PERIODIC MOTIONS IN SINGULAR ELECTROMAGNETIC FIELDS AND TWIST DYNAMICS

por

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A mis padres, Loles y Clodo,
por permitir que me equivoque,
y dejarme afrontar mis errores.
A mis hermanos, Juan y Lucía.
A mi abuela Ángeles.
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Resumen

La temática de esta tesis doctoral se enmarca en el análisis de *sistemas dinámicos no lineales* con origen físico. Para el lector, o lectora, no familiarizado con las Matemáticas, un sistema dinámico hace referencia a una ecuación, o conjunto de ecuaciones, cuya incógnita es una función que depende del tiempo. Es decir, a través del mismo se describe la evolución temporal o dinámica de la función en cuestión. El planteamiento de este tipo de ecuaciones, llamadas diferenciales, tienen su motivación natural en nuestro deseo de explicar y predecir multitud de fenómenos que ocurren en la naturaleza. Es por ello que poseen gran relevancia en campos de estudios dispares entre sí como son la Física, las distintas Ciencias Naturales o la Economía.

Gran parte de los problemas planteados en la presente memoria se centran en el estudio de dinámicas periódicas en la ecuación de la fuerza de Lorentz, la cual describe el movimiento de una partícula cargada en un campo electromagnetismo, según la Electrodinámica Clásica. Este área de la física describe las interacciones entre partículas y campos electromagnéticos, a una escala suficientemente grande tal que los efectos cuánticos son despreciables. De ese modo, notando por $q(t) : [0,T] \to \mathbb{R}^3$ a la posición de la partícula en el instante t, y por $\dot{q}(t)$ a su velocidad, dados los campo eléctrico y magnético E, $B : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, el sistema a estudiar es el siguiente:

$$\frac{d}{dt}\left(\frac{\dot{q}(t)}{\sqrt{1-|\dot{q}(t)|^2}}\right) = E(t,q(t)) + \dot{q}(t) \times B(t,q(t)),$$

donde masa y carga han sido normalizadas a 1 sin pérdida de generalidad. El término de la izquierda es la aceleración, en términos relativistas, de la partícula, mientras que el lado derecho denota la expresión clásica de la fuerza de Lorentz, que describe la fuerza que ejerce el campo electromagnético sobre la misma. En esta situación, se obtiene un movimiento periódico si existe una solución de la ecuación que cumpla las condiciones de contorno

$$q(0) - q(T) = \dot{q}(0) - \dot{q}(T) = 0.$$

De ese modo, cada solución se identifica con una órbita cerrada en el sistema.

La ecuación de la fuerza de Lorentz tiene su origen en los trabajos de Poincaré [101, 102] y Planck [100] a comienzos del siglo XX, y constituye, junto a las Ecuaciones de Maxwell, los pilares de la Electrodinámica Clásica. Ambos sistemas se consideran ecuaciones fundamentales de la Física Matemática en particular, y de la Ciencia en general, debido a su profunda relevancia teórica y experimentalmente. No obstante, pese a ello existen pocos resultados cualitativos y cuantitativos sobre la dinámica que describe la fuerza de Lorentz, siendo estos prácticamente inexistentes dos décadas años atrás. Hasta entonces, la mayoría de resultados conocidos

se limitan a identificar soluciones en situaciones simples, como en el caso de campos uniformes y constantes, o bajo simetrías circulares o elípticas. Uno de los principales motivos de esta ausencia es que las técnicas analíticas necesarias para ello se desarrollan en el último tercio del siglo XX. Además, estas herramientas se plantean en el marco abstracto del análisis no lineal, mientras que su aplicación en la ecuación de la fuerza de Lorentz no data más allá de los últimos 15 años. Concretamente, en 2008, Mawhin y Bereanu aplican argumentos de grado topológico para lograr resultados de existencia de solución en campos electromagnéticos continuos bajo condiciones de contorno periódicas, de Dirichlet y de Neumann [14]. Más adelante, en 2019, Arcoya, Bereanu y Torres desarrollan una Teoría de punto crítico para la ecuación de la fuerza de Lorentz en [7], siguiendo las ideas de los métodos variacionales de Szulkin [112]. En particular, prueban existencia de solución en una extensa clase de campos electromagnéticos continuos bajo condiciones de contorno periódicas y de Dirichlet. Además, en un segundo artículo obtienen resultados de multiplicidad en el mismo marco funcional de campos continuos [8].

En los trabajos anteriormente citados no se consideran campos que puedan admitir singularidades, manteniéndose como un problema abierto. La primera contribución de este manuscrito es la resolución parcial del mismo, para el caso de condiciones de contorno periódicas. Ello dio lugar a la siguiente publicación:

[62] M. Garzón and P.J. Torres, *Periodic solutions for the Lorentz force equation with singular potentials,* Nonlinear Analysis: Real World Applications, **56** (2020), 103162.

Mediante un argumento de grado topológico, en el artículo citado se prueba la existencia de órbitas cerradas asumiendo que el campo eléctrico es singular en el origen. Con respecto al magnético, también puede serlo siempre que la singularidad dominante sea la del campo eléctrico. Las hipótesis cubren casos físicamente relevantes como son los potenciales de tipo Coulomb (que representa a una partícula cargada y estática), los dipolos magnéticos, o los flujos ABC. Todo ello está extensamente detallado en la Introducción y en el capítulo 2.

Otros modelos electrodinámicos de interés son aquellos generados por corrientes eléctricas. Cabe decir que el concepto de corriente es una noción abstracta para ilustrar el efecto físico que produce el movimiento de las partículas cargadas, acorde a la ley de Maxwell-Faraday. De esta manera, si imaginamos un hilo que transporta una corriente de valor constante y uniforme, mediante las ecuaciones de Maxwell es sencillo probar que se produce un régimen magnetostático. Es decir, no existe campo eléctrico y la corriente genera un campo magnético autónomo que se obtiene acorde a ley de Biot-Savart. Como consecuencia, el módulo de la velocidad es una cantidad conservada del sistema, así como también lo es la energía. Ello implica que la ecuación de la Fuerza de Lorentz se reduce a una del tipo

$$\ddot{q} = \dot{q} \times B(q).$$

Luego, en Magnetostática, el marco newtoniano y el relativista son equivalentes. Este tipo de situaciones son muy relevantes en diversas ramas de la física experimental debido al gran numero de aplicaciones que suscita el movimiento de partículas cargadas en campos magnéticos, como la obtención de *trampas magnéticas* para encerrar electrones, entre otros ejemplos [18, 20, 23, 30, 59, 91, 99, 107]. No obstante, generalmente no es posible integrar analíticamente las ecuaciones de movimiento, con lo que muchos de los estudios existentes se reducen a aproximaciones numéricas de las mismas. No obstante, se han estudiado analíticamente las dinámicas inducidas por configuraciones de hilo que implican simetrías en el sistema, como el caso de una espira o cable circular, el hilo infinito de corriente, o la superposición de ambos [3, 64, 65]. De ese modo, acorde al teorema de Noether, cada simetría implica la existencia de una cantidad conservada, lo que permite reducir la dimensión del sistema dinámico, facilitando su resolución. En el caso del hilo infinito, la conservación de la energía y los momentos, angular y linear, implican que el sistema es integrable, quedando determinado por la dinámica radial (en coordenadas cilíndricas) de la partícula. En particular, se tiene que las partículas siguen un movimiento periódico en la variable radial, sin alcanzar a tocar el cable, y tal que existe un único punto de equilibrio en esta variable.

En contraposición, si pasamos de una corriente continua a una alterna, la dependencia temporal de la misma rompe el régimen magnetostático, generando un campo eléctrico no autónomo, que a su vez implica una dependencia temporal en el magnético. Debido a ello, no puede aplicarse la ley de Biot-Savart y el campo electromagnético ha de calcularse mediante la resolución rigurosa de las ecuaciones de Maxwell. En particular, asumiendo que la corriente, la cual fijamos en el eje z sin pérdida de generalidad, es una perturbación de la forma $I_0 + kI(t)$, con I_0, $k > 0$ constantes, e $I(t)$ siendo una función periódica de media nula:

$$\int_0^T I(t)dt = 0,$$

entonces el campo electromagnético generado viene dado por las identidades

$$B(t,q) = \nabla \times \vec{A}(t,q), \qquad E(t,q) = -\partial_t \vec{A}(t,q),$$

para el potencial $\vec{A}(t,q) = A(t,r)\mathbf{e}_z$, donde

$$A(t,r) = -\frac{\mu_0}{2\pi}[a_0(r) + ka(t,r)],$$

con

$$a_0(r) = I_0 \ln r, \quad a(t,r) = \int_0^\infty \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}} d\tau.$$

Se ha notado por r a la variable radial en el plano XY, $\mathbf{e}_z$ es el vector unitario positivo en la dirección z, y el corchete $[t,r,\tau] = \left(t - \sqrt{\tau^2+r^2}\right)$ muestra el efecto relativista de retardo que implican las ecuaciones de Maxwell. Nótese que para $k = 0$ estamos en la situación magnetostática. En esta línea, nuestra investigación ha contribuido con las dos siguientes publicaciones:

[61] M. Garzón and S. Marò, *Motions of a charged particle in the electromagnetic field created by a non-stationary current*, Physica D: Nonlinear Phenomena, **424** (2021), 132945.

[63] M. Garzón and P.J. Torres, *Relativistic dynamic in the electromagnetic field created by a non-stationary current.* Sometido a publicación, 2022.

En el artículo [61] planteamos el modelo electromagnético generado por el hilo infinito de corriente, deduciendo rigurosamente el potencial $\vec{A}$ mediante la resolución de las ecuaciones de Maxwell. Una vez obtenido, como continuación natural del modelo magnetostático estudiado en [3], nos centramos en el estudio de la dinámica inducida por la corriente alterna en el marco newtoniano, es decir, la descrita por la ecuación de Newton-Lorentz

$$\ddot{q} = E(t,q) + \dot{q} \times B(t,q).$$

Aunque la energía del sistema no se conserva para $k > 0$, sí lo hacen los momentos debido a las simetrías inducidas por el hilo, lo que permite reducir el sistema dinámico a la siguiente ecuación escalar para la variable radial

$$\ddot{r} = \frac{L^2}{r^3} - (p_z + I_0 \ln r + ka(t,r)) \left(\frac{I_0}{r} + k\partial_r a(t,r) \right),$$

donde las constantes $L, p_z \in \mathbb{R}$ denotan los momentos angular y linear respectivamente. En nuestro análisis, aplicamos el teorema clásico de continuación local de equilibrios [28], junto con el método de la tercera aproximación [114] (introducido originalmente por Ortega en [94,96]), para probar la existencia de soluciones periódicas en la ecuación radial que son de tipo twist, lo que implica que son soluciones estables con respecto de las condiciones iniciales. Además, probamos que la estabilidad se mantiene en el plano de fase de las cantidades conservadas. Debido a las técnicas empleadas, nuestro resultado cualitativo es válido para valores pequeños del parámetro de perturbación k. Por último, cabe decir que las soluciones periódicas de la ecuación escalar se corresponden con órbitas radialmente periódicas alrededor del hilo en la descripción tridimensional.

Por otra parte, en [63] estudiamos la alteración de la dinámica en el marco relativista para $k > 0$, ya que para $k = 0$ ambos enfoques son equivalentes. Nuevamente, las simetrías del hilo implican la conservación de los correspondientes momentos relativistas angular y linear, por lo que la ecuación de la fuerza de Lorentz se reduce al siguiente sistema diferencial de primer orden

$$\begin{cases} \dot{r} = \dfrac{p_r}{\sqrt{1 + (p_z - A)^2 + p_r^2 + L^2 r^{-2}}}, \\[2ex] \dot{p}_r = \dfrac{L^2 r^{-3} + (p_z - A)\partial_r A}{\sqrt{1 + (p_z - A)^2 + p_r^2 + L^2 r^{-2}}}, \end{cases}$$

el cual posee estructura hamiltoniana para $\mathcal{H}(t,r,p_r) = \sqrt{1+(p_z - A)^2 + p_r^2 + \frac{L^2}{r^2}}$. En este caso, obtenemos un resultado cuantitativo de existencia de soluciones radialmente periódicas para la ecuación de la fuerza de Lorentz, en un intervalo explícito del parámetro k, el cual depende de los valores de la corriente y los momentos. La demostración del mismo sigue un argumento de perturbación global mediante el grado topológico, donde son esenciales unas estimaciones explícitas del comportamiento asintótico del potencial $A(t,r)$. Estas son obtenidas previamente en el mismo trabajo, y pueden suscitar interés matemático por sí solas. Con respecto a la estabilidad de las soluciones, la obtención de un resultado cuantitativo se mantiene como un problema abierto.

Por último, esta tesis contribuye a la teoría de problemas periódicos en sistemas Hamiltonianos no lineales con la siguiente publicación, escrita durante mi estancia en la Universidad de Trieste (Italia) durante los meses de febrero, marzo y abril de 2022:

[42] A. Fonda, M. Garzón and A. Sfecci: *An extension of the Poincaré-Birkhoff Theorem coupling twist with lower and upper solutions.* Sometido a publicación, 2022.

El teorema de Poincaré-Birkhoff es uno de los teoremas de punto fijo más destacados. En su versión original, puede enunciarse como sigue:

Sea $\Psi : R \to R$ un homeomorfismo que preserva el área, donde $R \subset \mathbb{R}^2$ es el anillo formado por los círculos C_a y C_b concéntricos de radios a y b respectivamente, con $0 < a < b$. Supongamos que Ψ satisface la condición twist, es decir, los puntos de C_a rotan en sentido opuesto a los de C_b. Entonces, la aplicación Ψ tiene al menos dos puntos invariantes.

A pesar de su aparente sencillez, ha dado lugar a multitud de aplicaciones en el estudio de dinámicas periódicas en ecuaciones diferenciales escalares, particularmente en aquellas que tienen estructura hamiltoniana. Originalmente, es enunciado en 1912 por Poincaré, motivado por su investigación en el problema de 3 cuerpos, quien además lo demuestra para algunos casos particulares [103]. No obstante, tras su fallecimiento ese mismo año, el teorema permanece como conjetura abierta hasta que Birkhoff da una prueba del mismo en [16], pocos meses después. Concretamente, demuestra la existencia de un punto fijo, mientras que el segundo sigue de una observación de Poincaré, que establece que el número total de puntos fijos en la aplicación ha de ser par. Sin embargo, el razonamiento falla si el primer punto fijo tiene índice nulo, luego, como el propio Birkhoff observa en [17], la demostración para la existencia del segundo punto requiere ser más precisa. Por este motivo, a lo largo del siglo pasado se desarrollaron distintos resultados que prueban el teorema en condiciones más generales, véase [49, 106] para una revisión histórica más detallada.

Desde los trabajos de Birkhoff, varios autores han extendido el resultado planar a variedades diferenciables $2N$-dimensionales, con $N \geq 1$. Citando a Arnold, *los* *intentos por generalizar el resultado a dimensiones mayores son importantes para el estudio de soluciones periódicas en problemas con varios grados de libertad* [11, p. 416]. Sin embargo, *no existe una generalización apropiada del teorema de Poincaré-Birkhoff hasta la fecha*[i] [92].

No obstante, los resultados de Fonda y Ureña en la pasada década [55–57] produjeron un gran avance en esta dirección. En sus trabajos extienden el teorema clásico a sistemas hamiltonianos $2N$-dimensionales, asumiendo diferentes generalizaciones de la condición twist en dimensiones mayores. Esta línea de investigación ha sido continuada de forma exitosa por el primer autor y otros colaboradores, logrando resultados aún más generales y un gran número de aplicaciones, incluso para sistemas hamiltonianos de dimensión infinita.

El artículo [42] es una de las últimas contribuciones en esta línea, extendiendo estos resultados a sistemas hamiltonianos que acoplan la condición twist (en sus diferentes variantes) con una generalización de los conceptos de sub y súper solución en dimensiones mayores. Las definiciones dadas en [42] están basadas en los trabajos previos de Fonda, Klun, Sfecci y Toader [46, 52], que generalizan el método clásico de sub y súper soluciones en ecuaciones diferenciales de segundo orden a sistemas planares periódicos.

Estructura de la tesis.

- **Capítulo 1:** Introducción del documento. La primera sección se centra en la Electrodinámica, explicando con detalle los modelos de la ecuación de Lorentz y las ecuaciones de Maxwell. Respecto a estas, se plantean rigurosamente desde un punto matemático mediante la teoría de las distribuciones, así como se muestra su resolución según si el enfoque adoptado para ello es de carácter físico o matemático. Por otra parte, se hace una revisión detallada de los resultados existentes (y previos a este trabajo) sobre el problema periódico en campos electromagnéticos continuos. Por último, se resumen los resultados obtenidos para campos electromagnéticos que admiten singularidades puntuales, como también en el modelo del hilo.

  La segunda sección se centra en el teorema de Poincaré-Birkhoff, extendiendo lo comentado en este resumen sobre su historia y los resultados previos al trabajo aportado en esta memoria. Por último, se explica la situación más simple planteada en [42], consistente en un sistema diferencial en $\mathbb{R}^4$ que acopla el método planar de sub/súper solución con una generalización del concepto twist en $\mathbb{R}^2$. Dicho resultado es la base a partir del cual se producen las demás generalizaciones a dimensiones mayores en el citado artículo.

  Por último, se comentan las técnicas matemáticas de perturbación local y global empleadas en estas investigaciones, junto con el método de estabilidad

[i]Sepa el lector que las notas originales de Moser y Zhender datan de los 1979 y 1980, mientras que la última versión fue publicada en 2005.

de la tercera aproximación.

- **Capítulo 2:** Se corresponde con el artículo [62], que prueba la existencia de soluciones periódicas en la ecuación de la fuerza de Lorentz para campos electromagnéticos singulares, cubriendo diversos ejemplos físicamente relevantes.

- **Capítulo 3:** Se corresponde con el artículo [61]. En el mismo se deduce de manera rigurosa el campo electromagnético generado por el hilo infinito de corriente alterna, así como la existencia de soluciones radialmente periódicas en el marco no relativista para valores pequeños en la perturbación. Mediante el método de la tercera aproximación, se tiene que la dinámica radial de las mismas es estable en el sentido de Liapunov.

- **Capítulo 4:** Se corresponde con el artículo [63], que estudia el marco relativista del modelo del hilo infinito de corriente alterna. Mediante un argumento de grado topológico, se logra un resultado cuantitativo para la existencia de soluciones radialmente periódicas en un intervalos explícito del parámetro de perturbación. Este capítulo incluye también el estudio asintótico en cero y en infinito del campo electromagnético inducido por el hilo, para el cual se usan fuertemente ciertas propiedades de las funciones de Bessel.

- **Capítulo 5:** Se corresponde con el artículo [42]. En este trabajo se extienden las recientes generalizaciones del teorema de Poincaré-Birkhoff a sistemas hamiltonianos que acoplan variantes de la condición twist con el concepto de sub/súper solución en dimensiones mayores.

Contents

Chapter 1

Introduction

1.1 Electrodynamics

1.1.1 Preliminaries

As far as it is known by Physics, there are four fundamental interactions between particles in the universe, these are Gravity, Weak and Strong Nuclear forces and the Electromagnetism. In this thesis we focus on Classical Electrodynamics, that describes the relations between charged particles and electromagnetic fields, and holds when the length scale of interactions is large enough to avoid quantum effects. Mathematically, this is ruled by the Lorentz force equation (LFE) which, together with Maxwell's equations, compose the backbone of this branch of knowledge. Both systems are complementary in the following sense: while the LFE models the dynamics of a charged particle in an electromagnetic field, Maxwell's equations describe the geometry of the field and how it arises from distributions of charged particles. Therefore, in order to have a complete picture of these phenomena, and for a rigorous approach to them, it is necessary to understand these two systems of differential equations, which we introduce below.

The Lorentz force equation

According to Classical Electrodynamics, the motion of a charged particle in an electromagnetic field is ruled by the Lorentz force equation, where the influence of the particle on the field is assumed to be negligible, for instance, this includes the case of a *slowly accelerated motion*. The classical LFE has its origin in the pioneering works of Planck and Poincaré [100] - [101, 102] and, due to its relevance in theory and applications, it can be found in many classical textbooks on Physics and Electrodynamics, see for instance [72, Chapter 12], [67, Chapters 5 and 12], [77, Chapter 3] or [39]. Denoting by $q(t) : [0, T] \to \mathbb{R}^3$ the position of the particle at time t, and by $\dot{q}(t)$ its derivative, the dynamical system to be studied is the following:

$$\frac{d}{dt}\left(\frac{\dot{q}(t)}{\sqrt{1-|\dot{q}(t)|^2}}\right) = E(t,q(t)) + \dot{q}(t) \times B(t,q(t)), \tag{1.1}$$

where, without loss of generality, both the speed of light in vacuum and the charge-to-mass ratio are normalized to 1. The right-hand side of the equation is the common expression of the Lorentz force, with $E : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ and $B : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ being the electric and magnetic field respectively, while the left-hand side denotes the relativistic acceleration of the particle, implying the characteristic speed limitation of Special Relativity. Introducing the diffeomorphism

$$\phi : B(0,1) \to \mathbb{R}^3, \qquad \phi(v) = \frac{v}{\sqrt{1-|v|^2}},$$

the left-hand side is also known as a singular ϕ-Laplacian in the related (more mathematically oriented) literature, see for instance [14].

Although (1.1) is one of the fundamental equations of Mathematical Physics, there has been a remarkable lack of qualitative and quantitative results about its dynamics until recently. One of the main reasons is that the proper mathematical tools for it were developed during the last quarter of the last century. Besides, such techniques were conceived in an abstract mathematical framework, and their applications to (1.1) barely appears during the last 15 years. Nevertheless, let us focus on Maxwell's equations before to go into detail about these results.

Maxwell's equations

In 1865, J.C. Maxwell published the celebrated paper [90] where the Electromagnetism is unified in a set of coupled PDE, describing the nature of the electromagnetic field and how it arises from the source, the charged particles distribution. Originally, Maxwell stated 20 equations, but finally there were reduced to 4 by O. Heaviside, that we present here in the differential formulation in SI units [31, 39, 67, 72]:

$$\begin{cases} \nabla \cdot B = 0, & \text{(Gauss's law for magnetism)} \\ \nabla \times E + \partial_t B = 0, & \text{(Maxwell-Faraday equation)} \\ \nabla \times B = \mu_0 \vec{J} + \partial_t E, & \text{(Ampère's law)} \\ \nabla \cdot E = \mu_0 \rho. & \text{(Gauss's law)} \end{cases} \tag{1.2}$$

Remark 1. *As the speed light in the vacuum c has been normalized to* 1, (1.2) *is written only in term of the universal constant μ_0 of permeability of free space (for unitary c) by using the identity $c\sqrt{\epsilon_0\mu_0} = 1$. Then, the vacuum permittivity constant ε_0 is omitted.*

Remark 2. *Here the divergence and curl operators are defined only with respect of the variable q. This notation is adopted along this manuscript for any function,* *field or distribution $f(t,q)$, being similar for the laplacian Δ and the gradient ∇. In addition, the operator of partial derivative w.r.t an arbitrary variable x is identified as ∂_x.*

While the concept of electric charge density is naturally understood as the distribution of the particles in space-time, the notion of electric current is quite delicate. In particular, it represents a flow of charged particles crossing a particular section (like a wire, a circuit or a planar section of the plane) which creates an electromagnetic field according to the Ampère's law, it is an abstract notion defined to illustrate the physical effects of charges in motion. Mathematically, we define them as scalar and vectorial distributions in space-time, that is as elements of the duals for the compact supported functions spaces $\mathcal{C}_0^\infty(\mathbb{R}^4)$ and $\mathcal{C}_0^\infty(\mathbb{R}^4;\mathbb{R}^3)$. Concretely, $\rho \in \mathcal{D}(\mathbb{R}^4)$ and $\vec{J} \in [\mathcal{D}(\mathbb{R}^4)]^3$.

Remark 3. *Equivalently, from a geometrical perspective, $(\rho, \vec{J})$ can be defined by the theory of currents, i.e., weak duals of the space of the compactly supported smooth forms on the space-time manifold.*

Nevertheless, the pair $(\rho, \vec{J})$ cannot be chosen arbitrarily, there exists a necessary condition for it.

Definition 1. *A couple $\rho \in \mathcal{D}(\mathbb{R}^4)$, $\vec{J} \in [\mathcal{D}(\mathbb{R}^4)]^3$, denotes a **physically admissible** charge and current distributions if they satisfy the charge continuity equation*

$$\partial_t \rho + \nabla \cdot \vec{J} = 0. \tag{1.3}$$

Notice that (1.3) is a direct consequence of the last two equations in (1.2). We refer to them as the *non-homogeneous part* of Maxwell's equations because of the presence of the charge and current distributions in the differential system. This approach considering ρ and $\vec{J}$ as data of (1.2) is the natural focus from the perspective of physicists and engineers, in order to compute the electromagnetic field in a concrete physical situation. On the other hand, by applying formally the Helmholtz's theorem to the *homogeneous part* of (1.2), the scalar and vectorial potentials $\Phi \in \mathcal{D}(\mathbb{R}^4)$, $\vec{A} \in [\mathcal{D}(\mathbb{R}^4)]^3$ are introduced as follows:

$$B(t,q) = \nabla \times \vec{A}(t,q), \qquad E(t,q) = -\partial_t \vec{A}(t,q) - \nabla \Phi(t,q), \tag{1.4}$$

where the derivatives are in the distributional sense. Moreover, taking $f \in \mathcal{C}_0^\infty(\mathbb{R}^4)$ arbitrarily, observe that the couple $(\Phi', \vec{A}')$ defined as

$$\vec{A}' = \vec{A} - \nabla f, \qquad \Phi' = \Phi + \partial_t f,$$

gives by (1.4) the same electromagnetic field as $(\Phi, \vec{A})$, so both pairs describe the same physical situation. As a consequence, for any electromagnetic field there exists an infinite number of potentials that generates it, which is consistent since potentials are not physically observable. This mathematical equivalence is known as the *Gauge Invariance* of Maxwell's equations and it is the main tool for their resolution. So, introducing the potentials in the non-homogeneous part, (1.2) is reduced to

$$\begin{cases} \partial_t^2 \vec{A} - \Delta \vec{A} + \nabla\left(\partial_t \Phi + \nabla \cdot \vec{A}\right) = \mu_0 \vec{J}, \\ -\Delta \Phi - \partial_t\left(\nabla \cdot \vec{A}\right) = \mu_0 \rho, \end{cases} \tag{1.5}$$

that, in principle, does not seem easier to solve. However, assuming the *Lorenz Gauge* for $(\Phi, \vec{A})$:

$$\partial_t \Phi + \nabla \cdot \vec{A} = 0, \tag{1.6}$$

system (1.5) decouples into the following four-dimensional wave equation:

$$\begin{cases} \partial_t^2 \vec{A} - \Delta \vec{A} = \mu_0 \vec{J}, \\ \partial_t^2 \Phi - \Delta \Phi = \mu_0 \rho. \end{cases} \tag{1.7}$$

If the data is compactly supported, i.e. if every component of it belongs to $\mathcal{D}'(\mathbb{R}^4)$, the solution can be directly obtained by convolution with the fundamental solution of the wave operator in $\mathbb{R} \times \mathbb{R}^3$:

$$\mathcal{F}(t,q) = H(t)\frac{\delta_{|q|}(t)}{4\pi|q|}.$$

Here, $\delta_{|q|}(t)$ represents the Dirac's delta in the spherical radial variable at the point t and $H(t)$ is the Heaviside function. In this way, (1.7) is solved in the sense of the distributions by the retarded potentials:

$$\begin{aligned} \vec{A}(t,q) &= \left[\mathcal{F} * \vec{J}\,\right](t,q) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \frac{\vec{J}(t_r,q')}{|q-q'|}dq', \\ \Phi(t,q) &= \left[\mathcal{F} * \rho\,\right](t,q) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \frac{\rho(t_r,q')}{|q-q'|}dq', \end{aligned} \tag{1.9}$$

where

$$t_r = t - |q-q'|$$

is called retarded time and takes into account the finiteness of the speed of light. This shows that the Maxwell's equations are, in fact, relativistic. On the other hand, it is not difficult to see that if the datum $(\rho, \vec{J})$ is physically admissible, then the retarded potentials satisfy (1.6). To this aim, by the definition of convolution between distributions (see [32, Appendix], [75, Chapter 7]), for every $\vec{f} \in \mathcal{C}_0^\infty(\mathbb{R}^4;\mathbb{R}^3)$ we have that

$$\vec{A}\,(\vec{f}) = \left[\mathcal{F} * \vec{J}\,\right](\vec{f}) = \left[\vec{J} * \mathcal{F}\right](\vec{f}) = \left\langle \vec{J}(t,x), \left\langle \mathcal{F}(s,y), \vec{f}(t+s,x+y)\right\rangle\right\rangle,$$

where $\langle \cdot,\cdot \rangle$ denotes the scalar product in L^2. For the same reason,

$$\Phi(f) = \langle \rho(t,x), \langle \mathcal{F}(s,y), f(t+s,x+y)\rangle\rangle\,, \text{ for all } f \in \mathcal{C}_0^\infty\left(\mathbb{R}^4\right).$$

Then,

$$\left[\nabla \cdot \vec{A} - \partial_t \Phi\right](f) = \left\langle \nabla \cdot \vec{J}(t,x) - \partial_t \rho(t,x), \langle \mathcal{F}(s,y), f(t+s, x+y)\rangle \right\rangle = 0,$$

for every $f \in \mathcal{C}_0^\infty(\mathbb{R}^4)$, and the Lorenz Gauge is satisfied.

Remark 4. *In absence of charged particles,* (1.2) *can be easily reduced to the homogeneous system in* $\mathbb{R}^4$*:*

$$\partial_t^2 E - \Delta E = \partial_t^2 B - \Delta B = 0,$$

whose solutions are the called electromagnetic *waves, that are propagating indefinitely. Their consideration as solutions only has physical sense if the absense of charged particles is assumed in a certain region, and the electromagnetic field comes from the complementary set. If this is assumed in all the space-time, then electromagnetic fields cannot exist without sources that generate them. Mathematically, this can be avoided assuming boundary conditions in order to be physically realistic, like for instance*

$$\lim_{|q|\to\infty} |E(t,q)| = \lim_{|q|\to\infty} |B(t,q)| = 0, \quad \textit{for every } t \in \mathbb{R}. \tag{1.10}$$

In conclusion, despite the Gauge Invariance, the electromagnetic field given by (1.4) *is the unique distributional solution of* (1.2) *to be considered.*

Under this construction, different cases with physical relevance are covered; like a static charged particle at some point $x \in \mathbb{R}^3$, which is represented by $(\rho, \vec{J}) = (\delta_x, \vec{0})$ because the particle is fixed, or any finite configuration of a wire carrying a current along it. Regarding this last example, since to be physically realistic the current values must be bounded, it is natural to describe them by a compact support function $I(t,q) \in L^\infty(\mathbb{R}^4; \mathbb{R})$. Similarly, the wire is represented as a finite curve $W \subset \mathbb{R}^3$, and the current distribution $\vec{J} \in [\mathcal{D}'(\mathbb{R}^4)]^3$ is defined as

$$\vec{J}(\vec{f}) = \int_{\mathbb{R}} \int_W \vec{f}(t,q) I(t,q) dW dt, \quad \text{for all } \vec{f} \in \mathcal{C}_0^\infty\left(\mathbb{R}^4; \mathbb{R}^3\right).$$

Since there is no charge density, the scalar potential is assumed to be null and the electromagnetic field created by the current is given by the derivatives (1.4) of the retarded potential in (1.9), with $\vec{J}$ defined as above. Moreover, notice that the potential $\vec{A}(t,q)$ is singular along the wire, as it is the electromagnetic field.

On the other hand, if $(\rho, \vec{J}) \notin [\mathcal{D}'(\mathbb{R}^4)]^4$, the convolution with the fundamental solution (1.8) can be not well defined because both elements are not of compact support. One example of it is the model of an infinitely long, thin and straight wire carrying a current, which is commonly understood as a local approximation of the previous situations. It is a very classical physical model described in many textbooks of Electrodynamics, see for instance Chapter 5 in [72] and [67]. However, as the wire has not compact support, the derivation of the corresponding electromagnetic field cannot be computed directly and needs to be rigorously obtained as limit of approximated problems. This development has been done in Chapter 3, together with the Newtonian description of its induced dynamics. Additionally, the corresponding relativistic dynamics of a charged particle given by (1.1) is studied in Chapter 4.

Remark 5. *Observe that, for every* $(\Phi, \vec{A}) \in \mathcal{D}(\mathbb{R}^4) \times [\mathcal{D}(\mathbb{R}^4)]^3$*, a pair* $(\rho, \vec{J})$ *can be written depending of the potentials computing the corresponding derivatives in* (1.5)*. Therefore, the electromagnetic field given by* (1.4) *solves Maxwell's equations uniquely for this specific distribution of charge and current, that is physically admissible because the potentials satisfy the Lorenz Gauge* (1.6)*. Although this does not mean that the couple* $(\rho, \vec{J})$ *has any physical relevance in general, there is no lack of rigor in this approach, alternative to the one explained above.*

1.1.2 Periodic dynamics of charged particles

One of the research objectives of this thesis is to look for periodic motions of charged particles in different electromagnetic regimes ruled by (1.2), i.e., to prove the existence of periodic solutions of the Lorentz force equation for an extensive class of electromagnetic fields.

Definition 2. *Given* $T > 0$*, a* T***-periodic solution*** *of the Lorentz force equation is a function* $q(t)$ *in the Sobolev space* $W^{2,1}([0,T];\mathbb{R}^3)$ *such that* $|\dot{q}(t)| < 1$ *for all* $t \in [0,T]$*,*

$$q(0) - q(T) = \dot{q}(0) - \dot{q}(T) = 0,$$

and that satisfies (1.1) *in the Carathéodory sense, i.e, for almost every* $t \in [0,T]$.

As motivation, let us consider the case of unidirectional magnetic fields. Without loss of generality, consider $E = 0$ and $B = (0,0,1)$. The LFE is in that case

$$\frac{d}{dt}\left(\frac{\dot{q}}{\sqrt{1-|\dot{q}|^2}}\right) = \frac{(1-|\dot{q}|^2)\ddot{q} + (\dot{q}\cdot\ddot{q})\dot{q}}{(1-|\dot{q}|^2)^{3/2}} = \dot{q} \times B.$$

Taking the scalar product with $\dot{q}(t)$, it follows that $\dot{q}\ddot{q} = 0$ and then the modulus of the velocity $|\dot{q}| = r \in (0,1)$ is a conserved quantity. By this, (1.1) is reduced to

$$\ddot{q}_1 = \dot{q}_2\sqrt{1-r^2}, \quad \ddot{q}_2 = -\dot{q}_1\sqrt{1-r^2}, \quad \ddot{q}_3 = 0.$$

This system can be solved explicitly and, giving initial conditions, there are solutions of the form

$$q = \frac{r}{\sqrt{1-r^2}}\left(-\cos(t\sqrt{1-r^2}), \sin(t\sqrt{1-r^2}), 0\right).$$

Therefore, in a very simple electromagnetic situation there exist periodic motions of charged particles, so it is natural to ask for that in more general regimes.

Remark 6. *As the modulus $|\dot{q}|$ is a conserved quantity when the magnetic field is autonomous, in this case the Lorentz force equation is equivalent to a Newtonian system of the form $\ddot{q} - \dot{q} \times B(q) = 0$, without loss of generality. This ceases to be true when the magnetic field is time-dependent.*

Existence results in regular fields

As it was said before, even though the LFE is a very classical equation with a deep relevance in Physics and Engineering, until recently, most of the results about its dynamics were limited to identify exact solutions for particular cases of simple electromagnetic fields (uniform and static fields under symmetries and other variants [2, 3, 6, 64, 65, 77, 110]). In 2008, the existence of solutions was proven in [14], by topological degree methods, for periodic, Dirichlet and Neumann problems associated to the general equation

$$\frac{d}{dt}\phi(\dot{q}(t)) = f(t, q, \dot{q}),$$

with f continuous. In particular, they provide the next application to (1.1):

Theorem 1. [14, Partially from Theorem 8]. *Given $T > 0$, let $E, B \in \mathcal{C}\left(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}^3\right)$ be both T-periodic in the first variable. If there exists $R > 0$ such that*

$$q \cdot [E(t, q) + v \times B(t, q)] \neq 0,$$

for all $t \in [0, T]$, $|q| \geq R$ and $|v| < 1$, system (1.1) has at least one T-periodic solution.

A sufficient condition for it is the existence of $d > 0$, $\gamma \geq 1$, and $R > 0$ such that, for all $t \in [0, T]$ and $|q| \geq R$, one has

$$q \cdot E(t, q) \geq d|q|^{\gamma}, \quad |B(t, q)| < d|q|^{\gamma - 1}. \tag{1.11}$$

Thus, for all $t \in [0, T]$, $|q| \geq R$ and $|v| < 1$, it is satisfied that

$$q \cdot [E(t, q) + v \times B(t, q)] \geq d|q|^{\gamma} - d|v||q|^{\gamma} \geq d(1 - |v|)R^{\gamma} > 0.$$

More recently, in 2017, a different approach is given in [7], by developing a critical point theory for the Lorentz force equation that can be applied to the periodic and Dirichlet problem for continuous electromagnetic fields, i.e., when $\left(\Phi, \vec{A}\right) \in \mathcal{C}^1\left(\mathbb{R}^4; \mathbb{R}^4\right)$. Before [7], there was not a neat and rigourous variational approach for (1.1), even although its relativistic Lagrangian

$$\mathcal{L}(t, q, \dot{q}) = 1 - \sqrt{1 - |\dot{q}|^2} + \dot{q} \cdot \vec{A}(t, q) - \Phi(t, q),$$

dates back to Poincaré [101, 102], more than a century ago. This fact gives an important reason of the lack of qualitative results about the dynamics of charged particles. Let us show a schedule of the main difficulties for the variational approach and some applications to the periodic problem. At first, the usual critical point theory [5, 105] is not applicable because the corresponding action functional

$$I(q) := \int_0^T \mathcal{L}(t, q(t), \dot{q}(t))dt,$$

is not of class $\mathcal{C}^1$. However, when $\vec{A} = 0$, the Szulkin's critical point theory [112], for functionals having a regular part plus a lower semi-continuous term, can be applied. In particular, defining $I(q)$ in the space of T-periodic continuous functions, it can be proved that every Palais-Smale sequence (in the Szulkin' sense) has a convergent subsequence. Nevertheless, if the vectorial potential is not null, the functional action is not properly defined in that space and the problem needs a reformulation. Just focusing on the periodic case, let $W_T^{1,\infty}$ be the subspace of all the T-periodic functions in the Sobolev Space $W^{1,\infty}(\mathbb{R};\mathbb{R}^3)$, with its usual norm

$$\|q\|_{W_T^{1,\infty}} = \|q\|_\infty + \|\dot{q}\|_\infty,$$

and let K be the convex and closed set defined as

$$K = \{q \in W_T^{1,\infty} : \|\dot{q}\|_\infty \leq 1\}.$$

The Lagrangian action $I : W_T^{1,\infty} \to (-\infty, +\infty]$ associated with (1.1) is given by $I = [\Psi + F]$, where

$$\Psi(q) = \begin{cases} \displaystyle\int_0^T \left[1 - \sqrt{1 - |\dot{q}(t)|^2}\right] dt, & \text{if } q \in K, \\ +\infty, & \text{if } q \notin W_T^{1,\infty} \setminus K, \end{cases}$$

represents the relativistic term and is a proper convex function which is continuous in its closed domain K. On the other hand, F is the restriction to $W_T^{1,\infty}$ of the functional

$$\int_0^T \left[\dot{q}(t) \cdot \vec{A}(t, q(t)) - \Phi(t, q(t))\right] dt,$$

that is continuously differentiable by standard arguments, see for instance [89]. Therefore, the concept of critical points in the Szulkin's sense holds for this situation and the next result is obtained.

Definition 3. *A function $q \in W_T^{1,\infty}$ is a **critical point** of I if $q \in K$ and*

$$\Psi(\varphi) - \Psi(q) + F'(q)[\varphi - q] \geq 0, \quad \textit{for all } \varphi \in W_T^{1,\infty}.$$

Theorem 2. [7, Theorem 6]. *A function $q \in W_T^{1,\infty}$ is a critical point of I if and only if q is a T-periodic solution of the Lorentz force equation* (1.1).

By Theorem 2, a principle of least action for the periodic Lorentz force equation is provided in [7], where the solutions are minimizers of I.

