Problemas Variacionales Geométricos en Grupos de Lie Nilpotentes

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Problemas Variacionales Geométricos en Grupos de Lie Nilpotentes

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Contents

List of Symbols **ix**

Resumen **xi**

1 Introduction **1**
1.1 State of the art 1
1.2 Summary and conclusions 8

2 Nilpotent groups and sub-Finsler perimeter **19**
2.1 Carnot-Carathéodory spaces 19
2.1.1 Minkowski norms 20
2.1.2 Sub-Finsler Carnot-Carathéodory spaces 22
2.2 Nilpotent groups 22
2.2.1 Sub-Finsler Carnot groups 25
2.2.2 The Heisenberg group $\mathbb{H}^n$ 28
2.3 (X, K)-variation and (X, K)-Caccioppoli sets 29
2.3.1 Representations of the (X, K)-variation 32

3 Existence of isoperimetric regions in sub-Finsler nilpotent groups **35**
3.1 Preliminaries 37
3.1.1 Isoperimetric inequality for small volumes 38
3.2 Properties of isoperimetric regions 39
3.3 Existence of isoperimetric regions 43

4 The Brunn-Minkowski inequality in nilpotent groups **47**
4.1 The Brunn-Minkowski inequality 48
4.1.1 A sufficient condition for strict inequality in the Heisenberg group 53
4.2 Consequences 54

5 Pansu-Wulff shapes in $\mathbb{H}^1$ **59**

5.1 The pseudo-hermitian connection 61
5.2 First variation of sub-Finsler area 62
5.3 Pansu-Wulff spheres and examples 70
5.4 Geometric properties of the Pansu-Wulff spheres 77
5.5 Minimization property of the Pansu-Wulff shapes 85
5.5.1 A uniqueness result in $rK_0 \times \mathbb{R}$ 90

6 Area-minimizing graphs in $\mathbb{H}^1$ 97
6.1 The first variation formula and a stationary condition 97
6.2 Examples of minimizing graphs with one singular line 99
6.3 Area-Minimizing Cones in $\mathbb{H}^1$ 106

7 The PMC equation in $\mathbb{H}^n$ 109
7.1 Preliminaries 111
7.1.1 Finsler geometry of hypersurfaces in Euclidean space . 111
7.1.2 Sub-Finsler area 117
7.1.3 The sub-Finsler PMC equation 118
7.2 The Finsler approximation problem 119
7.2.1 The Finsler PMC equation 122
7.3 A priori estimates for the Finsler PMC equation 123
7.4 Existence of minimizer for the sub-Finsler functional 136

A Alternative proof of Theorem 2.3.3 139

List of Symbols

$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathbb{C}^n$	n-dimensional Complex space
$\mathbb{H}^n$	n-dimensional Heisenberg group
$\mathbb{S}^n$	n-dimensional Euclidean sphere
$\mathbb{B}_K$	Pansu-Wulff shape related to K
$\mathbb{S}_K$	Pansu-Wulff sphere related to K
$\mathfrak{g}$	Lie algebra of a Lie group G
TM, T_pM	tangent bundle and tangent space at p
$\mathcal{H}$, $\mathcal{H}_p$	horizontal distribution and horizontal subspace at p
d	Carnot-Carathéodory distance
d_K	sub-Finsler distance
$d_{K,\Sigma}$	Finsler distance from a hypersurface Σ
$B_{eu}(p,r)$	Euclidean open ball centered at p of radius r
$B(p,r)$	sub-Riemannian open ball centered at p of radius r
$B_K(p,r)$	Finsler open ball centered at p of radius r
$\bar{F}$	closure of a set F
$\#F$	cardinal of a set F
χ_F	characteristic function of a set F
∂F	topological boundary of a set F
F_s	set of points with density s with respect to F
$P(F;\cdot)$, $\lvert\partial F\rvert$	sub-Riemannian perimeter measure of a set F
$P_K(F;\cdot)$, $\lvert\partial F\rvert_K$	sub-Finsler perimeter measure of a set F
$\lvert F\rvert_n$	n-dimensional Lebesgue measure of a set F
$\lvert F\rvert$	d-dimensional Lebesgue measure of a set F in $\mathbb{R}^d$
$\mathcal{H}^s$	s-dimensional Hausdorff measure
μ	Haar measure on a Lie group

$\Subset$	compactly contained
Δ	symmetric difference of sets
$\oplus$	direct sum of vector spaces
$\times$	Cartesian product of sets
$\text{span}(A)$	subspace spanned by A
$[p, q]$	segment joining p and q
$p \cdot q$	group product between p and q in a group G
ℓ_p	left-translation in G by p
δ_r	dilation in a Carnot group G of factor r
$\|p\|$	Euclidean norm of p
$\|p\|_K$, $\|p\|_*$	Minkowski norm and dual norm of p associated to K
$\dot{\gamma}$, γ'	time derivative of a curve γ
$v^\top$	tangential part of v
$\frac{\partial f}{\partial v}$, $\frac{\partial}{\partial v}$, $\partial_v f$	partial derivative of f with respect to v
$g, \langle\cdot,\cdot\rangle$	Riemannian metric
D	Levi-Civita connection
$\frac{D}{ds}$	covariant derivative related to D
∇	pseudo-hermitian connection
$\frac{\nabla}{ds}$	covariant derivative related to ∇
$\exp(U)$	exponential of a vector field U in a Lie group
$[U, V]$	Lie bracket of the vector fields U and V
$\|U\|_{\infty,K}$	infinity norm related to $\|\cdot\|_K$
U_h	horizontal projection of U
N	unit normal vector
ν_h	horizontal unit normal vector
ν_K	horizontal unit normal vector related to K
$\mathcal{N}_K$	Gauss map of K
π_K, π	projection over K
h_K	support function of K
$H_{K,\Sigma}$	Finsler mean curvature of Σ
C^n_+	set of C^n convex bodies whose boundary has strictly positive curvature

Resumen

En 1982, Pierre Pansu terminó su tesis doctoral [136], centrada en la geometría de los grupos de Heisenberg. Estos grupos, denotados por $\mathbb{H}^n$, se pueden ver como $C^n \times \mathbb{R}$ junto con el producto de grupo dado por

$$(z,t) * (w,s) = \Big(z + w, t + s + \sum_{i=1}^{n} \operatorname{Im}(z_i \bar{w}_i)\Big),$$

y una base global de campos vectoriales invariantes a izquierdas dados por

$$X_i = \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

El grupo de Heisenberg riemanniano se obtiene al considerar una métrica riemanniana invariante a izquierda haciendo $X_1, Y_1, \ldots, X_n, Y_n, T$ ortonormales. Remarcamos que $X_1, Y_1, \ldots, X_n, Y_n$ satisfacen la llamada condición de Hörmander. En su trabajo, Pansu probó que la dimensión de Hausdorff de $\mathbb{H}^1$ con la métrica de Carnot-Carathéodory es 4, y obtuvo la desigualdad isoperimétrica para conjuntos abiertos con frontera C^1 dada por

$$\mathcal{H}^4(D) \leqslant \Big(\frac{12}{\pi}\Big)^{1/3} \mathcal{H}^3(D)^{3/4},$$

donde $\mathcal{H}^s$ es la medida de Hausdorff s-dimensional con respecto a la distancia de Carnot-Carathéodory. Además, Pansu probó que el exponente $3/4$ es óptimo mientras que la constante $12/\pi$ no lo es. Observó que, tomando una geodésica conectando el origen con un punto $(0,0,p)$, todas las rotaciones de γ sobre el eje vertical son también geodésicas conectando el origen y $(0,0,p)$ y la unión de γ y todas sus rotaciones forman una esfera con curvatura media constante. Pansu conjeturó que dichas esferas, posteriormente denominadas esferas de Pansu, son las únicas regiones isoperimétricas en $\mathbb{H}^1$, conjetura que sigue abierta en este momento.

El objetivo de esta tesis es el estudio de problemas variacionales geométricos en grupos de Lie nilpotentes con una estructura sub-finsleriana. Este ambiente extiende la geometría de finsleriana al ámbito de la geometría sub-riemanniana de los grupos de nilpotentes considerando una norma invariante a izquierda en la distribución horizontal del grupo.

En el capítulo 2 expondremos las principales características de los espacios de Carnot-Carathéodory y de los grupos de Lie nilpotentes. Diremos que una distribución $\mathcal{H}$ en un grupo de Lie nilpotente G es horizontal si al tomar sucesivos corchetes de Lie de campos en $\mathcal{H}$ se obtiene el álgebra de Lie $\mathfrak{g}$ de G. Fijada una distribución horizontal invariante a izquierda $\mathcal{H}$, dotaremos a G de estructura sub-finsleriana tomando una norma en $\mathcal{H}_0$. En el caso de tomar una norma euclídea se obtiene un a estructura sub-riemanniana. Estructuras sub-finslerianas simétricas han sido estudiadas recientemente, especialmente relacionadas con el estudio de las geodésicas, en los trabajos de Ardentov, Le Donne y Sachkov [8], y por Barilari, Boscain, Le Donne y Sigalotti [12]. Pueden verse algunas propiedades de las distancias asociadas a estructuras sub-finslerianas asimétricas en [119; 120] y [40]. Tomando una base X de $\mathcal{H}$ formada por campos invariantes a izquierda, daremos una noción de (X, K)-perímetro relacionado con la estructura sub-finsleriana mediante la variación de su función característica, siguiendo el procedimiento establecido con Ritoré en [142] (véase también [69]). La noción de contenido de Minkowski asociado a una estructura sub-finsleriana fue introducida por Sánchez [154]. Obtendremos una relación entre el (X, K)-perímetro con el perímetro sub-riemanniano en el teorema 2.3.3.

En el capítulo 3 consideraremos un grupo de Lie nilpotente G con una familia de campos vectoriales invariantes a izquierda X generando una distribución horizontal, y una norma $|\cdot|_K$ invariante a izquierda asociada a un cuerpo convexo K. Como parte de un trabajo en desarrollo, estudiaremos la existencia de regiones isoperimétricas para cualquier volumen siguiendo los argumentos de Galli y Ritoré en [77]. Además, se obtendrán propiedades geométricas de dichas regiones como su acotación o que su frontera topológica y esencial coinciden.

Algunos resultados en geometría sub-riemanniana sobre la existencia de dichas regiones isoperimétricas son los obtenidos por Galli y Ritoré en variedades de contacto sub-riemannianas [77] y el resultado de Rigot y Leonardi en grupos de Carnot [109]. De especial importancia es el lema de deformación, el cual aparece en una gran variedad de referencias en la literatura, algunas de ellas en [6; 7; 77; 114; 122; 131; 134].

En el capítulo 4 expondremos los resultados obtenidos en [141]. Generalizaremos la desigualdad de Brunn-Minkowski clásica sustituyendo la suma de Minkowski de conjuntos por un producto $* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ de la forma

$$z * w = z + w + (F_1, F_2(z, w), \ldots, F_d(z, w)) = z + w + F(z, w), \qquad (*)$$

donde F_1 es una constante y F_i son funciones continuas que dependen solo de $z_1, \ldots, z_{i-1}, w_1, \ldots, w_{i-1}$ $\forall i = 2, \ldots, d$. Aquí entendemos por un producto cualquier operación binaria sin asumir más propiedades como la asociatividad. Probaremos que para cada $A, B \subseteq \mathbb{R}^d$ conjuntos medibles tales que

$A * B$ es medible, se tiene que

$$|A * B|^{1/d} \geqslant |A|^{1/d} + |B|^{1/d}.$$

El producto de cualquier grupo de Lie nilpotente es de la forma (*) debido a la expresión del producto del grupo en coordenadas exponenciales de primer tipo. Este resultado generaliza al obtenido por Leonardi y Masnou [108] en grupos de Heisenberg.

La desigualdad de Brunn-Minkowski tiene especial relevancia en la geometría convexa [156; 81]. Algunas de sus consecuencias más relevantes están ligadas a las propiedades de las medidas de tipo gaussianas [21; 17], así como en la teoría de transporte óptimo [102; 9; 13; 121]. Algunas pruebas conocidas pueden verse en [81; 94; 104].

En geometría sub-riemanniana, Monti [124] probó que la generalización de la desigualdad de Brunn-Minkowski multiplicativa en $\mathbb{H}^n$ no puede tener exponente $(2n+2)^{-1}$, hecho que fue mejorado posteriormente por Juillet [102], quien probó que no puede darse la desigualdad usando exponente menor que $(2n+1)^{-1}$. Leonardi y Masnou [108] probaron que la desigualdad se da con exponente $(2n+1)^{-1}$, correspondiente a la dimensión topológica de $\mathbb{H}^n$, y finalmente Tao [160; 161] explicó en una entrada en su blog como producir una desigualdad de Prékopa-Leindler en cualquier grupo nilpotente, desigualdad equivalente a la de Brunn-Minkowski. Posteriormente a la escritura del artículo [141], el autor fue informado de que Bobkov probó idéntico resultado en [16].

En el capítulo 5 incluiremos los resultados obtenidos con Ritoré en [142]. Consideraremos puntos críticos del perímetro asociado a una estructura sub-finsleriana en el primer grupo de Heisenberg $\mathbb{H}^1$. En el caso de que la frontera S de E sea una superficie C^1 o lipschitziana, el perímetro de E viene dado por el funcional de área sub-finsleriana

$$A_K(S) = \int_S |N_h|_{K,*} \, dS,$$

donde $|\cdot|_{K,*}$ es la norma dual de $|\cdot|_K$, N_h es la proyección ortogonal sobre la distribución horizontal del vector unitario riemanniano N, y dS es la medida riemanniana de S.

Si consideramos un cuerpo convexo K con frontera de clase C^2 con curvatura geodésica positiva, podemos calcular la primera variación del funcional de área asociado a un campo vectorial U con soporte compacto en la parte regular de S como

$$A_K'(0) = \int_S u \left(\operatorname{div}_S \eta_K \right) dS.$$

La función $H_K = \operatorname{div}_S \eta_K$ se denomina la curvatura media de S. Comprobaremos que la curvatura media está localizada en las curvas horizontales de

S. Probaremos que dichas curvas satisfacen una ecuación diferencial, por lo que podremos clasificar superficies de curvatura media prescrita clasificando soluciones de una ecuación diferencial ordinaria y mirando la interacción de dichas curvas con el conjunto singular S_0 de S, compuesto por aquellos puntos donde el plano tangente es horizontal, como se hizo en [151] en el caso del perímetro sub-riemanniano.

Un observación clave es que las rectas horizontales son soluciones de $H_K = 0$ mientras que los levantamientos horizontales de ∂K son soluciones de $H_K = 1$. La convexidad estricta de K junto a la invarianza de la ecuación por traslaciones a izquierda y dilataciones implica que todas las soluciones son de dicho tipo.

De ésta manera, construiremos el conjunto $\mathbb{B}_K$ obtenido como el conjunto encerrado por los levantamientos horizontales de todas las traslaciones de la curva ∂K que contienen a 0. Comprobaremos que de esta manera se obtiene una esfera topológica $\mathbb{S}_K$ con dos polos sobre la misma recta vertical, que es la unión de dos grafos. Además, la frontera de $\mathbb{B}_K$ es C^2 fuera de los polos, mientras que tiene regularidad C^2 sobre los polos. Cuando $K = D$ es el disco unidad, estos conjuntos fueron construidos por P. Pansu [137] y son comúnmente denominadas esferas de Pansu. Dichas esferas son de clase C^2 pero no C^3 cerca de los puntos singulares, como se puede ver en la proposición 3.15 en [46].

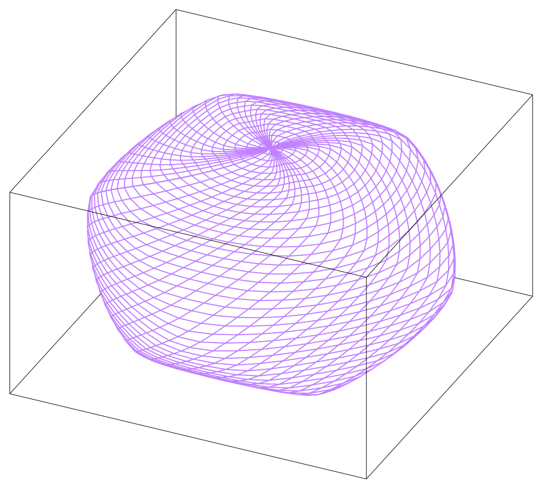

Figure 1: El conjunto $\mathbb{B}_K$ cuando K es la bola unidad de la r-norma $||(x,y)||_r = \left(|x|^r + |y|^r\right)^{1/r}$, $r = 1.5$

Observamos que estas esferas tienen curvatura media constante. Por tanto, son puntos críticos del funcional de área sub-finsleriano bajo una restricción de volumen. En la sección 5.5 probaremos que, bajo una condición geométrica, todo conjunto de perímetro finito con el mismo volumen que $\mathbb{B}_K$ tiene mayor perímetro o igual que dicho conjunto $\mathbb{B}_K$.

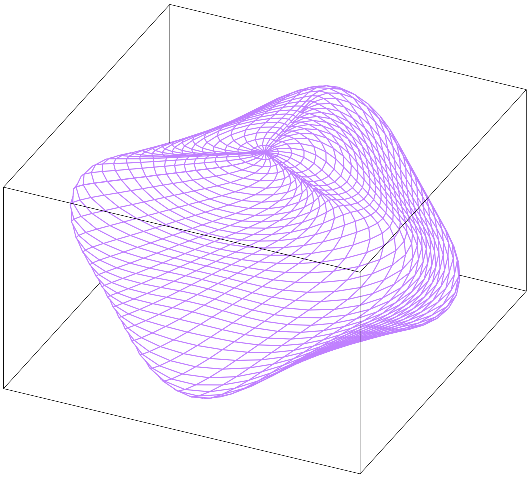

Figure 2: El conjunto $\mathbb{B}_K$ cuando K es una aproximación regular de la norma triangular

Algunos resultados parciales sobre la conjetura de Pansu pueden verse en [125; 126; 124; 151; 34; 147; 31; 33; 68; 108], asumiendo que los candidatos pertenecen a una cierta familia de conjuntos. La monografía [27] proporciona un estudio bastante completo de los resultados conocidos.

El capítulo 6 está dedicado a los resultados obtenidos con Giovannardi y Ritoré en [87]. En dicho capítulo, estudiaremos grafos horizontales enteros que sean area-minimizantes en $\mathbb{H}^1$ con una estructura sub-finsleriana invariante a izquierda. Dichos ejemplos están basados en los correspondientes en el caso sub-riemanniano obtenidos en [146], y la mayoría de ellos tienen una regularidad únicamente lipschitziana. Además se expondrán ejemplos de conos area-minimizantes, en el espíritu de [91].

