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C^1-Cubic Quasi-Interpolation Splines over a CT Refinement of a Type-1 Triangulation

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Abstract: C^1 continuous quasi-interpolating splines are constructed over Clough–Tocher refinement of a type-1 triangulation. Their Bernstein–Bézier coefficients are directly defined from the known values of the function to be approximated, so that a set of appropriate basis functions is not required. The resulting quasi-interpolation operators reproduce cubic polynomials. Some numerical tests are given in order to show the performance of the approximation scheme.

Keywords: Bernstein–Bézier coefficients; quasi-interpolation; type-1 triangulation; Clough–Tocher split

MSC: 41A15; 65D07

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1. Introduction

A novel, non-standard technique for constructing bivariate quasi-interpolating splines over uniform partitions was proposed by T. Sorokina and F. Zeilfelder in [1,2] (see also [3,4]). The essential idea of this methodology is to define the quasi-interpolant by directly providing the coefficients of the Bernstein–Bézier (BB-) form of its restriction to each of the subsets forming the partition.

In [2], the construction of C^1 quartic quasi-interpolants over a type-1 triangulation is addressed, so that the largest polynomial space is reproduced, namely the space $\mathbb{P}_3$ of polynomials of total degree less than or equal to three (see Figure 1). The coefficients of the quasi-interpolant on each triangle are linear combinations of function values at vertices and midpoints in a neighborhood of the triangle. The quasi-interpolant is constructed from them.

In [1], the same strategy is applied to construct C^1 quadratic quasi-interpolants on a triangulation which the authors called of type-2. Starting from a decomposition of the plane into squares, each of them is divided into eight micro-triangles by means of its diagonals and the straight lines parallel to the coordinate axes passing through the center of the square (see Figure 1).

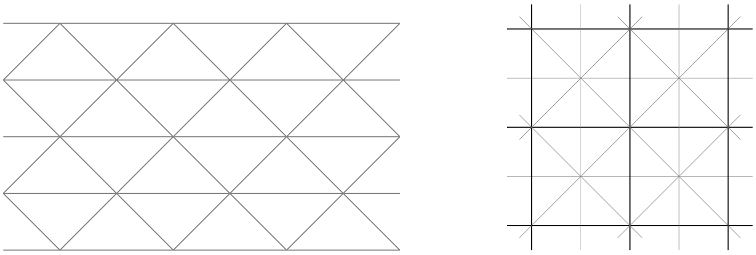

Figure 1. From left to right, type-1 and type-2 triangulations on which C^1-continuous quasi-interpolants are constructed in [1,2]: quartic and exact on $\mathbb{P}_3$, and quadratic and exact on $\mathbb{P}_2$, respectively.

The problem addressed in [2] is studied in detail in [5], proving that the approximation scheme proposed in [2] is a particular choice in a 19-parametric family of schemes. Moreover, different strategies for assigning values to the parameters are provided. In both [2] and [5], the quasi-interpolating splines interpolate the values at the vertices and the masks associated with the domain points that are key to the construction are applied taking into account the symmetries of the triangulation involved. Since the triangulation is uniform, these masks are independent of the specific triangle on which the quasi-interpolant is calculated (see also [6]).

Later, the cubic case was dealt with in [7], on the same triangulation used to construct quartic quasi-interpolants. The aim was to construct a C^1 cubic one, exact on $\mathbb{P}_2$, from the values at vertices and midpoints. Since it is not possible to define a quasi-interpolant that interpolates values at vertices, the authors opted to find specific masks for key domain points, including vertices, without imposing any symmetry. It was proved that there are unique masks that satisfy the required properties. Not being possible to achieve exactness on $\mathbb{P}_3$, this paper presents a construction on a refinement of the initial type-1 triangulation in order to achieve the optimal approximation order. Specifically, we work on a Clough–Tocher (CT-) refinement [8], which produces a subdivision into six micro-triangles of each square formed by two macro-triangles sharing an edge.

The rest of the paper is structured as follows. In Section 2, a type-1 triangulation endowed with a Clough–Tocher refinement is introduced, as well as the space of C^1 cubic splines defined over it. Further, a partition of the domain points associated with the micro-triangles is provided. In Section 3, the construction of quasi-interpolating splines is given and the general solution of the resulting problem. In Section 4, a method for selecting parameters based on the minimization of an upper bound of the quasi-interpolation error associated with the quartic monomials is proposed. In Section 5, the results of some numerical tests are given to illustrate the performance of the quasi-interpolation operator relative to the selected parameters. Finally, some details are included in Appendix A.

2. Bernstein–Bézier Form of Cubic Splines on a Type-1 Triangulation

Let us suppose that the triangulation is spanned by the vectors $e_1 := (h, h)$ and $e_2 := (h, -h)$, with $h > 0$. Its vertices are $v_{i,j} := ie_1 + je_2$, which define the lattice $\mathcal{V} := \{v_{i,j}, i, j \in \mathbb{Z}\}$. These vertices define squares which can be decomposed into the triangles $\mathbb{T}_{i,j}\langle v_{i,j}, v_{i+1,j+1}, v_{i+1,j}\rangle$ and $\mathbb{B}_{i,j}\langle v_{i,j}, v_{i+1,j+1}, v_{i,j+1}\rangle$ (see Figure 2). Therefore, a type-1 triangulation results:

$$\Delta := \bigcup_{i,j\in\mathbb{Z}} \left(\mathbb{T}_{i,j} \cup \mathbb{B}_{i,j}\right).$$

When there is no need to distinguish between the types of triangles in Δ, we denote by $\mathbb{T}$ any one of them.

To define the refinement of Δ to be used, let

$$t_{i,j} := \frac{1}{3}\left(v_{i,j} + v_{i+1,j+1} + v_{i+1,j}\right) \quad \text{and} \quad b_{i,j} := \frac{1}{3}\left(v_{i,j} + v_{i+1,j+1} + v_{i,j+1}\right)$$

be the barycenters of $\mathbb{T}_{i,j}$ and $\mathbb{B}_{i,j}$, respectively. Then, the CT-refinement of each triangle is obtained by joining its vertices with its barycenter [8]. Each macro-triangle $\mathbb{T}_{i,j}$ and $\mathbb{B}_{i,j}$ is, respectively, divided into the following micro-triangles:

$$\begin{array}{lll} t_1^+ = \langle v_{i,j}, v_{i+1,j+1}, t_{i,j}\rangle, & t_2^+ = \langle v_{i+1,j+1}, v_{i+1,j}, t_{i,j}\rangle, & t_3^+ = \langle v_{i+1,j}, v_{i,j}, t_{i,j}\rangle, \\ t_1^- = \langle v_{i,j}, v_{i,j+1}, b_{i,j}\rangle, & t_2^- = \langle v_{i,j+1}, v_{i+1,j+1}, b_{i,j}\rangle, & t_3^- = \langle v_{i+1,j+1}, v_{i,j}, b_{i,j}\rangle. \end{array} \tag{1}$$

They are shown in Figure 2, bottom, where any reference to the subscripts of the micro-triangles has been avoided. As in the case of macro-triangles, the lower case letter t will be used to represent any of the micro-triangles of Δ_{CT}.

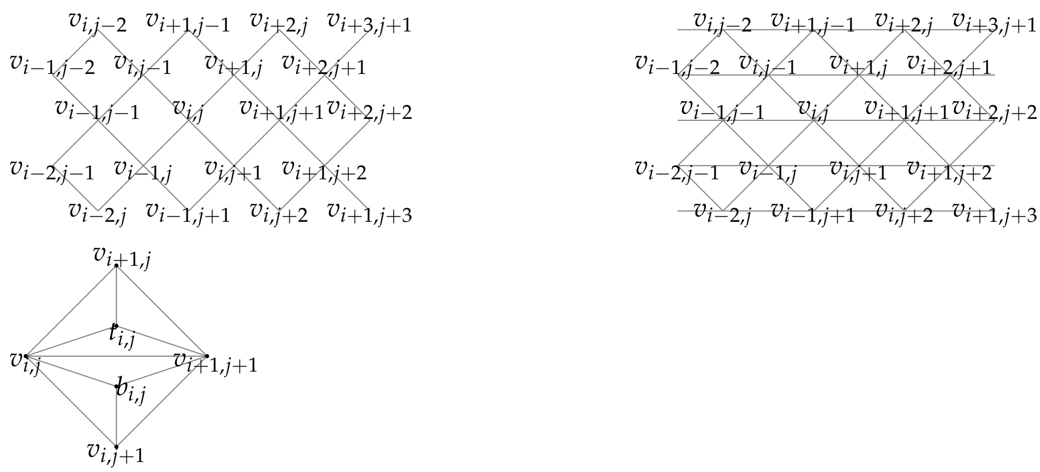

Figure 2. Top, from left to right, decomposition into squares induced by the vertices of Δ, type-1 triangulation. Bottom, CT-refinements of macro-triangles $T_{i,j}$ and $B_{i,j}$.

In this paper, we consider the space of C^1 cubic splines on Δ_{CT} defined by

$$S_3^1(\Delta_{\mathrm{CT}}) := \left\{ s \in C^1\left(\mathbb{R}^2\right) : s_{|t} \in \mathbb{P}_3 \quad \text{for all } t \in \Delta_{\mathrm{CT}} \right\}.$$

where the restriction is $s_{|t}$ of $s \in S_3^1(\Delta_{\mathrm{CT}})$ to a micro-triangle $t = \langle V_1, V_2, V_3 \rangle \in \Delta_{\mathrm{CT}}$ a cubic polynomial, it can be represented using the cubic Bernstein polynomials

$$B_{\beta,t}(p) := \frac{3!}{\beta!}\tau^{\beta} = \frac{6}{\beta_1!\beta_2!\beta_3!}\tau_1^{\beta_1}\tau_2^{\beta_2}\tau_3^{\beta_3},$$

where the multi-index notations $\beta := (\beta_1, \beta_2, \beta_3) \in \mathbb{N}_0^3$, $|\beta| := \beta_1 + \beta_2 + \beta_3$ and $\beta! := \beta_1!\beta_2!\beta_3!$ have been used, and $\tau := (\tau_1, \tau_2, \tau_3)$ provides the barycentric coordinates of point $p \in \mathbb{R}^2$ with respect to t, i.e., $p = \sum_{i=1}^{3} \tau_i V_i$ and $\sum_{i=1}^{3} \tau_i = 1$. The coordinates τ_1, τ_2 and τ_3 are non-negative whenever p belongs to t.

Every polynomial $q \in \mathbb{P}_3$ can be expressed on t in terms of the cubic Bernstein basis polynomials $B_{\beta,t}$, $|\beta| = 3$, i.e., there exist values b_β such that

$$q(x,y) = q(\tau) = \sum_{|\beta|=3} b_{\beta,t}\, B_{\beta,t}(\tau).$$

Coefficients in $D_t := \left\{ b_{\beta,t}, |\beta| = 3 \right\}$ are said to be the Bernstein–Bézier (BB-) coefficients of q. They are linked to the domain points $\xi_{\beta,t}$ determined by the barycentric coordinates $\left(\frac{\beta_{1,t}}{3}, \frac{\beta_{2,t}}{3}, \frac{\beta_{3,t}}{3}\right)$ with respect to t. They determine the lattice $L_3(t)$. The graph of q on t is included in the convex hull of $\left\{ \left(\xi_{\beta,t}, b_{\beta,t}\right), |\beta| = 3 \right\}$.