Theorem 3. [Theorem 7, [7]]. *If there exists $R > 0$ such that*

$$\inf_{K_R} I = \inf_{K} I,$$

then I is bounded from below on $W_T^{1,\infty}$ and attains its infimum at some $q \in K_R$, which is a solution of (1.1).

As an application, assume the existence of $\gamma > \beta \geq 1$ and $d, r > 0$, such that

$$\Phi(t,q) \leq -d|q|^{\gamma}, \text{ and } |\vec{A}(t,q)| \leq d|q|^{\beta}, \tag{1.12}$$

for every $(t,q) \in [0,T] \times \mathbb{R}^3$ with $|q| \geq r$, then the LFE (1.1) admits a T-periodic solution, which is a minimizer of I. See Theorem 8 in [7] for the computations.

Nevertheless, it cannot be assured that every Palais-Smale sequence in $W_T^{1,\infty}$ admits a convergent subsequence, so Sulzkin's critical point theory is not directly applicable to I in every case. However, the main result in [7] shows that, without any compactness assumption, there exist Palais-Smale sequences (in the Szulkin's sense) for functionals with this decomposition in general Banach spaces, relying essentially on the geometry and the continuity of the functional in its domain.

Theorem 4. [7, Theorem 1] *Assume that V is a Banach space and that the functional $I : V \to (-\infty, +\infty]$ is the sum of two functionals $I = \Psi + F$ such that*

(i) $\Psi : V \to (-\infty, +\infty]$ is convex and proper, with closed domain Dom $\Psi := \{v \in V : \Psi(v) < \infty\}$ in V. Moreover, Ψ is continuous in Dom Ψ.

(ii) $F : V \to \mathbb{R}$ is of class $\mathcal{C}^1$.

Let also M be a compact metric space, $M_0 \subset M$ a closed subset and $a_0 : M_0 \to V$ a continuous map. Consider the set $\Gamma = \{a : M \to V : a$ is continuous and $a|_{M_0} = a_0\}$. If

$$c_1 := \sup_{t \in M_0} I(a_0(t)) < c := \inf_{a \in \Gamma} \sup_{t \in K} I(a(t)) < \infty,$$

then, for every $\varepsilon > 0$ and $a \in \Gamma$ such that

$$c \leq \max_{t \in M} I(a(t)) \leq c + \frac{\varepsilon}{2},$$

there exists $a_\varepsilon \in \Gamma$ and $q_\varepsilon \in a_\varepsilon(M) \subset V$ satisfying:

$$c \leq \max_{t \in M} I(a_\varepsilon(t)) \leq \max_{t \in M} I(a(t)) \leq c + \frac{\varepsilon}{2},$$
$$\max_{t \in M} \| \bar{a}_\varepsilon(t) - a(t) \| \leq \sqrt{\varepsilon},$$
$$c - \varepsilon \leq I(q_\varepsilon) \leq c + \frac{\varepsilon}{2},$$

and

$$\Psi(\varphi) - \Psi(q_\varepsilon) + F'(q_\varepsilon)[\varphi - q_\varepsilon] \geq -\sqrt{\varepsilon} \| \varphi - q_\varepsilon \|, \quad \textit{for all } \varphi \in V.$$

The reader may find this result mathematically interesting by its own, we refer to Section 2 in [7] for the proof, other details and consequences. In particular, a non-smooth version of Rabinowitz's saddle point theorem is derived by choosing M as the closed ball B_R in $\bar{V}$ of center 0 and radius R, M_0 its boundary ∂B_R and a_0 the identity function.

Corollary 1. [7, Corollary 2]. *Let V be a Banach space such that $V = \bar{V} \oplus \widetilde{V}$, with $\dim \bar{V} < \infty$. If there exists $R > 0$ such that*

$$\sup_{\partial B_R} I < \inf_{\widetilde{V}} I, \tag{1.13}$$

and $\Gamma = \{a \in \mathcal{C}\left(\bar{B}_R, V\right) : a(q) = q, \text{ for all } q \in \partial B_R\}$, then there exists a sequence $\{q_n\} \subset E$ such that

$$\lim_{n\to\infty} I(q_n) = c := \inf_{a\in\Gamma} \sup_{q\in\bar{B}_R} I(a(q)), \tag{1.14}$$

and there exists $0 < \varepsilon_n \to 0$ such that

$$\Psi(\varphi) - \Psi(q_n) + F'(q_n)[\varphi - q_n] \geq -\varepsilon_n \|\varphi - q_n\|, \tag{1.15}$$

for all positive integer n and for all $\varphi \in Dom\Psi$.

Observe that, because of the usual decomposition: for every $q \in W_T^{1,\infty}$,

$$q(t) = \bar{q} + \tilde{q}(t), \quad \text{with} \quad \bar{q} = \frac{1}{T}\int_0^T q(t)dt, \quad \tilde{q}(t) = q(t) - \bar{q},$$

Corollary 1 can be applied to the periodic action functional and (1.13) is a sufficient condition for the existence of a Palais-Smale sequence (in the Szulkin's sense), according to (1.14) and (1.15). In addition, if the PS-sequence is bounded in $W_T^{1,\infty}$, then it has a subsequence converging in $\mathcal{C}([0,T];\mathbb{R}^3)$ to a critical point of I (see Lemma 6 in [7]), that is a T-periodic solution for the Lorentz force equation by Theorem 2.

Through this reasoning, two different applications are provided in [7]. At first, a situation symmetric to (1.12) is shown. Concretely, assuming that there exist $\gamma > \beta \geq 1$ and $d, r > 0$, such that

$$\Phi(t,q) \geq d|q|^\gamma, \text{ and } |\vec{A}(t,q)| \leq d|q|^\beta,$$

for every $(t,q) \in [0,T] \times \mathbb{R}^3$ with $|q| \geq r$, it follows that

$$I(\bar{q}) \to -\infty \text{ when } |\bar{q}| \to \infty, \quad \bar{q} \in \mathbb{R}^3.$$

Consequently, there exists a radius $R > 0$ such that (1.13) is satisfied and then (1.1) has at least one T-periodic solution, which is a saddle point of I. See Theorem 9 in [7] for the proof. On the other hand, consider the vectorial potential autonomous, i.e., $\vec{A}(t,q) = \vec{A}(q)$, for all $q \in \mathbb{R}^3$, and that both gradients are uniformly bounded by a constant $C > 0$. Thus, by (1.4),

$$B(q) = \nabla \times \vec{A}(q), \qquad E(t,q) = -\nabla\Phi(t,q),$$

and the electromagnetic field is uniformly bounded by C for all $(t,q) \in \mathbb{R}^4$. Then, if the scalar potential is such that

$$\lim_{|q|\to\infty} \int_0^T \Phi(t,q)dt = +\infty,$$

there exists at least a T-periodic solution of (1.1), that is a saddle point of the functional I. See Proposition 9 in [7].

To conclude, a last result is shown in [7] by applying minimax methods to the relativistic Hamiltonian $\mathcal{H} : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ of the Lorentz force equation, that is,

$$\mathcal{H}(t,q,p) = \sqrt{1 + |p - \vec{A}(t,q)|^2} - 1 + \Phi(t,q), \tag{1.16}$$

with the potentials assumed again of class $\mathcal{C}^1$, and the corresponding Hamiltonian formulation of (1.1) is

$$\dot{p} = \frac{p - \vec{A}(t,q)}{\sqrt{1 + |p - \vec{A}(t,q)|^2}} \cdot \nabla\vec{A}(t,q) - \nabla\Phi(t,q), \quad \dot{q} = \frac{p - \vec{A}(t,q)}{\sqrt{1 + |p - \vec{A}(t,q)|^2}}. \tag{1.17}$$

Historically, it seems that this formulation was presented for the first time by Planck in [100]. To see the equivalence, let $(q(t), p(t))$ be a continuous solution of the above system and observe that $\dot{q} = \phi^{-1}\left(p - \vec{A}(t,q)\right)$. Then, it is not difficult to see that $q(t)$ solves the Lorentz force equation, with relativistic momentum $p(t)$:

$$\begin{aligned}
\frac{d}{dt}\phi(\dot{q}) = \frac{d}{dt}\left(p - \vec{A}(t,q)\right) &= \dot{p} - \partial_t\vec{A}(t,q) - \nabla\vec{A}(t,q)\cdot\dot{q} \\
&= -\nabla\Phi(t,q) - \partial_t\vec{A}(t,q) + \left[\dot{q}\cdot\nabla\vec{A}(t,q) - \nabla\vec{A}(t,q)\cdot\dot{q}\right] \\
&= -\nabla\Phi(t,q) - \partial_t\vec{A}(t,q) + \dot{q}\times\left(\nabla\times\vec{A}(t,q)\right).
\end{aligned}$$

Moreover, the solution is of class $\mathcal{C}^2$ as a consequence of the continuity assumption for $(q(t), p(t))$. Concerning the last identity, this can be easily proved by a component by component development, in fact,

$$v \cdot \nabla F(q) - \nabla F(q) \cdot v = v \times (\nabla \times F(q)),$$

for any smooth map $F : \mathbb{R}^3 \to \mathbb{R}^3$ and any $q, v \in \mathbb{R}^3$.

Notice that the action functional associated to (1.16) is strongly indefinite, which is the main difficulty of the Hamiltonian approach. To overcome this, in [7] they modify the Hamiltonian in order to apply a suitable version of the well known generalized mountain pass theorem of Benci and Rabinowitz [105, Theorem 5.29]. This result, due to Felmer [38], holds for strongly indefinite functionals of class $\mathcal{C}^1$ in a Hilbert space E, that admit a certain "superquadratic" behaviour. In particular, $\Phi(t,q)$ is considered positive in $\mathbb{R}\times\mathbb{R}^3$, T-periodic with respect of t, and satisfying the following:

i) There exist constants $\gamma > 2$ and $d, r_0 > 0$ such that
$$\Phi(t,q) \leq d|q|^\gamma,$$
for all $(t,q)\in\mathbb{R}\times\mathbb{R}^3$ with $|q|\leq r_0$.

ii) There exist constants $\sigma > 2$ and $r_1 > 0$ such that
$$0 < \sigma\ \Phi(t,q) \leq q\cdot\nabla\Phi(t,q),$$
for all $(t,q)\in\mathbb{R}\times\mathbb{R}^3$ with $|q|\geq r_1$.

The above hypotheses were introduced by Rabinowitz in [104], where the existence of periodic solutions for second order differential systems of the form $\ddot{q}+\nabla\Phi(t,q) = 0$ is studied. On the other hand, the vectorial potential $\vec{A}(t,q)$ is also T-periodic with respect of t and satisfies:

iii) There exist constants $\beta > 1$ and $d, r_2 > 0$ such that
$$\left|\vec{A}(t,q)\right| \leq d|q|^\beta,$$
for all $(t,q)\in\mathbb{R}\times\mathbb{R}^3$ with $|q|\leq r_2$.

iv) There exist constants $d, r_3 > 0$ and $1\leq\theta<\min\{\sigma, 2\beta, \gamma\}$ such that
$$\left|\vec{A}(t,q)\right| \leq d|q|^\theta,$$
for all $(t,q)\in\mathbb{R}\times\mathbb{R}^3$ with $|q|\geq r_3$.

Theorem 5. [7, Theorem 10]. *If the potentials satisfy $i)-iv)$, then the Hamiltonian system* (1.17) *posseses a non-constant T-periodic solution (p,q) such that $q\in\mathcal{C}^2(\mathbb{R};\mathbb{R}^3)$, $|\dot{q}(t)|<1$, for every $t\in\mathbb{R}$, and q verifies the Lorentz force equation* (1.1).

All these applications have in common that, for high values of $|q|$, the vectorial potential is controlled in modulus by the absolute value of the scalar potential. This also happens in the first example (1.11), where the assumptions are taken over the fields instead of the potentials, and the electric field must be greater in modulus than the magnetic field outside some ball. In principle, this is just a mathematical requirement due to the techniques employed, because of the presence of the magnetic term $\dot{q}\times B(t,q)$ in the Lorentz force. So, we cannot say that these requirements are, in fact, optimal from the physical point of view. We consider that any achievement in this direction would be interesting.

Remark 7. *In a second paper [8], the authors of [7] provide a Lusternik-Schnirelmann theory for the periodic problem in* (1.1) *with continuous electromagnetic fields. However, their results have not been included in this manuscript.*

The singular Lorentz force equation

Despite the novelty of the previous results, they fail to cover the case of fields with singularities that are fundamental in Electromagnetism, like Coulomb potential or the magnetic dipole. In fact, the development of effective results based on a critical point theory for the Lorentz force equation with non-continuous electromagnetic fields remains an open problem to date. One possibility for it could be to try to extend the results developed in [4] for second order differential systems arising in Celestial Mechanics of the form $\ddot{q} + \nabla\Phi(t, q) = 0$, where Φ admits singularites. Again, the main difficulties in adapting these techniques stem from the presence of the magnetic term, thus requiring a new variational framework for the problem. Nevertheless, the qualitative results gap on periodic dynamics induced by singular electromagnetic fields has been partially filled in [62], by using topological degree methods and covering singular models with physical relevance as Coulomb potentials or the magnetic dipole. This paper was the first research of the thesis, published in 2020 in the journal *Nonlinear Analysis: Real World Applications*. Here we present a brief schedule of it, while the reader may find all the details and computations of [62] in Chapter 2.

Assume an electric field of the form

$$E(t, q) = -\nabla\Phi(q) + h(t),$$

with $\Phi \in \mathcal{C}^1(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$ and $h \in L^1([0, T]; \mathbb{R}^3)$, for which we denote its mean value as $\bar{h}$, that is $\bar{h} = \frac{1}{T}\int_0^T h(s)ds$. Moreover, the potential Φ satisfies the next assumptions:

h$_1$) $\lim_{|q|\to\infty} |\nabla\Phi(q)| = 0$.

h$_2$) $q \cdot \nabla\Phi(q)$ is negative for any q and there exist $c_0, \varepsilon_0 > 0$ and $\gamma \geq 1$ such that $q \cdot \nabla\Phi(q) \leq -c_0|q|^{-\gamma}$ for any $|q| < \varepsilon_0$.

Besides, the magnetic field $B \in \mathcal{C}([0, T] \times \mathbb{R}^3 \setminus \{0\}; \mathbb{R}^3)$ is required to satisfy the following:

h$_3$) $\limsup\limits_{|q|\to\infty} |B(t, q)| < C_B$, for some constant $C_B > 0$.

h$_4$) There exist $\varepsilon_1, c_1 > 0$ and $\beta \in (0, \gamma)$ such that $|B(t, q)| \leq c_1|q|^{-\beta-1}$, for all t when $|q| < \varepsilon_1$.

Notice that the electrical term controls the singularity point, establishing a similarity with the examples for regular fields provided above. In principle, as the reader can see in the proof, this is just a technical requirement. On the other hand, assumption h$_3$) allows to consider oscillating or bounded fields at infinity, in contrast with the conditions of decay (1.10). In these cases, as the magnetic field can have infinite energy, it only has a local physical meaning. However, as we shall see in Chapter 2, this covers a large class of magnetic field with physical relevance, like the well-known ABC flows.

Theorem 6. (Garzón, Torres, [62, Theorem 1]). *Assume* (h_1-h_4). *Then, for any* $h \in L^1([0,T];\mathbb{R}^3)$ *such that* $|\overline{h}| > C_B$, *the Lorentz force equation* (1.1) *admits at least one* T*-periodic solution.*

Remark 8. *Notice that, by the Maxwell-Faraday law in* (1.2), *the magnetic field must be autonomous because* $\nabla \times E = 0$. *Then, if* $B(q) = \nabla \times \vec{A}_0(q)$, *for some potential* $\vec{A}_0(q)$, *the pair* $(\Phi, \vec{A})$ *solves Maxwell's equations with*

$$\vec{A}(t,q) = \vec{A}_0(q) - \int_0^t h(s)ds.$$

This observation does not alter Theorem 6, since the time dependence of $B(t,q)$ *plays no role in the proof.*

Other models with singularities arise in Electrodynamics assuming a wire carrying a current along it, as discussed above. Concerning the dynamics induced by the wire, if the current is constant, Maxwell's equations can be rigorously reduced to the Biot-Savart law for the magnetic field, i.e, there is no electric field and $B(q)$ is autonomous. The study of the motion of a charged particle in a magnetic field has long been of interest in several areas of Physics [18, 20, 23, 30, 59, 91, 99, 107], and the corresponding magnetostatic dynamics has been studied extensively for some autonomous wire distributions, like the circular loop, the infinite wire or a particular coupling of both, see for instance [3, 64, 65]. The consideration of symmetries implies the existence of first integrals in the dynamical system, according to Noether's Theorem. In the case of the infinite wire, the conservation of the energy, linear and angular momenta imply that the system is totally integrable. In particular, the particles cannot reach the wire, every solution is radially periodic and there exists a unique radial equilibrium.

On the other hand, if we assume a time dependence in the current, the electromagnetic situation changes completely. At first, the Biot-Savart formula does not hold anymore and the electromagnetic field has to be obtained by solving Maxwell's equations for the corresponding non compact supported distribution $\vec{J}$. Concretely, considering a T-periodic current of the form $I_0 + kI(t)$ along the z-axis, with $k > 0$ and

$$\int_0^T I(t)dt = 0,$$

the associated current density is a vectorial distribution $\vec{J} = (0, 0, J)$ such that

$$J(f) = \int_{\mathbb{R}^2} [I_0 + kI(t)]\, f(t, 0, 0, z)\, dt\, dz, \qquad \text{for every } f \in \mathcal{C}_0^\infty\left(\mathbb{R}^4\right),$$

for which the PDE system (1.2) admits as unique distributional solution to the electromagnetic field given by (1.4), with $\Phi \equiv 0$ and $\vec{A}(t, q) = A(t, r)\mathbf{e}_z$, where

$$A(t, r) = -\frac{\mu_0}{2\pi}[a_0(r) + ka(t, r)],$$

and

$$a_0(r) = I_0 \ln r, \quad a(t, r) = \int_0^\infty \frac{I[t, r, \tau]}{\sqrt{\tau^2 + r^2}} d\tau.$$

Here r denotes the radial variable in the XY-plane, $\mathbf{e}_z$ is the positive unitary vector in the z-direction, and the bracket $[t, r, \tau] = \left(t - \sqrt{\tau^2 + r^2}\right)$ shows the delay effect of the potential.

The resolution of (1.2) for this specific distribution is done in [61], a joint work with Stefano Marò published in 2021 at *Physica D: Nonlinear Phenomena*, and it is included in Chapter 3 of this document. In the same paper the non-relativistic model for the above T-periodic time-dependent perturbation is studied, that is ruled by the Newton-Lorentz equation

$$\ddot{q} = E(t, q) + \dot{q} \times B(t, q),$$

and it holds when the velocity of the particle is small in comparison with the light speed constant. Although the energy is not conserved for $k > 0$, the moments still are, due to the symmetries induced by the wire. Through this, the Newton-Lorentz equation is reduced to the following second order differential equation for the radial component of the particle:

$$\ddot{r} = \frac{L^2}{r^3} - (p_z + I_0 \ln r + ka(t, r)) \left(\frac{I_0}{r} + k\partial_r a(t, r)\right).$$

The above equation has structure as a two-dimensional Hamiltonian system with one degree of freedom for

$$H(t, r, \dot{r}) = \frac{\dot{r}^2}{2} + \frac{L^2}{2r^2} + \frac{1}{2}(p_z + I_0 \ln r + ka(t, r))^2,$$

with $L, p_z \in \mathbb{R}$ denoting angular and linear momentum respectively. Properly applying the local continuation theorem [28], together with the third approximation method [114] (previously introduced by Ortega in [96]), it is shown the existence of solutions such that the radial component is T-periodic and stable in the Liapunov sense, not just with respect of the radial initial conditions, but also in the phase plane of conserved quantities. Because of the techniques employed, both results are qualitative and hold only for small values of the perturbation. It can be seen as a complementary study to [78], where the case of an infinite wire with no current density and a time-dependent charge density is considered. In that case, the dynamic equation for the radial component is of the form $\ddot{r} + g(t)r^{-3} = 0$, with $g(t)$ a T-periodic function, dealing with a different situation.

The last research on the Lorentz force equation included in this document corresponds to Chapter 4 and refers to the relativistic dynamics induced by the time-dependent wire, which was studied in [63], as a continuation of [61]. It is a paper developed together with P.J. Torres and currently submitted for publication.

Just like in the Newtonian case, the cylindrical symmetry is still present in (1.1) and the corresponding linear and angular relativistic momenta are conserved quantities when $k > 0$. Thought this, the Lorentz force equation is reduced to the following planar Hamiltonian system with one degree of freedom

$$\begin{cases} \dot{r} = \dfrac{p_r}{\sqrt{1 + (p_z - A)^2 + p_r^2 + L^2 r^{-2}}}, \\[2ex] \dot{p}_r = \dfrac{L^2 r^{-3} + (p_z - A)\partial_r A}{\sqrt{1 + (p_z - A)^2 + p_r^2 + L^2 r^{-2}}}, \end{cases}$$

whose Hamiltonian is $\mathcal{H}(t, r, p_r) = \sqrt{1 + (p_z - A)^2 + p_r^2 + L^2 r^{-2}}$. There is shown the existence of radially periodic solutions for an explicit interval of the perturbation, that depends on the values of the current and the conserved momenta. While [61] used a perturbation argument, here the mathematical procedure is different, based on a global continuation by using the topological degree, for which some delicate estimations of the asymptotic behaviour of $\vec{A}$ are essential. These were developed in the same paper and may have mathematical interest of their own. Regarding the radial stability, we are sure that it holds for small values of the perturbation, by using the third approximation method like in the Newtonian case. However, to provide a quantitative result in this regard remains an open problem.

1.2 Poincaré-Birkhoff Theorem and non-autonomous Hamiltonian systems

It could be asserted that the Poincaré-Birkhoff Theorem is one of the most remarkable fixed point results ever achieved. Despite the simplicity of the statements, there has been provided a large number of applications on the study of periodic dynamics in scalar differential equations, particularly for those with Hamiltonian formulation. Moreover, since Birkhoff's works, some authors have extended the planar result to certain $2N$-dimensional manifolds. Quoting Arnold, *attempts to generalize it to higher dimensions are important for the study of periodic solutions of problems with many degrees of freedom* [11, Page 416]. Nevertheless, as it was stated in [92], *there is no genuine generalization of the Poincaré-Birkhoff Theorem to higher dimension until the date*[i].

More recently, during the last decade, a further step in this direction has been made by Fonda and Ureña [55–57]. In their work, the classical result is generalized for $2N$-dimensional Hamiltonian systems with different natural variants of the twist condition in higher dimensions. This line has been succesfully continuated in several works by the first author and others, obtaining more general results and applications, even covering the cases of infinite-dimensional Hamiltonian systems.

This manuscript includes a recent contribution in this line. Coauthored by Fonda and Sfecci, in [42] we extend these generalizations of the Poincaré-Birkhoff Theorem for Hamiltonian systems coupling the twist with lower and upper solutions in higher dimensions. These are the first results obtained for the existence of periodic solutions in systems where this two different and very classical methods are linked. Concerning the case of upper and lower solutions, we have followed the recent works of Fonda, Klun, Sfecci and Toader [46, 54], that extend the classical concept for second order differential equations to periodic planar systems.

1.2.1 The original theorem

Also known as the Poincaré's last geometric theorem, it was conjetured in 1912 motivated by his research on the restricted three body problem [103], where Poincaré also proved the result in several cases. Unfortunately, he passed away shortly after, so the theorem remained an open conjecture until G.D. Birkhoff provided a proof for it in [16], just a few months later. Let us state it in a similar way as it was formulated originally:

Theorem 7. [Poincaré-Birkhoff Theorem.] *Let $\Psi : R \to R$ be an area-preserving homeomorphism, where $R \subset \mathbb{R}^2$ is the ring formed by concentric circles C_a and C_b of radii a and b respectively, with $0 < a < b$. Suppose that Ψ satisfies the twist condition, that is, the points in C_a rotate in the opposite direction as the points in C_b. Then, Ψ has at least two invariant points.*

In its proof, Birkhoff showed that the map has an invariant point, while the second exists as a consequence of an observation made by Poincaré in [103]. According to the index theory applying to Ψ, the number of fixed points must be even. However, this reasoning is true only if the first point has nonzero index, as Birkhoff himself declared in [17], so the proof for a second fixed point need to be made more precise. This fact has given rise to an extensive literature on proving the result under more general assumptions, we refer to the reader to [49, 106] for a complete historical review and more details.

[i]It is important to know that the original notes of Moser and Zhender come from the years 1979-80, while the last version was published in 2005.

1.2.2 The generalized Poincaré-Birkhoff Theorem for Hamiltonian flows in higher dimensions

At first, let us state Theorem 7 in a modern version like in [56]:

Theorem 8. [Poincaré-Birkhoff Theorem.] *Let $\mathcal{P} : \mathbb{R} \times [a, b] \to \mathbb{R} \times [a, b]$ be an area-preserving homeomorphism of the form*

$$\mathcal{P}(x, y) = (x + \vartheta(x, y), r(x, y)),$$

where the functions $\vartheta(x, y)$ and $r(x, y)$ are 2π-periodic in their first variable x, with $r(x, a) = a$ and $r(x, b) = b$, for every $x \in \mathbb{R}$. Assume the boundary twist condition

$$\vartheta(x, a)\vartheta(x, b) < 0, \text{ for every } x \in \mathbb{R}. \tag{1.18}$$

Then, $\mathcal{P}$ has at least two invariant points in $[0, 2\pi] \times (a, b)$.

Concerning the study of periodic dynamics in Hamiltonian systems, in 1983, C.C. Conley and E.J. Zehnder [29] proved two remarkable results on the periodic problem associated to the general system

$$S\dot{z} = \nabla H(t, z)\,, \tag{1.19}$$

giving a partial answer to a conjecture by V.I. Arnold [9,10]. Here, $S = \left(\begin{smallmatrix} 0 & -I_N \\ I_N & 0 \end{smallmatrix}\right)$ is the standard symplectic matrix, $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is T-periodic in t, and $\nabla H(t, z)$ denotes its gradient with respect to z. Let us write $z = (q, p)$, with $q = (q_1, \ldots, q_N)$ and $p = (p_1, \ldots, p_N)$, so that the system reads as

$$\dot{q} = \partial_p H(t, q, p)\,, \quad \dot{p} = -\partial_q H(t, q, p)\,.$$

In a first theorem, assuming the Hamiltonian function $H(t, z)$ to be periodic in all variables $q_1, \ldots, q_N$ and $p_1, \ldots, p_N$, they prove that system (1.19) has at least $2N + 1$ geometrically distinct T-periodic solutions. In a second theorem, they assume H to be periodic in $q_1, \ldots, q_N$ and to have a quadratic behaviour in p; namely, that there exist a constant $R > 0$ and a symmetric regular matrix $\mathbb{A}$ such that $H(t, q, p) = \frac{1}{2}\langle \mathbb{A}p, p\rangle$ + "lower order terms", when $|p| \geq R$. In this setting, they prove that system (1.19) has at least $N + 1$ geometrically distinct T-periodic solutions. They also mention a possible relation of this second result with the Poincaré-Birkhoff Theorem (the title of the paper is however a bit misleading).

The pioneering results in [29] have been extended in different directions by several authors, see for instance [25, 37, 48, 60, 74, 79, 86, 88, 105, 112].

More recently, a deeper relation between these results and the Poincaré-Birkhoff Theorem has been established by Fonda and Ureña in [56]. Taking $\mathcal{D} = \overline{B}(0, R)$ as the closed ball with radius R, and assuming H to be periodic in the q_i-variables, the existence of $N + 1$ geometrically distinct T-periodic solutions is established under the following hypothesis: there exist a constant $R > 0$ and a symmetric regular matrix $\mathbb{A}$ such that the solutions $z(t) = (q(t), p(t))$ of (1.19) starting with $p(0) \in \mathcal{D}$ are defined on $[0, T]$ and satisfy a "twist condition" like

$$\big\langle q(T) - q(0)\,, \mathbb{A}p(0)\big\rangle > 0\,, \quad \text{when } p(0) \in \partial\mathcal{D}\,. \tag{1.20}$$

Remark 9. *Observe that* (1.20) *generalizes to* $\mathbb{R}^{2N}$ *the twist boundary condition* (1.18)*. Moreover, it is easily checked that this result generalizes the second Conley-Zehnder Theorem described above. Variants of the twist condition were also proposed in [56, 57].*

The results in [56] have been extended in [43] by Fonda and Gidoni in order to include both the above quoted Conley-Zehnder theorems, assuming H to be periodic in $q_1, \dots, q_N$ and possibly also in $p_1, \dots, p_L$, for some $L \in \{1, \dots, N\}$, together with a very general twist condition, thus finding $N + L + 1$ periodic solutions. The same authors further extended the theory in [44] to the case when the Hamiltonian function includes a nonresonant quadratic term. Possible resonance has also been investigated in [26], assuming some Ahmad-Lazer-Paul conditions.

These general existence results have found so far several applications in [21, 41, 44, 50, 51, 53, 56], thus generalizing some previously established results for second order equations [24, 34, 35, 47, 52, 69, 73]. They have even been extended to the study of infinite-dimensional systems [19, 45].

1.2.3 Coupling Poincaré-Birkhoff Theorem with upper and lower solutions

In our particular case, the main tools for the achievement in [42] are provided by the results of [44], once we are able to truncate the Hamiltonian such that we obtain a bounded side plus a nonresonant quadratic term. However, the results of [44] are written in a very technical way and we have omitted them here.

To conclude, let us state the simplest situation considered in [42]. At first, we assume the four dimensional Hamiltonian system

$$\begin{cases} \dot q = \partial_p H(t, q, p, u, v)\,, & \dot p = -\partial_q H(t, q, p, u, v)\,, \\ \dot u = \partial_v H(t, q, p, u, v)\,, & \dot v = -\partial_u H(t, q, p, u, v)\,, \end{cases}$$

where the function $H(t, q, p, u, v)$ is continuous and T-periodic w.r.t. t, continuosly differentiable in the (q, p, u, v) variables, and 2π-periodic in q. As usual, we say that two T-periodic solutions are *geometrically distinct* if they do not differ in the q-variable by an integer multiple of 2π.

On the other hand, there exist some constants $\delta > 0$ and $\alpha \le \beta$ such that

$$v\,\partial_v H(t, q, p, u, v) > 0\,, \text{ when } u \in [\alpha - \delta, \alpha] \cup [\beta, \beta + \delta] \text{ and } v \neq 0\,,$$

and

$$\begin{cases} \partial_u H(t,q,p,u,0) \geq 0\,, & \text{when } u \in [\alpha-\delta,\alpha]\,,\\ \partial_u H(t,q,p,u,0) \leq 0\,, & \text{when } u \in [\beta,\beta+\delta]\,. \end{cases}$$

These considerations come from the definition of lower and upper solutions given in [46,54]. We require here that these lower and upper solutions are constant. More precisely, all constants in $[\alpha-\delta,\alpha]$ are lower solutions, and all constants in $[\beta,\beta+\delta]$ are upper solutions. Moreover, with the aim of the planar upper and lower solution referenced above, we assume the following Nagumo kind condition:

There exist $d>0$ and two continuous functions $f,\varphi : [d,+\infty) \to (0,+\infty)$, with

$$\int_d^{+\infty} \frac{f(s)}{\varphi(s)}\, ds = +\infty\,,$$

satisfying the following property. If $u \in [\alpha-\delta,\beta+\delta]$, then

$$\begin{cases} \partial_v H(t,q,p,u,v) \geq f(v)\,, & \text{when } v \geq d\,,\\ \partial_v H(t,q,p,u,v) \leq -f(-v)\,, & \text{when } v \leq -d\,, \end{cases}$$

and

$$|\partial_u H(t,q,p,u,v)| \leq \varphi(|v|)\,, \quad \text{when } |v| \geq d\,.$$

In addition, for every $K>0$ there is a constant $C_K>0$ such that

$$|\partial_q H(t,q,p,u,v)| \leq C_K(|p|+1)\,, \quad \text{when } u \in [\alpha-\delta,\beta+\delta] \text{ and } |v| \leq K\,.$$

Finally, assume also the existence of $a<b$ and $\rho>0$ such that, for any two continuous functions $U,V : [0,T] \to \mathbb{R}$ satisfying

$$\alpha-\delta \leq U(t) \leq \beta+\delta\,, \quad \text{for every } t \in [0,T]\,,$$

the solutions of the system

$$\dot q = \partial_p H(t,q,p,U(t),V(t))\,, \quad \dot p = -\partial_q H(t,q,p,U(t),V(t))$$

satisfy the twist condition:

$$\begin{cases} q(T)-q(0) < 0\,, & \text{when } p(0) \in [a-\rho,a]\,,\\ q(T)-q(0) > 0\,, & \text{when } p(0) \in [b,b+\rho]\,. \end{cases}$$

Notice that these solutions are well defined in $[0,T]$. Then, in the above situation, there exist at least two geometrically distinct T-periodic solutions of the Hamiltonian system such that

$$p(0) \in (a,b), \quad \text{and} \quad \alpha \leq u(t) \leq \beta\,, \quad \text{for every } t \in \mathbb{R}\,.$$

Moreover, the same conclusion holds if the twist condition is replaced by

$$\begin{cases} q(T)-q(0) > 0\,, & \text{when } p(0) \in [a-\rho,a]\,,\\ q(T)-q(0) < 0\,, & \text{when } p(0) \in [b,b+\rho]\,. \end{cases}$$

Other variants of this theorem are detailed in Chapter 5. Several higher-dimensional generalizations at different levels of abstraction are also included.

1.3 Perturbative methods and stability criteria in Dynamical Systems

Concerning the existence results developed in the papers [61–63], they have in common their achievent by continuation methods for periodic perturbations of nonlinear differential systems. However, the techniques applied in this monograph are essentially different between them from a mathematical point of view. At first, the classical local continuation [28, Theorem 1.1, Chapter 14] relies on the Implicit Function Theorem, while the topological degree of a nonlinear differential equation is commonly computed through its invariance by homotopy, previously obtaining an invariant bounded set for the solutions. Concretely, for periodic perturbations of autonomous systems, we use the celebrated global continuation theorems due to Capietto, Mawhin and Zanolin [22, Theorem 1-2]. As a consequence, a characterization of the Leray-Schauder degree for certain periodic Hamiltonian systems is given in [63], whose formulation generalizes the planar case of the relativistic dynamics induced by the current along the infinite wire.

Complementarily, a brief schedule of the third approximation method for the stability of elliptic periodic solutions in time-dependent Newtonian equation $\ddot{x} + f(t,x) = 0$ has been included. In these cases, the first method of Lyapunov fails, so the stability analysis relies on the nonlinear terms of the Taylor series for $f(t,x)$. This procedure is used in [61] to achieve a stability result in the non-relativistic regime induced by the time-dependent current along the wire. Although the method was originally introduced by Ortega in [94, 96], here we recall a generalization given in [114] that fits in the commented model.

In order to be mathematically rigourous, this section provides the necessary notions for their understanding and application.

1.3.1 Local continuation of periodic solutions

For the sake of convenience we recall here the classical theorem taken from [28]. To this aim, let us consider the differential equation

$$\dot{x} = f(t, x, \mu), \tag{1.21}$$

where $f : V \times B(\mu_0) \to \mathbb{R}^n$ having denoted by V a domain of $\mathbb{R} \times \mathbb{R}^n$ and by $B(\mu_0)$ the open ball of radius μ_0 in $\mathbb{R}^m$. The function f is continuous in (t, x, μ) and has first-order derivatives w.r.t. the components x_i of x. Moreover, assume $f(t+T, x, \mu) = f(t, x, \mu)$ and that, for $\mu = 0$, equation (1.21) admits a T-periodic solution $p(t)$ such that $(t, p(t)) \in V$ for every t. We also introduce the *first variation* of (1.21) with respect to the solution $p(t)$ as the differential system:

$$\dot{y} = \nabla_x f(t, p(t), \mu) \cdot y. \tag{1.22}$$

Under these assumptions, we can state the following.

Theorem 9. [28, Chapter 14, Theorem 1.1] *Suppose that the first variation* (1.22) *for* $\mu = 0$ *has no solution of period* T. *Then, there exists* $\mu_1 < \mu_0$ *such that for every* $|\mu| < \mu_1$, *equation* (1.21) *has a unique* T*-periodic and continuous solution* $q = q(t, \mu)$, *such that* $q(t, 0) = p(t)$.

Remark 10. *It comes from the proof of Theorem 9 that, for every* $|\mu| < \mu_1$, *the functions* $q(t, \mu)$ *are solutions of* (1.21) *with initial condition* $x(0) = \alpha(\mu)$ *where* $\alpha(\cdot)$ *is a continuous function defined in a neighbourhood of* 0 *such that* $\alpha(0) = p(0)$. *Since* $p(t)$ *is a solution of* (1.21) *for* $\mu = 0$ *and initial condition* $x(0) = p(0)$, *by continuous dependence with respect to parameters (see [70, Chapter V, Theorem 2.1]), we have that actually*

$$\lim_{\mu \to 0} q(t, \mu) = p(t), \quad \textit{uniformly in } t \in [0, T].$$

1.3.2 Stability of twist type solutions by KAM theory

Let us consider the scalar Newton's equation

$$\ddot{x} = f(t, x), \tag{1.23}$$

where $f \in \mathcal{C}(\mathbb{R}/T\mathbb{Z} \times \mathbb{R})$ is sufficiently smooth in x.

Let us also define, in some open subset $\mathcal{O} \subseteq \mathbb{R}^2$, the Poincaré map associated to (1.23) as the operator

$$\mathcal{P}(\upsilon) = (x(T; \upsilon), \dot{x}(T; \upsilon)), \tag{1.24}$$

where $x(t; \upsilon)$ is the solution of (1.23) with initial datum $\upsilon \in \mathcal{O}$. It is well known that (1.24) is area-preserving, see for instance [12, Chapter 1]. Moreover, for any non-resonant fixed point υ_0, there exist a $\beta \in \mathbb{R}$ and a sympletic map $\Psi \in \mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R}^2)$, with $\Psi(\upsilon_0) = \upsilon_0$, such that

$$\Psi^{-1} \circ \mathcal{P} \circ \Psi(\upsilon) = \mathbf{R}[\theta + \beta|\upsilon - \upsilon_0|^2](\upsilon - \upsilon_0) + \Theta\left(|\upsilon - \upsilon_0|^4\right),$$

when $\upsilon \to \upsilon_0$. Here we are denoting by $\mathbf{R}[\theta]$ to the rotation matrix of angle $\theta \in \mathbb{R}$. Furthermore, the number β does not depend on the map Ψ. More technical details are shown in [12, Theorem 6.1.2] or [111, Section 23].

Now, let $\mathbf{x}(t)$ be a non-resonant, T-periodic and elliptic solution of (1.23). As the initial data of the T-periodic solutions in (1.23) are fixed points in $\mathcal{P}$, we say that $\mathbf{x}$ is of *twist type* if the associated β is not null. In that case, by Moser twist theorem [111, Section 32] it follows that any T-periodic solution of twist type is accumulated by $\mathcal{P}$-invariant curves, and so it is Lyapunov stable. In particular, every point in the invariant curves of $\mathcal{P}$ corresponds to the initial datum for a quasiperiodic solution of (1.23). Moreover, KAM theory shows that the Poincaré-Birkhoff Theorem can be locally applied between two invariant curves, finding the existence of subharmonic solutions surrounding the fixed point. Consequently, every T-periodic solution of twist type in (1.23) is accumulated by this two kind of solutions. We refer to the reader to [111] for the mathematical procedure and the proper definitions.

To conclude, it comes from [94,96] that the twist character of a periodic solution can be deduced by the third approximation of the equation. To this aim, let us move the solution to the origin via the change of variable $y = x - \mathbf{x}(t)$ in (1.23) and compute the development up to third order around $x = 0$. This expression has the form

$$\ddot{y} + A(t)y + B(t)y^2 + C(t)y^3 = 0.$$

The results in [94, 96] do not fit in the model considered in Chapter 3, hence we recall the next theorem by Torres and Zhang.

Theorem 10. [114, Theorem 3.1] *The T-periodic solution $\boldsymbol{x}(t)$ is of twist type if the three following assertions are true:*

(i) $0 < A_* \leq A^* < \left(\frac{\pi}{2T}\right)^2$,

(ii) $C_* > 0$,

(iii) $10(B_*)^2(A_*)^{3/2} > 9C^*(A^*)^{5/2}$,

where f^* and f_* respectively represent the supremum and infimum of $f \in \mathcal{C}([0, T]; \mathbb{R})$.

