Periodica Mathematica Hungarica
https://doi.org/10.1007/s10998-022-00508-z

On two tensors associated to the structure Jacobi operator of a real hypersurface in complex projective space

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Accepted: 24 April 2022

Abstract
A real hypersurface M in a complex projective space inherits an almost contact metric structure from the Kählerian structure of the ambient space. This almost contact metric structure allows us to define, for any nonzero real number k, the so-called k-th generalized Tanaka–Webster connection. With this connection and the Levi-Civita one we can associate two tensors of type (1,2) to the structure Jacobi operator R_ξ of M. We classify real hypersurfaces in complex projective space for which such tensors satisfy a cyclic property.

Keywords k-th generalized Tanaka–Webster connection · Complex projective space · Real hypersurface · Lie derivative · Jacobi structure operator

Mathematics Subject Classification 53C15 · 53B25

1 Introduction

Let $(\mathbb{C}P^m, J, g)$, $m \geq 2$, be the complex projective space with the Kählerian structure (J, g), where J denotes its complex structure and g is the Fubini–Study metric of constant holomorphic sectional curvature 4. Suppose that M is a connected real hypersurface in $\mathbb{C}P^m$ without boundary and denote by N a unit local normal vector field on M. Let ∇ be the Levi-Civita connection on M and A the shape operator of M associated to N.

Let X be a vector field tangent to M. We write $JX = \phi X + \eta(X)N$, where ϕX is the tangential component of JX and $\eta(X) = g(X, \xi)$, $\xi = -JN$ being the structure (or Reeb) vector field on M. Then (ϕ, ξ, η, g) defines an almost contact metric structure on M. Using this almost contact metric structure Cho [1], defined for any nonzero real number k, the

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Published online: 29 December 2022

k-generalized Tanaka–Webster connection on M, $\hat{\nabla}^{(k)}$, by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$$

for any X, Y tangent to M. This generalizes the definition of Tanno's generalized Tanaka–Webster connection on a contact manifold, see [2–4].

The k-th Cho operator associated to the vector field X tangent to M will be given by

$$F^{(k)}_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$$

for any Y tangent to M. Then, if L denotes a tensor field of type (1,1) on M, for any nonzero real number k we can define a tensor field of type (1,2) on M, $L^{(k)}_F$, by

$$L^{(k)}_F(X, Y) = [F^{(k)}_X, L]Y = F^{(k)}_X LY - LF^{(k)}_X Y$$

for any X, Y tangent to M.

The torsion of the k-generalized Tanaka–Webster connection is given by $T^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$, for any X, Y tangent to M. We consider the k-th torsion operator associated to X, $T^{(k)}_X Y = T^{(k)}(X, Y)$. Then, if L is as above, we can define for any nonzero real number k, another tensor field of type (1,2), $L^{(k)}_T$, given by

$$L^{(k)}_T(X, Y) = [T^{(k)}_X, L]Y = T^{(k)}_X LY - LT^{(k)}_X Y$$

for any X, Y tangent to M.

We will say that a real hypersurface M in $\mathbb{C}P^m$ is Hopf if the structure vector field ξ is a principal vector of the shape operator, i.e. $A\xi = \alpha\xi$, for a certain function α on M. The maximal holomorphic distribution on M, $\mathbb{D}$, is defined by $\mathbb{D} = \mathrm{Ker}(\eta)$.

The best known examples of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 2$, are the homogeneous ones which were classified by Takagi into six types, see [5–7]. All these types are Hopf and Kimura, [8], proved that they are the unique Hopf real hypersurfaces that have constant principal curvatures in $\mathbb{C}P^m$. Such real hypersurfaces are:

- Type (A_1), geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$. They have 2 distinct constant principal curvatures, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$ and $\cot(r)$ with eigenspace $\mathbb{D}$.
- Type (A_2), tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m-1$. They have 3 distinct constant principal curvatures, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot(r)$ and $-\tan(r)$. The corresponding eigenspaces of $\cot(r)$ and $-\tan(r)$ are complementary and ϕ-invariant distributions in $\mathbb{D}$.
- Type (B), tubes of radius r, $0 < r < \frac{\pi}{4}$, over the complex quadric Q^{m-1}. They have 3 distinct constant principal curvatures, 2 $\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot(r - \frac{\pi}{4})$ and $-\tan(r - \frac{\pi}{4})$ whose corresponding eigenspaces are complementary and equal dimensional distributions in $\mathbb{D}$ such that $\phi V_{\cot(r-\frac{\pi}{4})} = V_{-\tan(r-\frac{\pi}{4})}$.
- Type (C), tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^n$, where $2n+1 = m$ and $m \geq 5$. They have 5 distinct constant principal curvatures, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot(r - \frac{\pi}{4})$ with multiplicity 2, $\cot(r - \frac{\pi}{2}) = -\tan(r)$ with multiplicity $m-3$, $\cot(r - \frac{3\pi}{4})$, with multiplicity 2 and $\cot(r - \pi) = \cot(r)$ with multiplicity $m-3$. Moreover $\phi V_{\cot(r-\frac{\pi}{4})} = V_{\cot(r-\frac{3\pi}{4})}$ and $V_{-\tan(r)}$ and $V_{\cot(r)}$ are ϕ-invariant.
- Type (D), tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Plücker embedding of the complex Grassmannian manifold $G(2, 5)$ in $\mathbb{C}P^9$. They have the same principal curvatures as type (C) real hypersurfaces, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, and the other 4 principal

curvatures have the same multiplicity 4 and their eigenspaces have the same behaviour with respect to ϕ as in type (C).

- Type (E), tubes of radius r, $0 < r < \frac{\pi}{4}$, over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^{15}$. They also have the same principal curvatures as type (C) real hypersurfaces, $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $\cot(r-\frac{\pi}{4})$ and $\cot(r-\frac{3\pi}{4})$ have multiplicities equal to 6 and $-\tan(r)$ and $\cot(r)$ have multiplicities equal to 8. Their corresponding eigenspaces have the same behaviour with respect to ϕ as in type (C).

We will call type (A) real hypersurfaces to both types (A_1) or (A_2).

The structure Jacobi operator R_ξ on M is defined as $R_\xi X = R(X,\xi)\xi$, where R is the Riemannian curvature tensor on M. It is given by

$$R_\xi X = X - \eta(X)\xi + g(A\xi,\xi)AX - g(A\xi,X)A\xi$$

for any X tangent to M.

In this paper we will consider the tensors $R^{(k)}_{\xi_F}$ and $R^{(k)}_{\xi_T}$ associated to R_ξ. The first one is related to the difference of the connections $\hat{\nabla}^{(k)} - \nabla$. In fact, $R^{(k)}_{\xi_F}(X,Y) = ((\hat{\nabla}^{(k)}_X - \nabla_X)R_\xi)Y$, for any X, Y tangent to M. In [9], first author proved non-existence of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $R^{(k)}_{\xi_F}$ identically vanishes. Now we generalize such a condition classifying real hypersurfaces in $\mathbb{C}P^m$ whose tensor $R^{(k)}_{\xi_F}$ satisfies the cyclic condition

$$g(R^{(k)}_{\xi_F}(X,Y),Z) + g(R^{(k)}_{\xi_F}(Y,Z),X) + g(R^{(k)}_{\xi_F}(Z,X),Y) = 0 \tag{1.1}$$

for any X, Y, Z tangent to M by the following

Theorem 1.1 *Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geq 3$, and k a nonzero real number such that $kg(A\xi,\xi) \neq 1$. Then M satisfies (1.1) if and only if M is locally congruent to a real hypersurface of type (A).*

If $\mathcal{L}$ denotes the Lie derivative on M, for any nonzero real number k, from the k-th generalized Tanaka–Webster connection we can define a differential operator of first order, called the derivative of Lie type $\mathcal{L}^{(k)}$, by

$$\mathcal{L}^{(k)}_X Y = \hat{\nabla}^{(k)}_X Y - \hat{\nabla}^{(k)}_Y X = \mathcal{L}_X Y + T^{(k)}_X Y \tag{1.2}$$

for any X, Y tangent to M. Then $R^{(k)}_{\xi_T}(X,Y) = ((\mathcal{L}^{(k)}_X - \mathcal{L}_X)R_\xi)Y$, for any X, Y tangent to M. In [10], the non-existence of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $R^{(k)}_{\xi_T}$ vanishes identically is also proved (see also [11]). We will generalize such a result studying real hypersurfaces M in $\mathbb{C}P^m$ whose tensor $R^{(k)}_{\xi_T}$ satisfies the cyclic condition

$$g(R^{(k)}_{\xi_T}(X,Y),Z) + g(R^{(k)}_{\xi_T}(Y,Z),X) + g(R^{(k)}_{\xi_T}(Z,X),Y) = 0 \tag{1.3}$$

for any X, Y, Z tangent to M.