On each micro-triangle, an element $s \in S_3^1(\Delta_{\mathrm{CT}})$ is uniquely determined by ten BB-coefficients, associated with the corresponding domain points. When all macro-triangles are taken into account, a subset of domain points is obtained, which we note $\mathcal{D}_3(\Delta_{\mathrm{CT}})$, i.e., $\mathcal{D}_3(\Delta_{\mathrm{CT}}) = \bigcup_{t \in \Delta_{\mathrm{CT}}} L_3(t)$, where the union is formed without taking repetitions into account. To determine s, it is necessary to give the BB-coefficients associated with all the points of $\mathcal{D}_3(\Delta_{\mathrm{CT}})$. As the triangulation is uniform, following the approach in [2,4–6], it is sufficient to establish a partition $\left\{ D_{i,j}, i,j \in \mathbb{Z} \right\}$ of $\mathcal{D}_3(\Delta_{\mathrm{CT}})$ and define the BB-coefficients linked to the domain points in $D_{i,j}$.

Figure 3 shows the twenty-seven domain points forming $D_{i,j}$, which are linked to vertex $v_{i,j}$. Each of them has the subscripts of $v_{i,j}$. The vertices and barycenter have already been defined. They remaining domain points in $D_{i,j}$ are given next:

$$
\begin{aligned}
u_{i,j}^{1,1} &:= \tfrac{1}{3}(2v_{i,j} + v_{i+1,j+1}), & u_{i,j}^{1,0} &:= \tfrac{1}{3}(2v_{i,j} + v_{i+1,j}),\\
u_{i,j}^{2,1} &:= \tfrac{1}{3}(2v_{i,j} + t_{i,j}), & u_{i,j}^{-1,-1} &:= \tfrac{1}{3}(2v_{i,j} + v_{i-1,j-1}),\\
u_{i,j}^{1,-1} &:= \tfrac{1}{3}(2v_{i,j} + b_{i,j-1}), & u_{i,j}^{0,-1} &:= \tfrac{1}{3}(2v_{i,j} + v_{i,j-1}),\\
u_{i,j}^{-1,-2} &:= \tfrac{1}{3}(2v_{i,j} + t_{i-1,j-1}), & u_{i,j}^{-2,-1} &:= \tfrac{1}{3}(2v_{i,j} + b_{i-1,j-1}),\\
u_{i,j}^{-1,0} &:= \tfrac{1}{3}(2v_{i,j} + v_{i-1,j}), & u_{i,j}^{-1,1} &:= \tfrac{1}{3}(2v_{i,j} + t_{i-1,j}),\\
u_{i,j}^{0,1} &:= \tfrac{1}{3}(2v_{i,j} + v_{i,j+1}), & u_{i,j}^{1,2} &:= \tfrac{1}{3}(2v_{i,j} + b_{i,j}),\\
x_{i,j}^{1,1} &:= \tfrac{1}{3}(v_{i,j} + v_{i+1,j+1} + b_{i,j}), & x_{i,j}^{1,0} &:= \tfrac{1}{3}(v_{i,j} + v_{i+1,j} + b_{i,j-1}),\\
x_{i,j}^{0,1} &:= \tfrac{1}{3}(v_{i,j} + v_{i,j+1} + b_{i,j}), & y_{i,j}^{2,1} &:= \tfrac{1}{3}(v_{i,j} + 2t_{i,j}),\\
y_{i,j}^{1,-1} &:= \tfrac{1}{3}(v_{i,j} + 2b_{i,j-1}), & y_{i,j}^{-1,-2} &:= \tfrac{1}{3}(v_{i,j} + 2t_{i-1,j-1}),\\
y_{i,j}^{-2,-1} &:= \tfrac{1}{3}(v_{i,j} + 2b_{i-1,j-1}), & y_{i,j}^{-1,1} &:= \tfrac{1}{3}(v_{i,j} + 2t_{i-1,j})\\
y_{i,j}^{1,2} &:= \tfrac{1}{3}(v_{i,j} + 2b_{i,j}), & z_{i,j}^{1,1} &:= \tfrac{1}{3}(v_{i,j} + v_{i+1,j+1} + t_{i,j}),\\
z_{i,j}^{1,0} &:= \tfrac{1}{3}(v_{i,j} + v_{i+1,j} + t_{i,j}), & z_{i,j}^{0,1} &:= \tfrac{1}{3}(v_{i,j} + v_{i,j+1} + t_{i-1,j}).
\end{aligned}
$$

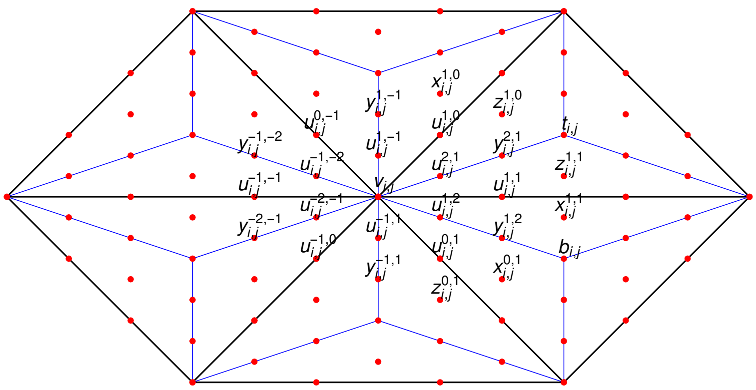

Figure 3. Domain points forming the subset $D_{i,j}$ corresponding to $v_{i,j}$.

Figure 4 shows the domain points in D lying in the hexagon formed by the six triangles sharing the vertex $v_{i,j}$.

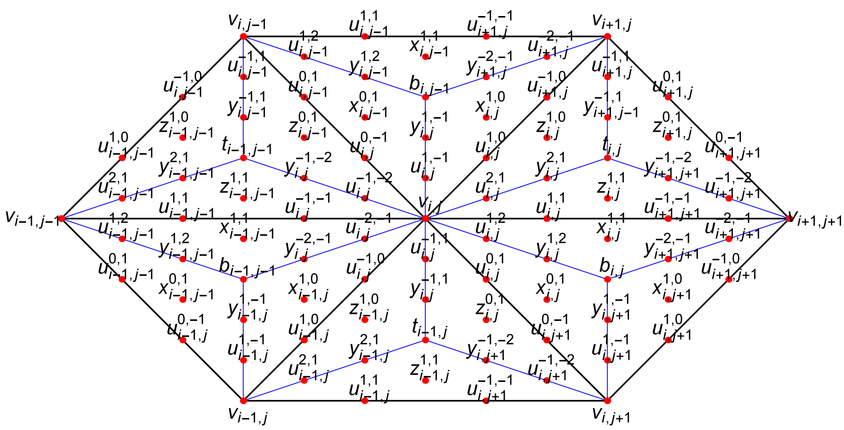

Figure 4. Domain points lying in the hexagon formed by the triangles sharing vertex $v_{i,j}$. Each shows the subscripts of the vertex to which it is linked.

3. C^1 Quasi-Interpolating Splines on a Clough–Tocher Refinement

The main objective of this work is to construct a quasi-interpolation operator for $S_3^1(\Delta_{\mathrm{CT}})$ that is exact on $\mathbb{P}_3$ in order to improve the result obtained in [7]. Let us denote it as $\mathcal{Q}$. It is assumed that the values of a function f are known at the domain points in $\mathcal{D}_3(\Delta_{\mathrm{CT}})$.

The quasi-interpolant $\mathcal{Q}f \in S_3^1(\Delta_{\mathrm{CT}})$ of f should be constructed in such a way that the BB-coefficients of the restriction $\mathcal{Q}f_{|t}$ to each micro-triangle $t \in \Delta_{\mathrm{CT}}$ are defined as combinations of those values of f. In other words, $\mathcal{Q}f_{|t}$ is written in the basis of Bernstein polynomials $B_{\beta,t}$, $|\beta| = 3$, as

$$\mathcal{Q}f_{|t} = \sum_{\gamma \in \Delta_3} P_\gamma B_{\gamma,t},$$

where P_γ denotes the BB-coefficient associated with the domain point $p_\gamma \in t$, Δ_3 is the set of indices with length equal to 3 written in the lexicographical order, i.e.,

$$\Delta_3 = \{(3,0,0),(2,1,0),(2,0,1),(1,2,0),(1,1,1),\\ (1,0,2),(0,3,0),(0,2,1),(0,1,2),(0,0,3)\}$$

and the vertices of each micro-triangle follow in the order they appear in (1).

For instance, with regard to the micro-triangle t_1^+ of $\mathbb{T}_{i,j}$ (see Figure 5) we write

$$\begin{aligned}\mathcal{Q}f_{|t_1^+} = {} & V_{i,j}B_{(3,0,0),t_1^+} + U_{i,j}^{1,1}B_{(2,1,0),t_1^+} + U_{i,j}^{2,1}B_{(2,0,1),t_1^+} + U_{i+1,j+1}^{-1,-1}B_{(1,2,0),t_1^+} \\ & + Z_{i,j}^{1,1}B_{(1,1,1),t_1^+} + Y_{i,j}^{2,1}B_{(1,0,2),t_1^+} + V_{i+1,j+1}B_{(0,3,0),t_1^+} \\ & + U_{i+1,j+1}^{-1,-2}B_{(0,2,1),t_1^+} + Y_{i+1,j+1}^{-1,-2}B_{(0,1,2),t_1^+} + T_{i,j}B_{(0,0,3),t_1^+}.\end{aligned}$$

Similar expressions are obtained for the restrictions of $\mathcal{Q}f$ to the other two micro-triangles of $\mathbb{T}_{i,j}$ and those three into which $\mathbb{B}_{i,j}$ is divided.

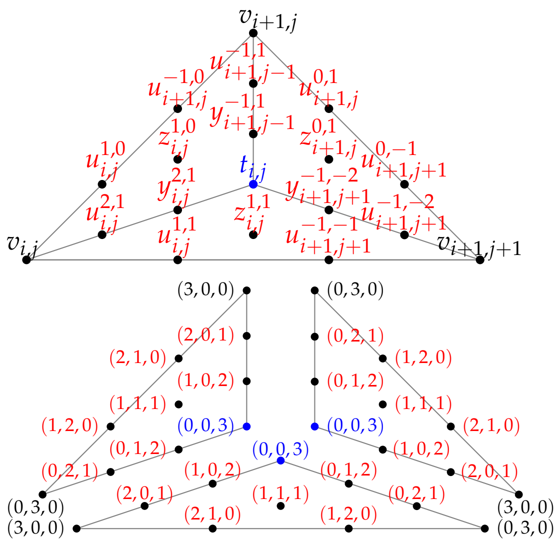

Figure 5. Top, the domain points associated with the three micro-triangles of the macro-triangle $\mathbb{T}_{i,j}$. They are denoted as shown in Figure 4 bottom; the indices corresponding to each micro-triangle, whose orientation is determined by the vertex ordering given by (1).

