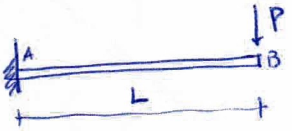
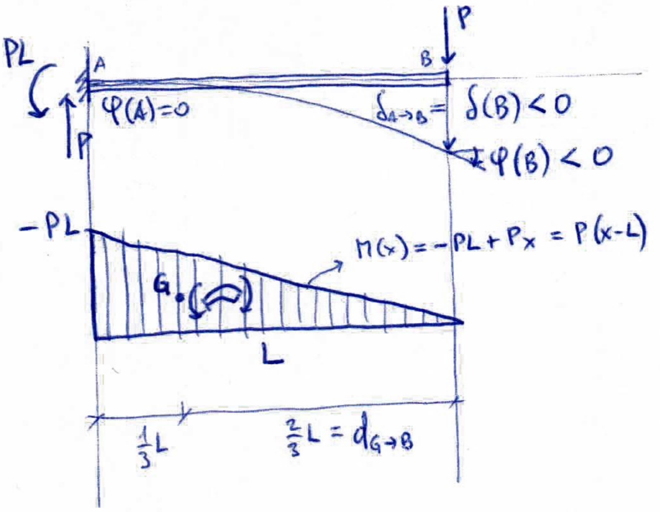
TEOREMAS DE MOHR 1 y 2: EJERCICIOS EN VIGAS

Los teoremas 1 y 2 se pueden aplicar gráficamente (calculando áreas y momentos estáticos) o analíticamente (integrando funciones). Siempre que sea posible (cuando las leyes de momentos sean rectas o parábolas con pendiente nula en un extremo) se recomienda aplicar los teoremas gráficamente.

Ej:



Calcular giro y desplazamiento vertical de B



Giro: Teorema 1

$$\varphi(B)=\varphi(B)-0=\varphi(B)-\varphi(A)=\int_A^B\frac{M(x)}{EI}dx$$

Gráficamente:

$$\int_A^B\frac{M(x)}{EI}=\frac{A_{M,A\to B}}{EI}=\frac{\frac{1}{2}L(-PL)}{EI}=\boxed{-\frac{PL^2}{2EI}}$$ (negativo ✓)

Analíticamente:

$$\int_A^B\frac{M(x)}{EI}=\frac{1}{EI}\int_0^L P(x-L)dx=\frac{P}{EI}\left[\frac{x^2}{2}-Lx\right]_0^L=$$

$$=\frac{P}{EI}\left(\frac{L^2}{2}-L^2-(0-0)\right)=-\frac{PL^2}{2EI}$$ ✓

Desplazamiento: Teorema 2

$\delta(B)=\delta_{A\to B}$ (paralela por A y vertical por B)

Gráficamente:

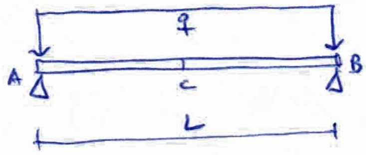
$$\delta_{A\to B}=\frac{S_{M,A\to B}}{EI}=\frac{A_{M,A\to B}\cdot\overbrace{d_{G\to B}}^{\text{con signo}}}{EI}=\frac{\left(-\frac{PL^2}{2}\right)\left(\frac{2}{3}L\right)}{EI}=\boxed{-\frac{PL^3}{3EI}}$$ (negativo) ✓

Analíticamente

$$\delta_{A\to B}=\frac{1}{EI}\int_A^B M(x)(x_B-x)\,dx=\frac{1}{EI}\int_0^L M(x)(L-x)dx=\frac{1}{EI}\int_0^L P(x-L)(L-x)dx=-\frac{P}{EI}\int_0^L(x-L)^2dx=$$

$$=-\frac{P}{EI}\int_0^L(x^2-2xL+L^2)dx=-\frac{P}{EI}\left[\frac{x^3}{3}-Lx^2+L^2x\right]_0^L=-\frac{P}{EI}\left(\frac{L^3}{3}-L^3+L^3\right)=-\frac{PL^3}{3EI}$$ ✓

Ej:

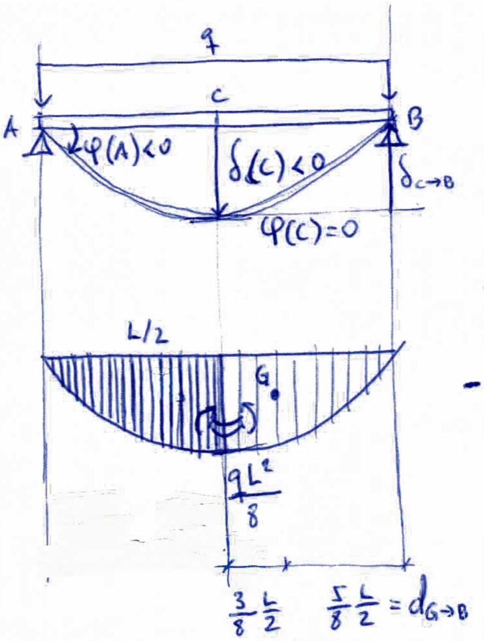


Calcular giro en A y flecha en C

(Aplicamos teoremas solo gráficamente)

$$\varphi(A) = \varphi(A) - 0 = \varphi(A) - \varphi(C) = \int_C^A \frac{M(x)}{EI}\,dx = \frac{A_{M,C\to A}}{EI} =$$

$$= \frac{-A_{M,A\to C}}{EI} = -\frac{\frac{2}{3}\frac{L}{2}\frac{qL^2}{8}}{EI} = \boxed{-\frac{qL^3}{24EI}}$$ (negativo ✓)

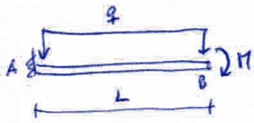


$\delta_C = -\delta_{C\to B}$ (paralela por C y vertical por B, cambiada de signo)

$$-\delta_{C\to B} = -\frac{S_{M,C\to B}}{EI} = -\frac{A_{M,C\to B}\cdot d_{G\to B}}{EI} = -\frac{\frac{2}{3}\frac{L}{2}\frac{qL^2}{8}\cdot\frac{5}{8}\frac{L}{2}}{EI} =$$

$$= \boxed{-\frac{5}{384}\frac{qL^4}{EI}}$$ (negativo ✓)

Ej:

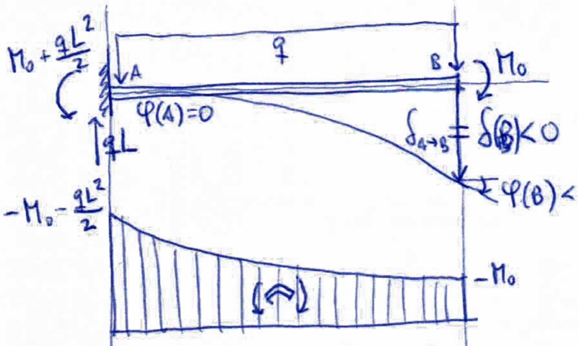


Calcular giro y flecha en B. (Solo se puede aplicar teorema analíticamente)

$$\varphi(B) = \varphi(B) - 0 = \varphi(B) - \varphi(A) = \int_A^B \frac{M(x)}{EI}\,dx =$$

$$= \frac{1}{EI}\int_0^L \left(-\frac{q}{2}x^2 + qLx - \left(M_0 + \frac{qL^2}{2}\right)\right)dx = \frac{1}{EI}\left[-\frac{q}{6}x^3 + \frac{qL}{2}x^2 - \left(M_0 + \frac{qL^2}{2}\right)x\right]_0^L$$

$$= \frac{1}{EI}\left(-\frac{qL^3}{6} + \frac{qL^3}{2} - M_0L - \frac{qL^3}{2}\right) = \boxed{\underbrace{-\frac{qL^3}{6EI}}_{\text{giro debido a: } q} \underbrace{-\frac{M_0L}{EI}}_{M_0}}$$ (negativo ✓)



$$M(x) = -M_0 - \frac{qL^2}{2} + qLx - \frac{qx^2}{2} =$$

$$= -\frac{q}{2}x^2 + qLx - \left(M_0 + \frac{qL^2}{2}\right)$$

$$\delta(B) = \delta_{A\to B} = \int_A^B \frac{M(x)(x_B - x)}{EI}\,dx = \frac{1}{EI}\int_0^L \left(-\frac{qx^2}{2} + qLx - M_0 - \frac{qL^2}{2}\right)(L-x)\,dx =$$

$$= \frac{1}{EI}\int_0^L \left(-\frac{qLx^2}{2} + qL^2x - M_0L - \frac{qL^3}{2} + \frac{qx^3}{2} - qLx^2 + M_0x + \frac{qL^2x}{2}\right)dx =$$

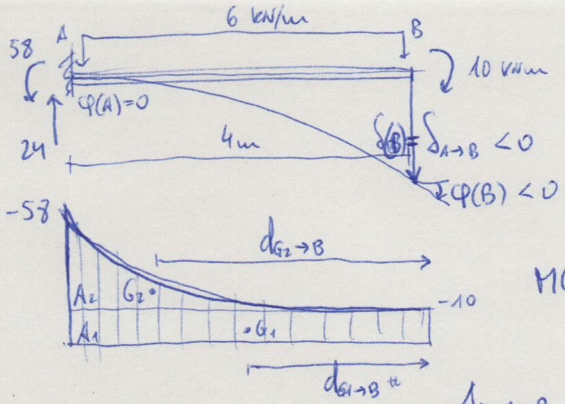
$$= \frac{1}{EI}\int_0^L \left(\frac{qx^3}{2} - \frac{3qL}{2}x^2 + \left(\frac{3qL^2}{2} + M_0\right)x - \left(M_0L + \frac{qL^3}{2}\right)\right)dx = \frac{1}{EI}\left[\frac{qx^4}{8} - \frac{qLx^3}{2} + \left(\frac{M_0}{2} + \frac{3qL^2}{4}\right)x^2 - \left(M_0L + \frac{qL^3}{2}\right)x\right]_0^L =$$

$$= \frac{qL^4}{8} - \frac{qL^4}{2} + \frac{M_0L^2}{2} + \frac{3qL^4}{4} - M_0L^2 - \frac{qL^4}{2} = qL^4\left(\frac{1}{8} - \frac{1}{2} + \frac{3}{4} - \frac{1}{2}\right) - \frac{M_0L^2}{2} = qL^4\,\frac{1-4+6-4}{8} - \frac{M_0L^2}{2} =$$

$$= \boxed{\underbrace{-\frac{qL^4}{8}}_{q} \underbrace{-\frac{M_0L^2}{2}}_{M_0}}$$ (negativo ✓)

flecha debida a: q M₀

Ej:



$$A_1 = 4(-10) = -40$$

$$A_2 = \frac{1}{3}4(-48) = -64$$

$$\left.\right\} A_{M,A\to B} = -104$$

$$M(x) = -58 + 24x - 6\frac{x^2}{2} = -3x^2 + 24x - 58$$

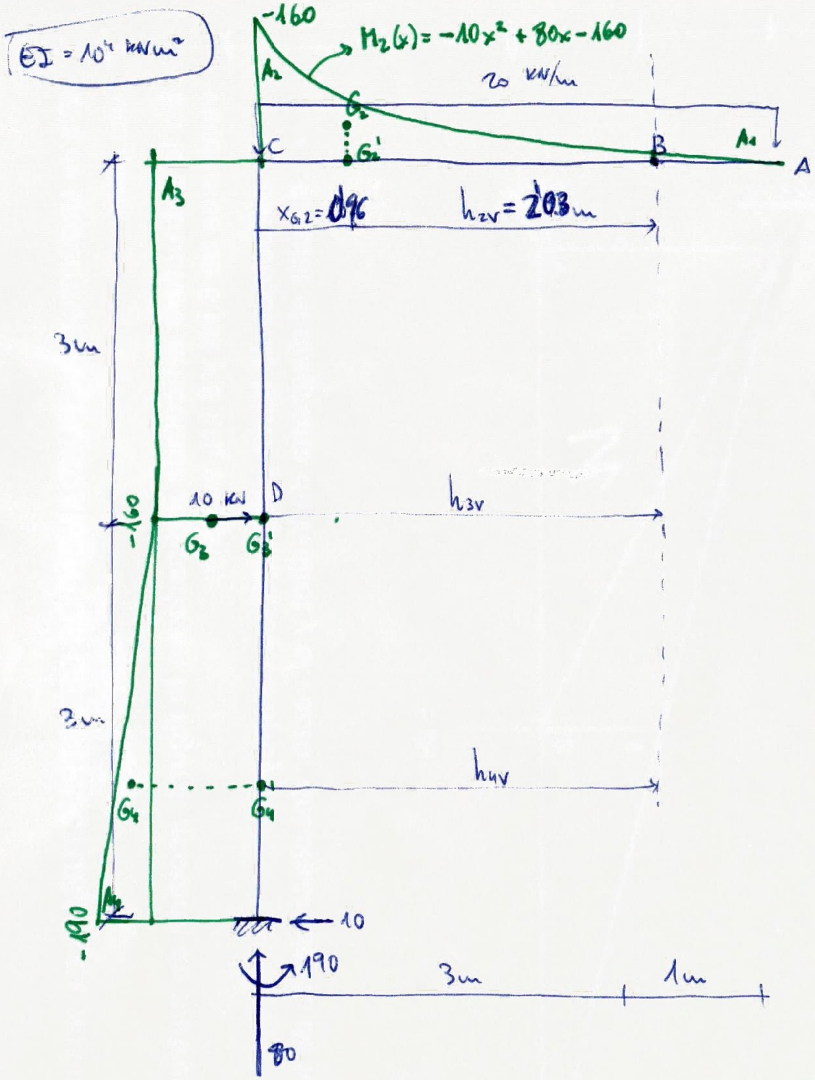
$$\varphi(B) = \varphi(B) - 0 = \varphi(B) - \varphi(A) = \frac{A_{M,A\to B}}{EI} = -\frac{104}{EI}$$