Theorem 1.2 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonzero real number such that $kg(A\xi,\xi) \neq 1$. Then $R^{(k)}_{\xi_T}$ satifies (1.3) if and only if either*

1. *$k < 0$ and M is locally congruent to a geodesic hypersphere of radius r, $0 < r < \frac{\pi}{2}$, with $\cot(r) = -k$, or*
2. *$k = -\frac{(1+\lambda^2)(2+\alpha\lambda)\lambda+\alpha}{\lambda(\lambda-\alpha)}$, with $\alpha = 2\cot(2r)$, $\lambda = \cot(r-\frac{\pi}{4})$, and M is locally congruent to a real hypersurface of type (B) and radius $0 < r < \frac{\pi}{4}$.*

2 Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi-Civita connection on M and (J, g) the Kählerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M (see [12]). That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any tangent vectors X, Y to M. From (2.1) we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \;\text{ and }\; \nabla_X \xi = \phi AX \tag{2.3}$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ -2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.4}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \tag{2.5}$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M.

In the sequel we need the following results.

Theorem 2.1 (Maeda [13]) *Let M be a Hopf real hypersurface in $\mathbb{C}P^m$, $m \geq 2$. Then $\alpha = g(A\xi, \xi)$ is constant and if W is a vector field which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then $2\lambda - \alpha \neq 0$ and $A\phi W = \mu\phi W$, where $\mu = \frac{\alpha\lambda + 2}{2\lambda - \alpha}$.*

Theorem 2.2 ([14]) *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 2$. Then the following facts are equivalent:*

1. *$\phi A = A\phi$.*
2. *M is locally congruent to a real hypersurface of type (A).*

The following non-existence Theorems will also be applied.

Theorem 2.3 ([15]) *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, whose shape operator satisfies $A\xi = \alpha\xi + U$, $AU = \xi$, $A\phi U = -\frac{1}{\alpha}\phi U$, where U is a unit vector field in $\mathbb{D}$ and α a nonzero function defined on M.*

Theorem 2.4 ([15]) *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, whose shape operator is given by $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi + \frac{\beta^2 - 1}{\alpha}U$, $A\phi U = -\frac{1}{\alpha}\phi U$, the eigenvalues of A in $\mathbb{D}_U = \mathrm{Span}\{\xi, U, \phi U\}^\perp$ are different from 0, $-\frac{1}{\alpha}$ and $\frac{\beta^2 - 1}{\alpha}$ and if $Z \in \mathbb{D}_U$ satisfies $AZ = \lambda Z$, then $A\phi Z = \lambda\phi Z$, where U and α are as in Theorem 2.3 and β is a nonzero function defined on M.*

Theorem 2.5 *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, whose shape operator is given by $A\xi = \xi + \beta U$, $AU = \beta\xi + (\beta^2 - 1)U$, $A\phi U = -\phi U$, and there exists $Z \in \mathbb{D}_U$ such that $AZ = -Z$, $A\phi Z = -\phi Z$, where U, $\mathbb{D}_U$ and β are as in Theorem 2.4.*

3 Proof of Theorem 1.1

Let M be a real hypersurface in $\mathbb{C}P^m$ satisfying (1.1). This yields

$$\begin{aligned} &g(\phi AX, R_\xi Y)\eta(Z) - k\eta(X)g(\phi R_\xi Y, Z) + \eta(Y)g(R_\xi\phi AX, Z) + k\eta(X)g(R_\xi\phi Y, Z) \\ &\quad + g(\phi AY, R_\xi Z)\eta(X) - k\eta(Y)g(\phi R_\xi Z, X) + \eta(Z)g(R_\xi\phi AY, X) + k\eta(Y)g(R_\xi\phi Z, X) \\ &\quad + g(\phi AZ, R_\xi X)\eta(Y) - k\eta(Z)g(\phi R_\xi X, Y) + \eta(X)g(R_\xi\phi AZ, Y) \\ &\quad + k\eta(Z)g(R_\xi\phi X, Y) = 0 \end{aligned} \tag{3.1}$$

for any X, Y, Z tangent to M.