1.3.3 Global continuation of equilibria in autonomous Hamiltonian systems

Here we particularize the Leray-Schauder degree theory for periodic Hamiltonian systems that are homotopic with an autonomous one. Concretely, consider a system of the form

$$S\dot{x}(t) = \nabla\mathcal{H}_\lambda(t, x(t)), \quad x(0) = x(T), \quad \lambda \in [0, 1],$$

where S is the standard symplectic matrix in $\mathbb{R}^{2N}$, $\mathcal{H}_\lambda : \mathbb{R} \times \mathcal{O} \to \mathbb{R}$ is T-periodic in t, $\mathcal{O} \subseteq \mathbb{R}^{2N}$ is an open set, and the gradient operator is computed in x. Furthermore, $\mathcal{H}_\lambda$ is a homotopy in λ verifying that:

i) $\mathcal{H}_1(t, x)$ is such that its gradient is a Carathéodory function.

ii) $\mathcal{H}_0(x)$ is autonomous and of class $\mathcal{C}^2$ in $\mathcal{O}$.

The construction of the topological degree for this kind of systems is a particular case of the general theory for nonlinear perturbations of Fredholm operator, we refer to the reader to Section 2 in [87] for more details. To this aim, let X be the

Banach space of T-periodic functions in $\mathcal{C}\left(\mathbb{R};\mathbb{R}^{2N}\right)$ with the uniform norm, and consider the metric space

$$X_{\mathcal{O}} = \{x \in X : \mathrm{Im}(x) \subset \mathcal{O}\}.$$

As usual, the subspace of constant functions in X is naturally identified with $\mathbb{R}^{2N}$. In addition, given any subset Ω of X, we denote by $\mathbf{cl}_{\mathbb{R}^{2N}}\Omega$ to the closure of Ω in $\mathbb{R}^{2N}$, and by $\partial_X\Omega$ to the boundary of the set in X. Moreover, let $Z_\Omega(\mathcal{H}_0)$ be the set of critical points of $\mathcal{H}_0$ in $\Omega\cap\mathbb{R}^{2N}$. Then, denoting by $\mathbf{sg}$ to the sign function and by $|\mathrm{Hess}f(x)|$ to the determinant of the Hessian matrix of $f(x)$, we present the next result, which we prove at the end of the section.

Theorem 11. (Garzón, Torres, [63, Theorem 4]) *Let $\Omega \subset X_{\mathcal{O}}$ be open, bounded and such that:*

a) There is no $x \in \partial_X\Omega$ solving (1.25)*, for any λ.*

b) All the critical points of $\mathcal{H}_0$ in $\boldsymbol{cl}_{\mathbb{R}^{2N}}\Omega$ are non-degenerate.

c) $\sum\limits_{x\in Z_\Omega(\mathcal{H}_0)} \boldsymbol{sg}\,|Hess\mathcal{H}(x)| \neq 0.$

Then, (1.25) *admits at least one T-periodic solution $x_\lambda(t)$, for any $\lambda \in [0,1]$.*

Remark 11. *Observe that the last two assertions hold when the set $Z_\Omega(\mathcal{H}_0)$ has an odd number of points and they are all non-degenerate. Moreover, b) is equivalent to assume that 0 is a regular value for $\nabla\mathcal{H}_0$ in $\boldsymbol{cl}_{\mathbb{R}^{2N}}\Omega$, which is a necessary condition to define the Brouwer degree of a function of class $\mathcal{C}^1$ in $\Omega\cap\mathbb{R}^{2N}$.*

On the other hand, if the homotopy is of the form

$$\mathcal{H}_\lambda(,t,x) = f_\lambda(t,x)^\alpha, \text{ with } \alpha \in \mathbb{R}\setminus\{0\},$$

Theorem 11 is adapted as follows.

Corollary 2. (Garzón, Torres, [63, Corollary 2]) *Let $\Omega \subset X_{\mathcal{O}}$ be open, bounded and such that there is no $x \in \partial_X\Omega$ solving* (1.25) *for any λ. Moreover, let $f_0(t,x)$ be a function without critical points at level 0 in $\boldsymbol{cl}_{\mathbb{R}^{2N}}\Omega$ and that verifies b and c in Theorem 11. Then,* (1.25) *admits at least one T-periodic solution $x_\lambda(t)$, for any $\lambda \in [0,1]$.*

Proof. The proof relies on the fact that the hypotheses about $f_0(x)$ are equivalent to assume that $\mathcal{H}_0(x) = f_0^\alpha(x)$ verifies *b* and *c*. To this aim, it is enough to compute the Hessian of $\mathcal{H}_0(x)$ and make some observations. Firstly, computing the partial derivatives of first order, we have that

$$\partial_{x_i}\mathcal{H}_0 = \partial_{x_i}f_0^\alpha = \alpha f_0^{\alpha-1}\partial_{x_i}f_0 = \alpha\mathcal{H}_0^{1-\alpha^{-1}}\partial_{x_i}f_0.$$

Then, if there is no $x \in \mathcal{Z}_\Omega(f_0)$ such that $f_0(x) = 0$, the set of critical point $\mathcal{Z}_\Omega(\cdot)$ coincides for $f_0(x)$ and $\mathcal{H}_0(x)$. On the other hand,

$$\partial^2_{x_j x_i}\mathcal{H}_0 = \alpha(\alpha-1)\mathcal{H}_0^{1-2\alpha^{-1}}\partial_{x_j}f_0\partial_{x_i}f_0 + \alpha\mathcal{H}_0^{1-\alpha^{-1}}\partial^2_{x_j x_i}f_0,$$

and

$$\mathrm{Hess}\mathcal{H}_0(x) = \alpha\mathcal{H}_0^{1-\alpha^{-1}}(x)\mathrm{Hess}f_0(x).$$

Therefore,

$$\mathrm{Hess}\mathcal{H}_0(x) = 0 \Leftrightarrow \mathrm{Hess}f_0(x) = 0,$$

and Corollary 2 follows directly from this identity.

■

To conclude, we prove Theorem 11.

Proof. Firstly, some definitions are needed in order to rewrite (1.25) in an abstract form. To this aim, let L be the derivative operator, that is Fredholm and of index 0, and is defined in the subspace $D(L) = X \cap \mathcal{C}^1\left(\mathbb{R};\mathbb{R}^{2N}\right)$. We also introduce the mean value projector in X as Q, i.e.

$$Qx = \frac{1}{T}\int_0^T x(t)dt,$$

which verifies that $L+Q : D(L) \to X$ is a bijection. To calculate $(L+Q)^{-1}$, let $\mathcal{K}$ be the inverse of $L|_{D(L)\cap\mathrm{Ker}Q}$, defined in the subspace of X of functions with null mean value. Then, it is standard to see that $(L+Q)^{-1} = Q + \mathcal{K}(I-Q)$, where I is the identity projector.

On the other hand, the Nemitsky operators $N_\lambda : X_{\mathcal{O}} \to X$ associated to (1.25) are the symplectic orthogonal gradients of $\mathcal{H}_\lambda(t,x)$, i.e.

$$N_\lambda x = -S\nabla\mathcal{H}_\lambda(t,x), \quad \forall x \in X_{\mathcal{O}}.$$

For any λ, the good properties of $\mathcal{H}_\lambda$ and the Arzelà-Ascoli Theorem imply the L-compactness of N_λ and then (1.25) can be reformulated as the fixed point problem:

$$x = \mathcal{F}_\lambda x := Q(I+N_\lambda)x + \mathcal{K}(I-Q)N_\lambda x, \quad x \in D(L)\cap X_{\mathcal{O}}.$$

Note that $\mathcal{F}_\lambda : X_{\mathcal{O}} \to X$ is completely continuous by the same arguments. Therefore, for any λ and any open and bounded subset Ω in $X_{\mathcal{O}}$ such that $0 \notin (I-\mathcal{F}_\lambda)\left[D(L)\cap\partial_X\Omega\right]$, the coincidence degree of $L-N_\lambda$ in $D(L)\cap\Omega$ is well defined as the Leray-Schauder degree of $I-\mathcal{F}_\lambda$:

$$d\left(L-N_\lambda, D(L)\cap\Omega\right) := d\left(I-\mathcal{F}_\lambda, D(L)\cap\Omega\right). \tag{1.26}$$

Furthermore, if Ω is such that $0 \notin \cup_{\lambda\in[0,1]}(I-\mathcal{F}_\lambda)\left[D(L)\cap\partial_X\Omega\right]$, (1.26) is well defined for all λ. Then, as the degree is invariant by homotopy, we have that

$$d\left(L-N_\lambda, D(L)\cap\Omega\right) = d\left(L-N_0, D(L)\cap\Omega\right) = d_B\left(g, \mathbb{R}^{2N}\cap\Omega\right), \tag{1.27}$$

where the right term denotes the $2N$-dimensional Brouwer degree of the function $g(x) = -S\nabla\mathcal{H}_0(x)$. Moreover, the last equality follows from [22, Theorem 1]. Assume now that 0 is a regular value of g, i.e. $\mathcal{H}_0$ satisfies $b)$. Then (1.27) is easily computed by classical properties of Brouwer degree:

$$d_B\left(g, \mathbb{R}^{2N} \cap \Omega\right) = \sum_{x \in \mathcal{Z}_\Omega(\mathcal{H}_0)} \mathbf{sg}|\nabla g(x)| = -\sum_{x \in \mathcal{Z}_\Omega(\mathcal{H}_0)} \mathbf{sg}|\mathrm{Hess}\mathcal{H}_0(x)|.$$

To conclude, if this sum is not null, $0 \in (L - N_\lambda)\,[D(L) \cap \Omega]$ for all $\lambda \in [0,1]$ and the theorem is proven. ■

Chapter 2

Periodic solutions in the Lorentz force equation with isolated singularities

The present chapter corresponds to the first article of this manuscript: *Periodic solutions for the Lorentz force equation with singular potentials*, published in the scientific journal *Nonlinear Analysis: Real worlds applications*, [62]. The cited research, co-authored with Pedro J. Torres, provides sufficient conditions for the existence of periodic solutions of the Lorentz force equation for a long class of electromagnetic fields assuming isolated singularities at the origin. The basic assumptions cover relevant physical models with singularities like Coulomb-like electric potentials or the magnetic dipole.

2.1 The main result.

We consider the electric field such that

$$E(t, q) = -\nabla\Phi(q) + h(t),$$

with $\Phi \in \mathcal{C}^1(\mathbb{R}^3 \setminus \{0\}, \mathbb{R})$ and $h \in L^1([0, T], \mathbb{R}^3)$. Moreover, the scalar potential Φ satisfies the assumptions:

h$_1$) $\lim_{|q|\to\infty} |\nabla\Phi(q)| = 0$.

h$_2$) $q \cdot \nabla\Phi(q)$ is negative for any q and there exist $c_0, \varepsilon_0 > 0$ and $\gamma \geq 1$ such that $q \cdot \nabla\Phi(q) \leq -c_0|q|^{-\gamma}$ for any $|q| < \varepsilon_0$.

The canonical model verifying both hypothesis is the Coulomb potential $\Phi(q) = c_0|q|^{-1}$, which solves (1.2) for $\vec{J} = 0$ and $\rho = c_0\delta_0$, i.e. it gives the electromagnetic field generated by a static charge placed at the origin. On the other hand, the magnetic field $B \in \mathcal{C}([0, T] \times \mathbb{R}^3 \setminus \{0\})$ is required to satisfy the next hypotheses:

h$_3$) $\limsup_{|q|\to\infty} |B(t,q)| < C_B$, for some constant $C_B > 0$.

h$_4$) There exist $\varepsilon_1, c_1 > 0$ and $\beta \in (0,\gamma)$ such that $|B(t,q)| \leq c_1 |q|^{-\beta-1}$, for all t when $|q| < \varepsilon_1$.

These conditions are fulfilled in particular by a bounded magnetic force, like for instance an ABC magnetic field. In fields like Astrophysics and Plasma Physics, ABC magnetic fields play an important role and may serve as a relevant example of bounded magnetic fields. Even in the absence of electric field, it is known that ABC magnetic fields may generate complex dynamics, including chaotic motion and Arnold diffusion, see for instance [36,80,81]. Other examples of bounded magnetic fields have been studied in the literature [13, 27], revealing a rich dynamics. Moreover, condition h$_4$ enables a singularity of the magnetic field near the origin, and for instance the magnetic dipole field (in its many variants [109]) is covered by our assumptions.

Theorem 12. *Assume (h_1-h_4). Then, for any $h \in L^1([0,T];\mathbb{R}^3)$ such that $\left|\overline{h}\right| > C_B$, the Lorentz force equation* (1.1) *admits at least one T-periodic solution.*

The proof relies on the previously named global continuation theorem for periodic perturbations of autonomous system due to Capietto, Mawhin and Zanolin [22]. To this purpose, it is necessary to derive a priori bounds for the position and momentum of any eventual T-periodic solution of a suitable homotopic system that drives the original problem to an autonomous system. This is developed in Section 2.2 by means of similar techniques to that employed in [68, 115]. In our case, the dissipative effect assumed in the cited references is replaced by the relativistic effect. The proof finishes in Section 2.3 with the computation of the Brouwer degree of the vector field.

2.2 A priori bounds

We begin by defining the homotopic system:

$$\frac{d}{dt}\phi(\dot{q}(t)) = -\nabla\Phi_\lambda(q(t)) + h_\lambda(t) + \lambda\left(\dot{q}(t) \times B(t,q(t))\right), \quad \lambda \in [0,1], \tag{2.1}$$

where $\Phi_\lambda(q) = \lambda\Phi(q) + (1-\lambda)c_0|q|^{-1}$, $h_\lambda(t) = \lambda h(t) + (1-\lambda)\overline{h}$. It is clear that the original LFE corresponds to $\lambda = 1$.

The objective of this section is to find uniform (not depending on λ) a priori bounds for the T-periodic solutions of (2.1).

Upper bound

By h_1) and h_3), there exists $R > 0$ (not depending on λ) such that

$$|B(t,q)| < C_B \quad \text{and} \quad |\nabla\Phi_\lambda(q)| < |\overline{h}| - C_B$$

for all $t \in [0,T]$ and any $|q| > R$. Suppose that $q(t)$ is a solution of (2.1) such that doesn't belong to $B_R(0)$, for all t. Note that $B_R(0)$ denotes the ball of radius R and centered at the origin. Then, by integrating (2.1) in the whole period, we obtain

$$0 = -\int_0^T \nabla\Phi_\lambda(q(t))\ dt + \lambda \int_0^T \dot{q}(t) \times B(t,q(t))\ dt + T\overline{h},$$

so

$$\left|\int_0^T \nabla\Phi_\lambda(q(t))dt\right| = \left|T\overline{h} + \lambda\int_0^T \dot{q}(t)\times B(t,q(t))\ dt\right|.$$

Now, bounding both sides we get a contradiction:

$$\left|\int_0^T \nabla\Phi_\lambda(q(t))dt\right| \le \int_0^T |\nabla\Phi_\lambda(q(t))|\,dt < T\left(|\overline{h}| - C_B\right),$$

$$\left|T\overline{h} + \lambda\int_0^T \dot{q}(t)\times B(t,q(t))\ dt\right| \ge T\left|\overline{h}\right| - \lambda\left|\int_0^T \dot{q}(t)\times B(t,q(t))\ dt\right| > T\left(|\overline{h}| - C_B\right).$$

Here we have used that $|\dot{q}(t)| < 1$ for all t. Thus, there exists at least an instant $\tilde{t}$ such that $|q(\tilde{t})| \le R$. By using again the bound for the derivative, we get

$$|q(t)| = \left|q(\tilde{t}) + \int_{\tilde{t}}^t \dot{q}(s)ds\right| < R + T := M.$$

Hence every T-periodic solution of (2.1) belongs to the open ball $B_M(0)$.

Lower bound

It's clear that, by condition h_2), exists a number $\varepsilon > 0$ small enough such that

$$-q\cdot\nabla\Phi(q) \ge c_0|q|^{-1} + c_1|q|^{-\alpha}, \quad \text{for } |q| < \varepsilon,$$

with $\gamma > \alpha > 0$. Note that ε depends of the constants α, c_0 and c_1. In particular, taking $\alpha = \beta$ we get:

$$-q\cdot\nabla\Phi_\lambda(q) + \lambda q\cdot v\times B(t,q) \ge c_0\left(\lambda\,|q|^{-1} + (1-\lambda)\,|q|^{-1}\right) = c_0\,|q|^{-1}, \qquad (2.2)$$

for $|q| \le \varepsilon$ and any $|v| \le 1$.

On the other hand, integrating the scalar product of (2.1) with a solution $q(t)$ over $[0,T]$, we obtain

$$\int_0^T q(t)\cdot \frac{d}{dt}\frac{\dot q(t)}{\sqrt{1-|\dot q(t)|^2}}dt = \int_0^T q(t)\cdot [h_\lambda(t)+\lambda \dot q(t)\times B(t,q(t))]\,dt \\ -\int_0^T q(t)\cdot \nabla\Phi_\lambda(q(t))dt.$$

Integrating by parts on the left-hand side and using the periodicity of the solution, we see that it is a negative number, i.e:

$$0\geq \int_0^T [-q(t)\cdot\nabla\Phi_\lambda(q(t))+\lambda q(t)\cdot \dot q(t)\times B(t,q(t))]\,dt \\ +\int_0^T q(t)\cdot\left[\lambda h(t)+(1-\lambda)\overline{h}\right]dt.$$

Thus, applying inequality (2.2) we can write the following relation:

$$\begin{aligned} I_\varepsilon(q(t)) &= \int_{|q(t)|\leq\varepsilon} q(t)\cdot[-\nabla\Phi_\lambda(q(t))+\lambda\dot q(t)\times B(t,q(t))]\,dt \\ &\leq \left|\int_{|q(t)|>\varepsilon} q(t)\cdot[-\nabla\Phi_\lambda(q(t))+\lambda\dot q(t)\times B(t,q(t))]\,dt \right.\\ &\qquad \left. +\int_0^T q(t)\cdot\left[\lambda h(t)+(1-\lambda)\overline{h}\right]dt\right| \\ &< \; Tc_o\varepsilon^{-1}+TC_{\nabla\Phi,B}+M\|h\|_1, \end{aligned}$$

where

$$C_{\nabla\Phi,B} := \max_{\varepsilon<|q|<M}\left(|\nabla\Phi(q)|+|B(t,q)|\right),$$

and $\|\cdot\|_1$ denotes the L_1 norm on $[0,T]$. This quantity is finite because ε is fixed and the fields $\nabla\Phi$ and B are continuous in that set. Furthermore, this estimation is independent of λ.

Assume now that there is an interval $[t_1,t_2]$ where the particle enters the ball $B_\varepsilon(0)$, i.e:

$$\big|q(t_1)\big| = \varepsilon, \quad |q(t)|<\varepsilon,\ \forall t\in(t_1,t_2].$$

Then, taking $t\in(t_1,t_2)$ we can write:

$$\begin{aligned} \Big|\ln|q(t)|\Big| &= \left|\ln|q(t_1)|+\int_{t_1}^t \frac{q(t)\cdot\dot q(t)}{|q(t)|^2}dt\right| \leq \Big|\ln|\varepsilon|\Big| + \int_{t_1}^t |q(t)|^{-1}\,dt \\ &\leq |\ln\varepsilon| + \int_{|q(t)|\leq\varepsilon} |q(t)|^{-1}\,dt \leq |\ln\varepsilon| + \frac{1}{c_0}I_\varepsilon(q(t)) \\ &< |\ln\varepsilon| + T\varepsilon^{-1} + \frac{TC_{\nabla\Phi,B}+M\|h\|_1}{c_0}. \end{aligned}$$

Since it is a finite bound, it follows the existence of a strictly positive lower estimate for the module of $q(t)$. More concretely,

$$|q(t)| > m := \exp\left[-|\ln \varepsilon| - T\varepsilon^{-1} - \frac{TC_{\nabla\Phi,B} + M\|h\|_1}{c_0}\right].$$

Hence, we conclude that every periodic solution of (2.1) is bounded from below by m and, in particular, the singularity point is always avoided.

Bound for the momentum

For a given T-periodic solution $q(t)$ of the homotopic system (2.1), the quantity $p(t) = \phi(\dot{q}(t))$ has a neat physical interpretation as the relativistic momentum of the particle. For our purposes, explicit bounds for the momentum will be needed as well.

First, it is easy to verify that

$$|-\nabla\Phi_\lambda(q) + \lambda\left(\dot{q}(t) \times B(t, q(t))\right)| \leq |\nabla\Phi(q)| + \frac{c_0}{|q|^2} + |B(t,q)| =: H(t,q).$$

Then, by making use of the bounds obtained for the position of the particle $q(t)$, one has

$$|-\nabla\Phi_\lambda(q) + \lambda\left(\dot{q}(t) \times B(t, q(t))\right)| \leq L =: \max_{\substack{m\leq|q|\leq M,\\ t\in[0,T].}} H(t,q).$$

Assume now that $t_0 \in [0,T]$ is a critical point of $q(t)$. Then,

$$\begin{aligned}
|p(t)| &= |\phi(\dot{q}(t))| = \left|\int_{t_0}^{t} \left[-\nabla\Phi_\lambda(q) + h_\lambda(s) + \lambda\left(\dot{q} \times B(s,q)\right)\right] ds\right| \\
&\leq \int_0^T |-\nabla\Phi_\lambda(q) + h_\lambda(s) + \lambda\left(\dot{q} \times B(s,q)\right)| \, dt \\
&< TL + 2\|h\|_1 =: P.
\end{aligned}$$

Of course, this estimate is independent of λ.

2.3 Global continuation and topological degree

In this section, we are going to prove the main result by the global continuation theorem [22, Th 2]. Firstly, let us write (2.1) as a first-order system with position $q(t)$ and relativistic momentum $p(t) = \phi(\dot{q}(t))$ as the new coordinates,

$$\begin{aligned}
\dot{q}(t) &= \phi^{-1}(p(t)), \\
\dot{p}(t) &= -\nabla\Phi_\lambda(q(t)) + h_\lambda(t) + \lambda\left(\phi^{-1}(p(t)) \times B(t,q(t))\right).
\end{aligned} \tag{2.4}$$

This is a system of six differential equations. On the other hand, let us define the Banach space $X = \{x \in C([0,T],\mathbb{R}^6) : x(0) = x(T)\}$. By the results of Section 2.2, every T-periodic solution of (2.4) is contained into the open and bounded set

$$\Omega = \{(p,q) \in X : m < |q(t)| < M, |p(t)| < P \text{ for all } t \in [0,T]\}.$$

Now, by defining $x = (p,q)$ and the function

$$f(t,x;\lambda) = (\phi^{-1}(p), -\nabla\Phi_\lambda(q) + h_\lambda(t) + \lambda\left(\phi^{-1}(p) \times B(t,q)\right),$$

system (2.4) reads simply

$$\dot{x}(t) = f(t,x(t);\lambda).$$

To apply the theorem, it remains to prove that

$$d_B(f_0, \Omega \cap \mathbb{R}^6, 0) \neq 0,$$

where

$$f_0(x) = f(t,x;0) = \left(\phi^{-1}(p), \overline{h} + c_0\frac{q}{|q|^3}\right).$$

Note that

$$\phi^{-1}(p) = \frac{p}{\sqrt{1+|p|^2}},$$

hence f_0 is of class C^∞ in the domain $\Omega \cap \mathbb{R}^6$, and its partial derivatives can be calculated explicitly:

$$\begin{aligned}
\frac{\partial f_0^i}{\partial p_j}(p,q) &= \delta_{ij}\left(1+|p|^2\right)^{-1/2} - p_i p_j\left(1+|p|^2\right)^{-3/2}, \\
\frac{\partial f_0^{1+i}}{\partial q_j}(p,q) &= \delta_{ij}|q|^{-3} - 3q_i q_j |q|^{-5}, \qquad i,j = 1,2,3,
\end{aligned}$$

where δ_{ij} denotes the Kronecker delta function. As the other derivatives are identically zero, the Jacobian matrix ∇f_0 is a block matrix with determinant

$$|\nabla f_0|(p,q) = -2|q|^{-9}\left[\left(1+|p|^2\right)^{-3/2} - |p|^2\left(1+|p|^2\right)^{-5/2}\right]. \tag{2.5}$$

On the other hand, it is easy to check that $x_0 := \left(0, -\overline{h}\left|\overline{h}\right|^{-3/2}\sqrt{c_0}\right)$ is the unique zero of f_0. Replacing it in (2.5), we see that the determinant is negative, which implies that 0 is a regular value for f_0. Then, by a classical property of the Brouwer degree, we get that

$$d_B(f_0, \Omega \cap \mathbb{R}^6, 0) = \mathbf{sg}\,|\nabla f_0(x_0)| = -1 \neq 0.$$

Finally, applying the global continuation theorem [22, Theorem 2], we conclude the proof.

Chapter 3

The wire model. The Newtonian approach and the existence of radially periodic solutions of twist type

Essentially, this chapter corresponds to the paper *Motions of a charged particle in the electromagnetic field induced by a non-stationary current*, co-authored with Stefano Marò and published in the scientific journal *Physica D, Nonlinear Phenomena*, [61]. There it is studied the non-relativistic dynamics of a charged particle in the electromagnetic field induced by a time-periodic oscillating (AC-DC) current $\vec{J}$ along an infinitely long and infinitely thin straight wire. The motions are described by the Newton-Lorentz equation

$$\ddot{q} = E(t, q) + \dot{q} \times B(t, q). \tag{3.1}$$

We shall prove that many features of the integrable time independent case are preserved. More precisely, introducing cylindrical coordinates, we obtain the existence of (non-resonant) radially periodic motions that are also of twist type. In particular, these solutions are Lyapunov stable and accumulated by subharmonic and quasiperiodic motions. Furthermore, the corresponding electromagnetic field is rigourously obtained by solving the Maxwell's equations with the noncompact supported current distribution $\vec{J}$ as data.

The chapter is organized as follows. In Section 3.2 we present the problem in a rigorous way and state our main results in Proposition 1 and Theorems 13-14. Section 3.3 is dedicated to prove Proposition 1, where we get a formulation and some properties of the electromagnetic field for the time dependent case, in terms of the vectorial potential, that solves Maxwell's equations (1.2) in the distributional sense. The Hamiltonian structure of the problem is discussed in Section 3.4 together with the reduction to a time dependent problem with one degree of freedom. Finally, the proofs of the theorems are given in Section 3.5.

3.1 Introduction

Given a current density, finding an explicit formulation for the associated electromagnetic field is a major problem. However, assuming stationary currents in electrically neutral wires, it turns out that the electric field vanishes and the magnetic field can be computed via the Biot-Savart law. In particular, the field $B(q)$ has an explicit formulation in the case of a straight wire, and the magnetic lines are circles around it. In this case, equation (3.1) turns out to be an autonomous integrable Hamiltonian system and the corresponding dynamics have been studied extensively in [3, 64, 65]. More precisely, the conservation of energy, linear and angular momenta implies that the particle cannot collide with the wire. Moreover, the motions of particles with non-null angular momentum are helicoidal: radially periodic, turning around the wire and linearly definitely increasing in the direction parallel to the wire, having that each motion is confined between two cylinders and radially stable.

The introduction of a periodic dependence upon time in the current produces complications both in the computation of the electromagnetic field and in the dynamics given by the Newton-Lorentz equation. First of all, even if there is no charge density, the oscillations of the current generate an electric field and, therefore, the regime is no longer magnetostatic. In particular, Biot-Savart law does not hold in the non stationary case. Solutions of time dependent Maxwell's equations can be rigorously obtained by (1.4) and (1.9) for compactly supported current distributions, via the introduction of retarded potentials. In our case the wire does not have compact support being infitely long, but we shall prove that the corresponding retarded potential still gives a solution of Maxwell's equations, at least in the distributional sense. This is performed by an approximation procedure and gives a rigorous justification of the model we are considering.

From the point of view of dynamics, equation (3.1) is still Hamiltonian, but time dependent and no more integrable. As a consequence of the periodic dependence, resonances are introduced and collisions with the wire cannot, in principle, be avoided. In general, the motions induced by a time dependent electromagnetic field have not been studied extensively. In the first chapter of this manuscript were discussed some of the results obtained through variational techniques in relativistic regimes in [7, 8], where critical point and Lusternik-Schnirelman theories are developed for periodic and Dirichlet boundary conditions. However, in these works, the electromagnetic fields are assumed to be regular. This is not satisfied by our problem since collisions with the wire produce singularities. Concerning singular fields, in the second chapter we developed the paper [62], where periodic solutions are found. However, only isolated singularities are considered there, not covering the present case of an infinite wire. This case has been studied in [71, 76, 78] for a polarized neutral atom under the presence of an electromagnetic field, which is given by a time dependent charge density without current.

In the present research, we show that several aspects of the integrable dynamics can be recovered in the time dependent case, at least when the current is supposed to be a small perturbation of a constant one. We prove that there exist many motions of the particle with periodic and Lyapunov stable radial component. As a byproduct of our analysis we will also get the existence of solutions that are subharmonic or quasiperiodic in the radial component.

The cylindrical symmetry of the problem is still present in the time dependent case, so that angular and linear momentum are still preserved. Therefore, equation (3.1) can be reduced to a time dependent Hamiltonian system with one degree of freedom describing the radial component of the solutions. Considering this fact, stability will be understood as Lyapunov stability of the radial component in the whole phase space, not restricting to fixed levels of angular and linear momenta.

To prove the result we will first work on the reduced system describing the radial component of the motion for fixed values of the momenta. We get the existence of periodic solutions for the time dependent current, through local continuation of equilibria for the (integrable) case of constant current. At this stage we will exclude some resonances and it will be important the assumption of the time dependent current being a small perturbation of a constant one. The stability of the just obtained periodic solutions is proved via the third approximation method, introduced by Ortega in [96] and generalized in [114] to a version that will fit in our problem, that it is included in Chapter 1.3.2 of this manuscript. More precisely, we shall get that the periodic solutions are of twist type, getting also the existence of subharmonic and quasiperiodic solutions. At this stage, it will be necessary to exclude other resonances. We recall here that a periodic solution of twist type corresponds to an elliptic fixed point of the associated Poincaré map, that is accumulated by KAM invariant curves. Hence, it is Lyapunov stable and accumulated by subharmonic and quasiperiodic solutions, see [111]. This technique has also been used in [78] for the case of a time dependent charge density in the wire. Finally, adapting an argument in [111] we also prove that stability of twist type for fixed values of the momenta implies stability in the whole phase space, allowing variations of the momenta. In conclusion, we gave an analytical study of the dynamics and some stability properties induced by a time dependent current. This represents a counterpart of the well known results for constant currents. We believe that our techniques could be adapted to the study of different configurations of the wire, such as the case of a circular wire.

3.2 Statement of the main results

Let us consider an infinitely long straight wire carrying a current of the form $I_0 + kI(t)$, where $I_0 > 0$, $k \geq 0$ are constants and $I(t) \in \mathcal{C}^n(\mathbb{R}/T\mathbb{Z})$, with arbitrary

$n \in \mathbb{N}$, satisfies

$$\int_0^T I(t)dt = 0. \tag{3.2}$$

Without loss of generality, let us assume the wire centered in the z-axis. From a mathematical perspective, it is natural to define the current density $\vec{J} = (0, 0, J)$ as the following vectorial distribution in the space-time:

$$J(f) = \int_{\mathbb{R}^2} [I_0 + kI(t)] f(t, 0, 0, z)\, dt\, dz, \qquad \text{for every } f \in \mathcal{C}_0^\infty\left(\mathbb{R}^4\right). \tag{3.3}$$

Remark 12. *Observe that* (3.3) *can be understood as a generalization of the Dirac delta concept for the present situation of a straight thin wire.*

On the other hand, we also assume the wire electrically neutral, i.e., at every time, every segment of it contains as many electrons as protons, and so the charge density ρ is null. Regarding the charge continuity equation (1.3), we have

$$\nabla \cdot \vec{J}(f) = \partial_z J(f) = J(\partial_z f) = 0, \qquad \text{for every } f \in \mathcal{C}_0^\infty\left(\mathbb{R}^4\right),$$

then the couple $(0, \vec{J})$ is physically admissible. According to this, Maxwell's equations are reduced to

$$\begin{cases} \nabla \cdot B = 0, & \nabla \times E + \partial_t B = 0, \\ \nabla \cdot E = 0, & \nabla \times B = \mu_0 \vec{J} + \partial_t E, \end{cases} \tag{3.4}$$

with the boundary conditions of decaying at infinity:

$$\lim_{|q|\to\infty} E(t, q) = \lim_{|q|\to\infty} B(t, q) = 0, \quad \text{for every } t \in \mathbb{R}.$$

As said in Chapter 1, since $\vec{J}$ is not compactly supported, (3.4) cannot be solved directly by its convolution with (1.8). Here we overcome this through a sequence of approximated problems with compact supported data that gives (1.2) in the limit, see Section 3.3 for the details.

To state our results, let us introduce the cylindrical coordinates $q = (r\cos\theta, r\sin\theta, z)$ and its associated basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. Moreover, we shall use the next notation to illustrate the delay effects:

$$f\left[\alpha\left(t, r, \tau\right)\right] := f\left(\alpha\left(t - \sqrt{\tau^2 + r^2}\right)\right), \quad \text{for } \alpha \in \mathbb{R},$$

denoting by $f[t, r, \tau]$ the case $\alpha = 1$.

Proposition 1. *Suppose $I(t) \in \mathcal{C}^n(\mathbb{R}/T\mathbb{Z})$ with $n \geq 1$, and satisfying* (3.2)*. Then,* (3.4) *admits as unique distributional solution to the electromagnetic field given by* (1.4)*, with $\Phi \equiv 0$ and $\vec{A}(t, q) = A(t, r)\boldsymbol{e}_z$, where*

$$A(t, r) = -\frac{\mu_0}{2\pi}[a_0(r) + ka(t, r)],$$

and

$$a_0(r) = I_0 \ln r, \quad a(t,r) = \int_0^\infty \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}} d\tau. \tag{3.5}$$

Moreover, the function $a(t,r)$ in (3.5):

- *is well defined and belongs to $\mathcal{C}^n(\mathbb{R}\times\mathbb{R}^+)$;*
- *is T-periodic in t;*
- *if $n \geq 2$, satisfies the following estimate: there exists a constant $C > 0$ such that*
$$|a(t,r)| \leq C|\ln(r)| \text{ for } r \ll 1.$$

Concerning the dynamical system (3.1), it is well known that in the stationary case corresponding to $k = 0$, there exist solutions that move around the wire along cylindrical helices, see for instance [3, 64, 65]. More generally, the motions of the particle with non-null angular momentum are helicoidal: periodic in the r-coordinate, turning around the z-axis and linearly increasing in the z-coordinate as $t \to \infty$. This comes from the fact that the angular momentum $L = r^2\dot{\theta}$, the linear momentum $p_z = \dot{z} - \mu_0(2\pi)^1 a_0(r)$ and the energy $E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$ are first integrals. In particular, the motion is confined between two cylinders of radii r_1, r_2 depending on the value of the integrals L, p_z, E and the period of the radial variable depends continuously on these values.

We are going to show how this picture is modified in the non stationary case $k > 0$. To state the results, we note that the energy is no more preserved, while the angular momentum $L = r^2\dot{\theta}$ and the linear momentum $p_z = \dot{z} + A(t,q)$ are first integrals in the present case also (see Section 3.4). Note that in the expression of p_z we have introduced the potential $A(t,q)$ such that $B = \nabla \times \vec{A}$ (see Section 3.3).

Let us first introduce the kind of solutions that we shall study.

Definition 4. *A solution $q(t) = (r(t), \theta(t), z(t))$ of (3.1) with angular momentum L and linear momentum p_z is called a (L, p_z)-solution. Moreover it is*

(i) radially T-periodic if $r(t+T) = r(t)$ for every $t \in \mathbb{R}$;

(ii) radially stable if for every $\epsilon > 0$ there exists a neighborhood $\mathcal{U}$ of $(r(0), \dot{r}(0), L, p_z)$ such that, for every $(\tilde{r}_0, \dot{\tilde{r}}_0, \tilde{L}, \tilde{p}_z) \in \mathcal{U}$, each $(\tilde{L}, \tilde{p}_z)$-solution $\tilde{q}(t)$ with $\tilde{r}(0) = \tilde{r}_0, \dot{\tilde{r}}(0) = \dot{\tilde{r}}_0$ satisfies

$$|\tilde{r}(t) - r(t)| + |\dot{\tilde{r}}(t) - \dot{r}(t)| < \epsilon, \qquad \text{for every } t > 0.$$

(iii) radially $(\bar{r}, p, q)$-subharmonic, for given $(q,p) \in \mathbb{N}^2$ and $\bar{r} > 0$, if $r(t)$ is qT-periodic, not lT-periodic for every $l = 1, \ldots, q-1$ and such that the function $r(t) - \bar{r}$ has $2p$ zeros in $[0, qT]$;

(iv) *radially (generalized) quasiperiodic with frequencies* $(1, \omega)$, *for some positive* $\omega \in \mathbb{R}$, *if the function* $r(t)$ *is (generalized) quasiperiodic.*

Remark 13. 1. *Condition (ii) means Lyapunov stability in the radial direction w.r.t. small variations of the initial radial coordinates* $(r_0, \dot{r}_0)$ *and values of the integrals* L, p_z. *Since* L *and* p_z *do not depend on the variables* θ, z, *stability is also guaranteed w.r.t. arbitrary variations of* (θ_0, z_0).

2. *The standard definition of quasiperiodic solution can be found in [111]. We decided not to include it to avoid some technicality. We just say that quasiperiodic solutions come in families parametrised by some* $\xi \in \mathbb{R}$. *The solution is generalized quasiperiodic if the dependence on* ξ *is not continuous and this depends on the arithmetic properties of the number* ω. *See [84, 85, 95, 97] for further discussions on it.*

Moreover, we will have to restrict the set of parameters, so that we introduce also the following:

Definition 5. *The triplet* $(\bar{r}, L, p_z)$ *with* $\bar{r} > 0$ *and* $L \neq 0$ *is admissible if*

$$\bar{r}^2 \left[p_z + \frac{\mu_0}{2\pi} I_0 \ln \bar{r} \right] = \frac{2\pi}{\mu_0} \frac{L^2}{I_0}.$$

An admissible triplet is non-resonant if

$$T \notin \left\{ \frac{n}{\bar{r}} \sqrt{\frac{2L^2}{\bar{r}^2} + \frac{\mu_0^2}{4\pi^2} I_0^2},\ n \in \mathbb{N} \right\}.$$

If in addition

$$\frac{\sqrt{2L^2 + \boldsymbol{C}^2 \bar{r}^2}}{\bar{r}^2} < \frac{\pi}{2T}, \tag{3.6}$$

with $\boldsymbol{C} = \mu_0 I_0 (2\pi)^{-1}$, *then the admissible resonant triplet* $(\bar{r}, L, p_z)$ *is said strongly non-resonant.*

Remark 14. *Fixing* $\bar{r} > 0$, *it is easy to show that* $L^2 = \boldsymbol{C}^2 \bar{r}^2$, $p_z = \boldsymbol{C}(1 - \ln \bar{r})$ *complete an admissible triplet. In this case, the non resonant condition reads as*

$$\bar{r} \notin \left\{ n \frac{\sqrt{3} \mu_0 I_0}{T},\ n \in \mathbb{N} \right\}$$

and the strong non-resonant condition becomes

$$\frac{\bar{r}}{\mu_0 I_0} > \sqrt{3} \frac{T}{2\pi^2}.$$

Now we are ready to state our first result concerning the existence of radially T-periodic solutions

Theorem 13. *Consider a current density $\vec{J}$ of the form* (3.3) *with $I(t) \in \mathcal{C}^2(\mathbb{R}/T\mathbb{Z})$ satisfying* (3.2).
Then, for every admissible non-resonant triplet $(\bar{r}, L, p_z)$ there exists a number $k_0 > 0$ such that, for every $|k| < k_0$, equation (3.1) *admits a radially T-periodic (L, p_z)-solution $(r_k(t), \theta_k(t), z_k(t))$ continuous in (t, k) with $r_0(t) = \bar{r}$ for every $t \in \mathbb{R}$. Moreover, $\dot{z}(t) = I_0 + \xi_k(t)$ where $\xi_k(t) \to 0$ as $k \to 0$ uniformly in $t \in [0, T]$.*

We will also show that under the strong non-resonant condition we can describe the dynamics close to the just introduced solutions.