$$\hookrightarrow = \frac{1}{EI}\int_A^B M(x)\,dx = \frac{1}{EI}\left[-3\frac{x^3}{3} + 24\frac{x^2}{2} - 58x\right]_0^4 = \frac{1}{EI}\left(-4^3 + 12\cdot 4^2 - 58\cdot 4\right) = -\frac{104}{EI}$$

$$\delta(B) = \delta_{A\to B} = \frac{S_{M,A\to B}}{EI} = \frac{A_1\cdot d_{G1\to B} + A_2\cdot d_{G2\to B}}{EI} = \frac{-40\cdot 2 - 64\cdot 3}{EI} = -\frac{272}{EI}$$

$$\hookrightarrow = \frac{1}{EI}\int_A^B M(x)(x_B - x)\,dx = \frac{1}{EI}\int_0^4 (-3x^2 + 24x - 58)(4-x)\,dx = \frac{1}{EI}\int_0^4 (-12x^2 + 96x - 232 + 3x^3 - 24x^2 + 58x)\,dx =$$

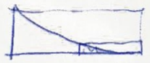
$$= \frac{1}{EI}\int_0^4 (3x^3 - 36x^2 + 154x - 232)\,dx = \frac{1}{EI}\left[3\frac{x^4}{4} - 36\frac{x^3}{3} + 154\frac{x^2}{2} - 232x\right]_0^4 = -\frac{272}{EI}$$



$$A_2 = \int_C^B M_2(x)\,dx = \int_0^3 (-10x^2 + 80x - 160)\,dx = \left[-10\frac{x^3}{3} + 80\frac{x^2}{2} - 160x\right]_0^3 = -210 \;;\; |A_2| = 210$$

↓

Comprobación:



$$A_2 = \frac{1}{3}4 \cdot 160 - \frac{1}{3}1 \cdot 10 = 210 \checkmark$$

$$x_{G_2} = \frac{\int_C^B M_2(x) \cdot x\,dx}{A_2} = \frac{\left[-10\frac{x^4}{4} + 80\frac{x^3}{3} - 160\frac{x^2}{2}\right]_0^3}{-210} = 0'96\,m$$

$$|A_3| = 6 \cdot 160 = 960 \;;\; |A_4| = \frac{1}{2} 3 \cdot 30 = 45$$

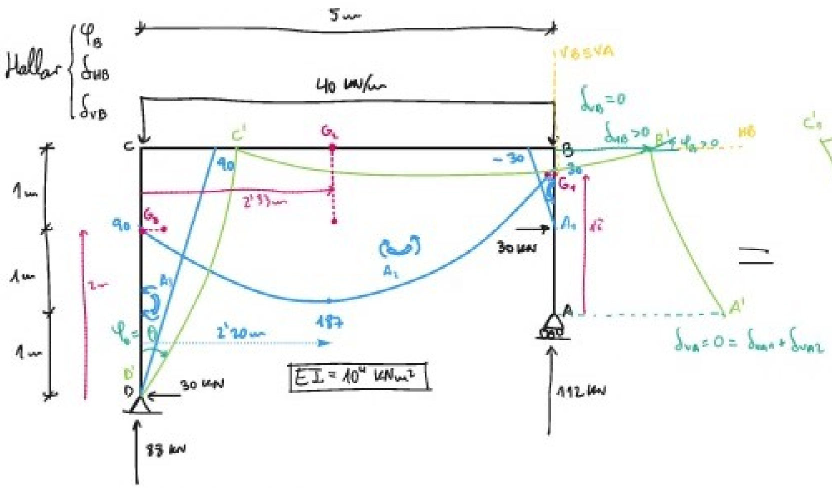
$$\varphi_B = \frac{1}{EI}(-A_2 - A_3 - A_4) = -0'122\ rad$$

$$\delta_{HB} = \frac{1}{EI}(A_3\, h_{3H} + A_4 \cdot h_{4H}) = 0'31\,m$$

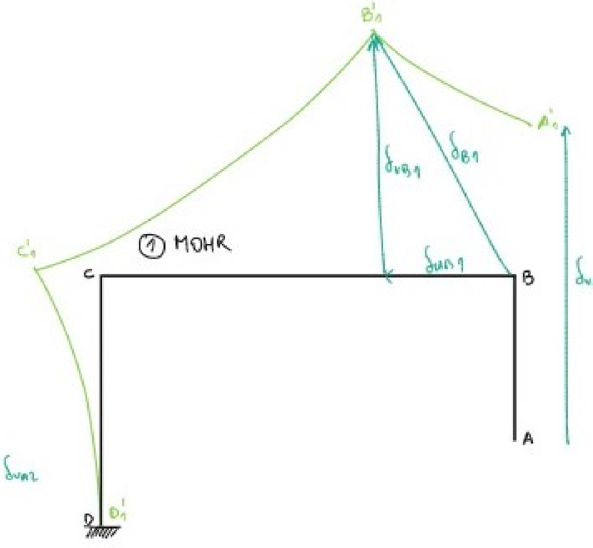
$$\delta_{VB} = \frac{1}{EI}(-A_2\, h_{2V} - A_3\, h_{3V} - A_4\, h_{4V}) = -0'34\,m$$

	φ_B	δ_{HB}	δ_{VB}	h_H	h_V	A
A_2	−	∅	−	0	2'03	210
A_3	−	+	−	3'00	3'00	960
A_4	−	+	−	5'00	3'00	45

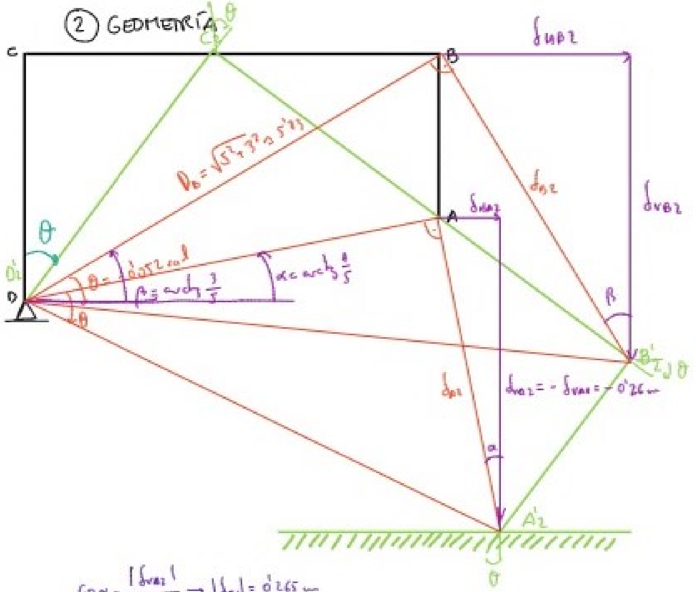
Hallar $\begin{cases} \varphi_B \\ \delta_{HB} \\ \delta_{VB} \end{cases}$