Suppose that M is Hopf, that is, $A\xi = \alpha\xi$, and take $X = \xi$, $Y, Z \in \mathbb{D}$ in (3.1). We obtain

$$-kg(\phi R_\xi Y, Z) + kg(R_\xi\phi Y, Z) + g(R_\xi\phi AY, Z) - g(A\phi R_\xi Y, Z) = 0$$

for any $Y, Z \in \mathbb{D}$. Thus for any $Y \in \mathbb{D}$ we have $-k\phi R_\xi Y + kR_\xi\phi Y + R_\xi\phi AY - A\phi R_\xi Y = 0$. If $Y \in \mathbb{D}$ satisfies $AY = \lambda Y$, from Theorem 2.1 we have $A\phi Y = \mu\phi Y$, $R_\xi Y = (1 + \alpha\lambda)Y$ and $R_\xi\phi Y = (1 + \alpha\mu)\phi Y$, where $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Therefore, we get $-k(1 + \alpha\lambda) + k(1 + \alpha\mu) + \lambda(1 + \alpha\mu) - \mu(1 + \alpha\lambda) = 0$, that is, $(\lambda - \mu)(1 - k\alpha) = 0$. As we suppose $k\alpha \neq 1$, we arrive to $\lambda = \mu$. This yields $\phi A = A\phi$ and by Theorem 2.2 M is locally congruent to a real hypersurface of type (A).

Now we suppose that M is non-Hopf and we write $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathbb{D}$ and α and β are functions on M, and β does not vanish at least on a neighborhood of a point $p \in M$. We will make the calculations on such a neighborhood.

Take $X = Y = \xi$, $Z \in \mathbb{D}$ in (3.1). It follows $g(R_\xi\phi A\xi, Z) = 0$, that is, $\beta g(R_\xi\phi U, Z) = 0$, for any $Z \in \mathbb{D}$. As $\beta \neq 0$ this yields $R_\xi\phi U = 0$. Therefore, $\alpha \neq 0$ and

$$A\phi U = -\tfrac{1}{\alpha}\phi U. \tag{3.2}$$

Putting $X = \xi$, $Y, Z \in \mathbb{D}$ in (3.1) we obtain

$$-kg(\phi R_\xi Y, Z) + kg(R_\xi\phi Y, Z) + g(R_\xi\phi AY, Z) - g(A\phi R_\xi Y, Z) = 0 \tag{3.3}$$

for any $Y, Z \in \mathbb{D}$. Then $-k\phi R_\xi Y + kR_\xi\phi Y + R_\xi\phi AY - A\phi R_\xi Y = -g(A\phi R_\xi Y, \xi)\xi = \beta g(R_\xi Y, \phi U)\xi = 0$, for any $Y \in \mathbb{D}$. If we take $Y = \phi U$ we get $-k\phi R_\xi\phi U - kR_\xi U + R_\xi\phi A\phi U - A\phi R_\xi\phi U = 0$, and this yields $(\frac{1}{\alpha} - k)R_\xi U = 0$. As we suppose $k\alpha \neq 1$, we obtain $R_\xi U = 0$. That is,

$$AU = \beta\xi + \tfrac{\beta^2-1}{\alpha}U. \tag{3.4}$$

From (3.2) and (3.4) we have that $\mathrm{Span}\{\xi, U, \phi U\}$ is A-invariant. Therefore, its orthogonal complementary $\mathbb{D}_U = \{X \in \mathbb{D} \mid g(X, U) = g(X, \phi U) = 0\}$ is also A-invariant. Take $Y \in \mathbb{D}_U$ such that $AY = \lambda Y$. Then from (3.3) we obtain $-k\phi R_\xi Y + (k+\lambda)R_\xi\phi Y - A\phi R_\xi Y = 0$. As $R_\xi Y = (1+\alpha\lambda)Y$ and $R_\xi\phi Y = \phi Y + \alpha A\phi Y$ it yields $-k\alpha\lambda\phi Y + \lambda\phi Y + \alpha(k+\lambda)A\phi Y - (1+\alpha\lambda)A\phi Y = 0$, that is, $\lambda(1-\alpha k)\phi Y + (\alpha k - 1)A\phi Y = 0$. As $\alpha k \neq 1$ we get $A\phi Y = \lambda\phi Y$ and then the eigenspaces in $\mathbb{D}_U$ are ϕ-invariant.

