Λ_0 be a connected component of $\partial\Sigma$ such that $\sigma|_{\Lambda_0} \ne 0$, and consider a smooth neighborhood U. By the Uniformization Theorem for annuli (see for instance [20, IV.7]) we can pass via a conformal map to a problem in the annulus $A(0; r, 1)$ for some $r > 0$. That is, we need to consider:

$$\begin{cases} -\Delta_g u_n + 2\tilde{K}_n(x) = 2K_n(x)e^{u_n}, & \text{in } A(0; r, 1), \\ \dfrac{\partial u_n}{\partial n} + 2\tilde{h}_n(x) = 2h_n(x)e^{u_n/2}, & \text{on } \Lambda_0, \end{cases}$$

where g is a metric conformal to the standard one g_0, that is, $g = e^v g_0$. Here we identify $\Lambda_0 = \{|x| = 1\}$.

As a consequence, the function $\hat{u}_n = u_n + v$ satisfies the following equation with respect to g_0:

$$(6.1) \quad \begin{cases} -\Delta \hat{u}_n + 2\hat{K}_n(x) = 2K_n(x)e^{\hat{u}_n}, & \text{in } A(0; r, 1), \\ \dfrac{\partial u_n}{\partial n} + 2\hat{h}_n(x) = 2h_n(x)e^{\hat{u}_n/2}, & \text{on } \Lambda_0, \end{cases}$$

for some smooth functions $\hat{K}_n$, $\hat{h}_n$. In what follows we can consider problem (6.1) in a flat annulus and prove the statement of Proposition 6.2.

Take any analytic real function f defined in Λ_0, and consider $F : \Lambda_0 \to \mathbb{C}$ defined by $F(p) = f(p)\tau(p)$, where $\tau(p)$ is the tangent unit vector. By analytic continuation we can extend F to a holomorphic function $F : A(0; r, 1) \to \mathbb{C}$, by taking r closer to 1 if necessary. We define $\tilde{F}(p) = F(p)\phi(p)$, where

ϕ is a cut-off such that $\phi = 1$ in $A(0; r_0, 1)$ and $\phi = 0$ in $A(0; r, r_1)$ with $r < r_1 < r_0$. We apply Lemma 5.5 to the field F to obtain:

$$\oint_{\Lambda_0} 4f(h_n e^{\hat{u}_n/2} - \hat{h}_n)(\hat{u}_n)_\tau$$
$$= \int_{A(0;r_0,1)} [4\hat{K}_n \nabla \hat{u}_n \cdot F + 4e^{\hat{u}_n}(\nabla K_n \cdot F + K_n \nabla \cdot F)] + O(1).$$

Observe that here we are using Lemma 5.4, c). Let us also point out here that $2DF(\nabla \hat{u}_n, \nabla \hat{u}_n) - \nabla \cdot F|\nabla \hat{u}_n|^2 = 0$ by the Cauchy-Riemann equations. We now get rid of

$$\int_{A(0;r_0,1)} 4\hat{K}_n \nabla \hat{u}_n \cdot F = \int_{A(0;r_0,1)} 4\hat{K}_n (\nabla \hat{u}_n^+ + \nabla \hat{u}_n^-) \cdot F.$$

The term $\int_{A(0;r_0,1)} 4\hat{K}_n \nabla \hat{u}_n^- \cdot F$ is bounded by Lemma 5.4, whereas the one with the positive part of $\hat{u}_n$ can be estimated via an integration by parts:

$$\left| \int_{A(0;r_0,1)} 4\hat{K}_n \nabla \hat{u}_n^+ \cdot F \right| \leq \left| \oint_{\Lambda_0} 4\hat{K}_n \hat{u}_n^+ F \cdot \nu - \int_{A(0;r_0,1)} \hat{u}_n^+ \nabla \cdot (\hat{K}_n F) \right| + O(1)$$
$$\leq C \left(\oint_{\Lambda_0} \hat{u}_n^+ + \int_{A(0;r_0,1)} \hat{u}_n^+ \right) + O(1) = o(\rho_n).$$

In the same way we can estimate the term:

$$\oint_{\Lambda_0} f\hat{h}_n(\hat{u}_n)_\tau = \oint_{\Lambda_0} f\hat{h}_n(\hat{u}_n^+ + \hat{u}_n^-)_\tau.$$

Indeed,

$$\oint_{\Lambda_0} f\hat{h}_n(\hat{u}_n^-)_\tau = \oint_{\Lambda_0} f\hat{h}_n \left(\hat{u}_n^- - \fint_{\Lambda_0} u_n^- \right)_\tau =$$
$$-\oint_{\Lambda_0} (f\hat{h}_n)_\tau \left(\hat{u}_n^- - \fint_{\Lambda_0} u_n^- \right) = O(1),$$

by Lemma 5.4. Moreover,

$$\oint_{\Lambda_0} f\hat{h}_n(\hat{u}_n^+)_\tau = -\oint_{\Lambda_0} (f\hat{h}_n)_\tau \hat{u}_n^+ = o(\rho_n).$$

Integrating by parts we find

$$4\oint_{\Lambda_0} h_n f e^{\hat{u}_n/2}(\nabla \hat{u}_n \cdot F) = -8\oint_{\Lambda_0} (h_n f)_\tau e^{\hat{u}_n/2}.$$

Then,

$$-8\oint_{\Lambda_0} ((h_n)_\tau f + h_n f_\tau)e^{\hat{u}_n/2} = \int_{A(0;r_0,1)} (4\nabla K_n \cdot F + 4K_n \nabla \cdot F)e^{\hat{u}_n} + o(\rho_n).$$

Observe now that on Λ_0, $\nabla \cdot F = 2f_\tau$. By Proposition 6.1, we can divide by ρ_n and pass to the limit to obtain:

$$-8 \oint_{\Lambda_0} \left(\frac{h_\tau}{h} f + f_\tau \right) d\sigma = - \oint_{\Lambda_0} \left(4f \frac{K_\tau}{K} + 8 f_\tau \right) d\sigma.$$

Recall that in the support of σ we have the inequality $\mathfrak{D}(p) \geq 1$, which implies that $h(p)$ is positive: this allows us to write h in the denominator.