Theorem 14. *Consider a current density $\vec{J}$ of the form* (3.3) *with $I(t) \in \mathcal{C}^4(\mathbb{R}/T\mathbb{Z})$ satisfying* (3.2).
Then, for every admissible strongly non-resonant triplet $(\bar{r}, L, p_z)$ there exists a number $k_1 > 0$ such that, for every $|k| < k_1$, the radially T-periodic (L, p_z)-solution $(r_k(t), \theta_k(t), z_k(t))$ coming from Theorem 13 is radially stable. Moreover, there exists $h > 0$ such that

- *for every $(q, p) \in \mathbb{N}^2$ with $p/q < h$, there exists a radially $(\bar{r}, p, q)$-subharmonic (L, p_z)-solution;*
- *for every positive irrational $\omega < h$, there exists a radially (generalized) quasiperiodic (L, p_z)-solution with frequencies $(1, \omega)$.*

The radial component $r(t)$ of all these solutions converge uniformly to $\bar{r}$ as $k \to 0$.

Remark 15. *The following picture comes from Theorem 14. For every admissible strongly non-resonant triplet $(\bar{r}, L, p_z)$ there exist a number $k_1 > 0$ such that:*

- *There exist two sequences $r_k^{(0)} \to \bar{r}$ and $\dot{r}_k^{(0)} \to 0$, defined for $|k| < k_1$, such that every (L, p_z)-solution $(r_k(t), \theta_k(t), z_k(t))$ of* (3.1) *with initial position in the cylinder $\mathcal{C}_k = \{(r, \theta, z) \in \mathbb{R}^3 : \ r = r_k^{(0)}\}$ and initial radial velocity equal to $\dot{r}_k^{(0)}$ is radially T-periodic.*
- *The cylinders $\mathcal{C}_k$ define trapping regions in the following sense. For every $\varepsilon > 0$ there exists δ such that all the $(\tilde{L}, \tilde{p}_z)$-solutions with $|L - \tilde{L}|, |p_z - \tilde{p}_z| < \delta$, initial position in the cylinder $\mathcal{C}_k^\delta = \{(r, \theta, z) \in \mathbb{R}^3 : \ |r - r_k^{(0)}| < \delta\}$ and initial radial velocity $\dot{r}_0$ satisfying $|\dot{r}_0 - \dot{r}_k^{(0)}| < \delta$, are contained in the cylinder $\mathcal{C}_k^\varepsilon = \{(r, \theta, z) \in \mathbb{R}^3 : \ |r - r_k^{(0)}| < \varepsilon\}$ for every time.*
- *Each cylinder $\mathcal{C}_k$ is accumulated by radially subharmonic and radially quasiperiodic solutions.*

Remark 16. *In the unperturbed case $k = 0$, the conservation of the energy E implies that, for fixed values of the momenta p_z, L, there exists one radially constant* *solution and all the other solutions are radially periodic with period depending continuously on the values of E, L, p_z (see [3]). In the perturbed case, the T-periodic dependence on time implies that the only admissible periods are integer multiples of T. Theorems 13 and 14 show that the radially constant solution is continued for $k > 0$ to a radially T-periodic solution. Moreover, there exist radially qT-periodic (for large $q \in \mathbb{N}$) and quasiperiodic solutions. This latter class of solutions is not present in the unperturbed case.*

3.3 The electromagnetic potential

In this section, the electric and magnetic fields generated by the current density $\vec{J}$ are deduced. At first, introducing the potential notation (1.4)-(1.6) in (3.4), the system is uncoupled into the wave equation (1.7), where we can take $\Phi \equiv 0$ because there is no charge density. Consequently, we are left to solve

$$\partial_t^2 \vec{A} - \Delta \vec{A} = \mu_0 \vec{J}, \quad \nabla \cdot \vec{A} = 0, \tag{3.7}$$

with the condition of decaying at infinity

$$\lim_{|q|\to\infty} |\nabla_q \vec{A}(t,q)| = 0, \quad \text{for every } t \in \mathbb{R}.$$

As it was explained in Chapter 1, we cannot proceed here like for compact supported data. Actually, using (1.9) we get

$$\vec{A}(t,r) = \frac{\mu_0}{2\pi}\left[\int_0^\infty \frac{I_0}{\sqrt{\tau^2+r^2}} d\tau + k\int_0^\infty \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}} d\tau\right] \mathbf{e}_z,$$

and the first integral is clearly divergent. On the other hand, by linearity of Maxwell's equations, the solution for (3.4) can be expressed as the sum of the particular solutions for the constant and non constant part in (3.3), that is:

$$J_0(f) = \int_{\mathbb{R}^2} I_0 f(t,0,0,z)dtdz, \quad J_k(f) = k\int_{\mathbb{R}^2} I(t) f(t,0,0,z)dtdz,$$

for every $f \in \mathcal{C}_0^\infty(\mathbb{R}^4)$. Therefore, we shall solve separately the magnetostatic and the time-dependent regime.

3.3.1 Magnenostatic regime. The Biot-Savart law.

Firstly, computing the rotational in the Maxwell-Faraday equation, as the term $\nabla \cdot E$ is zero, one has that

$$-\Delta E(t,q) = -\partial_t (\nabla \times B(t,q)) = -\partial_t^2 E(t,q),$$

where the second equality follows from the Ampère's law for $\vec{J_0} = (0, 0, J_0)$. Therefore, E satisfies the homogeneous wave equation, whose unique distributional solution decaying at infinity is the zero. Consequently, (3.4) is reduced to the next curl-divergence system

$$\nabla \cdot B = 0, \qquad \nabla \times B = \mu_0 \vec{J_0}, \qquad \lim_{|q| \to \infty} |B(q)| = 0. \tag{3.8}$$

Remark 17. *Observe that we just assumed that the current is autonomous, thus system* (3.8) *is common to any magnetostatic situation. Moreover, without loss of generality we can redefine J_0 as follows:*

$$J_0(f) = \int_{\mathbb{R}} I_0 f(0, 0, z) dz, \quad \textit{for every } f \in \mathcal{C}_0^\infty\left(\mathbb{R}^3\right).$$

Concerning (3.8), the magnetic field $B(q)$ can be formally obtained via the Biot-Savart law for the Newtonian potential in $\mathbb{R}^3$, that is

$$B(q) = \nabla \times \vec{A}(q), \quad \text{where} \quad \vec{A}(q) = \frac{-\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\vec{J_0}(q')}{|q - q'|} dq', \tag{3.9}$$

and it solves Poisson's equation assuming certain conditions of decaying at infinity. See for instance [31, Chapter II-Laplace Operator] or [82, Chapter 2.4] for more mathematical details. Nevertheless, applying (3.9) for the non-compact supported distribution $\vec{J_0}$, we get $\vec{A}(q) = (0, 0, A(r))$ with

$$A(r) = \frac{-\mu_0}{2\pi} \int_0^\infty \frac{I_0}{\sqrt{\tau^2 + r^2}} d\tau,$$

that gives a divergent integral for any $r \geq 0$. However, in this case the magnetic field $B(q)$ can be measured and it is well known that

$$B(q) = \frac{\mu_0}{2\pi r^2}(-x_2, x_1, 0), \tag{3.10}$$

which can be obtained by the Biot-Savart formula in $\mathbb{R}^2$ for $A(r) = -\frac{\mu_0}{2\pi} \ln r$. In fact,

$$\begin{aligned} -\frac{2\pi}{I_0 \mu_0} \Delta A(f) &= \int_{\mathbb{R}^3} \ln(r) \Delta f(q) dq = \int_{\mathbb{R}^3} \ln(r) \Delta_{x,y} f(q) dq \\ &= 2\pi \int_{\mathbb{R}} f(0, 0, z) dz = \frac{2\pi}{I_0} J_0(f), \quad \text{for every } f \in \mathcal{C}_0^\infty\left(\mathbb{R}^3\right), \end{aligned}$$

where we have used the logarithm as fundamental solution for the Laplace operator in $\mathbb{R}^2$.

Despite the above heuristic check, let us deduce (3.10) properly using the potential theory in $\mathbb{R}^3$. To this aim, we normalize the constants $I_0, \mu_0 = 1$, without loss of generality. Then, for any $m > 0$, we define the map $\delta_m : \mathcal{C}(\mathbb{R}^3) \to \mathbb{R}$ as follows:

$$\delta_m(f) = \int_{-m}^{m} f(0,0,z)dz.$$

Clearly, $\{\delta_m\}_{m>0}$ is a sequence in $\mathcal{D}'(\mathbb{R}^3)$ that tends to J_0 in the sense of distributions when $m \to \infty$. Besides, let $\vec{A}_m = (0,0,A_m)$ be the sequence of potentials given by the convolution of the Newtonian potential in $\mathbb{R}^3$ with $(0,0,\delta_m)$. Distributionally, that is

$$4\pi A_m(f) = -\delta_m\left(\int_{\mathbb{R}^3} \frac{f(q+q')}{|q'|}dq'\right) = -\int_{[-m,m]}\int_{\mathbb{R}^3} \frac{f(x',y',z+z')}{|q'|}dq'dz,$$

for every $f \in \mathcal{C}_0^\infty(\mathbb{R}^3)$. As f has compact support, from Fubini and the Fundamental theorem of calculus, it follows that $\partial_z A_m \to 0$ when $m \to \infty$.

On the other hand, consider the rotational $B_m = (\partial_y A_m, -\partial_x A_m, 0)$. Computing (3.9), we have that

$$\begin{aligned}\nabla \times B_m(f) &= \left(\partial_{x_3}\partial_{x_1}A_m(f), \partial_{x_3}\partial_{x_2}A_m(f), -\sum_{i=1}^{2}\partial^2_{x_i}A_m(f)\right)\\ &= \nabla(\partial_{x_3}A_m(f)) - (0,0,\Delta A_m(f)),\end{aligned}$$

for every $f \in \mathcal{C}_0^\infty(\mathbb{R}^3)$. Assuming that it exists the limit when $m \to \infty$, then

$$\lim_{m\to\infty} B_m(f) = \lim_{m\to\infty}(0,0,-\Delta A_m(f)) = \vec{J}_0(f), \quad \text{for every } f \in \mathcal{C}_0^\infty\left(\mathbb{R}^3\right), \tag{3.11}$$

and $B(q) = \lim_{m\to\infty} B_m(q)$ would solve (3.8) in the sense of distributions.

In order to justify the above limit, observe that A_m has associated the following function in $L^1_{loc}(\mathbb{R}^3)$:

$$A_m(q) = -\frac{1}{4\pi}\int_{-m}^{m} \frac{dz'}{\sqrt{x^2+y^2+(z-z')^2}}.$$

This means $A_m(f) = \langle A_m, f\rangle$, for every $f \in \mathcal{C}_0^\infty(\mathbb{R}^3)$. Concerning its partial derivatives, it is not difficult to see that they exist for almost every $q \in \mathbb{R}^3$ and that they are locally integrable functions in $\mathbb{R}^3$, so they can also be defined distributionally. Furthermore,

$$
\begin{aligned}
A_m(q) =& -\frac{1}{4\pi}\left[\int_z^m \frac{dz'}{\sqrt{r^2+(z-z')^2}} + \int_{-m}^z \frac{dz'}{\sqrt{r^2+(z-z')^2}}\right]\\
=&\frac{1}{4\pi}\left[\int_0^{z-m} \frac{d\bar{z}}{\sqrt{r^2+\bar{z}^2}} + \int_0^{-z-m} \frac{d\bar{z}}{\sqrt{r^2+\bar{z}^2}}\right]\\
=&\frac{1}{4\pi}\left[\int_r^{\sqrt{r^2+(z-m)^2}} \frac{d\tau}{\sqrt{\tau^2-r^2}} + \int_r^{\sqrt{r^2+(z+m)^2}} \frac{d\tau}{\sqrt{\tau^2-r^2}}\right]\\
=&\frac{1}{4\pi}\left[\ln\left(\sqrt{\tau^2-r^2}+\tau\right)\Big|_r^{\sqrt{r^2+(z-m)^2}} + \ln\left(\sqrt{\tau^2-r^2}+\tau\right)\Big|_r^{\sqrt{r^2+(z+m)^2}}\right]\\
=&\frac{1}{4\pi}\left[\ln\left(z-m+\sqrt{r^2+(z-m)^2}\right)+\ln\left(z+m+\sqrt{r^2+(z+m)^2}\right)\right]-\frac{\ln(r)}{2\pi}.
\end{aligned}
$$

Clearly, the terms depending on m diverge when $m\to\infty$, and so does $A_m(q)$. Nevertheless, observe that their partial derivatives are of the order of $O(m^{-1})$, and then

$$\partial_x A_m \longrightarrow \frac{-1}{2\pi}\partial_x \ln(r), \quad \partial_y A_m \longrightarrow \frac{-1}{2\pi}\partial_y \ln(r), \quad \text{when } m\to\infty,$$

puntually for $r>0$, and also in the sense of distributions. In particular,

$$B_m(x) \longrightarrow B(x) = \frac{1}{2\pi r^2}(-x_2, x_1, 0), \quad \text{when } m\to\infty,$$

recovering then (3.10), that solves (3.8) for $\vec{J}_0$ according to (3.11).

3.3.2 Proof of Proposition 1

Let us begin by studying the properties of $a(t,r)$. Firstly, to prove that the function is well defined, let us see that the integral converges writing it as

$$a(t,r) = \int_0^\infty \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}}d\tau = \int_0^r \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}}d\tau + \int_r^\infty \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}}d\tau.$$

So, the first integral is finite since

$$\left|\int_0^r \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}}d\tau\right| \le \|I\|_\infty \int_0^r \frac{1}{\sqrt{\tau^2+r^2}}d\tau = \|I\|_\infty \ln\left(1+\sqrt{2}\right)$$

and I is bounded. For the second, through integration by parts and denoting $\mathcal{I}(t)$ as a primitive of $I(t)$, we get

$$\int_r^\infty \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}}d\tau = -\left[\frac{\mathcal{I}[t,r,\tau]}{\tau}\right]_r^\infty + \int_r^\infty \frac{\mathcal{I}[t,r,\tau]}{\tau^2}d\tau.$$

Since $I(t)$ has null average, then $\mathcal{I}(t)$ is periodic, so that

$$\left|\int_r^\infty \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}}d\tau\right| = \left|-\left[\frac{\mathcal{I}[t,r,\tau]}{\tau}\right]_r^\infty + \int_r^\infty \frac{\mathcal{I}[t,r,\tau]}{\tau^2}d\tau\right|$$
$$\leq \frac{|\mathcal{I}(t-\sqrt{2}r)|}{r} + \left|\int_r^\infty \frac{\mathcal{I}[t,r,\tau]}{\tau^2}d\tau\right| \leq \frac{2\|\mathcal{I}\|_\infty}{r}. \quad (3.12)$$

Hence, $a(t,r)$ is well defined for $(t,r) \in [0,T] \times (0,+\infty)$. Let us now consider the regularity, by proving that $a(t,r)$ is $\mathcal{C}^k$ on $[0,T] \times (a,b)$ for every $0 < a < b < +\infty$. The integrand function in $a(t,r)$ is continuous for $(t,r,\tau) \in [0,T] \times [a,b] \times [0,+\infty)$ hence, to prove the continuity of $a(t,r)$ is enough to prove that the improper integral is uniformly convergent on $[0,T] \times [a,b]$. This follows integrating by parts, actually, for any $m > 0$ one has:

$$\left|\int_m^{+\infty} \frac{I\left[t,r,\tau\right]}{\sqrt{\tau^2+r^2}}d\tau\right| = \left|\left[\frac{\mathcal{I}[t,r,\tau]}{\tau}\right]_m^\infty + \int_m^\infty \frac{\mathcal{I}[t,r,\tau]}{\tau^2}d\tau\right| \leq \frac{2\|\mathcal{I}\|_\infty}{m}. \quad (3.13)$$

Concerning the derivatives, let us denote by $F_{ij}(t,r,\tau)$ the derivatives of the integrand function w.r.t (t,r) of order (i,j) such that $i+j \leq n$. It can be checked that they are all continuous in $(t,r,\tau) \in [0,T] \times [a,b] \times [0,+\infty)$. As before, we need to check that the corresponding improper integrals converge uniformly. This can be seen by the following observations. First of all, by considering the cases F_{i0} (i.e. the derivatives involve the sole variable t) the uniform convergence is proved as in (3.13), since

$$F_{i0}(t,r,\tau) = \frac{I^{(i)}\left[t,r,\tau\right]}{\sqrt{\tau^2+r^2}}.$$

On the other hand, if a derivative involves the variable r, fixing t and r we have:

$$F_{ij}(t,r,\tau) \sim O\left(\tau^{-2}\right) \qquad \text{when } \tau \to \infty,$$

with $j > 0$. Therefore, uniform convergence follows from the fact that r belongs to a compact set. Finally, the periodicity in t of $a(t,r)$ follows from the periodicity of I.

Note that estimate (3.12) does not imply that $a(t,r)$ is a locally integrable function in r, however, it allows to define it as a distribution in $\mathbb{R}^4$. As usual, considering $f \in \mathcal{C}_0^\infty(\mathbb{R}^4)$, we establish $a(f)$ as the scalar product in $L^2(\mathbb{R}^4)$ of $a(t,r)$ with $f(t,q)$, then:

$$a(f) = \int_{\mathbb{R}^4} f(t,q) \int_0^r \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}}d\tau dt dq + \int_{\mathbb{R}^4} f(t,q) \int_r^\infty \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}}d\tau dt dq$$
$$\leq \ln\left(1+\sqrt{2}\right)\|I\|_\infty \|f\|_{L^1} + 2\|\mathcal{I}\|_\infty \int_{\mathbb{R}^4} \frac{|f(t,q)|}{r}dt dq.$$

Passing to cylindrical coordinates, we obtain the bound

$$a(f) \leq C_f \|f\|_{L^\infty}\left(\|I\|_\infty + \|\mathcal{I}\|_\infty\right),$$

where C_f is a constant that depends on the measure of the support of f.

To study the asymptotic behaviour of $a(t,\cdot)$ near to the singularity, assume $I \in \mathcal{C}^2$ and consider its (uniformly absolutely-convergent) Fourier expansion

$$I(t) = \sum_{j\geq 1} [\alpha_j \cos \lambda_j t + \beta_j \sin \lambda_j t]$$

where $\lambda_j = j2\pi T^{-1}$ and $\{\alpha_j, \beta_j\}_{j\geq 1}$ are the corresponding Fourier coefficients that, by the regularity of I satisfy

$$|\alpha_j|, |\beta_j| \leq \frac{C}{j^2} \text{ for some } C > 0. \tag{3.14}$$

Moreover, $\alpha_0 = 0$ because I has null mean value. Inserting this expansion in the definition of $a(t,r)$ and using the absolute convergence, we get

$$\begin{aligned} a(t,r) &= \int_0^\infty \frac{\sum_{j\geq 1} [\alpha_j \cos[\lambda_j (t,r,\tau)] + \beta_j \sin[\lambda_j (t,r,\tau)]]}{\sqrt{\tau^2+r^2}} d\tau \\ &= \sum_{j\geq 1} \left[\alpha_j \int_0^\infty \frac{\cos[\lambda_j (t,r,\tau)]}{\sqrt{\tau^2+r^2}} d\tau + \beta_j \int_0^\infty \frac{\sin[\lambda_j (t,r,\tau)]}{\sqrt{\tau^2+r^2}} d\tau \right]. \end{aligned}$$

Note that these integrals can be solved explicitly. Actually, performing the change of variable $\tau^2 + r^2 = r^2 \cosh^2 \xi$:

$$\begin{aligned} \int_0^\infty & \frac{\cos[\lambda_j (t,r,\tau)]}{\sqrt{\tau^2+r^2}} d\tau = \int_0^\infty \cos(\lambda_j (t - r\cosh\xi))\, d\xi \\ &= \cos(\lambda_j t) \int_0^\infty \cos(\lambda_j r \cosh\xi)\, d\xi + \sin(\lambda_j t) \int_0^\infty \sin(\lambda_j r\cosh\xi)\, d\xi \\ &= \frac{\pi}{2} [\cos(\lambda_j t)\mathcal{Y}_0(\lambda_j r) + \sin(\lambda_j t)\mathcal{J}_0(\lambda_j r)], \end{aligned}$$

where $\mathcal{J}_0(x), \mathcal{Y}_0(x)$ are the Bessel functions of order zero of first and second kind respectively. Reasoning analogously for the second integral, we obtain:

$$\begin{aligned} a(t,r) = {} & \frac{\pi}{2} \sum_{j\geq 1} [\alpha_j \cos(\lambda_j t) + \beta_j \sin(\lambda_j t)]\, \mathcal{Y}_0(\lambda_j r) \\ & + \frac{\pi}{2} \sum_{j\geq 1} [\alpha_j \sin(\lambda_j t) - \beta_j \cos(\lambda_j t)]\, \mathcal{J}_0(\lambda_j r). \end{aligned}$$

Using (3.14),

$$|a(t,r)| \leq \frac{C\pi}{2} \sum_{j\geq 1} \frac{1}{j^2} [|\mathcal{Y}_0(jr)| + |\mathcal{J}_0(jr)|], \tag{3.15}$$

where the constant $\frac{2\pi}{T}$ in λ_j has been normalized without loss of generality. Inequality (3.15) gives the desired estimate. Actually, $\mathcal{J}_0$ is bounded so that it is enough to estimate, for fixed $r << 1$,

$$\sum_{j\geq 1} \frac{|\mathcal{Y}_0(jr)|}{j^2}.$$

Since $\mathcal{Y}_0(x) \sim \ln(x)$ for $x \ll 1$, if $jr << 1$ we have that

$$\frac{|\mathcal{Y}_0(jr)|}{j^2} \leq \tilde{C}\frac{|\ln(jr)|}{j^2} \leq \tilde{C}\frac{|\ln r|}{j^2}, \quad \text{for some } \tilde{C} > 0.$$

Using also the fact that $\mathcal{Y}_0(x)$ is bounded in the rest of its domain and the convergence of the series $\sum j^{-2}$, we can prove the existence of a constant C such that

$$|a(t,r)| \leq C|\ln(r)|, \quad \text{for all } t \text{ and } r \ll 1.$$

Finally, we deduce that the potential generated by the time-dependent current distribution is $a(t,r)$. We recall that it the solution of (3.7) for $\vec{J_k} = (0,0,J_k)$ such that

$$J_k(f) = k\int_{\mathbb{R}^2} I(t)f(t,0,0,z)dtdz, \quad \text{for every } f \in \mathcal{C}_0^\infty\left(\mathbb{R}^4\right).$$

Again, we cannot compute directly the solution for J_k as a convolution with (1.8), because the data is not of compact support, as it was said many times before. For that reason, we define the sequence $\{J_{k,m}\}_{m\in\mathbb{N}}$ in $\mathcal{D}'\left(\mathbb{R}^4\right)$ as

$$J_{k,m}(f) = k\int_{[-m,m]^2} f(t,0,0,z)J(t)dtdz,$$

which clearly converges to J_k when $m \nearrow \infty$. Fixed $k > 0$, we also denote by a_m the corresponding solution of the wave equation (3.7). By computing (1.9) we can deduce the expression

$$a_m(t,r,z) = \frac{\mu_0}{4\pi}\int_{-m}^{m} \frac{I\left(t-\sqrt{r^2+(z'-z)^2}\right)}{\sqrt{r^2+(z'-z)^2}}dz'.$$

Again, by the change of variable $\tau = z - z'$ and integrating by parts, it is proved that a_m converges uniformly to a when $m \nearrow \infty$, which implies its convergence in the distributional sense. Due to this, the function $a(t,r)$ solves the wave equation for the datum J_k. To conclude, we have to see that the retarded potential $\vec{A}(t,r)$ defined in (3.5) satisfies the Lorenz Gauge, and then it is a solution of Maxwell's equations (3.4). But this is trivial because the two first components of $\vec{A}$ and the scalar potential Φ are zero. So, we only has to compute the derivative $\partial_z A$, which is clearly null.

Remark 18. *Note that when $I(t)$ is a linear combination of sines and cosines, then the function $a(t,r)$ has an explicit representation by means of Bessel functions.*

3.4 Hamiltonian formulation and reduction

From Proposition 1, system (3.1) reduces to

$$\ddot{q}(t) = -\partial_t\vec{A}(t,q(t)) + \dot{q}(t)\times\left[\nabla\times\vec{A}(t,q(t))\right], \quad (t,q) \in \mathbb{R}^4,$$

that is Hamiltonian with

$$H(t,q,p) = \frac{1}{2}\left|p - \vec{A}(t,q)\right|^2.$$

From $\dot{q} = \partial_p H$ we get that the momenta $p = (p_x, p_y, p_z)$ are

$$p = \dot{q} + \vec{A}(t,q).$$

With these assumptions, the Hamilton equations read

$$2H(t,q,p) = p_x^2 + p_y^2 + \left(p_z + \frac{\mu_0}{2\pi}\left(I_0 \ln r + ka(t,r)\right)\right)^2,$$

and

$$\begin{cases} \dot{p}_x = -\frac{\mu_0}{2\pi}\left(p_z + \frac{\mu_0}{2\pi}\left(I_0 \ln r + ka(t,r)\right)\right)\left(\frac{I_0}{r^2} + k\frac{\partial_r a(t,r)}{r}\right)x, \\ \dot{p}_y = -\frac{\mu_0}{2\pi}\left(p_z + \frac{\mu_0}{2\pi}\left(I_0 \ln r + ka(t,r)\right)\right)\left(\frac{I_0}{r^2} + k\frac{\partial_r a(t,r)}{r}\right)y, \\ \dot{p}_z = 0, \\ \dot{x} = p_x, \\ \dot{y} = p_y, \\ \dot{z} = p_z + \frac{\mu_0}{2\pi}\left(I_0 \ln r + ka(t,r)\right). \end{cases}$$

From this,

$$\begin{cases} \ddot{x} = -\frac{\mu_0}{2\pi}\left(p_z + \frac{\mu_0}{2\pi}\left(I_0 \ln r + ka(t,r)\right)\right)\left(\frac{I_0}{r^2} + k\frac{\partial_r a(t,r)}{r}\right)x, \\ \ddot{y} = -\frac{\mu_0}{2\pi}\left(p_z + \frac{\mu_0}{2\pi}\left(I_0 \ln r + ka(t,r)\right)\right)\left(\frac{I_0}{r^2} + k\frac{\partial_r a(t,r)}{r}\right)y, \\ \dot{z} = p_z + \frac{\mu_0}{2\pi}\left(I_0 \ln r + ka(t,r)\right). \end{cases}$$

Observe that the linear momentum p_z is a first integral. Passing in polar coordinates, $(x,y) = r(\cos\theta, \sin\theta)$, we get that the norm of the angular momentum is another first integral:

$$\dot{y}x - \dot{x}y = r^2\dot{\theta} = L.$$

Differentiating twice the equation $r^2 = x^2 + y^2$, and noting that $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$, we get

$$\ddot{r} = \frac{L^2}{r^3} - \frac{\mu_0}{2\pi}\left(p_z + \frac{\mu_0}{2\pi}\left(I_0 \ln r + ka(t,r)\right)\right)\left(\frac{I_0}{r} + k\partial_r a(t,r)\right). \tag{3.16}$$

Hence, we will consider the equations

$$\begin{cases} \ddot{r} = \frac{L^2}{r^3} - \frac{\mu_0}{2\pi}\left(p_z + \frac{\mu_0}{2\pi}\left(I_0 \ln r + ka(t,r)\right)\right)\left(\frac{I_0}{r} + k\partial_r a(t,r)\right), \\ r^2\dot{\theta} = L, \\ \dot{z} = p_z + \frac{\mu_0}{2\pi}\left(I_0 \ln r + ka(t,r)\right). \end{cases} \tag{3.17}$$

Note that $\ddot{r} = -\partial_r V(t,r)$ with $V(t,r) = \frac{L^2}{2r^2} + \frac{1}{2}(p_z + \frac{\mu_0}{2\pi}(I_0 \ln r + ka(t,r)))^2$ and, if $k = 0$, then also the energy

$$E = \frac{1}{2}\dot{r}^2 + V_0(r) = \frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2} + \frac{1}{2}\left(p_z + \frac{\mu_0}{2\pi} I_0 \ln r\right)^2$$

is preserved. If $L \neq 0$, then $V_0(r)$ has only one minimum $\bar{r}$ and $V_0(r) \to \infty$ as $r \to 0, +\infty$. Hence, in the magnetostatic regime, every solution of (3.16) with nonzero angular momentum is periodic. In that case, using the equilibrium $r(t) = \bar{r}$ in system (3.17), we get $\dot{\theta}(t) = L/\bar{r}^2$ and $\dot{z}(t) = p_z + \frac{\mu_0}{2\pi} I_0 \ln \bar{r}$. Consequently, the particle moves on a cylindrical helix when $L \neq 0$ and $\dot{z}(0) = p_z + \frac{\mu_0}{2\pi} I_0 \ln \bar{r} \neq 0$.

We want to study how this dynamics is perturbed for $k > 0$.

Remark 19. *In [78] it is considered the induced motion of a polarized neutral atom (i.e. a dipole) under the electromagnetic field generated by a time dependent charge density (and no current) along the infinity wire. As in our case, the motion can be reduced to the radial component given a singular equation of the form*

$$\ddot{r} = \frac{L^2 - C + \rho(t)}{r^3}$$

where L is the angular momentum, C a constant depending on the physical properties of the atom and $\rho(t)$ represents the charge density. Note that this equation and (3.16) *have different singularities.*

3.5 Existence of solutions with periodic radial oscillations of twist type

Here we conclude proving the two theorems in [61] about the Newtonian dynamics of charged particles induced by the time-dependent current (3.3).

3.5.1 Proof of Theorem 13

At first, we shall apply the local continuation theorem (see Theorem 9 in 1.3.1) to equation (3.16), in order to get the following result.

Proposition 2. *Suppose $I(t) \in \mathcal{C}^2(\mathbb{R}/T\mathbb{Z})$ satisfying* (3.2). *Then, for every admissible non-resonant triplet $(\bar{r}, L, p_z)$ there exists a positive k_0 such that, for every $|k| < k_0$, equation* (3.16) *admits a unique positive T-periodic (L, p_z)-solution $r_k(t)$. Moreover, it is continuous in (t, k) and such that*

$$r_k(t) \to \bar{r}, \quad \textit{uniformly in } t \in [0,T], \ \textit{when } |k| \to 0.$$

Consequently, for any admissible non-resonant triplet $(\bar{r}, L, p_z)$, Proposition 2 guarantees the existence of a positive solution $r_k(t)$ of (3.16) continuous in (t, k) such that $r_k(t+T) = r_k(t)$, $r_0(t) = \bar{r}$. Inserting this solution in (3.17) we get the thesis. Concerning the sign of $\dot{z}$ we have

$$\dot{z} = I_0 + I_0[\ln(r_k(t)) - \ln \bar{r}] + ka(t, r_k(t)).$$

Hence, since $r_k(t) \to \bar{r}$ uniformly in t as $k \to 0$ and $a(t, r)$ is continuous in a neighbourhood of $\bar{r}$, we get the result for some eventually smaller k_0 recalling that $I_0 \neq 0$.

Proof of Proposition 2. Let us fix an admissible non-resonant triplet $(\bar{r}, L, p_z)$ and consider only the case $I_0 > 0$, being the other case similar. In order to apply Theorem 9 to equation (3.16), we note that it can be written in the form (1.21) as a system of first order, where the regularity assumptions are satisfied in a neighbourhood of $\bar{r}$ by Proposition 1. For $k = 0$, equation (3.16) reduces to

$$\ddot{r} = \frac{L^2}{r^3} - \frac{\mathbf{C}}{r}\left(p_z + \mathbf{C} \ln r\right),$$

that admits the constant solution $r(t) = \bar{r}$ since $\bar{r}, L, p_z$ belong to an admissible triplet. Here we are denoting $\mathbf{C} = \mu_0 I_0 (2\pi)^{-1}$.

Writing $p_z = p_z(\bar{r}, L)$, the first variation w.r.t. r of (3.16) at the point $(r, k) = (\bar{r}, 0)$ can be written as

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{\bar{r}^2}\left(\frac{2L^2}{\bar{r}^2} + \mathbf{C}^2\right) & 0 \end{pmatrix} y,$$

whose solutions are T_0-periodic with

$$T_0 = \frac{1}{\bar{r}}\sqrt{\frac{2L^2}{\bar{r}^2} + \mathbf{C}^2}.$$

Since by hypothesis $T \notin \{nT_0,\ n \in \mathbb{N}\}$, Theorem 9 and Remark 10, give the existence of $k_0 > 0$ such that for every $|k| < k_0$ there exists a unique T-periodic solution $r_k(t)$ of (3.16), and $r_k(t) \to \bar{r}$ as $k \to 0$ uniformly in t. Finally, by the uniform convergence to $\bar{r} > 0$ one can eventually decrease the value of k_0 in order to guarantee that, for every fixed $|k| < k_0$, $r_k(t) > 0$ for every $t \in [0, T]$. □

3.5.2 Proof of Theorem 14. Stability by twist analysis.

At first, we shall study when the T-periodic solution $r_k(t)$ of (3.16) coming from Proposition 2 is of twist type.

It comes from [96] that the twist character of a periodic solution can be deduced by the third approximation of the equation. To this aim, let us move the solution to the origin via the change of variable $y = r - r_k(t)$ in (3.16) and compute the development up to third order around $y = 0$. This expression has the form

$$\ddot{y} + A_k(t)y + B_k(t)y^2 + C_k(t)y^3 = 0.$$

By Theorem 10 in Chapter 1.3.2, we get the following:

Proposition 3. *Suppose $I(t) \in \mathcal{C}^4(\mathbb{R}/T\mathbb{Z})$ satisfying* (3.2). *For every admissible strongly non-resonant triplet $(\bar{r}, L, p_z)$ there exists k_1, such that the solution $r_k(t)$ of* (3.16) *coming from Proposition 2 is of twist type for $k < k_1$.*

Proof. As before, let us consider only the case $I_0 > 0$, for which we are denoting again $\mathbf{C} = \mu_0 I_0 (2\pi)^{-1}$. Fixing the strongly non resonant triplet $(\bar{r}, L, p_z)$ we can apply Theorem 13 and consider, for $k < k_0$, the T-periodic solutions $r_k(t)$ of (3.16). We recall that $r_k(t) = \bar{r} + \xi_r(t,k)$ where $\xi_r(t,k)$ is a continuous function, T-periodic in t and such that $\xi_r(t,k) \to 0$ as $k \to 0$ uniformly in t.

To compute the coefficients $A_k(t), B_k(t), C_k(t)$ of Theorem 10, it is convenient to rewrite equation (3.16) as:

$$\ddot{r} - \frac{L}{r^3} + \frac{\mathbf{C}}{r}(p_z + \mathbf{C}\ln r) - k\mathbf{C}\partial_r\left(a(t,r)g(t,r,k)\right) = 0,$$

where $g(t,r,k) := p_z + \mathbf{C}\ln r + k\frac{\mathbf{C}}{I_0}a(t,r)$. Using this, the calculus is simplified and we obtain the expressions:

$$\begin{aligned}
A_k(t) &= \frac{3L^2}{r_k^4(t)} - \frac{\mathbf{C}p_z}{r_k^2(t)} + \frac{\mathbf{C}^2(1-\ln r_k(t))}{r_k^2(t)} + k\partial_{rr}\left(a(t,r_k(t))g(t,r_k(t),k)\right);\\
B_k(t) &= -\frac{6L^2}{r_k^5(t)} + \frac{\mathbf{C}p_z}{r_k^3(t)} - \frac{\mathbf{C}^2(3-2\ln r_k(t))}{2r_k^3(t)} + k\partial_{rrr}\left(a(t,r_k(t))g(t,r_k(t),k)\right);\\
C_k(t) &= \frac{10L^2}{r_k^6(t)} - \frac{\mathbf{C}p_z}{r_k^4(t)} + \frac{\mathbf{C}^2(11-6\ln r_k(t))}{6r_k^4(t)} + k\partial_{rrrr}\left(a(t,r_k(t))g(t,r_k(t),k)\right).
\end{aligned}$$

Since the triplet is admissible $\mathbf{C}p_z = \frac{L^2}{\bar{r}^2} - \mathbf{C}^2\ln\bar{r}$, so that we can write the coefficients as follows:

$$\begin{aligned}
A_k(t) &= \bar{A} + \xi_A(t,r_k(t),k), &\quad \text{with } \bar{A} &= \frac{2L^2}{\bar{r}^4} + \frac{\mathbf{C}^2}{\bar{r}^2},\\
B_k(t) &= \bar{B} + \xi_B(t,r_k(t),k), &\quad \text{with } \bar{B} &= -\frac{5L^2}{\bar{r}^5} - \frac{3\mathbf{C}^2}{2\bar{r}^3},\\
C_k(t) &= \bar{C} + \xi_C(t,r_k(t),k), &\quad \text{with } \bar{C} &= 9\frac{L^2}{\bar{r}^6} + \frac{11}{6}\frac{\mathbf{C}^2}{\bar{r}^4}.
\end{aligned}$$

Note that, as $r_k(t)$ converges to $\bar{r}$ uniformly in time, then the residual functions ξ_A, ξ_B, ξ_C converge to 0 uniformly in t as $k \to 0$. Moreover, $\bar{C}$ is trivially positive, the strong non-resonant condition implies that $\bar{A} < \frac{\pi^2}{4T^2}$ and a direct computation shows that $10\bar{B}^2\bar{A}^{3/2} > 9\bar{C}\bar{A}^{5/2}$ (note that it is enough to check that $10\bar{B}^2 > 9\bar{C}\bar{A}$).

Therefore, using the fact that the remainders ξ_A, ξ_B, ξ_C converge to 0 uniformly in t as $k \to 0$, we have that there exists $k_1 > 0$ such that the coefficients $A_k(t), B_k(t), C_k(t)$ satisfy the hypothesis of Theorem 10 for $k < k_1$.

□

We conclude proving the result.

Proof of Theorem 14. Let us fix an admissible strongly non-resonant triplet $(\bar{r}, L, p_z)$. From Proposition 3 we consider for every $k < k_0$ the T-periodic solution of twist type of (3.16). Lyapunov stability is a known property of solutions of twist type. Moreover there exist, for ω and p/q small, (generalized) quasi periodic solutions with frequencies $(1, \omega)$ and solutions $r(t)$ with minimal period qT. These periodic solutions are such that the function $r(t) - r_k(t)$ has $2p$ zeros in a period $[0, qT]$.

Note that, since $r_k(t) \to \bar{r}$ uniformly for $k \to 0$, we also have that $r(t) - \bar{r}$ has $2p$ zeros in a period $[0, qT]$.

Inserting these solutions into system (3.17) we get the thesis, for fixed values of L and p_z. In particular we get stability restricted to the integral levels. It remains to prove the stability in the whole phase space. We will proceed adapting an argument in [111]. Consider the map $P(r_0, \dot{r}_0, L, p_z) = (r(T), \dot{r}(T), L, p_z)$ that maps one of these solutions with initial condition $(r_0, \dot{r}_0)$ and integrals L, p_z to the corresponding values at time T. To prove stability it is enough to prove that $q_0 = (r_0, \dot{r}_0, L_0, p_{z_0})$ is a stable fixed point for the map P. Let us fix a neighbourhood $\mathcal{U}$ of q_0. Since the point $(r_0, \dot{r}_0)$ is the initial condition of a periodic solution of (3.16) of twist type for fixed values of L, p_z, we can find in $\mathcal{U}$ an invariant planar region bounded by a closed curve surrounding $(r_0, \dot{r}_0)$ of the form

$$\mathcal{U}_1 = \{(r - r_0)^2 + (\dot{r} - \dot{r}_0)^2 \leq R(\Theta),\ L = L_0,\ p_z = p_{z_0}\} \subset \mathcal{U},$$

where $\Theta = \Theta(r, \dot{r})$ represents the angle cantered in $(r_0, \dot{r}_0)$. Equation (3.16) depends continuously on the parameters L, p_z, hence by continuous dependence, we can find a family of curves

$$(r - r_0)^2 + (\dot{r} - \dot{r}_0)^2 \leq R(\Theta, L, p_z)$$

depending continuously on L, p_z and with the properties above. Note that here we have used the fact that the strongly non resonant condition (3.6) is an open condition (for fixed $\bar{r}$). Therefore, for sufficiently small δ, the region

$$\mathcal{U}_2 = \{(r - r_0)^2 + (\dot{r} - \dot{r}_0)^2 \leq R(\Theta, L, p_z),\ |L - L_0| < \delta,\ |p_z - p_{z_0}| < \delta\}$$

is invariant and contained in $\mathcal{U}$. Therefore, the point q_0 is stable under the map P and the solutions are radially stable. □

Chapter 4

A quantitative result about the existence of relativistic charged particles with radially periodic orbits

Complementarily to Chapter 3, here we focus on the relativistic dynamics induced by the infinite wire model previously introduced. This research was done together with P.J. Torres and it corresponds to the paper *Periodic dynamics in the relativistic regime of an electromagnetic field induced by a time-dependent wire* [63], actually submitted for publication.