El capítulo 7 está dedicado a algunos de los resultados obtenidos durante la estancia en la Università di Trento en colaboración con Giovannardi, Pinamonti y Verzellesi. Estudiaremos la ecuación de curvatura media prescrita para t-grafos en un grupo de Heisenberg $\mathbb{H}^n$ con una estructura sub-finsleriana invariante a izquierda. Consideraremos el funcional

$$\mathcal{I}(u) = \int_\Omega |\nabla u + F|_{K_0,*} \, dxdy + \int_\Omega H u \, dxdy, \tag{i}$$

donde $|\cdot|_{K_0,*}$ denota la norma dual de $|\cdot|_{K_0}$. En particular, cuando $F(x,y) = (-y,x)$ el primer término de (i) coincide con el área sub-finsleriana del t-grafo de u. Además, si K_0 es la bola euclídea centrada en el origen y $H = 0$ entonces (i) expresa el funcional de área sub-riemanniano para t-grafos en el grupo de Heisenberg, como se puede ver en [33; 99] y las referencias contenidas. Decimos que el grafo de u tiene K_0-curvatura media prescrita H en Ω si u es un minimizante de $\mathcal{I}$. La ecuación de Euler-Lagrange asociada a fuera del conjunto singular, ésto es, los puntos donde $\nabla u + F$ se anula, viene dada por

$$\operatorname{div}(\pi_{K_0}(\nabla u + F)) = H, \tag{ii}$$

donde π_{K_0} es una cierta función 0-homogénea definida en (2.1.5). Nuestro resultado más importante es el teorema 7.4.1, donde probamos que, bajo ciertas condiciones de regularidad en el dato al borde, existe una solución lipschitziana al problema de Dirichlet para la ecuación de curvatura media prescrita siempre que H sea constante y verifique que

$$|H| < H_{K_0,\partial\Omega}(z_0) \tag{iii}$$

para cada z_0 en $\partial\Omega$, donde $H_{K_0,\partial\Omega}$ denota la curvatura media finsleriana a la frontera $\partial\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$. La acotación (iii) de H en términos de la curvatura media finsleriana del borde del dominio es la condición análoga a la condición clásica de solución del problema de Dirichlet para la ecuación de curvatura media en el espacio euclídeo, como puede verse en [158], [84] o [83] (véase también [85, Theorem 16.11]). Éste problema ha sido estudiado en el primer grupo de Heisenberg riemanniano en [2] bajo la misma condición de H. En el caso sub-riemanniano, dicho problema de Dirichlet con $H = 0$ ha sido estudiado en [138; 35; 33; 32; 58; 139].

La demostración del teorema 7.4.1 sigue el esquema desarrollado en [2]. Estudiaremos la familia de ecuaciones elípticas

$$\operatorname{div}\left(\pi_{K_0}(\nabla u + F)\frac{|\nabla u+F|_*^2}{(\varepsilon^3+|\nabla u+F|_*^3)^{\frac{2}{3}}}\right) = H, \tag{iv}$$

donde $0 < \varepsilon < 1$. Dicha familia de ecuaciones se obtienen considerando una sucesión de cuerpos convexos K_ε en $\mathbb{R}^{2n+1}$ conteniendo el origen y convergiendo a K_0 con la distancia de Hausdorff. La elección de K_ε no es arbitraria. Consideramos para cada $0 < \varepsilon < 1$ la norma invariante a izquierda en $T\mathbb{H}^n$ asociada a K_ε, cuyo funcional del área finsleriana viene dado por

$$\mathcal{I}_\varepsilon(u) = \int_\Omega \left(\varepsilon^3 + |\nabla u + F|_{K_0,*}^3\right)^{\frac{1}{3}} \, dxdy + \int_\Omega Hu \, dxdy.$$

Dado un dato al borde $\varphi \in C^{2,\alpha}(\bar{\Omega})$, la resolución del problema de Dirichlet asociado a la ecuación res:I se reduce a probar estimaciones $C^1(\bar{\Omega})$ *a priori*. Dichas estimaciones se reducen de forma habitual a probar tres pasos: estimar el supremo de $|u|$, estimar el gradiente de u la frontera y estimar el gradiente de u en el interior. Las dos primeras estimaciones se siguen usando un argumento de barreras, las cuales dependen de la distancia finsleriana desde la frontera $\partial\Omega$. Remarcamos que dichas estimaciones se siguen en el caso de considerar la curvatura H no constante y lipschitziana. En el caso de que H no sea constante, para obtener la estimación de $|u|$ asumiremos que existe $\delta \in (0, 1]$ tal que

$$\left|\int_\Omega Hv \, dxdy\right| \leqslant (1-\delta) \int_\Omega |\nabla v|_{K_0,*} \, dxdy \tag{v}$$

para cada $v \in C_c^\infty(\Omega)$. La condición (v) con $\delta = 0$ puede verse que es necesaria al integrar por partes la ecuación (iv). Además, en el espacio euclídeo, Giusti [90] probó que es también suficiente para la existencia de soluciones para la ecuación de curvatura media prescrita. Además, probamos que para el caso $\delta > 0$, la condición (v) es redundante cuando $H = 0$ es constante. El único paso en el que usaremos de manera crucial que H es constante es al usar un principio del máximo para el gradiente de una solución que permite reducir la estimación del gradiente en el interior a su estimación en la frontera. Una observación clave es que las estimaciones C^1 obtenidas son independientes de ε, por lo que podremos usar el teorema de Arzelà-Ascoli para obtener la existencia de un minimizante lipschitziano del problema de Dirichlet para la curvatura media.

Chapter 1

Introduction

1.1 State of the art

The decade of 50 saw the dawn of the branch in mathematics known nowadays as Geometric Measure Theory. In 1952, Renato Caccioppoli published his work *Misura e integrazione sugli insiemi dimensionalmente orientati* [23], where among other results, he studied the family of sets approximable by polyhedral domains of finite perimeter and called a set F of finite perimeter whenever the minimum limit of the perimeters of the polyhedra approximating F in media is finite. Caccioppoli also defined the perimeter using the notion of function of bounded variation on several variables developed by Lamberto Cesari [29] in 1936. Thus a set F has finite perimeter if, and only if, its characteristic function has bounded variation. In 1954 Ennio de Giorgi published *Su una teoria generale della misura (r-1)-dimensionale in uno spazio ad r dimensioni* [50], where he proved that smoothing the characteristic function of a set F, the limit of the variation of the approximations is precisely the perimeter of F. Moreover, he established a divergence theorem for F as a limit in the divergence formula of the approximations. In the following years, De Giorgi developed the theory of finite perimeter sets in [51; 52], proving that the perimeter is supported in the so called reduced boundary. In 1958, after the death of Caccioppoli, he started to use the name Caccioppoli sets for finite perimeter sets.

Later on in the 60s, another keystone on Geometric Measure Theory was developed, the introduction of rectifiable and integral currents. In the paper *Normal and Integral Currents* [63], Federer and Fleming used the definition of currents given by Rham in 1955, to show that Plateau's problem has solution in the class of integral currents. The name Geometric Measure Theory was probably first used by Federer, and in 1969, he published his famous book *Geometric Measure Theory* [61], one of the most cited text books in mathematics, devoted to rectifiability and integral currents. Federer showed that Caccioppoli sets are normal currents of dimension n in n-dimensional space. It is worth mentioning that, although the theory of Caccioppoli sets can be studied as part of the theory of currents, it is usual to use the approach of functions of bounded variation.

Soon after these results, contemporary authors encountered the difficult problem of regularity. This problem consists in proving that Caccioppoli sets or k-dimensional currents satisfying an area minimizing condition are necessarily locally smooth manifolds of dimension k, except for a set of singular points with small Hausdorff dimension. The first important results on this problem are due to Reifenberg [144] and De Giorgi [53], who proved that the singular set of a locally area minimizing set has k-dimensional Hausdorff measure 0. Almgren proved [3] in the decade of the 60s almost everywhere regularity for Plateau's problem and a broader class of geometric variational problems. In the particular case of $k = n - 1$, the existence of singular points is closely related to the question of whether hyperplanes are the only $n-1$-dimensional cones in $\mathbb{R}^n$ which are locally minimal. It is natural to think that the answer is positive, and Fleming proved in [65] that it is true in dimension $n = 3$, a result later generalized by De Giorgi, Almgren and Simons to dimension $n \leqslant 7$, so that area minimizer Caccioppoli sets have no singular points in dimension $n \leqslant 7$. However, in dimension 8 Bombieri, De Giorgi and Giusti provided a counterexample in [18], the Simons' cone. This cone is given in $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4$ by $|x| = |y|$, where x and y are in $\mathbb{R}^4$. As it was showed in [18], Simons' cone is locally area minimizing with a singular point at its vertex 0. This result also implies a negative answer for the Bernstein problem in dimension 8. By these contributions to the Bernstein problem, and contributions on number theory, Enrico Bombieri was awarded the Fields medal in 1974. Federer generalized this result to dimension > 8 in [62], proving that the singular set has Hausdorff dimension at most $n-8$.

At the same time of the advances in the problem of regularity, Lars Hörmander started his study on hypoelliptic operators. In 1967, he published *Hypoelliptic second order differential equations* [98], where he proved that, given a family of vector fields $X_1, \ldots, X_k$ under the so called Hörmander's condition, then the following Cauchy problem has a fundamental solution

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = F(u(t,x)), & t > 0, x \in \mathbb{R}^d, \\ u(0,\cdot) = f(\cdot), \end{cases}$$

where $F = \frac{1}{2}\sum_{i=1}^{n} X_i^2 + X_0$ and X_0 is a vector field. The main goal of [98] is that, even when this operator is not elliptic, there exists a fundamental solution under the weaker Hörmander's condition. His contributions on partial differential equations and, in particular, hypoelliptic operators, led him to win the Fields Medal in 1962. It was in the year 1985 when Hörmander's condition regained interest due to the work of Nagel, Stein and Wainger *Balls and metrics defined by vector fields I: Basic properties* [133]. In this paper, they consider a family of vector fields $X_1, \ldots, X_k$ in $\mathbb{R}^n$ satisfying Hörmander's condition. Due to the previous results of Carathéodory [28], Chow [37] and Rashevskii [143], it is possible to connect any pair of points p and q by a curve moving in the directions of the distribution generated by $X_1, \ldots, X_k$. Thus, taking the infimum of the length of unit-speed curves joining two given points we obtain a distance, called the Carnot-Carathéodory distance. The relevance of this distance is exposed by the property that, for any compact set K, there exists two constants $C_1 > 0$, $C_2 > 0$ and $m > 1$ such that for any two points p and q in K, there holds

$$C_1|q-p| \leqslant d(p,q) \leqslant C_2|q-p|^m,$$

where $|\cdot|$ stands for the Euclidean norm. In particular, Carnot-Carathéodory distance does not come from any Riemannian metric on $\mathbb{R}^n$.

We stop the timeline to introduce some isoperimetric problems, central in this thesis. The classical isoperimetric problem in Euclidean space consists on proving existence and characterizing those sets E in $\mathbb{R}^n$ such that $P(E) \leqslant P(F)$ for all F in $\mathbb{R}^n$ with finite perimeter and the same volume as E. Those sets satisfying the previous condition are called isoperimetric regions. The problem is harder when we choose a more general definition of perimeter. Some of the best known proofs of the solution are the proofs of Steiner [159] and Schwarz [157] by means of symmetrization procedures and the direct proof based on the Brunn-Minkowski inequality with the notion of perimeter given by the Minkowski content. Two classical proofs of the Brunn-Minkowski inequality, are the one by Hadwiger and Ohman [94] and the one by Knothe [104]. This inequality was used by Borell in [19] to solve this isoperimetric problem in $\mathbb{R}^n$ with the Gaussian area. A function related to isoperimetric problems in a Riemannian manifold M is the isoperimetric profile in M, defined as the function $I_M : (0, |M|) \to \mathbb{R}$ given by

$$I_M(v) = \inf\{P(F) : F \subseteq M,\ |F| = v\}.$$

A lower bound on the isoperimetric profile is called an isoperimetric inequality. Throughout analytic properties of I_M it is possible to obtain information on isoperimetric regions, as stated by Chavel [30] and Gallot [80] (see also [148]). We remark this relation stating a result proven by Christophe Bavard and Pierre Pansu in 1986 [15]: in a Riemannian manifold M, of volume v with $\Omega \subset M$ isoperimetric region of volume $0 < t < v$, then the isoperimetric profile I_M has left and right derivatives everywhere. Moreover, if H is the mean curvature of $\partial\Omega$, then

$$I'_+(t) \leqslant 2H \leqslant I'_-(t).$$

The Wulff problem constitutes another problem closely related to the classical isoperimetric problem. Considering a norm $\|\cdot\|$ in Euclidean space with dual norm $\|\cdot\|_*$ and a C^1 surface S, we consider the integral

$$\int_S \|N\|_* dS,$$

where N is a unit normal vector to S and dS is the Riemannian area element. This Functional represents the Gibbs free energy, which is proportional to the area of the surface of contact and the surface tension of an anisotropic interface separating two fluids or gases. Minimizing the free energy for a drop of given volume it is obtained an equilibrium state. Solutions of this problem where described by G. Wulff in 1895: they are translations and dilations of the unit ball related to the norm $\|\cdot\|$, known as Wulff shapes of the free energy. The first proof of this fact was given by Dinghas in [56] and, later on, by Busemann [22] and Taylor [162]. More recent proofs are due to Fonseca [66] and Fonseca and Müller [67].

In 1982, Pansu completed his Ph.D. [136], focused on the geometry of Heisenberg groups. These groups, denoted by $\mathbb{H}^n$, can be seen as $\mathbb{C}^n \times \mathbb{R}$ together with the group product $*$ given by

$$(z,t) * (w,s) = \Big(z + w, t + s + \sum_{i=1}^{n} \operatorname{Im}(z_i \bar{w}_i)\Big),$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$, and a global basis of left-invariant vector fields is given by

$$X_i = \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

where $i = 1, \ldots, n$. The Riemannian Heisenberg group is obtained by taking a left-invariant Riemannian metric g making $X_1, Y_1, \ldots, X_n, Y_n, T$ orthonormal. Notice that the vector fields $X_1, Y_1, \ldots, X_n, Y_n$ satisfy Hörmander's condition. On this work [136], Pansu proved that the Hausdorff dimension of $\mathbb{H}^1$ with the Carnot-Caratheodory metric is 4, and obtained an isoperimetric inequality for open sets D with C^1 boundary

$$\mathcal{H}^4(D) \leqslant \Big(\frac{12}{\pi}\Big)^{1/3} \mathcal{H}^3(D)^{3/4},$$

where $\mathcal{H}^s$ is the s-dimensional Hausdorff measure with respect to the Carnot Carathéodory distance. Pansu also proved that the exponent $3/4$ is optimal but the constant $(12/\pi)^{1/3}$ is not. He noticed that, taking a geodesic conecting the origin in the first Heisenberg group with a point $(0,0,p)$, all the rotations of γ about the vertical axis are also geodesics conecting the origin and $(0,0,p)$ and the union of γ and all its rotations forms a ball with constant mean curvature. Pansu conjetured that this balls, later called Pansu balls, are the only isoperimetric regions in $\mathbb{H}^1$, a conjeture that remains open at this moment.

The decade of the 90s was marked by the blossom of Differential Geometry. Using their tools, new proofs of classical isoperimetric inequalities were provided. By means of the regularity results obtained by Almgren [3], Federer [62], and Gonzalez, Massari and Tamanini [92; 117], the isoperimetric problem can be restricted to sets with smooth boundary up to dimension $n \leqslant 7$, which makes it a problem more suited to study in Riemannian manifolds. The study of the first and second variation of perimeter can be used to characterize minima of perimeter with smooth boundary under a volume constraint up to second order. This pathway was first used by Do Carmo and Barbosa [10] to solved the isoperimetric problem in Euclidean space, and in the sphere and hyperbolic space by Barbosa, Do Carmo and Eschenburg in [11]. Ritoré and Ros studied stable surfaces in Space forms in [149], obtaining a classification of isoperimetric sets in the 3-dimensional projective space. These advances continued in the 2000s, and among the results we remark the characterization of isoperimetric regions on rotationally symmetric surfaces by Ritoré [145], existence and characterization of isoperimetric regions in Euclidean cones by Ritoré and Rosales in [150] and the general proof by Frank Morgan in [131] of existence of isoperimetric regions of any volume in Riemannian manifolds which have compact quotient under the action of the isometry group. The monograph [148] presents a quite complete collection of different existence results.

At the same time, Bruno Franchi, Raul Serapioni and Francesco Serra Cassano started their work on the theory of BV functions depending on vector fields. In [70], they proposed a notion of the space BV depending on a family of vector fields, not necessarily satisfying a ellipticity condition as Hörmander's condition, and prove that any function in this space can be approximated by smooth functions. In [71], they studied the Heisenberg group with the Carnot-Carathéodory metric and the intrinsic perimeter, and dealt with the difficult problem of rectifiability of Caccioppoli sets from an intrinsic perspective. Later on, they continued working in Carnot groups [72], that is, Lie groups with a family of vector fields $X_1, \ldots, X_k$ forming a stratification on the Lie algebra $\mathfrak{g}$, that is, a decomposition as a direct sum of subspaces

$$\mathfrak{g} = V_0 \oplus \ldots \oplus V_r.$$

where $V_0 = \operatorname{span}\{X_1, \ldots, X_k\}$ and $V_{i+1} = [V_0, V_i]$. Carnot groups can be seen as $\mathbb{R}^n$ with a family of polynomial vector fields satisfying a Hörmander condition, those that can be equipped with a family of dilations. An isoperimetric inequality for BV spaces related to a family of vector fields satisfying a Hörmander condition was proven in [82] by Garofalo and Nhieu.

The Heisenberg group $\mathbb{H}^1$ has regained relevance in the context of Riemannian geometry. It is one of the model spaces of the classification up to isometries of 3-dimensional homogeneous manifolds with isometry group of dimension 4, the so called $E(\kappa, \tau)$-spaces. An intense work is been developed on this spaces to clasify constant mean curvature surfaces (CMC) and minimal surfaces since the work of Daniel [43] and Abresch and Rosenberg [1]. For 3-dimensional homogeneous manifolds with 3-dimensional isometry group, Meeks, Mira, Pérez and Ros in [118] proved uniqueness of CMC spheres and studied the values of the curvature for this sphere to exist.

The sub-Riemannian Plateau's problem was first considered by Pauls [138]. Under given Dirichlet conditions on p-convex domains, Cheng, Hwang and Yang [35] proved existence and uniqueness of t-graphs which are Lipschitz continuous weak solutions of the minimal surface equation in $\mathbb{H}^1$. Later on, Pinamonti, Serra Cassano, Treu and Vittone [140] obtained existence and uniqueness of t-graphs on domains with boundary data satisfying a bounded slope condition, thus showing that Lipschitz regularity is optimal at least in the first Heisenberg group $\mathbb{H}^1$. Capogna, Citti and Manfredini [24] established that intrinsic graphs of a Lipschitz continuous function which are viscosity solutions of the sub-Riemannian minimal surface equation in $\mathbb{H}^1$ are of class $C^{1,\alpha}$, with higher regularity in the case of $\mathbb{H}^n$, $n > 1$, see [25]. It was shown in [36] that the regular part of a t-graph of class C^1 with continuous prescribed Sub-Riemannian mean curvature in $\mathbb{H}^1$ is foliated by C^2 characteristic curves. Furthermore, in [79] the authors generalized the previous result when the boundary S is a general C^1 surface in a three-dimensional contact sub-Riemannian manifold. Later on, Galli in [76] improved the result in [79] only assuming that the boundary S is Euclidean Lipschitz and $\mathbb{H}$-regular in the sense of [71]. Recently, in [88] the authors extended the result in [76] to the sub-Finsler Heisenberg group.

Bernstein type problems for surfaces in $\mathbb{H}^1$ have also received a special attention. The nature of the sub-Riemannian Bernstein's problem in the Heisenberg group is completely different from the Euclidean one even for graphs. On the one hand the area functional for t-graphs is convex as in the Euclidean setting. Therefore the critical points of the area are automatically minimizers for the area functional. However, since t-graphs admit singular points where the horizontal gradient vanishing their classification is not an easy task. Thanks to a deep study of the singular set for C^2 surfaces in $\mathbb{H}^1$, Cheng, Hwang, Malchiodi, and Yang [33] showed that minimal t-graphs of class C^2 are congruent to the hyperbolic paraboloid $u(x, y) = xy$ or to Euclidean planes. The same result was also obtained by Ritoré and Rosales in [151]. If we consider the huge class of Euclidean Lipschitz t-graphs, the previous classification does not hold true since there are several examples of area-minimizing surfaces of low regularity, see [146]. The complete classification for C^2 surfaces was established by Hurtado, Ritoré and Rosales in [99] where they showed that a complete, orientable, connected, stable area-stationary surface is congruent to $u(x, y) = xy$ or to a Euclidean plane. As well as in the Euclidean setting the stability condition is crucial in order to discard some minimal surfaces such as helicoids and catenoids.