The terms in f_τ cancel and we can rewrite this expression as:

$$\oint_{\Lambda_0} \left(2 \frac{h_\tau}{h} - \frac{K_\tau}{K} \right) f \, d\sigma = 0.$$

Let us define the measure $\mu = \left(2 \frac{h_\tau}{h} - \frac{K_\tau}{K} \right) \sigma$. Then, we have obtained that $\oint_{\Lambda_0} f \, d\mu = 0$ for any analytic real function f. Since the analytic functions form a dense subset of the space of continuous functions, we conclude that $\mu = 0$.

Finally, notice that

$$2 \frac{h_\tau}{h} - \frac{K_\tau}{K} = 2 \frac{\mathfrak{D}_\tau}{\mathfrak{D}}.$$

Hence $\mathfrak{D}_\tau(p) = 0$ for any $p \in \operatorname{supp} \sigma$. $\qquad\square$

7. Blow-up with bounded mass or Morse index

In this section we consider the case in which the sequence u_n has either bounded mass or bounded Morse index, and we will exploit this information to give a more complete description of the blow-up phenomena, proving *(2)* and *(4)* in Theorem 1.4. The key ingredient here is that, if $\mathfrak{D}(p) > 1$, the only limit profiles with bounded mass or bounded Morse index are the bubbles in the form (4.4).

Let us start with the following fact:

Lemma 7.1. *Assume that $\int_\Sigma e^{u_n}$ is bounded in n. Then,*

$$S \subset \{ p \in \partial \Sigma : \mathfrak{D}(p) > 1 \} .$$

Proof. Let p be a point in S, and consider the function v_n defined in (5.2). By Fatou's lemma, the classification of the entire solutions on the upper half-plane and Remark 5.3, we find that for some $r > 0$

$$C \geq \lim_{n \to +\infty} \int_{B_p^+(r)} |K_n| e^{u_n} = \lim_{n \to +\infty} \int_{\tilde{B}_n} |K_n| e^{v_n} \geq \int_{\mathbb{R}_+^2} |K(p_0)| e^v,$$

where $\tilde{B}_n = B_{\frac{x_n - p}{\delta_n}}^+ \left(\frac{r}{\delta_n} \right)$ and v is an entire solution of the limit problem (5.6). Observe that if $\mathfrak{D}(p) = 1$, v is of the form (4.2) (see Section 4), so the upper bound on the volume is violated. Therefore $\mathfrak{D}(p) > 1$.

$\qquad\square$

The main result of the section is the following proposition, which completes the proof of Theorem 1.4 and includes the asymptotic behavior of the blowing-up sequences of solutions near the singular points.

Proposition 7.2. *Let u_n be a blowing-up sequence (namely, $\sup_\Sigma u_n \to +\infty$) of solutions to (1.7), under the conditions of Theorem 1.4. Assume also that:*

$$(7.1) \qquad\qquad \text{either } \int_\Sigma e^{u_n} \text{ or } ind(u_n) \text{ is bounded.}$$

Then $S = S_0 \cup S_1$, with:
$$S_0 \subset \{p \in \partial\Sigma : \mathfrak{D}(p) = 1, \ \mathfrak{D}_\tau(p) = 0\},$$
$$S_1 = \{p_1, \ldots p_m\} \subset \{p \in \partial\Sigma : \mathfrak{D}(p) > 1\}.$$
Moreover, the following asymptotics hold:

    a) *If $p \in S_0$, there exists $x_n \to p$, $\lambda_n \to 0$, $R_n \to +\infty$ such that $d(x_n, \partial\Sigma) = o(\lambda_n)$ and*

$$(7.2) \qquad\qquad u_n(y) = v_{\lambda_n}(0, t) + o(1), \quad y \in B_{x_n}(R_n \delta_n),$$

    where v_λ is the $1 - D$ solution given in (4.2), and $t = d(y, \partial\Sigma)$.
    b) *If $p \in S_1$, there exists $(s_n, t_n) \to p$ and $\lambda_n \to 0$*

$$(7.3) \qquad u_n(s, t) = 2 \log\left(\frac{2\lambda_n}{(s - s_n)^2 + (t + \mathfrak{D}(p)\lambda_n)^2 - \lambda_n^2}\right) + O(1),$$

    in $B_p^+(r) \subset \mathbb{R}^2$ with $d(p, p_j) > 2r$ for any $j = 1, \ldots, m$, where (s, t) is an isothermal coordinate centered at p. Moreover,

$$(7.4) \quad \nabla u_n(x) = -4\frac{x - p}{|x - p|^2} + o\left(\frac{1}{|x - p|^2}\right), \quad \text{in } B_p^+(r) \setminus B_p^+(\lambda_n \log \tfrac{1}{\lambda_n}).$$

    c) *Finally, if $\chi_n \le 0$, then S_1 is empty.*

This result follows from the lemmas below.