The BB-coefficients involved in the definition of $\mathcal{Q}f$ on each micro-triangle of $\mathbb{T}_{i,j}$ and $\mathbb{B}_{i,j}$ will be linear combinations of f at the specific domain points for cubic polynomials lying in the hexagon defined by the triangles sharing vertex $v_{i,j}$. Specifically, the union without repetitions $\mathcal{D}_3(\Delta) := \bigcup_{\mathbb{T}\in\Delta} L_3(\mathbb{T})$ is formed and decomposed as

$$\mathcal{D}_3(\Delta) = \bigcup_{i,j\in\mathbb{Z}} S_{i,j},$$

where the ordered subset $S_{i,j}$ consists of the thirty-seven domain points given below:

$$\begin{aligned} S_{i,j} := \Big\{ & v_{i,j}, u_{i,j}^{1,1}, u_{i,j}^{1,0}, u_{i,j}^{0,-1}, u_{i,j}^{-1,-1}, u_{i,j}^{-1,0}, u_{i,j}^{0,1}, u_{i+1,j+1}^{-1,-1}, t_{i,j}, u_{i+1,j}^{-1,0}, b_{i,j-1}, \\ & u_{i,j-1}^{0,-1}, t_{i-1,j-1}, u_{i-,j-1}^{-1,-1}, b_{i-1,j-1}, u_{i-1,j}^{1,0}, t_{i-1,j}, u_{i,j+1}^{0,-1}, b_{i,j}, v_{i+1,j+1}, u_{i+1,j+1}^{0,-1}, \\ & u_{i+1,j}^{0,1}, v_{i+1,j}, u_{i+1,j}^{-1,-1}, u_{i,j-1}^{1,1}, v_{i,j-1}, u_{i,j-1}^{-1,0}, u_{i-1,j-1}^{1,0}, v_{i-1,j-1}, u_{i-1,j-1}^{0,1}, u_{i-1,j}^{0,-1}, \\ & v_{i-1,j}, u_{i-1,j}^{1,1}, u_{i,j+1}^{-1,-1}, v_{i,j+1}, u_{i,j+1}^{1,0}, u_{i+1,j+1}^{-1,0} \Big\}. \end{aligned}$$

The BB-coefficient P of a domain point p is a linear combination of values of f at points in $S_{i,j}$, its coefficients give rise to a vector $M(p)$, ordered as $S_{i,j}$, which is said to be the mask of p. If $f(S_{i,j}) := \{f(p), p \in S_{i,j}\}$ is also ordered as $S_{i,j}$, then

$$P = M(p) \cdot f(S_{i,j}) := \sum_{\ell=1}^{37} M(p)_\ell f(S_{i,j})_\ell,$$

where $M(p)_\ell$ and $f(S_{i,j})_\ell$ stand for the ℓ-th entries of $M(p)$ and $f(S_{i,j})$, respectively.

In the following, we state the problem that is the object of this work.

Problem 1. *Find masks for the domain points in $D_{i,j}$ such that the associated quasi-interpolation operator $\mathcal{Q}$ is exact on $\mathbb{P}_3$ and produces C^1 quasi-interpolating splines.*

The following result holds.

Proposition 2. *Problem 1 has a 17-parametric family of solutions.*

Proof. Given an arbitrary function f, C^1 continuity of $\mathcal{Q}f$ across segment $[v_{i,j}, v_{i+1,j}]$ is equivalent to the following conditions [9] (Thm. 2.28) (see Figure 6 and the notations used for the domain points in Figures 3 and 4):

$$\begin{aligned} V_{i,j} + U_{i,j}^{1,0} - U_{i,j}^{1,-1} - U_{i,j}^{2,1} &= 0, \\ U_{i,j}^{1,0} + U_{i+1,j}^{-1,0} - X_{i,j}^{1,0} - Z_{i,j}^{1,0} &= 0, \\ U_{i+1,j}^{-1,0} + V_{i+1,j} - U_{i+1,j}^{2,1} - U_{i+1,j}^{-1,1} &= 0. \end{aligned}$$

For $[v_{i,j}, v_{i+1,j+1}]$,

$$\begin{aligned} V_{i,j} + U_{i,j}^{1,1} - U_{i,j}^{2,1} - U_{i,j}^{1,2} &= 0, \\ U_{i,j}^{1,1} + U_{i+1,j+1}^{-1,-1} - X_{i,j}^{1,1} - Z_{i,j}^{1,1} &= 0, \\ U_{i+1,j+1}^{-1,-1} + V_{i+1,j+1} - U_{i+1,j+1}^{-1,-2} - U_{i+1,j+1}^{2,1} &= 0. \end{aligned}$$

And for $[v_{i,j}, v_{i,j+1}]$,

$$\begin{aligned} V_{i,j} + U_{i,j}^{0,1} - U_{i,j}^{-1,1} - U_{i,j}^{1,2} &= 0, \\ U_{i,j}^{0,1} + U_{i,j+1}^{0,-1} - Z_{i,j}^{0,1} - X_{i,j}^{0,1} &= 0, \\ U_{i,j+1}^{0,-1} + V_{i,j+1} - U_{i,j+1}^{-1,-2} - U_{i,j+1}^{1,-1} &= 0. \end{aligned}$$

Regarding micro-edges, C^1 continuity across $[v_{i,j}, t_{i,j}]$ is equivalent to conditions

$$U_{i,j}^{2,1} - \frac{1}{3}\Big(V_{i,j} + U_{i,j}^{1,0} + U_{i,j}^{1,1}\Big) = 0,\ Y_{i,j}^{2,1} - \frac{1}{3}\Big(U_{i,j}^{2,1} + Z_{i,j}^{1,0} + Z_{i,j}^{1,1}\Big) = 0.$$

Similarly, it is satisfied across $[v_{i+1,j}, t_{i,j}]$ and $[v_{i+1,j+1}, t_{i,j}]$, respectively, if and only if

$$U_{i+1,j}^{-1,1} - \frac{1}{3}\Big(V_{i+1,j} + U_{i+1,j}^{-1,0} + U_{i+1,j}^{0,1}\Big) = 0,$$
$$Y_{i+1,j+1}^{-1,-1} - \frac{1}{3}\Big(U_{i+1,j}^{-1,1} + Z_{i,j}^{1,0} + Z_{i+1,j}^{0,1}\Big) = 0,$$

and

$$U_{i+1,j+1}^{-1,-2} - \frac{1}{3}\Big(V_{i+1,j+1} + U_{i+1,j+1}^{-1,-1} + U_{i+1,j+1}^{0,-1}\Big) = 0,$$
$$Y_{i+1,j+1}^{-1,-2} - \frac{1}{3}\Big(U_{i+1,j+1}^{-1,-2} + Z_{i,j}^{1,1} + Z_{i+1,j}^{0,1}\Big) = 0.$$

For the micro-sides of macro-triangle $\mathbb{B}_{i,j}$, six new conditions are involved. For $[v_{i,j}, b_{i,j}]$, $[v_{i,j+1}, b_{i,j}]$ and $[v_{i+1,j+1}, b_{i,j}]$, C^1 regularity is equivalent to

$$U_{i,j}^{1,2} - \frac{1}{3}\Big(V_{i,j} + U_{i,j}^{0,1} + U_{i,j}^{1,1}\Big) = 0,\ Y_{i,j}^{1,2} - \frac{1}{3}\Big(U_{i,j}^{1,2} + X_{i,j}^{1,1} + X_{i,j}^{0,1}\Big) = 0,$$

$$U_{i,j+1}^{1,-1} - \frac{1}{3}\Big(V_{i,j+1} + U_{i,j+1}^{0,-1} + U_{i,j+1}^{1,0}\Big) = 0,$$
$$Y_{i,j+1}^{1,-1} - \frac{1}{3}\Big(U_{i,j+1}^{1,-1} + X_{i,j}^{0,1} + X_{i,j+1}^{1,0}\Big) = 0,$$

and

$$U_{i+1,j+1}^{2,-1} - \frac{1}{3}\Big(V_{i+1,j+1} + U_{i+1,j+1}^{-1,-1} + U_{i+1,j+1}^{-1,0}\Big) = 0,$$
$$Y_{i+1,j+1}^{-2,-1} - \frac{1}{3}\Big(U_{i+1,j+1}^{2,1} + X_{i,j}^{1,1} + X_{i,j+1}^{1,0}\Big) = 0,$$

respectively. Finally, C^1 continuity at the barycenters of $\mathbb{T}_{i,j}$ and $\mathbb{B}_{i,j}$ is obtained if and only if

$$T_{i,j} - \frac{1}{3}\Big(Y_{i,j}^{2,1} + Y_{i+1,j+1}^{-1,-2} + Y_{i+1,j-1}^{-1,1}\Big) = 0,$$
$$B_{i,j} - \frac{1}{3}\Big(Y_{i,j}^{1,2} + Y_{i+1,j+1}^{-2,-1} + Y_{i,j+1}^{-1,-1}\Big) = 0.$$

These are all equalities involving the values $f(p)$, $p \in S_{i,j}$, so $\mathcal{Q}f$ is C^1 continuous if and only if all the coefficients of the f-values in these equalities are zero. Therefore, the requirements on the C^1 continuity are equivalent to a system of equations having a 122-parametric family of solutions. To these equations must be added those related to the exactness of the operator on $\mathbb{P}_3$. They are obtained by imposing that the BB-coefficients on each microtriangle of the monomials of degree less than or equal to three and those of their quasi-interpolants are equal. The resulting system can be solved with a Computer Algebra System, namely, Mathematica, obtaining the existence of a 17-parametric family of solutions. The free parameters are entries with indices 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 18, 19, 20, 21, and 22 of the mask $M(b_{i,j})$. $\square$

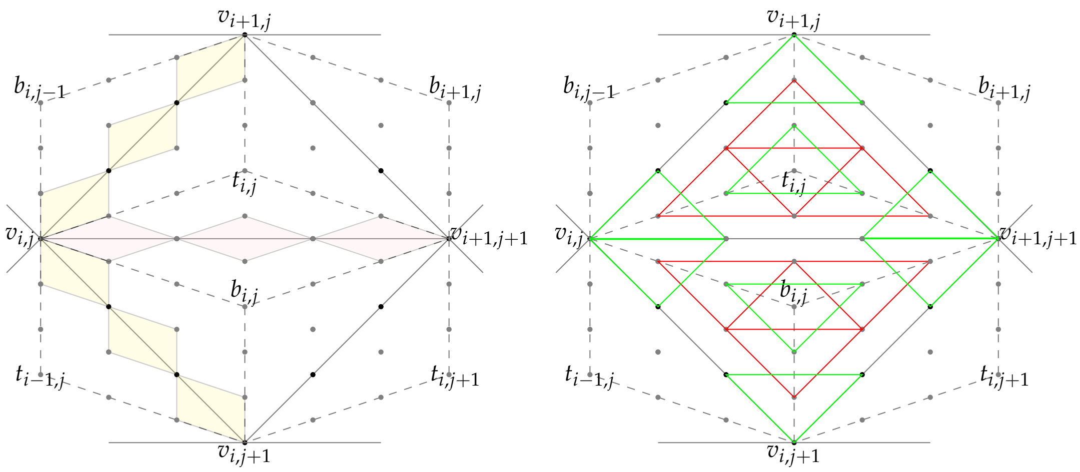

Figure 6. Schematic representation of the conditions to be imposed to achieve C^1 continuity on the macro-interval edges (top) and on the micro-edges and at barycenters (bottom). In each of the shaded parallelograms in the figure on the left, it must be fulfilled that the sum of the BB-coefficients of two opposite domain points must be equal to that of the other two. The C^1 continuity across the micro-edges of the triangle $\mathbb{T}_{i,j}$ is obtained if, in each of the two green and red $\triangle$-triangles closest to each vertex, it is satisfied that the BB-coefficient corresponding to the interior domain point is equal to one-third of the sum of those of the three vertices of the triangle. The same condition must be fulfilled for the $\triangledown$-triangles of $\mathbb{B}_{i,j}$.