On the other hand, the situation for intrinsic graphs is completely different since their associated area functional is not convex. Indeed Danielli, Garofalo, Nhieu in [45] discovered that the family of graphs $u_\alpha(x, t) = \frac{\alpha x t}{1+2\alpha x^2}$ for $\alpha > 0$ are area-stationary but *unstable*. In [128], Monti, Serra Cassano and Vittone provided an example of an area-minimizing intrinsic graph of regularity $C^{1/2}(\mathbb{R}^2)$ that is an intrinsic cone. Therefore the Euclidean threshold of dimension $n = 8$ fails in the sub-Riemannian setting. In [14], Barone Adesi, Serra Cassano and Vittone classified complete C^2 area-stationary intrinsic graphs. Later Danielli, Garofalo, Nhieu and Pauls in [48] showed that a C^2 complete stable embedded minimal surface in $\mathbb{H}^1$ with empty characteristic set must be a plane. In [78] Galli and Ritoré proved that a complete, oriented and stable area-stationary C^1 surface without singular points is a vertical plane. Later, Nicolussi Golo and Serra Cassano [135] showed that Euclidean Lipschitz stable area-stationary intrinsic graphs are vertical planes. Recently, Giovannardi and Ritoré [89] showed that in the Heisenberg group $\mathbb{H}^1$ with a sub-Finsler structure, a complete, stable, Euclidean Lipschitz surface without singular points is a vertical plane and Young [167] proved that a ruled area-minimizing entire intrinsic graph in $\mathbb{H}^1$ is a vertical plane by introducing a family of deformations of graphical strips based on variations of a vertical curve. See also [24; 47; 128].

For minimal surfaces in sub-Riemannian geometry, existence was proved by Garofalo and Nhieu [82] while Ritoré [146] gave examples in $\mathbb{H}^1$ of area-minimizers of the perimeter with low regularity. Nevertheless, for sets with boundaries with prescribed curvature and C^1 regularity, Cheng, Hwang and Yang [36] proved that they are indeed C^2. This result was extended by Galli and Ritoré in [79] to contact manifolds, while Galli [76] improved it assuming only Euclidean Lipschitz regularity of the boundary.

Area-stationary surfaces in sub-Riemannian spaces are usually classified assuming a priori some regularity of the surface. In $\mathbb{H}^1$ it was studied by Ritoré and Rosales [151] for C^2 surfaces and completed by Hurtado, Ritoré and Rosales in [99] while, later on, Galli and Ritoré gave the classification assuming C^1 regularity in [78]. More classification of area-stationary stable surfaces in different sub-Riemannian spaces are in [75; 100]. Rosales [153] studied CMC surfaces with empty singular set in sub-Riemannian Sasakian 3-manifolds. Closely related to the study of minimal and stable surfaces is the computation of variational formulas for the area, some of them can be found in [38; 97; 74]. The works [44] and [5] present several results on sub-Riemannian calculus for hypersurfaces in Carnot groups and intrinsic hypersurfaces in Heisenberg groups respectively.

Regarding Pansu's conjeture, the boundaries of the conjectured solutions to the isoperimetric problem in $\mathbb{H}^1$ are of class C^2. While the characterization of isoperimetric regions is still open, there are several partial results assuming that the candidates are in a given family. The monograph [27] provides a quite complete survey of progress on the subject. We remark the works of Monti, where in [125] he solved the conjeture assuming the candidates are radially symmetric, in [126] and together with Rickly, under the condition of convexity, and in [124], where he proved that it is not possible to expect a

Brunn-Minkowski inequality to hold with the exponent of the homogeneous dimension, 4, otherwise geodesic balls would be isoperimetric regions, and it is easy to check that it is false. A major advance in the conjeture was the work by Ritoré and Rosales [151], where they solve the conjeture among sets with C^2 boundary. This proof, supported in the works of Cheng, Hwang, Malchiodi and Yang [34] about the singular set of C^1, would constitute a complete proof if it were not for the difficult problem of regularity in sub-Riemannian geometry. We also remark the proof of Ritoré [147] that constitutes the only known proof of Pansu's conjeture that does not restrict to a particular family of sets, but to be enclosed on a vertical circular cylinder and containing the disk at the base. See also the works [31; 33; 68; 108]. For isoperimetric inequalities, or the equivalent Sobolev inequalities, we remark the articles by Garofalo and Nhieu [82] and Capogna, Danielli and Garofalo [26].

Finally, we mention that sub-Riemannian geometry has been applied in the study of the perceptual completion and formation of subjective surfaces [39; 155].

1.2 Summary and conclusions

The aim of this thesis is to study variational geometric problems in nilpotent Lie groups with a sub-Finsler structure. This setting extends Finsler geometry to the sub-Riemannian nilpotent groups by considering a left-infariant norm in the horizontal distribution. We will give a notion of (X, K)-perimeter related to a sub-Finsler structure and provide examples of area-minimizing surfaces in $\mathbb{H}^1$ with low regularity, construct CMC spheres in $\mathbb{H}^1$ and prove a minimizing property together with regularity of such spheres, and study the existence of solutions to the prescribed curvature equation with Dirichlet conditions.

The structure of the manuscript is the following.

In Chapter 2 we state the main features of Carnot-Carathéodory spaces (CC) and nilpotent groups. A sub-Finsler structure in a Carnot-Carathéodory manifold with a completely non-integrable distribution $\mathcal{H}$ is defined by a smooth norm on $\mathcal{H}$. The case of a Euclidean norm is that of sub-Riemannian geometry. Symmetric sub-Finsler structures in $\mathbb{H}^1$ have received intense interest recently, specially the study of geodesics by Ardentov, Le Donne and Sachkov [8] and by Barilari, Boscain, Le Donne and Sigalotti [12], see [123] for the classical sub-Riemannian case. General asymmetric sub-Finsler structures have an associated asymmetric distance and might have different metric properties, see [119; 120] and [40].

In Section 2.3 we shall define a notion of (X, K)-perimeter as the by means of the variation of its characteristic function, following the procedure established with Ritoré in [142]. The perimeter associated to the Euclidean norm $|\cdot|$ is the sub-Riemannian perimeter as it is defined in [82; 71; 70]. A set has finite perimeter for a given norm if and only if it has finite perimeter for the standard sub-Riemannian perimeter. Hence all known results in the standard case apply to the sub-Finsler perimeter. We prove in 2.3.1 a representation of the (X, K)- perimeter in terms of the sub-Riemannian one as the dual norm of the horizontal unit normal. The notion of sub-Finsler Minkowski content was introduced by Sánchez [154] in his Ph.D. thesis, while the perimeter as the variation of the characteristic function in $\mathbb{H}^1$ was defined by Pozuelo and Ritoré [142] and Franceschi et al. [69].

Different notion of perimeter of a submanifold of fixed degree immersed in a graded manifold, can be found in [38; 86; 115].

In Chapter 3 we shall consider a nilpotent group G with a set of left-invariant vector fields X satisfying a Hörmander condition, and an asymmetric left-invariant norm $|\cdot|_K$ associated to a convex body K, without the assumption of been equipped with a family of dilations. The main result of the chapter is the existence of isoperimetric regions for any given volume. Moreover, any isoperimetric region has a finite number of connected components.

This result is an extension of the existence result of Leonardi and Rigot for nilpotent groups with no dilations and a sub-Finsler norm. The proof follows the arguments in [77].

In sub-Riemannian geometry, apart from the compact case, there are only two known results. Galli and Ritoré proved in [77] an existence result in contact sub-Riemannian manifolds. The argument followed Morgan's structure and can be seen in [148]: they pick a minimizing sequence of sets of volume v whose perimeters approach the infimum of the perimeters of sets of volume v. This sequence can be splitted into two subsequences. The first subsequence is converging to a set, and it is proved that is isoperimetric for its volume and bounded. Nevertheless, it might be a loss of mass at infinity. In this case, they use isometries to translate the second subsequence, which is diverging, to recover some of the lost volume. An essential point is that they always recover a fixed fraction of the volume. In Carnot groups, existence of isoperimetric regions was proven by Leonardi and Rigot in [109]. Dilations in a Carnot group $\mathbb{G}$ plays a key role, since from them it is direct that the isoperimetric profile has the form $I_{\mathbb{G}}(v) = Cv^q$, where C is a positive constant and $q \in (0, 1)$. In particular, the function $I_{\mathbb{G}}$ is concave, a crucial property to deduce that there is no loss of mass at infinity.

At this point, we shall assume that given a finite perimeter $F \subseteq G$ with finite volume with nonempty interior, there exist $C_3 > 0$ and a family of finite perimeter sets $\{F^\lambda\}$ such that $|F^\lambda| \geqslant |F| + \lambda C_3$ and

$$P_K(F^\lambda) - P_K(F) \leqslant C_3\lambda.$$

This result, called deformation lemma, can be seen in a variety of references in literature, some of them in [6; 7; 77; 114; 122; 131; 134]. The proof of this result usually relies on taking a vector field U and use the formulas for the first variation of the volume and the area. In [77], Galli and Ritoré used a calibration argument, exploiting that Pansu spheres in $\mathbb{H}^1$ have constant mean curvature to construct, in a neighborhood of a given point, a horizontal vector field with bounded divergence.

By a calibration argument, we obtain in Proposition 3.2.1 that the isoperimetric profile is non-decreasing. The sub-additiveness is proven in Corollary 3.3.3. We shall also extend the properties obtained in Carnot groups by Leonardi and Rigot in [109], that isoperimetric regions are bounded and its topological boundary and the essential boundary coincide.

In Chapter 4, we show the results obtained in [141]. We have generalized the classical Brunn-Minkowski inequality to be suited to nilpotent groups. The classical Brunn-Minkowski inequality in Euclidean space asserts that, given $A, B \subset \mathbb{R}^d$ measurable sets such that $A + B$ is also measurable, we have

$$|A + B|^{1/d} \geqslant |A|^{1/d} + |B|^{1/d},$$

where $|\cdot|$ indicates the volume of a set, and $A + B = \{a + b : a \in A, b \in B\}$ is the classical Minkowski addition of sets. Taking $\lambda \in [0, 1]$, and replacing A by λA and B by $(1-\lambda)B$, we get the equivalent inequality

$$|\lambda A + (1-\lambda)B|^{1/d} \geqslant \lambda|A|^{1/d} + (1-\lambda)|B|^{1/d}.$$

First connected to the isoperimetric theorem, this inequality is a cornerstone in convex geometry [156; 81]. Through the equivalent functional formulation of the Brunn-Minkowski inequality, the Prékopa-Leindler inequality, we can see some of the implications in the preservation of logarithmic concavity under convolutions noticed by Brascamp and Lieb [21], as well as in the work of Bobkov and Ledoux [17] where it is derived the concentration of measure of Gaussian-like measures, Brascamp-Lieb and logarithmic Sobolev inequalities.

There are several ways of generalizing the Brunn-Minkowski inequality. In Lie groups we can define the Minkowski addition of sets using the group product and take as volume the Haar measure of the group. The Brunn-Minkowski inequality obtained this way is called the *multiplicative* Brunn-Minkowski inequality. In general metric measure spaces the notion of s-intermediate points can be used to replace the convex combination of points in Euclidean space, see [152]. This leads to the *geodesic* Brunn-Minkowski inequality.

A large number of proofs for the Brunn-Minkowski inequality in Euclidean space are known, some of them can be found in [81; 94; 104]. Ritoré and Yepes [152] proved the geodesic Brunn-Minkowski inequality for products of metric measures spaces. For Riemannian manifolds with a lower bound on the Ricci curvature this inequality is proven in [41] employing techniques of optimal transport. This techniques where latter applied to prove this inequality for CD spaces (see [64]). While Juillet [102] proved that no CD condition holds in sub-Riemannian Heisenberg groups $\mathbb{H}^n$, the optimal transport approach was followed by Balogh, Kristály and Sipos [9] and by Barilari and Rizzi [13] to prove geodesic Brunn-Minkowski inequalities in the sub-Riemannian setting (see also [121]).

In 2003, Monti [124] observed that the multiplicative Brunn-Minkowski inequality in $\mathbb{H}^n$ cannot hold with exponent $(2n+2)^{-1}$, corresponding to the homogeneous dimension of $\mathbb{H}^n$, since otherwise Carnot-Carathéodory balls would be isoperimetric sets.

Leonardi and Masnou [108] proved in 2005 that this inequality holds with exponent $(2n+1)^{-1}$, corresponding to the topological dimension of $\mathbb{H}^n$. Their proof was based on Hadwiger-Ohmann's proof of the classical Brunn-Minkowski inequality given in [94].

Later on, Tao [160; 161] posted an entry in his blog in 2011 explaining how to produce a Prékopa-Leindler inequality in any nilpotent Lie group of topological dimension d, which provides a natural way to prove the multiplicative Brunn-Minkowski inequality with exponent d^{-1}.

Juillet [102] gave examples of sets for which the multiplicative Brunn-Minkowski inequality in $\mathbb{H}^n$ does not hold with exponent smaller than $(2n+1)^{-1}$.

In this chapter, we prove a generalization of the Brunn-Minkowski inequality in Euclidean space where the Minkowski addition of sets is replaced by any product $* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ of the form

$$z * w = z + w + (F_1, F_2(z,w), \ldots, F_d(z,w)) = z + w + F(z,w), \qquad (*)$$

where F_1 is a constant and F_i are continuous functions that depend only on $z_1, \ldots, z_{i-1}, w_1, \ldots, w_{i-1}$ $\forall i = 2, \ldots, d$. By a product here we mean a binary operation without assuming any further properties such as associativity. We prove that for any $A, B \subset \mathbb{R}^d$ be measurable sets such that $A * B$ is measurable, we have

$$|A * B|^{1/d} \geqslant |A|^{1/d} + |B|^{1/d}.$$

The product in any nilpotent Lie group is of the form * because of the expression of the group product in exponential coordinates of the first kind.

This result is an extension of the result obtained by Leonardi and Masnou [108] in Heisenberg groups. While the proof of Leonardi and Masnou only works in Heisenberg groups, their arguments can be seen as the first step of an induction argument developed in this chapter. In this chapter, we shall consider a product $*$ of the form (*), that not necessarily comes from a group product, and change $*$ for another one $*_{z_1,w_1}$ of the form (*), depending on the sets A and B, that allows us to compare the volume of the Minkowski addition of sets for the products $*$ and $*_{z_1,w_1}$, as a consequence of Lemma 4.1.1. When the product $*$ comes from a nilpotent group it is not true that $*_{z_1,w_1}$ can define a group product. Then, by an induction argument, we will compare the volume of the Minkowski addition of sets A and B with the volume of the Euclidean Minkowski addition of A and B, and establish in Proposition 4.1.5 a sufficient condition in $\mathbb{H}^1$ for the strict inequality in (4.0.1).

At the end of the chapter, we state several classical variations of inequality (4.0.1) in the case of Carnot groups, where dilations can be defined.

After [141] was completed, the author was informed that Theorem 4.0.1 was also proven by Bobkov [16] in 2011, where he used Knothe's map to get the Brunn-Minkowski inequality for convex sets and obtained the general result after proving the equivalent analytic version of the theorem, the Prékopa-Leindler inequality.

In Chapter 5, we gather the results obtained with Ritoré in [142]. We consider critical points of the perimeter associated to an asymmetric sub-Finsler structure in the first Heisenberg group $\mathbb{H}^1$. Such a structure is defined by means of an asymmetric left-invariant norm $|\cdot|_K$ associated to a convex body $K \subset \mathbb{R}^2$ containing 0 in its interior.

In case the boundary S of E is a C^1 or Euclidean lipschitz surface, the perimeter of E is given by the sub-Finsler area functional

$$A_K(S) = \int_S |N_h|_{K,*} \, dS, \tag{*}$$

where $|\cdot|_{K,*}$ is the dual norm of $|\cdot|_K$, N_h is the orthogonal projection to the horizontal distribution of the Riemannian unit normal N, and dS is the Riemannian measure on S.

If we consider a convex set K with boundary of class C^2_+ (i.e., so that ∂K is of class C^2 and ∂K has positive geodesic curvature everywhere), we may compute the first variation of the area functional associated to a vector field U with compact support in the regular part of S to get

$$A_K'(0) = \int_S u \left(\operatorname{div}_S \eta_K \right) dS.$$

In this formula $u = \langle U, N \rangle$ is the normal component of the variation and $\operatorname{div}_S \eta_K$ is the divergence on S of the vector field $\eta_K = \pi_K(\nu_h)$, where $\nu_h =$

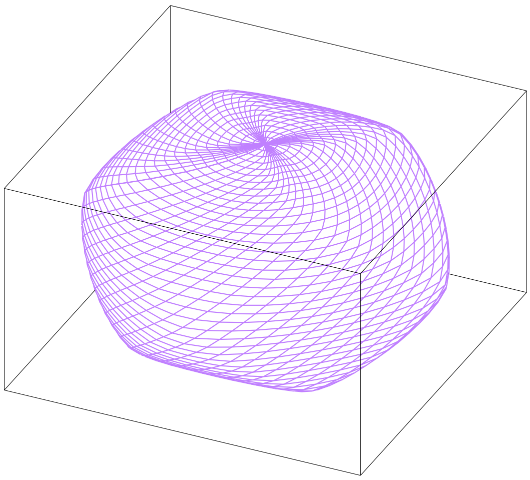

Figure 1.1: The set $\mathbb{B}_K$ when K is the unit ball of the r-norm $||(x,y)||_r = \big(|x|^r + |y|^r\big)^{1/r}$, $r = 1.5$

$N_h/|N_h|$ is the horizontal unit normal and π_K is the map projecting any vector $v \neq 0$ to the intersection of the supporting line in the direction of v with $|\cdot|_K = 1$ (the boundary of K). The strict convexity of $|\cdot|_K$ implies that this map is well-defined.

The function $H_K = \operatorname{div}_S \eta_K$ appearing in the first variation of perimeter is called the *mean curvature* of S. Further calculations imply that H_K is equal to $\langle D_Z \eta_K, Z\rangle$, where $Z = -J(\nu_h)$ is the horizontal direction on the regular part of S. Hence the mean curvature function is localized on the horizontal curves of S. It is not difficult to check that a horizontal curve in a surface with mean curvature H_K must satisfy a differential equation depending on H_K. Hence we can reconstruct the regular part of a surface with prescribed mean curvature by taking solutions of this differential equation. Furthermore, we might be able classify surfaces with prescribed mean curvature by classifying solutions of this ordinary differential equation and by looking at the interaction of these curves with the singular set S_0 of S composed of the points where the tangent plane is horizontal, as was done in [151] for the standard sub-Riemannian perimeter.

Key observations are that horizontal straight lines are solutions of the differential equation for $H_K = 0$ and that horizontal liftings of ∂K are solutions for $H_K = 1$. The strict convexity of K together with the invariance of the equation by left-translations and dilations imply that all solutions are of this type.