Lemma 7.3. *Under assumption (7.1), the set S_1 is finite. Moreover, let $p \in S_1$ and consider a blow-up profile v at p constructed as in the proof of Proposition 5.1. Then v has finite volume.*

Proof. Recalling (1.7), the Morse index bound is related to the quadratic form (1.9). Recall also that the limit profile v is defined as the limit of the sequence

$$v_n(x) = u_n(\delta_n x + x_n) + 2 \log \delta_n; \qquad \delta_n = e^{-u_n(x_n)/2}.$$

We claim that (4.3) is satisfied for v. Otherwise, by Fatou's lemma, $\int_\Sigma e^{u_n}$ is unbounded. Moreover, by Theorem 4.2 *b)* we can find compactly-supported functions (with disjoint supports) $\psi_1, \ldots, \psi_{m+1}$ such that

$$a_i := \int_{\mathbb{R}_+^2} [|\nabla(\psi_i)|^2 + 2e^v(\psi_i)^2] - \int_{\partial\mathbb{R}_+^2} \frac{1}{\mathfrak{D}(p)} e^{v/2}(\psi_i)^2 < 0$$

for all $i = 1, \ldots, m + 1$.
    Define then

$$\psi_{i,n} = \psi_i\left(\frac{x - x_n}{\delta_n}\right).$$

By scaling variables, recalling the expression in (1.9), it is then easy to see that

$$Q_n(\psi_{i,n}) = a_i + o_n(1) < 0; \qquad i = 1, \ldots, m + 1.$$

Since also the $\psi_{i,n}$'s have disjoint supports, we deduce that $ind(u_n) \geq m+1$ for n large. In this way we arrive to a contradiction with (7.1).

We know now that for any point in S_1 we can construct a blow-up profile v that satisfies (4.3), and in particular its Morse index is equal to 1. Moreover, by (4.5),

$$\liminf_{n \to +\infty} \int_\Sigma |K_n| e^{u_n} \geq \int_{\mathbb{R}^2_+} |K(p_0)| e^v = 2\pi \left(\frac{h(p_0)}{\sqrt{h^2(p_0) + K(p_0)}} - 1 \right) > \delta,$$

for a suitable $\delta > 0$. Hence condition (7.1) implies that S is finite. $\qquad \square$

Lemma 7.4. *Let $p \in S_1$. Then there exists fixed constants $r, C > 0$ such that*

$$\int_{B_p(r) \cap \Sigma} e^{u_n} \leq C; \qquad \oint_{B_p(r) \cap \partial\Sigma} e^{u_n/2} \leq C, \quad u_n \to -\infty \text{ on } \partial B_p(r) \cap \Sigma.$$

Proof. We will follow a modification of the strategy in [34], and hence we will be sketchy in some parts. First, we can conformally deform a geodesic ball centred at p into a planar half-ball $B_0^+(r_0)$, reducing ourselves to the same equation as in (6.1). It is then convenient to divide the proof into several steps.

Step 1: blowing a first bubble. We already know from Lemma 7.3 that the subset S_1 in the blow-up set S is finite and therefore, by sub-harmonicity, for n large (and up to a subsequence) u_n has local maxima x_n on $\Gamma_0^+(r)$, converging to p.

We rescale the solutions u_n as follows

$$v_n(x) = u_n(x_n + \delta_n x) + 2\log \delta_n; \qquad e^{-2\delta_n} = u_n(x_n).$$

Reasoning as in the proof of Proposition 5.1, we can find $R_n \to +\infty$ (slowly) such that $\|v_n - v\|_{C^2(B_0^+(2R_n))} \to 0$, with v a solution of (5.6). By the classification results given in Section 4, v must be of the form

$$v(s, t) = 2\log \left(\frac{2\lambda}{-\lambda^2 + (s - s_0)^2 + (t + t_0)^2} \right)$$

for some $\lambda > 0$, $s_0 \in \mathbb{R}$ and t_0 determined by $t_0 = \mathfrak{D}(p)\lambda$. Notice that, at infinity

(7.5) $$v(s, t) \simeq -4\log |(s, t) - (s_0, -t_0)|.$$

We also recall that this solution has Morse index one, by Theorem 4.2.

Step 2: blowing (possibly) other bubbles. Letting r be as in the previous step and $R_n \to +\infty$ as in Step 1, we consider the maximization problem

$$\sup_{R_n \delta_n \leq |x - x_n| \leq r/2} (u_n + 2\log |x - x_n|).$$

If this supremum tends to infinity, reasoning as in the proof of Lemma 4 in [34], one could again rescale u_n near a maximum point of the function

$u_n + 2 \log |x - x_n|$ to obtain another limiting profile. Since all limiting profiles have the expression of the above function v and have Morse index equal to one and fixed mass, (7.1) implies that continuing the procedure there is a finite number of bubbles. Therefore, we can find an integer k for $1 \leq k \leq m$, sequences of boundary points $x_n^i \in B_0^+(r)$, $i = 1, \ldots, k$, sequences $(\delta_n^i)_n$, $\delta_n^i \to 0$ for $i = 1, \ldots, k$ sequences $(R_n^i)_n$, $R_n^i \to \infty$, $i = 1, \ldots, k$ and a fixed constant C with the following properties

(i): $v_n^i(x) := u_n(x_n^i + \delta_n^i x) + 2 \log \delta_n^i \to v$ in $C^2(B_0^+(R_n^i))$;

(ii): $B_{x_n^i}^+(4R_n^i \delta_n^i) \cap B_{x_n^j}^+(4R_n^j \delta_n^j) = \emptyset$ for $i \neq j$;

(iii): $\sup_{B_0^+(r/2) \setminus \cup_{i=1}^k B_{x_n^i}^+(R_n^i \delta_n^i)} \left(u_n + 2 \log \min_i |x - x_n^i| \right) \leq C$.

Step 3: Harnack inequality. For $s_n^i \in [2R_n^i \delta_n^i, 1/8 \min_{j \neq i} |x_n^i - x_n^j|]$, $s_n^i \leq \frac{r}{8}$, one can consider the rescaled function w_n^i given by

$$w_n^i(x) = u_n(x_n^i + s_n^i x) + 2 \log s_n^i; \qquad x \in B_0^+(4) \setminus B_0^+(1/2).$$

By the above bound **(iii)** on u_n^i, w_n^i is uniformly controlled from above, and satisfies an equation with uniformly bounded data in $B_0^+(4) \setminus B_0^+(1/2)$. Reasoning as for (5.5), we can deduce a Harnack inequality for $e^{w_n^i}$ in $B_0^+(2) \setminus B_0^+(1)$, i.e., w_n has uniformly bounded oscillation there. Define the (semi-circular) average

$$\overline{u}_n^i(s) := \frac{1}{|\partial^+ B_{x_n^i}(s)|} \fint_{\partial^+ B_{x_n^i}(s)} u_n.$$

The Harnack inequality implies that there exists a fixed $C > 0$ such that

$$u_n(x) \leq \overline{u}_n^i(s) + C, \qquad x \in B_{x_n^i}^+(2s) \setminus B_{x_n^i}^+(s/2),$$

provided that $s \in [2R_n^i \delta_n^i, 1/8 \min_{j \neq i} |x_n^i - x_n^j|]$.