We shall consider the Lorentz force equation (1.1), where the electromagnetic field is given by the derivatives (1.4) of the vectorial potential

$$\vec{A}(t,r) = -\frac{\mu_0}{2\pi}[a_0(r) + ka(t,r)]\mathbf{e}_z,$$

which solves the wave equation (3.7) for the time-dependent distribution (3.3) associated to the current along the infinite wire. Just like in the Newtonian case, the cylindrical symmetries are still present in our model and the corresponding linear and angular relativistic momenta are conserved quantities. Therefore, we are able to reduce (1.1) to a planar hamiltonian system with one degree of freedom.

By using global continuation and topological degree, we identify a bi-parametric family of radially periodic motions in (1.1) for an explicit interval of the perturbation parameter k. Moreover, the proofs involve some delicate estimations of the induced potential $\vec{A}$, which can be of independent interest, see Section 4.2 for details.

Regarding the radial stability, we are sure that it holds for small values of k reasoning like in [61], where the third approximation method is used [96, 114]. However, we have not been able to obtain a quantitative result, which remains as an open problem.

The chapter is structured as follows. In Section 4.1 we construct rigorously the model from physical principles and the main results are presented. Section 4.2 collects some delicate estimations about the asymptotic behaviour of the potential $\vec{A}$. In Section 4.3, the symmetries of the system are used to deduce the corresponding conserved quantities (linear and relativistic angular momentum) and reduce the problem to a Hamiltonian system with one degree of freedom. To conclude, Section 4.4 contains the main proof for the existence of radially periodic solutions. It relies on the topological results developed in Section 1.3.3, that resemble the celebrated global continuation theorem by Capietto-Mawhin-Zanolin [22], adapted to our context. Additionally, it is included a brief appendix about the Bessel functions that are essential to describe the asymptotic behaviour of $\vec{A}(t,q)$ in the singularity points.

4.1 Statements of the main results

As in Chapter 3, we consider an infinitely long and infinitely thin straight wire carrying a current of the form $I_0 + kI(t)$, where $I_0 > 0$, $k \geq 0$ are constants and $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfies

$$\int_0^T I(t)dt = 0.$$

Notice that (3.2) implies the existence of a T-periodic primitive of $I(t)$, which we denote by $\mathcal{I}(t)$. Without loss of generality, let us assume the wire centered in the z-axis. Then, the corresponding current density is a vectorial distribution $\vec{J} = (0,0,J)$ such that

$$J(f) = \int_{\mathbb{R}^2} [I_0 + kI(t)]\, f(t,0,0,z)\, dt\, dz, \qquad \text{for every } f \in \mathcal{C}_0^\infty\left(\mathbb{R}^4\right),$$

Furthermore, the wire is also assumed to be electrically neutral, i.e., at any time, every segment of it contains as many electrons as protons, so the charge density ρ is null. In this situation, (3.4) admits as unique distributional solution to the electromagnetic field given by (1.4), with $\Phi \equiv 0$ and $\vec{A}(t,q) = A(t,r)\mathbf{e}_z$, where

$$A(t,r) = -\frac{\mu_0}{2\pi}[a_0(r) + ka(t,r)], \tag{4.1}$$

and

$$a_0(r) = I_0 \ln r, \quad a(t,r) = \int_0^\infty \frac{I[t,r,\tau]}{\sqrt{\tau^2 + r^2}} d\tau.$$

Here r denotes the radial variable in the XY-plane, $\mathbf{e}_z$ is the positive unitary vector in the z-direction, and the bracket $[t,r,\tau] = \left(t - \sqrt{\tau^2 + r^2}\right)$ shows the delay effect of the potential. We refer the reader to Proposition 1 in Chapter 3 (or Proposition 1 in [61]) for the mathematical details. By the same result, (4.1) is T-periodic and such that

$$\text{if } I(t) \in \mathcal{C}^n([0,T];\mathbb{R}), \text{ then } a(t,r) \in \mathcal{C}^n\left([0,T] \times \mathbb{R}^+;\mathbb{R}\right). \tag{4.2}$$

Moreover, there exists a constant $C > 0$ for which

$$|a(t,r)| \leq C|\ln r|, \quad \text{when } r << 1.$$

Therefore, the blow up sign of (4.1) when r tends to 0 is controlled for all $t \in [0,T]$ by the logarithm term $a_0(r)$ if k is small enough. In this chapter, we shall improve Proposition 1 in [61] by giving a quantitative version, being able to estimate explicitly not just C but also describing the asymptotic behaviour of $a(t,r)$ and its partial derivative $\partial_r a(t,r)$ when r tends to 0 or to $+\infty$. Concretely, for any $\hat{r} > 0$, let $K_{\hat{r}}$ be the quantity

$$K_{\hat{r}} := \frac{4I_0}{\max\left\{[2\left(1+\sqrt{2}\right)+\pi]\|I\|_\infty + \hat{r}\pi\|\dot{I}\|_\infty; \frac{4T^2}{3\pi}\|\ddot{I}\|_\infty\right\}}.$$

Theorem 15. *Let I_0 be positive and $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying* (3.2). *Then* (4.1) *is such that:*

i) For any $k \geq 0$, $\lim_{r\to+\infty} A(t,r) = -\infty$, uniformly in t.

ii) For any $k \in [0, K_{\hat{r}})$, $\lim_{r\to 0^+} A(t,r) = +\infty$, uniformly in t.

iii) For any $k \in [0, K_{\hat{r}})$,

$$0 > -\frac{I_0}{r}\left[1 - \frac{k}{K_{\hat{r}}}\right] > \frac{2\pi}{\mu_0}\partial_r A(t,r) > -\frac{I_0}{r}\left[1 + \frac{k}{K_{\hat{r}}}\right], \tag{4.4}$$

for all $(t,r) \in [0,T] \times [0,\hat{r}]$.

Moreover,

iv) If $k > 0$, then

$$\partial_r A(t,r) \sim r^{-1/2}\mathcal{G}(t,r), \quad \textit{when } r >> 1,$$

with $\mathcal{G}(t,r)$ being an oscillating function such that, given any $(t,\mathrm{r}) \in [0,T] \times (0,+\infty)$, there exists a infinitely number of points in $[\mathrm{r},+\infty)$ where $\mathcal{G}(t,r)$ changes its sign.

Notice that the first three assertions are satisfied by the logarithm $a_0(r)$. Therefore, the modification of the logarithm picture is explicitly controlled with respect of the perturbation parameter k. Through this we are able to find invariant sets for the solutions in order to apply a fixed point argument. The proof of Theorem 15 is given in Section 4.2.

In our particular conditions, the dynamical system (1.1) is

$$\frac{d}{dt}\phi(\dot{q}(t))' = -\partial_t \vec{A}(t,q(t)) + \dot{q}(t) \times \left[\nabla_q \times \vec{A}(t,q(t))\right]. \tag{4.5}$$

Again, induced by the wire symmetries, let us consider the cylindrical coordinates $q = (r\cos\theta, r\sin\theta, z)$. We also denote by p_r, L and p_z the relativistic radial, angular and linear momenta respectively, defined as

$$\begin{aligned} p_r &= \frac{\dot{r}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-\dot{z}^2}}, \qquad L = \frac{r^2\dot{\theta}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-\dot{z}^2}}, \\ p_z &= \frac{\dot{z}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-\dot{z}^2}} + A(t,r). \end{aligned} \tag{4.6}$$

Both L and p_z are first integrals of (4.5) for every $k \geq 0$. We rely on this fact to introduce the following definition, analogous to the one established in the previous chapter.

Definition 6. *A solution $q(t) = (r(t), \theta(t), z(t))$ of (4.5) with angular momentum L and linear momentum p_z is called a (L, p_z)-solution. Moreover, it is radially T-periodic if $r(t+T) = r(t)$, for every $t \in \mathbb{R}$.*

Now, we are ready to set up the main result about the existence of radially periodic solutions. Given T, $k > 0$, let us define

$$\mathbf{P}(k) = -\frac{\mu_0}{2\pi}\left[I_0 \ln T + k\frac{T^2}{3}\|\ddot{I}\|_\infty\right]. \tag{4.7}$$

Theorem 16. *Let I_0, L be positive, $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying (3.2) and $P_0 < \boldsymbol{P}(K_T)$. Take $\varepsilon > 0$ and $\bar{k} \in [0, K_{r_\varepsilon+T+\varepsilon}]$, where r_ε is the unique point satisfying the identity*

$$r^2\left[P_0 + \frac{\mu_0}{2\pi}\left(I_0 \ln r - K_{r+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right)\right] = \frac{2\pi}{\mu_0}\frac{L^2}{I_0}\frac{C_1 + C_2(r+T+\varepsilon)}{C_2\varepsilon}, \tag{4.8}$$

with

$$C_1 = \|I\|_\infty\left[2\left(1+\sqrt{2}\right)+\pi\right], \quad \text{and } C_2 = \pi\,\|\dot{I}\|_\infty.$$

Then, there exists a positive constant $r_m < r_\varepsilon$, depending on $\bar{k}$ and $\boldsymbol{P}(\bar{k})$, such that for every $(k, p_z) \in [0, \bar{k}] \times \left[P_0, \boldsymbol{P}(\bar{k})\right]$, the Lorentz force equation (4.5) admits at least one radially T-periodic (L, p_z)-solution with

$$r_m < r(t) < r_\varepsilon, \qquad \text{for all } t \in \mathbb{R}.$$

Moreover,

$$\lim_{\varepsilon\to 0} r_\varepsilon = +\infty\,, \qquad \lim_{\varepsilon\to 0} K_{r_\varepsilon} = 0.$$

4.2 Estimations for the potential

In this section we prove Theorem 15 as a consequence of two previous results. The first one describes explicitly the asymptotic behaviour of (4.1) close to the wire and also very far from it. In particular, we have the following.

Proposition 4. *Let I_0 be positive and $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying* (3.2). *Then, for every $k \geq 0$,*

$$\lim_{r \to +\infty} A(t,r) = -\infty, \text{ uniformly in } t.$$

Moreover, let k_0 be the constant

$$k_0 = \frac{I_0}{\frac{T^2}{3\pi} \|\ddot{I}\|_\infty}.$$

Then, for any $k \in [0, k_0)$,

$$\lim_{r \to 0^+} A(t,r) = +\infty, \text{ uniformly in } t.$$

On the other hand, as the potential (4.1) is regular, we can also proceed in a similar way with its derivative $\partial_r A(t,r)$.

Lemma 1. *Let I_0 be positive and $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying* (3.2). *Given any radius $\hat{r} > 0$, define the constant*

$$k_{\hat{r}} = \frac{4 I_0}{[2\left(1+\sqrt{2}\right) + \pi]\|I\|_\infty + \hat{r}\pi \|\dot{I}\|_\infty}. \tag{4.9}$$

If $k \in [0, k_{\hat{r}})$, then

$$0 > -\frac{I_0}{r}\left[1 - \frac{k}{k_{\hat{r}}}\right] > \frac{2\pi}{\mu_0}\partial_r A(t,r) > -\frac{I_0}{r}\left[1 + \frac{k}{k_{\hat{r}}}\right],$$

for all $(t,r) \in [0,T] \times [0,\hat{r}]$. Moreover, if $k > 0$, then

$$\partial_r A(t,r) \sim r^{-1/2}\mathcal{G}(t,r), \quad \text{when } r >> 1,$$

where $\mathcal{G}(t,r)$ is a bounded and oscillating function such that, given any $(t,r) \in [0,T] \times (0,+\infty)$, changes sign infinitely times in $[r,+\infty)$.

Once we prove both results, it is clear that Theorem 15 follows combining them.

4.2.1 Proof of Proposition 4.

By Proposition 1 in [61], we know that

$$|a(t,r)| \leq \|I\|_\infty \ln\left(1+\sqrt{2}\right) + 2r^{-1}\|I\|_\infty,$$

for all $(t,r) \in [0,T] \times \mathbb{R}^+$. Consequently, for every $k \geq 0$,

$$-\frac{2\pi}{\mu_0}A(t,r) \geq I_0 \ln r - k|a(t,r)| \geq I_0 \ln r - k\|I\|_\infty \left(\ln\left(1+\sqrt{2}\right) + 2r^{-1}\right),$$

and the first assertion follows taking limits when $r \to +\infty$.

Complementary, recalling again Proposition 1 in [61], the Fourier series of $I(t)$ allows to write $a(t,r)$ as

$$\begin{aligned} a(t,r) = &- \frac{\pi}{2} \sum_{j\geq 1} \left[\alpha_j \cos\left(\lambda_j t\right) + \beta_j \sin\left(\lambda_j t\right)\right] \mathcal{Y}_0\left(\lambda_j r\right) \\ &+ \frac{\pi}{2} \sum_{j\geq 1} \left[\alpha_j \sin\left(\lambda_j t\right) - \beta_j \cos\left(\lambda_j t\right)\right] \mathcal{J}_0\left(\lambda_j r\right), \end{aligned} \tag{4.10}$$

where $\mathcal{J}_0$, $\mathcal{Y}_0$ are the Bessel functions of first and second kind with zero index respectively, $\lambda_j = j2\pi T^{-1}$ and $\{\alpha_j, \beta_j\}_{j\geq 1}$ are the Fourier coefficients:

$$\alpha_j = \frac{2}{T}\int_0^T I(t)\cos(\lambda_j t)dt, \quad \beta_j = \frac{2}{T}\int_0^T I(t)\sin(\lambda_j t)dt.$$

Due to the regularity of I, integrating by parts twice we get the following:

$$\alpha_j = -\frac{T}{2\pi^2 j^2}\int_0^T \ddot{I}(t)\cos(\lambda_j t)dt, \quad \beta_j = -\frac{T}{2\pi^2 j^2}\int_0^T \ddot{I}(t)\sin(\lambda_j t)dt.$$

It is clear that $|\alpha_j|, |\beta_j| \leq \frac{T^2}{2\pi^2 j^2}\|\ddot{I}\|_\infty$, for all $j \geq 1$. However, let us be more precise:

$$\begin{aligned} \int_0^T |\sin(\lambda_j t)|dt &= \int_0^T \left|\sin\left(\tfrac{j2\pi}{T}t\right)\right|dt = j\int_0^{\frac{T}{j}} \left|\sin\left(\tfrac{j2\pi}{T}t\right)\right|dt \\ &= j\left[\int_0^{\frac{T}{2j}} \sin\left(\tfrac{j2\pi}{T}t\right)dt - \int_{\frac{T}{2j}}^{\frac{T}{j}} \sin\left(\tfrac{j2\pi}{T}t\right)dt\right] \\ &= \frac{T}{2\pi}\left[-\cos\left(\tfrac{j2\pi}{T}t\right)\Big|_0^{\frac{T}{2j}} + \cos\left(\tfrac{j2\pi}{T}t\right)\Big|_{\frac{T}{2j}}^{\frac{T}{j}}\right] = \frac{2T}{\pi}. \end{aligned}$$

Similarly, $\int_0^T |\cos(\lambda_j t)|dt = \frac{2T}{\pi}$ and

$$|\alpha_j|, |\beta_j| \leq \frac{T^2}{\pi^3 j^2}\|\ddot{I}\|_\infty, \text{ for all } j \geq 1.$$

Therefore, as $\|\mathcal{J}_0\|_\infty = \mathcal{J}_0(0) = 1$ and $\sum_{j\geq 1} j^{-2} = \pi^2/6$, then

$$|a(t,r)| < \frac{T^2}{\pi^2}\|\ddot{I}\|_\infty \left[\sum_{j\geq 1} \frac{1}{j^2}|\mathcal{Y}_0\left(\lambda_j r\right)| + \frac{\pi^2}{6}\right], \quad \text{for all } (t,r) \in [0,T]\times\mathbb{R}^+.$$

Concerning $\mathcal{Y}_0(x)$, it is known that $|\mathcal{Y}_0(x)| < 1$, when $x > \sigma$, where σ[i] is the unique real root of the equation $\mathcal{Y}_0(x) + 1 = 0$. Thus, defining

$$\mathbf{r} = \sigma\frac{T}{2\pi}, \tag{4.11}$$

[i] $\sigma \approx 0,2258374310$

if $r \geq \mathbf{r}$, it is clear that $\lambda_j r > \sigma$, for every $j \in \mathbb{N}$. Furthermore,

$$|a(t,r)| < \frac{T^2}{3}\|\ddot{I}\|_\infty, \quad \text{for all } (t,r) \in [0,T] \times [\mathbf{r},\infty). \tag{4.12}$$

On the other hand, assume now $r \in (0,\mathbf{r})$ and let us define the number

$$j_r = \left\lfloor \frac{\mathbf{r}}{r} \right\rfloor \geq 1,$$

where $\lfloor \cdot \rfloor$ denotes the integer part function. Moreover, note that $\lambda_j r < \sigma$ when $j \leq j_r$, and then $|\mathcal{Y}_0(\lambda_j r)| \geq 1$ if, and only if, $j \leq j_r$. Therefore,

$$\begin{aligned}\sum_{j\geq 1}\frac{1}{j^2}|\mathcal{Y}_0(\lambda_j r)| &= \sum_{j\geq 1}^{j_r}\frac{1}{j^2}|\mathcal{Y}_0(\lambda_j r)| + \sum_{j>j_r}\frac{1}{j^2}|\mathcal{Y}_0(\lambda_j r)| \\ &\leq \sum_{j\geq 1}^{j_r}\frac{1}{j^2}|\mathcal{Y}_0(\lambda_j r)| + \sum_{j>j_r}\frac{1}{j^2} < \sum_{j\geq 1}^{j_r}\frac{1}{j^2}|\mathcal{Y}_0(\lambda_j r)| + \sum_{j\geq 1}\frac{1}{j^2}.\end{aligned}$$

Now we recall the inequality (4.19), developed in Appendix 4.5:

$$|\mathcal{Y}_0(x)| < \frac{2}{\pi}\left[\left|\ln\left(\frac{|x|}{2}\right)\right| + \gamma\right] + \frac{2}{\pi}\exp\left(\frac{x^2}{4}\right), \quad x \in \mathbb{R},$$

with γ[ii] denoting the well known Euler-Masheroni constant, see [40, Section 1.5] and the references therein. In particular,

$$|\mathcal{Y}_0(\lambda_j r)| < \frac{2}{\pi}\left[\left|\ln\left(\frac{r\pi}{T}\right)\right| + \ln j + \gamma + \exp\left(\lambda_j^2\frac{r^2}{4}\right)\right], \quad r > 0,$$

and

$$\begin{aligned}\frac{\pi}{2}\sum_{j\geq 1}^{j_r}\frac{1}{j^2}|\mathcal{Y}_0(\lambda_j r)| &< \sum_{j\geq 1}^{j_r}\frac{1}{j^2}\left[\left|\ln\left(\frac{r\pi}{T}\right)\right| + \ln j + \gamma + \exp\left(\frac{\sigma^2}{4}\right)\right] \\ &< \left[\left|\ln\left(\frac{r\pi}{T}\right)\right| + \gamma + \exp\left(\frac{\sigma^2}{4}\right)\right]\sum_{j\geq 1}\frac{1}{j^2} + \sum_{j\geq 1}\frac{\ln j}{j^2}.\end{aligned}$$

The second sum is also convergent and can be computed explicitly by using the derivative of the Riemann zeta function, see for instance [40, p. 135]. More concretely,

$$\zeta'(2) = \sum_{j\geq 2}\frac{\ln j}{j^2} = \frac{\pi^2}{6}\left(12\ln\mathcal{A} - \gamma - \ln(2\pi)\right),$$

where $\mathcal{A}$[iii] denotes the Glaisher–Kinkelin constant. Using this,

$$\sum_{j\geq 1}^{j_r}\frac{1}{j^2}|\mathcal{Y}_0(\lambda_j r)| < \frac{\pi}{3}\left[\left|\ln\left(\frac{r}{T}\right)\right| + \exp\left(\frac{\sigma^2}{4}\right) + 12\ln\mathcal{A} - \ln 2\right],$$

[ii] $\gamma = 0,5772156649...$

[iii] $\mathcal{A} = 1,2824271291...$

and then, for any $(t,r) \in [0,T] \times (0,\mathbf{r})$, we conclude that

$$|a(t,r)| < \frac{T^2}{3\pi}\|\ddot{I}\|_\infty \left(|\ln r| + \widetilde{C}\right), \tag{4.13}$$

with $\widetilde{C} = |\ln T| + \exp\left(\frac{\sigma^2}{4}\right) + 12\ln\mathcal{A} - \ln 2 + \pi$. Finally, by (4.13),

$$\begin{aligned}\frac{2\pi}{\mu_0}A(t,r) &= -I_0 \ln r - ka(t,r) \geq -I_0 \ln r - k|a(t,r)|,\\ &\geq -I_0 \ln r + k\frac{T^2}{3\pi}\|\ddot{I}\|_\infty \left(|\ln r| + \widetilde{C}\right)\end{aligned}$$

when $(t,r) \in [0,T] \times (0,\mathbf{r})$, and the second assertion follows taking limit when $r \to 0$.

4.2.2 Proof of Lemma 1

Due to the smoothness of $a(t,r)$, by the Leibniz's rule we obtain $\partial_r a(t,r)$ deriving under the integral sign

$$\partial_r a(t,r) = -r\int_0^\infty \frac{1}{\tau^2+r^2}\left[\dot{I}[t,r,\tau] + \frac{I[t,r,\tau]}{\sqrt{\tau^2+r^2}}\right]d\tau.$$

Fix $m > 0$ arbitrarily and let us consider the following decomposition:

$$\int_0^\infty \frac{\dot{I}[t,r,\tau]}{\tau^2+r^2}d\tau = \int_0^m \frac{\dot{I}[t,r,\tau]}{\tau^2+r^2}d\tau + \int_m^\infty \frac{\dot{I}[t,r,\tau]}{\tau^2+r^2}d\tau.$$

Integrating by parts, we get that

$$\begin{aligned}\int_m^\infty \frac{\dot{I}[t,r,\tau]}{\tau^2+r^2}d\tau &= -\frac{I[t,r,\tau]}{\tau\sqrt{\tau^2+r^2}}\Bigg|_m^\infty + \int_m^\infty I[t,r,\tau]\frac{d}{d\tau}\frac{1}{\tau\sqrt{\tau^2+r^2}}d\tau\\ &= \frac{I[t,r,m]}{m\sqrt{m^2+r^2}} - \int_m^\infty \frac{I[t,r,\tau]}{\tau^2\left(\tau^2+r^2\right)}\left[\sqrt{\tau^2+r^2} + \frac{\tau^2}{\sqrt{\tau^2+r^2}}\right]d\tau\\ &= \frac{I[t,r,m]}{m\sqrt{m^2+r^2}} - \int_m^\infty \frac{I[t,r,\tau]}{\tau^2\sqrt{\tau^2+r^2}}d\tau - \int_m^\infty \frac{I[t,r,\tau]}{(\tau^2+r^2)^{3/2}}d\tau.\end{aligned}$$

Therefore, we can write

$$\begin{aligned}\partial_r a(t,r) = r\Bigg[&-\int_0^m \frac{\dot{I}[t,r,\tau]}{\tau^2+r^2}d\tau - \int_0^m \frac{I[t,r,\tau]}{(\tau^2+r^2)^{3/2}}\\ &- \frac{I[t,r,m]}{m\sqrt{m^2+r^2}} + \int_m^\infty \frac{I[t,r,\tau]}{\tau^2\sqrt{\tau^2+r^2}}d\tau d\tau\Bigg].\end{aligned}$$

With respect of the first line, both terms are bounded because

$$\left|\int_0^m \frac{f(\tau)}{\tau^2+r^2}d\tau\right| \leq \|f\|_\infty \int_0^m \frac{1}{\tau^2+r^2}d\tau = \frac{\|f\|_\infty}{r}\arctan(m/r),$$

for any $f \in L^\infty(\mathbb{R})$. On the other hand,

$$\left|\int_m^\infty \frac{I[t,r,\tau]}{\tau^2\sqrt{\tau^2+r^2}}d\tau\right| < \|I\|_\infty \int_m^\infty \frac{1}{\tau^3}d\tau = \frac{\|I\|_\infty}{2m^2}.$$

Putting together all this, and choosing $m = r$, it follows that

$$|\partial_r a(t,r)| < \|I\|_\infty \frac{2\left(1+\sqrt{2}\right)+\pi}{4r} + \|\dot{I}\|_\infty \frac{\pi}{4}, \quad \text{for all } (t,r) \in [0,T]\times\mathbb{R}^+.$$

Moreover, fix $\hat{r} > 0$ arbitrarily. Then, by (4.9),

$$|\partial_r a(t,r)| < \frac{1}{r}\frac{I_0}{k_{\hat{r}}}, \quad \text{for all } (t,r) \in [0,T]\times[0,\hat{r}].$$

From this, the inequalities of Lemma 1 are obtained easily because

$$\partial_r A(t,r) = -\frac{I_0}{r} - k\partial_r a(t,r) \leq -\frac{I_0}{r} + k\,|\partial_r a(t,r)| < -\frac{I_0}{r}\left[1-\frac{k}{k_{\hat{r}}}\right]$$

and

$$\partial_r A(t,r) \geq -\frac{I_0}{r} - k\,|\partial_r a(t,r)| > -\frac{I_0}{r}\left[1+\frac{k}{k_{\hat{r}}}\right].$$

To conclude, let us focus again in the Bessel functions. By 9.2.1 and 9.2.2 in [1], for $|x|$ large we have that

$$\mathcal{J}_0(x) = \sqrt{\frac{2}{\pi x}}\cos(x-\pi/4) + O(|x|^{-3/2}),$$
$$\mathcal{Y}_0(x) = \sqrt{\frac{2}{\pi x}}\sin(x-\pi/4) + O(|x|^{-3/2}),$$

where $O(f(x))$ means that the remaining term is of the order of a certain function $f(x)$. Then, applying this in (4.10), we obtain that, for r large enough,

$$a(t,r) = r^{-1/2}\mathcal{F}(t,r) + O(r^{-3/2}),$$

with

$$\mathcal{F}(t,r) = -\sqrt{\frac{\pi}{2}}\sum_{j\geq 1}\left[\alpha_j\cos(\lambda_j t) + \beta_j\sin(\lambda_j t)\right]\sin(\lambda_j r - \pi/4)\frac{1}{\lambda_j}$$
$$+\sqrt{\frac{\pi}{2}}\sum_{j\geq 1}\left[\alpha_j\sin(\lambda_j t) - \beta_j\cos(\lambda_j t)\right]\cos(\lambda_j r - \pi/4)\frac{1}{\lambda_j}.$$

Reasoning like we did above, it is not difficult to see that $\mathcal{F}(t,r)$ is well defined in $\mathbb{R}\times\mathbb{R}^+$ and that satisfies the properties of the statement. Finally, as $a(t,r)$ is regular, for r large enough we can write

$$\frac{2\pi}{\mu_0}\partial_r A(t,r) = -\frac{I_0}{r} + k\partial_r a(t,r) = \frac{k}{\sqrt{r}}\mathcal{G}(t,r) + \Theta(r^{-1}),$$

with $\mathcal{G}(t,r) = \partial_r \mathcal{F}(t,r)$, and the lemma is proven.

4.3 Hamiltonian structure and magnetostatic regime

In general electrodynamics situations, the Lorentz Force equation (1.1) admits the Hamiltonian formulation (1.17), according to [7]. In our particular case, as Section 4.1 stated, (1.1) is written as (4.5) and the corresponding hamiltonian function is

$$H(t,p,q) = \sqrt{1+|p-\vec{A}(t,q)|^2}.$$

Again, the cylindrical symmetries induce us to consider the cylindrical change of variable $q = (r\cos\theta, r\sin\theta, z)$ and its associated basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, where

$$\mathbf{e}_r = (\cos\theta, \sin\theta, 0)\,,\ \ \mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0)\,.$$

In this coordinates, the time-derivative is $\dot{q} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z$ and, after some basic computations (see [113] for a similar procedure), we are able to rewrite the Lorentz force equation (4.5) as the following system:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\dot{r}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-\dot{z}^2}}\right) - \dfrac{r\dot{\theta}^2}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-\dot{z}^2}} = \dot{z}\partial_r A(t,r), \\ \dfrac{d}{dt}\left(\dfrac{r^2\dot{\theta}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-\dot{z}^2}}\right) = 0, \\ \dfrac{d}{dt}\left(\dfrac{\dot{z}}{\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2-\dot{z}^2}} + A(t,r)\right) = 0. \end{cases}$$

Remark 20. *In this coordinates system, the characteristic Special Relativity effect for the limitation of the particle velocities is the vector $(\dot{r}, r\dot{\theta}, \dot{z})$ strictly contained in the interior of the unitary ball.*

Observe that, as in the previous chapter, every symmetry implies a conservation law, being consistent with Noether's Theorem, while there is a dynamical equation for the radial component $r(t)$. Consequently, this induces the definition (4.6) of the corresponding relativistic momenta (p_r, L, p_z) that, after some algebra, can be reversed obtaining:

$$\dot{r} = \frac{p_r}{\sqrt{1+(p_z-A)^2+p_r^2+L^2r^{-2}}}, \quad r^2\dot{\theta} = \frac{L}{\sqrt{1+(p_z-A)^2+p_r^2+L^2r^{-2}}},$$
$$\dot{z} = \frac{p_z - A(t,r)}{\sqrt{1+(p_z-A)^2+p_r^2+L^2r^{-2}}}.$$

From these identities, it is clear that the motion of a charged particle is described by its radial component, with implicit dependence on the constant L, p_z and k. In particular, this allows to reduce (4.5) to the planar system

$$\begin{cases} \dot{r} = \dfrac{p_r}{\sqrt{1+(p_z-A)^2+p_r^2+L^2r^{-2}}}, \\[2ex] \dot{p}_r = \dfrac{L^2r^{-3}+(p_z-A)\partial_r A}{\sqrt{1+(p_z-A)^2+p_r^2+L^2r^{-2}}}, \end{cases} \tag{4.14}$$

that is Hamiltonian for the energy function

$$\mathcal{H}(t,r,p_r) = \sqrt{1+(p_z-A)^2+p_r^2+L^2r^{-2}}. \tag{4.15}$$

In summary, by the wire symmetries, the dynamics described in (4.5) and in (4.14) are equivalent. Therefore, we are reducing the Lorentz Force equation (1.1) to a planar Hamiltonian system with one degree of freedom for the radial component of the solutions, which is a similarity with the Newtonian approach.

Remark 21. Magnetostatic dynamics.

As (4.15) *is integrable when* $k = 0$*, the energy of the system is conserved in that case. Physically, this induces a magnetostatic regime, because of the vanishing of the electric field, and the particles cannot collide with the wire. It is not difficult to see that, for any pair* $(L, p_z) \in \mathbb{R}^+ \times \mathbb{R}$, (4.14) *admits a unique equilibrium* $(\bar{r}, 0)$*. We call it* (L, p_z)*-equilibrium and is given by the identity*

$$\bar{r}^2\left[p_z + \frac{\mu_0}{2\pi} I_0 \ln \bar{r}\right] = \frac{2\pi}{\mu_0}\frac{L^2}{I_0}. \tag{4.16}$$

Furthermore, this point is the same that in the Newtonian dynamics (see Definition 5 in Chapter 3). By (4.16)*, observe that the* (L, p_z)*-equilibria are close to the wire only for* p_z *large. Also,*

$$\bar{r} \in \left(\exp\left(-p_z \frac{2\pi}{\mu_0 I_0}\right), 1\right), \quad \textit{when } p_z > 0,$$

while $\bar{r} \geq 1$ *in the complementary case. However, regardless of the sign of* p_z*, a particle with radially constant motion is always accelerated in the current direction because*

$$\dot{z} = H_0^{-1}\left(p_z + \frac{\mu_0}{2\pi} I_0 \ln \bar{r}\right) > 0.$$

Again, this is also a similarity with the non-relativistic approach, where the unique difference is the presence of the Hamiltonian constant H_0. In fact, as $H_0 > 1$, $\dot{z}$ is smaller than in the Newtonian case, which is something expected.

4.4 Existence of radially periodic solutions

In this section we apply the topological degree arguments developed in the Section 1.3.3 to the planar Hamiltonian system (4.14) in order to prove Theorem 16. Before of that, *suitable a priori bounds* are obtained using the asymptotic estimations of $A(t,r)$ developed in Section 4.2. As Theorem 16 was formulated in terms of equation (4.5), the equivalent version for (4.14) reads:

Theorem 17. *Let I_0, L be positive, $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying* (3.2) *and $P_0 < \boldsymbol{P}(K_T)$. Take $\varepsilon > 0$ and $\bar{k} \in [0, K_{r_\varepsilon+T+\varepsilon}]$, with r_ε defined in* (4.8). *Then, there exists a positive constant $r_m < r_\varepsilon$, depending on $\bar{k}$ and $\boldsymbol{P}(\bar{k})$, such that for all $(k, p_z) \in [0,\bar{k}] \times \left[P_0, \boldsymbol{P}(\bar{k})\right]$,* (4.14) *admits at least one T-periodic solution with*

$$r_m < r(t) < r_\varepsilon, \qquad \textit{for every } t \in \mathbb{R}.$$

Moreover,

$$\lim_{\varepsilon\to 0} r_\varepsilon = +\infty\,, \qquad \lim_{\varepsilon\to 0} K_{r_\varepsilon} = 0.$$

From here to the end, we will refer to $(r(t), p_r(t))$ as a T-periodic solution of (4.14), where L, k and p_z will be specified depending of the case. By Remark 21, we recall that $|\dot{r}(t)| < 1$ for all $t \in \mathbb{R}$, thus the oscillation of $r(t)$ is bounded by the period, i.e.,

$$\max_{t\in[0,T]} r(t) - \min_{t\in[0,T]} r(t) < T.$$

On the other hand, the periodicity of the solution in (4.14) implies the existence of $t_0 \in [0,T]$ such that

$$r^3\left(t_0\right)\left[A\left(t_0, r\left(t_0\right)\right) - p_z\right] \partial_r A\left(t_0, r\left(t_0\right)\right) = L^2.$$

We begin with the lower bound.

Lemma 2. *Let I_0, L be positive, $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying* (3.2). *Then, for any $\bar{k} \in [0, K_T]$, there exists a constant $r_m > 0$, depending on $\bar{k}$ and $\boldsymbol{P}(\bar{k})$, such that*

$$r_m < \min\left\{r(t) : t \in [0,T],\ (k,p_z) \in [0,\bar{k}] \times \left(-\infty, \boldsymbol{P}(\bar{k})\right]\right\}.$$

Proof. Let us fix $(k, p_z) \in [0,\bar{k}] \times \left(-\infty, \mathbf{P}(\bar{k})\right]$. Recalling that $T > \mathbf{r}$, with $\mathbf{r}$ defined in (4.11), using (4.12) we get

$$A(t,T) > -\frac{\mu_0}{2\pi}\left[I_0 \ln T + k\frac{T^2}{3}\|\ddot{I}\|_\infty\right] \geq \mathbf{P}(\bar{k}), \quad \text{for all } t \in \mathbb{R}.$$

So, in particular, $A(t_0, T) - p_z > 0$. Furthermore, Theorem 15 state that $A(t, r)$ is strictly decreasing when $r \le T$, and, because of the signs in (4.17), $r(t_0) > T$ necessarily.

Finally, define the set

$$\mathcal{Z}_m(\bar{k}) = \{r \in \mathbb{R}^+ : \mathbf{P}(\bar{k}) = A(t,r), \text{ for some } (t,k) \in [0,T] \times [0,\bar{k}]\},$$

that is closed because A is continuous in (t, r, k). Therefore, it has a minimum $m(\bar{k}) := \min \mathcal{Z}_m(\bar{k}) > T$, and

$$\min_{t\in[0,T]} r(t) > \max_{t\in[0,T]} r(t) - T > m(\bar{k}) - T := r_m > 0.$$

■

The above result gives an explicit set for k and p_z such that the particles with radially periodic motion do not collide with the wire. This cannot be doing for the upper bound, at least using the identity (4.17), due to the oscillations of $A(t, r)$ for any $k > 0$ when r is large. More precisely, by iv) in Theorem 15,

$$r^3 \left[A(t,r) - p_z\right] \partial_r A(t,r) = -p_z r^{5/2} \mathcal{G}(t,r) + O(r^2), \quad r >> 1.$$

Therefore, for any $L > 0$, there exists a sequence $\{(t_n, r_n)\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} r_n = +\infty$, and

$$r_n^3 \left[A(t_n,r_n) - p_z\right] \partial_r A(t_n,r_n) = L^2, \quad \text{for all } n \in \mathbb{N}.$$

However, under suitable conditions, the control for the decreasing of $A(t, r)$, stated in iii)-Theorem 15, allows to find explicit radius where there are no periodic solutions of (4.14). To this aim, let us recall the identity (4.8):

$$r^2 \left[P_0 + \frac{\mu_0}{2\pi}\left(I_0 \ln r - K_{r+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right)\right] = \frac{2\pi}{\mu_0}\frac{L^2}{I_0}\frac{C_1 + C_2(r+T+\varepsilon)}{C_2\varepsilon}.$$

Lemma 3. *Let I_0, L be positive, $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying* (3.2) *and $P_0 < \boldsymbol{P}(K_T)$. Then, for any $\varepsilon > 0$ there exists a unique r_ε satisfying* (4.8)*. Moreover,*

i) $\lim_{\varepsilon\to 0} r_\varepsilon = +\infty$.

ii) $r_\varepsilon \notin \{r(t) : t \in [0,T],\ (k,p_z) \in [0, K_{r_\varepsilon+T+\varepsilon}] \times [P_0, \boldsymbol{P}(K_T)]\}$.

Proof. Fix $\varepsilon > 0$ arbitrarily and let us write (4.8) as $f_\varepsilon(r) = g_\varepsilon(r)$. Firstly, it is not difficult to see that $f(r)$ has a unique critical point $\widetilde{r}$, which is a global minimum. Furthermore,

$$\lim_{r\to 0^+} f_\varepsilon(r) = 0^- \text{ and } \lim_{r\to+\infty} f_\varepsilon(r) = +\infty.$$

Then, as $g_\varepsilon(r)$ is an increasing linear function with $g_\varepsilon(0) > \frac{2\pi}{\mu_0}\frac{L^2}{I_0}$, the intersection point r_ε is unique.