Figure 7 shows the mask relative to vertex $v_{i,j}$. The entries of the masks for $u_{i,j}^{0,-1}$, $u_{i,j}^{-1,-2}$, $u_{i,j}^{-2,-1}$ and $u_{i,j}^{-1,0}$ are almost all zero, and the following expressions for their BB-coefficients are found:

$$\begin{aligned}
U_{i,j}^{-1,0} &= -\frac{5}{6}f(v_{i,j}) + 3f\left(u_{i,j}^{-1,0}\right) - \frac{3}{2}f\left(u_{i,j-1}^{0,-1}\right) + \frac{1}{3}f(v_{i,j-1}),\\
U_{i,j}^{0,-1} &= -\frac{5}{6}f(v_{i,j}) - \frac{3}{2}f\left(u_{i,j}^{-1,0}\right) + 3f\left(u_{i,j-1}^{0,-1}\right) + \frac{1}{3}f(v_{i+1,j+1}),\\
U_{i,j}^{-1,-2} &= -\frac{5}{6}f(v_{i,j}) + f\left(u_{i,j}^{-1,0}\right) + 2f\left(u_{i,j}^{0,-1}\right) - \frac{1}{2}f\left(u_{i,j-1}^{0,-1}\right)\\
&\quad - f\left(u_{i+1,j}^{-1,0}\right) + \frac{2}{9}f(v_{i+1,j+1}) + \frac{1}{9}f(v_{i,j-1}),\\
U_{i,j}^{-2,-1} &= -\frac{5}{6}f(v_{i,j}) + 2f\left(u_{i,j}^{-1,0}\right) + f\left(u_{i,j}^{0,-1}\right) - f\left(u_{i,j-1}^{0,-1}\right)\\
&\quad - \frac{1}{2}f\left(u_{i+1,j}^{-1,0}\right) + \frac{1}{9}f(v_{i+1,j+1}) + \frac{2}{9}f(v_{i,j-1}).
\end{aligned}$$

Fourteen of the masks relative to the remaining twenty-two domain points in $D_{i,j}$ do not depend on any parameters and appear in Appendix A. Those of $t_{i,j}$, $b_{i,j}$, $y_{i,j}^{1,2}$, $y_{i,j}^{2,1}$, $y_{i,j}^{-1,-2}$, $y_{i,j}^{-2,-1}$, $x_{i,j}^{1,1}$, and $z_{i,j}^{1,1}$ have very long entries and will not be given.

$$\begin{array}{c}
\frac{4\,484\,399\,224\,513}{6\,514\,447\,513\,140} \quad -\frac{2\,552\,223\,421\,511}{1\,085\,741\,252\,190} \quad \frac{5\,169\,718\,992\,073}{1\,085\,741\,252\,190} \quad -\frac{8\,031\,525\,079}{232\,658\,839\,755}\\
-\frac{64\,504\,080\,325}{868\,593\,001\,752} \quad -\frac{3\,505\,392\,583\,699}{4\,342\,965\,008\,760} \quad -\frac{27\,952\,071\,335\,903}{2\,895\,310\,005\,840} \quad -\frac{3\,505\,392\,583\,699}{4\,342\,965\,008\,760} \quad \frac{5\,169\,718\,992\,073}{1\,085\,741\,252\,190}\\
\frac{36\,832\,622\,942}{542\,870\,626\,095} \quad \frac{120\,096\,488\,429}{482\,551\,667\,640} \quad \frac{519\,419\,554\,543}{310\,211\,786\,340} \quad -\frac{533\,680\,265\,081}{1\,447\,655\,002\,920} \quad \frac{120\,096\,488\,429}{482\,551\,667\,640} \quad -\frac{2\,552\,223\,421\,511}{1\,085\,741\,252\,190}\\
-\frac{6\,955\,013\,524\,199}{13\,028\,895\,026\,280} \quad -\frac{144\,225\,313\,868}{1\,628\,611\,878\,285} \quad \frac{382\,180\,214\,323}{100\,531\,597\,425} \quad -\frac{176\,728\,557\,928}{232\,658\,839\,755} \quad -\frac{285\,439\,445\,629}{1\,085\,741\,252\,190} \quad \frac{6\,856\,655\,665\,381}{8\,685\,930\,017\,520} \quad \frac{4\,484\,399\,224\,513}{6\,514\,447\,513\,140}\\
\frac{2\,095\,907\,565\,527}{2\,171\,482\,504\,380} \quad -\frac{2\,725\,154\,777\,111}{4\,342\,965\,008\,760} \quad \frac{519\,419\,554\,543}{310\,211\,786\,340} \quad -\frac{533\,680\,265\,081}{1\,447\,655\,002\,920} \quad -\frac{64\,504\,080\,325}{868\,593\,001\,752} \quad \frac{343\,799\,911\,081}{1\,085\,741\,252\,190}\\
-\frac{2\,725\,154\,777\,111}{4\,342\,965\,008\,760} \quad \frac{2\,095\,907\,565\,527}{2\,171\,482\,504\,380} \quad -\frac{6\,955\,013\,524\,199}{13\,028\,895\,026\,280} \quad \frac{36\,832\,622\,942}{542\,870\,626\,095} \quad \frac{343\,799\,911\,081}{1\,085\,741\,252\,190}\\
-\frac{31\,932\,271\,589}{120\,637\,916\,910} \quad -\frac{31\,932\,271\,589}{120\,637\,916\,910} \quad \frac{73\,887\,390\,529}{5\,211\,558\,010\,512} \quad -\frac{25\,733\,502\,500\,393}{130\,288\,950\,262\,800}
\end{array}$$

Figure 7. Mask $M(v_{i,j})$.

Remark 1. *It can be proved that it is not possible to obtain quasi-interpolants with the required characteristics if the BB-coefficients are linear combinations of function values at the vertices lying in the hexagon $H_{i,j}$ determined by the six triangles sharing vertex $v_{i,j}$, and the midpoints of the edges of $H_{i,j}$. Neither is it possible to construct C^1 cubic quasi-interpolants exact on $\mathbb{P}_3$ in this way if function values at $v_{i,j}$ and at the eighteen vertices closest to it are used.*

Moreover, quasi-interpolation error estimates are found using a standard procedure [2].

Proposition 3. *There exists an absolute constant K such that for every $f \in C^{m+1}(\mathbb{R}^2)$, $0 \le m \le 2$,*

$$\|D^\gamma(f - \mathcal{Q}f)\|_{\infty,\mathbb{T}} \le K h^{m+1-|\gamma|} \left\| D^{m+1} f \right\|_{\infty,\Omega_{\mathbb{T}}}, \tag{2}$$

for all $0 \le |\gamma| \le 1$, $\gamma = (\gamma_1, \gamma_2)$, with $\Omega_{\mathbb{T}}$ denoting the union of the triangles in Δ having a non-empty intersection with T.

4. Selecting Parameters

An obvious choice is to make all parameters equal to zero. However, a reasonable strategy is to minimize an upper bound of the quasi-interpolation error for monomials of smaller degree non reproduced by the quasi-interpolation operator, namely $m_{k,4-k}(x,y) := x^k y^{4-k}$, $k = 0,1,2,3,4$. Let us suppose that the BB-coefficients of $m_{k,4-k}$ relative to each micro-triangle t_ℓ^+, $\ell = 1,2,3$, of $\mathbb{T}_{i,j}$ are μ_{k,β,t_ℓ^+}, $|\beta| = 4$, and that those of the cubic quasi-interpolant $Qm_{k,4-k}$ are b_{k,γ,t_ℓ^+}, $|\gamma| = 4$. By degree elevation, $Qm_{k,4-k|t_\ell^+}$ can be represented as a quartic polynomial having BB-coefficients b_{k,β,t_ℓ^+}, $|\beta| = 4$, which depend on parameters $z_r := M(b_{i,j})_r$, $1 \le r \le 12$, and $z_r := M(b_{i,j})_{r+5}$, $13 \le r \le 17$. Therefore, the BB-coefficients of the restriction of $m_{k,4-k} - Qm_{k,4-k}$ to t_ℓ^+ have the form

$$\sigma_{k,t_\ell^+}(z) = c_{k,t_\ell^+} + \sum_{r=1}^{17} c_{k,t_\ell^+}^{(r)} z_r$$

for real values c_{k,t_ℓ^+} and $c_{k,t_\ell^+}^{(r)}$, where $z := (z_1, \ldots, z_{17})$. Since the Bernstein polynomials relative to t_ℓ^+ form a partition of unity, then the infinity norm of $m_{k,4-k} - Qm_{k,4-k}$ is bounded by

$$\max\left\{ \left| \sigma_{k,t_\ell^+}(z) \right|,\ \ell = 1,2,3 \right\}.$$

Consequently, an upper bound for the quasi-interpolation errors for quartic monomials in the macro-triangle $\mathbb{T}_{i,j}$ is

$$U_+(z) := \max\left\{ \left| \sigma_{k,t_\ell^+}(z) \right|,\ \ell = 1,2,3;\ k = 0,1,2,3,4 \right\}.$$

Analogously, an upper bound of such errors in the macro-triangle $\mathbb{B}_{i,j}$ is written as

$$U_-(z) := \max\left\{ \left| \sigma_{k,t_\ell^-}(z) \right|,\ \ell = 1,2,3;\ k = 0,1,2,3,4 \right\},$$

where

$$\sigma_{k,t_\ell^-}(z) = c_{k,t_\ell^-} + \sum_{r=1}^{17} c_{k,t_\ell}^{(r)} z_r,$$

for real values c_{k,t_ℓ^-} and $c_{k,t_\ell}^{(r)}$. In short, the function

$$U(z) := \max\{U_+(z), U_-(z)\}$$

is an upper bound for the quasi-interpolation errors for quartic monomials in the square $\mathbb{T}_{i,j} \bigcup \mathbb{B}_{i,j}$.

Function U can be rewritten as

$$U(z) = \max_{1 \leq \alpha \leq 30} \frac{1}{c_\alpha} \left(d_\alpha + \sum_{\beta=1}^{17} e_{\alpha,\beta} \left| f_{\alpha,\beta} \cdot z \right| \right),$$

where $c_\alpha, d_\alpha, e_{\alpha,\beta} \in \mathbb{N}$, $f_{\alpha,\beta} \in \mathbb{Z}^{17}$ and $A \cdot B := \sum_{s=1}^{17} A_s B_s$. The number of terms involved in each sum depends on α, because some of them will be zero. Therefore, the minimization of U is equivalent to the following linear programming problem:

$$\begin{array}{ll} \text{Minimize} & \mu \\ \text{such that} & \left\{ \begin{array}{ll} d_\alpha + \sum_{\beta=1}^{17} e_{\alpha,\beta} \left(u_{\alpha,\beta} + v_{\alpha,\beta} \right) - c_\alpha \mu \leq 0, & 1 \leq \alpha \leq 30, \\ f_{\alpha,\beta} \cdot (Z^+ - Z^-) - u_{\alpha,\beta} + v_{\alpha,\beta} = 0, & 1 \leq \alpha \leq 30,\ 1 \leq \beta \leq 17, \\ u_{p,n}, v_{p,n}, X_1, X_2, Y_1, Y_2, Z_1, Z_2, \mu \geq 0, & \end{array} \right. \end{array}$$

where it has been used that each variable z_r can be written as $z_r = z_r^+ - z_r^-$, $z_r^+, z_r^- \geq 0$, therefore $Z = Z^+ - Z^-$, with $Z^+ := \left(z_1^+, \ldots, z_{17}^+ \right)$ and $Z^- := \left(z_1^-, \ldots, z_{17}^- \right)$. The solution of this problem has been exactly determined by using Mathematica, and the minimum value $\mu = \frac{35971348390906381}{87945041427390}$ is reached at