Hence, given a convex body $K \subset \mathbb{R}^2$ containing 0 in its interior and its associated left–invariant norm $|\cdot|_K$, we consider the set $\mathbb{B}_K$ obtained as the ball enclosed by the horizontal liftings of all translations of the curve ∂K containing 0. It is not difficult to prove that this way we obtain a topological

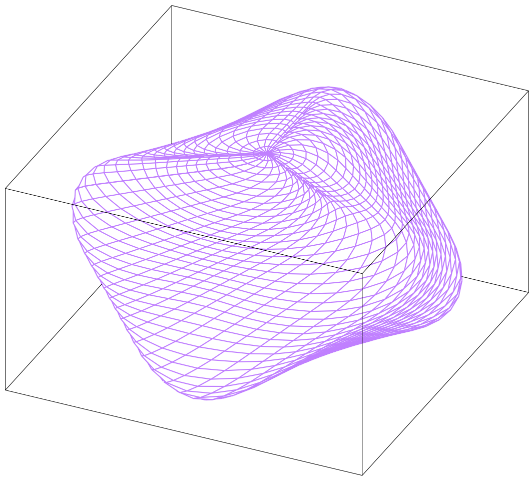

Figure 1.2: The set $\mathbb{B}_K$ when K is a smooth approximation of the triangular norm

sphere $\mathbb{S}_K$ with two poles on the same vertical line, that is the union of two graphs. Moreover the boundary of $\mathbb{B}_K$ is C^2 outside the poles (indeed C^ℓ if the boundary of K is of class C^ℓ, $\ell \geqslant 2$) and of regularity C^2 around the poles. When $K = D$, these sets were build by Pansu [137] and are frequently referred to as Pansu spheres. We remark that Pansu spheres' $\mathbb{B}_D$ are of class C^2 but not C^3 near the singular points, see Proposition 3.15 in [46] and Example 3.3 in [151].

We observe that these objects have constant mean curvature. Hence they are critical points of the sub-Finsler area functional under a volume constraint. Further evidence that they have stronger minimization properties is given in Section 5.5, where it is proven that, under a geometric condition, a set of finite perimeter E with volume equal to the volume of $\mathbb{B}_K$ has perimeter larger than or equal to the one of the ball $\mathbb{B}_K$. A slightly weaker result for the Euclidean norm was proven in [147].

Chapter 6 is devoted to the results obtained with Giovannardi and Ritoré given in [87]. In this note, we provide examples of entire perimeter-minimizing horizontal graphs for a fixed but arbitrary left-invariant sub-Finsler structure in the first Heisenberg group $\mathbb{H}^1$. Our examples are inspired by the corresponding sub-Riemannian ones in [146]. Of particular interest are the conical examples invariant by the non-isotropic dilations of $\mathbb{H}^1$. In the sub-Riemannian case these examples were investigated in [91] and [146]. In Theorem 5.2.1 of Section 6.1 we obtain a necessary and sufficient condition, based on Theorem 3.1 in [142], for a surface to be a critical point of the sub-Finsler area. We assume that the surface is piecewise C^2, composed of pieces meeting in a C^1 way along C^1 curves. This condition will allow us to construct area-minimizing examples in Proposition 6.2.3 of Section 6.2, and examples with low regularity in Proposition 6.2.4. The same construction, keeping fixed the angle at one side of the singular line, provides examples of area-minimizing cones, see Corollary 6.2.5. Finally, in Section 6.3 we provide examples of area-minimizing cones in the spirit of [91]. These examples are obtained in Theorem 6.3.2 from circular sectors of the area-minimizing cones with one singular line obtained in Corollary 6.2.5.

Chapter 7 exposes some of the results obtained during the stay at Università di Trento in colaboration with Giovannardi, Pinamonti and Verzellesi. We study the prescribed mean curvature equation for t-graphs in the Heisenberg group $\mathbb{H}^n$ with an asymmetric left-invariant norm $|\cdot|_{K_0}$ on the horizontal distribution of $\mathbb{H}^n$ associated to a convex body $K_0 \subset \mathbb{R}^{2n}$ containing the origin in its interior. Let $\Omega \subset \mathbb{R}^{2n}$ be a bounded open set, $H \in L^\infty(\Omega)$, $F \in L^1(\Omega, \mathbb{R}^{2n})$ and $u \in W^{1,1}(\Omega)$. We consider the functional

$$\mathcal{I}(u) = \int_\Omega |\nabla u + F|_{K_0,*}\, dxdy + \int_\Omega H u\, dxdy, \tag{i}$$

where $|\cdot|_{K_0,*}$ denotes the dual norm of $|\cdot|_{K_0}$. In particular, when $F(x,y) = (-y, x)$ the first term in (i) coincides with the sub-Finsler area of the t-graph of u. Moreover, if K_0 is the Euclidean unit ball centered at the origin and $H = 0$ then (i) boils down to the classical area functional for t-graphs in Heisenberg group, see [33; 99] and references therein. We say that the graph of u has prescribed K_0-mean curvature H in Ω if u is a minimizer of $\mathcal{I}$. Indeed, the Euler-Lagrange equation associated to $\mathcal{I}$ out of the singular set Ω_0, i.e. the set of points where $\nabla u + F$ vanishes, is given by

$$\operatorname{div}(\pi_{K_0}(\nabla u + F)) = H, \tag{ii}$$

where π_{K_0} is a suitable 0-homogeneous function defined in (2.1.5). When we fix a boundary datum $\varphi \in W^{1,1}(\Omega)$, a solution to the *Dirichlet problem* for the prescribed K_0-mean curvature equation is a minimizer u of $\mathcal{I}$ such that $u - \varphi$ belongs to the Sobolev space $W^{1,1}_0(\Omega)$. Our main result is Theorem 7.4.1, where we prove, under suitable regularity assumptions on the data, that there exists a Lipschitz solution to the Dirichlet problem for the prescribed K_0-mean curvature equation when H is *constant* and satisfies

$$|H| < H_{K_0,\partial\Omega}(z_0) \tag{iii}$$

for each $z_0 = (x_0, y_0) \in \partial\Omega$, where $H_{K_0,\partial\Omega}$ denotes the Finsler mean curvature of the boundary $\partial\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$. Notice that the mean curvature of the graph of u is computed with respect to the downward pointing unit normal and the Finsler mean curvature of $\partial\Omega$ is computed with respect to the inner unit normal. The upper bound (iii) of H in terms of the Finsler mean curvature of the boundary is the Finsler analogous of the standard assumption for the solution to the Dirichlet problem for the classical mean curvature equation in the Euclidean setting as stated in [158], [84] or [83] (see also [85, Theorem 16.11]). The Dirichlet problem for constant mean curvature in the first Riemannian Heisenberg group has been studied in [2] under the same condition on the mean curvature. It is worth mentioning that this is the first time that the existence of solutions to the sub-Finsler Dirichlet problem has been studied when $H \neq 0$, even in the particular case in which K_0 is the unit disk centered at 0, where the sub-Finsler and the sub-Riemannian frameworks coincide. Indeed, as far as we know, the sub-Riemannian Dirichlet problem has been studied in [138; 35; 33; 32; 58; 139] only in the case of minimal surface under the bounded slope condition or the p-convexity assumption on Ω. In particular, we point out that when $n = 1$ our assumption (iii) implies that $\Omega \subset \mathbb{R}^2$ is strictly convex, see Remark 7.3.7. It is easy to check that our sub-Finsler functional $\mathcal{I}$ for $H = 0$ satisfies the hypothesis of the area functional considered in [58]. Thus, assuming the bounded slope condition we directly obtain the existence of Euclidean Lipschitz minimizer for Plateau's problem. The approach of the present chapter, based on the Schauder fixed-point theory, follows the scheme developed in [35] and extends its results both to the case of prescribed constant mean curvature $H \neq 0$ and to the sub-Finsler setting. In Theorem 7.4.1 we can not expect better regularity than Lipschitz.

Since equation (ii) is sub-elliptic degenerate and it is singular next the singular set, inspired by [35; 138], we introduce a family of elliptic approximating equations given by

$$\operatorname{div}\left(\pi_{K_0}(\nabla u + F)\frac{|\nabla u+F|_*^2}{(\varepsilon^3+|\nabla u+F|_*^3)^{\frac{2}{3}}}\right) = H \tag{iv}$$

for each $0 < \varepsilon < 1$. A similar approximation scheme was considered in the sub-Riemannian setting by [25; 24] to study the Lipschitz regularity for non-characteristic minimal surfaces. To obtain this family of equations we consider a $2n+1$ dimensional convex body K_ε containing the origin in its interior, that converges in the Hausdorff sense to the $2n$ dimensional convex body K_0 as ε tends to 0. The choice of the convex body K_ε is not arbitrary. Indeed, we need a specific shape in order to obtain an approximating equation well defined in the classical sense in the singular set. It is interesting to point out that the Riemannian approximation of [35; 138; 25; 24] produces an approximation of the unit disk $D \subseteq \mathbb{R}^{2n}$ by ellipsoids in the sub-Riemannian setting, and this approximation does not work in the sub-Finsler context. For $0 < \varepsilon < 1$, the convex body K_ε defines a Finsler norm on $T\mathbb{H}^n$ whose associated Finsler area functional is given by

$$\mathcal{I}_\varepsilon(u) = \int_\Omega \left(\varepsilon^3 + |\nabla u + F|_{K_0,*}^3\right)^{\frac{1}{3}}\, dxdy + \int_\Omega Hu\, dxdy.$$

It is easy to see that the Euler-Lagrange equation associated to this functional is elliptic and avoids singularities. Given a boundary datum $\varphi \in C^{2,\alpha}(\bar{\Omega})$, the solvability of the Dirichlet problem associated to (iv) is reduced by [85, Theorem 13.8] to *a priori* estimates in $C^1(\overline{\Omega})$ of a related family of problems. As usual the *a priori* estimates in $C^1(\overline{\Omega})$ consist of three parts: estimates of the supremum of $|u|$, boundary estimates of the gradient of u and interior estimates of the gradient of u. Both the estimates of the supremum and the boundary estimates of the gradient are obtained by a barriers argument that depends on the Finsler distance from the boundary $\partial\Omega$. Due to technical reasons in the construction of the barriers we need to assume the strict inequality in (iii), avoiding the optimal case when H coincides with $H_{K_0,\partial\Omega}(z_0)$ at a given point $z_0 \in \partial\Omega$. We emphasize that these results hold even if the prescribed curvature H is non-constant and Lipschitz. When H is non-constant, in order to obtain the estimates of the supremum we assume that there exists $\delta \in (0,1]$ such that

$$\left| \int_\Omega H v \, dxdy \right| \leqslant (1-\delta) \int_\Omega |\nabla v|_{K_0,*} \, dxdy \tag{v}$$

for each $v \in C_c^\infty(\Omega)$. Assumption (v) is a standard sufficient condition for the estimates of the supremum of $|u|$ (see [83] or [85]). Notice that (v) with $\delta = 0$ is a necessary condition obtained integrating by parts the elliptic equation (iv). In the Euclidean space, Giusti [90] proved that (v) with $\delta = 0$ is also a sufficient condition for the existence of solutions to the prescribed mean curvature equation. This result was later generalized to weakly regular domains in [111]. Moreover, in analogy with the Euclidean case, we show that the sufficient condition (v) with $\delta > 0$, which is *a priori* stronger than the necessary condition with $\delta = 0$, is redundant when H is constant. The only crucial step where we need that H is constant is the maximum principle for the gradient of the solution that allows us to reduce the interior estimates of the gradient to its boundary estimates. Finally, once we realize that C^1 estimates are independent of the approximation parameter ε, passing to the limit as ε tends to 0 and using Arzelà-Ascoli Theorem we get the existence of a Lispchitz minimizer for the sub-Finsler Dirichlet problem.

The analysis of the Dirichlet problem with $H \neq 0$ constant for t-graphs is essential since it is strictly related to the isoperimetric problem in $\mathbb{H}^n$. Recently, similar results concerning CMC graphs and surfaces in the Euclidean setting with an anisotropic norm have been obtained by [55; 54].

Appendix A contains an alternative proof of the representation of the perimeter given in Theorem 2.3.3.

Chapter 2

Nilpotent groups and sub-Finsler perimeter

The aim of this chapter is twofold. The first one is to introduce some background and notation that will be used throughout the thesis. The second is to give different notions of the (X, K)-perimeter as introduced in [142] and [69].

In Section 2.1 it will be introduced the Carnot-Carathéodory spaces (CC) and, with the notion of Minkowski norms of Subsection 2.1.1, the sub-Finsler Carnot-Carathéodory spaces in Subsection 2.1.2. Relevant cases of sub-Finsler CC spaces are sub-Finsler nilpotent groups, where X is taken left-invariant and left-translations are isometries that preserves the perimeter. This groups are introduced in Section 2.2, while in Subsection 2.2.1 we study sub-Finsler nilpotent groups with a suitable family of dilations.

In the second part of the chapter we introduce in Section 2.3 the notion of (X, K)-variation of a function and proof the Divergence's theorem 2.3.2. In Subsection 2.3.1 we give a proof of the representation of the (X, K)-perimeter given in Theorem 2.3.3.

2.1 Carnot-Carathéodory spaces

Given a family of smooth vector fields $X = \{X_1, \ldots, X_k\}$ in $\mathbb{R}^d$, we consider the distribution generated by X, called the horizontal distribution $\mathcal{H}$, defined in a point p of $\mathbb{R}^d$ as

$$\mathcal{H}_p = \operatorname{span}\{X_1(p), \ldots, X_k(p)\}.$$

A smooth curve $\gamma : [0, T] \to \mathbb{R}^d$ is called *horizontal* if $\dot\gamma(t) \in \mathcal{H}_{\gamma(t)}$ for all t. In that case, we can write

$$\dot\gamma(t) = \sum_{i=1}^{k} \dot\gamma_i(t) X_i.$$

Given p and q in $\mathbb{R}^d$, we consider the quantity

$$d(p,q) = \inf\left\{ \int_0^T |\dot\gamma(t)|dt : \gamma \text{ is horizontal}, \ \gamma(0) = p, \ \gamma(T) = q \right\}.$$

If there is no horizontal curve joining p and q, we define $d(p,q) = +\infty$.

In case that d is finite for every p and q in $\mathbb{R}^d$, it defines a distance [93; 123] called the *Carnot-Carathéodory* distance. A standard condition on X that guarantees that d_X is finite is *Hörmander's condition*, also called the *bracket generating condition* or *Chow's condition* [37; 143]. We denote by $[U,V]$ the Lie bracket of two C^1 vector fields U, V on $\mathbb{H}^n$. We consider

$$\begin{aligned} &\mathcal{H}^1(p) := \mathcal{H}(p) + [\mathcal{H}(p), \mathcal{H}(p)], \\ &\mathcal{H}^2(p) := \mathcal{H}^1(p) + [\mathcal{H}^1(p), \mathcal{H}(p)], \\ &\ldots, \end{aligned} \tag{2.1.1}$$

then X satisfies Hörmander's condition if there exists $r(p) > 0$ such that $\mathcal{H}_p^{r-1} \neq T_p\mathbb{R}^d$ and $\mathcal{H}_p^r = T_p\mathbb{R}^d$.

2.1.1 Minkowski norms

We say that $|\cdot| : \mathbb{R}^d \to \mathbb{R}^+$ is a norm if verifies

1. $|x| = 0 \Leftrightarrow x = 0$.
2. $|sx| = s|x| \ \forall s > 0$ and $\forall x \in \mathbb{R}^d$.
3. $|x+y| \leqslant |x| + |y| \ \forall x, y \in \mathbb{R}^d$.

We stress the fact that we are not assuming the symmetry property $|-v| = |v|$.

The unit ball $K = \{x : |x| \leqslant 1\}$ is a convex, compact set such that $0 \in \text{int}(K)$. Reciprocally, given a convex compact set K such that $0 \in K$, we can define the Minkowski norm associated to K as

$$|x|_K := \min\{t : x \in tK\}.$$

It is immediate that $|\cdot|_K$ verifies (1) and (2). Let us show that (3) is also verified. Given x and y in $\mathbb{R}^d$ we denote $t_1 = |x|_K$ and $t_2 = |y|_K$. Let us see that $\frac{x+y}{t_1+t_2}$ is in K, and hence $|x+y|_K \leqslant t_1 + t_2$ which is equivalent to (3).

$$\frac{x}{t_1+t_2} + \frac{y}{t_1+t_2} = \frac{t_1}{t_1+t_2}\frac{x}{t_1} + \frac{t_2}{t_1+t_2}\frac{y}{t_2}.$$

Since $\frac{x}{t_1}$ and $\frac{y}{t_2}$ are in K and $\frac{t_1}{t_1+t_2} + \frac{t_2}{t_2+t_1} = 1$, we have that $\frac{x+y}{t_1+t_2}$ is a convex combination of elements of K and thus $\frac{x+y}{t_1+t_2}$ is in K.

We write $|\cdot|_K$ to indicate the dependence of the norm on K. The case of a symmetric norm corresponds to a centrally symmetric convex body. The norm associated to the closed unit disc D centered at 0 coincides with the Euclidean norm and is denoted by $|\cdot|$.

We say that a set K is a convex body if it is convex, compact and has non-empty interior. We say that a convex body K is in $C^{k,\alpha}_+$, for $k \in \mathbb{N}$ and $\alpha \in [0,1]$, if ∂K is of class $C^{k,\alpha}$ with strictly positive principal curvatures.

It is well known that any norm is equivalent to the Euclidean norm $|\cdot|$, that is, given a norm $|\cdot|_K$ in $\mathbb{R}^d$ there exist constants $0 < c < C$ such that

$$c\,|\cdot| \leqslant |\cdot|_K \leqslant C\,|\cdot|\,. \tag{2.1.2}$$

Given a norm $|\cdot|_K$ and a scalar product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^d$, we consider the dual norm $|\cdot|_{K,*}$ of $|\cdot|_K$ with respect to $\langle\cdot,\cdot\rangle$, defined by

$$|u|_{K,*} = \sup_{v\in K}\langle u, v\rangle. \tag{2.1.3}$$

The dual norm is the support function of the unit ball F with respect to the scalar product $\langle\cdot,\cdot\rangle$. Moreover, thanks to the above definitions the following Cauchy-Schwarz formula holds:

$$\langle u, v\rangle \leqslant |u|_{K,*}|v|_K \tag{2.1.4}$$

for any $u, v \in \mathbb{R}^d$. If in addition we assume K to be strictly convex and $u \neq 0$, then the compactness and strict convexity of K guarantee the existence of a unique vector $\pi_K(u)$ in ∂K where the supremum in (2.1.3) is attained, i.e.

$$|u|_{K,*} = \langle u, \pi_K(u)\rangle. \tag{2.1.5}$$

It is easy to see that π_K is a positively 0-homogeneous map, i.e. $\pi_K(\lambda u) = \pi_K(u)$ for any $\lambda > 0$ and $u \in \mathbb{R}^d \setminus \{0\}$, and that $|\pi_K(u)|_K = 1$ for any $u \in \mathbb{R}^d \setminus \{0\}$. Moreover, if we assume that K is C^2_+, then $\pi_K|_{\mathbb{S}^{d-1}} : \mathbb{S}^{d-1} \to \partial K$ is a C^1 diffeomorphism whose inverse is the Gauss map $\mathcal{N}_K$ of ∂K with respect to the outer unit normal. Furthermore, we have that the norms $|\cdot|_K$ and $|\cdot|_{K,*}$ belongs to $C^{k,\alpha}(\mathbb{R}^d \setminus \{0\})$ if and only if ∂K is $C^{k,\alpha}$ for $k \in \mathbb{N}$ and $0 \leqslant \alpha \leqslant 1$. For further details see [156, Section 2.5]. The relation between the dual norm and the map π_K is given by

$$\nabla|u|_{K,*} = \pi_K(u). \tag{2.1.6}$$

Indeed, for any $u \in \mathbb{R}^d \setminus \{0\}$

$$\nabla|u|_{K,*} = \nabla\langle u, \pi_K(u)\rangle = \pi_k(u) + u \cdot D\pi_K(u) = \pi_K(u),$$

where the last equality follows from the fact that 0-homogeneous functions are radial.