Step 4: local radial decay. Let us assume first that there exists $x_n^j \neq x_n^i$ as in Step 2 such that $|x_n^i - x_n^j| \to 0$. We will show that the functions u_n (properly rescaled) keep the profile as in (7.5) for $|x - x_n^i| \leq O(\min_{j \neq i} |x_n^i - x_n^j|)$. This will imply in particular the finiteness of accumulation of volume up to that scale.

If ν_s stands for the outer unit normal to $\partial^+ B_{x_n^i}(s)$, one has the formula

$$(7.6) \qquad |\partial^+ B_{x_n^i}(s)| \frac{d}{ds} \overline{u}_n^i(s) = \fint_{\partial^+ B_{x_n^i}(s)} \frac{\partial u_n}{\partial \nu_s}.$$

Fix now a number δ small and positive: we claim that

$$(7.7) \qquad \overline{u}_n^i(s) \leq \overline{u}_n^i(R_n^i \delta_n^i) - (4 - \delta) \left(\log s - \log(R_n^i \delta_n^i) \right)$$

for all $s \in [2R_n^i \delta_n^i, 1/8 \min_{j \neq i} |x_n^i - x_n^j|]$. Assuming by contradiction that this is false, define t_i^n to be the infimum of the radii s such that

$$\overline{u}_n^i(s) > \overline{u}_n^i(R_n^i \delta_n^i) - (4 - \delta) \left(\log s - \log(R_n^i \delta_n^i) \right).$$

Notice that at this first value t_n^i we must also have

$$t_i^n \frac{d}{ds}\big|_{s=t_n^i} \overline{u}_n^i(s) \geq \delta - 4.$$

(7.6), the fact that $|\partial^+ B_{x_n^i}(s)| = \pi s$ and the latter formula imply

$$\oint_{\partial^+ B_{x_n^i}(t_i^n)} \frac{\partial u_n}{\partial \nu_{t_i^n}} \geq (\delta - 4)\pi(1 + O((t_i^n)^2)).$$

On the other hand, from the convergence in Step 1 and the limiting behavior in (7.5) we find that

$$\oint_{\partial^+ B_{x_n^i}(R_n^i \delta_n^i)} \frac{\partial u_n}{\partial \nu_{R_n^i \delta_n^i}} \to -4\pi \quad \text{as } n \to +\infty.$$

Integrating (1.7) and using the last two formulas, together with the fact that $t_i \to 0$, we obtain

$$2 \int_{B_{x_n^i}^+(t_i^n) \setminus B_{x_n^i}^+(R_n^i \delta_n^i)} (K_n e^{u_n} - \tilde{K}_n u_n) + 2 \oint_{\Gamma_{x_n^i}^+(t_i^n) \setminus \Gamma_{x_n^i}^+(R_n^i \delta_n^i)} (h_n e^{u_n/2} - \tilde{h}_n u_n)$$

$$(7.8) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq -\delta\pi + o_n(1).$$

However, by the definition of t_i and by the Harnack inequality, in the above regions we have that

$$(7.9) \qquad e^{u_n}(x) \leq C(\delta_n^i)^{-2}(R_n^i)^{-4}\left(\frac{R_n^i \delta_n^i}{|x|}\right)^{4-\delta} \qquad |x| \in [R_n^i \delta_n^i, t_i^n]$$

and moreover, by the asymptotics of u_n

$$|u(x) - u(\delta_n^i R_n^i/|x|\, x)| \leq C \log \frac{|x|}{\delta_n^i R_n^i}; \qquad |x| \in [R_n^i \delta_n^i, t_i^n].$$

The function u_n, where it is positive, can be estimated from above by its exponential. Where it is negative, the previous formula allows to estimate it in absolute value by $C \log |x|$. The last two formulas and the latter argument imply that the integrals on the l.h.s. of (7.8) converge to zero, giving a contradiction.

Assuming next that there is no $x_n^j \neq x_n^i$ as above, i.e. that there is only one bubble, we choose the upper bound for t_i (in the formula after (7.7)) to be a small but fixed number $\tilde{r} > 0$. In this way, all the above arguments hold true, with the exception that we need to replace the r.h.s. of (7.8) by $-\delta\pi + O(\tilde{r}^2) + o_n(1)$. It is then sufficient to choose $\tilde{r}$ small compared to δ to reach again a contradiction.

Step 5: global radial decay. This step can be skipped if in the previous one only one bubble appears. If we have more bubbles instead, we assume for simplicity that there are only two of them: x_n^1 and x_n^2. The general case can be dealt with in a similar way, properly grouping points in clusters.

By the previous step, fixing any small $\delta > 0$, we have the inequality (7.7), and similarly

$$\overline{u}_n^2(s) \leq \overline{u}_n^2(R_n^2 \delta_n^2) - (4 - \delta)\left(\log s - \log(R_n^2 \delta_n^2)\right),$$

for all $s \in [2R_n^2 \delta_n^2, 1/8|x_n^1 - x_n^2|]$.