$$\begin{array}{llll} Z_3^+ & = \frac{33654106472661220639}{246477118305507944440}, & Z_6^- & = \frac{28931119278287059059781}{79874153306877553434720}, \\ Z_7^- & = \frac{1477134152647983512 89}{4929542366110158888 0}, & Z_9^- & = \frac{716874104646429666 11}{4929542366110158888 0}, \\ Z_{10}^- & = \frac{37235621945453397190951 99}{11182381462962857480860 80}, & Z_{12}^- & = \frac{345992170811097165 2593}{122883312779811620668 80}, \\ Z_{13}^- & = \frac{14379153233222450221212 77}{18637302438271429134768 00}, & Z_{15}^+ & = \frac{933461094140338011503538 1}{1006414331666657173277472 0}, \end{array}$$

being equal to zero all the remaining values. Therefore, the minimum is attained at point $z^*$with components $z_r^* = 0$ for $r \in \{1, 2, 4, 5, 8, 11, 14, 16, 17\}$, and

$$\begin{array}{llll} z_3^* & = \frac{33654106472661220639}{246477118305507944440}, & z_6^* & = -\frac{28931119278287059059781}{79874153306877553434720}, \\ z_7^* & = -\frac{147713415264798351289}{49295423661101588880}, & z_9^* & = -\frac{71687410464642966611}{49295423661101588880}, \\ z_{10}^* & = -\frac{3723562194545339719095199}{1118238146296285748086080}, & z_{12}^* & = -\frac{3459921708110971652593}{12288331277981162066880}, \\ z_{13}^* & = -\frac{1437915323322245022121277}{1863730243827142913476800}, & z_{15}^* & = \frac{9334610941403380115035381}{10064143316666571732774720}. \end{array}$$

5. Numerical Tests

In this section, the performance of the quasi-interpolation operator Q^* defined by the masks provided by the solution above is tested. To perform this, we consider Franke's function

$$f_1(x_1, x_2) = \frac{3}{4} \exp\left(-\frac{(9x_1 - 2)^2}{4} - \frac{(9x_2 - 2)^2}{4} \right) + \frac{3}{4} \exp\left(-\frac{(9x_1 + 1)^2}{49} - \frac{9x_2 + 1}{10} \right)$$
$$+ \frac{1}{2} \exp\left(-\frac{(9x_1 - 7)^2}{4} - \frac{(9x_2 - 3)^2}{4} \right) - \frac{1}{5} \exp\left(-(9x_1 - 4)^2 - (9x_2 - 7)^2 \right)$$

and Nielson's function

$$f_2(x_1, x_2) = \frac{x_2}{2} \cos^4\left(4(x_1^2 + x_2 - 1) \right)$$

to produce quasi-interpolants on the unit square [10,11]. The plots of f_1 and f_2 are shown in Figure 8, together with those of their quasi-interpolants obtained by diving the unit interval into 256 equal parts.

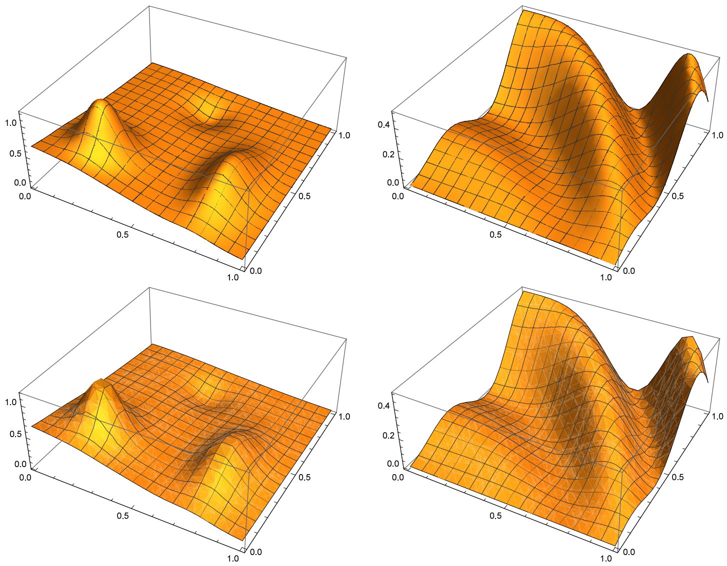

Figure 8. Top, from left to right, plots of the test functions. Bottom, the ones of their respective quasi-interpolants $Q^* f_1$ and $Q^* f_2$ with $h = 1/256$.

The quasi-interpolation error is estimated as

$$\max_{k,\ell=1,\dots,400} |\mathbb{Q}^* f(x_k, y_\ell) - f(x_k, y_\ell)|,$$

x_k and y_ℓ being equally spaced points in $[0, 1]$. The numerical convergence order (NCO) is given by the rate

$$\text{NCO} := \log\left(\frac{\mathtt{E}(h_2)}{\mathtt{E}(h_1)}\right) / \log\left(\frac{h_2}{h_1}\right),$$

where $\mathtt{E}(h)$ stands for the estimated error associated with the step length h.

The quasi-interpolation errors are estimated for different values of the step length h and the NCO are calculated. The results are shown in Table 1. They confirm the theoretical ones.

Table 1. Errors and NCOs for functions f_1 and f_2 with $h = 1/n$, $n = 20, 40, 80, 160$.

	f_1		f_2	
n	**Estimated Error**	**NCO**	**Estimated Error**	**NCO**
16	7.07377×10^{-1}	–	1.47146×10^{-1}	–
32	4.49051×10^{-2}	3.97753	1.44799×10^{-2}	3.34512
64	3.14830×10^{-3}	3.83423	8.62813×10^{-4}	4.06886
128	1.76965×10^{-4}	4.15304	5.36388×10^{-5}	4.00770
256	1.07615×10^{-5}	4.03951	3.48823×10^{-6}	3.94271

6. Conclusions

In this work, C^1 cubic quasi-interpolants have been defined on a Clough–Tocher refinement of a type-1 triangulation, providing directly their BB-coefficients on each of the micro-triangles of the sub-triangulation, which are linear combinations of the values taken by the approximated function at specific points in a neighborhood of each macro-triangle. Cubic polynomials are reproduced. The general problem has a 17-parametric family of solutions and a specific solution has been chosen, which minimizes an upper bound of the quasi-interpolation errors associated with the quartic monomials.

The results improve on those available for cubic quasi-interpolation over a type-1 triangulation since the quasi-interpolation operator is now exact on $\mathbb{P}_3$ instead of $\mathbb{P}_2$.

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Appendix A. Masks

This appendix includes the masks provided by Proposition 2 and which do not depend on the parameters indicated. Further, the remaining ones corresponding to the parameters values z_r^* were performed. They have been obtained by minimizing the considered upper bound of the quasi-interpolation errors of the quartic monomials. They are very lengthy expressions, but are included to provide the reader with as much information as possible.

Appendix A.1. Masks That Do Not Depend on Parameters

Mask of $u_{i,j}^{1,1}$:

$$\Big(-\frac{47473646953}{77552946585}, -\frac{285439445629}{361913750730}, -\frac{533680265081}{482551667640}, \frac{209207768203}{103403928780}, \frac{382180214323}{33510532475},$$
$$\frac{209207768203}{103403928780}, -\frac{533680265081}{482551667640}, \frac{6856655665381}{2895310005840}, \frac{120096488429}{160850555880}, -\frac{1333910079319}{1447655002920},$$
$$-\frac{27952071335903}{965103335280}, -\frac{1333910079319}{1447655002920}, \frac{120096488429}{160850555880}, -\frac{144225313868}{542870626095}, -\frac{2725154777111}{1447655002920},$$
$$\frac{2095907565527}{723827501460}, -\frac{6955013524199}{4342965008760}, \frac{36832622942}{180956875365}, -\frac{64504080325}{289531000584}, \frac{3760571723053}{2171482504380},$$
$$-\frac{2552223421511}{361913750730}, \frac{5169718992073}{361913750730}, -\frac{8031525079}{77552946585}, \frac{5169718992073}{361913750730}, -\frac{2552223421511}{361913750730},$$
$$\frac{3760571723053}{2171482504380}, -\frac{64504080325}{289531000584}, \frac{36832622942}{180956875365}, -\frac{6955013524199}{4342965008760}, \frac{2095907565527}{723827501460},$$
$$-\frac{2725154777111}{1447655002920}, -\frac{31932271589}{40212638970}, -\frac{31932271589}{40212638970}, \frac{73887390529}{1737186003504}, -\frac{25733502500393}{43429650087600},$$
$$\frac{343799911081}{361913750730}, \frac{343799911081}{361913750730}\Big)$$

Mask of $u_{i,j}^{-1,-1}$:

$$\Big(-\frac{211036174997}{232658839755}, \frac{285439445629}{1085741252190}, \frac{533680265081}{1447655002920}, \frac{411215804477}{310211786340}, -\frac{382180214323}{100531597425},$$
$$\frac{411215804477}{310211786340}, \frac{533680265081}{1447655002920}, -\frac{6856655665381}{8685930017520}, -\frac{120096488429}{482551667640}, -\frac{3009054929441}{4342965008760},$$
$$\frac{27952071335903}{2895310005840}, -\frac{3009054929441}{4342965008760}, -\frac{120096488429}{482551667640}, \frac{144225313868}{1628611878285}, \frac{2725154777111}{4342965008760},$$
$$-\frac{2095907565527}{2171482504380}, \frac{6955013524199}{13028895026280}, -\frac{36832622942}{542870626095}, \frac{64504080325}{868593001752}, -\frac{2312916720133}{6514447513140},$$
$$\frac{2552223421511}{1085741252190}, -\frac{5169718992073}{1085741252190}, \frac{8031525079}{232658839755}, -\frac{5169718992073}{1085741252190}, \frac{2552223421511}{1085741252190},$$
$$-\frac{2312916720133}{6514447513140}, \frac{64504080325}{868593001752}, -\frac{36832622942}{542870626095}, \frac{6955013524199}{13028895026280}, -\frac{2095907565527}{2171482504380},$$
$$\frac{2725154777111}{4342965008760}, \frac{31932271589}{120637916910}, \frac{31932271589}{120637916910}, -\frac{73887390529}{5211558010512}, \frac{25733502500393}{130288950262800},$$
$$-\frac{343799911081}{1085741252190}, -\frac{343799911081}{1085741252190}\Big)$$

Mask of $u_{i,j}^{2,1}$:

$$\left(-\frac{319149498787}{465317679510}, -\frac{285439445629}{542870626095}, -\frac{533680265081}{723827501460}, \frac{364313661373}{155105893170}, \frac{764360428646}{100531597425},\right.$$
$$\frac{209207768203}{155105893170}, -\frac{533680265081}{723827501460}, \frac{6856655665381}{4342965008760}, \frac{120096488429}{241275833820}, -\frac{2419651331509}{2171482504380},$$
$$-\frac{27952071335903}{1447655002920}, -\frac{1333910079319}{2171482504380}, \frac{120096488429}{241275833820}, -\frac{288450627736}{1628611878285}, -\frac{2725154777111}{2171482504380},$$
$$\frac{2095907565527}{1085741252190}, -\frac{6955013524199}{6514447513140}, \frac{73665245884}{542870626095}, -\frac{64504080325}{434296500876}, \frac{4122485473783}{3257223756570},$$
$$-\frac{2552223421511}{542870626095}, \frac{5169718992073}{542870626095}, -\frac{16063050158}{232658839755}, \frac{5169718992073}{542870626095}, -\frac{2552223421511}{542870626095},$$
$$\frac{3760571723053}{3257223756570}, -\frac{64504080325}{434296500876}, \frac{73665245884}{542870626095}, -\frac{6955013524199}{6514447513140}, \frac{2095907565527}{1085741252190},$$
$$-\frac{2725154777111}{2171482504380}, -\frac{31932271589}{60318958455}, -\frac{31932271589}{60318958455}, \frac{73887390529}{2605779005256}, -\frac{25733502500393}{65144475131400},$$
$$\left.\frac{343799911081}{542870626095}, \frac{343799911081}{542870626095}\right)$$