Take now $Y \in \mathbb{D}_U$ such that $AY = \lambda Y$, $A\phi Y = \lambda\phi Y$. The Codazzi equation $(\nabla_Y A)\phi Y - (\nabla_{\phi Y} A)Y = -2\xi$ implies $\nabla_Y(\lambda\phi Y) - A\nabla_Y\phi Y - \nabla_{\phi Y}(\lambda Y) + A\nabla_{\phi Y} Y = -2\xi$, that is,

$Y(\lambda)\phi Y+\lambda\nabla_Y\phi Y-A\nabla_Y\phi Y-(\phi Y)(\lambda)Y-\lambda\nabla_{\phi Y}Y+A\nabla_{\phi Y}Y=-2\xi$. Its scalar product with ξ gives $-\lambda g(\phi Y,\phi AY)-g(\nabla_Y\phi Y,\alpha\xi+\beta U)+\lambda g(Y,\phi A\phi Y)+g(\nabla_{\phi Y}Y,\alpha\xi+\beta U)=-2$. Thus

$$\beta g([\phi Y,Y],U)=2\lambda^2-2\alpha\lambda-2 \tag{3.5}$$

and its scalar product with U implies $\lambda g(\nabla_Y\phi Y,U)-g(\nabla_Y\phi Y,\beta\xi+\frac{\beta^2-1}{\alpha}U)-\lambda g(\nabla_{\phi Y}Y,U)+g(\nabla_{\phi Y}Y,\beta\xi+\frac{\beta^2-1}{\alpha}U)=0$. Then $(\frac{\beta^2-1}{\alpha}-\lambda)g([\phi Y,Y],U)+\beta g(\phi Y,\phi AY)-\beta g(Y,\phi A\phi Y)=0$ and

$$\left(\frac{\beta^2-1}{\alpha}-\lambda\right)g([\phi Y,Y],U)=-2\lambda\beta. \tag{3.6}$$

If we suppose that $\lambda=\frac{\beta^2-1}{\alpha}$, from (3.6) we have $2\lambda\beta=0$, and as $\beta\neq0$, this yields $\lambda=0$ and $\beta^2=1$. Maybe after changing ξ by $-\xi$ we can suppose $\beta=1$. Therefore $A\xi=\alpha\xi+U$, $AU=\beta\xi$ and by Theorem 2.3 this kind of real hypersurfaces does not exist. Thus we have proved that $\lambda\neq\frac{\beta^2-1}{\alpha}$.

If $\lambda=0$, (3.5) and (3.6) become, respectively, $\beta g([\phi Y,Y],U)=-2$ and $\frac{\beta^2-1}{\alpha}g([\phi Y,Y],U)=0$. As we suppose $0=\lambda\neq\frac{\beta^2-1}{\alpha}$, we arrive to a contradiction.

Suppose then that $\lambda=-\frac{1}{\alpha}$. Then (3.5) gives $\beta g([\phi Y,Y],U)=\frac{2}{\alpha^2}$ and (3.6) implies $g([\phi Y,Y],U)=\frac{2}{\beta}$. Thus $\beta=\beta\alpha^2$ yields $\alpha^2=1$. Maybe after changing ξ by $-\xi$ we can suppose $\alpha=1$, $A\xi=\xi+\beta U$, $AU=\beta\xi+(\beta^2-1)U$ and $A\phi U=-\phi U$. Moreover, there exists $Z\in\mathbb{D}_U$ such that $AZ=-Z$ and $A\phi Z=-\phi Z$. From Theorem 2.5 such real hypersurfaces do not exist and the Theorem follows from Theorem 2.4.