=



+



Resultados finales

Nudo Ⓑ
$$\begin{cases} \varphi_B = \overset{①}{\varphi_{B1}} + \underbrace{\overset{②}{\varphi_{B2}}}_{\theta} = 0'035 - 0'052 = \boxed{0'033 \text{ rad } \circlearrowright} \\ \delta_{HB} = \delta_{HB1} + \delta_{HB2} = -0'0135 + 0'156 = \boxed{0'142 \text{ m} \rightarrow} \\ \delta_{VB} = \delta_{VB1} + \delta_{VB2} = 0'26 - 0'26 = \boxed{0} \end{cases}$$

$$M_2(x) = -40\frac{x^2}{2} + 87x + 90 = -20x^2 + 87x + 90$$

$$A_2 = \int_0^5 M(x)\,dx = \left[-20\frac{x^3}{3} + 87\frac{x^2}{2} + 90x\right]_0^5 = 716'6 \quad > \; 623 \quad \boxed{2/3}$$

$$x_{G2} = \frac{S_2}{A_2} = \frac{\int_0^5 M(x)\cdot x\,dx}{716'6} = \frac{\left[-20\frac{x^4}{4} + 87\frac{x^3}{3} + 90\frac{x^2}{2}\right]_0^5}{716'6} = 2'33 \text{ m}$$

	|A| [kNm²]	Nudo apoyado (A)		Nudo de interés (B)				
		δ_{VA}	h_{VA}	φ_B	δ_{HB}	δ_{VB}	h_{HB}	h_{VB}
A_1	15	X	X	X	X	X	X	X
A_2	716'6	+	2'67	+	∅	+	∅	2'67
A_3	135	+	5	+	−	+	1	5

Nudo de interés (B)

$$\begin{cases} \varphi_{B1} = \frac{1}{EI}(A_2 + A_3) = 0'035 \text{ rad} \circlearrowright \\ \delta_{HB1} = \frac{1}{EI}(-A_3 \cdot h_{HB3}) = -0'0135 \text{ m} \leftarrow \\ \delta_{VB1} = \frac{1}{EI}(+A_2 \cdot h_{VB2} + A_3 \cdot h_{VB3}) = 0'26 \text{ m} \uparrow \end{cases}$$

Nudo apoyado (A)

$$\delta_{VA1} = \frac{1}{EI}(+A_2 \cdot h_{VA2} + A_3 \cdot h_{VA3}) = 0'26 \text{ m} \uparrow$$

$$\cos\alpha = \frac{|\delta_{VA2}|}{|\delta_{A2}|} \Rightarrow |\delta_{A2}| = 0'265 \text{ m}$$

$$\text{tg}|\theta| \simeq |\theta| = \frac{|\delta_{A2}|}{D_A} = 0'052 \text{ rad} \Rightarrow \theta = -0'052 \text{ rad}$$

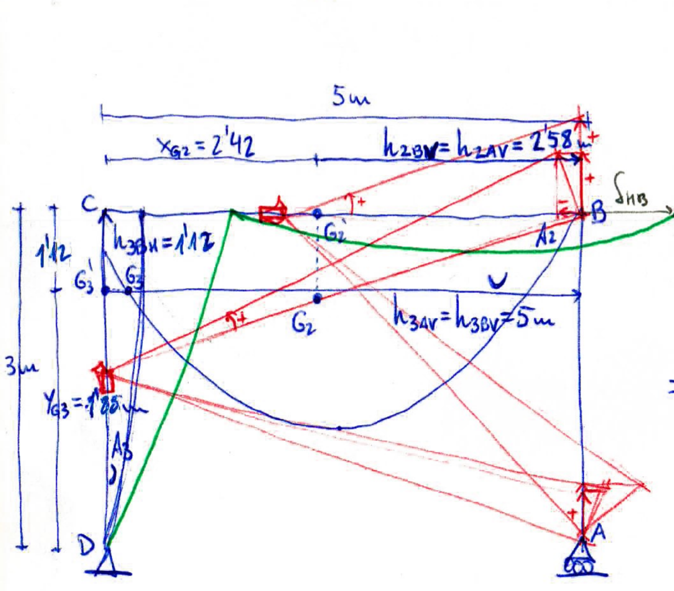
$$\text{tg}|\theta| \simeq |\theta| = \frac{|\delta_{B2}|}{D_B} \Rightarrow |\delta_{B2}| = 0'303 \text{ m}$$

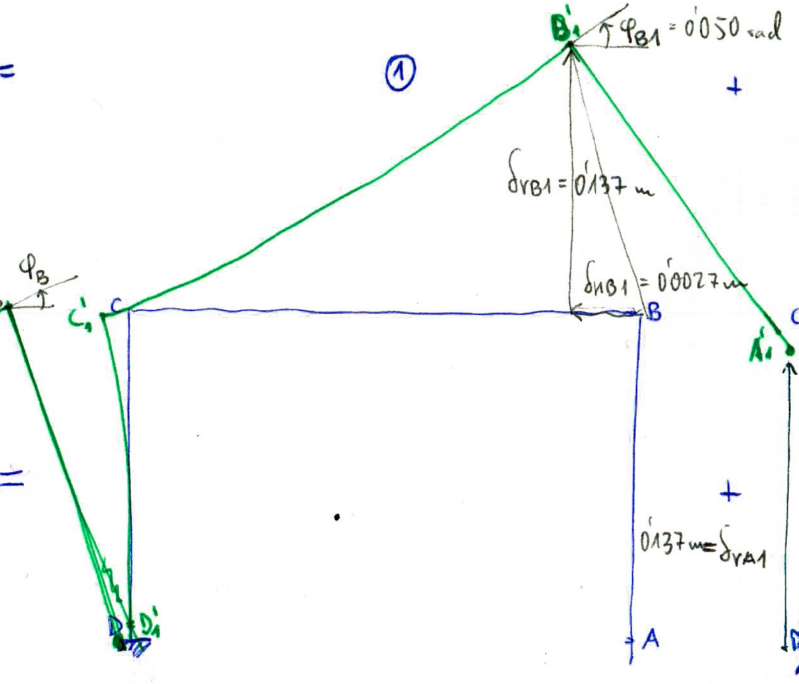
$$\text{sen}\,\beta = \frac{|\delta_{HB2}|}{|\delta_{B2}|} \Rightarrow \delta_{HB2} = 0'156 \text{ m} \rightarrow$$