Mask of $u_{i,j}^{1,0}$:

$$\left(-\frac{319149498787}{465317679510}, -\frac{285439445629}{542870626095}, -\frac{533680265081}{723827501460}, \frac{519419554543}{155105893170}, \frac{764360428646}{100531597425},\right.$$
$$\frac{54101875033}{155105893170}, -\frac{533680265081}{723827501460}, \frac{6856655665381}{4342965008760}, \frac{120096488429}{241275833820}, -\frac{3505392583699}{2171482504380},$$
$$-\frac{27952071335903}{1447655002920}, -\frac{248168827129}{2171482504380}, \frac{120096488429}{241275833820}, -\frac{288450627736}{1628611878285}, -\frac{2725154777111}{2171482504380},$$
$$\frac{2095907565527}{1085741252190}, -\frac{6955013524199}{6514447513140}, \frac{73665245884}{542870626095}, -\frac{64504080325}{434296500876}, \frac{4484399224513}{3257223756570},$$
$$-\frac{2552223421511}{542870626095}, \frac{5169718992073}{542870626095}, -\frac{16063050158}{232658839755}, \frac{5169718992073}{542870626095}, -\frac{2552223421511}{542870626095},$$
$$\frac{3398657972323}{3257223756570}, -\frac{64504080325}{434296500876}, \frac{73665245884}{542870626095}, -\frac{6955013524199}{6514447513140}, \frac{2095907565527}{1085741252190},$$
$$-\frac{2725154777111}{2171482504380}, -\frac{31932271589}{60318958455}, -\frac{31932271589}{60318958455}, \frac{73887390529}{2605779005256}, -\frac{25733502500393}{65144475131400},$$
$$\left.\frac{343799911081}{542870626095}, \frac{343799911081}{542870626095}\right)$$

Mask of $u_{i,j}^{1,-1}$:

$$\left(-\frac{176728557928}{232658839755}, -\frac{285439445629}{1085741252190}, -\frac{533680265081}{1447655002920}, \frac{829631340883}{310211786340}, \frac{382180214323}{100531597425},\right.$$
$$\frac{209207768203}{310211786340}, -\frac{533680265081}{1447655002920}, \frac{6856655665381}{8685930017520}, \frac{120096488429}{482551667640}, -\frac{5676875088079}{4342965008760},$$
$$-\frac{27952071335903}{2895310005840}, -\frac{1333910079319}{4342965008760}, \frac{120096488429}{482551667640}, -\frac{144225313868}{1628611878285}, -\frac{2725154777111}{4342965008760},$$
$$\frac{2095907565527}{2171482504380}, -\frac{6955013524199}{13028895026280}, \frac{36832622942}{542870626095}, -\frac{64504080325}{868593001752}, \frac{5208226725973}{6514447513140},$$
$$-\frac{2552223421511}{1085741252190}, \frac{5169718992073}{1085741252190}, -\frac{8031525079}{232658839755}, \frac{5169718992073}{1085741252190}, -\frac{2552223421511}{1085741252190},$$
$$\frac{3760571723053}{6514447513140}, -\frac{64504080325}{868593001752}, \frac{36832622942}{542870626095}, -\frac{6955013524199}{13028895026280}, \frac{2095907565527}{2171482504380},$$
$$-\frac{2725154777111}{4342965008760}, -\frac{31932271589}{120637916910}, -\frac{31932271589}{120637916910}, \frac{73887390529}{5211558010512}, -\frac{25733502500393}{130288950262800},$$
$$\left.\frac{343799911081}{1085741252190}, \frac{343799911081}{1085741252190}\right)$$

Mask of $u_{i,j}^{-1,1}$:

$$\left(-\frac{176728557928}{232658839755}, -\frac{285439445629}{1085741252190}, -\frac{533680265081}{1447655002920}, \frac{209207768203}{310211786340}, \frac{382180214323}{100531597425}, \frac{829631340883}{310211786340}, -\frac{533680265081}{1447655002920}, \frac{6856655665381}{8685930017520}, \frac{120096488429}{482551667640}, -\frac{1333910079319}{4342965008760}, -\frac{27952071335903}{2895310005840}, -\frac{5676875088079}{4342965008760}, \frac{120096488429}{482551667640}, -\frac{144225313868}{1628611878285}, -\frac{2725154777111}{4342965008760}, \frac{2095907565527}{2171482504380}, -\frac{6955013524199}{13028895026280}, \frac{36832622942}{542870626095}, -\frac{64504080325}{868593001752}, \frac{3760571723053}{6514447513140}, -\frac{2552223421511}{1085741252190}, \frac{5169718992073}{1085741252190}, -\frac{8031525079}{232658839755}, \frac{5169718992073}{1085741252190}, -\frac{2552223421511}{1085741252190}, \frac{5208226725973}{6514447513140}, -\frac{64504080325}{868593001752}, \frac{36832622942}{542870626095}, -\frac{6955013524199}{13028895026280}, \frac{2095907565527}{2171482504380}, -\frac{2725154777111}{4342965008760}, -\frac{31932271589}{120637916910}, -\frac{31932271589}{120637916910}, \frac{73887390529}{5211558010512}, -\frac{25733502500393}{130288950262800}, \frac{343799911081}{1085741252190}, \frac{343799911081}{1085741252190}\right)$$

Mask of $u_{i,j}^{0,1}$:

$$\left(-\frac{319149498787}{465317679510}, -\frac{285439445629}{542870626095}, -\frac{533680265081}{723827501460}, \frac{54101875033}{155105893170}, \frac{764360428646}{100531597425}, \frac{519419554543}{155105893170}, -\frac{533680265081}{723827501460}, \frac{6856655665381}{4342965008760}, \frac{120096488429}{241275833820}, -\frac{248168827129}{2171482504380}, -\frac{27952071335903}{1447655002920}, -\frac{3505392583699}{2171482504380}, \frac{120096488429}{241275833820}, -\frac{288450627736}{1628611878285}, -\frac{2725154777111}{2171482504380}, \frac{2095907565527}{1085741252190}, -\frac{6955013524199}{6514447513140}, \frac{73665245884}{542870626095}, -\frac{64504080325}{434296500876}, \frac{3398657972323}{3257223756570}, -\frac{2552223421511}{542870626095}, \frac{5169718992073}{542870626095}, -\frac{16063050158}{232658839755}, \frac{5169718992073}{542870626095}, -\frac{2552223421511}{542870626095}, \frac{4484399224513}{3257223756570}, -\frac{64504080325}{434296500876}, \frac{73665245884}{542870626095}, -\frac{6955013524199}{6514447513140}, \frac{2095907565527}{1085741252190}, -\frac{2725154777111}{2171482504380}, -\frac{31932271589}{60318958455}, -\frac{31932271589}{60318958455}, \frac{73887390529}{2605779005256}, -\frac{25733502500393}{65144475131400}, \frac{343799911081}{542870626095}, \frac{343799911081}{542870626095}\right)$$

Mask $u_{i,j}^{1,2}$:

$$\left(-\frac{319149498787}{465317679510}, -\frac{285439445629}{542870626095}, -\frac{533680265081}{723827501460}, \frac{209207768203}{155105893170}, \frac{764360428646}{100531597425}, \frac{364313661373}{155105893170}, -\frac{533680265081}{723827501460}, \frac{6856655665381}{4342965008760}, \frac{120096488429}{241275833820}, -\frac{1333910079319}{2171482504380}, -\frac{27952071335903}{1447655002920}, -\frac{2419651331509}{2171482504380}, \frac{120096488429}{241275833820}, -\frac{288450627736}{1628611878285}, -\frac{2725154777111}{2171482504380}, \frac{2095907565527}{1085741252190}, -\frac{6955013524199}{6514447513140}, \frac{73665245884}{542870626095}, -\frac{64504080325}{434296500876}, \frac{3760571723053}{3257223756570}, -\frac{2552223421511}{542870626095}, \frac{5169718992073}{542870626095}, -\frac{16063050158}{232658839755}, \frac{5169718992073}{542870626095}, -\frac{2552223421511}{542870626095}, \frac{4122485473783}{3257223756570}, -\frac{64504080325}{434296500876}, \frac{73665245884}{542870626095}, -\frac{6955013524199}{6514447513140}, \frac{2095907565527}{1085741252190}, -\frac{2725154777111}{2171482504380}, -\frac{31932271589}{60318958455}, -\frac{31932271589}{60318958455}, \frac{73887390529}{2605779005256}, -\frac{25733502500393}{65144475131400}, \frac{343799911081}{542870626095}, \frac{343799911081}{542870626095}\right).$$

Mask of $y_{i,j}^{1,-1}$:

$$\Big(-\frac{176522697979}{827231430240}, -\frac{285439445629}{6514447513140}, \frac{1755007242871}{13028895026280}, \frac{43682365976659}{52115580105120}, -\frac{3914887523587}{26057790052560}, 0,$$
$$0, \frac{6856655665381}{52115580105120}, \frac{23609319453817}{52115580105120}, -\frac{4979513337263}{52115580105120}, \frac{8038657526531}{26057790052560}, 0, 0,$$
$$-\frac{72112656934}{4885835634855}, -\frac{2725154777111}{26057790052560}, \frac{2095907565527}{13028895026280}, -\frac{41401785965929}{78173370157680}, \frac{48774214262167}{52115580105120},$$
$$-\frac{231734735789}{325722375657}, \frac{12969882253997}{156346740315360}, \frac{7618392504781}{26057790052560}, -\frac{8209241769433}{52115580105120}, -\frac{2520438176729}{19543342539420},$$
$$0, 0, 0, 0, 0, 0, 0, 0, -\frac{31932271589}{723827501460}, 0, 0, 0, -\frac{26064240688}{180956875365}, 0\Big)$$

Mask of $y_{i,j}^{-1,1}$:

$$\Big(-\frac{176522697979}{827231430240}, -\frac{285439445629}{6514447513140}, 0, 0, -\frac{3914887523587}{26057790052560}, \frac{43682365976659}{52115580105120}, \frac{1755007242871}{13028895026280},$$
$$\frac{6856655665381}{52115580105120}, 0, 0, \frac{8038657526531}{26057790052560}, -\frac{4979513337263}{52115580105120}, \frac{23609319453817}{52115580105120}, -\frac{72112656934}{4885835634855},$$
$$0, 0, 0, 0, 0, 0, 0, 0, -\frac{2520438176729}{19543342539420}, -\frac{8209241769433}{52115580105120}, \frac{7618392504781}{26057790052560}, \frac{12969882253997}{156346740315360},$$
$$-\frac{231734735789}{325722375657}, \frac{48774214262167}{52115580105120}, -\frac{41401785965929}{78173370157680}, \frac{2095907565527}{13028895026280},$$
$$-\frac{2725154777111}{26057790052560}, 0, -\frac{31932271589}{723827501460}, 0, 0, 0, -\frac{26064240688}{180956875365}\Big).$$