4 Proof of Theorem 1.2

If M satisfies (1.3) we have

$$\begin{aligned}
&g(\phi AX,R_\xi Y)\eta(Z)-k\eta(X)g(\phi R_\xi Y,Z)-\eta(Z)g(\phi AR_\xi Y,X)+\eta(X)g(\phi AR_\xi Y,Z)\\
&\quad+\eta(Y)g(R_\xi\phi AX,Z)+k\eta(X)g(R_\xi\phi Y,Z)-\eta(X)g(R_\xi\phi AY,Z)-k\eta(Y)g(R_\xi\phi X,Z)\\
&\quad+g(\phi AY,R_\xi Z)\eta(X)-k\eta(Y)g(\phi R_\xi Z,X)-\eta(X)g(\phi AR_\xi Z,Y)+\eta(Y)g(\phi AR_\xi Z,X)\\
&\quad+\eta(Z)g(R_\xi\phi AY,X)+k\eta(Y)g(R_\xi\phi Z,X)-\eta(Y)g(R_\xi\phi AZ,X)-k\eta(Z)g(R_\xi\phi Y,X)\\
&\quad+g(\phi AZ,R_\xi X)\eta(Y)-k\eta(Z)g(\phi R_\xi X,Y)-\eta(Y)g(\phi AR_\xi X,Z)+\eta(Z)g(\phi AR_\xi X,Y)\\
&\quad+\eta(X)g(R_\xi\phi AZ,Y)+k\eta(Z)g(R_\xi\phi X,Y)-\eta(Z)g(R_\xi\phi AX,Y)-k\eta(X)g(R_\xi\phi Z,Y)=0
\end{aligned} \tag{4.1}$$

for any X,Y,Z tangent to M.

Let us suppose that M is Hopf, that is, $A\xi=\alpha\xi$. Take $X=\xi$, $Y,Z\in\mathbb{D}$ in (4.1). We obtain $-kg(\phi R_\xi Y,Z)+g(\phi AR_\xi Y,Z)+kg(R_\xi\phi Y,Z)-g(R_\xi\phi AY,Z)+g(\phi AY,R_\xi Z)-g(\phi AR_\xi Z,Y)+g(R_\xi\phi AZ,Y)-kg(R_\xi\phi Z,Y)=0$ for any $Y,Z\in\mathbb{D}$. This yields $\phi AR_\xi Y+kR_\xi\phi Y+R_\xi A\phi Y-A\phi R_\xi Y=0$, for any $Y\in\mathbb{D}$. Take $Y\in\mathbb{D}$ satisfying $AY=\lambda Y$. Then $A\phi Y=\mu\phi Y$, $R_\xi Y=(1+\alpha\lambda)Y$ and $R_\xi\phi Y=(1+\alpha\mu)\phi Y$. Last expression implies $(1+\alpha\lambda)\phi AY+k(1+\alpha\mu)\phi Y+\mu(1+\alpha\mu)\phi Y-(1+\alpha\lambda)A\phi Y=0$. Then

$$(1+\alpha\lambda)(\lambda-\mu)+(1+\alpha\mu)(k+\mu)=0. \tag{4.2}$$

As $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, (4.2) becomes

$$2(1+\alpha\lambda)(2\lambda-\alpha)(\lambda^2-\alpha\lambda-1)+(\alpha^2\lambda+2\lambda+\alpha)(2k\lambda-k\alpha+\alpha\lambda+2)=0. \quad (4.3)$$

From (4.3) all the principal curvatures in $\mathbb{D}$ are constant and, at most, there are four distinct. Therefore M can be locally congruent to any real hypersurface in Takagi's list.

If M is a geodesic hypersphere, $\lambda = \mu$ and from (4.2) either $\lambda = -k$ or $\lambda = -\frac{1}{\alpha}$. In the first case $\cot(r) = -k$, $0 < r < \frac{\pi}{2}$, and therefore $k < 0$. In the second case, as $\lambda = \mu$, $-\frac{1}{\alpha} = \frac{1}{-\frac{2}{\alpha}-\alpha}$, that is, $\frac{1}{\alpha} = \frac{\alpha}{\alpha^2+2}$, which implies $2 = 0$, that is impossible.

If M is of type (A_2), then either $\cot(r) = -k$ and $-\tan(r) = -\frac{1}{\alpha}$ or $\cot(r) = -\frac{1}{\alpha}$ and $-\tan(r) = -k$. In both cases $-1 = \frac{k}{\alpha}$, that is, $\alpha = -k$. As $\alpha = 2\cot(2r) = \cot(r) - \tan(r) = -k$, and in the first case we obtain $-k = -k + \frac{1}{\alpha}$, which is impossible. Thus $-k = -\frac{1}{\alpha} + k$, that is, $-k = \alpha = \frac{1}{2k}$, also impossible.