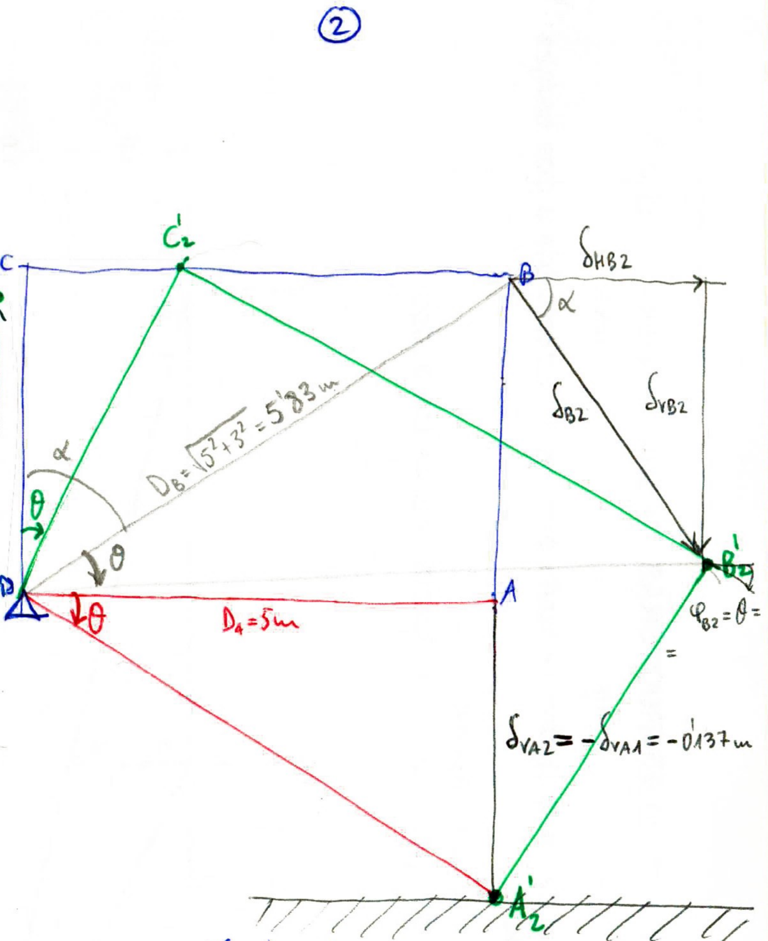
$$\cos\beta = \frac{|\delta_{VB2}|}{|\delta_{B2}|} \Rightarrow \delta_{VB2} = -0'26 \text{ m} \downarrow$$

$\boxed{EJ = 10^4\ kNm^2}$

Ⓡ =







	A	Nudo A			Nudo B			Nudo C			Nudo D					
	[kNm²]	φ_A	δ_{HA}	δ_{VA}	φ_B	δ_{HB}	δ_{VB}	φ_C	δ_{HC}	δ_{VC}	φ_D	δ_{HD}	δ_{VD}	h_{AV}	h_{BH}	h_{BV}
A_2	461'6			+	+	∅	+							2'58	0	2'58
A_3	36			+	+	−	+							5'00	1'12	5'00

$$A_2 = \int_C^B M_2(x)\,dx = \int_0^5 (-20x^2 + 96'4x + 18)\,dx = \left[-20\frac{x^3}{3} + 96'4\frac{x^2}{2} + 18x\right]_0^5 = 461'\hat{6}\ kNm^2$$

$$A_3 = \frac{2}{3}\,3\cdot 18 = 36\ kNm^2$$

$$x_{G2} = \frac{\int_C^B M_2(x)\cdot x\,dx}{A_2} = \frac{\int_0^5 (-20x^3 + 96'4x^2 + 18x)\,dx}{461'\hat{6}} = \frac{\left[-20\frac{x^4}{4} + 96'4\frac{x^3}{3} + 18\frac{x^2}{2}\right]_0^5}{461'\hat{6}} = 2'42\ m$$

$$Y_{G3} = \frac{5}{8}\,3 = 1'88\ m$$

$$\delta_{VA1} = \frac{1}{EJ}\left(A_2\cdot h_{2AV} + A_3\cdot h_{3AV}\right) =$$

$$= \frac{1}{EJ}\left(461'6\cdot 2'58 + 36\cdot 5\right) =$$

$$= 0'137\ m\ \uparrow$$

$$\varphi_{B1} = \frac{1}{EJ}\left(A_2 + A_3\right) = 0'050\ rad\ ↺$$

$$\delta_{HB1} = \frac{1}{EJ}\left(-A_3\cdot h_{3BH}\right) = -0'004\ m\ \leftarrow$$

$$\delta_{VB1} = \frac{1}{EJ}\left(A_2\,h_{2BV} + A_3\,h_{3BV}\right) = 0'137\ m\ \uparrow$$

$$|\theta| \simeq tg\,\theta = \frac{|\delta_{VA1}|}{D_A} = \frac{0'137}{5} = 0'0274\ rad\,;$$

$$\theta = \varphi_{A2} = \varphi_{B2} = -0'0274\ rad\ ↻$$

$$tg\,\theta \simeq \theta = \frac{\delta_{B2}}{D_B} \Rightarrow \delta_{B2} = 0'160\ m$$

$$\alpha = arctg\frac{5}{3}$$

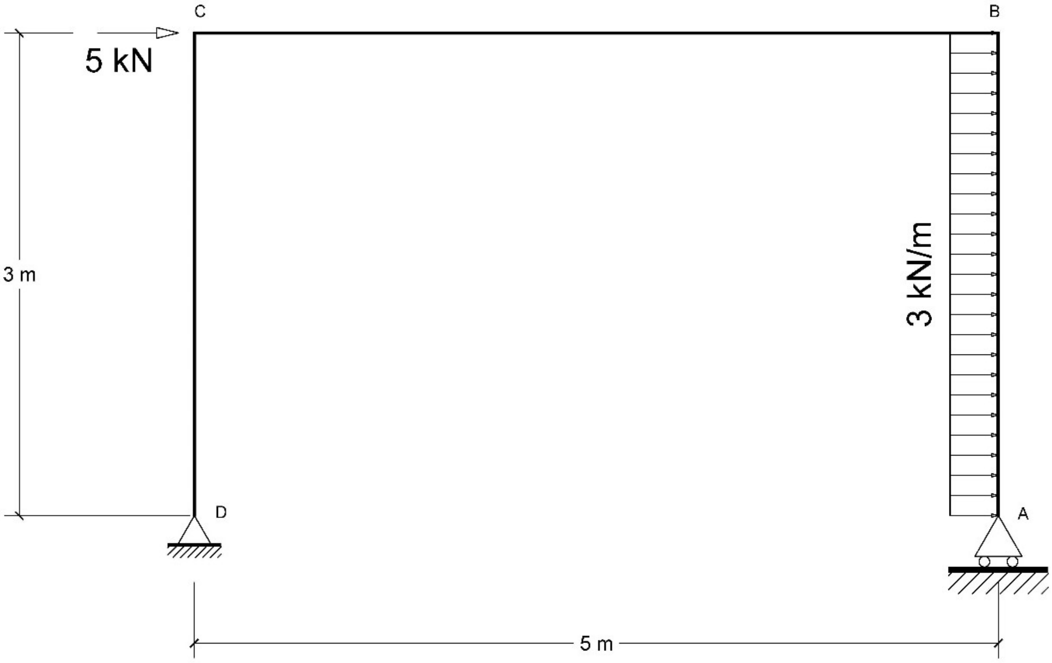
$$\cos\alpha = \frac{|\delta_{HB2}|}{\delta_{B2}} \Rightarrow \delta_{HB2} = 0'082\ m\ \rightarrow$$

$$sen\,\alpha = \frac{|\delta_{VB2}|}{\delta_{B2}} \rightarrow \delta_{VB2} = -0'137\ m\ \downarrow$$

$$\boxed{\begin{aligned} \varphi_B &= \varphi_{B1} + \varphi_{B2} = 0'050 - 0'0274 = 0'0226\ rad\ ↺ \\ \delta_{HB} &= \delta_{HB1} + \delta_{HB2} = -0'004 + 0'0820 = 0'078\ m \rightarrow \\ \delta_{VB} &= \delta_{VB1} + \delta_{VB2} = 0'137 - 0'137 = 0 \end{aligned}}$$