By the Harnack inequality in Step 3 we have that the differences of the radial averages $|\bar{u}_n^2(1/8|x_n^1 - x_n^2|) - \bar{u}_n^1(1/8|x_n^1 - x_n^2|)|$ are uniformly bounded and that moreover (using also (7.9))

$$|u_n(x) - \bar{u}_n^1(1/8|x_n^1 - x_n^2|)| \leq C \quad \text{for evey } x \in \Omega_n;$$

$$\int_{\Omega_n} e^{u_n} \to 0; \qquad \oint_{\Xi_n} e^{u_n/2} \to 0,$$

where

$$\Omega_n = B_{x_n^1}^+(8|x_n^1 - x_n^2|) \setminus \left(B_{x_n^1}^+(1/8|x_n^1 - x_n^2|) \cup B_{x_n^2}^+(1/8|x_n^1 - x_n^2|) \right),$$

$$\Xi_n = \Gamma_{x_n^1}^+(8|x_n^1 - x_n^2|) \setminus \left(\Gamma_{x_n^1}^+(1/8|x_n^1 - x_n^2|) \cup \Gamma_{x_n^2}^+(1/8|x_n^1 - x_n^2|) \right).$$

Integrating (1.7) and reasoning as in the previous step it follows that

$$\oint_{\partial^+ B_{x_n^1}(8|x_n^1 - x_n^2|)} \frac{\partial u_n}{\partial \nu_n} \to -8\pi \quad \text{as } n \to +\infty,$$

since the total flux of the gradient over the boundary of Ω_n tends to zero. One can then repeat the reasoning of Step 4 to show that

$$\bar{u}_n^1(s) \leq \bar{u}_n^1(8|x_n^1 - x_n^2|) - 2(4 - \delta)\left(\log s - \log(8|x_n^1 - x_n^2|)\right) + C$$

for all $s \in [8|x_n^1 - x_n^2|, \tilde{r}]$, where C is a fixed positive constant and $\tilde{r}$ is a small, but fixed, positive constant. From these facts, one can pass to the next (and last) step. Notice that these latter estimates imply that in the above set Ω_n the integral of e^{u_n} tends to zero (as well as the boundary integral of $e^{u_n/2}$).

Step 6: conclusion. From Step 4 and Step 5 it follows that the integral of e^{u_n} (respectively, the boundary integral of $e^{u_n/2}$) are bounded in a fixed neighborhood of p, which is the first conclusion of the proposition. It readily follows also that u_n diverges negatively on $\partial^+ B_p(r)$.

$\square$

Lemma 7.5. *Under the conditions of Proposition 7.2, the asymptotic behavior given in (7.2) and (7.3) hold. If moreover $\chi_n \leq 0$, then S_1 is empty.*

Proof. If $p \in S_0$, the statement follows from Case 1 in the proof of Theorem 5.2. Recall that if $\mathfrak{D}(p) = 1$, the only solution of (5.6) is given by (4.2) (see Section 4). In this regard, take also into account Remark 5.3.

If $p \in S_1$, our main concern is to rule out the blow-up cluster phenomena. Via a conformal map, we can work in $B_0^+(r) \subset \mathbb{R}^2$, where p is mapped to the origin. One has the distributional convergence
(7.10)
$$(|K_n|e^{u_n})|_{B_p(r)\cap\Sigma} \rightharpoonup \tilde{m}\beta\delta_p \quad \text{and} \quad h_n e^{u_n/2}|_{\partial B_p(r)\cap\Sigma} \rightharpoonup \tilde{m}(\beta + 2\pi)\delta_p,$$

where $\tilde{m} \in \mathbb{N}$ and

(7.11)
$$\beta = 2\pi\left(\frac{h(p)}{\sqrt{h^2(p) + K(p)}} - 1\right)$$

In order to derive the result, we use a Pohozaev identity. Take $r > 0$ such that $B_0^+(r) \cap S = \{0\}$ and apply Lemma 5.5 on $B_0^+(r)$ with $F(x) = x$ to obtain

$$\oint_{\partial B_0^+(r)} [4K_n e^{u_n}(x \cdot \nu) + 2(\nabla u_n \cdot x)(\nabla u_n \cdot \nu) - |\nabla u_n|^2 x \cdot \nu]$$
$$= \int_{B_0^+(r)} [4\tilde{K}_n \nabla u_n \cdot x + 8K_n e^{u_n} + (\nabla K_n \cdot x)e^{u_n}].$$

Now, we split the boundary integrals into $\Gamma_0^+(r)$ and $\partial^+ B_0(r)$, to find:

$$(7.12) \qquad \oint_{\Gamma_0^+(r)} \left[4h_n e^{u_n/2}(\nabla u_n \cdot x) - 4\tilde{h}_n(\nabla u_n \cdot x) \right] +$$
$$r \oint_{\partial^+ B_0(r)} \left[4K_n e^{u_n} + 2\left(\frac{\nabla u_n \cdot x}{r}\right)^2 - |\nabla u_n|^2 \right]$$
$$= \int_{B_0^+(r)} \left[4\tilde{K}_n \nabla u_n \cdot x + 8K_n e^{u_n} + (\nabla K_n \cdot x)e^{u_n} \right].$$

Taking into account that $|\nabla K_n| \leq C$ in $B_0^+(r)$, then

$$(7.13) \qquad \int_{B_0^+(r)} (\nabla K_n \cdot x)e^{u_n} \leq r \int_{B_0^+(r)} |\nabla K_n|e^{u_n} = O(r).$$

On the other hand, integrating by parts

$$\oint_{\Gamma_0^+(r)} 4h_n e^{u_n/2}(\nabla u_n \cdot x) = \left[8h_n e^{u_n/2} x_1 \right]_{-r}^r - 8\oint_{\Gamma_0^+(r)} \nabla h_n \cdot x\, e^{u_n/2} - 8\oint_{\Gamma_0^+(r)} h_n e^{u_n/2}.$$