Regarding the properties, $i)$ is trivial because g_ε tends to $+\infty$ when $\varepsilon \to 0$. On the other hand, as K_r is decreasing in r, by (4.7) we have that

$$-\mathbf{P}(K_T) > \frac{\mu_0}{2\pi}\left(I_0 \ln T + K_{r+T}\frac{T^2}{3}\|\ddot{I}\|_\infty\right),$$

for any $r > 0$. As a consequence,

$$P_0 + \frac{\mu_0}{2\pi}\left(I_0 \ln T - K_{r+T}\frac{T^2}{3}\|\ddot{I}\|_\infty\right) < P_0 - \mathbf{P}(K_T) < 0,$$

for any $r > 0$ and then $f(T) < 0$, from where it follows that $r_\varepsilon > T$. In particular, $r_\varepsilon > \mathbf{r}$ and, by (4.12),

$$|a(t,r)| < \frac{T^2}{3}\|\ddot{I}\|_\infty, \text{ for all } (t,r) \in [0,T] \times [r_\varepsilon, +\infty). \tag{4.18}$$

Now fix $(k, p_z) \in (0, K_{r_\varepsilon+T+\varepsilon}] \times [P_0, \mathbf{P}(K_T)]$. Firstly, let us see that r_ε cannot satisfy (4.17). To this aim, defining the function

$$F(t,r) = r^3\left[-p_z + A(t,r)\right]\partial_r A(t,r),$$

we can write (4.17) as $F(t_0, r(t_0)) = L^2$. By (4.18) and (4.8), we have that

$$\begin{aligned} p_z - A(t, r_\varepsilon) &= p_z + \frac{\mu_0}{2\pi}\left[I_0 \ln r_\varepsilon + k a(t, r_\varepsilon)\right] \\ &> P_0 + \frac{\mu_0}{2\pi}\left[I_0 \ln r_\varepsilon - K_{r_\varepsilon+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right] > 0, \end{aligned}$$

for every $t \in \mathbb{R}$. On the other hand, as $k \leq K_{r_\varepsilon+T+\varepsilon}$, then $\partial_r A(t, r_\varepsilon)$ is strictly negative and we can write

$$F(t, r_\varepsilon) = r_\varepsilon^3\left[p_z - A(t, r_\varepsilon)\right]\left|\partial_r A(t, r_\varepsilon)\right|.$$

Using (4.4),

$$\begin{aligned} F(t, r_\varepsilon) &> \frac{I_0\mu_0}{2\pi} r_\varepsilon^2\left(P_0 + \frac{\mu_0}{2\pi}\left[I_0 \ln r_\varepsilon - K_{r_\varepsilon+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right]\right)\left[1 - \frac{K_{r_\varepsilon+T+\varepsilon}}{K_{r_\varepsilon+T}}\right] \\ &= \frac{I_0\mu_0}{2\pi} r_\varepsilon^2\left(P_0 + \frac{\mu_0}{2\pi}\left[I_0 \ln r_\varepsilon - K_{r_\varepsilon+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right]\right)\frac{C_2\varepsilon}{C_1 + C_2(r_\varepsilon + T + \varepsilon)}, \end{aligned}$$

where in the last equality we have used the definition (4.3) of K_r. Then, from (4.8) it follows that $F(t, r_\varepsilon) > L^2$ for all $t \in [0,T]$ and, consequently,

$$r_\varepsilon \notin \{r(t_0) : r(t)\ T\text{-periodic with } (k, p_z) \in (0, K_{r_\varepsilon+T+\varepsilon}] \times [P_0, \mathbf{P}(K_T)]\}.$$

However, if there exists a $t_1 \neq t_0$ such that $r(t_1) = r_\varepsilon$, condition (4.17) must be satisfied at some point $r \in (r_\varepsilon, r_\varepsilon + T)$. Reasoning for any r in this set as we just did above for r_ε, we obtain that

$$\begin{aligned} F(t,r) &> \frac{I_0\mu_0}{2\pi} r^2\left(P_0 + \frac{\mu_0}{2\pi}\left[I_0 \ln r - K_{r_\varepsilon+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right]\right)\left[1 - \frac{K_{r_\varepsilon+T+\varepsilon}}{K_{r_\varepsilon+T}}\right] \\ &> \frac{I_0\mu_0}{2\pi} r_\varepsilon^2\left(P_0 + \frac{\mu_0}{2\pi}\left[I_0 \ln r_\varepsilon - K_{r_\varepsilon+T+\varepsilon}\frac{T^2}{3}\|\ddot{I}\|_\infty\right]\right)\left[1 - \frac{K_{r_\varepsilon+T+\varepsilon}}{K_{r_\varepsilon+T}}\right], \end{aligned}$$

and therefore $F(t,r) > L^2$, for all $(t,r) \in [0,T] \times [r_\varepsilon, r_\varepsilon + T]$, which proves $ii)$. ■

Corollary 3. *Let I_0, L be positive, $I(t) \in C^2(\mathbb{R}/T\mathbb{Z})$ satisfying* (3.2) *and $P_0 < \boldsymbol{P}(K_T)$. Then,*

$$r_\varepsilon > \bar{r}, \quad \text{for any } (\varepsilon, p_z) \in \mathbb{R}^+ \times [P_0, \boldsymbol{P}(K_T)],$$

where $\bar{r}$ is the corresponding (L, p_z)-equilibrium defined in (4.16).

Proof. Fix $\varepsilon > 0$, $p_z \in [P_0, \mathbf{P}(K_T)]$ arbitrarily and let us write again (4.8) as $f_\varepsilon(r) = g_\varepsilon(r)$. By (4.16),

$$f_\varepsilon(\bar{r}) < \bar{r}^2 \left[p_z + \frac{\mu_0}{2\pi} I_0 \ln \bar{r}\right] = \frac{2\pi}{\mu_0} \frac{L^2}{I_0} < g_\varepsilon(0) < g_\varepsilon(r_\varepsilon) = f_\varepsilon(r_\varepsilon).$$

Then, as $f_\varepsilon(r)$ is increasing for all $r \geq r_\varepsilon$, the result is proven.

∎

Finally, we prove Theorem 17.

Proof. Let us obtain a bound for $p_r(t)$. To this aim, fix $\varepsilon > 0$ and take $(\bar{k}, p_z) \in [0, K_{r_\varepsilon + T + \varepsilon}] \times [P_0, \mathbf{P}(K_T)]$. Because of Lemmas 2 and 3, we assume the existence of a T-periodic solution such that $r(t) \in \{r \in \mathbb{R} : r_m < r < r_\varepsilon\}$. Then, by periodicity, there exists a $\bar{t} \in [0, T]$ such that $p_r(\bar{t}) = 0$ and, integrating $\dot{p}_r$ in $[\bar{t}, t]$, with t arbitrary, we obtain the a priori bound:

$$\begin{aligned}
|p_r(t)| &= \left| \int_{\bar{t}}^{t} \frac{L^2 r^{-3}(s) + (p_z - A(s, r(s)))\partial_r A(s, r(s))}{\sqrt{1 + (p_z - A(s, r(s)))^2 + p_r^2 + L^2 r^{-2}(s)}} ds \right| \\
&< \int_{\bar{t}}^{t} L^2 r^{-3}(s) ds + \int_{\bar{t}}^{t} \frac{|p_z - A(s, r(s))| \, |\partial_r A(s, r(s))|}{\sqrt{1 + (p_z - A(s, r(s)))^2 + p_r^2 + L^2 r^{-2}(s)}} ds \\
&< T \frac{L^2}{r_m^3} + \int_{\bar{t}}^{t} |\partial_r A(s, r(s))| \, ds < T \frac{L^2}{r_m^3} + C(\bar{k}) := P(\bar{k}) < \infty,
\end{aligned}$$

with $C(\bar{k}) = \max\{|\partial_r A(t, r)| \, ; \, (t, r) \in [0, T] \times [r_m, r_\varepsilon]\}$. Due to this, we define the sets:

$$\begin{aligned}
&\Omega_{\varepsilon, \bar{k}} = \{x \in \mathbb{R}^2 : r_m < x_1 < r_\varepsilon, \ |x_2| < P(\bar{k})\}; \\
&\Omega = \{x \in X : \ \mathrm{Im}(x) \subset \Omega_{\varepsilon, \bar{k}}\}.
\end{aligned}$$

On the other hand, take $k = \lambda \bar{k}$, with $\lambda \in [0, 1]$, and let us consider the corresponding potential $A_\lambda(t, r)$ and hamiltonian function $\mathcal{H}_\lambda(t, r, p_r)$ giving by (4.1) and (4.15) respectively. By (4.2), $\mathcal{H}_\lambda(t, r, p_r)$ is T-periodic and regular in its domain for any $\lambda \in [0, 1]$. Furthermore, as $\mathcal{H}_0$ is autonomous, hypotheses i)-ii) of Section 1.3.3 are verified and the homotopic problem associated to (4.14) satisfies (1.25). Therefore, we conclude applying Corollary 2 in Section 1.3.3 to $f_0(r, p_r) = \mathcal{H}_0^2(r, p_r)$, for which it only remain to study its critical points in $\Omega_{\bar{k}}$:

$$\partial_r \mathcal{H}_0^2 = \left(p_z + \frac{I_0 \mu_0}{2\pi} \ln r\right) \frac{I_0 \mu_0}{\pi} \frac{1}{r} - 2\frac{L^2}{r^3}; \quad \partial_{p_r} \mathcal{H}_0^2 = 2p_r.$$

After basic computations, it follows that the equilibrium $(\bar{r}, 0)$, which belongs to the interior of $\Omega_{\bar{k}}$ by Corollary 3, is the unique critical point of $\mathcal{H}_0^2(r, p_r)$. Concerning the derivatives of second order, clearly

$$\partial^2_{r\,p_r}\mathcal{H}_0^2(r,p_r) = \partial^2_{p_r\,r}\mathcal{H}_0^2(r,p_r) = 0, \quad \text{and} \quad \partial^2_{p_r p_r}\mathcal{H}_0^2(r,p_r) = 2.$$

Moreover,

$$\begin{aligned}
\partial^2_{rr}\mathcal{H}_0^2(r,p_r) &= \frac{I_0^2\mu_0^2}{2\pi^2}\frac{1}{r^2} - \left(p_z + \frac{I_0\mu_0}{2\pi}\ln r\right)\frac{I_0\mu_0}{\pi}\frac{1}{r^2} + 6\frac{L^2}{r^4} \\
&= -\frac{\partial_r\mathcal{H}_0^2}{r}(r,p_r) + \frac{1}{r^2}\left(\frac{I_0^2\mu_0^2}{2\pi^2} + 4\frac{L^2}{r^2}\right).
\end{aligned}$$

Finally, as $\partial_r\mathcal{H}_0^2(\bar{r},0) = 0$,

$$\left|\mathrm{Hess}\mathcal{H}_0^2(\bar{r},0)\right| = \frac{2}{\bar{r}^2}\left(\frac{I_0^2\mu_0^2}{2\pi^2} + 4\frac{L^2}{\bar{r}^2}\right) > 0,$$

and the result is proven. ■

4.5 Appendix on Bessel functions

As some of the bounds obtained in Section 4.2 are computed using properties of the Bessel functions, we consider necessary to include this appendix just to comment some basic aspects of them. In particular, we need to justify the inequality

$$|\mathcal{Y}_0(z)| \leq \frac{2}{\pi}\left[\left|\ln\left(\frac{z}{2}\right)\right| + \gamma\right] + \frac{2}{\pi}\exp(z^2/4), \quad z \geq 0. \tag{4.19}$$

Before to define them, let us recall some brief notions about the Gamma function $\Gamma(x)$, which we use in our development.

Gamma function

Defined as the improper integral

$$\Gamma(x) = \int_0^{+\infty} s^{x-1}e^{-s}ds, \quad x > 0.$$

It is clear that Γ diverges in the origin, moreover

$$\Gamma(m) = (m-1)!, \quad m \in \mathbb{N}.$$

On the other hand, we denote by ψ_0 the Digamma function, defined as the logarithmic derivative of Γ, i.e. $\frac{\Gamma'(x)}{\Gamma(x)} = \psi_0(x)$. About this, we just remark the following property

$$\psi_0(x+1) = \psi_0(x) + \frac{1}{x}, \quad x > 0.$$

In particular, this implies:

$$\psi_0(m+1) = \psi_0(1) + \sum_{n=1}^{m}\frac{1}{n} = -\gamma + \sum_{n=1}^{m}\frac{1}{n}, \quad m \in \mathbb{N}, \tag{4.20}$$

where γ is commonly known as the Euler-Masheroni constant.

Bessel functions

Given $\nu \in \mathbb{R}$ and $z \geq 0$, the Bessel function of first kind, order ν and argument z, is defined as

$$\mathcal{J}_\nu(z) = \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+\nu}}{m!\Gamma(m+\nu+1)}.$$

This expression is a linear combination of solutions for the Bessel's equation

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)y = 0,$$

which is also a solution by linearity. In particular, when $\nu = 0$ we get

$$\mathcal{J}_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m}}{(m!)^2}. \tag{4.21}$$

One of its basic properties of is that $\mathcal{J}_\nu(z)$ is analytic for any $\nu \in \mathbb{R}$ and any $z \neq 0$. Therefore, the Bessel function of second kind and natural order $n \in \mathbb{N}$ is defined as the derivative

$$\mathcal{Y}_n(z) = \frac{1}{\pi}\left[\frac{\partial J_\nu}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}}{\partial \nu}\right]_{\nu=n}, \quad n \in \mathbb{N}.$$

It is not difficult to see that $\frac{\partial J_\nu}{\partial \nu}(z)|_{\nu=0} = -\frac{\partial J_{-\nu}}{\partial \nu}(z)|_{\nu=0}$. By this,

$$\begin{aligned}\mathcal{Y}_0(z) =& \frac{2}{\pi}\frac{\partial J_\nu}{\partial \nu}(z)\Big|_{\nu=0} = \frac{2}{\pi}\left[\frac{\partial}{\partial \nu}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+\nu}}{m!\Gamma(m+\nu+1)}\right]_{\nu=0} \\ =& \frac{2}{\pi}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+\nu}}{m!\Gamma(m+\nu+1)}\ln\left(\frac{z}{2}\right) - \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+\nu}}{m!\Gamma(m+\nu+1)}\psi_0(m+\nu+1)\right]_{\nu=0}.\end{aligned}$$

Then, using (4.21) and (4.20), we obtain

$$\mathcal{Y}_0(z) = \frac{2}{\pi}\left[\ln\left(\frac{z}{2}\right) + \gamma\right]\mathcal{J}_0(z) - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m}}{(m!)^2}\left[1 + ... + \frac{1}{m}\right].$$

Finally, as $|\mathcal{J}_0(z)| \leq 1$ for all z, we get (4.19):

$$\begin{aligned}|\mathcal{Y}_0(z)| \;&\leq\; \frac{2}{\pi}\left[\left|\ln\left(\frac{z}{2}\right)\right| + \gamma\right] + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(z^2/4)^m}{m!} \\ &=\; \frac{2}{\pi}\left[\left|\ln\left(\frac{z}{2}\right)\right| + \gamma\right] + \frac{2}{\pi}\exp(z^2/4).\end{aligned}$$

Chapter 5

Coupling Poincaré-Birkhoff with upper and lower solutions

This last chapter corresponds with the paper *An extension of the Poincaré–Birkhoff Theorem coupling twist with lower and upper solutions* [42], coauthored by Alessandro Fonda and Andrea Sfecci, from the University of Trieste, and that is actually submitted for publication.

As it was detailed in the first chapter, in [42] we extend some famous results of C.C. Conley and E.J. Zehnder in [29], where they gave a partial answer to a conjecture by V.I. Arnold [9, 10]. Concretely, their result consider a periodic problem for general Hamiltonian system

$$S\dot{z} = \nabla H(t, z)\,. \tag{5.1}$$

where $S = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ is the standard symplectic matrix, $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is T-periodic in t, and $\nabla H(t, z)$ denotes its gradient with respect to z. Writing $z = (q, p)$, with $q = (q_1, \ldots, q_N)$ and $p = (p_1, \ldots, p_N)$, the system reads as

$$\dot{q} = \partial_p H(t, q, p)\,, \quad \dot{p} = -\partial_q H(t, q, p)\,.$$

Let us recall again the two main results of [29].

In a first theorem, assuming the Hamiltonian function $H(t, z)$ to be periodic in all variables $q_1, \ldots, q_N$ and $p_1, \ldots, p_N$, they prove that system (5.1) has at least $2N + 1$ geometrically distinct T-periodic solutions. In a second theorem, they assume H to be periodic in $q_1, \ldots, q_N$ and to have a quadratic behaviour in p such that there exist a constant $R > 0$ and a symmetric regular matrix $\mathbb{A}$ for which $H(t, q, p) = \frac{1}{2}\langle \mathbb{A}p, p\rangle$ + “lower order terms”, when $|p| \geq R$. In this setting, they prove that system (5.1) has at least $N + 1$ geometrically distinct T-periodic solutions. They also mention a possible relation of this second result with the Poincaré-Birkhoff Theorem.

These pioneering results in [29] have been extended in different directions, like in [25, 37, 44, 48, 55–57, 60, 74, 79, 86, 88, 105, 112].

Our approach is strongly based on the higher-dimensional generalizations of the Poincaré-Birkhoff Theorem established by Fonda and Ureña [55–57], where some variants of the twist condition in higher dimensions are provided. These works have also been extended by Fonda and Gidoni in [43], assuming H to be periodic in $q_1, \ldots, q_N$ and possibly also in $p_1, \ldots, p_L$, for some $L \in \{1, \ldots, N\}$, together with a very general twist condition, thus finding $N + L + 1$ periodic solutions. In a second paper [44], they obtain a most general result assuming that the Hamiltonian function includes a nonresonant quadratic term. As it was commented in Chapter 1.2.2, several applications of this theory have been provided [19, 21, 41, 44, 45, 50, 51, 53, 56], generalizing some previously established results for second order equations [24, 34, 35, 47, 52, 69, 73].

It is the aim of this chapter to further extend this fertile theory to systems that, besides the periodicity-twist conditions illustrated above, also present a pair of well-ordered lower and upper solutions. In order to better explain this situation, let the considered Hamiltonian system be of the type

$$\begin{cases} \dot{q} = \partial_p H(t, q, p, u, v)\,, & \dot{p} = -\partial_q H(t, q, p, u, v)\,, \\ \dot{u} = \partial_v H(t, q, p, u, v)\,, & \dot{v} = -\partial_u H(t, q, p, u, v)\,. \end{cases} \tag{5.2}$$

Here $H : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ is T-periodic in t, periodic in q, and has a twist involving the (q, p) variables. At the same time, we also assume that there exist some constant lower/upper solutions $\alpha \leq \beta$ involving the (u, v) variables. In this situation we are able to prove the existence of two geometrically distinct T-periodic solutions. The result will then be extended to higher dimensional systems.

The reader is surely familiar with the method of lower/upper solutions in the case of scalar equations like, for instance,

$$\ddot{u} = f(t, u)\,.$$

This method has a long history, dating back to the pioneering papers [93, 98, 108], see also the book [33] for a detailed exposition. We recall that the T-periodic functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are said to be a lower solution and an upper solution, respectively, if

$$\ddot{\alpha}(t) \geq f(t, \alpha(t))\,, \qquad \ddot{\beta}(t) \leq f(t, \beta(t))\,,$$

for every $t \in [0, T]$. Recently the theory has been extended to periodic planar systems in [46, 54]. This will be the approach adopted here, even if we will only be able to deal with the case of constant lower and upper solutions.

Let us briefly explain why we need to reduce to constant lower/upper solutions. The standard proof procedure in lower/upper solutions theorems is as follows: a) modify the problem below $\alpha(t)$ and above $\beta(t)$, in the u component, by the use of a truncating function; b) show that the modified problem has a solution; c) prove that the u component of this solution stays between $\alpha(t)$ and $\beta(t)$. The technical difficulty encountered in the present work is that we need to maintain the Hamiltonian structure, hence the modification of the problem has to be made in the Hamiltonian function itself, being careful to preserve the differentiability of the new function. Once the modification has been made, we are allowed to apply the results in [44]. We believe that this technical difficulty could be overcome, but for now the case of nonconstant lower/upper solutions remains an open problem.

Another open problem arises in the case of non-well-ordered lower and upper solutions. Assuming some nonresonance conditions with respect to the higher part of the spectrum, this case is usually treated by topological degree methods. We do not know how to adapt this type of technique to our situation.

In order to maintain a friendly exposition, the chapter is written following an increasing order of complexity, first presenting the main ideas in the simplest situation, then extending them to more general systems. The chapter is thus organized as follows.

In Section 5.1 we state our result in the simple case of system (5.2). The proof is provided in Section 5.2. Then, in Section 5.3, we provide some variants of the first theorem. In particular, we state a version of the theorem involving a topological annulus, in the spirit of Poincaré's original statement. In Section 5.4 we illustrate several examples of applications.

In Section 5.5 we extend the result to higher dimensions, thus generalizing both the Conley–Zehnder theorems presented above. The proof is provided in Section 5.6. In Section 5.7 we extend the higher dimensional result to systems whose Hamiltonian function further involves a quadratic term and some examples of possible applications are given in Section 5.8. In Section 5.9 we provide a further application to the study of periodic solutions to perturbations of completely integrable systems.

Finally, in Section 5.10 we establish the most general result of [42], where the twist condition is stated as an "avoiding cones condition".

5.1 Statement of the first result

In this section we consider the Hamiltonian system (5.2), where the Hamiltonian function $H : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ is assumed to be continuous, T-periodic in t, and continuously differentiable in the q, p, u, v variables.

Let us state our assumptions.

Assumption 1 (Periodicity). *The function $H(t, q, p, u, v)$ is periodic in q.*

To fix the ideas, we will assume the period in q to be equal to 2π. Under this setting, T-periodic solutions of (5.2) appear in equivalence classes made of those solutions whose components $q(t)$ differ by an integer multiple of 2π. We say that two T-periodic solutions are *geometrically distinct* if they do not belong to the same equivalence class.

Assumption 2 (Lower and upper solutions)**.** *There exist some constants $\delta > 0$ and $\alpha \leq \beta$ such that*

$$v\,\partial_v H(t,q,p,u,v) > 0\,, \text{ when } u \in [\alpha-\delta,\alpha] \cup [\beta,\beta+\delta] \text{ and } v \neq 0\,, \tag{5.3}$$

and

$$\begin{cases} \partial_u H(t,q,p,u,0) \geq 0\,, & \text{when } u \in [\alpha-\delta,\alpha]\,, \\ \partial_u H(t,q,p,u,0) \leq 0\,, & \text{when } u \in [\beta,\beta+\delta]\,. \end{cases} \tag{5.4}$$

The above assumption comes from the definition of lower and upper solutions given in [46,54]. We require here that these lower and upper solutions are constant. More precisely, all constants in $[\alpha-\delta,\alpha]$ are lower solutions, and all constants in $[\beta,\beta+\delta]$ are upper solutions. In the sequel, the constant $\delta > 0$ provided by Assumption 2 will be used without further mention.

Assumption 3 (Nagumo condition)**.** *There exist $d > 0$ and two continuous functions $f,\varphi : [d,+\infty) \to (0,+\infty)$, with*

$$\int_d^{+\infty} \frac{f(s)}{\varphi(s)}\,ds = +\infty\,,$$

satisfying the following property. If $u \in [\alpha-\delta,\beta+\delta]$, then

$$\begin{cases} \partial_v H(t,q,p,u,v) \geq f(v)\,, & \text{when } v \geq d\,, \\ \partial_v H(t,q,p,u,v) \leq -f(-v)\,, & \text{when } v \leq -d\,, \end{cases}$$

and

$$|\partial_u H(t,q,p,u,v)| \leq \varphi(|v|)\,, \quad \text{when } |v| \geq d\,.$$

Assumption 4 (Linear growth)**.** *For every $K > 0$ there is a constant $C_K > 0$ such that*

$$|\partial_q H(t,q,p,u,v)| \leq C_K(|p|+1)\,, \quad \text{when } u \in [\alpha-\delta,\beta+\delta] \text{ and } |v| \leq K\,.$$

Remark 22. *Notice that, under the above assumption, for any two continuous functions $U,V : [0,T] \to \mathbb{R}$, with*

$$\alpha-\delta \leq U(t) \leq \beta+\delta\,, \quad \text{for every } t \in [0,T]\,, \tag{5.5}$$

the solutions of the system

$$\dot q = \partial_p H(t,q,p,U(t),V(t))\,, \quad \dot p = -\partial_q H(t,q,p,U(t),V(t)) \tag{5.6}$$

are defined on $[0,T]$. *Indeed, let* $(q(t),p(t))$ *be a solution of system* (5.6) *starting at time* $t=0$ *from some* $(q(0),p(0))=(q_0,p_0)$. *This solution is defined on a maximal interval of future existence* $[0,T]\cap[0,\omega)$. *Set* $K=\|V\|_\infty$. *By Assumption 4,*

$$|\dot p(t)|\le C_K(|p(t)|+1)\,,\qquad \textit{for every } t\in[0,T]\cap[0,\omega)\,,$$

which, combined with the Gronwall Lemma, yields

$$|p(t)|\le(|p_0|+1)\,e^{C_KT}\,,\qquad \textit{for every } t\in[0,T]\cap[0,\omega)\,.$$

Then, since H *is periodic in* q, *we have that there is a constant* $C>0$ *depending only on* U, V *and* $|p_0|$, *such that*

$$|\dot q(t)|\le C\,,\qquad \textit{for every } t\in[0,T]\cap[0,\omega)\,.$$

Hence,

$$|q(t)|\le|q_0|+CT\,,\qquad \textit{for every } t\in[0,T]\cap[0,\omega)\,.$$

This implies that the solution $(q(t),p(t))$ *must be defined on* $[0,T]$, *i.e., that* $\omega>T$.

Here is our first result.

Theorem 18. *Let Assumptions 1, 2, 3 and 4 hold. Assume that there exist* $a<b$ *and* $\rho>0$ *with the following property: For any two continuous functions* $U,V:[0,T]\to\mathbb{R}$ *satisfying* (5.5), *the solutions of system* (5.6) *are such that*

$$\begin{cases} q(T)-q(0)<0\,, & \textit{when } p(0)\in[a-\rho,a]\,,\\ q(T)-q(0)>0\,, & \textit{when } p(0)\in[b,b+\rho]\,. \end{cases}\tag{5.7}$$

Then, there exist at least two geometrically distinct T*-periodic solutions of system* (5.2), *such that* $p(0)\in(a,b)$ *and*

$$\alpha\le u(t)\le\beta\,,\quad \textit{for every } t\in\mathbb{R}\,.\tag{5.8}$$

The same conclusion holds if (5.7) *is replaced by*

$$\begin{cases} q(T)-q(0)>0\,, & \textit{when } p(0)\in[a-\rho,a]\,,\\ q(T)-q(0)<0\,, & \textit{when } p(0)\in[b,b+\rho]\,. \end{cases}$$

5.2 The proof of Theorem 18

The proof is based on [44, Corollary 2.4], and will be divided in two steps. In the first step we analyze the dynamics focusing on the (u,v) variables. In the second one, we draw our attention on the (q,p) variables.

5.2.1 Working with the (u, v) coordinates

Let us focus our attention on the couple of variables (u, v). At first, we are going to modify the original problem (5.2) outside some suitably chosen set $\mathcal{V} \subseteq \mathbb{R}^4$. We will then prove that all the T-periodic solutions of the modified system must be such that $z(t) = (q(t), p(t), u(t), v(t)) \in \mathcal{V}$ for every $t \in [0, T]$. Hence, such solutions will solve the original problem (5.2), too.

Assumption 3 permits us to apply the reasoning in [54, Theorem 3.1] (see also [46, Lemma 15]) in order to find two continuously differentiable functions $\gamma_\pm : \mathbb{R} \to \mathbb{R}$ satisfying

$$\gamma_-(s) < -d < d < \gamma_+(s)\,,$$

for every $s \in [\alpha - \delta, \beta + \delta]$, and

$$-\partial_u H\big(t, q, p, u, \gamma_+(u)\big) > \partial_v H\big(t, q, p, u, \gamma_+(u)\big)\ \gamma_+'(u)\,, \tag{5.9}$$

$$-\partial_u H\big(t, q, p, u, \gamma_-(u)\big) < \partial_v H\big(t, q, p, u, \gamma_-(v)\big)\ \gamma_-'(u)\,, \tag{5.10}$$

for every $(t, q, p, u) \in [0, T] \times [0, 2\pi] \times \mathbb{R} \times [\alpha - \delta, \beta + \delta]$. Correspondingly, we introduce the set

$$\mathcal{V} = \{z = (q, p, u, v) \mid \alpha \le u \le \beta\,,\ \gamma_-(u) < v < \gamma_+(u)\}\,.$$

Now we can choose a constant $\widehat{d} > d$ satisfying

$$-\widehat{d} < \gamma_-(s) < \gamma_+(s) < \widehat{d}\,,$$

for every $s \in [\alpha - \delta, \beta + \delta]$.

We consider a continuously differentiable function $\zeta : \mathbb{R} \to \mathbb{R}$ such that

$$\zeta(u) = \begin{cases} \alpha - \delta\,, & \text{if } u \le \alpha - 2\delta\,, \\ u\,, & \text{if } \alpha \le u \le \beta\,, \\ \beta + \delta\,, & \text{if } u \ge \beta + 2\delta\,, \end{cases} \tag{5.11}$$

and

$$\zeta'(u) > 0\,,\ \text{ when } u \in\ (\alpha - 2\delta, \beta + 2\delta)\,. \tag{5.12}$$

Notice that

$$\alpha - \delta \le \zeta(u) \le \beta + \delta\,,\quad \text{for every } u \in \mathbb{R}\,.$$

Then, we introduce a continuously differentiable function $\chi : [\alpha-\delta, \beta+\delta] \times \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

$$\chi(u, v) = \begin{cases} -\widehat{d}\,, & \text{if } v < -\widehat{d} - 1\,, \\ v\,, & \text{if } \gamma_-(u) \le v \le \gamma_+(u)\,, \\ \widehat{d}\,, & \text{if } v > \widehat{d} + 1\,. \end{cases} \tag{5.13}$$

Moreover, we assume that

$$\partial_v \chi(u,v) > 0\,, \quad \text{when } |v| < \widehat{d} + 1\,. \tag{5.14}$$

Let $\widehat{H} : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ be defined as

$$\widehat{H}(t,q,p,u,v) = H\big(t,q,p,\zeta(u),\chi(\zeta(u),v)\big) + \mathcal{H}(u,v)\,,$$

with

$$\mathcal{H}(u,v) = \frac{1}{2}\left[\Big[(v-\widehat{d})^+\Big]^2 + \Big[(v+\widehat{d})^-\Big]^2 - \big[(u-\beta)^+\big]^2 - \big[(u-\alpha)^-\big]^2\right]\,,$$

where, as usual, $\xi^+ = \max\{\xi, 0\}$ and $\xi^- = \max\{-\xi, 0\}$. We consider the modified Hamiltonian system

$$\begin{cases} \dot q = \partial_p \widehat{H}(t,q,p,u,v)\,, & \dot p = -\partial_q \widehat{H}(t,q,p,u,v)\,, \\ \dot u = \partial_v \widehat{H}(t,q,p,u,v)\,, & \dot v = -\partial_u \widehat{H}(t,q,p,u,v)\,. \end{cases} \tag{5.15}$$

Notice that $\widehat{H} = H$ in the closure of the set $\mathcal{V}$. Our aim is to prove the following a priori bound.

Lemma 4. *If $z = (q,p,u,v)$ is a T-periodic solution of* (5.15)*, then $z(t) \in \mathcal{V}$ for every $t \in [0,T]$, hence it solves* (5.2)*.*

The proof of this lemma needs some preparation, hence it will be provided at the end of the section.

In what follows, for every $z = (q,p,u,v)$ we introduce the notation

$$\tilde z = (q,p,\zeta(u),\chi(\zeta(u),v))\,.$$

We can explicitly compute

$$\partial_q \widehat{H}(t,z) = \partial_q H\big(t,\tilde z\big)\,, \qquad \partial_p \widehat{H}(t,z) = \partial_p H\big(t,\tilde z\big)\,, \tag{5.16}$$

$$\partial_u \widehat{H}(t,z) = \big[\partial_u H\big(t,\tilde z\big) + \partial_v H\big(t,\tilde z\big)\,\partial_u \chi(\zeta(u),v)\big]\,\zeta'(u) + \partial_u \mathcal{H}(u,v)\,, \tag{5.17}$$

$$\partial_v \widehat{H}(t,z) = \partial_v H\big(t,\tilde z\big)\,\partial_v \chi(\zeta(u),v) + \partial_v \mathcal{H}(u,v)\,. \tag{5.18}$$

Proposition 5. *Any solution of* (5.15) *satisfies, for every $t_0 \in \mathbb{R}$,*

$$\begin{aligned} &[u(t_0) < \alpha \;\; \textit{and} \;\; v(t_0) = 0] \quad \Rightarrow \quad \dot v(t_0) < 0\,, \\ &[u(t_0) > \beta \;\; \textit{and} \;\; v(t_0) = 0] \quad \Rightarrow \quad \dot v(t_0) > 0\,. \end{aligned}$$

Proof. We first note that, as an immediate consequence of (5.3), we have

$$\partial_v H(t,q,p,u,0) = 0\,, \quad \text{when } u \in [\alpha-\delta,\alpha]\cup[\beta,\beta+\delta]\,. \tag{5.19}$$

We prove the first implication, the second one being similar. Let $t_0 \in \mathbb{R}$ be such that $u(t_0) < \alpha$ and $v(t_0) = 0$. Let $z(t)$ be a solution of (5.15), with $z(t_0) = (q(t_0), p(t_0), u(t_0), 0)$. Then, $\tilde{z}(t_0) = (q(t_0), p(t_0), \zeta(u(t_0)), 0)$. By (5.17) and (5.19) we have that

$$\begin{aligned}\dot v(t_0) = -\partial_u \widehat{H}(t_0, z(t_0)) &= -\partial_u H(t_0, \tilde z(t_0))\zeta'(u(t_0)) - \partial_u \mathcal{H}(u(t_0), 0)\\ &= -\partial_u H(t_0, \tilde z(t_0))\zeta'(u(t_0)) + u(t_0) - \alpha\,,\end{aligned} \tag{5.20}$$

so that

$$\dot v(t_0) < -\partial_u H(t_0, \tilde z(t_0))\zeta'(u(t_0))\,.$$

Then, (5.4) and (5.12) give the negative sign when $u(t_0) \in (\alpha - 2\delta, \alpha)$. On the other hand, $\zeta'(u)$ vanishes when $u \le \alpha - 2\delta$ and the conclusion easily follows from (5.20). □

Proposition 6. *Any solution of* (5.15) *satisfies, for every* $t_0 \in \mathbb{R}$*,*

$$\begin{array}{lll}
[u(t_0) \le \alpha \;\; \textit{and} \;\; v(t_0) < 0] & \Rightarrow & \dot u(t_0) < 0\,,\\
[u(t_0) \le \alpha \;\; \textit{and} \;\; v(t_0) > 0] & \Rightarrow & \dot u(t_0) > 0\,,\\
[u(t_0) \ge \beta \;\; \textit{and} \;\; v(t_0) < 0] & \Rightarrow & \dot u(t_0) < 0\,,\\
[u(t_0) \ge \beta \;\; \textit{and} \;\; v(t_0) > 0] & \Rightarrow & \dot u(t_0) > 0\,.
\end{array}$$

Proof. Let us prove the first assertion, the proof of the others being similar. Let $t_0 \in \mathbb{R}$ be such that $u(t_0) \le \alpha$ and $v(t_0) < 0$. Since $\zeta(u) \in [\alpha - \delta, \alpha]$ when $u \le \alpha$ and $\chi(u, v) < 0$ when $v < 0$, recalling (5.3), (5.14), and (5.18), if $-\widehat{d} - 1 < v(t_0) < 0$ we get

$$\begin{aligned}\dot u(t_0) &= \partial_v H(t_0, \tilde z(t_0))\partial_v \chi(\zeta(u(t_0)), v(t_0)) - (v(t_0) + \widehat{d}\,)^-\\ &\le \partial_v H(t_0, \tilde z(t_0))\partial_v \chi(\zeta(u(t_0)), v(t_0)) < 0\,.\end{aligned}$$

On the other hand, if $v(t_0) \le -\widehat{d} - 1$, then $\partial_v \chi(\zeta(u(t_0)), v(t_0)) = 0$, so that $\dot u(t_0) = -(v(t_0) + \widehat{d}\,)^- < 0$. □

We define the open sets

$$\begin{aligned}A_{NW} &= \{z \in \mathbb{R}^4 \mid u < \alpha\,,\, v > 0\}\,, & A_{NE} &= \{z \in \mathbb{R}^4 \mid u > \beta\,,\, v > 0\}\,,\\ A_{SW} &= \{z \in \mathbb{R}^4 \mid u < \alpha\,,\, v < 0\}\,, & A_{SE} &= \{z \in \mathbb{R}^4 \mid u > \beta\,,\, v < 0\}\,.\end{aligned}$$

As a consequence of the previous propositions, the following can be easily proved.

Proposition 7. *For every solution z of* (5.15) *the following assertions hold:*

if there is $t_0 \in \mathbb{R}$ such that $z(t_0) \in A_{NW}$ then $z(t) \in A_{NW}$ for every $t < t_0$,
if there is $t_0 \in \mathbb{R}$ such that $z(t_0) \in A_{NE}$ then $z(t) \in A_{NE}$ for every $t > t_0$,
if there is $t_0 \in \mathbb{R}$ such that $z(t_0) \in A_{SW}$ then $z(t) \in A_{SW}$ for every $t > t_0$,
if there is $t_0 \in \mathbb{R}$ such that $z(t_0) \in A_{SE}$ then $z(t) \in A_{SE}$ for every $t < t_0$.

Hence, A_{NE} and A_{SW} are positively invariant sets, while A_{SE} and A_{NW} are negatively invariant.

Proposition 8. *Any solution of* (5.15) *satisfies, for every $t_0 \in \mathbb{R}$,*

$$[\alpha \le u(t_0) \le \beta \ \ \textit{and} \ \ v(t_0) > d] \quad \Rightarrow \quad \dot{u}(t_0) > 0\,,$$
$$[\alpha \le u(t_0) \le \beta \ \ \textit{and} \ \ v(t_0) < -d] \quad \Rightarrow \quad \dot{u}(t_0) < 0\,.$$

Proof. Let us prove the first implication. Since $u(t_0) \in [\alpha, \beta]$ and $v(t_0) > d$, we have that $\zeta(u(t_0)) = u(t_0)$ and

$$\dot{u}(t_0) = \partial_v H(t_0, \tilde{z}(t_0))\partial_v \chi(u(t_0), v(t_0)) + (v(t_0) - \widehat{d}\,)^+\,.$$

If $v(t_0) \ge \widehat{d} + 1$, then $\partial_v \chi(u(t_0), v(t_0)) = 0$ and $(v(t_0) - \widehat{d}\,)^+ > 0$. On the other hand, if $d < v(t_0) < \widehat{d} + 1$, then $\partial_v \chi(u(t_0), v(t_0)) > 0$ and $(v(t_0) - \widehat{d}\,)^+ \ge 0$. Moreover, $\partial_v H(t_0, \tilde{z}(t_0)) > 0$ by Assumption 3, thus proving that $\dot{u}(t_0) > 0$ in both cases. □

Proof of Lemma 4. Let z be a T-periodic solution of (5.15). We begin proving that $\alpha \le u(t) \le \beta$ for every t. By contradiction, assume that there exists $t_0 \in \mathbb{R}$ such that $u(t_0) > \beta$. If $z(t_0) \in A_{NE}$, by Proposition 7, since the function is periodic, $z(t) \in A_{NE}$ for every $t \in \mathbb{R}$. Then, we get a contradiction using Proposition 6. Similarly we cannot have $z(t_0) \in A_{SE}$. Finally, if $v(t_0) = 0$, Proposition 5 takes us to the previous contradicting situations. We have thus proved that $u(t) \le \beta$ for every $t \in \mathbb{R}$. A similar argument proves that $u(t) \ge \alpha$ for every $t \in \mathbb{R}$.

Now we show that $v(t) < \gamma_+(u(t))$ for every $t \in \mathbb{R}$. Let us define the function $G_+(t) = v(t) - \gamma_+(u(t))$. By Proposition 8, it cannot be that $G_+(t) > 0$ for every $t \in [0, T]$. Assume by contradiction the existence of $t_0 \in \mathbb{R}$ such that $G_+(t_0) = 0$. Then, since $\nabla H(t, z) = \nabla \widehat{H}(t, z)$ for z in the closure of $\mathcal{V}$, by (5.9) we have

$$G_+'(t_0) = -\partial_u H(t_0, z(t_0)) - \gamma_+'(u(t_0))\ \partial_v H(t_0, z(t_0)) > 0\,,$$

where $z(t_0) = \big(q(t_0), p(t_0), u(t_0), \gamma_+(u(t_0))\big)$. This implies that $G_+(t) > 0$ for every $t > t_0$, which is in contradiction with the periodicity of z. We have thus proved that $G_+(t) < 0$ for every $t \in [0, T]$.