APELLIDOS: NOMBRE: D.N.I.:

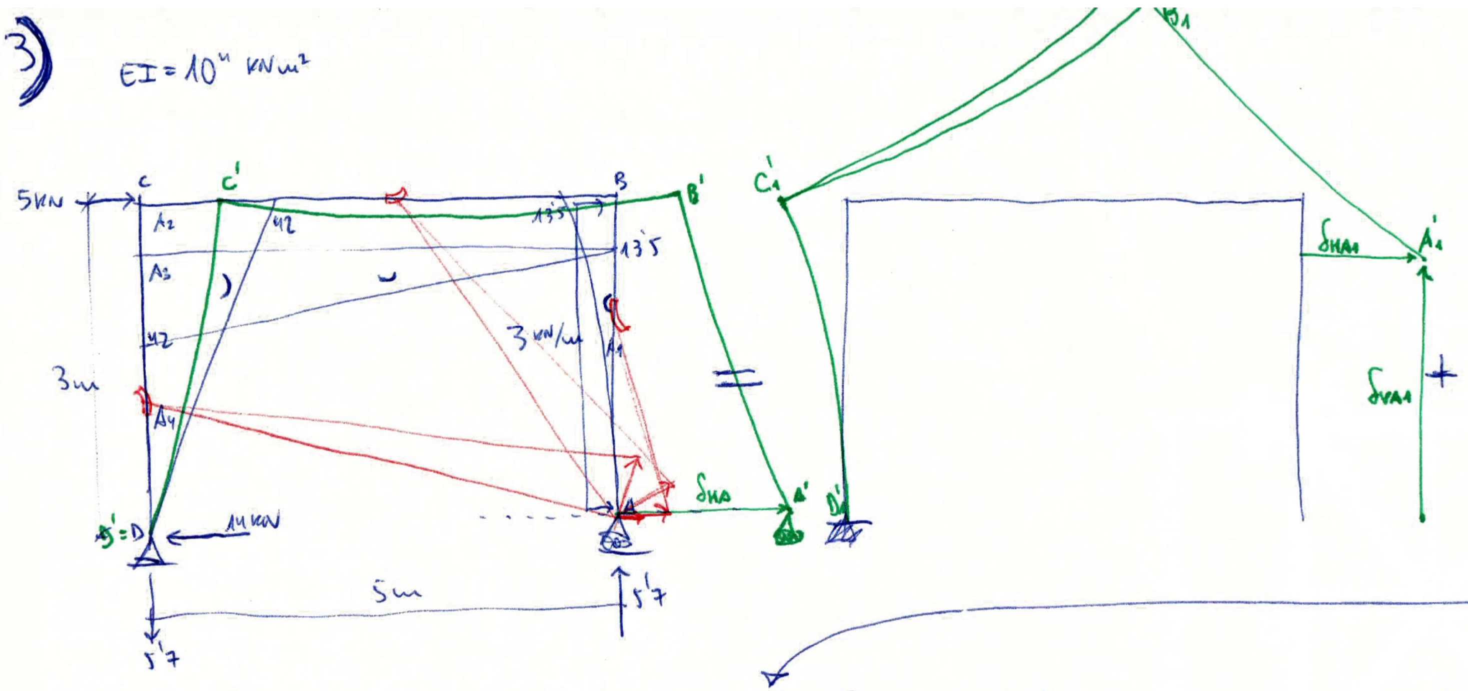
En la estructura indicada en la figura, dibujar la deformada aproximada y determinar el desplazamiento horizontal del nudo A:

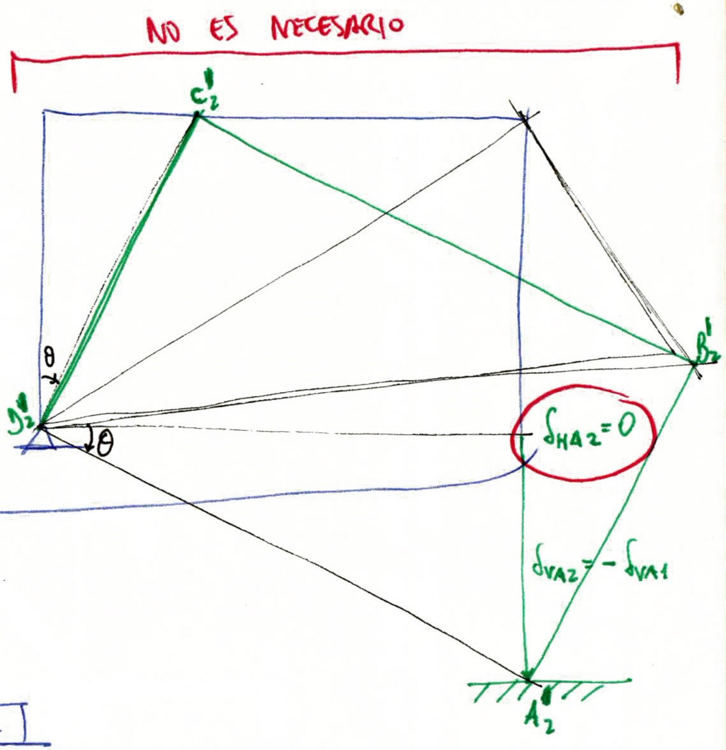


Barras: EI = 10^4 kNm^2

3)