Again, since $|\nabla h_n| \leq C$, we have

$$(7.14) \qquad \oint_{\Gamma_0^+(r)} \nabla h_n \cdot x\, e^{u_n/2} = O(r).$$

Moreover, by Lemma 7.4 one deduces

$$(7.15) \qquad \left[h_n e^{u_n/2} x_1 \right]_{-r}^r \to 0, \qquad \int_{\partial^+ B_0(r)} K_n e^{u_n} \to 0, \qquad \text{as } n \to +\infty.$$

Next, it is needed to estimate the gradient terms. In order to do it, we introduce the function $v_n = u_n - \inf_{\partial^+ B_0(r)} u_n - w_n$, where v_n, w_n satisfy

$$\begin{cases} -\Delta v_n + 2\tilde{K}_n = 2K_n e^{u_n}, & \text{in } B_0^+(r), \\ \dfrac{\partial v_n}{\partial n} + 2\tilde{h}_n = 2h_n e^{u_n/2}, & \text{on } \Gamma_0^+(r), \\ v_n = 0, & \text{on } \partial^+ B_0(r), \end{cases}$$

$$\begin{cases} \Delta w_n = 0, & \text{in } B_0^+(r), \\ \dfrac{\partial w_n}{\partial n} = 0, & \text{on } \Gamma_0^+(r), \\ w_n = u_n - \inf_{\partial^+ B_0(r)} u_n, & \text{on } \partial^+ B_0(r). \end{cases}$$

Recall the Green's representation formula for v_n

$$
\begin{aligned}
v_n(x) &= \frac{-1}{\pi} \int_{B_0^+(r)} \log|x - y| \, (2K_n(y)e^{u_n(y)} - 2\tilde{K}_n(y)) dy \\
(7.16) &\quad - \frac{1}{\pi} \oint_{\Gamma_0^+(r)} \log|x - y| \left(2h_n(y)e^{u_n(y)/2} - 2\tilde{h}_n(y)\right) dy + R_n(x),
\end{aligned}
$$

where R_n is uniformly bounded.

Let us define the function

$$
\varphi_n = u_n - \inf_{B_0^+(r)} u_n \geq 0.
$$

By Lemma (5.4), *c)*, we know that $\varphi_n \leq C$ on $\partial^+ B_0(r)$, so using the Green's representation formula for φ_n we obtain that

$$
\begin{aligned}
\varphi_n(x) &= \frac{-1}{\pi} \int_{B_0^+(r)} \log|x - y|(2K_n(y)e^{u_n(y)} - 2\tilde{K}_n(y)) \, dy \\
&\quad - \frac{1}{\pi} \oint_{\Gamma_0^+(r)} \log|x - y|(2h_n(y)e^{u_n(y)/2} - 2\tilde{h}_n(y)) \, dy \; + O(1).
\end{aligned}
$$

Therefore

$$
\varphi_n \to \varphi = -4\tilde{m} \log|x| + \phi \qquad \text{uniformly in } C^2(B_0^+(r) \setminus \{0\}),
$$

where ϕ is a regular function on $B_0^+(r)$. As a consequence, we obtain that

$$
(7.17) \qquad\qquad \nabla u_n = \nabla \varphi_n \to -4\tilde{m}\frac{x}{|x|^2} + \nabla \phi.
$$

Now, by the boundedness of $\tilde{K}_n$, $\tilde{h}_n$ and (7.17) one gets

$$
(7.18) \qquad \int_{B_0^+(r)} \tilde{K}_n \nabla u_n \cdot x = O(r), \qquad \int_{\Gamma_0^+(r)} \tilde{h}_n \nabla u_n \cdot x = O(r).
$$

By (7.13), (7.14), (7.15), (7.18) and (7.17), we can pass to the limit in (7.12) as $r \to 0$ to conclude that $\tilde{m} = 1$.

We are now concerned with pointwise estimates for the blow–up profile, namely (7.3). For this it suffices to follow the arguments for the proof of Theorem 1.6 in [3], which exploits the Green's representation formula and a Pohozaev type identity, and also to use (5.5) and the first bound in Lemma 5.4 *c)*. Taking the limit problem (5.6) into account, for the function v_n defined in (5.2) one gets

$$
v_n(s, t) = 2 \log\left(\frac{2\lambda_0}{s^2 + (t + t_0)^2 - \lambda_0^2}\right) + O(1), \qquad \text{in } B_0^+(r/\delta_n),
$$

where $t_0 = \mathfrak{D}(0)\lambda_0$ and, since $v(0) = 0$, $\lambda_0 = \frac{2}{\mathfrak{D}^2(0)-1}$. In this way, taking into account Remark 5.3 and that $x_n \in \Gamma_0^+(r)$, the previous profile estimates

can be recast for u_n to obtain (7.3), where $\lambda_n = \lambda_0 \delta_n$. The gradient estimate (7.4) follows from similar arguments.

Finally, as a consequence of (7.3) and (7.4), we obtain that

$$
\begin{aligned}
\int_{B_0^+(r)} |\nabla u_n^-|^2 \;&\geq\; 16 \int_{B_0^+(r) \setminus B_0^+(\delta_n \log \delta_n)} \frac{1}{|x|^2} + o\left(\frac{1}{|x|^2}\right) \\
&=\; -4 \log(\delta_n \log \delta_n) + o(\log(\delta_n \log \delta_n)).
\end{aligned}
$$

where δ_n is defined in (5.1), so

$$
\text{(7.19)} \qquad\qquad \int_{B_0^+(r)} |\nabla u_n^-|^2 \to +\infty.
$$

Obviously, (7.19) contradicts property *b)* in Lemma 5.4 and we conclude the proof.

$\square$

We finish the proof of Proposition 7.2 with the following two lemmas:

Lemma 7.6. *For every* $p \in S_0$, $\mathfrak{D}_\tau(p) = 0$.

Proof. If $\mathfrak{D}_\tau(p) \neq 0$, with $p \in S_0$, then p is isolated in S_0. Recalling that S_1 is finite by the previous lemma, p is also isolated in S. By a conformal map we can pass to a problem in $B_p^+(r) \subset \mathbb{R}^2$, where p is the unique singular point.