2.1.2 Sub-Finsler Carnot-Carathéodory spaces

Let X be a family of k linearly-independent vector fields in $\mathbb{R}^d$ and K a convex body in $\mathbb{R}^k$ with $0 \in \text{int}(K)$.

We extend the Minkowski norm $|\cdot|_K$ to each fiber of $\mathcal{H}_0$ as

$$\left(\Big|\sum_{i=1}^{k} f_i(p) X_i(p)\Big|_K\right)_p = \big|\big(f_1(p), \ldots, f_k(p)\big)\big|_K.$$

Similarly, we extend $|\cdot|_{K,*}$ and π_K to $\mathcal{H}_0$. When $|\cdot|_{K_0}$ is C^l with $l \geqslant 2$, all norms $(|\cdot|_{K_0})_p$ are C^l.

We define the sub-Finsler length of a piecewise smooth horizontal curve $\gamma : I \to \mathbb{R}^d$ by the formula

$$L(\gamma) = \int_I |\gamma'(s)|_K \, ds.$$

Under the assumption that any two given points $p, q \in \mathbb{R}^d$ can be joined by a horizontal curve, we can define the (asymmetric) distance $d_K(p,q)$ between p and q as the infimum of the length of horizontal piecewise C^1 curves joining p and q. This distance satisfies the properties

1. $d_K(p,q) \leqslant d_K(p,r) + d_K(r,q)$, for all $p, q, r \in \mathbb{H}^1$, and
2. $d_K(p,q) = 0$ if and only if $p = q$.

The second property follows from comparison with the standard Carnot-Carathéodory distance. This asymmetric distance does not satisfy the property $d_K(p,q) = d_K(q,p)$. We refer the reader to [119; 120] for properties of asymmetric distances and their geodesics.

2.2 Nilpotent groups

We recall some results on nilpotent groups. For a quite complete description of nilpotent Lie groups the reader is referred to Section 1.13 in [103].

The exponential map in a Lie group G of a left-invariant vector field X will be denoted by $\exp(X)$, writing $\exp_G$ if specifying the group is needed. Unless otherwise specified, we shall write $\cdot$ for the group product of G. The left-translation in G by q is the map $\ell_q(p) = q \cdot p$.

Let $\mathfrak{g}$ be a Lie algebra. We define recursively $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i] = \text{span}\{[X,Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}_i\}$. The decreasing series

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots$$

is called the *lower central series of* $\mathfrak{g}$. If $\mathfrak{g}_r = 0$ and $\mathfrak{g}_{r-1} \neq 0$ for some r, we say that $\mathfrak{g}$ is *nilpotent* of step r. A connected Lie group is said to be *nilpotent* if its Lie algebra is nilpotent.

The following Lemma will be used in the sequel.

Lemma 2.2.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then there exists a basis $\{Y_1, \dots, Y_d\}$ of $\mathfrak{g}$ such that*

1. *for each $1 \leqslant n \leqslant d$, $\mathfrak{h}_n = span\{Y_{d-n+1}, \dots, Y_d\}$ is an ideal of $\mathfrak{g}$,*
2. *for each $0 \leqslant i \leqslant r-1$, $\mathfrak{h}_{n_i} = \mathfrak{g}_i$.*

A basis verifying this is called a *Malcev basis.* This construction is adapted from § 1.2 in [42]. Fixed a Malcev basis, the exponential map centered at 0 provides a diffeomorphism between $\mathbb{R}^d$ and G, given by the map

$$x = (x_1, \dots, x_d) \mapsto \exp(x_1 Y_1 + ... + x_d Y_d).$$

This result can be found as Theorem 1.127 in [103]. The inverse of this map provides coordinates called *canonical coordinates of the first kind.* The group product can be recovered by the Hausdorff-Campbell-Baker formula as

$$(x_1, \dots, x_d) \cdot (y_1, \dots, y_d) = \exp^{-1}(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \dots),$$

where $X = \sum x_i Y_i$, $Y = \sum y_i Y_i$, $(x_1, \dots, x_d) = \exp(X)$ and $(y_1, \dots, y_d) = \exp(Y)$.

The structure of this product is given by the following theorem. It was first proved by Malcev in 1949 [116], and a proof can be found as Theorem 4.1 in [165], or with some modification as Proposition 1.2.7 in [42].

Theorem 2.2.2. *Let G be a simply connected nilpotent group. Then the multiplication map takes the following form:*

$$z * w = z + w + (P_1(z, w), \dots, P_d(z, w)), \tag{2.2.1}$$

where $z = (z_1, \dots, z_d)$, $w = (w_1, \dots, w_d)$, P_1 is a constant and P_i is a polynomial in the variables $z_1, \dots, z_{i-1}, w_1, \dots, w_{i-1}$ $\forall\ i = d - n_1 + 1, \dots, d$.

The next result we show that, slightly refining Theorem 2.2.2, the multiplication map acts as a sum in the coordinates corresponding to the complement of $\mathfrak{g}_1$. This arguments can be seen also in [107], Proposition 6.0.16.

Theorem 2.2.3. *Let G be a simply connected nilpotent group and let $n_1 = \dim(\mathfrak{g}_1)$. Then the multiplication map takes the following form:*

$$z * w = z + w + (0, \dots, 0, P_{d-n_1+1}(z, w), \dots, P_d(z, w))$$

where $z = (z_1, \dots, z_d)$, $w = (w_1, \dots, w_d)$ and P_i is a polynomial in the variables $z_1, \dots, z_{i-1}, w_1, \dots, w_{i-1}$ $\forall\ i = d - n_1 + 1, \dots, d$.

Proof. Let $Z = \sum_{i=1}^d z_i X_i$, $W = \sum_{i=1}^d w_i X_i$. Since $\mathfrak{g}_1$ is an ideal in $\mathfrak{g}$, there is a normal Lie subgroup $G_1 \subseteq G$ whose Lie algebra is $\mathfrak{g}_1$. Let $q : G \to G/G_1$ denote the projection over the quotient, $\tilde{z} = q(z)$, $\tilde{w} = q(w)$,

$\tilde{Z} = (dq)_0(Z)$, $\tilde{W} = (dq)_0(W)$. Notice that $\ker(dq)_0 = \mathfrak{g}_1$ and $\mathfrak{g}/\mathfrak{g}_1$ is a trivial Lie algebra with the induced product. Therefore, by the Baker-Campbell-Hausdorff formula,

$$\tilde{z} * \tilde{w} = \tilde{z} + \tilde{w}. \tag{2.2.2}$$

On the other hand, by Theorem 2.2.2 it holds that

$$\begin{aligned} &\exp_{G/G_1}(\tilde{Z}) \exp_{G/G_1}(\tilde{W}) = q(\exp_G(Z) \exp_G(W)) = \\ &q\big(\exp_G\big(Z + W + \sum_{i=1}^{d} P_i(z,w)X_i\big)\big) = \exp_{G/G_1}\big(\tilde{Z} + \tilde{W} + \sum_{i=1}^{d-n_1} P_i(z,w)X_i\big). \end{aligned} \tag{2.2.3}$$

Taking $\log_{G/G_1}$ in (2.2.3), we obtain

$$\tilde{z} * \tilde{w} = \tilde{z} + \tilde{w} + (P_1(z,w), \ldots, P_{d-n_1}(z,w), 0, \ldots, 0). \tag{2.2.4}$$

From (2.2.2) and (2.2.4), we obtain that $P_i = 0$ $\forall i = 1, \ldots, d - n_1$. □

From Theorem 2.2.3 it can be proved that right translations are maps whose Jacobian determinant is equal to 1 at any point, and the change of variables gives us the following theorem. The interested reader can find the details as Theorem 1.2.9 and Theorem 1.2.10 in [42].

Proposition 2.2.4. *Let G be a simply connected nilpotent group. Then, after having chosen a strong Malcev basis on $\mathfrak{g}$, the exponential takes the Lebesgue measure on $\mathbb{R}^d$ to a Haar measure μ on G, that is, for any $A \subset G$ measurable and any $f : G \to \mathbb{R}$ integrable, one has*

$$\mu(A) = |\log(A)| \quad \text{and} \quad \int_G f d\mu = \int_{\mathbb{R}^d} (f \circ \exp)(x) dx.$$

The Lebesgue measure of $\mathbb{R}^d$ will be denoted as $|\cdot|$ and, as we can see Theorem 1.2.10 in [42], it coincides with the Haar measure on $\mathbb{R}^d$ with this product.

From now on, we shall denote a simply connected nilpotent group as $(\mathbb{R}^d, \cdot)$.

The *dimension at infinity* D of a simply connected nilpotent group $(\mathbb{R}^d, \cdot)$ is defined by

$$D = \sum i m_i,$$

where $m_i := \dim(\mathfrak{g}_{i-1}) - \dim(\mathfrak{g}_i)$.

Given a simply connected nilpotent group $(\mathbb{R}^d, \cdot)$ and a system of linearly-independent left-invariant vector fields $X = \{X_1, \ldots, X_k\}$, we define the distributions

$$\mathcal{H} := \operatorname{span}(X) \qquad \mathcal{H}^n := \mathcal{H}^{n-1} + [\mathcal{H}^{n-1}, \mathcal{H}^0]$$

where $\mathcal{H}^0 = \mathcal{H}$. A vector field U is said to be *horizontal* if $U(x) \in \mathcal{H}_x$ for all x in $\mathbb{R}^d$. The *local dimension* of $\mathbb{R}^d$ and X, denoted by l, is defined as

$$l = \sum i m_i',$$

where $m_i' := \dim(\mathcal{H}^{i-1}) - \dim(\mathcal{H}^{i-2})$ and $m_1' := \dim(\mathcal{H})$.

We consider a left-invariant Riemannian metric $g := \langle \cdot, \cdot \rangle$ forming X an orthonormal basis of $\mathcal{H}$, and making orthogonal the subbundles $\mathcal{H}$ and V, where V is a complementary subbundle of $\mathcal{H}$.

The following result can be seen in Section IV.5 of [164].

Theorem 2.2.5 (Proposition IV.5.6 and Proposition IV.5.7 in [164])**.** *Let $(\mathbb{R}^d, \cdot)$ be a simply connected nilpotent group with X a bracket-generating system and D and l the dimension at infinity and the local dimension respectively. There exist positive constants α and β such that*

$$\begin{aligned} \alpha^{-1} t^l \leqslant &|B(0,t)| \leqslant \alpha t^l \quad 0 \leqslant t \leqslant 1, \\ \beta^{-1} t^D \leqslant &|B(0,t)| \leqslant \beta t^D \quad t > 1. \end{aligned} \tag{2.2.5}$$

In particular, we obtain the following inequality.

$$\mu(B(x,s)) \leqslant C\Big(\frac{s}{r}\Big)^l \mu(B(x,r)), \tag{2.2.6}$$

where $x \in \mathbb{R}^d$, $0 < r \leqslant s \leqslant 1$.

We shall denote by $(\mathbb{R}^d, \cdot, X, K)$ a simply connected sub-Finsler nilpotent group, where $\cdot$ is the group product, X is a family of k linearly-independent vector fields and $K \subset \mathbb{R}^k$ is a convex body with $0 \in \operatorname{int}(K)$.

2.2.1 Sub-Finsler Carnot groups

We refer the reader to [106] for the details on the rest of this section.

A *stratification* of a Lie algebra $\mathfrak{g}$ is a direct-sum decomposition

$$\mathfrak{g} = V_0 \oplus \ldots \oplus V_r,$$

for some integer $r \geqslant 1$, where $V_r \neq \{0\}$, $[V_0, V_i] = V_{i+1}$ for all $i \in \{1, \ldots, r\}$ and $V_{r+1} = \{0\}$. We say that a Lie algebra is *stratifiable* if there exists a stratification on it. We say that a Lie algebra is *stratified* when it is stratifiable and endowed with a fixed stratification. We say that a Lie group is *stratifiable* if it is connected and simply connected and its Lie algebra is stratifiable.

The following lemma assures that any stratifiable group is a nilpotent group.

Lemma 2.2.6. *Let* $\mathfrak{g} = V_0 \oplus \ldots \oplus V_r$ *be a stratified Lie algebra. Then*

$$\mathfrak{g}_{k-1} = V_k \oplus \ldots \oplus V_r.$$

In particular, $\mathfrak{g}$ *is a simply connected nilpotent Lie algebra of step* r*, and* $\mathfrak{g} = V_0 \oplus \mathfrak{g}_1$.

It is worth noting that Theorem 2.2.3 manifests that the multiplication map acts as a sum in the coordinates corresponding to V_0.

Remark 2.2.7. In a stratifiable group, not every subspace V_0 such that $\mathfrak{g} = V_0 \oplus \mathfrak{g}_1$ generates a stratification. Moreover, not every nilpotent group is stratifiable (see Example 1.8 and 1.9 [106]).

Proposition 2.2.8. *Let* $\mathfrak{g}$ *be a stratifiable Lie algebra with stratifications*

$$\mathfrak{g} = V_0 \oplus \ldots \oplus V_r = W_0 \oplus \ldots \oplus W_s.$$

Then $r = s$ *and there exists a Lie algebra automorphism* $A : \mathfrak{g} \to \mathfrak{g}$ *such that* $A(V_i) = W_i$ *for* $i = 1, \ldots, r$.

Proposition 2.2.8 guarantees that for a stratifiable group $(\mathbb{R}^d, \cdot)$, the natural number

$$Q = \sum_{i=1}^{r} i \dim(V_i),$$

does not depends on the particular stratification. Q is called the *homogeneous dimension* of $(\mathbb{R}^d, \cdot)$.

Remark 2.2.9. Given a simply connected nilpotent group $(\mathbb{R}^d, \cdot)$ with a system of left-invariant vector fields X, dimension at infinity D and local dimension l, it is clear that $D \geqslant l$. Moreover, $D = l$ if and only if $(\mathbb{R}^d, \cdot)$ is stratifiable and X generates a stratification on $\mathfrak{g}$. Moreover, $D = l = Q$.

For $\lambda > 0$ we define the *dilation on* $\mathfrak{g}$ *of factor* λ as the unique linear map $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ such that

$$\delta_\lambda(X) = \lambda^t X \quad \forall X \in V_t \ \forall t \in \{1, \ldots, r\}.$$

Remark 2.2.10. Dilations $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ are Lie algebra isomorphisms.

The fact that $(\mathbb{R}^d, \cdot)$ is simply connected certifies that there exists a unique Lie groups automorphism $\delta_\lambda : \mathbb{R}^d \to \mathbb{R}^d$ (denoted as the dilation on the Lie algebra) whose differential at 0 is the dilation on $\mathfrak{g}$ of factor λ. This automorphism is called *dilation on* $(\mathbb{R}^d, \cdot)$ *of factor* λ.

Proposition 2.2.11. *Let* $(\mathbb{R}^d, \cdot)$ *be a stratified group and let* $\lambda > 0$*. Then*

$$\int_{\mathbb{R}^d} f dx = \lambda^Q \int_{\mathbb{R}^d} (f \circ \delta_\lambda) dx,$$

where Q *is the homogeneous dimension of* $(\mathbb{R}^d, \cdot)$.

Let $(\mathbb{R}^d, \cdot)$ be a stratified group with stratification

$$\mathfrak{g} = V_0 \oplus V_2 \oplus \ldots \oplus V_r,$$

We take the Malcev basis $\{Y_1, \ldots, Y_d\}$ given by Lemma 2.2.1. In particular,

$$\text{span}\{Y_1 \ldots, Y_k\} = V_0,$$

where $k = d - \dim(\mathfrak{g}_1)$.

We can extend $Y_1, \ldots, Y_k$ to a family of linearly-independent left-invariant vector fields, also denoted by $Y_1, \ldots, Y_k$. A stratified group $(\mathbb{R}^d, \cdot)$ with $X = \{Y_1, \ldots, Y_k\}$ and a convex body K in $\mathbb{R}^k$ with $0 \in \text{int}(K)$ is called *sub-Finsler Carnot group*, and is denoted by $(\mathbb{R}^d, \cdot, K)$. The sub-Finsler CC distance is denoted by d_K.

It is easy to check that d_K is homogeneous with respect to δ_λ, that is,

$$d_K(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d_K(p, q) \qquad \forall \lambda > 0 \;\; \forall p, q \in \mathbb{R}^d. \tag{2.2.7}$$

Moreover, the sub-Finsler CC distance of two given points p and p' is preserved by left translations. That is, for any q in $\mathbb{R}^d$, we have

$$d_K(\ell_q(p), \ell_q(p')) = d_K(p, p'),$$

where $\ell_q(p) = q \cdot p$.

Given $0 < L \leqslant 1$, we denote g_L the Riemannian metric making orthonormal the basis

$$\tilde{Y}_i = L^{\frac{j}{2}} Y_i,$$

where $Y_i \in V_j$. When L is 1 we shall omit the subindex L.

Thanks to Theorem 2.2.3, the projection over the first k coordinates q_1 of the product is given by

$$q_1\Big(p \exp(t \sum_{i=1}^{k} v_i Y_i)\Big) = p_1 + tv,$$

where $v = (v_1, \ldots, v_k) \in \mathbb{R}^k$, $p = (p_1, p_2) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ and $t \in \mathbb{R}$. In particular, it is easy to check that the projection of $(d\ell_p)_0(v)$ over $V = \text{span}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\}$, q_V, is given by

$$q_V\Big((d\ell_p)_0(\sum_{i=1}^{k} v_i Y_i)\Big) = \sum_{i=1}^{k} v_i \frac{\partial}{\partial x_i}.$$

In particular, we get

$$Y_i = \frac{\partial}{\partial x_i} + \sum_{j=k+1}^{d} f_{i,j} \frac{\partial}{\partial x_j}$$

for some smooth functions $f_{i,j}$.

2.2.2 The Heisenberg group $\mathbb{H}^n$

In this subsection we follow the notation and background given in [151]. We define the product $*$ in $\mathbb{R}^{2n+1} \equiv \mathbb{C}^n \times \mathbb{R}$ given by

$$(z,t)\cdot(z',t') := (z+z', t+t' + \sum_{i=1}^{n} \mathrm{Im}(z_i \overline{z_i}')),$$

where $z = (z_1, \ldots, z_n)$ and $z' = (z'_1, \ldots, z'_n)$. The Lie group $(\mathbb{R}^{2n+1}, \cdot)$ is referred to as the n *Heisenberg group* and denoted by $\mathbb{H}^n$. For $p \in \mathbb{H}^n$, the *left translation* by p is the diffeomorphism $\ell_p(q) = p \cdot q$. A basis of left-invariant vector fields is given by

$$X_i = \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial t}, \qquad Y_i = \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t},$$

where $i = 1 \ldots, n$ and $z_i = (x_i, y_i) \in \mathbb{C}$. The *horizontal distribution* $\mathcal{H}$ in $\mathbb{H}^n$ is the smooth planar distribution generated by $X_1, Y_1, \ldots, X_n, Y_n$. The *horizontal projection* of a vector U onto $\mathcal{H}$ will be denoted by U_h. A vector field U is called *horizontal* if $U = U_h$. A *horizontal curve* is a C^1 curve whose tangent vector lies in the horizontal distribution.

Note that $[X_i, T] = [Y_i, T] = 0$, while $[X_i, Y_i] = -2T$. The last equality implies that $\mathcal{H}$ is a bracket generating distribution. Moreover, by Frobenius Theorem we have that $\mathcal{H}$ is nonintegrable. The horizontal distribution $\mathcal{H}$ is the kernel of the (contact) 1-form $\omega := \sum_{i=1}^{n} -y_i\, dx_i + x_i\, dy_i + dt$.