We can similarly prove that $\gamma_-(u(t)) < v(t)$ for every $t \in \mathbb{R}$, using (5.10). □

5.2.2 Working with the (q,p) coordinates

Let us fix $K > 0$ such that

$$|\chi(\zeta(u), v)| < K\,, \text{ for every } (u,v) \in \mathbb{R}^2\,.$$

Let $z(t) = (q(t), p(t), u(t), v(t))$ be a solution of system (5.15) starting at time $t = 0$ with $p(0) \in [a - \rho, b + \rho]$. This solution is defined on a maximal interval of future existence $[0, \omega)$. By Assumption 4, recalling (5.16),

$$|\dot{p}(t)| \le C_K(|p(t)| + 1)\,, \qquad \text{for every } t \in [0, \omega)\,,$$

hence, setting

$$c = (\max\{|a|, |b|\} + \rho + 1)\; e^{C_K T},$$

by Gronwall Lemma we have that

$$|p(t)| \le c\,, \qquad \text{for every } t \in [0, T] \cap [0, \omega)\,.$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^∞-smooth nonincreasing function such that

$$\eta(s) = \begin{cases} 1\,, & \text{if } s \le c\,, \\ 0\,, & \text{if } s \ge c+1\,. \end{cases} \tag{5.21}$$

We can rewrite the Hamiltonian $\widehat{H}$ as

$$\widehat{H}(t, q, p, u, v) = \frac{1}{2}(v^2 - u^2) + \widehat{h}(t, q, p, u, v)\,, \tag{5.22}$$

and define the new Hamiltonian function

$$\widetilde{H}(t, q, p, u, v) = \frac{1}{2}(v^2 - u^2) + \widetilde{h}(t, q, p, u, v)\,, \tag{5.23}$$

where

$$\widetilde{h}(t, q, p, u, v) = \eta(|p|)\, \widehat{h}(t, q, p, u, v)\,.$$

Notice that

$$|p| \ge c + 1 \quad \Rightarrow \quad \widetilde{h}(t, q, p, u, v) = 0 \quad \text{for every } (t, q, u, v) \in \mathbb{R} \times \mathbb{R}^3\,.$$

The Hamiltonian $\widetilde{H}$ induces the system

$$\begin{cases} \dot{q} = \partial_p \widetilde{h}(t, q, p, u, v)\,, & \dot{p} = -\partial_q \widetilde{h}(t, q, p, u, v)\,, \\ \dot{u} = v + \partial_v \widetilde{h}(t, q, p, u, v)\,, & \dot{v} = u - \partial_u \widetilde{h}(t, q, p, u, v)\,. \end{cases} \tag{5.24}$$

Since the Hamiltonian function is periodic in q and the functions ζ, ζ', χ, $\nabla\chi$, η and η' are all bounded, there exists a constant $\widetilde{C} > 0$ such that

$$\big|\partial_q \widetilde{h}(t, z)\big| + \big|\partial_p \widetilde{h}(t, z)\big| + \big|\partial_u \widetilde{h}(t, z)\big| + \big|\partial_v \widetilde{h}(t, z)\big| \le \widetilde{C}\,,$$

for every $(t, z) \in \mathbb{R} \times \mathbb{R}^4$. Hence, the Hamiltonian function $\widetilde{H}$ is the sum of a nonresonant quadratic term and a function with bounded gradient $\nabla\widetilde{h}(t, z)$. In particular, all solutions of (5.24) are globally defined on $[0, T]$.

We now verify the twist condition for system (5.24). Let $z = (q,p,u,v)$ be a solution of (5.24) such that $p(0) \in [a-\rho, b+\rho]$. As long as $|p(t)| \leq c$, we have

$$|\dot p(t)| = |\partial_q H(t, q(t), p(t), \zeta(u(t)), \chi(\zeta(u(t)), v(t)))| \leq C_K(|p(t)|+1)\,.$$

Hence, by Gronwall Lemma, we conclude that $|p(t)| \leq c$ for every $t \in [0,T]$. So, $z = (q,p,u,v)$ is a solution of (5.15). Then, (q,p) is a solution of (5.6) with $U(t) = \zeta(u(t))$ and $V(t) = \chi(\zeta(u(t)), v(t))$, condition (5.5) is verified and, by (5.7),

$$\begin{cases} p(0) \in [a-\rho, a] \quad \Rightarrow \quad q(T) < q(0)\,, \\ p(0) \in [b, b+\rho] \quad \Rightarrow \quad q(T) > q(0)\,. \end{cases}$$

Hence, we can apply [44, Corollary 2.4] so to find two T-periodic solutions $z = (q,p,u,v)$ of (5.24) such that $p(0) \in (a,b)$. By the above estimates, these are indeed solutions of (5.15) and, recalling Lemma 4, we conclude that they are the T-periodic solutions of the original system (5.2) we were looking for. $\square$

5.3 Some variants of Theorem 18

Let us start with two observations.

Remark 23. *Assumption 2 can be generalized by asking that there further exist v_α, v_β such that*

$$\begin{cases} (v - v_\alpha)\,\partial_v H(t,q,p,u,v) > 0\,, & \text{when } u \in [\alpha-\delta, \alpha] \text{ and } v \neq v_\alpha\,, \\ (v - v_\beta)\,\partial_v H(t,q,p,u,v) > 0\,, & \text{when } u \in [\beta, \beta+\delta] \text{ and } v \neq v_\beta\,, \end{cases}$$

and

$$\begin{cases} \partial_u H(t,q,p,u,v_\alpha) \geq 0\,, & \text{when } u \in [\alpha-\delta, \alpha]\,, \\ \partial_u H(t,q,p,u,v_\beta) \leq 0\,, & \text{when } u \in [\beta, \beta+\delta]\,. \end{cases}$$

Remark 24. *Instead of a fixed interval $[a,b]$, we could have a varying interval $[a(q), b(q)]$, where $a, b : \mathbb{R} \to \mathbb{R}$ are continuous 2π-periodic functions. Indeed, if a and b are continuously differentiable, this case can be reduced to the previous one by the symplectic change of variables*

$$\Psi(q,p,u,v) = \left(\int_0^q \frac{b(s)-a(s)}{2}\,ds\,,\ \frac{2p - b(q) - a(q)}{b(q)-a(q)}, u, v \right),$$

cf. [92, Exercise 1, p. 132]. If a and b are just continuous, they can be replaced by smooth functions by the use of Fejer Theorem. Notice that the new Hamiltonian $\widetilde H(t,\tilde q,\tilde p,u,v) = H(t, \Psi^{-1}(\tilde q, \tilde p, u, v))$ is periodic in $\tilde q$, with period $\tau := \frac{1}{2}\int_0^{2\pi}(b(s) - a(s))\,ds$.

We now propose a variant of our result which is more in the spirit of the Poincaré–Birkhoff Theorem as originally stated by Poincaré [103]. We first recall the definition of rotation number. For $t_1 < t_2$, let $\eta : [t_1, t_2] \to \mathbb{R}^2$ be a continuous curve such that $\eta(t) \neq (0,0)$ for every $t \in [t_1, t_2]$. Writing $\eta(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, with $\rho : \mathbb{R} \to (0, +\infty)$ and $\theta : \mathbb{R} \to \mathbb{R}$ continuous, we define

$$\mathrm{Rot}\,(\eta; [t_1, t_2]) = -\frac{\theta(t_2) - \theta(t_1)}{2\pi}\,.$$

We need to suitably modify Assumption 4, in the following way

Assumption 5 (Energy growth)**.** *For every $K > 0$ there is a constant $C_K > 0$ such that*

$$\begin{aligned}\big|q\,\partial_p H(t,q,p,u,v) - {}& p\,\partial_q H(t,q,p,u,v)\big| \le C_K(q^2 + p^2 + 1)\,,\\ & \textit{when } u \in [\alpha - \delta, \beta + \delta] \textit{ and } |v| \le K\,.\end{aligned}$$

In the sequel, we denote by $\mathcal{D}(\Gamma)$ the open bounded region delimited by a planar Jordan curve Γ.

Theorem 19. *Let Assumptions 2, 3, and 5 hold. Let k be an integer and assume that there exist $\rho > 0$, $\tilde\rho > 0$ and two planar Jordan curves Γ_1, Γ_2, strictly star-shaped with respect to the origin, with*

$$0 \in \mathcal{D}(\Gamma_1) \subseteq \overline{\mathcal{D}(\Gamma_1)} \subseteq \mathcal{D}(\Gamma_2)\,,$$

such that, for any two continuous functions $U, V : [0,T] \to \mathbb{R}$ satisfying (5.5)*, the solutions of system* (5.6) *with* $\mathrm{dist}\big((q(0),p(0)), \overline{\mathcal{D}(\Gamma_2)} \setminus \mathcal{D}(\Gamma_1)\big) \le \rho$ *which are defined on* $[0,T]$ *satisfy*

$$q(t)^2 + p(t)^2 \ge \tilde\rho\,, \quad \textit{for every } t \in [0,T]\,,$$

and, if $(q(0),p(0)) \notin \mathcal{D}(\Gamma_2) \setminus \overline{\mathcal{D}(\Gamma_1)}$,

$$\begin{aligned}&\mathrm{Rot}\,((q,p);[0,T]) < k\,, \quad \textit{when } \mathrm{dist}\big((q(0),p(0)), \Gamma_1\big) \le \rho\,,\\ &\mathrm{Rot}\,((q,p);[0,T]) > k\,, \quad \textit{when } \mathrm{dist}\big((q(0),p(0)), \Gamma_2\big) \le \rho\,.\end{aligned}$$

Then, the Hamiltonian system (5.2) *has at least two T-periodic solutions $z_i(t)$, $i = 1, 2$, satisfying* (5.8)*, with*

$$(q_i(0), p_i(0)) \in \mathcal{D}(\Gamma_2) \setminus \overline{\mathcal{D}(\Gamma_1)}\,,$$

and

$$\mathrm{Rot}\,((q_i,p_i);[0,T]) = k.$$

The same is true if (19) is replaced by

$$\begin{aligned}&\mathrm{Rot}\,((q,p);[0,T]) > k\,, \quad \textit{when } \mathrm{dist}\big((q(0),p(0)), \Gamma_1\big) \le \rho\,,\\ &\mathrm{Rot}\,((q,p);[0,T]) < k\,, \quad \textit{when } \mathrm{dist}\big((q(0),p(0)), \Gamma_2\big) \le \rho\,.\end{aligned}$$

Proof. At first, let $R_0 > 0$ be such that

$$\operatorname{dist}\big((q(0), p(0)), \overline{\mathcal{D}(\Gamma_2)}\big) \le \rho \quad \Rightarrow \quad q(0)^2 + p(0)^2 \le R_0 \,.$$

We modify the Hamiltonian H introducing the function $\widehat{H}$, arguing as in Section 5.2.1, thus finding the a priori bound in Lemma 4. Let $K > 0$ be such that

$$|\chi(\zeta(u), v)| < K \,, \text{ for every } (u, v) \in \mathbb{R}^2 \,.$$

Let $z(t) = (q(t), p(t), u(t), v(t))$ be any solution of system (5.15) starting at time $t = 0$ from a point $z(0)$ satisfying $q(0)^2 + p(0)^2 \le R_0$. This solution is defined on a maximal interval of future existence $[0, \omega)$. Defining $r(t) = q(t)^2 + p(t)^2$, by Assumption 5,

$$|\dot{r}(t)| \le C_K(r(t) + 1) \,, \qquad \text{for every } t \in [0, \omega) \,,$$

hence, setting

$$c = (R_0 + 1)\, e^{C_K T},$$

by Gronwall Lemma we have that

$$r(t) \le c \,, \qquad \text{for every } t \in [0, T] \cap [0, \omega) \,.$$

Let η be the function introduced in (5.21). Arguing as above, we can write $\widehat{H}$ as in (5.22) and define $\widetilde{H}$ as in (5.23), where now

$$\widetilde{h}(t, q, p, u, v) = \eta(q^2 + p^2)\, \widehat{h}(t, q, p, u, v) \,.$$

The Hamiltonian function $\widetilde{H}$ induces the system (5.24), whose solutions are globally defined on $[0, T]$.

Let $z(t) = (q(t), p(t), u(t), v(t))$ be any solution of system (5.24) with starting point $z(0)$ satisfying $\operatorname{dist}\big((q(0), p(0)), \overline{\mathcal{D}(\Gamma_2)} \setminus \mathcal{D}(\Gamma_1)\big) \le \rho$. Set $U(t) = \zeta(u(t))$ and $V(t) = \chi(\zeta(u(t), v(t))$. Notice that (5.5) is satisfied. It can be verified that $r(t) = q(t)^2 + p(t)^2 \le c$ for every $t \in [0, T]$, with the same constant c as above. Therefore, the couple $(q(t), p(t))$ is a solution of (5.6). So, by the assumption of the theorem, we get

$$r(t) \ge \tilde{\rho} \,, \qquad \text{for every } t \in [0, T] \,.$$

We now introduce a new smooth cut-off increasing function $\widetilde{\eta} : \mathbb{R} \to [0, 1]$ such that

$$\widetilde{\eta}(s) = \begin{cases} 0 \,, & \text{if } r \le \tilde{\rho}/2 \,, \\ 1 \,, & \text{if } r \ge \tilde{\rho} \,. \end{cases}$$

and the new Hamiltonian function

$$\overline{H}(t, q, p, u, v) = \frac{1}{2}(v^2 - u^2) + \overline{h}(t, q, p, u, v) \,,$$

where $\overline{h}(t, q, p, u, v) = \widetilde{\eta}(q^2 + p^2)\widetilde{h}(t, q, p, u, v)$.

For the new Hamiltonian system $S\dot{z} = \nabla\overline{H}(t,z)$, we introduce the symplectic change of variables

$$q(t) = \sqrt{2r(t)}\cos\left(\theta(t) - \tfrac{2\pi}{T}\,kt\right), \quad p(t) = \sqrt{2r(t)}\sin\left(\theta(t) - \tfrac{2\pi}{T}\,kt\right),$$

so to get a Hamiltonian system

$$\begin{cases} \dot{\theta} = \partial_r\mathcal{H}(t,\theta,r,u,v)\,, & \dot{r} = -\partial_\theta\mathcal{H}(t,\theta,r,u,v)\,, \\ \dot{u} = \partial_v\mathcal{H}(t,\theta,r,u,v)\,, & \dot{v} = -\partial_u\mathcal{H}(t,\theta,r,u,v)\,, \end{cases} \tag{5.25}$$

defined for $r > 0$. This system can now be extended also for $r \le 0$, preserving the regularity. Now, the periodicity Assumption 1 is recovered in the variable θ, while Assumption 5 implies that the linear growth Assumption 4 holds for the new variables (θ, r) instead of (q,p). Then Theorem 18 applies, in view of Remark 24, providing the existence of two geometrically distinct T-periodic solutions of (5.25), which are then translated, by the inverse change of variables $(\theta,r) \mapsto (q,p)$, into the T-periodic solutions of (5.2) we are looking for. □

5.4 Examples of applications

Both Theorem 18 and Theorem 19 open the way to a multitude of applications. We will just sketch here a few.

Let us consider for example a system of the type

$$\begin{cases} -\ddot{q} = g(t,q) - e(t) + \partial_q P(t,q,u)\,, \\ -\ddot{u} = -f(u) + \partial_u P(t,q,u)\,, \end{cases} \tag{5.26}$$

where all functions are continuous and T-periodic in t, with $\int_0^T e(t)\,dt = 0$. Let $P(t,q,u)$ be 2π-periodic in q and continuously differentiable in (q,u). Assume moreover $g(t,q)$ to be 2π-periodic in q, with $\int_0^{2\pi} g(t,s)\,ds = 0$.

Then system (5.26) is of the type (5.2), with

$$H(t,q,p,u,v) = \frac{1}{2}(p+E(t))^2 + \frac{1}{2}v^2 + \int_0^q g(t,s)\,ds - \int_0^u f(\sigma)\,d\sigma + P(t,q,u)\,,$$

where $E(t) = \int_0^t e(s)\,ds$. Precisely, we have

$$\begin{cases} \dot{q} = p + E(t)\,, & \dot{p} = -g(t,q) - \partial_q P(t,q,u)\,, \\ \dot{u} = v\,, & \dot{v} = f(u) - \partial_u P(t,q,u)\,. \end{cases}$$

Corollary 4. *In the above setting, assume moreover that there exist two constants $\alpha < \beta$ such that*

$$f(\alpha) < \partial_u P(t,q,\alpha), \qquad \partial_u P(t,q,\beta) < f(\beta), \tag{5.27}$$

for all $(t, q) \in [0, T] \times [0, 2\pi]$. *Then, system* (5.26) *has at least two geometrically distinct* T*-periodic solutions* (q, u), *with* $\alpha \leq u \leq \beta$.

Proof. Notice that $H(t, q, p, u, v)$ is 2π-periodic in q. By continuity, there is a sufficiently small $\delta > 0$ such that Assumption 2 is satisfied. Also Assumption 3 is easily verified, taking $f(s) = s$ and $\varphi(s)$ constant. Moreover, there exist $R > 0$ with the following property: For any continuous function $U : [0, T] \to \mathbb{R}$ such that $\alpha - \delta \leq U(t) \leq \beta + \delta$ for every $t \in [0, T]$, the solutions of

$$-\ddot{q} = g(t, q) - e(t) + \partial_q P(t, q, U(t))$$

satisfy

$$\begin{cases} q(T) - q(0) < 0\,, & \text{when } \dot{q}(0) < -R\,, \\ q(T) - q(0) > 0\,, & \text{when } \dot{q}(0) > R\,. \end{cases}$$

Then, taking $b = -a = R + \|E\|_\infty$, Theorem 18 applies. □

As an immediate consequence, we have the following.

Corollary 5. *The system of coupled pendulums of the form*

$$\begin{cases} \ddot{q} + a \sin q = e(t) - \partial_q P(t, q, u), \\ \ddot{u} + b \sin u = -\partial_u P(t, q, u), \end{cases}$$

where $P(t, q, u)$ *and* $e(t)$ *are as above, has at least two geometrically distinct* T*-periodic solutions, for any* $a > 0$, *if* $\|\partial_u P\|_\infty < b$.

We thus extend two classical results on the pendulum equation (cf. [89]). The first one states that, if $e(t)$ is T-periodic and $\int_0^T e(t)\,dt = 0$, then

$$\ddot{q} + a \sin q = e(t)$$

has at least two geometrically distinct T-periodic solutions, for any $a > 0$. The second one states that, if $\hat{e}(t)$ is T-periodic and $\|\hat{e}\|_\infty \leq b$, then

$$\ddot{u} + b \sin u = \hat{e}(t)$$

has at least two geometrically distinct T-periodic solutions.

Here are some other possible examples of applications.

1. Let $f(u) = |u|^{\ell-1}u$, with $\ell > 0$, and let P be a function as above, and such that

$$\lim_{u \to \pm\infty} \frac{\partial_u P(t, q, u)}{|u|^\ell} = 0\,, \qquad \text{uniformly in } (t, q) \in [0, T] \times [0, 2\pi]\,.$$

Then, there exist two constants $\beta = -\alpha > 0$ large enough such that (5.27) is satisfied. In particular, if $\partial_u P$ is bounded, it is enough to take

$$\alpha < -\|\partial_u P\|_\infty^{1/\ell}, \qquad \beta > \|\partial_u P\|_\infty^{1/\ell}.$$

Notice that this includes the case of $\partial_u P$ being periodic in u. Hence, in this situation, there exist at least two geometrically distinct T-periodic solutions of system (5.26).

2. Let $f(u) = |u|^\ell \sin(u)$, with $\ell > 0$, and let P be as in the previous example. Then (5.27) is satisfied for an infinite number of positive and negative pairs (α_i, β_i). Hence, there exist infinitely many T-periodic solutions of system (5.26) with positive u component and infinitely many T-periodic solutions with negative u component.

3. Consider $f(u) = \arctan(u)$. Then, if $\partial_u P$ is bounded, with

$$\|\partial_u P\|_\infty < \frac{\pi}{2}\,,$$

there exist at least two geometrically distinct T-periodic solutions of system (5.26). It is sufficient to take $\beta = -\alpha > 0$ large enough.

When $P(t,q,u)$ is *not* periodic in q, we can appeal to Theorem 19. Consider for example the system

$$\begin{cases} -\ddot{q} = g(q) + \partial_q P(t,q,u)\,, \\ -\ddot{u} = -f(u) + \partial_u P(t,q,u)\,, \end{cases} \tag{5.28}$$

where $P(t,q,u)$ is continuously differentiable in (q,u). We can prove the following.

Corollary 6. *Assume that*

$$\lim_{|q|\to\infty} \frac{g(q)}{q} = +\infty\,, \tag{5.29}$$

and that $\partial_q P$ is bounded. If there exist two constants $\alpha < \beta$ such that (5.27) *holds, then system* (5.28) *has infinitely-many T-periodic solutions.*

Proof. We will briefly explain the main arguments. At first we modify system (5.28) by introducing the functions

$$\begin{aligned} \widehat{f}(u) &= (u-\beta)^+ - (u-\alpha)^- + f(\zeta(u))\zeta'(u)\,, \\ \widehat{P}(t,q,u) &= P(t,q,\zeta(u))\,, \end{aligned}$$

thus obtaining

$$\begin{cases} \dot{q} = p\,, \qquad \dot{p} = -g(q) - \partial_q \widehat{P}(t,q,u)\,, \\ \dot{u} = v\,, \qquad \dot{v} = \widehat{f}(u) - \partial_u \widehat{P}(t,q,u)\,, \end{cases} \tag{5.30}$$

and denoting by $\widehat{H}(t,z) = \frac{1}{2}(v^2 - u^2) + \widehat{h}(t,z)$ its Hamiltonian, for a suitable choice of $\widehat{h}$. Let $B(0,R) \subseteq \mathbb{R}^2$ be the ball of radius $R > 0$ centered at the origin, and set $\mathcal{B}_R := B(0,R) \times \mathbb{R}^2$. Then, following the arguments in [34, Section 2], we can prove that any solution of system (5.30) is globally defined in the interval $[0,T]$. Moreover, by the elastic property stated in [34, Lemma 1] we can find $r_3 > r_2 > r_1 > 0$ such that any solution $z(t) = (q(t), p(t), u(t), v(t))$ of (5.30) starting at $t = 0$ from a point belonging to $\partial\mathcal{B}_{r_2}$ satisfies $z(t) \in \mathcal{B}_{r_3} \setminus \mathcal{B}_{r_1}$ for every $t \in [0,T]$. For such solutions we can find $k > 0$ such that $\mathrm{Rot}\,((q,p);[0,T]) < k$.

Let us fix any $k' \geq k$. Assumption (5.29) provides the existence of $R_1 > 0$ such that every solution z of (5.30), with $z(t) \notin \mathcal{B}_{R_1}$ for any $t \in [0, T]$, satisfies $\mathrm{Rot}\,((q,p);[0,T]) > k'$. Then, using again [34, Lemma 1], we can find $R_3 > R_2 > R_1$ such that any solution z of (5.30) starting at $t = 0$ from a point belonging to $\partial\mathcal{B}_{R_2}$ satisfies $z(t) \in \mathcal{B}_{R_3} \setminus \mathcal{B}_{R_1}$ for every $t \in [0, T]$. Hence, we have for such solutions $\mathrm{Rot}\,((q,p);[0,T]) > k'$.

Now, we introduce a smooth cut-off function $\widehat{\eta} : [0, +\infty) \to [0, 1]$ such that

$$\widehat{\eta}(r) = \begin{cases} 0 & \text{if } r \in [0, r_1/2] \cup [2R_3, +\infty)\,, \\ 1 & \text{if } r \in [r_1, R_3]\,. \end{cases}$$

and the new Hamiltonian $\widetilde{H}(t,z) = \frac{1}{2}(v^2 - u^2) + \widetilde{h}(t,z)$ where now

$$\widetilde{h}(t,z) = \widehat{\eta}\big(\sqrt{q(t)^2 + p(t)^2}\big)\widehat{h}(t,z)\,.$$

Finally, we are able to apply Theorem 19 to system (5.30) thus obtaining two T-periodic solutions satisfying $\mathrm{Rot}\,((q,p);[0,T]) = k'$ with starting point $z(0) \in \mathcal{B}_{R_2} \setminus \overline{\mathcal{B}_{r_2}}$. Then, we can easily see that they are indeed solutions of (5.28).

Since the above construction can be made for every $k' \geq k$, we have proved the existence of infinitely many T-periodic solutions. □

We thus extend a result by Ding and Zanolin [34], stating that, if $g(q)$ satisfies (5.29) and $e(t)$ is T-periodic, the scalar equation

$$\ddot{q} + g(q) = e(t)$$

has infinitely-many T-periodic solutions. A similar result for system (5.26) remains an open problem, because global existence of the solutions is not guaranteed. However, one could follow the lines of [50, 66, 69, 73] by assuming that the first equation has 0 as an equilibrium point, and then prove that there are infinitely many T-periodic solutions.

Whenever the function g has a sublinear growth at infinity, the existence of *subharmonic* solutions has been investigated in [35]. We could also state such kind of result here, but we avoid the details, for briefness.

Another situation where our results can be applied is when the system in (q, p) has a different rotational behaviour at zero and at infinity. There are many papers dealing with such a problem, see for instance [83] and the references therein. This type of situation can be also exploited when dealing with some kind of asymmetric oscillators like in [21], where the so-called *jumping nonlinearities* are treated. We do not enter into details, again, to be brief.

5.5 Going to higher dimensions

We consider system (5.1), assuming the Hamiltonian function $H\colon \mathbb{R}\times\mathbb{R}^{2N}\to\mathbb{R}$ to be continuous, T-periodic in its first variable t, and continuously differentiable with respect to its second variable z, with corresponding gradient $\nabla H(t,z)$.

For $z\in\mathbb{R}^{2N}$, we write $z=(\phi,\psi,q,p,u,v)$, where, for some nonnegative integers L, M and D,

$$\begin{array}{ll} \phi=(\phi_1,\dots,\phi_L)\in\mathbb{R}^L, & \psi=(\psi_1,\dots,\psi_L)\in\mathbb{R}^L,\\ q=(q_1,\dots,q_M)\in\mathbb{R}^M, & p=(p_1,\dots,p_M)\in\mathbb{R}^M,\\ u=(u_1,\dots,u_D)\in\mathbb{R}^D, & v=(v_1,\dots,v_D)\in\mathbb{R}^D. \end{array}$$

Notice that one or more of these integers could be equal to zero, in which case the corresponding group will not be taken into account. For example, if $L=0$, then ϕ and ψ will disappear from the list.

Our system (5.1) then reads as

$$\begin{cases} \dot\phi=\partial_\psi H(t,z)\,, & \dot\psi=-\partial_\phi H(t,z)\,,\\ \dot q=\partial_p H(t,z)\,, & \dot p=-\partial_q H(t,z)\,,\\ \dot u=\partial_v H(t,z)\,, & \dot v=-\partial_u H(t,z)\,. \end{cases} \tag{5.31}$$

Let us introduce our assumptions.

Assumption 6 (Periodicity)**.** *The function $H(t,z)$ is periodic in each of the variables included in ϕ,ψ,q.*

To fix the ideas, we assume that all periods are equal to 2π. The total number of variables in which our Hamiltonian function is 2π-periodic is thus $2L+M$. Under this setting, T-periodic solutions $z(t)$ of (5.1) appear in equivalence classes made of those solutions whose components in $\phi(t)$, $\psi(t)$, $q(t)$, differ by an integer multiple of 2π. We say that two T-periodic solutions are *geometrically distinct* if they do not belong to the same equivalence class.

In the sequel, inequalities $\le$ involving vectors are to be interpreted componentwise. Moreover, for any $\sigma\in\mathbb{R}$, we use the notation $\bar\sigma=(\sigma,\dots,\sigma)\in\mathbb{R}^D$.

Assumption 7 (Lower and upper solutions)**.** *There exist some constants $\delta>0$, $\alpha=(\alpha_1,\dots,\alpha_D)$ and $\beta=(\beta_1,\dots,\beta_D)$ with $\alpha\le\beta$, having the following property. If $\alpha-\bar\delta\le u\le\beta+\bar\delta$, then, for every $j\in\{1,\dots,D\}$,*

$$v_j\,\partial_{v_j}H(t,z)>0\,,\ \text{ when } u_j\in[\alpha_j-\delta,\alpha_j]\cup[\beta_j,\beta_j+\delta] \text{ and } v_j\neq0\,,$$

and

$$\begin{cases} \partial_{u_j}H(t,z)\ge0\,, & \text{when } u_j\in[\alpha_j-\delta,\alpha_j] \text{ and } v_j=0\,,\\ \partial_{u_j}H(t,z)\le0\,, & \text{when } u_j\in[\beta_j,\beta_j+\delta] \text{ and } v_j=0\,. \end{cases}$$

In the sequel, the constant $\delta > 0$ provided by Assumption 7 will be used without further mention.

Assumption 8 (Nagumo condition)**.** *For every $j \in \{1, \ldots, D\}$ there exist $d_j > 0$ and two continuous functions $f_j, \varphi_j : [d_j, +\infty) \to (0, +\infty)$, with*

$$\int_{d_j}^{+\infty} \frac{f_j(s)}{\varphi_j(s)}\, ds = +\infty\,,$$

satisfying the following property. If $\alpha - \bar{\delta} \le u \le \beta + \bar{\delta}$, then

$$\begin{cases} \partial_{v_j} H(t,z) \ge f_j(v_j)\,, & \text{when } v_j \ge d_j\,, \\ \partial_{v_j} H(t,z) \le -f_j(-v_j)\,, & \text{when } v_j \le -d_j\,, \end{cases}$$

and

$$\big|\partial_{u_j} H(t,z)\big| \le \varphi_j(|v_j|)\,, \quad \text{when } |v_j| \ge d_j\,.$$

Assumption 9 (Linear growth)**.** *For every $K > 0$ there is a constant $C_K > 0$ with the following property. If $\alpha - \bar{\delta} \le u \le \beta + \bar{\delta}$ and $|v| \le K$, then*

$$|\partial_q H(t,z)| \le C_K(|p| + 1)\,.$$

Adapting the argument in Remark 22, under the above assumption, for any two continuous functions $U, V : [0,T] \to \mathbb{R}^D$ with

$$\alpha - \bar{\delta} \le U(t) \le \beta + \bar{\delta}\,, \quad \text{for every } t \in [0,T]\,, \tag{5.32}$$

the solutions of the system

$$\begin{cases} \dot{\phi} = \partial_\psi H(t,\phi,\psi,q,p,U(t),V(t))\,, \\ \dot{\psi} = -\partial_\phi H(t,\phi,\psi,q,p,U(t),V(t))\,, \\ \dot{q} = \partial_p H(t,\phi,\psi,q,p,U(t),V(t))\,, \\ \dot{p} = -\partial_q H(t,\phi,\psi,q,p,U(t),V(t))\,, \end{cases} \tag{5.33}$$

are defined on $[0,T]$ and, setting $K = \|V\|_\infty$,

$$|p(t)| \le (|p(0)| + 1)\, e^{C_K T}, \quad \text{for every } t \in [0,T]\,. \tag{5.34}$$

In the following, we consider a convex body $\mathcal{D}$ of $\mathbb{R}^M$, i.e., a closed convex bounded set with nonempty interior. We denote by $\pi_{\mathcal{D}} \colon \mathbb{R}^M \setminus \mathring{\mathcal{D}} \to \partial\mathcal{D}$ the projection on the convex set $\mathcal{D}$ and by $\nu_{\mathcal{D}}(\zeta)$ the unit outward normal at $\zeta \in \partial\mathcal{D}$, assuming that $\mathcal{D}$ has a smooth boundary. We say that $\mathcal{D}$ is *strongly convex* if, for any $p \in \partial\mathcal{D}$, the function $\mathcal{F} : \mathcal{D} \to \mathbb{R}$ defined as $\mathcal{F}(\eta) = \langle \eta - p, \nu_{\mathcal{D}}(p)\rangle$ has a unique maximum point at $\eta = p$.

Here is our first result in this higher dimensional setting, generalizing Theorem 18.

Theorem 20. *Let Assumptions 6, 7, 8, and 9 hold. Assume that there exist* $\rho > 0$*, a symmetric regular* $M \times M$ *matrix* $\mathbb{A}$ *and a strongly convex body* $\mathcal{D}$ *of* $\mathbb{R}^M$*, having a smooth boundary, with the following property: For any two continuous functions* $U, V : [0,T] \to \mathbb{R}^D$ *satisfying* (5.32)*, the solutions of* (5.33) *with* $p(0) \notin \mathring{\mathcal{D}}$ *and* $\operatorname{dist}(p(0), \partial\mathcal{D}) \leq \rho$ *are such that*

$$\big\langle q(T) - q(0)\,,\, \mathbb{A}\nu_{\mathcal{D}}\big(\pi_{\mathcal{D}}(p(0))\big)\big\rangle > 0\,.$$

Then, system (5.31) *has at least* $2L + M + 1$ *geometrically distinct* T*-periodic solutions, satisfying*

$$p(0) \in \mathring{\mathcal{D}}\,,$$

and

$$\alpha \leq u(t) \leq \beta\,, \quad \text{for every } t \in \mathbb{R}\,. \tag{5.35}$$

5.6 Proof of Theorem 20

Since the arguments will be similar to those provided in Section 5.2, we will try to be brief.

At first we need to suitably modify the Hamiltonian system working componentwise in the (u,v) variables.

For every $j \in \{1, \ldots, D\}$, from Assumption 8 we can find some continuously differentiable functions $\gamma_j^\pm : \mathbb{R} \to \mathbb{R}$ and then $\widehat{d}_j > 0$ such that

$$-\widehat{d}_j < \gamma_j^-(s) < -d_j\,, \quad d_j < \gamma_j^+(s) < \widehat{d}_j\,,$$

for every $s \in [\alpha_j - \delta, \beta_j + \delta]$, satisfying

$$\begin{aligned} -\partial_{u_j} H\big(t,z\big) &> \partial_{v_j} H\big(t,z\big)(\gamma_j^+)'(u_j)\,, \quad \text{when } v_j = \gamma_j^+(u_j)\,,\\ -\partial_{u_j} H\big(t,z\big) &< \partial_{v_j} H\big(t,z\big)(\gamma_j^-)'(u_j)\,, \quad \text{when } v_j = \gamma_j^-(u_j)\,, \end{aligned}$$

for every $(t,z) \in [0,T] \times \mathbb{R}^{2N}$ with $u_j \in [\alpha_j - \delta, \beta_j + \delta]$.

We then define $\gamma^\pm : \mathbb{R}^D \to \mathbb{R}^D$ as

$$\gamma^\pm(u) = (\gamma_1^\pm(u_1), \ldots, \gamma_D^\pm(u_D))\,,$$

and introduce the set

$$\mathcal{V} = \{z = (\phi, \psi, q, p, u, v) \mid \alpha \leq u \leq \beta\,,\, \gamma^-(u) < v < \gamma^+(u)\}\,,$$

recalling that we need to check the inequalities componentwise. We can then introduce some functions $\zeta_j : \mathbb{R} \to \mathbb{R}$ and $\chi_j : [\alpha_j - \delta, \beta_j + \delta] \times \mathbb{R} \to \mathbb{R}$ similarly as in (5.11) and (5.13), define

$$\zeta(u) = (\zeta_1(u_1), \dots, \zeta_D(u_D)) ,$$
$$\chi(u, v) = (\chi_1(u_1, v_1), \dots, \chi_D(u_D, v_D)) ,$$

and consider the modified Hamiltonian $\widehat{H} : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ as

$$\widehat{H}(t, \phi, \psi, q, p, u, v) = H\big(t, \phi, \psi, q, p, \zeta(u), \chi(\zeta(u), v)\big) + \mathcal{H}(u, v) ,$$

with

$$\mathcal{H}(u, v) = \frac{1}{2} \sum_{j=1}^{D} \bigg[\Big[(v_j - \widehat{d}_j)^+\Big]^2 + \Big[(v_j + \widehat{d}_j)^-\Big]^2 - \big[(u_j - \beta_j)^+\big]^2 - \big[(u_j - \alpha_j)^-\big]^2 \bigg].$$

The modified Hamiltonian system

$$S\dot{z} = \nabla \widehat{H}(t, z) \tag{5.36}$$

complies the following a priori bound, whose proof can be given adapting the one of Lemma 4, arguing separately on every couple of variables (u_j, v_j), and verifying the validity of some analogues of Propositions 5, 6, 7, and 8.

Lemma 5. *If $z = (\phi, \psi, q, p, u, v)$ is a T-periodic solution of* (5.36)*, then $z(t) \in \mathcal{V}$ for every $t \in [0, T]$, hence it solves* (5.31).

The next step involves the (q, p) variables. The reasoning in Section 5.2.2 can be adapted to higher dimension. We fix $K > 0$ such that

$$|\chi(\zeta(u), v)| < K , \text{ for every } (u, v) \in \mathbb{R}^{2D}, \tag{5.37}$$

and, given the convex body $\mathcal{D}$ and the constant $\rho > 0$ as in the statement of the theorem, we set

$$P_0 = \max\{|p| : p \in \mathcal{D}\} , \quad \text{and} \quad c = (P_0 + \rho + 1)e^{C_K T} .$$

Let $z(t)$ be a solution of system (5.36) having $[0, \omega)$ as its maximal interval of future existence, starting with $p(0)$ satisfying $\mathrm{dist}(p(0), \mathcal{D}) \le \rho$. Arguing as at the beginning of Section 5.2.2, from Assumption 9 and (5.34), we have that

$$|p(t)| \le c , \quad \text{for every } t \in [0, T] \cap [0, \omega) . \tag{5.38}$$

So, we can introduce the cut-off function η as in (5.21) in order to change the Hamiltonian function

$$\widehat{H}(t, z) = \frac{1}{2}(|v|^2 - |u|^2) + \widehat{h}(t, z) .$$

into the new Hamiltonian function

$$\widetilde{H}(t, z) = \frac{1}{2}(|v|^2 - |u|^2) + \widetilde{h}(t, z) .$$

with $\widetilde{h}(t, z) = \eta(|p|)\, \widehat{h}(t, z)$. Notice that $|\nabla \widetilde{h}(t, z)| \le \widetilde{C}$ for a certain positive constant $\widetilde{C}$.

Our aim is now to apply [44, Corollary 2.3] to the modified system

$$S\dot{z} = \nabla \widetilde{H}(t,z)\,. \tag{5.39}$$

Any solution $z = (\phi, \psi, q, p, u, v)$ of (5.39) is defined on $[0,T]$. As in the proof of Theorem 18, we can prove that, if $\operatorname{dist}(p(0), \mathcal{D}) \le \rho$, then $|p(t)| \le c$ for every $t \in [0,T]$. In particular it is a solution of (5.36). Then, if we set $U(t) = \zeta(u(t))$ and $V(t) = \chi(\zeta(u(t)), v(t))$, we see that (ϕ, ψ, q, p) is a solution of (5.33). By the assumption of the theorem, if the solution starts with $\operatorname{dist}(p(0), \partial\mathcal{D}) \le \rho$ and $p(0) \notin \mathring{\mathcal{D}}$, then

$$\big\langle q(T) - q(0)\,,\, \mathbb{A}\nu_{\mathcal{D}}\big(\pi_{\mathcal{D}}(p(0))\big)\big\rangle > 0\,.$$

The application of [44, Corollary 2.3] provides us $2L + M + 1$ geometrically distinct T-periodic solutions of (5.39) satisfying $p(0) \in \mathring{\mathcal{D}}$. Then, by the above estimates, they are solutions of (5.36). Finally, from Lemma 5, these are indeed solutions of the original Hamiltonian system (5.31), and they satisfy $\alpha \le u(t) \le \beta$, for every $t \in \mathbb{R}$. □

5.7 Variants in higher dimensions

Here are two variants of Theorem 20. In the first one, the twist is formulated as an *avoiding rays condition.*

Theorem 21. *Let Assumptions 6, 7, 8, and 9 hold. Assume that there exist $\rho > 0$ and a convex body $\mathcal{D}$ of $\mathbb{R}^M$, having a smooth boundary, with the following property: For any two continuous functions $U, V : [0,T] \to \mathbb{R}^D$ satisfying (5.32), the solutions of system (5.33) with $p(0) \notin \mathring{\mathcal{D}}$ and $\operatorname{dist}(p(0), \partial\mathcal{D}) \le \rho$ are such that*

$$q(T) - q(0) \notin \{\lambda \nu_{\mathcal{D}}\big(\pi_{\mathcal{D}}(p(0))\big) : \lambda \ge 0\}\,. \tag{5.40}$$

Then, the same conclusion of Theorem 20 holds.

Proof. The argument follows the lines of the proof of Theorem 20, with the only difference of applying [44, Corollary 2.1] instead of [44, Corollary 2.3]. □

Notice that the twist condition (5.40) may as well be replaced by

$$q(T) - q(0) \notin \{\lambda \nu_{\mathcal{D}}\big(\pi_{\mathcal{D}}(p(0))\big) : \lambda \le 0\}\,,$$

and the same conclusion of Theorem 20 still holds.

We now consider the case when $\mathcal{D}$ is a M-cell, namely

$$\mathcal{D} = [a_1, b_1] \times \cdots \times [a_M, b_M]\,.$$

Theorem 22. *Let Assumptions 6, 7, 8, and 9 hold. Assume that there exist* $\rho > 0$ *and a* M*-tuple* $\sigma = (\sigma_1, \dots, \sigma_M) \in \{-1, 1\}^M$*, with the following property: For any two continuous functions* $U, V : [0, T] \to \mathbb{R}^D$ *satisfying* (5.32)*, the solutions of* (5.33) *with* $p(0) \notin \mathring{\mathcal{D}}$ *and* $\operatorname{dist}(p(0), \partial\mathcal{D}) \leq \rho$ *are such that, for every* $i \in \{1, \dots, M\}$*,*

$$\begin{cases} \sigma_i(q_i(T) - q_i(0)) < 0\,, & \text{when } p_i(0) \in [a_i - \rho, a_i]\,, \\ \sigma_i(q_i(T) - q_i(0)) > 0\,, & \text{when } p_i(0) \in [b_i, b_i + \rho]\,. \end{cases}$$

Then, the same conclusion of Theorem 20 holds.