If M is of type (B), the principal curvatures in $\mathbb{D}$ are $\lambda = \cot(r - \frac{\pi}{4})$ and $\mu = -\tan(r - \frac{\pi}{4})$, $0 < r < \frac{\pi}{4}$. If in (4.2) we suppose $\mu = -\frac{1}{\alpha}$ we get $(1+\alpha\lambda)(\lambda-\mu) = 0$ and as $\lambda \neq \mu$, we obtain $\lambda = -\frac{1}{\alpha}$, a contradiction. Thus $\mu \neq -\frac{1}{\alpha}$ and k looks like item 2 of Theorem 1.2.

If M is either of type (C) or (D) or (E), we have that in $\mathbb{D}$, $\lambda_1 = -\tan(r)$ and $\lambda_2 = \cot(r)$ are two of the distinct principal curvatures and their corresponding eigenspaces are ϕ-invariant. That means that $\mu_1 = \lambda_1$ and $\mu_2 = \lambda_2$. Then (4.2) for $\lambda_1 = \tan(r)$ yields $(1 - \alpha\tan(r))(k - \tan(r)) = 0$. If $\alpha\tan(r) = 1$, as $\alpha = 2\cot(2r) = \cot(r) - \tan(r)$, we should have $1 - tan^2(r) = 1$. This yields $\tan(r) = 0$, which is impossible because $0 < r < \frac{\pi}{4}$. Therefore $\tan(r) = k$.

For $\lambda_2 = \cot(r)$, (4.2) implies $(1+\alpha\cot(r))(k+\cot(r)) = 0$. If $1 + \alpha\cot(r) = 0$, $\alpha\cot(r) = -1$. As above this gives $cot^2(r) - 1 = -1$, that is, $\cot(r) = 0$ for $0 < r < \frac{\pi}{4}$, which is impossible. Then $k = -\cot(r)$, that contradicts the fact of $k = \tan(r)$ and finishes the proof in this case.

If M is non-Hopf, we write $A\xi = \alpha\xi + \beta U$, as in the proof of Theorem 1.1. If we take $X = Y = \xi$, $Z \in \mathbb{D}$ in (4.1) we obtain $g(\phi A\xi, R_\xi Z) = \beta g(R_\xi\phi U, Z) = 0$ for any $Z \in \mathbb{D}$. As we suppose $\beta \neq 0$, we get $R_\xi\phi U = 0$. This yields $\alpha \neq 0$ and

$$A\phi U = -\frac{1}{\alpha}\phi U. \quad (4.4)$$

Now, if we take $X = \xi$, $Y, Z \in \mathbb{D}$ in (4.1) we obtain $g(\phi AR_\xi Y, Z) + kg(R_\xi\phi Y, Z) - g(\phi AR_\xi Z, Y) + g(R_\xi\phi AZ, Y) = 0$ for any $Y, Z \in \mathbb{D}$. This yields

$$-R_\xi A\phi Z - k\phi R_\xi Z - \phi AR_\xi Z + R_\xi\phi AZ = 0 \quad (4.5)$$

for any $Z \in \mathbb{D}$. If we take $Z = \phi U$ in (4.5) we have $R_\xi AU + R_\xi\phi A\phi U = 0$ and from (4.4) we get

$$R_\xi AU + \frac{1}{\alpha}R_\xi U = 0. \quad (4.6)$$

But from the expression used to obtain (4.5) we also obtain

$$\phi AR_\xi Y + kR_\xi\phi Y + R_\xi A\phi Y - A\phi R_\xi Y = 0 \quad (4.7)$$

for any $Y \in \mathbb{D}$. If we take $Y = \phi U$ in (4.7) we have

$$-kR_\xi U - R_\xi AU = 0. \quad (4.8)$$

From (4.6) and (4.8) it follows $(\frac{1}{\alpha} - k)R_\xi U = 0$. As we suppose $k\alpha \neq 1$ we have $R_\xi U = 0$ and this yields

$$AU = \beta\xi + \frac{\beta^2-1}{\alpha}U. \quad (4.9)$$

Thus, as in the proof of Theorem 1.1, $\mathbb{D}_U$ is A-invariant. Take a unit $Y \in \mathbb{D}_U$ such that $AY = \lambda Y$. If we introduce such a Y in (4.7) and take $Z = Y$ in (4.5) we obtain

$$k(R_\xi \phi Y - \phi R_\xi Y) + R_\xi \phi AY - A\phi R_\xi Y = 0. \tag{4.10}$$

Now $R_\xi Y = (1+\alpha\lambda)Y$ and $R_\xi \phi Y = \phi Y + \alpha A\phi Y$. From (4.10) we have $k(\phi Y + \alpha A\phi Y - (1+\alpha\lambda)\phi Y) + \lambda R_\xi \phi Y - (1+\alpha\lambda)A\phi Y = 0$, which yields $\lambda(1-k\alpha)\phi Y + (k\alpha - 1)A\phi Y = 0$. As we suppose $k\alpha \neq 1$, we obtain $A\phi Y = \lambda\phi Y$ and any eigenspace in $\mathbb{D}_U$ is ϕ-invariant.