We shall consider on $\mathbb{H}^n$ the (left invariant) Riemannian metric $g = \langle \cdot, \cdot \rangle$ so that $\{X_1, Y_1, \ldots, X_n, Y_n, T\}$ is an orthonormal basis at every point, and the associated Levi-Civitá connection D. The modulus of a vector field U with respect to this Riemannian metric will be denoted by $|U|$. The following derivatives can be easily computed

$$\begin{aligned} &D_{X_i}X_j = 0, && D_{Y_i}Y_i = 0, && D_T T = 0,\\ &D_{X_i}Y_j = -\delta_{i,j}T, && D_{X_i}T = Y_i, && D_{Y_i}T = -X_i,\\ &D_{Y_i}X_j = \delta_{i,j}T, && D_T X = Y, && D_T Y_i = -X_i \end{aligned} \tag{2.2.8}$$

for any $1 \leqslant i, j \leqslant n$, where $\delta_{i,j}$ is the Kronecker delta. Setting $J(U) = D_U T$ for any vector field U in $\mathbb{H}^n$ we get $J(X_i) = Y_i$, $J(Y_i) = -X_i$ and $J(T) = 0$. Therefore $-J^2$ coincides with the identity when restricted to the horizontal distribution.

The Riemannian volume of a set E is, up to a constant, the Haar measure of the group and is denoted by $|E|$. The integral of a function f with respect to the Riemannian measure by $\int f \, d\mathbb{H}^n$.

2.2.2.1 Immersed surfaces in $\mathbb{H}^1$

We consider oriented surfaces of class C^2 immersed in $\mathbb{H}^1$ and we shall choose a unit normal to S. In case S is the boundary of a domain $\Omega \subset \mathbb{H}^1$, we always choose the *outer* unit normal. The *singular set* of S is denoted by S_0 and it is composed of the points in $p \in S$ where the tangent space T_pS coincides with the horizontal distribution $\mathcal{H}_p$. The *horizontal* unit normal ν_h is defined in $S \setminus S_0$ by

$$\nu_h = \frac{N_h}{|N_h|}.$$

The vector field Z is defined by

$$Z = -J(\nu_h).$$

The vector field Z is defined on $S \setminus S_0$ and it is tangent to S and horizontal. It generates at every point $p \in S \setminus S_0$ the subspace $T_pS \cap \mathcal{H}_p$.

2.3 (X,K)-variation and (X,K)-Caccioppoli sets

Let $X = \{X_1, \ldots, X_k\}$ in $\mathbb{R}^d$ be a family of linearly-independent smooth vector fields and $|\cdot|_K$ the norm associated to a convex body $K \subset \mathbb{R}^k$ with $0 \in \mathrm{int}(K)$.

Given a horizontal vector field U in an open set $\Omega \subset \mathbb{R}^d$ is said *horizontal* if $U(p) \in \mathcal{H}_p$ for all $p \in \Omega$. The set of horizontal vector fields in Ω of class $C^1_c(\Omega)$ is denoted by $\mathcal{H}^1_c(\Omega)$.

Taking X' a family of d linearly-independent smooth vector fields that extends X, we consider the Riemannian metric g making X' an orthonormal basis. We write the divergence of $U = \sum_{i=1}^k u_i X_i \in \mathcal{H}^1(\Omega)$ as

$$\operatorname{div} U = \sum_{i=1}^{k} X_i(u_i).$$

Remark 2.3.1. Fixed a distribution $\mathcal{H}$ and a Riemannian metric g in $\mathcal{H}$, div is independent of the chosen family X orthonormal at every point.

Given an open set $\Omega \subset \mathbb{R}^d$ and a function $u : \mathbb{R}^d \to \mathbb{R}$ in $L^1(\Omega)$, we define say that u has *locally finite* (X,K)*-variation* in Ω if for any relatively compact open set $V \subset \Omega$ we have

$$V_K(u;V) = \sup\left\{ \int_{\mathbb{R}^d} u \operatorname{div} U \, dx \, : \, U \in \mathcal{H}^1_c(V), \; |U|_{K,\infty} \leqslant 1 \right\} < +\infty.$$

In this expression, $|U|_{K,\infty} = \sup_{p\in V} |U_p|_K$. The integral is computed with respect to the Lebesgue measure dx on $\mathbb{R}^d$. If u is the characteristic function of a measurable set E, we say that E is a *locally* (X,K)*-Caccioppoli set* or a *locally* (X,K)*-finite perimeter set.*

The quantity $V_K(u;\Omega)$ is called the (X,K)*-variation* of u in Ω. Whenever u is the characteristic function of a measurable set E, $V_K(u;\Omega)$ is called the

(X, K)*-perimeter of* E in Ω and is denoted by $P_K(E;\Omega)$. When X defines a structure of Carnot group, we call P_K the K-perimeter. We shall write $P_K(E)$ whenever $\Omega = \mathbb{R}^d$.

Let K, K' convex bodies with 0 in its interior. Then there exist constants $\alpha, \beta > 0$ such that

$$\alpha|x|_{K'} \leqslant |x|_K \leqslant \beta|x|_{K'}, \quad \text{for all } x \in \mathbb{R}^k.$$

Let $\Omega \subset \mathbb{R}^d$ an open set, $V \subset \Omega$ a relatively open set and $u : \mathbb{R}^d \to \mathbb{R}$ in $L^1(\Omega)$. Take $U \in \mathcal{H}^1_c(V)$ with $|U|_{K,\infty} \leqslant 1$. Hence $|\alpha U|_{K'} \leqslant |U|_K \leqslant 1$ and

$$\int_E u \operatorname{div}(U) dx = \frac{1}{\alpha} \int_E u \operatorname{div}(\alpha U)\, dx \leqslant \frac{1}{\alpha} V_K(u; V),$$

Taking supremum over the vector fields in $\mathcal{H}^1_c(V)$ with $|\cdot|_K \leqslant 1$, we get $V_K(u;V) \leqslant \frac{1}{\alpha} V_{K'}(u;V)$. In a similar way we get $\frac{1}{\beta} V_{K'}(u;V) \leqslant V_K(u;V)$, so that we have

$$\tfrac{1}{\beta} V_{K'}(u;V) \leqslant V_K(u;V) \leqslant \tfrac{1}{\alpha} V_{K'}(u;V). \tag{2.3.1}$$

As a consequence, u has locally finite (X, K)-variation if and only if it has locally finite (X, K')-variation.

Let $E \subset \mathbb{R}^d$ be a set with locally finite (X, K)-perimeter in Ω. We can define a linear functional $L : C^1_c(\Omega, \mathbb{R}^k) \to \mathbb{R}$ by

$$L(h) = L((h_1, \ldots, h_k)) = \int_E \operatorname{div}(h_1 X_1 + \ldots + h_k U_k)\, dx.$$

For any relatively compact open set $V \subset \Omega$ we have

$$C(V) := \sup\{L(h) : h \in C^1_c(V, \mathbb{R}^k),\ |h|_{K,\infty} \leqslant 1\} < +\infty,$$

We fix any compact subset $C \subset \Omega$ and take a relatively compact open set V such that $C \subset V \subset \Omega$. For each $h \in C_c(\Omega, \mathbb{R}^k)$ with support in K we can find a sequence of C^1 functions $(h_i)_{i\in\mathbb{N}}$ with support in V such that h_i converges uniformly to h. Hence equality

$$\overline{L}(h) = \lim_{i\to\infty} L(h_i)$$

allows to extend L to a linear functional $\overline{L} : C_c(\Omega, \mathbb{R}^k) \to \mathbb{R}$ satisfying

$$\sup\{\overline{L}(h) : h \in C_c(\Omega, \mathbb{R}^k), \operatorname{supp}(h) \subset C, |h|_{K,\infty} \leqslant 1\} \leqslant C(V) < +\infty.$$

The proof of the Riesz Representation Theorem, see § 1.8 in [59], can be adapted to obtain the existence of a Radon measure μ_K on Ω and a μ_K-measurable function $\nu_K = (\nu_1, \ldots, \nu_k) : \Omega \to \mathbb{R}^k$ satisfying

$$\overline{L}(h) = \int_\Omega \langle h, \nu_K\rangle\, d\mu_K.$$

The measure μ_K is the total variation measure

$$\mu_K(V) = \sup\{\bar{L}(h) : h \in C_c(\Omega, \mathbb{R}^k),\ \operatorname{supp}(h) \subset V,\ |h|_{K,\infty} \leqslant 1\}$$

that coincides with $P_K(E;V)$ since $\overline{L}$ is a continuous extension of L. Henceforth we denote μ_K by $|\partial E|_K$ and we shall drop the subscript K when the convex is the unit disk D.

Let us check that

$$|(\nu_K)_p|_{K,*} = 1 \text{ for } |\partial E|_K\text{-}a.e.\ p. \tag{2.3.2}$$

Here $|\cdot|_{K,*}$ is the dual norm of $|\cdot|_K$. To prove (2.3.2) we take a relatively compact open set $V \subset \Omega$ and $h \in C_c(\Omega, \mathbb{R}^k)$ with $\operatorname{supp}(h) \subset V$ and $|h|_{K,\infty} \leqslant 1$. Since $\langle h, \nu_K \rangle \leqslant |\nu_K|_{K,*}$ we have

$$\overline{L}(h) \leqslant \int_V |\nu_K|_{K,*} d|\partial E|_K.$$

Taking supremum over such g we have

$$|\partial E|_K(V) \leqslant \int_V |\nu_K|_{K,*} d|\partial E|_K.$$

On the other hand, we can take a sequence $(\varphi_i) = ((\varphi_1)_i, \dots, (\varphi_k)_i)$ in $C_c^1(V)$ such that $|\varphi_i|_K \leqslant 1$ and $\langle \varphi_i, \nu_K \rangle$ converges to $|\nu_K|_{K,*}$ $|\partial E|_K$-a.e. This is a consequence of Lusin's Theorem, see § 1.2 in [59], and follows by approximating the measurable function $\pi_K(\nu_K)$ by continuous uniformly bounded functions. Then we would have

$$\int_V |\nu_K|_{K,*} d|\partial E|_K = \lim_{i\to\infty} \langle \varphi_i, \nu_K \rangle d|\partial E|_K \leqslant |\partial E|_K(V).$$

So we would have

$$|\partial E|_K(V) = \int_V |\nu_K|_{K,*} d|\partial E|_K$$

and so $|\nu_K|_{K,*} = 1$ for $|\partial E|_K$-a.e. Observe that $h = (h_1, \dots, h_k)$ and $\nu_K = (\nu_1, \dots, \nu_k)$ can be canonically identified with the vector fields $\sum_{i=1}^k h_i X_i$ and $\sum_{i=1}^k \nu_i X_i$. Thus we obtained the following result.

Theorem 2.3.2. *Let E be a finite perimeter set in $\mathbb{R}^d$ and $U \in \mathcal{H}_c^1(\Omega)$. Then, there exists a measure $|\partial E|_K$ and a $|\partial E|_K$-measurable function ν_K with $|\nu_K|_{K,*} = 1$ such that*

$$\int_E \operatorname{div}(U) dx = \int_V \langle U, \nu_K \rangle |\partial E|_K. \tag{2.3.3}$$

2.3.1 Representations of the (X, K)-variation

Given two convex sets $K, K' \subset \mathbb{R}^k$ containing 0 in their interiors, We shall obtain the following representation formula for the sub-Finsler perimeter measure $|\partial E|_K$ and the vector field ν_K in terms of K' and ν_K.

Theorem 2.3.3. *Given two convex sets $K, K' \subset \mathbb{R}^k$ with 0 in their interiors,*

$$|\partial E|_K = |\nu_{K'}|_{K,*}|\partial E|_{K'}, \quad \nu_K = \frac{\nu_{K'}}{|\nu_{K'}|_{K,*}}. \tag{2.3.4}$$

In particular, considering the unit disk D and its associated sub-Finsler perimeter measure $|\partial E|$, we have

$$P_K(E) = \int_{\mathbb{R}^d} |\nu|_{K,*}|\partial E|, \tag{2.3.5}$$

where $\nu = \nu_D$ and $|\partial E| = |\partial E|_K$.

Proof. From (2.3.1), there exist two positive constants λ, Λ such that

$$\lambda|\partial E|_K \leqslant |\partial E|_{K'} \leqslant \Lambda|\partial E|_K.$$

This implies that each of the Radon measures $|\partial E|_K, |\partial E|_{K'}$ is absolutely continuous with respect to the other one. Hence both Radon-Nikodym derivatives exist. Take a relatively compact open set $V \subset \Omega$ and $U \in \mathcal{H}_0^1(V)$. Then we have

$$\begin{aligned}\int_V \langle U, \nu_{K'}\rangle \, d|\partial E|_{K'} &= \int_V \chi_E \operatorname{div}(U) \, d\mathbb{H}^1 \\ &= \int_V \langle U, \nu_K\rangle \, d|\partial E|_K = \int_V \langle U, \frac{d|\partial E|_K}{d|\partial E|_{K'}} \nu_K\rangle \, d|\partial E|_{K'}.\end{aligned} \tag{2.3.6}$$

By the uniqueness of $\nu_{K'}$ we have

$$\nu_{K'} = \frac{d|\partial E|_K}{d|\partial E|_{K'}} \nu_K, \quad |\partial E|_{K'}\text{-}a.e. \tag{2.3.7}$$

On the other hand, inserting $U \in \mathcal{H}_0^1(V)$ in (2.3.6) with $|U|_K \leqslant 1$ we get

$$\int_V \langle U, \nu_K\rangle d|\partial E|_K = \int_V \langle U, \nu_{K'}\rangle \, d|\partial E|_{K'} \leqslant \int_V |\nu_{K'}|_{K,*} d|\partial E|_{K'}.$$

Taking supremum over U we obtain

$$\int_V \frac{d|\partial E|_K}{d|\partial E|_{K'}} d|\partial E|_{K'} = |\partial E|_K(V) \leqslant \int_V |\nu_{K'}|_{K,*} d|\partial E|_{K'}$$

and, since V is arbitrary, we have

$$\frac{d|\partial E|_K}{d|\partial E|_{K'}} \leqslant |\nu_{K'}|_{K,*} \quad |\partial E|_K\text{-}a.e. \tag{2.3.8}$$

Substituting (2.3.7) into (2.3.8) we have

$$\frac{d|\partial E|_K}{d|\partial E|_{K'}} \leqslant |\nu_{K'}|_{K,*} = \frac{d|\partial E|_K}{d|\partial E|_{K'}} \quad |\partial E|_K\text{-}a.e.$$

Hence we have equality and so

$$\frac{d|\partial E|_K}{d|\partial E|_{K'}} = |\nu_{K'}|_{K,*} \quad |\partial E|_K\text{-}a.e. \tag{2.3.9}$$

Hence we get from equation (2.3.4) from (2.3.9) and (2.3.7). □

For the closed unit disk $D \subset \mathbb{R}^k$ centered at 0 we know that in the C^1 case $\nu_D = \nu_h$ and $|N_h| = |N_h|_{D,*}$. Hence we have

$$|\partial E|_K = |\nu_h|_{K,*} d|\partial E|, \quad \nu_K = \frac{\nu_h}{|\nu_h|_{K,*}}.$$

Here $|\partial E|$ is the standard sub-Riemannian perimeter measure.

Proposition 2.3.4. *Let $(\mathbb{R}^d, \cdot, K)$ be a sub-Finsler Carnot group. Given Ω and E open sets in $\mathbb{R}^d$ with $S = \partial E$ Euclidean Lipschitz, then*

$$P_K(E; \Omega) = \int_{S \cap \Omega} |N_h(p)|_{K,*} dS(p), \tag{2.3.10}$$

where N_h is the horizontal projection of the unit normal to S and dS is the Riemannian measure on S.

Notes

Notes of § 2.3 1. In [115] and in [38] can be seen definitions of area for immersed submanifold of fixed degree in stratified groups and in graded manifolds respectively. A notion of perimeter for submanifolds with intrinsic regularity in Heisenberg groups can be found in [73]. See also [86].

Notes of § 2.3.1 1. An alternative proof of (2.3.5) is given in Appendix A using techniques of convex analysis developed by Bouchitté and Valadier in [20]. This approach can be adapted to generalize the result to a CC space with a different norm in any subspace $\mathcal{H}_p$, although we will restrict to the case with only one fixed norm.

Chapter 3

Existence of isoperimetric regions in sub-Finsler nilpotent groups

This chapter contains the results of a work in progress.

We consider a sub-Finsler nilpotent group $(\mathbb{R}^d, \cdot, X, K)$. In this chapter we prove the existence of minimizers of the perimeter functional P_K associated to $|\cdot|_K$ under a volume (Haar measure) constraint.

The privileged position of Carnot groups within geometric measure theory is revealed by its characterization as the only metric spaces that are locally compact, geodesic, isometrically homogeneous, and self-similar (i.e. admitting a dilation). This characterization can be found as Theorem 1.1 in [105]. Removing the self-similarity condition, sub-Finsler nilpotent groups acquire relevant importance.

In sub-Riemannian geometry, apart from the compact case, there are few known results. Galli and Ritoré proved in [77] an existence result in contact sub-Riemannian manifolds. The argument followed Morgan's structure: they pick a minimizing sequence of sets of volume v whose perimeters approach the infimum of the perimeters of sets of volume v. This sequence can be splitted into two subsequences. The first subsequence is converging to a set, and it is proved that is isoperimetric for its volume and bounded. Nevertheless, it might be a loss of mass at infinity. In this case, they use isometries to translate the second subsequence, which is diverging, to recover some of the lost volume. An essential point is that they always recover a fixed fraction of the volume.

The main result of the chapter is the following theorem.

Theorem 3.0.1 (Existence of isoperimetric regions)**.** *Let (G, X, K) be a sub-Finsler nilpotent group. Then, for any $v > 0$, there exists a finite perimeter set E such that $|E| = v$ and $I_K(v) = P_K(E)$. Moreover, E has a finite* *number of connected components.*

This result is an extension of the existence result of Leonardi and Rigot for nilpotent groups with no dilations and a sub-Finsler norm. The proof follows the arguments in [77]. As in [77], one of the main difficulties is to prove a Deformation Lemma that allow us to increase the volume of any finite perimeter set while modifying the perimeter in a controlled way, more precisely, the difference of the perimeters is linear with respect to the volume we are adding.

We shall assume the validity of the following result.

Theorem 3.0.2 (Deformation Lemma)**.** *Let (G, X, K) be a sub-Finsler nilpotent group. Let $F \subseteq G$ be a finite perimeter and finite volume set and suppose that there exists $p \in int(F)$. Then there exist $C_3 > 0$ and a family of finite perimeter sets $\{F^\lambda\}$ such that $|F^\lambda| \geqslant |F| + \lambda C$ and*

$$P_K(F^\lambda) - P_K(F) \leqslant C_3\lambda. \tag{3.0.1}$$

There are a large number of deformation lemmas in literature, some of them in [7; 6; 77; 114; 122; 131; 134]. The proof of Theorem 3.0.2 usually relies on taking a vector field U and use the formulas for the first variation of the volume and the area. In [77], Galli and Ritoré used a calibration argument, exploiting that Pansu spheres in $\mathbb{H}^1$ have constant mean curvature to construct, in a neighborhood of a given point, a horizontal vector field with bounded divergence.