Notice also that, at points in S_0, the limit profiles of blowing-up solutions are given by formula (4.2), and therefore the local accumulation of local volume must tend to $+\infty$. This also implies that $\rho_n(u_n) \to +\infty$.

We choose now $F(s,t) = (1,0)$ in Lemma 5.5 and cut it off in a neighborhood of p. We can then continue with the proof of Proposition 6.2, replacing the *global mass* ρ_n with a *localized version*, namely $\int_{B_p^+(r)} e^{\hat{u}_n}$ for some small but fixed r. Take also into account Lemma 5.4, *c)*. In this way we still obtain $\mathfrak{D}_\tau(p) = 0$, as desired.

$\square$

8. APPENDIX: TEST FUNCTIONS

Consider a point p located on $\partial\Sigma$. Let $q = p + q_2 n(p)$, where n is the outward normal vector to $\partial\Sigma$ and $q_2 > 0$ is small enough such that q belongs to a regular extension of Σ. Given a parameter μ such that $\mu d(x,q) > 1$ for every $x \in \Sigma$, we define the functions

$$
\varphi_{\mu,p} : \Sigma \to \mathbb{R} \qquad \varphi_{\mu,p}(x) = \log \frac{4\mu^2}{(\mu^2 d^2(x,q) - 1)^2},
$$

$$
\text{(8.1)} \qquad\qquad \tilde{\varphi}_{\mu,p} : \Sigma \to \mathbb{R} \qquad \tilde{\varphi}_{\mu,p}(x) = \varphi_{\mu,p} - \log|K(x)|.
$$

Lemma 8.1. *Let* $p \in \partial\Sigma$ *such that* $\mathfrak{D}(p) > 1$, *let* I *be as in* (1.4), *and let the function* $\tilde{\varphi}_{\mu,p}$ *be defined in* (8.1). *Then*

$$I(\tilde{\varphi}_{\mu,p}) \to -\infty, \quad \oint_{\partial\Sigma} e^{\frac{\tilde{\varphi}_{\mu,p}}{2}} \to +\infty \quad as \quad \mu \to \frac{1}{d(p,q)} = \frac{1}{q_2}.$$

Proof. First of all, notice that

$$I(\tilde{\varphi}_{\mu,p}) = \int_{\Sigma} \left(\frac{1}{2}|\nabla\tilde{\varphi}_{\mu,p}|^2 + 2\tilde{K}\tilde{\varphi}_{\mu,p} + 2e^{\varphi_{\mu,p}} \right) - 4\oint_{\partial\Sigma} \mathfrak{D}e^{\varphi_{\mu,p}/2}.$$

Letting $\varepsilon > 0$, using the Young's inequality we can estimate the first term as

$$\frac{1}{2}\int_{\Sigma} |\nabla\tilde{\varphi}_{\mu,p}|^2 \le \left(\frac{1}{2} + \varepsilon\right)\int_{\Sigma} |\nabla\varphi_{\mu,p}|^2 + O(1).$$

We consider each term of the functional I_μ separately, and claim that the following estimates hold:

$$(8.2) \qquad \int_{\Sigma} |\nabla\varphi_{\mu,p}|^2 \le \frac{8\pi}{\sqrt{\mu^2 q_2^2 - 1}} + o\left(\frac{1}{\sqrt{\mu^2 q_2^2 - 1}}\right),$$

$$(8.3) \qquad \int_{\Sigma} e^{\varphi_{\mu,p}} \le \frac{2\pi\mu q_2}{\sqrt{\mu^2 q_2^2 - 1}} + o\left(\frac{1}{\sqrt{\mu^2 q_2^2 - 1}}\right),$$

$$(8.4) \qquad \oint_{\partial\Sigma} \mathfrak{D}e^{\frac{\varphi_{\mu,p}}{2}} \ge \min_{B_p(r)\cap\partial\Sigma} \mathfrak{D}\frac{2\pi}{\sqrt{\mu^2 q_2^2 - 1}} + o\left(\frac{1}{\sqrt{\mu^2 q_2^2 - 1}}\right),$$

$$(8.5) \qquad \int_{\Sigma} \tilde{K}\,\tilde{\varphi}_{\mu,p} \le O(1).$$

From these, the assertion of Lemma 8.1 follows immediately.

Proof of (8.2).

Letting $r > q_2$, we divide the integral into two parts as follows

$$\int_{\Sigma} |\nabla\varphi_{\mu,p}|^2 \, dV_g = \int_{\Sigma\setminus B_q(r)} |\nabla\varphi_{\mu,p}|^2 \, dV_g + \int_{B_q(r)\cap\Sigma} |\nabla\varphi_{\mu,p}|^2 \, dV_g.$$

Applying the inequality $|\nabla d(x,z)^2| \le 2d(x,z)$, we have

$$|\nabla\varphi_{\mu,p}(x)| = 2\mu^2 \frac{|\nabla d^2(x,q)|}{\mu^2 d^2(x,q) - 1} \le 4\mu^2 \frac{d(x,q)}{\mu^2 d^2(x,q) - 1} \qquad \text{for every } x \in \Sigma.$$

By the use of the last inequality, normal coordinates centred at q and the estimate

$$dV_g = (1 + o_r(1))\,dx, \quad d(q,x) = |x - q|, \text{ for } x \in B_q(r) \cap \Sigma,$$

with r sufficiently small, one finds that

$$\int_{B_q(r)\cap\Sigma} |\nabla\varphi_{\mu,p}|^2 \, dV_g \le 16\mu^4 \int_{B_q^+(r)} (1 + o_r(1))\frac{|x - q|^2 \, dx}{(\mu^2|x - q|^2 - 1)^2},$$

where $B_q^+(r) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 + q_2)^2 < r^2 \text{ and } x_2 > 0\}$.