$EI = 10^4 \ kNm^2$





	A [kNm²]	δ_{HA1}	h_{AH}
A_1	13'5	+	2'25
A_2	67'5	+	3
A_3	71'25	+	3
A_4	63	+	2

$$\delta_{HA} = \delta_{HA1} + \cancel{\delta_{HA2}} = \delta_{HA1} = \frac{1}{EI}\left(A_1 \cdot h_{AH1} + A_2 \cdot h_{AH2} + A_3 \cdot h_{AH3} + A_4 \cdot h_{AH4}\right) =$$

$$= 10^{-4}\left(13'5 \cdot 2'25 + 67'5 \cdot 3 + 71'25 \cdot 3 + 63' \cdot 2\right) = 0'057 \ m = \boxed{57 \ mm}$$

$$A_1 = \frac{1}{3} 3 \cdot 13'5 = 13'5$$

$$A_2 = 5 \cdot 13'5 = 67'5$$

$$A_3 = \frac{1}{2} 5 \cdot (42 - 13'5) = 71'25$$

$$A_4 = \frac{1}{2} 3 \cdot 42 = 63$$

$$h_{AH1} = \frac{3}{4} 3 = 2'25 \ m$$

$$h_{AH2} = 3 \ m$$

$$h_{AH3} = 3 \ m$$

$$h_{AH4} = \frac{2}{3} 3 = 2 \ m$$

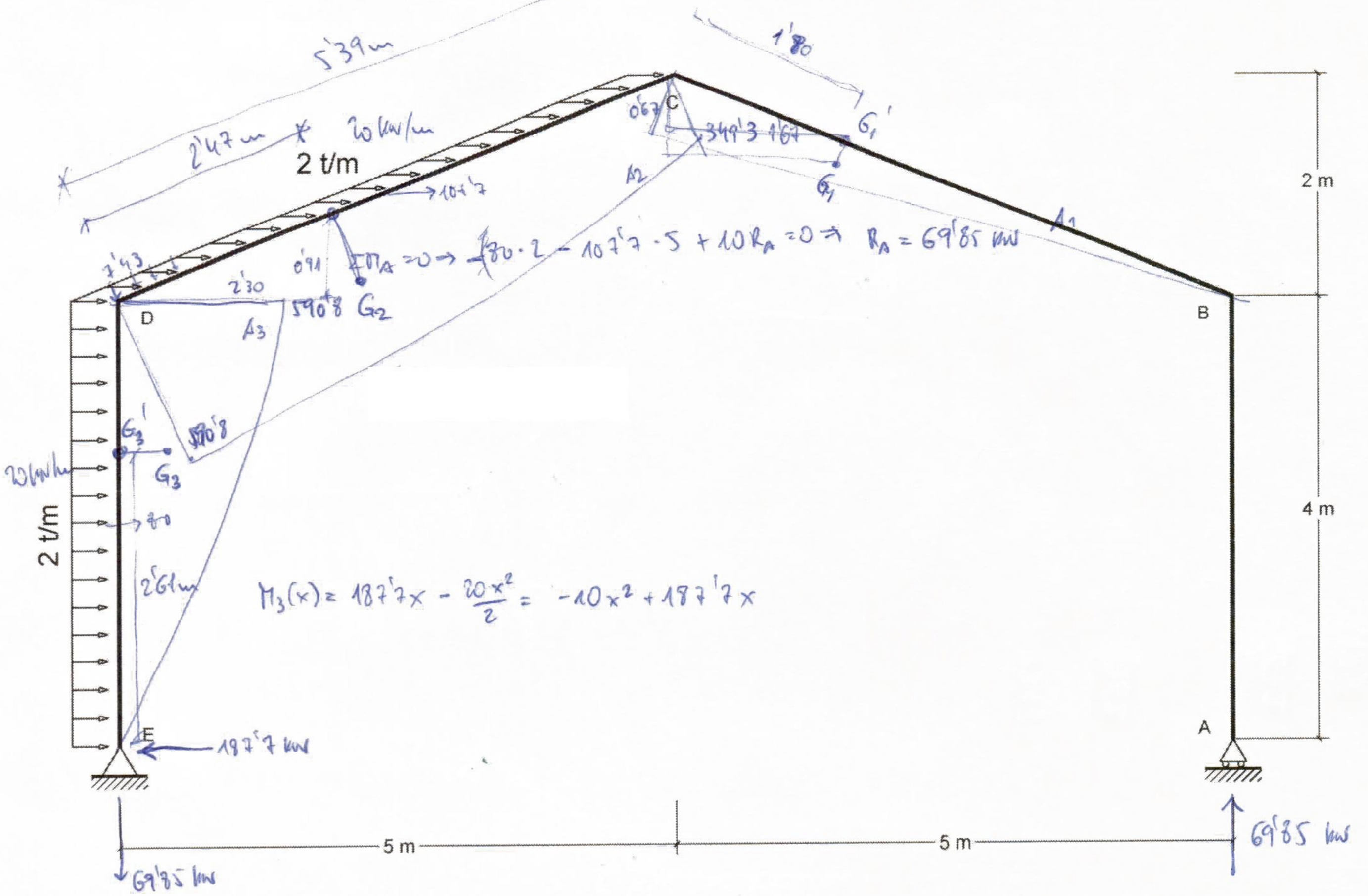
ETS de ARQUITECTURA. FUNDAMENTOS DE ESTRUCTURAS. CA 2019/20 ENERO Ej. nº 2

APELLIDOS: NOMBRE: D.N.I.:

En la estructura indicada en la figura, mediante aplicación de los teoremas de Mohr generalizados, determinar:

- **Reacciones en apoyos**
- **Ley de momentos flectores**
- **Deformada aproximada**
- **Desplazamientos y giros de los puntos A y C**

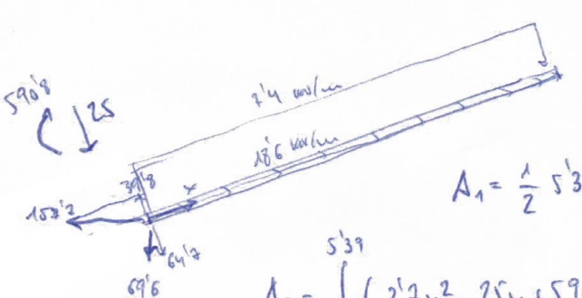
$\alpha = \operatorname{arctg} \frac{2}{5} \simeq 21'8^\circ \begin{cases} \cos\alpha = 0'93 \\ \operatorname{sen}\alpha = 0'37 \end{cases}$



$\Sigma M_A = 0 \rightarrow -\cancel{80} \cdot 2 - 107'7 \cdot 5 + 10 R_A = 0 \Rightarrow R_A = 69'85$ kN

$M_3(x) = 187'7x - \frac{20x^2}{2} = -10x^2 + 187'7x$

Barras: EI = 10^4 Tn m^2 = 10^5 kNm^2



$M_2(x) = 590'8 - 25x - \frac{7'4x^2}{2} = -3'7x^2 - 25x + 590'8$

$A_1 = \frac{1}{2} 5'39 \cdot 349'3 = 941'4$ kNm^2

$A_2 = \int_0^{5'39} (-3'7x^2 - 25x + 590'8)\,dx = \left[\frac{-3'7x^3}{3} - \frac{25x^2}{2} + 590'8x\right]_0^{5'39} = 2628$ kNm^2 ✓

$A_3 = \int_0^4 (-10x^2 + 187'7x)\,dx = \left[\frac{-10x^3}{3} + \frac{187'7x^2}{2}\right]_0^4 = 1288$ kNm^2 ✓

$x_{G2} = \frac{\int_0^{5'39} (-3'7x^3 - 25x^2 + 590'8x)}{2628} = \frac{\left[\frac{-3'7x^4}{4} - 25\frac{x^3}{3} + \frac{590'8x^2}{2}\right]_0^{5'39}}{2628} = 2'47$ m ✓