By a calibration argument, we obtain in Proposition 3.2.1 that the isoperimetric profile is non-decreasing. One advantage of our proof of Theorem 3.0.2 is that the constant C depends only on the radius of a ball inside F. The property of sub-additiveness of the isoperimetric profile is proven in Corollary 3.3.3. We shall also extend the properties obtained in Carnot groups by Leonardi and Rigot in [109], that isoperimetric regions are bounded and its topological and essential boundaries coincide.

This chapter is organized as follows. In Section 3.1, we fix some notation and give some background on sub-Finsler nilpotent groups and the notion of K-perimeter. In Section 3.2, we prove Theorem 3.0.2 and study some properties of the isoperimetric regions such as that they are open up to a nullset (Corollary 3.2.3), bounded (Theorem 3.2.9) and its essential and topological boundaries coincide (Theorem 3.2.8), and prove in Proposition 3.2.1 that the isoperimetric profile is non-decreasing. In Section 3.3, we prove Theorem 3.0.1, the existence of isoperimetric regions and deduce that the isoperimetric profile is a sub-additive function in Corollary 3.3.3.

3.1 Preliminaries

We consider a left-invariant Riemannian metric $g := \langle\cdot,\cdot\rangle$ forming X an orthonormal basis of $\mathcal{H}^0$, and making orthogonal the subbundles $\mathcal{H}^0$ and V, where V is a complementary subbundle of $\mathcal{H}^0$.

From the definition and that the Lebesgue measure is invariant by left-translations it follows that, for any $p \in \mathbb{R}^d$,

$$P_{(X,K)}(\ell_p \cdot E; \ell_p \cdot \Omega) = P_{(X,K)}(E; \Omega).$$

We shall use exhaustively the following decomposition of the K-perimeter

$$P_K(E) \geqslant P_K(E \cap B) + P_K(E \setminus B) - 2C_1 P(E \cap B; \partial B), \tag{3.1.1}$$

where B is any sub-Riemannian ball in $\mathbb{R}^d$ and P is the sub-Riemannian perimeter. Indeed, we have

$$P_K(E\cap B) = P_K(E\cap B; B) + P_K(E\cap B; B^c) = P_K(E; B) + P_K(E\cap B; \partial B).$$

From the above equation and the relation between the Euclidean and the Minkowski norm (2.3.1), we obtain

$$\begin{aligned} P_K(E; B) &= P_K(E \cap B) - P_K(E \cap B; \partial B) \\ &\geqslant P_K(E \cap B) - C_1^{-1} P(E \cap B; \partial B). \end{aligned} \tag{3.1.2}$$

Similarly, it holds

$$\begin{aligned} P_K(E; \bar{B}^c) &= P_K(E \setminus B) - P_K(E \setminus B; \partial B) \\ &\geqslant P_K(E \setminus B) - C_1^{-1} P(E \setminus B; \partial B) \\ &= P_K(E \setminus B) - C_1^{-1} P(E \cap B; \partial B). \end{aligned} \tag{3.1.3}$$

Adding (3.1.2) and (3.1.3), we obtain (3.1.1).

The following relation between the sub-Riemannian perimeter and the derivative of the volume can be found as Lemma 3.5 in [4].

Lemma 3.1.1. *Let $(\mathbb{R}^d, \cdot, X, K)$ be a sub-Finsler nilpotent group. Let $F \subseteq \mathbb{R}^d$ be a finite (sub-Riemannian) perimeter set and B_r the sub-Riemannian ball of radius r centered in* 0*. Then for a.e. $r > 0$, we have*

$$\max\{P(F \cap B_r; \partial B_r), P(F \setminus B_r; \partial B_r)\} \leqslant -\frac{d}{ds}\Big|_{s=r} |F \setminus B_s|. \tag{3.1.4}$$

A measurable set F is said to have density s at a point x provided the following limit exists and equals s

$$\lim_{t\to 0^+} \frac{|F \cap B(x,r)|}{|B(x,r)|}.$$

The set of points where de density of F is s is denoted by F_s. The *essential boundary* of F is $\mathbb{R}^d \setminus (F_1 \cup F_0)$.

3.1.1 Isoperimetric inequality for small volumes

A fundamental tool in a metric measure space (X, d, μ) is the existence of a $(1,1)$-Poincaré Inequality, that is, the existence of constants $C \geqslant 0$ and $\lambda \geqslant 1$ such that

$$\int_{B(p,r)} |f - f_{p,r}| d\mu \leqslant Cr \int_{B(p,\lambda r)} |\nabla f| d\mu, \tag{3.1.5}$$

for all f locally Lipschitz, where $B(p,r)$ is the metric ball of center p and radius r, $f_{p,r} := 1/|B(p,r)| \int_{B(p,r)} f d\mu$ and $|\nabla f|$ is an upper gradient of f in the sense of Heinonen and Koskela [96]. In the context of connected Lie groups with polynomial volume growth, a $(1,1)$-Poincaré inequality was proven by Varopoulos in [163], and in $\mathbb{R}^d$ with a Lie bracket generating system by Jerison in [101]. As stated by Hajlasz and Koskela in Theorem 5.1 and 9.7 in [95], the $(1,1)$-Poincaré inequality (3.1.5) together with (2.2.6) implies the following Sobolev inequality.

$$\left(\fint_{B(p,r)} |f - f_{p,r}|^{l/(l-1)} d\mu \right)^{(l-1)/l} \leqslant \tilde{C} r \fint_{B(p,r)} |\nabla f| d\mu. \tag{3.1.6}$$

From Inequality (3.1.6) and (2.3.1), the relative isoperimetric inequality easily follows (see also Theorem 1.18 in [82]).

Theorem 3.1.2 (Relative Isoperimetric inequality). *Let $(\mathbb{R}^d, \cdot, X, K)$ be a sub-Finsler nilpotent group with local dimension l. There exists $C > 0$ such that if $F \subset \mathbb{R}^d$ is any finite perimeter, then*

$$C \min\{|F \cap B(p,r)|, |B(p,r) \setminus F|\}^{(l-1)/l} \leqslant P_K(F; B(p,r)), \tag{3.1.7}$$

for any $p \in \mathbb{R}^d$ and $0 < r \leqslant 1$.

From a classical covering argument, we obtain the following isoperimetric inequality for small volumes. The proof follows identically as in Lemma 3.10 of [77]. It is also proven in Proposition 3.20 [7], in the context of PI-spaces, that is, metric measure spaces satisfying a weak $(1,1)$-Poincaré inequality and doubling property, which are uniformly s-Ahlfors regular.

Theorem 3.1.3 (Isoperimetric inequality for small volumes). *Let $(\mathbb{R}^d, \cdot, X, K)$ be a sub-Finsler nilpotent group with local dimension l. There exists $C_2 > 0$ and $v_0 > 0$ such that if $F \subseteq \mathbb{R}^d$ is any finite perimeter set and $|F| < v_0$, then*

$$C_2 |F|^{l-1/l} \leqslant P_K(F). \tag{3.1.8}$$

3.2 Properties of isoperimetric regions

Throughout this section, $(\mathbb{R}^d, \cdot, X, K)$ denotes a nilpotent group with X a Lie bracket generating system, K denotes a convex body in $\mathcal{H}_0^0$ containing 0 in its interior, $B(p,r)$ the sub-Riemannian ball centered in p of radius $r > 0$. We shall see that an isoperimetric region E is open up to a nullset and its topological and essential boundary coincide, using the arguments developed in [109; 49]. We shall also prove that isoperimetric regions are bounded. Moreover, we shall prove a Deformation Lemma.

The isoperimetric profile is defined as

$$I_K(v) := \inf\{P_K(E) : E \subseteq \mathbb{R}^d \textit{ is a finite perimeter set and } |E| = v\}.$$

Proposition 3.2.1. *Let $(\mathbb{R}^d, \cdot, X, K)$ be a sub-Finsler nilpotent group. The isoperimetric profile is non-decreasing.*

Proof. We recall from the previous section that we can take X_1 in $X \setminus \mathfrak{g}_1$, $\mathcal{D}$ be the distribution orthogonal to X_1 and S a hypersurface passing through 0 with $TS = \mathcal{D}$. Moreover, S is orientable. Let S^+ and S^- the open regions in $\mathbb{R}^d$ with boundary S and horizontal normal vectors X_1 and $-X_1$ respectively.

Fix $v > w > 0$ and let $E_n \subseteq \mathbb{R}^d$ such that $|E_n| = v$ and $P_K(E_n) = I_K(v) + \frac{1}{n}$. Let p_n be such that $|\ell_{p_n} E_n \cap S^+| = w$. By abuse of notation, we will write E_n and E_n^- for $\ell_{p_n} E_n$ and $\ell_{p_n} E_n \cap S^-$ respectively. Let $U \in \Pi_K(X_1)$ be a projection over K, that is, satisfying (**??**). Applying (2.3.3) to E_n^- and U, we get

$$\int_{E_n^-} \operatorname{div} U dx = -\int_{E_n \cap S} \langle \pi_K(X_1), X_1 \rangle d|\partial E_n^-|_K + \int_{E_n^-} \langle U, \nu \rangle d|\partial E_n^-|_K.$$

Since U has constant coordinates in X, $\operatorname{div} U = 0$. Therefore, from (2.3.5) we get

$$P_K(E_n \cap S^+; S) = \int_{E_n^-} \langle U, \nu \rangle d|\partial E_n^-|_K \leqslant P_K(E_n; S^-). \tag{3.2.1}$$

Adding $P_K(E_n; S^+)$ to both sides of Equation (3.2.1), we get

$$I_K(w) \leqslant P_K(E_n \cap S^+) \leqslant P_K(E_n) = I_K(v) + \frac{1}{n}. \qquad \square$$

We shall need the following Lemma proven in [109] for Carnot groups.

Lemma 3.2.2. *Let E be an isoperimetric region, $p \in \mathbb{R}^d$ and $0 < r \leqslant 1$. Then there exists $\varepsilon > 0$ such that if $r^{-l}|B(p,r) \setminus E| \leqslant \varepsilon$, then*

$$|B(p, r/2) \setminus E| = 0,$$

where is $B(p,r)$ the sub-Riemannian ball centered in p of radius $r > 0$.

Proof. Let $|E| = v$. Suppose that $r^{-l}|B(p,r) \setminus E| \leqslant \varepsilon$. Let $t > 0$, $B := B(p,t)$, $E_t := E \cup B$ and $m(t) := |B \setminus E|$. It is clear that

$$\begin{aligned} P_K(E) &\geqslant P_K(E;B) + P_K(E;\bar{B}^c) \\ &= P_K(E;B) + P_K(E_t;\bar{B}^c). \end{aligned}$$

It follows that

$$P_K(E_t;\bar{B}^c) \leqslant P_K(E) - P_K(E;B).$$

Then

$$\begin{aligned} P_K(E_t) &= P_K(E_t;B) + P_K(E_t;\bar{B}^c) + P_K(E_t;\partial B) \\ &\leqslant P_K(E) - P_K(E;B) + P_K(E_t;\partial B), \end{aligned} \tag{3.2.2}$$

where we used that $P_K(E_t;B) = 0$. On the other hand, using (3.1.4) and that $-|E^c \setminus B|' = m'(t)$, we get

$$P_K(E_t;\partial B) \leqslant C_1 P(E_t;\partial B) = C_1 P(E^c \setminus B;\partial B) \leqslant C_1 m'(t). \tag{3.2.3}$$

Using again (3.1.4),

$$P_K(E;B) = P_K(B \setminus E) - P_K(B \setminus E;\partial B) \geqslant P_K(B \setminus E) - m'(t). \tag{3.2.4}$$

Since $|E_t| \geqslant |E|$ and E is an isoperimetric region, Proposition 3.2.1 gives us

$$P_K(E) \leqslant P_K(E_t). \tag{3.2.5}$$

Substituting (6.2.4), (6.2.5) and (6.2.6) in (3.2.2), it follows that

$$P_K(B \setminus E) \leqslant (1 + C_1) m'(t).$$

From the Isoperimetric Inequality (3.1.8),

$$C m(t)^{l-1/l} \leqslant m'(t). \tag{3.2.6}$$

The above inequalities holds for a.e. $t > 0$. Suppose that $m(t) > 0$ for all $t \in [r/2, r]$, otherwise there is nothing to prove. Then we can rewrite Inequality (3.2.6) as

$$C \leqslant \frac{m'(t)}{m(t)^{l-1/l}},$$

and integrating between $r/2$ and r,

$$r \leqslant C(m(r)^{1/l} - m(r/2)^{1/l}) \leqslant C m(r)^{1/l} \leqslant C\varepsilon^{1/l} r.$$

This is impossible for ε small enough, and we get a contradiction. Therefore $|B(x,r/2) \setminus E| = 0$. □

Corollary 3.2.3. *Let E be an isoperimetric region. Then there exists an open set E_0 that coincides with E almost everywhere.*

Proof. Let p be in E_1. By (2.2.6) we get

$$\frac{|B \setminus E|}{|B|} \geqslant \tilde{C}\frac{|B \setminus E|}{r^l} \geqslant 0$$

for r small enough. Since the left hand side of the above inequality converges to 0, we can apply Lemma 3.2.2. □

Remark 3.2.4. From now on we shall assume that an isoperimetric region E is exactly E_1, and therefore an open set.

Remark 3.2.5. Notice that in the proof of Corollary 3.2.4 is important to have the same exponent l in (2.2.6) and in (3.1.8).

Remark 3.2.6. The constant C_3 depends on the radius of the ball $B(p,r)$ taken inside F. Therefore, if $r > 0$ and F_1 and F_2 are two finite perimeter sets with finite volume such that $B(p_1,r) \subseteq F_1$ and $B(p_2,r) \subseteq F_2$ for some p_1 and p_2, then we can take $C_3 > 0$ satisfying (3.0.1).

In Lemma 3.2.2 we proved that if we have a ball that is almost contained in an isoperimetric region E, then the ball of half the radius is in E. Following again the arguments in [109] and using the Deformation Lemma 3.0.2, we prove in Lemma 3.2.7 the analog result when the starting ball is almost outside E.

Lemma 3.2.7. *Let E be an isoperimetric region, $p \in \mathbb{R}^d$ and $0 < r \leqslant 1$. Then there exists $\varepsilon > 0$ such that if $r^{-l}|E \cap B(p,r)| \leqslant \varepsilon$, then*

$$|E \cap B(p, r/2)| = 0.$$

where is $B(p,r)$ the sub-Riemannian ball centered in x of radius $r > 0$.

Proof. Let $t > 0$, $B := B(p,t)$ and $m(t) = |E \cap B|$. Under the assumption that $r^{-l}|E \cap B(x,r)| \leqslant \varepsilon$, $m(t)$ is small enough to define $E_t := (E \setminus B)^{m(t)}$ as the set given by the Deformation Lemma 3.0.2 with volume $|E_t| = |E|$. Thus

$$P_K(E) \leqslant P_K(E_t).$$

On the other hand, reasoning as in Lemma 3.2.2 and using the Deformation Lemma, we get

$$\begin{aligned} P_K(E_t) &\leqslant P_K(E \setminus B) + C_3 m(t) \\ &\leqslant P_K(E) - P_K(E \cap B) + (1 + C_1)m'(t) + C_3 m(t) \\ &\leqslant P_K(E) - m(t)^{l-1/l} + (1 + C_1)m'(t) + C_3 m(t). \end{aligned}$$

The above inequalities gives us

$$m(t)^{l-1/l} - (1 + C_1)m(t) \leqslant C_3 m'(t).$$

For $m(t)$ small enough, there exists $C > 0$ such that

$$Cm(t)^{l-1/l} \leqslant m(t)^{l-1/l} - (1 + C_1)m(t),$$

and

$$C'm(t)^{l-1/l} \leqslant m'(t).$$

Again, supposing that $m(t) > 0$ for all $t \in [r/2, r]$, we have

$$C' \leqslant \frac{m'(t)}{m(t)^{l-1/l}},$$

and integrating over $r/2$ and r, we get a contradiction for $\varepsilon > 0$ small enough. □

Let E be an isoperimetric region, we define the sets

$$\begin{aligned} E_1 &= \{p \in \mathbb{R}^d : \exists r > 0 \text{ such that } |B(p,r) \setminus E| = 0\} \\ E_0 &= \{p \in \mathbb{R}^d : \exists r > 0 \text{ such that } |E \cap B(p,r)| = 0\} \\ S &= \{p \in \mathbb{R}^d : \min\{|E \cap B(p,r)|, |B(p,r) \setminus E|\} > \varepsilon \ \forall r \leqslant 1\}. \end{aligned}$$

Theorem 3.2.8. *Let $(\mathbb{R}^d, \cdot, X, K)$ be a sub-Finsler nilpotent group and let E be a isoperimetric region. Then the topological and essential boundaries of E coincide.*

Proof. By Lemma 3.2.2 and 3.2.7, the sets E_0, E_1 and S form a partition of $\mathbb{R}^d$. Since E_1 and E_0 are open and disjoint, $\partial E_1 \cup \partial E_0 \subseteq S$. On the other hand, if $p \in S$ and $r > 0$, $B(p,r) \cap E_1 \neq \emptyset$ then $B(p,r) \cap E_0 \neq \emptyset$, otherwise $p \in \text{int}(E_1)$, and $p \in \partial E_1 \cap \partial E_0$. □

Theorem 3.2.9 (Boundedness)**.** *Any isoperimetric region in a sub-Finsler nilpotent group $(\mathbb{R}^d, \cdot, X, K)$ is bounded.*

Proof. Let E be an isoperimetric set of volume v, B the sub-Riemannian ball centered in 0 of radius $r > 0$, and $m(r) = |E \setminus B|$, and $(E \cap B)^{m(r)}$ be the family given by Lemma 3.0.2. Since $|(E \cap B)^{m(r)}| = v$, we have

$$P_K(E) \leqslant P_K((E \cap B)^{m(r)}). \tag{3.2.7}$$

Using the Deformation Lemma (3.0.1), the Isoperimetric inequality (3.1.8) and Equations (3.1.4) and (3.1.1), we get

$$\begin{aligned} P_K((E \cap B)^{m(r)}) &\leqslant P_K(E \cap B) + C_3 m(r) \\ &\leqslant P_K(E) - P_K(E \setminus B) + 2C_1 P(E \cap B; \partial B) + C_3 m(r) \\ &\leqslant P_K(E) - P_K(E \setminus B) - 2C_1 m'(r) + C_3 m(r) \\ &\leqslant P_K(E) - C_2 m(r)^{l-1/l} - 2C_1 m'(r) + C_3 m(r). \end{aligned} \tag{3.2.8}$$

Subtracting Inequality (3.2.7) in (3.2.8), we get

$$-2C_1m'(r) \geqslant C_2m(r)^{l-1/l} - C_3m(r).$$

As $m(r)$ tends to 0 as r tends to ∞, for r big enough there exists $C > 0$ such that

$$C_2m(r)^{l-1/l} - C_3m(r) \geqslant Cm(r)^{l-1/l}.$$

Let $r > 1$ and suppose that $m(r) > 0$. Then

$$\frac{-m'(r)}{m(r)^{l-1/l}} \geqslant C, \tag{3.2.9}$$

Since

$$\int_1^r \frac{-m'(s)}{m(s)^{l-1/l}} = -\int_{m(1)}^{m(r)} \frac{1}{s^{l-1/l}} ds = m(1)^{1/l} - m(r)^{1/l}. \tag{3.2.10}$$

Integrating (3.2.9) between 1 and r and using (3.2.10), we get

$$m(1)^{1/l} \geqslant Cr + m(r)^{1/l}$$

and r is bounded. □

3.3 Existence of isoperimetric regions

Throughout this section, K shall denote a convex body in $\mathcal{H}_0^0$ containing 0 in its interior and $B(p,r)$ the sub-Riemannian ball centered in p of radius $r > 0$. We shall follow the arguments of Galli and Ritoré [77].