Mask of $z_{i,j}^{1,0}$:

$$\Big(-\frac{16044311288503}{10423116021024}, -\frac{285439445629}{434296500876}, -\frac{19207698623}{32170111176}, \frac{7517711285213}{1737186003504}, \frac{13541457918013}{1206379169100},$$
$$\frac{105803839423}{103403928780}, -\frac{533680265081}{482551667640}, \frac{6856655665381}{3474372007008}, \frac{948342242035}{1158124002336}, -\frac{1604093855459}{496338858144},$$
$$-\frac{2302892888971}{80425277940}, -\frac{610082577859}{1447655002920}, \frac{120096488429}{160850555880}, -\frac{72112656934}{325722375657}, -\frac{2725154777111}{1737186003504},$$
$$\frac{2095907565527}{868593001752}, -\frac{5562426555557}{5211558010512}, -\frac{1095500413283}{1737186003504}, \frac{1089486754079}{868593001752}, \frac{505619067587}{1302889502628},$$
$$-\frac{521386636583}{144765500292}, \frac{2085546546332}{180956875365}, \frac{364490450669}{542870626095}, \frac{5169718992073}{361913750730}, -\frac{2552223421511}{361913750730},$$
$$\frac{3519295889233}{2171482504380}, -\frac{64504080325}{289531000584}, \frac{36832622942}{180956875365}, -\frac{6955013524199}{4342965008760}, \frac{2095907565527}{723827501460},$$
$$-\frac{2725154777111}{1447655002920}, -\frac{31932271589}{48255166764}, -\frac{31932271589}{40212638970}, \frac{73887390529}{1737186003504}, -\frac{25733502500393}{43429650087600},$$
$$\frac{521386636583}{723827501460}, \frac{343799911081}{361913750730}\Big)$$

Mask of $z_{i,j}^{0,1}$:

$$\Big(\frac{61848672613411}{52115580105120}, \frac{285439445629}{2171482504380}, \frac{533680265081}{1447655002920}, -\frac{209207768203}{310211786340}, -\frac{4369132774261}{1206379169100},$$
$$- -\frac{8501061371657}{8685930017520}, \frac{339213799501}{206807857560}, -\frac{6856655665381}{17371860035040}, -\frac{120096488429}{482551667640}, \frac{1333910079319}{4342965008760},$$
$$\frac{2700000133115}{289531000584}, \frac{28100144271473}{17371860035040}, \frac{15512464547161}{5790620011680}, \frac{72112656934}{1628611878285}, \frac{2725154777111}{4342965008760},$$
$$-\frac{2095907565527}{2171482504380}, \frac{6955013524199}{13028895026280}, -\frac{36832622942}{542870626095}, \frac{64504080325}{868593001752}, -\frac{3760571723053}{6514447513140},$$
$$\frac{2552223421511}{1085741252190}, -\frac{5169718992073}{1085741252190}, -\frac{1205912703113}{1628611878285}, -\frac{1086920646923}{542870626095}, -\frac{2388094137299}{2171482504380},$$
$$\frac{6440703111091}{6514447513140}, -\frac{1218494914729}{868593001752}, \frac{6656146000559}{8685930017520}, -\frac{21722746362811}{26057790052560}, -\frac{2095907565527}{4342965008760},$$
$$\frac{2725154777111}{8685930017520}, \frac{31932271589}{120637916910}, \frac{31932271589}{241275833820}, -\frac{73887390529}{5211558010512}, \frac{25733502500393}{130288950262800},$$
$$-\frac{343799911081}{1085741252190}, -\frac{37792053085}{434296500876}\Big)$$

Mask of $x_{i,j}^{1,0}$:

$$\left(\frac{61848672613411}{52115580105120}, \frac{285439445629}{2171482504380}, -\frac{339213799501}{206807857560}, -\frac{8501061371657}{8685930017520}, -\frac{4369132774261}{1206379169100},\right.$$
$$-\frac{209207768203}{310211786340}, \frac{533680265081}{1447655002920}, -\frac{6856655665381}{17371860035040}, \frac{15512464547161}{5790620011680}, \frac{28100144271473}{17371860035040},$$
$$\frac{2700000133115}{289531000584}, \frac{1333910079319}{4342965008760}, -\frac{120096488429}{482551667640}, \frac{72112656934}{1628611878285}, \frac{2725154777111}{8685930017520},$$
$$-\frac{2095907565527}{4342965008760}, -\frac{21722746362811}{26057790052560}, \frac{6656146000559}{8685930017520}, -\frac{1218494914729}{868593001752}, \frac{6440703111091}{6514447513140},$$
$$-\frac{2388094137299}{2171482504380}, -\frac{1086920646923}{542870626095}, -\frac{1205912703113}{1628611878285}, -\frac{5169718992073}{1085741252190}, \frac{2552223421511}{1085741252190},$$
$$-\frac{3760571723053}{6514447513140}, \frac{64504080325}{868593001752}, -\frac{36832622942}{542870626095}, \frac{6955013524199}{13028895026280}, -\frac{2095907565527}{2171482504380},$$
$$\frac{2725154777111}{4342965008760}, \frac{31932271589}{241275833820}, \frac{31932271589}{120637916910}, -\frac{73887390529}{5211558010512}, \frac{25733502500393}{130288950262800},$$
$$\left.-\frac{37792053085}{434296500876}, -\frac{343799911081}{1085741252190}\right)$$

Mask of $x_{i,j}^{0,1}$:

$$\left(-\frac{16044311288503}{10423116021024}, -\frac{285439445629}{434296500876}, -\frac{533680265081}{482551667640}, \frac{105803839423}{103403928780}, \frac{13541457918013}{1206379169100},\right.$$
$$\frac{7517711285213}{1737186003504}, -\frac{19207698623}{32170111176}, \frac{6856655665381}{3474372007008}, \frac{120096488429}{160850555880}, -\frac{610082577859}{1447655002920},$$
$$-\frac{2302892888971}{80425277940}, -\frac{1604093855459}{496338858144}, \frac{948342242035}{1158124002336}, -\frac{72112656934}{325722375657}, -\frac{2725154777111}{1447655002920},$$
$$\frac{2095907565527}{723827501460}, -\frac{6955013524199}{4342965008760}, \frac{36832622942}{180956875365}, -\frac{64504080325}{289531000584}, \frac{3519295889233}{2171482504380},$$
$$-\frac{2552223421511}{361913750730}, \frac{5169718992073}{361913750730}, \frac{364490450669}{542870626095}, \frac{2085546546332}{180956875365}, -\frac{521386636583}{144765500292},$$
$$\frac{505619067587}{1302889502628}, \frac{1089486754079}{868593001752}, -\frac{1095500413283}{1737186003504}, -\frac{5562426555557}{5211558010512}, \frac{2095907565527}{868593001752},$$
$$-\frac{2725154777111}{1737186003504}, -\frac{31932271589}{40212638970}, -\frac{31932271589}{48255166764}, \frac{73887390529}{1737186003504}, -\frac{25733502500393}{43429650087600},$$
$$\left.\frac{343799911081}{361913750730}, \frac{521386636583}{723827501460}\right).$$

Masks associated with the parameter values z_r^*

Mask of $t_{i,j}$:

$$\left(-\frac{4531890127703}{13028895026280}, \frac{36006119259559}{39086685078840}, -\frac{20242568383524532937}{7042203380157369840}, \frac{17297520671281}{13028895026280},\right.$$
$$\frac{29593026919579}{5428706260950}, \frac{2699484999225562140276 5}{1597483066137551068694 4}, \frac{9165456190681132751}{6161927957637698610}, \frac{30904875065308}{24429178174275},$$
$$\frac{64028235288551979169}{24647711830550794440}, \frac{57633462010750450089 6935}{22364762925925714961 7216}, -\frac{361472787357233}{26057790052560},$$
$$-\frac{5791606797469437299171}{1228833127798116206688 0}, -\frac{9645124685875037347 9}{4929542366110158888 0}, \frac{9488658832619183459}{7783487946489724560},$$
$$\frac{399428393604596767}{379195566623858376 0}, -\frac{2120621509846280547 1}{4929542366110158888 0}, -\frac{888284192303734585}{155669758929794491 2},$$
$$\frac{1354491661355237551055957}{186373024382714291347680 0}, \frac{1387013003869}{8685930017520}, -\frac{28780833585309605134943 89}{10064143316666571732774720},$$
$$-\frac{1681864799399}{558381215412}, \frac{4586462085623}{697976519265}, \frac{17955480525488661251 3}{2218294064749571499600},$$
$$\frac{5674250490552090470842 19}{1490984195061714330781 44}, \frac{7269217075992785869551 67}{7987415330687755343472 00}, -\frac{10872297460717193051971 27}{629008957291660733298420},$$
$$\frac{7013088865869552773812 9}{3194966132275102137388 8}, -\frac{5574602285095942873064 3}{2795595365740714370215 20}, -\frac{292198896309538327}{972935993311215570},$$
$$\frac{148884558838306413569}{49295423661101588880}, -\frac{7135795471566067417}{1895977833119291880}, -\frac{111252026570678852880 0107}{335471443888885724425 8240},$$
$$\frac{1767938963792804738010533}{6709428877777714488516480}, -\frac{232977362971875229}{341276009961472538 40}, -\frac{4855184873014742947401 1}{2764874537545761465048 00},$$
$$\left.-\frac{2318866060366412313392359}{838678609722214311064560}, -\frac{7657959982383155961476047}{3354714438888857244258240}\right)$$

Mask of $b_{i,j}$:

$$\Big(0,0,\frac{3365410647266122063 9}{2464771183055079444 0},0,0,-\frac{28931119278287059059781}{7987415330687755343472 0},-\frac{14771341526479835128 9}{4929542366110158888 0},$$

$$0,-\frac{71687410464642966611}{4929542366110158888 0},-\frac{372356219454533971909519 9}{111823814629628574808608 0},0,-\frac{34599217081109716525 93}{122883312779811620668 80},$$

$$\frac{10915736212229383229}{3521101690078684920},-\frac{17837846075447753051}{7783487946489724560},-\frac{2964129356331035}{27085397615989884},$$

$$\frac{80572693526252875501}{49295423661101588880},-\frac{5840838214799719 33}{1945871986622431140},-\frac{1437915323322245022121277}{1863730243827142913476800},0,$$

$$\frac{9334610941403380115035381}{10064143316666571732774720},0,0,\frac{6547984526167163665}{8873176258998285998 4},\frac{2061565679251584881570009}{7454920975308571653907 20},$$

$$-\frac{313276064201823150241956 7}{7987415330687755343472 00},\frac{1490762720001245530293439}{6290089572916607332984 20},-\frac{325145030004122832756977}{1597483066137551068694 40},$$

$$\frac{8646494711181661614169}{5591190731481428740430 4},-\frac{4440165076962254041}{7783487946489724560},-\frac{8951808041051634353 9}{4929542366110158888 0},$$

$$\frac{14256041226850386701}{3791955666238583760},-\frac{2818467836914786993979 89}{33547144388888572442582 40},-\frac{911334612517867838881345}{1341885775555428977032 96},$$

$$\frac{352952140065170249}{6825520199229450768},-\frac{606565515626167530063 1}{5529749075091522930096 0},\frac{26810651096423644005742 51}{83867860972221431106456 0},$$

$$\frac{1821351235897392862040723}{670942887777771448851648}\Big)$$

Mask of $y_{i,j}^{2,1}$:

$$\Big(-\frac{12271780430239}{11167624308240},\frac{23115036621721}{13028895026280},-\frac{697752284589655474}{146712570419945205},\frac{19233346304581}{5211558010512},$$

$$\frac{29593026919579}{1809568753650},\frac{26994849992255621402765}{5324943553791836895648},\frac{31613815182159137939}{16431807887033862960},\frac{123384121964447}{65144475131400},$$

$$\frac{7928485881111597527}{2053975985879232870},\frac{29403880266065435767818 49}{372746048765428582695360},-\frac{361472787357233}{8685930017520},$$

$$-\frac{579160679746943729917 1}{4096110425993720688960},-\frac{896353880326065349 69}{32863615774067725920},\frac{23919500601269859 55}{1037798392865296608},$$

$$-\frac{833558368032964139}{631992611039763960},\frac{2966315162024509075}{6572723154813545184},\frac{4586462085623}{232658839755},-\frac{869086434956909083}{648623995540810380},$$

$$\frac{14478264410145206439705 77}{6212434146090476378256 00},\frac{444445950829}{2605779005256},-\frac{24454372885588166201409 01}{33547144388888572442582 40},$$

$$-\frac{1681864799399}{186127071804},\frac{32802569068644561035 3}{14788627098330476664 00},\frac{56742504905520904708421 9}{49699473168723811026048},$$

$$\frac{7269217075992785869551 67}{2662471776895918447824 00},-\frac{10872297460717193051971 27}{20966965243055357776614 0},\frac{70130888658695527738129}{10649887107583673791296},$$

$$-\frac{5574602285095942873064 3}{9318651219135714567384 0},-\frac{2370991526403268243}{2594495982163241520},\frac{17805640243921011986 9}{32863615774067725920},$$

$$-\frac{400370254455630256 1}{631992611039763960},-\frac{95145411039889983536881 1}{11182381462962857480860 80},\frac{17679389637928047380105 33}{22364762925925714961721 60},$$

$$\frac{8956702739347499}{5687933499357875640},-\frac{4855184873014742947401 1}{9216248458485871550160 0},-\frac{23188660603664123133923 59}{27955953657407143702152 0},$$

$$-\frac{76579599823831559614760 47}{11182381462962857480860 80}\Big),$$

Mask of $y_{i,j}^{-1,-2}$:

$$\Big(\frac{279021003241424923}{148256913266470944},0,-\frac{2095907565527}{13028895026280},\frac{2987873000594748871}{2527970444159055840},\frac{251548442921233}{260577900525600},$$

$$-\frac{391511615876220607}{78999076379970495},0,0,\frac{2725154777111}{26057790052560},-\frac{61670773096205223361}{32863615774067725920},\frac{11080236034699}{6514447513140},$$

$$\frac{119712715237402707269}{32863615774067725920},0,-\frac{35841538350081461}{1625123856959393040},0,0,\frac{72112656934}{4885835634855},-\frac{6856655665381}{52115580105120},$$

$$\frac{285439445629}{6514447513140},-\frac{822872808792542327}{5188991964326483040},\frac{20342292042566996155}{6572723154813545184},-\frac{123959409326402622509}{32863615774067725920},$$

$$-\frac{1437666942653}{52115580105120},\frac{41709834343289924069}{16431807887033862960},-\frac{103267105684894211989}{32863615774067725920},\frac{33400355926961627}{2594495982163241520},$$

$$0,0,0,0,0,0,0,0,\frac{31083919823327614673}{1478862709833047666400},\frac{31932271589}{723827501460},0\Big)$$

Mask of $y_{i,j}^{-2,-1}$:

$$\Big(-\frac{3512153958923412841}{1037798392865296608}, 0, 0, \frac{132824112520673051}{252797044415905584}, -\frac{5398434624971}{5790620011680}, \frac{3368897166768107761}{505594088831811168},$$
$$-\frac{2095907565527}{13028895026280}, 0, 0, \frac{5140084992534916920 1}{32863615774067725920}, \frac{2771608293079}{8685930017520}, -\frac{12998263840825876142 9}{32863615774067725920},$$
$$\frac{2725154777111}{26057790052560}, \frac{747978263198293 01}{1137586699871575128}, 0, 0, 0, 0, 0, \frac{34028298205171069}{2594495982163241520},$$
$$-\frac{7850326929078680512 1}{32863615774067725920}, \frac{48377645066533267 13}{20539759858792328 70}, \frac{5150734764713}{10423116021024}, -\frac{12997484590652924323 9}{32863615774067725920},$$
$$\frac{18067899515277483949}{4694802253438246560}, -\frac{821616924236123443}{5188991964326483040}, \frac{285439445629}{6514447513140}, -\frac{6856655665381}{52115580105120},$$
$$\frac{72112656934}{4885835634855}, 0, 0, 0, 0, 0, \frac{3045745544524018757}{29577254196660953328 0}, 0, \frac{31932271589}{723827501460}\Big)$$

Mask of $y_{i,j}^{1,2}$:

$$\Big(-\frac{22557725931173}{39086685078840}, \frac{10223953983883}{26057790052560}, \frac{1430304844604791378 7}{8215903943516931480}, 0, 0, -\frac{734305235019830442889 1}{532494355379183689564 8},$$
$$-\frac{811559741641047234 53}{16431807887033862960}, -\frac{47075659357}{13028895026280}, -\frac{648715516384991281 01}{32863615774067725920},$$
$$-\frac{3723562194545339719095199}{372746048765428582695360}, 0, -\frac{281470274031953129767 9}{40961104259937206889 60}, \frac{7580980524594648365}{1643180788703386296},$$
$$-\frac{3073350921486401347}{1037798392865296608}, -\frac{215819734497942031}{252797044415905584}, \frac{10974453712715658180 1}{32863615774067725920},$$
$$-\frac{2370363584125058801}{2594495982163241520}, -\frac{143791532332224502212127 7}{621243414609047637825600}, 0, \frac{933461094140338011503538 1}{335471443888885724425824 0},$$
$$0, 0, \frac{8919157102449659699}{4225322028094421904 0}, \frac{206156567925158488157000 9}{248497365843619055130240}, -\frac{313276064201823150241956 7}{266247177689591844782400},$$
$$\frac{379450774843626130907008}{524174131076383944415 35}, -\frac{683144264060623578050 65}{106498871075836737912 96}, \frac{2861634525240038600401 9}{465932560956785728369 20},$$
$$-\frac{347571779754942689}{259449598216324152}, -\frac{534802895019309926 93}{32863615774067725920}, \frac{52611896805569666 89}{12639852220795279 20},$$
$$-\frac{2818467836914786993979 89}{111823814629628574808608 0}, -\frac{4234540751973561807544133}{223647629259257149617216 0}, \frac{203356487425511647}{2275173399743150256},$$
$$-\frac{606565515626167530063 1}{184324969169717431003 20}, \frac{268106510964236440057425 1}{279559536574071437021520}, \frac{1821351235897392862040723}{223647629259257149617216}\Big)$$

Mask of $z_{i,j}^{1,1}$:

$$\Big(-\frac{55838625716687}{52115580105120}, \frac{28252946643043}{4342965008760}, -\frac{35419776850673763263}{2738634647838977160}, \frac{340862589821}{77552946585},$$
$$\frac{36472270777393}{1206379169100}, \frac{2278456774419731489904 5}{17749811845972789652 16}, \frac{41709834343289924069}{5477269295677954320}, \frac{184986982915643}{86859300175200},$$
$$\frac{16061835113666299907}{1564934084479415520}, \frac{6960779820048589013942 27}{248497365843619055130 24}, -\frac{222664500682471}{2895310005840},$$
$$-\frac{43774743823286244292 91}{136537014199790689632 0}, -\frac{10326710568489421198 9}{10954538591355908640}, \frac{1264903422827789726 3}{172966398810882768 0},$$
$$-\frac{954829982212037 57}{842656814719685 28}, -\frac{32748200305393967077}{10954538591355908640}, -\frac{16299690896471675 23}{864831994054413840},$$
$$\frac{1550315458679682817567037}{207081138203015879275200}, -\frac{64504080325}{108574125219}, -\frac{4294689070651846383297973}{1118238146296285748086080},$$
$$-\frac{10208893686044}{542870626095}, \frac{20678875968292}{542870626095}, \frac{310839198233276146 73}{492954236611015888800}, \frac{17302121078937403294988 3}{165664910562412703420 16},$$
$$\frac{1770022110879746120192767}{887490592298639482608 00}, -\frac{3202974039104927951977 03}{17472471035879464813845}, \frac{714490379953317517391 29}{354996236919455793043 2},$$
$$-\frac{6628353554071922205368 3}{31062170730452381891280}, -\frac{626920878568621 33}{864831994054413840}, \frac{125189984533080661589}{10954538591355908640},$$
$$-\frac{417844641336210772}{26333025459990165}, -\frac{5074657885263635628527 63}{372746048765428582695360}, \frac{2754579679065107565823973}{745492097530857165390720},$$
$$-\frac{35841538350081461}{541707952319797680}, -\frac{182133393026644271751 01}{3072082819495290516720 0}, -\frac{24450050757017497294589 23}{93186512191357145673840},$$
$$-\frac{8248110067234692639264367}{372746048765428582695360}\Big)$$

Mask of $x_{i,j}^{1,1}$:

$$\Big(\frac{25735396581967}{52115580105120}, \frac{20499774026527}{8685930017520}, \frac{48377645066533326713}{68465866195974429}, -\frac{147165410935}{62042357268}, -\frac{4542756612353}{241275833820},$$
$$-\frac{95967048423019818088417}{88749059229863948260800}, -\frac{10549610284487262413}{7824670422397077600}, -\frac{61898203625857}{17371860035040},$$
$$-\frac{7850326929078680512 1}{10954538591355908640}, -\frac{35948761447675257479361 19}{12424868292180952756512 0}, \frac{69404143337381}{1447655002920},$$
$$\frac{31193838906447646384 67}{13653701419979068963 20}, \frac{13719668218977150621 7}{10954538591355908640}, -\frac{14677470679789039247}{17296639881088276 80},$$
$$\frac{48558431048704523}{84265681471968528}, \frac{56878119221027123521}{10954538591355908640}, -\frac{62064145578652691}{86483199405441384 0},$$
$$-\frac{150816540792064364427487 7}{207081138203015879275200}, \frac{322520401625}{868593001752}, \frac{62312530282793501055364 21}{111823814629628574808608 0}, \frac{2552223421511}{217148250438},$$
$$-\frac{5169718992073}{217148250438}, \frac{30457455445240187 57}{98590847322203177760}, \frac{31810546085063487765359 3}{82832455281206351710080},$$
$$-\frac{23958823528480266401353 27}{88749059229863948260800}, \frac{28044497259873803268614 3}{13977976828703571851076},$$
$$-\frac{72239927597313486139729}{35499623691945579304 32}, \frac{72606043154575098047507}{31062170730452381891280}, -\frac{1629341147368958081}{864831994054413840},$$
$$-\frac{20212013123489501029}{219090771827118172 8}, \frac{3225445712758438091}{2106642036799213 20}, \frac{2114735739446727145087 31}{3727460487654285826953 60},$$
$$-\frac{33465641082284892625120 37}{74549209753085716539072 0}, \frac{74797826319829301}{379195566623858376}, \frac{20467292349251591 51}{61441656389905810334 40},$$
$$\frac{25335275884294802311271 71}{93186512191357145673840}, \frac{86022001181456146459373 59}{3727460487654285826953 60}\Big)$$

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