Proof. Again the proof is similar to the one of Theorem 20, just applying [44, Corollary 2.4] instead of [44, Corollary 2.3]. □

Remark 25. *As noticed in Remark 24, instead of the fixed intervals* $[a_i, b_i]$ *defining the* M*-cell* $\mathcal{D}$*, we could have varying intervals* $[a_i(q_i), b_i(q_i)]$*, where* $a_i, b_i : \mathbb{R} \to \mathbb{R}$ *are continuous* 2π*-periodic functions.*

We now generalize Theorem 19. We thus drop the periodicity in the q-variables, still maintaining it in the ϕ and ψ variables, as stated below.

Assumption 10 (Periodicity)**.** *The function* $H(t, z)$ *is periodic in each of the variables included in* ϕ, ψ*.*

We also need to suitably modify Assumption 4, in the following way

Assumption 11 (Energy growth)**.** *For every* $K > 0$ *there is a constant* $C_K > 0$ *such that, for every* $i \in \{1, \dots, M\}$*,*

$$\left| q_i\, \partial_{p_i} H(t, q, p, u, v) - p_i\, \partial_{q_i} H(t, q, p, u, v) \right| \leq C_K (q_i^2 + p_i^2 + 1)\,,$$

when u

Here is the generalization of Theorem 19.

Theorem 23. *Let Assumptions 7, 8, 10 and 11 hold. Let* $k_1, \dots, k_M$ *be integers and assume that that there exist* $\rho > 0$*,* $\tilde{\rho} > 0$ *and, for each* $i \in \{1, \dots, M\}$ *there exist two planar Jordan curves* Γ_1^i*,* Γ_2^i*, strictly star-shaped with respect to the origin, with*

$$0 \in \mathcal{D}(\Gamma_1^i) \subseteq \overline{\mathcal{D}(\Gamma_1^i)} \subseteq \mathcal{D}(\Gamma_2^i)\,,$$

such that, for any two continuous functions $U, V : [0, T] \to \mathbb{R}$ *satisfying* (5.32)*, the solutions of system* (5.33) *with*

$$\operatorname{dist}\big((q_i(0), p_i(0)), \overline{\mathcal{D}(\Gamma_2^i)} \setminus \mathcal{D}(\Gamma_1^i)\big) \leq \rho \quad \text{for every } i \in \{1, \dots, M\}$$

which are defined on $[0, T]$*, satisfy*

$$q_i(t)^2 + p_i(t)^2 \geq \tilde{\rho}\,, \text{ for every } t \in [0, T]\,,$$

and, if $(q_i(0), p_i(0)) \notin \mathcal{D}(\Gamma_2^i) \setminus \overline{\mathcal{D}(\Gamma_1^i)}$,

$$\begin{aligned} \mathrm{Rot}\,((q_i, p_i); [0,T]) < k_i\,, \quad &\textit{when } \mathrm{dist}\big((q_i(0), p_i(0)), \Gamma_1^i\big) \le \rho\,, \\ \mathrm{Rot}\,((q_i, p_i); [0,T]) > k_i\,, \quad &\textit{when } \mathrm{dist}\big((q_i(0), p_i(0)), \Gamma_2^i\big) \le \rho\,. \end{aligned}$$

Then, system (5.31) *has at least* $2L + M + 1$ *distinct* T*-periodic solutions*

$$z^{(1)}(t)\,,\ \dots\,,\ z^{(2L+M+1)}(t)\,,$$

satisfying (5.35)*, with*

$$(q_i^{(n)}(0), p_i^{(n)}(0)) \in \mathcal{D}(\Gamma_2^i) \setminus \overline{\mathcal{D}(\Gamma_1^i)}\,,$$

and

$$\mathrm{Rot}\,((q_i^{(n)}, p_i^{(n)}); [0,T]) = k_i\,,$$

for every $i = 1, \dots, M$ *and* $n = 1, \dots, 2L + M + 1$*. The same is true if, for some* $i \in \{1, \dots, M\}$*, assumption (23) is replaced by*

$$\begin{aligned} \mathrm{Rot}\,((q_i, p_i); [0,T]) > k_i\,, \quad &\textit{when } \mathrm{dist}\big((q_i(0), p_i(0)), \Gamma_1^i\big) \le \rho\,, \\ \mathrm{Rot}\,((q_i, p_i); [0,T]) < k_i\,, \quad &\textit{when } \mathrm{dist}\big((q_i(0), p_i(0)), \Gamma_2^i\big) \le \rho\,. \end{aligned}$$

Proof. It is perfectly analogous to the one of Theorem 19, the only difference being the use of Theorem 22 (and Remark 25) instead of Theorem 18. □

Let us now add a further equation, and consider the more general system

$$\begin{cases} \dot\phi = \partial_\psi H(t,z)\,, & \dot\psi = -\partial_\phi H(t,z)\,, \\ \dot q = \partial_p H(t,z)\,, & \dot p = -\partial_q H(t,z)\,, \\ \dot u = \partial_v H(t,z)\,, & \dot v = -\partial_u H(t,z)\,, \\ J\dot w = \partial_w H(t,z)\,. \end{cases} \tag{5.41}$$

Here, and in the following, the symbol J is always used as the standard symplectic matrix, in different dimensions. Moreover, now $z = (\phi, \psi, q, p, u, v, w)$. We will generalize Theorem 20. Assumptions 6, 7 and 8 will remain the same, while Assumption 9 needs to be modified as follows.

Assumption 12 (Linear growth)**.** *Let*

$$H(t,z) = \tfrac{1}{2}\big\langle \mathbb{B}(t)w, w\big\rangle + \mathcal{H}(t,z)\,,$$

where the symmetric matrix $\mathbb{B}(t)$ *is continuous,* T*-periodic, and such that*

$$w(t) \equiv 0 \quad \textit{is the only } T\textit{-periodic solution of } S\dot w = \mathbb{B}(t)w\,.$$

The function $\mathcal{H}$ *is such that for every* $K > 0$ *there is a constant* $C_K > 0$ *with the following property: If* $\alpha - \bar{\bar\delta} \le u \le \beta + \bar{\bar\delta}$ *and* $|v| \le K$*, then*

$$|\partial_q \mathcal{H}(t,z)| + |\partial_w \mathcal{H}(t,z)| \le C_K(|p| + 1)\,.$$

Under the above assumption, given any two continuous functions $U, V : [0,T] \to \mathbb{R}^D$, with

$$\alpha - \bar{\delta} \le U(t) \le \beta + \bar{\delta}\,, \quad \text{for every } t \in [0,T]\,, \tag{5.42}$$

the solutions of the system

$$\begin{cases} \dot{\phi} = \partial_\psi H(t, \phi, \psi, q, p, U(t), V(t), w)\,, \\ \dot{\psi} = -\partial_\phi H(t, \phi, \psi, q, p, U(t), V(t), w)\,, \\ \dot{q} = \partial_p H(t, \phi, \psi, q, p, U(t), V(t), w)\,, \\ \dot{p} = -\partial_q H(t, \phi, \psi, q, p, U(t), V(t), w)\,, \\ J\dot{w} = \partial_w H(t, \phi, \psi, q, p, U(t), V(t), w)\,, \end{cases} \tag{5.43}$$

are defined on $[0,T]$, cf. Remark 22.

Theorem 24. *Let Assumptions 6, 7, 8 and 12 hold. Assume that there exist $\rho > 0$, a symmetric regular $M \times M$ matrix $\mathbb{A}$ and a strongly convex body $\mathcal{D}$ of $\mathbb{R}^M$, having a smooth boundary, with the following property: For any two continuous functions $U, V : [0,T] \to \mathbb{R}^D$ satisfying* (5.42)*, the solutions of* (5.43) *with $p(0) \notin \mathring{\mathcal{D}}$ and* $\operatorname{dist}(p(0), \partial\mathcal{D}) \le \rho$ *are such that*

$$\big\langle q(T) - q(0)\,,\, \mathbb{A}\nu_{\mathcal{D}}\big(\pi_{\mathcal{D}}(p(0))\big)\big\rangle > 0\,. \tag{5.44}$$

Then, system (5.41) *has at least $2L + M + 1$ geometrically distinct T-periodic solutions, such that $p(0) \in \mathring{\mathcal{D}}$, and $\alpha \le u(t) \le \beta$, for every $t \in \mathbb{R}$.*

Proof. Following the lines of the proof of Theorem 20, we can introduce a modified system ruled by a Hamiltonian function of the type

$$\widehat{H}(t,z) = \tfrac{1}{2}\big\langle \mathbb{B}(t)w, w\big\rangle + \frac{1}{2}(|v|^2 - |u|^2) + \widehat{h}(t,z)\,.$$

Moreover, $\widehat{H}$ can be introduced so to guarantee that every T-periodic solution of $S\dot{z} = \nabla \widehat{H}(t,z)$ satisfies an a priori bound as in Lemma 5, in particular $\alpha \le u \le \beta$. Then, from Assumption 12, we can introduce $K > 0$ as in (5.37) such that, if $\alpha - \bar{\delta} \le u \le \beta + \bar{\delta}$ and $|v| \le K$, then

$$|\partial_q \widehat{h}(t,z)| + |\partial_w \widehat{h}(t,z)| \le \widehat{C}_K(|p| + 1)\,.$$

Hence, we can find a constant $c > 0$ such that any solution of $S\dot{z} = \nabla \widehat{H}(t,z)$ starting with $z(0)$ satisfying $\operatorname{dist}(p(0), \mathcal{D}) \le \rho$ is such that (5.38) holds.

Again, we can introduce the cut-off function η, as in (5.21), and the Hamiltonian function

$$\widetilde{H}(t,z) = \tfrac{1}{2}\big\langle \mathbb{B}(t)w, w\big\rangle + \frac{1}{2}(|v|^2 - |u|^2) + \widetilde{h}(t,z)\,,$$

where $\widetilde{h}(t,z) = \eta(|p|)\widehat{h}(t,z)$. Using Assumption 12 we get

$$|\nabla\widetilde{h}(t,z)| \le \widetilde{C}\,,$$

for a certain positive constant $\widetilde{C}$. We can rewrite the previous Hamiltonian as

$$\widetilde{H}(t,z) = \tfrac{1}{2}\Big\langle \mathbb{M}(t)\big(w,(u,v)\big), \big(w,(u,v)\big)\Big\rangle + \widetilde{h}(t,z)\,,$$

where

$$\mathbb{M}(t)\big(w,(u,v)\big) = \big(\mathbb{B}(t)w,(-u,v)\big)\,.$$

Our aim is to apply [44, Corollary 2.3] to the modified system $S\dot{z} = \nabla\widetilde{H}(t,z)$. We easily verify the nonresonance condition

$$\omega(t) \equiv 0 \quad \text{is the only } T\text{-periodic solution of } S\dot{\omega} = \mathbb{M}(t)\omega\,,$$

where $\omega = (w,(u,v))$.

We finally verify, with the usual argument, that any solution z of the previous system with $0 < \mathrm{dist}(p(0),\mathcal{D}) \le \rho$ is such that

$$\langle q(T) - q(0), \mathbb{A}p(0)\rangle > 0\,.$$

The application of [44, Corollary 2.3] provides us $2L + M + 1$ geometrically distinct T-periodic solutions of $S\dot{z} = \nabla\widetilde{H}(t,z)$ satisfying $p(0) \in \mathring{\mathcal{D}}$. Then, as in the previous proofs we end showing that they are solutions of (5.41), too, and they satisfy $\alpha \le u(t) \le \beta$, for every $t \in \mathbb{R}$. □

Clearly enough, the twist condition (5.44) could be replaced by an avoiding rays condition, as in Theorem 21, or by a sign condition on the edges of an M-cell, like in Theorem 22. Also, we could provide a statement on an annulus, similarly as in Theorem 23. Or even some combination of these could be considered. We avoid the details, for briefness.

5.8 Examples in higher dimensions

In this section we just briefly mention how the examples given in Section 5.4 generalize to higher dimensions applying the results of Sections 5.5 and 5.7. To this aim, consider a system of the form

$$\begin{cases} \dot{\phi} = \partial_\psi P(t,\phi,\psi,q,u)\,, \quad \dot{\psi} = -\partial_\phi P(t,\phi,\psi,q,u)\,, \\ -\ddot{q}_i = \; g_i(t,q_i) - e_i(t) + \partial_{q_i} P(t,\phi,\psi,q,u)\,, \quad i = 1,\dots,M, \\ -\ddot{u}_j = -h_j(u_j) + \partial_{u_j} P(t,\phi,\psi,q,u)\,, \quad j = 1,\dots,D, \end{cases} \tag{5.45}$$

where all functions are continuous and T-periodic in t. For simplicity, we assume that $P(t,\phi,\psi,q,u)$ is 2π-periodic in each of the variables included in ϕ, ψ and q, so, due to this, we fix an arbitrary cube in $\mathbb{R}^{2L+M}$ of length $[0,2\pi]$, denoting it by Θ. Moreover, let $P(t,\phi,\psi,q,u)$ be continuously differentiable in (ϕ,ψ,q,u) and assume that, for every $i \in \{1,\dots,M\}$,

$$\int_0^T e_i(t)dt = 0 \quad \text{and} \quad \int_0^{2\pi} g_i(t,s)ds = 0\,.$$

Then system (5.45) is of the type (5.31), with

$$\begin{aligned} H(t,z) = \frac{1}{2}|p+E(t)|^2 + \frac{1}{2}|v|^2 + \sum_{i=1}^M \int_0^{q_i} g(t,s)\,ds \\ - \sum_{j=1}^D \int_0^{u_j} h_j(\sigma)\,d\sigma + P(t,\phi,\psi,q,u)\,, \end{aligned}$$

where $z = (\phi,\psi,q,p,u,v)$ and $E = (E_1,\dots,E_M) : [0,T] \to \mathbb{R}^M$ is a primitive of the field $e(t) = (e_1(t),\dots,e_M(t))$.

Let us state the analogues of Corollaries 4 and 5.

Corollary 7. *In the above setting, assume moreover that there exist two vectors $\alpha,\beta \in \mathbb{R}^D$, with $\alpha \le \beta$, such that, for all $j \in \{1,\dots,D\}$,*

$$\begin{aligned} h_j(\alpha_j) < \partial_{u_j} P(t,\phi,\psi,q,u), \qquad \text{when } u_j = \alpha_j\,, \\ h_j(\beta_j) > \partial_{u_j} P(t,\phi,\psi,q,u), \qquad \text{when } u_j = \beta_j\,, \end{aligned}$$

for all $(t,\phi,\psi,q) \in [0,T]\times\Theta$ and all $u \in \mathbb{R}^D$ such that $\alpha \le u \le \beta$. Then, system (5.26) *has at least $2L+M+1$ geometrically distinct T-periodic solutions, with $\alpha \le u \le \beta$.*

Proof. All the assumptions of Theorem 22 are easily verified, whence the conclusion. □

As an immediate consequence, we have the following.

Corollary 8. *The system*

$$\begin{cases} \dot\phi = \partial_\psi P(t,\phi,\psi,q,u)\,, \quad \dot\psi = -\partial_\phi P(t,\phi,\psi,q,u)\,, \\ \ddot q_i + a_i \sin q_i = e_i(t) - \partial_{q_i} P(t,\phi,\psi,q,u)\,, \quad i = 1,\dots,M, \\ \ddot u_j + b_j \sin u_j = -\partial_{u_j} P(t,\phi,\psi,q,u)\,, \quad j = 1,\dots,D, \end{cases}$$

where $P(t,\phi,\psi,q,u)$ and $e(t)$ are as above, has at least $2L+M+1$ geometrically distinct T-periodic solutions, for any $a_i > 0$, if $\|\partial_{u_j} P\|_\infty < b_j$ for every $j \in \{1,\dots,D\}$.

All the other examples presented in Section 5.4 can be displayed in this more general setting, also with a mixing of assumptions on each component. In particular, we thus generalize the results in [21, 24, 26, 34, 35, 50, 53, 58, 69, 73, 86, 88]. Also equations with singularities could be considered, as in [47, 51]. We will avoid entering further into details, for briefness.

A further example of application is analyzed in the next section.

5.9 Periodic perturbations of completely integrable systems

In [15, 44], perturbations of completely integrable Hamiltonian systems were studied, see also [45], and the references therein. We will now add an extra term to the Hamiltonian function, involving lower and upper solutions.

We consider the system

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) + \varepsilon\partial_I P(t,\varphi,I,u)\,, \quad \dot I = -\varepsilon\partial_\varphi P(t,\varphi,I,u)\,, \\ -\ddot u = \partial_u G(t,u) + \varepsilon\partial_u P(t,\varphi,I,u)\,. \end{cases} \tag{5.46}$$

Here, we have $(\varphi, I) \in \mathbb{R}^{2M}$ and $u \in \mathbb{R}^D$. We assume that $\mathcal{K} : \mathbb{R}^M \to \mathbb{R}$ is continuously differentiable and the same for $G : \mathbb{R}\times\mathbb{R}^D \to \mathbb{R}$ with respect to the second variable. The perturbation function $P : \mathbb{R}\times\mathbb{R}^{2M+D} \to \mathbb{R}$ is continuous, T-periodic in t, and continuously differentiable with respect to all the other variables, with a bounded gradient. Moreover, the function P is τ_i-periodic in each φ_i, i.e., for every $i \in \{1,\dots,M\}$,

$$P(\dots,\varphi_i+\tau_i,\dots) = P(\dots,\varphi_i,\dots)\,.$$

We also assume that there exist some integers $m_1,\dots,m_M$ for which

$$T\nabla\mathcal{K}(I^0) = (m_1\tau_1,\dots,m_M\tau_M)\,.$$

The expert reader will recognize that we are dealing with a *completely resonant torus.* Here is our result.

Theorem 25. *In the above setting, assume that there exist $I^0 \in \mathbb{R}^M$, a symmetric invertible $M\times M$ matrix $\mathbb{A}$ and $\bar\rho > 0$ such that*

$$0 < |I - I^0| \le \bar\rho \quad \Rightarrow \quad \langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0), \mathbb{A}(I-I^0)\rangle > 0\,. \tag{5.47}$$

Moreover, let there exist some constants $\delta > 0$, $\varsigma > 0$, $\alpha = (\alpha_1,\dots,\alpha_D)$ and $\beta = (\beta_1,\dots,\beta_D)$ with $\alpha \le \beta$, having the following property: If $\alpha - \bar\delta \le u \le \beta + \bar\delta$, for every $j \in \{1,\dots,D\}$ one has

$$\begin{cases} \partial_{u_j} G(t,u) \ge \varsigma\,, & \textit{when } u_j \in [\alpha_j - \delta, \alpha_j]\,, \\ \partial_{u_j} G(t,u) \le -\varsigma\,, & \textit{when } u_j \in [\beta_j, \beta_j + \delta]\,. \end{cases}$$

Then, for every $\sigma > 0$ there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \le \bar{\varepsilon}$, there are at least $M+1$ geometrically distinct solutions of system (5.46) *satisfying*

$$\varphi(t+T) = \varphi(t) + T\nabla\mathcal{K}(I^0), \quad I(t+T) = I(t),$$

$$u(t+T) = u(t), \quad v(t+T) = v(t),$$

and such that

$$|\varphi(t) - \varphi(0) - t\,\nabla\mathcal{K}(I^0)| + |I(t) - I^0| < \sigma,$$

and

$$\alpha \le u(t) \le \beta,$$

for every $t \in \mathbb{R}$.

Proof. Since we are looking for solutions with $I(t)$ in the open ball $B(I^0, \sigma)$, we can suitably modify the function $\mathcal{K}$ outside this set and assume that it has a bounded gradient. We perform the change of variables

$$q(t) = \varphi(t) - t\,\nabla\mathcal{K}(I^0), \quad p(t) = I(t) - I^0,$$

thus obtaining the new Hamiltonian function

$$\mathcal{H}(t,q,p,u,v) = \mathcal{K}(p) + \tfrac{1}{2}|v|^2 + G(t,u) + \varepsilon Q(t,q,p,u),$$

where the functions

$$\mathcal{K}(p) = \mathcal{K}(I) - \left\langle \nabla\mathcal{K}(I^0), I - I^0 \right\rangle, \qquad Q(t,q,p,u) = P(t,\varphi,I,u)$$

are implicitly defined. We notice that Q is periodic in $q_1, \ldots, q_M$ and both Q and $\mathcal{K}$ have bounded gradient: so, there are $C_{\mathcal{K}}, C_Q > 0$ such that

$$|\nabla\mathcal{K}(p)| \le C_{\mathcal{K}}, \qquad |\nabla Q(t,q,p,u)| \le C_Q,$$

for every $(t,q,p,u) \in [0,T] \times \mathbb{R}^{2M+D}$. Our goal is now to prove that for every $\overline{\sigma} > 0$ there exists $\overline{\varepsilon} > 0$ such that, if $|\varepsilon| \le \overline{\varepsilon}$, there are $M+1$ geometrically distinct T-periodic solutions $z = (q,p,u,v)$ of the system

$$S\dot{z} = \nabla\mathcal{H}(t,z), \tag{5.48}$$

satisfying $\max\{|q(t) - q(0)|, |p(t)|\} < \overline{\sigma}$ and $\alpha \le u(t) \le \beta$, for every $t \in \mathbb{R}$.

Let us fix $\overline{\sigma} \in (0, \overline{\rho})$. We are in the setting of Theorem 24. Assumptions 6, 7, 8 and 12 can be easily checked. We need to verify the twist condition (5.44).

Since $\nabla\mathcal{K}(0) = 0$, we can choose $r < \overline{\sigma}/4$ such that

$$|p| \le 4r \quad \Rightarrow \quad 2T|\nabla\mathcal{K}(p)| < \overline{\sigma}, \tag{5.49}$$

and by (5.47) there is $\ell > 0$ such that

$$r \leq |p| \leq 4r \quad \Rightarrow \quad \langle \nabla \mathcal{K}(p), \mathbb{A}p \rangle > 4\ell \,. \tag{5.50}$$

Let us fix $\delta > 0$ satisfying

$$\delta < \min\left\{ r \,, \frac{2\ell}{C_{\mathcal{K}} \|\mathbb{A}\|} \right\}.$$

Reducing $\overline{\varepsilon}$ if necessary, every solution of (5.48), with $|\varepsilon| \leq \overline{\varepsilon}$, is such that $|p(t) - p(0)| < \delta$ for every $t \in [0, T]$.

Let us consider a solution z of (5.48) with $|\varepsilon| \leq \overline{\varepsilon}$ and initial condition satisfying $2r \leq |p(0)| \leq 3r$. Then, $r \leq |p(t)| \leq 4r$ for every $t \in [0, T]$, and so, by (5.50),

$$\langle \nabla \mathcal{K}(p(t)), \mathbb{A}p(t) \rangle > 4\ell \,, \quad \text{for every } t \in [0, T] \,.$$

Reducing $\overline{\varepsilon}$ if necessary, we have that

$$\begin{aligned}
&\langle \partial_p H(t, z(t)), \mathbb{A}p(0) \rangle = \\
&\quad = \langle \nabla \mathcal{K}(p(t)), \mathbb{A}p(t) \rangle - \langle \nabla \mathcal{K}(p(t)), \mathbb{A}(p(t) - p(0)) \rangle + \varepsilon \langle \partial_p Q(t, z(t)), \mathbb{A}p(0)) \rangle \\
&\quad > 4\ell - C_{\mathcal{K}} \|\mathbb{A}\| \delta - 3\varepsilon C_Q \|\mathbb{A}\| r > \ell \,.
\end{aligned}$$

Integrating the previous estimate in the interval $[0, T]$, we get

$$\langle q(T) - q(0), \mathbb{A}p(0) \rangle > \ell T > 0 \,.$$

We can thus apply Theorem 24, choosing $\mathcal{D} = \overline{B}(0, 2r)$ and $\rho = r$. We have that

$$|p(t)| \leq |p(0)| + \delta < 2r + \delta < 3r < \overline{\sigma} \,, \qquad \text{for every } t \in [0, T] \,.$$

Moreover, by (5.49), we deduce that $|q(t) - q(0)| < \overline{\sigma}$, for every $t \in [0, T]$.

The T-periodic solutions we have found are then translated, by the inverse change of variables, into the solutions of (5.46) we are looking for. □

Remark 26. *The twist condition* (5.47) *is surely verified if* $\mathcal{K}$ *is twice continuously differentiable with* $\det \mathcal{K}''(I^0) \neq 0$, *by taking* $\mathbb{A} = \mathcal{K}''(I^0)$.

Remark 27. *A more general twist condition (see [41]) can be considered, for instance,*

$$0 \in cl\left\{ r \in (0, +\infty) \;:\; \min_{|I - I^0| = r} \langle \nabla \mathcal{K}(I) - \nabla \mathcal{K}(I^0), \mathbb{A}(I - I^0) \rangle > 0 \right\}.$$

Remark 28. *The same type of result holds, with the due changes, for a more general system of the type*

$$\begin{cases}
\dot{\phi} = \varepsilon \partial_\psi P(t, \phi, \psi, \varphi, I, u, w) \,, \quad \dot{\psi} = -\varepsilon \partial_\phi P(t, \phi, \psi, \varphi, I, u, w) \,, \\
\dot{\varphi} = \nabla \mathcal{K}(I) + \varepsilon \partial_I P(t, \phi, \psi, \varphi, I, u, w) \,, \quad \dot{I} = -\varepsilon \partial_\varphi P(t, \phi, \psi, \varphi, I, u, w) \,, \\
-\ddot{u} = \partial_u G(t, u) + \varepsilon \partial_u P(t, \phi, \psi, \varphi, I, u, w) \,, \\
S\dot{w} = \mathbb{B}(t) w + \varepsilon \partial_w P(t, \phi, \psi, \varphi, I, u, w) \,,
\end{cases}$$

when P is also periodic in $\phi_1, \dots, \phi_L$ and $\psi_1, \dots, \psi_L$, and $\mathbb{B}(t)$ is a symmetric matrix, continuous and T-periodic, such that $w(t) \equiv 0$ is the only T-periodic solution of $S\dot{w} = \mathbb{B}(t)w$.

Remark 29. *It would be interesting to see how Theorem 25 could be extended to infinite dimensions, in the spirit of [45].*

5.10 The general result

We consider system (5.1), assuming the Hamiltonian function $H\colon \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ to be continuous, T-periodic in its first variable t, and continuously differentiable with respect to the variable z, with corresponding gradient $\nabla H(t, z)$.

For $z \in \mathbb{R}^{2N}$, we use the notation $z = (x, y)$, with $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $y = (y_1, \dots, y_N) \in \mathbb{R}^N$. Moreover, we gather into five groups the variables of x and y, respectively, thus writing

$$x = (x^a, x^b, x^c, x^d, x^e), \quad y = (y^a, y^b, y^c, y^d, y^e),$$

where, for some nonnegative integers N^a, N^b, N^c, N^d, N^e,

$$\begin{aligned}
x^a &= (x_1^a, \dots, x_{N^a}^a) \in \mathbb{R}^{N^a}, & y^a &= (y_1^a, \dots, y_{N^a}^a) \in \mathbb{R}^{N^a}, \\
x^b &= (x_1^b, \dots, x_{N^b}^b) \in \mathbb{R}^{N^b}, & y^b &= (y_1^b, \dots, y_{N^b}^b) \in \mathbb{R}^{N^b}, \\
x^c &= (x_1^c, \dots, x_{N^c}^c) \in \mathbb{R}^{N^c}, & y^c &= (y_1^c, \dots, y_{N^c}^c) \in \mathbb{R}^{N^c}, \\
x^d &= (x_1^d, \dots, x_{N^d}^d) \in \mathbb{R}^{N^d}, & y^d &= (y_1^d, \dots, y_{N^d}^d) \in \mathbb{R}^{N^d}, \\
x^e &= (x_1^e, \dots, x_{N^e}^e) \in \mathbb{R}^{N^e}, & y^e &= (y_1^e, \dots, y_{N^c}^e) \in \mathbb{R}^{N^e}.
\end{aligned}$$

and we also introduce the notation

$$z^a = (x^a, y^a)\,,\ z^b = (x^b, y^b)\,,\ z^c = (x^c, y^c)\,,\ z^d = (x^d, y^d)\,,\ z^e = (x^e, y^e)\,.$$

Notice that one or more of these integers could be equal to zero, in which case the corresponding group will not be taken into account. For example, if $N^a = 0$, then x^a, y^a and z^a will disappear from the list. In the following, for simplicity we sometimes write (u, v) instead of (x^e, y^e), and $D = N^e$.

Let us introduce our assumptions in this general setting.

Assumption 13 (Periodicity)**.** *The function $H(t, z)$ is periodic in each of the variables included in x^a, x^b, y^a, y^c.*

The total number of variables in which our Hamiltonian function is periodic is thus $2N^a + N^b + N^c$. Under this setting, T-periodic solutions $z(t)$ of (5.1) appear in equivalence classes made of those solutions whose components in $x^a(t), x^b(t), y^a(t), y^c(t)$ differ by an integer multiple of the corresponding periods. We say that two T-periodic solutions are *geometrically distinct* if they do not belong to the same equivalence class.

Assumption 14 (Lower and upper solutions). *There exist some constants $\delta > 0$, $\alpha = (\alpha_1, \dots, \alpha_D)$ and $\beta = (\beta_1, \dots, \beta_D)$ with $\alpha \leq \beta$, having the following property. If $\alpha - \bar{\delta} \leq u \leq \beta + \bar{\delta}$, then, for every $j \in \{1, \dots, D\}$,*

$$v_j \, \partial_{v_j} H(t, z) > 0 \,, \;\; \text{when } u_j \in [\alpha_j - \delta, \alpha_j] \cup [\beta_j, \beta_j + \delta] \text{ and } v_j \neq 0 \,,$$

and

$$\begin{cases} \partial_{u_j} H(t, z) \geq 0 \,, & \text{when } u_j \in [\alpha_j - \delta, \alpha_j] \text{ and } v_j = 0 \,, \\ \partial_{u_j} H(t, z) \leq 0 \,, & \text{when } u_j \in [\beta_j, \beta_j + \delta] \text{ and } v_j = 0 \,. \end{cases}$$

Assumption 15 (Nagumo condition). *For every $j \in \{1, \dots, D\}$ there exist $d_j > 0$ and two continuous functions $f_j, \varphi_j : [d_j, +\infty) \to (0, +\infty)$, with*

$$\int_{d_j}^{+\infty} \frac{f_j(s)}{\varphi_j(s)} \, ds = +\infty \,,$$

satisfying the following property. If $\alpha - \bar{\delta} \leq u \leq \beta + \bar{\delta}$, then

$$\begin{cases} \partial_{v_j} H(t, z) \geq f_j(v_j) \,, & \text{when } v_j \geq d_j \,, \\ \partial_{v_j} H(t, z) \leq -f_j(-v_j) \,, & \text{when } v_j \leq -d_j \,, \end{cases}$$

and

$$\big| \partial_{u_j} H(t, z) \big| \leq \varphi_j(|v_j|) \,, \quad \text{when } |v_j| \geq d_j \,.$$

Assumption 16 (Linear growth). *There exists a symmetric $2N^d \times 2N^d$ matrix $\mathbb{B}(t)$, T-periodic and continuous in t, satisfying the nonresonance condition*

$$z^d(t) \equiv 0 \quad \text{is the only } T\text{-periodic solution of } S\dot{z}^d(t) = \mathbb{B}(t) z^d(t) \,,$$

and such that, writing

$$H(t, z) = \tfrac{1}{2} \big\langle \mathbb{B}(t) z^d, z^d \big\rangle + \mathcal{H}(t, z) \,,$$

for every $K > 0$ there is a $C_K > 0$ with the following property: If $\alpha - \bar{\delta} \leq u \leq \beta + \bar{\delta}$ and $|v| \leq K$, then

$$|\partial_{x^b} \mathcal{H}(t, z)| + |\partial_{y^c} \mathcal{H}(t, z)| + |\partial_{x^d} \mathcal{H}(t, z)| + |\partial_{y^d} \mathcal{H}(t, z)| \leq C_K (|y^b| + |x^c| + 1) \,.$$

The above assumption guarantees that, for any two continuous functions $U, V : [0, T] \to \mathbb{R}^D$, with

$$\alpha - \bar{\delta} \leq U(t) \leq \beta + \bar{\delta} \,, \qquad \text{for every } t \in [0, T] \,, \tag{5.51}$$

setting $W(t) = (U(t), V(t))$ and

$$\overline{H}(t, z) = H(t, z^a, z^b, z^c, z^d, W(t)) \,,$$

the solutions of

$$S\dot{z} = \nabla \overline{H}(t, z) \tag{5.52}$$

are defined on $[0, T]$, cf. Remark 22.

Let us also introduce a C^1-function $h\colon \mathbb{R}^{N^b+N^c} \to \mathbb{R}$ and a regular symmetric $(N^b+N^c)\times(N^b+N^c)$ matrix $\mathbb{S}$ such that, for some positive constants C_1, C_2,

$$|h(v) - \tfrac{1}{2}\langle \mathbb{S}v, v\rangle| \le C_1 \quad \text{and} \quad |\nabla h(v) - \mathbb{S}v| \le C_2\,, \qquad \text{for every } v \in \mathbb{R}^{N^b+N^c},$$

and let

$$\mathcal{D} = \{v \in \mathbb{R}^{N^b+N^c} : \nabla h(v) = 0\}\,.$$

We assume that such a set is compact. Our main result is the following.

Theorem 26. *Let Assumptions 13, 14, 15, and 16 hold. Assume moreover that there exists $\rho > 0$ such that, for any two continuous functions $U, V : [0,T] \to \mathbb{R}^D$ satisfying* (5.51)*, the solutions of* (5.52) *with $(y^b(0), x^c(0)) \notin \mathring{\mathcal{D}}$ and* $\operatorname{dist}((y^b(0), x^c(0)), \mathcal{D}) \le \rho$ *are such that*

$$(x^b(T) - x^b(0), y^c(T) - y^c(0)) \notin \{\lambda J \nabla h((y^b(0), x^c(0))) : \lambda \ge 0\}\,.$$

Then, system (5.1) *has at least $2N^a + N^b + N^c + 1$ geometrically distinct T-periodic solutions $z(t)$, such that*

$$(y^b(0), x^c(0)) \in \mathring{\mathcal{D}}\,,$$

and

$$\alpha \le u(t) \le \beta\,, \quad \textit{for every } t \in \mathbb{R}\,.$$

Proof. With the same procedure adopted in the proof of Theorem 21 provided in Section 5.6, dealing with the $z^e = (u, v)$ coordinates, we can introduce a modified problem ruled by a Hamiltonian of the type

$$\widehat{H}(t,z) = \tfrac{1}{2}\big\langle \mathbb{B}(t) z^d, z^d \big\rangle + \frac{1}{2}(|v|^2 - |u|^2) + \widehat{h}(t,z)\,.$$

Moreover, $\widehat{H}$ can be introduced so to guarantee that every T-periodic solution of $S\dot{z} = \nabla \widehat{H}(t,z)$ satisfies an a priori bound as in Lemma 5, in particular $\alpha \le u \le \beta$. Then, from Assumption 16, we can introduce $K > 0$ as in (5.37) so to obtain

$$|\partial_{x^b} \widehat{h}(t,z)| + |\partial_{y^c} \widehat{h}(t,z)| \le C_K(|y^b| + |x^c| + 1)\,.$$

when $\alpha - \bar{\delta} \le u \le \beta + \bar{\delta}$ and $|v| \le K$. Hence, as in (5.38), we can find a constant $c > 0$ such that, if $z(t)$ is a solution of $S\dot{z} = \nabla \widehat{H}(t,z)$ starting with $\operatorname{dist}((y^b(0), x^c(0), \mathcal{D}) \le \rho$, with maximal interval of future existence $[0, \omega)$, then

$$\max\{|y^b(t)|\,,\, |x^c(t)|\} \le c\,, \qquad \text{for every } t \in [0,T] \cap [0,\omega)\,.$$

Again, we can introduce the cut-off function η as in (5.21), and the Hamiltonian

$$\widetilde{H}(t,z) = \tfrac{1}{2}\big\langle \mathbb{B}(t) z^d, z^d \big\rangle + \frac{1}{2}(|v|^2 - |u|^2) + \widetilde{h}(t,z)\,,$$

where $\widetilde{h}(t,z) = \eta(|y^b|)\eta(|x^c|)\widehat{h}(t,z)$. Using Assumption 16 we get

$$|\nabla \widetilde{h}(t,z)| \le \widetilde{C}\,,$$

for a certain positive constant $\widetilde{C}$.

We can rewrite the previous Hamiltonian as

$$\widetilde{H}(t,z) = \tfrac{1}{2}\big\langle \mathbb{M}(t)(z^d, z^e), (z^d, z^e)\big\rangle + \widetilde{h}(t,z)\,,$$

where

$$\mathbb{M}(t)\big(z^d, z^e\big) = \mathbb{M}(t)\big(z^d, (u,v)\big) = \big(\mathbb{B}(t)z^d, (-u,v)\big)\,.$$

In particular, we can verify the nonresonance condition

$$z^{d,e}(t) \equiv 0 \;\text{ is the only } T\text{-periodic solution of }\; S\dot{z}^{d,e}(t) = \mathbb{M}(t)z^{d,e}(t)\,,$$

where $z^{d,e} = (z^d, z^e)$. Our aim is to apply [44, Theorem 1.1] to the modified system $S\dot{z} = \nabla\widetilde{H}(t,z)$. Its solutions are globally defined on $[0,T]$.

Let $z = (z^a, z^b, z^c, z^d, z^e)$ be a solution of $S\dot{z} = \nabla\widetilde{H}(t,z)$. As in the above proofs we can show that if $\mathrm{dist}((y^b(0), x^c(0), \mathcal{D}) \le \rho$, then $\max\{|y^b(t)|\,, |x^c(t)|\} \le c$ for every $t \in [0,T]$. Hence, z solves $S\dot{z} = \nabla\widehat{H}(t,z)$ and, setting $W(t) = \big(\zeta(u(t)), \chi(\zeta(u(t)), v(t))\big)$, we see that (z^a, z^b, z^c, z^d) is a solution of (5.52).

If $0 < \mathrm{dist}((y^b(0), x^c(0)), \mathcal{D}) \le \rho$, then

$$(x^b(T) - x^b(0), y^c(T) - y^c(0)) \notin \{\lambda J\nabla h((y^b(0), x^c(0))) : \lambda \ge 0\}\,,$$

by the hypothesis of the theorem.

The application of [44, Theorem 1.1] provides us $2N^a + N^b + N^c + 1$ geometrically distinct T-periodic solutions of $S\dot{z} = \nabla\widetilde{H}(t,z)$ satisfying $(y^b(0), x^c(0)) \in \mathring{\mathcal{D}}$. Then, following the argument of the previous proofs, it can be seen that they are solutions of $S\dot{z} = \nabla H(t,z)$, as well, and $\alpha \le u \le \beta$. □

Remark 30. *Theorem 26 generalizes all three Theorems 20, 21 and 22 previously stated. For example, let the assumptions of Theorems 20 hold. We consider a smooth function* $\sigma\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$\sigma(s) = \begin{cases} 0\,, & \text{if } s \le 0\,,\\ 1\,, & \text{if } s \ge 1\,, \end{cases} \qquad \text{and} \qquad \sigma'(s) > 0 \;\text{ if } s \in (0,1)\,,$$

and, adapting the notations, we define the function $h\colon \mathbb{R}^{N^b+N^c} \to \mathbb{R}$ *by*

$$h(p) = -\xi(p)\langle \mathbb{A}(p - \pi_{\mathcal{D}}(p)), p - \pi_{\mathcal{D}}(p)\rangle\,,$$

where

$$\xi(p) = \begin{cases} 0\,, & \text{if } p \in \mathcal{D}\,,\\ \tfrac{1}{2}\sigma(|p - \pi_{\mathcal{D}}(p)|)\,, & \text{if } p \notin \mathcal{D}\,. \end{cases}$$

Following the proof of [44, Corollary 2.3] one can verify that h satisfies the assumptions in Theorem 26.

Having extended with Theorem 26 the main theorem in [44], we thus have generalized, for instance, the results in [25, 29, 43, 56, 60, 74, 79, 86].

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