Now, writing $\mu^2 |x - q|^2 = 1 + o_r(1)$, we can transform the integral

$$(8.6) \quad (A) := \int_{B_q^+(r)} \frac{\mu^4 |x - q|^2}{(\mu^2 |x - q|^2 - 1)^2} \, dx = \int_{B_q^+(r)} \frac{\mu^2 (1 + o_r(1))}{(\mu^2 |x - q|^2 - 1)^2} \, dx.$$

Next, take polar coordinates (ρ, θ) with center in $(0, -q_2)$ and consider the change of variable $t = \mu^2 \rho^2 - 1$ to get

$$\int_{B_q^+(r)} \frac{\mu^2 \, dx}{(\mu^2 |x - q|^2 - 1)^2} = 2 \int_{\arcsin(\frac{q_2}{r})}^{\pi/2} \int_{\frac{q_2}{\sin \theta}}^{r} \frac{\mu^2 \rho}{(\mu^2 \rho^2 - 1)^2} \, d\rho \, d\theta$$

$$= \int_{\arcsin(\frac{q_2}{r})}^{\pi/2} \int_{\frac{\mu^2 q_2^2}{\sin^2 \theta} - 1}^{\mu^2 r^2 - 1} \frac{dt \, d\theta}{t^2} = \int_{\arcsin(\frac{q_2}{r})}^{\pi/2} \frac{\sin^2 \theta}{\mu^2 q_2^2 \sin^2 \theta - \sin^2 \theta} \, d\theta + O(1)$$

$$(8.7)$$

$$= \frac{\mu q_2}{\sqrt{\mu^2 q_2^2 - 1}} \left[\arctan \left(\frac{\sqrt{\mu^2 q_2^2 - 1}}{\mu q_2} \tan \theta \right) \right]_{\arcsin(\frac{q_2}{r})}^{\pi/2} + O(1) = \frac{\mu \pi q_2}{2 \sqrt{\mu^2 q_2^2 - 1}} + O(1).$$

Observe that for any $x \in \Sigma \setminus B_q(r)$ one has

$$|\nabla \varphi_{\mu,p}(x)| \le \frac{diam(\Sigma)}{\mu^2 r^2 - 1} \qquad \text{for any } x \in \Sigma \setminus B_q(r).$$

From this inequality, since $\mu r > \mu q_2 > 1$, we obtain that

$$(8.8) \qquad \int_{\Sigma \setminus B_q(r)} |\nabla \varphi_{\mu,p}(x)|^2 = O(1).$$

So, we have proved (8.2) by (8.7) and (8.8).

Proof of (8.3).

In order to compute the exponential terms, we divide the integral as

$$\int_{\Sigma} e^{\varphi_{\mu,p}} = \int_{B_q(r) \cap \Sigma} e^{\varphi_{\mu,p}} + \int_{\Sigma \setminus B_q(r)} e^{\varphi_{\mu,p}}.$$

We focus on the first integral. Again, using normal coordinates, we should compute

$$\int_{B_q^+(r)} e^{\varphi_{\mu,p}} \, dx = 4 \int_{B_q^+(r)} \frac{\mu^2 \, dx}{(\mu^2 |x - q|^2 - 1)^2} = 4 \, (A),$$

where (A) is defined in (8.6).

Using the fact that

$$e^{\varphi_{\mu,p}} \le \frac{4\mu^2}{(\mu^2 r^2 - 1)^2} \qquad \text{for any } x \in \Sigma \setminus B_q(r),$$

then

$$(8.9) \qquad \fint_{\Sigma \setminus B_q(r)} e^{\varphi_{\mu,p}} = O(1).$$

The estimates (8.7) and (8.9) conclude the proof.

Proof of (8.4).

First, we split the integral as

$$\oint_{\Sigma} \mathfrak{D} e^{\frac{\varphi_{\mu,p}}{2}} = \oint_{B_p(r) \cap \partial\Sigma} \mathfrak{D} e^{\frac{\varphi_{\mu,p}}{2}} + \oint_{\partial\Sigma \setminus B_p(r)} \mathfrak{D} e^{\frac{\varphi_{\mu,p}}{2}},$$

and focus on the first term. Next,

$$\oint_{B_p(r) \cap \partial\Sigma} \mathfrak{D} e^{\frac{\varphi_{\mu,p}}{2}} \geq \min_{B_p(r) \cap \partial\Sigma} \mathfrak{D}(x) \oint_{B_p(r) \cap \partial\Sigma} e^{\frac{\varphi_{\mu,p}}{2}}.$$

Taking r small enough and using normal coordinates, we compute

$$(8.10) \qquad \oint_{B_p(r) \cap \partial\Sigma} e^{\frac{\varphi_{\mu,p}}{2}} = 2 \oint_0^r \frac{\mu \, dx_1}{\mu^2(x_1^2 + q_2^2) - 1} = \frac{\pi}{\sqrt{\mu^2 q_2^2 - 1}} + O(1).$$

In addition, we have

$$|e^{\frac{\varphi_{\mu,p}}{2}}| \leq O(1) \qquad \text{in } \partial\Sigma \setminus B_p(r),$$

so that

$$(8.11) \qquad \left| \oint_{\partial\Sigma \setminus B_p(r)} \mathfrak{D} e^{\frac{\varphi_{\mu,p}}{2}} \right| \leq O(1).$$

The estimates (8.10) and (8.11) complete the proof of (8.4).

Proof of (8.5).

By the definition of $\tilde{\varphi}_{\mu,p}$, it is direct to check that

$$\int_{\Sigma} \tilde{K} \, \tilde{\varphi}_{\mu,p} \leq -C \int_{\Sigma} \varphi_{\mu,p} + O(1).$$

Since

$$\log \frac{4\mu^2}{(\mu^2 q_2^2 - 1)^2} \leq \varphi_{\mu,p} \qquad \text{for any } x \in \Sigma,$$

the claim is proved.

$\square$

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