From (4.5) if $Z \in \mathbb{D}_U$ satisfies $AZ = \lambda Z$, $A\phi Z = \lambda\phi Z$, we get $\phi AR_\xi Z = -k\phi R_\xi Z$, that is, $(1+\alpha\lambda)\phi AZ = -k(1+\alpha\lambda)\phi Z$. Therefore, either $\lambda = -\frac{1}{\alpha}$ or $\lambda = -k$.

Let us suppose that $-\frac{1}{\alpha}$ is a principal curvature in $\mathbb{D}_U$ and take $Y \in \mathbb{D}_U$ such that $AY = -\frac{1}{\alpha}Y$, $A\phi Y = -\frac{1}{\alpha}\phi Y$. The Codazzi equation $(\nabla_Y A)\phi Y - (\nabla_{\phi Y} A)Y = -2\xi$ yields $-Y(\frac{1}{\alpha})\phi Y - \frac{1}{\alpha}\nabla_Y \phi Y - A\nabla_Y \phi Y + (\phi Y)(\frac{1}{\alpha})Y + \frac{1}{\alpha}\nabla_{\phi Y} Y + A\nabla_{\phi Y} Y = -2\xi$. Its scalar product with ξ gives $\frac{1}{\alpha}g(\phi Y, \phi AY) - g(\nabla_Y \phi Y, \alpha\xi + \beta U) - \frac{1}{\alpha}g(Y, \phi A\phi Y) - g(\nabla_{\phi Y} Y, \alpha\xi + \beta U) = -2$. That is, $-\frac{2}{\alpha^2} - 2 + \beta g([\phi Y, Y], U) = -2$. Therefore

$$g([\phi Y, Y], U) = \frac{2}{\alpha^2\beta}. \tag{4.11}$$

Its scalar product with U yields $-\frac{1}{\alpha}g(\nabla_Y \phi Y, U) - g(\nabla_Y \phi Y, \beta\xi + \frac{\beta^2-1}{\alpha}U) + \frac{1}{\alpha}g(\nabla_{\phi Y} Y, U) + g(\nabla_{\phi Y} Y, \beta\xi + \frac{\beta^2-1}{\alpha}U) = 0$. That is, $\frac{\beta^2}{\alpha}g([\phi Y, Y], U) = \frac{2\beta}{\alpha}$. Thus

$$g([\phi Y, Y], U) = \frac{2}{\beta}. \tag{4.12}$$

From (4.11) and (4.12) we have $\alpha^2 = 1$. Maybe after changing ξ by $-\xi$ we can suppose $\alpha = 1$. From Theorem 2.5 these real hypersurfaces do not exist.

Thus the unique principal curvature in $\mathbb{D}_U$ is $-k$ and $-k \neq 0$, $-k \neq -\frac{1}{\alpha}$. Let us suppose $-k = \frac{\beta^2-1}{\alpha}$. Take a unit $Y \in \mathbb{D}_U$ such that $AY = -kY$, $A\phi Y = -k\phi Y$. From the equation of Codazzi and the same calculations as in the proof of Theorem 1.1 we obtain

$$\beta g([\phi Y, Y], U) = 2k^2 - 2 + 2k\alpha \tag{4.13}$$

and

$$\left(\frac{\beta^2-1}{\alpha} + k\right) g([\phi Y, Y], U) = 2k\beta. \tag{4.14}$$

As $\frac{\beta^2-1}{\alpha} + k = 0$, (4.14) yields $2k\beta = 0$, which is impossible. The result follows as in the proof of Theorem 1.1.

Acknowledgements First author is partially supported by MICINN Project PID 2020-11 612GB-I00 and Project PY20-01391 from Junta de Andalucía. The authors want to thank the referee for valuable comments that have improved the paper.

Funding Funding for open access publishing: Universidad de Granada/CBUA

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