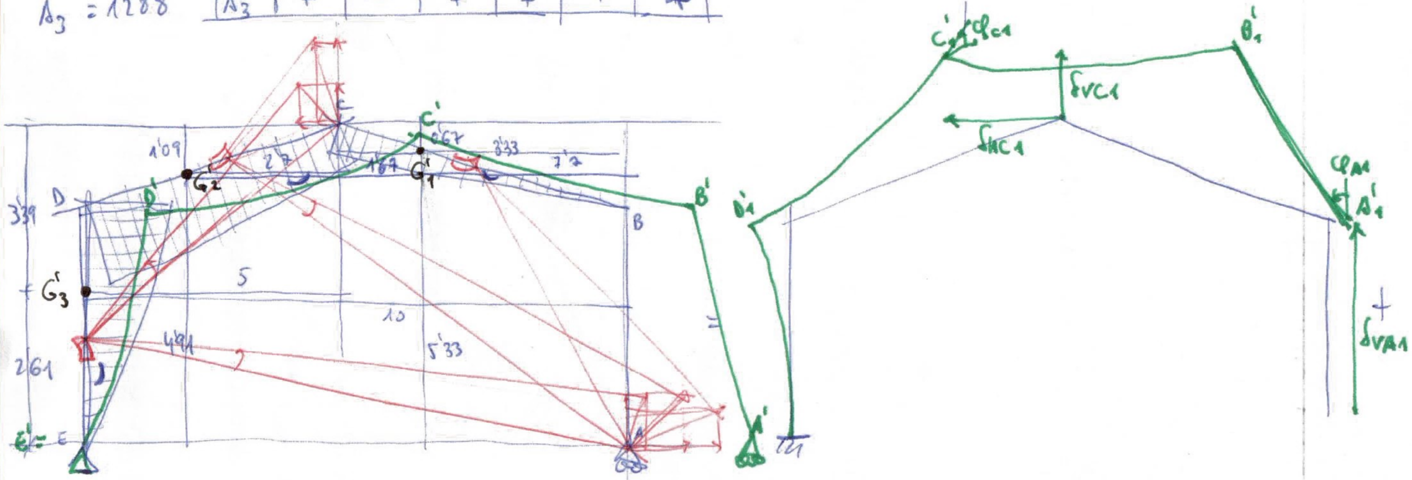
$x_{G3} = \frac{\int_0^4 (-10x^3 + 187'7x^2)\,dx}{1288} = \frac{\left[\frac{-10x^4}{4} + \frac{187'7x^3}{3}\right]_0^4}{1288} = 2'61$ m

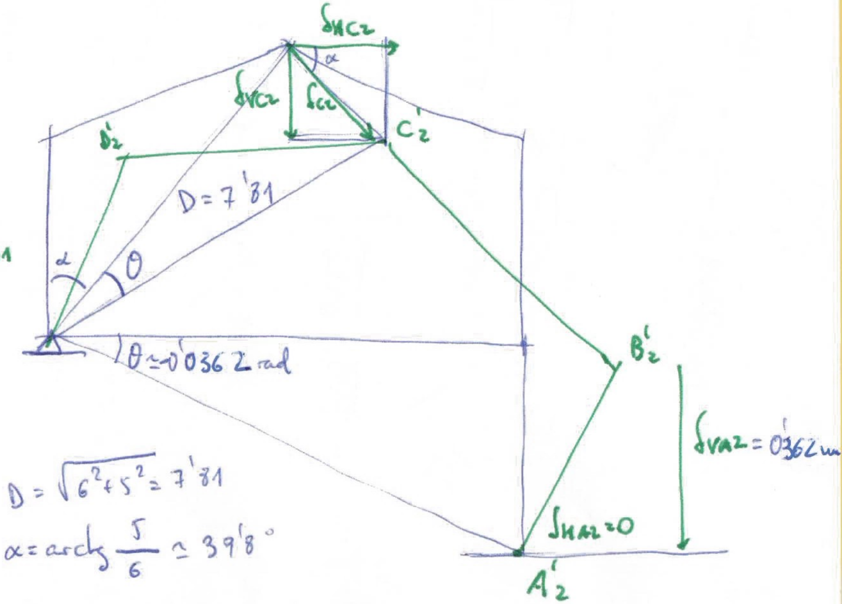
$A_1 = 941'4$

$A_2 = 2628$

$A_3 = 1288$

	C			A		
	φ	δ_H	δ_V	φ	δ_H	δ_V
A_1	~~0~~	~~0~~	~~0~~	+	+	+
A_2	+	–	+	+	+	+
A_3	+	–	+	+	+	+





$h_{1AH} = 5'33$

$h_{1AV} = 3'33$

$h_{2AH} = 4'91$

$h_{2AV} = 7'7$

$h_{3AH} = 2'61$

$h_{3AV} = 10$

$h_{1CH} = 0'67$

$h_{1CV} = 1'67$

$h_{2CH} = 1'09$

$h_{2CV} = 2'7$

$h_{3CH} = 3'39$

$h_{3CV} = 5$

$$\varphi_{A1} = \frac{1}{EI}(A_1 + A_2 + A_3) = 0'0486 \text{ rad}$$

$$\delta_{HA1} = \frac{1}{EI}(A_1 h_{1AH} + A_2 h_{2AH} + A_3 h_{3AH}) = 0'213 \text{ m}$$

$$\delta_{VA1} = \frac{1}{EI}(A_1 h_{1AV} + A_2 h_{2AV} + A_3 h_{3AV}) = 0'362 \text{ m}$$

$$\varphi_{C1} = \frac{1}{EI}(A_2 + A_3) = 0'0392 \text{ rad}$$

$$\delta_{HC1} = \frac{1}{EI}(-A_2 h_{2CH} - A_3 h_{3CH}) = -0'072 \text{ m}$$

$$\delta_{VC1} = \frac{1}{EI}(A_2 h_{2CV} + A_3 h_{3CV}) = 0'135 \text{ m}$$

$$D = \sqrt{6^2 + 5^2} = 7'81$$

$$\alpha = \arctan\frac{5}{6} \simeq 39'8^\circ$$

$$\delta_{C2} = 7'81 \cdot 0'0362 \simeq 0'283 \text{ m}$$

$$\delta_{HC2} = 0'217 \text{ m}$$

$$\delta_{VC2} = -0'181 \text{ m}$$

$$A \begin{cases} \varphi_A = \varphi_{A1} + \theta = 0'0124 \text{ rad} \\ \delta_{HA} = \delta_{HA1} + \delta_{HA2} = 0'213 \text{ m} \\ \delta_{VA} = \delta_{VA1} + \delta_{VA2} = 0 \text{ m} \end{cases}$$

$$B \begin{cases} \varphi_B = \varphi_{B1} + \theta = 0'0030 \text{ rad} \\ \delta_{HB} = \delta_{HB1} + \delta_{HB2} = 0'145 \text{ m} \\ \delta_{VB} = \delta_{VB1} + \delta_{VB2} = -0'046 \text{ m} \end{cases}$$