The following lemma can be found in [109] for Carnot groups, and in the context of sub-Finsler nilpotent groups the proof can be done *mutatis mutandis.*

Lemma 3.3.1 (Concentration Lemma)**.** *Let F be a set with finite perimeter and volume. Suppose that there exists $m \in (0, |B(0,1)|/2)$ such that $|F \cap B(p,1)| < m$ for all $p \in \mathbb{R}^d$. Then there exists $C > 0$ depending only on l such that*

$$C|F|^l P_K(F)^{-l} \leqslant m.$$

The following Lemma can be found in [110].

Lemma 3.3.2. *Let $\{E_n\}$ be a sequence of uniformly bounded perimeter sets of volumes $\{v_n\}$ converging to $v > 0$. Let E be the limit in $L^1_{loc}(\mathbb{R}^d)$ of E_n. Then there exists a divergence sequence of radii $\{r_n\}$ such that, setting $F_n = E_n \setminus B(0,r_n)$ and up to a subsequence, it is satisfied*

$$\begin{aligned} |E| + \liminf_{n\to\infty} |F_n| &= v, \\ P_K(E) + \liminf_{n\to\infty} P_K(F_n) &\leqslant \liminf_{n\to\infty} P_K(E_n). \end{aligned} \tag{3.3.1}$$

Proof. Take $\{s_n\}$ increasing with $s_n - s_{n+1} > n$. We claim that there exists r_n in $[s_n, s_{n+1}]$ such that

$$P(E_n \cap \partial B(0,r_n); \partial B(0,r_n)) < v_n/n,$$

where P and $B(0,r)$ are the sub-Riemannian perimeter and ball of center 0 and radius r respectively. Otherwise, by Inequality (3.1.4) we have

$$v_n < \int_{s_n}^{s_{n+1}} P(E_n \cap \partial B(0,t))dt \leqslant \int_{s_n}^{s_{n+1}} \frac{d}{ds}|_{s=t}|E_n \cap B(0,s)|dt \leqslant v_n.$$

Therefore, by Inequality (3.1.1) we get

$$\begin{aligned} P_K(E_n) \geqslant& P_K(E_n \cap B(0,r_n)) + P_K(E_n \setminus B(0,r_n)) \\ &- 2C_1 P(E_n \cap \partial B(0,r_n); \partial B(0,r_n)) \\ \geqslant& P_K(E_n \cap B(0,r_n)) + P_K(F_n) - 2C_1 v_n/n. \end{aligned} \tag{3.3.2}$$

On the other hand,

$$|E_n| = |E_n \cap B(0,r_n)| + |E_n \setminus B(0,r_n)|. \tag{3.3.3}$$

Taking inferior limits in n in (3.3.2) and (3.3.3), and using the lower semi-continuity, we have the result. □

Proof of Theorem 3.0.1. Let $\{E_n\}_{k\in\mathbb{N}}$ be a minimizing sequence of sets with $|E_n| = v$ and $P_K(E_n) \leqslant I_K(v) + \frac{1}{n}$. By compactness, the sequence converges in $L^1_{loc}(\mathbb{R}^d)$ to a set E^0. Let $v_0 := |E^0|$. By Lemma 3.3.2, we can find a sequence of divergence radii r_n such that, denoting $F_n := E_n \setminus B(0,r_n)$, we have

$$\begin{aligned} v_0 + \liminf_{n\to\infty} |F_n| &= v, \\ P_K(E^0) + \liminf_{n\to\infty} P_K(F_n) &\leqslant I_K(v). \end{aligned} \tag{3.3.4}$$

If $v_0 = v$, then the Theorem is proven. If $v_0 < v$, we claim that E_0 is isoperimetric for its volume. Otherwise, we can find $O \subseteq G$ such that $|O| = v_0$ and $P_K(O) < P_K(E^0)$. By Theorem 3.2.8, O is bounded and by definition of F_n, we can find n_0 such that $\forall n > n_0$, O and F_n are disjoint. Then

$$\liminf_{n\to\infty} |O \cup F_n| = |O| + \liminf_{n\to\infty} |F_n| = v.$$

By Equation (3.3.4),

$$\begin{aligned} I_K(v) &\leqslant \liminf_{n\to\infty} P_K(O \cup F_n) \\ &= P_K(O) + \liminf_{n\to\infty} P_K(F_n) \\ &< P_K(E^0) + \liminf_{n\to\infty} P_K(F_n) \\ &\leqslant I_K(v), \end{aligned}$$

and we have a contradiction.

Step two. If $v_0 < v$ we apply Lemma 3.3.1 to find a divergent sequence of points $\{p_n^1\}$ such that $|F_n \cap B(p_n,1)| \geqslant m_0|F_n|$. The sets $\ell_{-p_n}F_n$ converge in $L^1_{loc}(\mathbb{R}^d)$ to a set E^1 of volume $v_1 \leqslant \lim_{n\to\infty}|F_n| = v - v_0$. By Lemma 3.3.2, we can find a divergent sequence $\{r_n'\}$ of radii so that the sets $F_n' := (\ell_{-p_n}F_n) \setminus B(0,r_n')$ verifies

$$\begin{aligned} v_1 + \liminf_{n\to\infty} |F_n'| &= v - v_0, \\ P_K(E^1) + \liminf_{n\to\infty} P_K(F_n') &\leqslant \liminf_{n\to\infty} P_K(F_n). \end{aligned} \tag{3.3.5}$$

Since E^0 in bounded, we can suppose that $E^0 \cap E^1 = \emptyset$. If $v_1 = v - v_0$, then $|E^0 \cup E^1| = |E^0| + |E^1| = v$ and

$$P_K(E^0 \cup E^1) = P_K(E^0) + P_K(E^1) \leqslant P_K(E^0) + \lim_{n\to\infty} P_K(F_n) \leqslant I(v).$$

Thus $E^0 \cup E^1$ is the isoperimetric region of volume v. If $v_1 < v - v_0$, then $E^0 \cup E^1$ is isoperimetric for its volume. Otherwise there exists $O \subset G$ such that $|O| = v_0 + v_1$ and $P_K(O) < P_K(E^0) + P_K(E^1)$. Then

$$\begin{aligned} I_K(v) &\leqslant \liminf_{n\to\infty} P_K(O \cup F_n') \\ &= P_K(O) + \liminf_{n\to\infty} P_K(F_n') \\ &< P_K(E^0) + P_K(E^1) + \liminf_{n\to\infty} P_K(F_n') \\ &\leqslant P_K(E^0) + \liminf_{n\to\infty} P_K(F_n) \\ &\leqslant I_K(v), \end{aligned}$$

and we have a contradiction.

By induction, we get a sequence of sets $E^0, \dots, E^n$ pairwise disjoint of volumes $v_0, \dots, v_n$ whose union is isoperimetric for its volume $\sum_{i=1}^n v_i$. Suppose that there exists a infinite number of pieces E^i. Then $\sum_{i=0}^\infty v_i \leqslant v$. Let j and k with $v_j \geqslant v_i$ for all i and v_k small enough so we can take $(E^j)^{v_k}$ the family defined in 3.0.2 and there exists $C > 0$ with $Cv_k^{l-1/l} > C_3 v_k$. Then, by the Deformation Lemma (3.0.1) and the Isoperimetric Inequality (3.1.3), we get

$$\begin{aligned} I_K(\sum_i v_i) &\leqslant \sum_{i\neq j,k} P_K(E^i) + P_K((E^j)^{v_k}) \leqslant \sum_{i\neq k} P_K(E^i) + C_3 v_k \\ &< \sum_{i\neq k} P_K(E^i) + Cv_k^{l-1/l} \leqslant \sum_{i\neq k} P_K(E^i) + P_K(E^k) = I_K(\sum_i v_i), \end{aligned}$$

which is a contradiction. Therefore there are a finite number of pieces, r, until $\sum_{i=1}^r v_i \geqslant v$, and $E^0 \cup E^1 \dots \cup E^r$ is the isoperimetric region of volume v. □

Corollary 3.3.3. *Let* $(\mathbb{R}^d, \cdot, X, K)$ *be a sub-Finsler nilpotent group. The isoperimetric profile* I_K *is sub-additive.*

Proof. Let $v_0, \ldots, v_n \geqslant 0$. Take E_k isoperimetric region of volume v_k. By Theorem 3.2.9, E_k is bounded and we can take $E_j \cap E_i = \emptyset$. Therefore

$$I_K(v_0 + \ldots + v_n) \leqslant P_K(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} P_K(E_k) = \sum_{k=1}^{n} I_K(v_k). \qquad \square$$

Chapter 4

The Brunn-Minkowski inequality in Nilpotent Lie groups

In this chapter we show the results obtained in [141]. We shall prove a generalization of the Brunn-Minkowski inequality in Euclidean space where the Minkowski addition of sets is replaced by any product $* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ of the form

$$z * w = z + w + (F_1, F_2(z,w), \ldots, F_d(z,w)) = z + w + F(z,w), \qquad (*)$$

where F_1 is a constant and F_i are continuous functions that depend only on $z_1, \ldots, z_{i-1}, w_1, \ldots, w_{i-1}$ $\forall i = 2, \ldots, d$. By a product here we mean a binary operation without assuming any further properties such as associativity.

Theorem 4.0.1 (Brunn-Minkowski inequality for (*) products)**.** *Let* $*$: $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *be a product of the form* (*) *and let* $A, B \subset \mathbb{R}^d$ *be measurable sets such that* $A * B$ *is measurable. Then we have*

$$|A * B|^{1/d} \geqslant |A|^{1/d} + |B|^{1/d}. \qquad (4.0.1)$$

Any nilpotent Lie group verifies the hypothesis of Theorem 4.0.1 because of the expression of the group product in exponential coordinates of the first kind. In this chapter, we shall consider a product $*$ of the form (*), that not necessarily comes from a group product, and change $*$ for another one $*_{z_1,w_1}$ of the form (*), depending on the sets A and B, that allows us to compare the volume of the Minkowski addition of sets for the products $*$ and $*_{z_1,w_1}$, as a consequence of Lemma 4.1.1. When the product $*$ comes from a nilpotent group it is not true that $*_{z_1,w_1}$ can define a group product. Then, by an induction argument, we will compare the volume of the Minkowski addition of sets A and B with the volume of the Euclidean Minkowski addition of A

and B, and establish in Proposition 4.1.5 a sufficient condition in $\mathbb{H}^1$ for the strict inequality in (4.0.1).

At the end of the chapter, we state several classical variations of inequality (4.0.1) in the case of Carnot groups, where dilations can be defined.

4.1 The Brunn-Minkowski inequality

We have seen that any simply connected nilpotent group is isomorphic to $\mathbb{R}^d$ with a product of the form (2.2.1). Now we prove the Brunn-Minkowski inequality for any product $* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ of the form (*). This product does not necessarily defines a group structure in $\mathbb{R}^d$. Given such a map F and $z_1', w_1' \in \mathbb{R}$, we can define another product $*_{z_1',w_1'} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, by

$$z *_{z_1',w_1'} w = z + w + F((z_1', \tilde{z}), (w_1', \tilde{w})),$$

where $\tilde{z} = (z_2, \dots, z_d)$, $\tilde{w} = (w_2, \dots, w_d)$. We define the map $F_{(z_1',w_1')} : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ by

$$F_{(z_1',w_1')}(\tilde{z}, \tilde{w}) := (F_2, \dots, F_d)((z_1', \tilde{z}), (w_1', \tilde{w})). \tag{4.1.1}$$

Notice that $F_i((z_1', \tilde{z}), (w_1', \tilde{w}))$ only depends on the first $i-2$ variables of $\tilde{z}$ and $\tilde{w}$ and so $F_2((z_1', \tilde{z}), (w_1', \tilde{w}))$ is constant. Thus the product $\tilde{*} : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ given by

$$\tilde{z} \,\tilde{*}\, \tilde{w} = \tilde{z} + \tilde{w} + F_{(z_1',w_1')}(\tilde{z}, \tilde{w}), \tag{4.1.2}$$

has the form (*). Notice that the product $\tilde{*}$ depends on the choice of z_1', w_1'.

We hall denote $|\cdot|_n$ the Lebesgue measure in $\mathbb{R}^d$. We will drop the subscript when n coincides with the topological dimension.

Lemma 4.1.1. *Let $* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a product of the form (*) and let $A, B \subset \mathbb{R}^d$ be $A = I \times \tilde{A}$, and $B = J \times \tilde{B}$, where I, J are compact intervals in $\mathbb{R}$ and $\tilde{A}, \tilde{B} \subset \mathbb{R}^{d-1}$ are compact. Then*

$$|A * B| \geqslant |I + J|_1 |\tilde{A} \,\tilde{*}\, \tilde{B}|_{d-1}, \tag{4.1.3}$$

where $\tilde{*}$ is the product described in (4.1.2) for certain $z_1' \in I$ and $w_1' \in J$. Moreover, if F does not depends on z_1, w_1, then equality holds in (4.1.3).

Proof. Notice that $A * B$ and $\tilde{A} \,\tilde{*}\, \tilde{B}$ are compact, and so measurable. Let $I = [a, b]$, $J = [a', b']$ and $l = b - a$, $l' = b' - a'$. The product is

$$A * B = \left\{ z + w + F(z, w) : z_1 \in I,\ w_1 \in J,\ \tilde{z} \in \tilde{A},\ \tilde{w} \in \tilde{B} \right\}.$$

We define a diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ by $(s, z) \mapsto (z, s - z)$. The inverse $\phi^{-1}(z, w) = (z + w, z)$ is a diffeomorphism between the sets $I \times J$ and $\{(s_1, z_1) : s_1 \in I + J,\ z_1 \in I \cap (s_1 - J) = K(s_1)\}$. Hence, we clearly have

$$A * B = \Big\{(s_1, \tilde{z} + \tilde{w}) + (F_1, F_{\phi(s_1,z_1)})(\tilde{z}, \tilde{w}) : s_1 \in I + J, \\ \tilde{z} \in \tilde{A}, z_1 \in K(s_1), \tilde{w} \in \tilde{B}\Big\}.$$

Now we use Fubini's theorem and we obtain

$$|A * B| = \int_{I+J} h(s_1) ds_1, \tag{4.1.4}$$

where $h : I + J \to \mathbb{R}_0^+$ is the function

$$h(s_1) = \big|\{\tilde{p} \in \mathbb{R}^{d-1} : (s_1 + F_1, \tilde{p}) \in A * B\}\big|_{d-1} = \Bigg|\bigcup_{z_1 \in K(s_1)} D_{(s_1,z_1)}\Bigg|_{d-1}, \tag{4.1.5}$$

and

$$D_{(s_1,z_1)} = \{\tilde{z} + \tilde{w} + F_{\phi(s_1,z_1)}(\tilde{z}, \tilde{w}) :\ \tilde{z} \in \tilde{A},\ \tilde{w} \in \tilde{B}\}. \tag{4.1.6}$$

Now we compare $h(s_1)$ with the measure of $D_{(s_1,z_1)}$ for some z_1. Let $z_1 : I + J \to \mathbb{R}$ be the function

$$z_1(s_1) = tl + a,$$

where $t = \frac{s_1 - (a+a')}{l+l'}$. It is clear that $0 \leqslant t \leqslant 1$, hence $z_1(s_1) \in I$. Moreover,

$$tl + a = s_1 - tl' - a',$$

and therefore $z_1(s_1) \in s_1 - J$. Then $z_1(s_1) \in K(s_1)$.

Let $f : I + J \to \mathbb{R}_0^+$ be the map given by $f(s_1) = \big|D_{(s_1,z_1(s_1))}\big|_{d-1}$. It is easy to check that f is continuous, and hence f reaches its minimum at a certain value s_1'. Thus, we get

$$\int_{I+J} h(s_1) ds_1 \geqslant \int_{I+J} f(s_1) ds_1 \geqslant \int_{I+J} f(s_1') ds_1 = |I + J|_1 f(s_1'). \tag{4.1.7}$$

Denoting by $z_1' := z_1(s_1')$ and $w_1' := s_1' - z_1'$, we can write $F_{\phi(s_1',z_1')} = F_{(z_1',w_1')}$. Hence we have that $D_{(s_1',z_1')} = \tilde{A} \,\tilde{*}\, \tilde{B}$ and

$$f(s_1') = |\tilde{A} \,\tilde{*}\, \tilde{B}|_{d-1}. \tag{4.1.8}$$

From (4.1.4), (4.1.7) and (4.1.8) we obtain (4.1.3).

Suppose that F does not depend on z_1, w_1, let us prove that equality holds in (4.1.3). It is enough to prove equality in (4.1.7). For any $s_1 \in I + J$ and $z_1 \in K(s_1)$, we have that

$$F_{\phi(s_1', z_1')} = F_{z_1', w_1'} = F_{z_1, w_1} = F_{\phi(s_1, z_1)},$$

where $w_1 = s_1 - z_1$. Therefore

$$D_{(s_1', z_1')} = D_{(s_1, z_1)} = \bigcup_{z_1 \in K(s_1)} D_{(s_1, z_1)}. \tag{4.1.9}$$

Hence, from (4.1.5) and (4.1.9) we get that $f(s_1') = h(s_1)$ for all $s_1 \in I + J$. Thus equality holds in (4.1.7) and the result follows. □

Remark 4.1.2. The product $*_{z_1', w_1'}$ does not depend on z_1, w_1 and Lemma 4.1.1 guarantees

$$|A * B| \geqslant |I + J|_1 |\tilde{A} \,\tilde{*}\, \tilde{B}|_{d-1} = |A *_{z_1', w_1'} B|. \tag{4.1.10}$$

Recall that $*_{z_1', w_1'}$ acts as a sum in the first two coordinates, and someway (4.1.10) allows us to compare the measure of $A * B$ with the measure of a set more similar to the Euclidean Minkowski addition of A and B.

Proof of Theorem 4.0.1. The proof is divided into three steps.

Step 1. We first claim that (4.0.1) holds for a pair of d-rectangles A and B, that is,

$$A = I_1 \times \cdots \times I_d$$
$$B = J_1 \times \cdots \times J_d,$$

where I_i, J_j are compact intervals $\forall\ 1 \leqslant i, j \leqslant d$. We shall see that

$$|A * B| \geqslant |I_1 + J_1|_1 \ldots |I_d + J_d|_1 = |A + B|, \tag{4.1.11}$$

and the classical Brunn-Minkowski inequality in $\mathbb{R}^d$ would imply (4.0.1).

In order to prove (4.1.11), we use Lemma 4.1.1 to obtain

$$|A * B| \geqslant |I_1 + J_1|_1 |\tilde{A} \,\tilde{*}\, \tilde{B}|_{d-1},$$

but now $\tilde{A} = I_2 \times (I_3 \times \ldots \times I_d), \tilde{B} = J_2 \times (J_3 \times \ldots \times J_d)$ and $\tilde{*}$ has the form (*), and so we can apply Lemma 4.1.1 to the sets $\tilde{A}$ and $\tilde{B}$. Iterating this process, we get (4.1.11).

Step 2. Now we consider the case where A and B are finite unions of dyadic d-rectangles, that is, $A = A_1 \cup \ldots \cup A_n$, $B = B_1 \cup \ldots \cup B_m$ where $A_i = I_1^i \times \ldots \times I_d^i$, $B_j = J_1^j \times \ldots \times J_d^j$ and, for any $k = 1, \ldots, d$ and $r \neq s$ ($p \neq q$), it is satisfied that either $\operatorname{int}(I_k^r) \cap \operatorname{int}(I_k^s) = \emptyset$ or $I_k^r = I_k^s$ (either $\operatorname{int}(J_k^p) \cap \operatorname{int}(J_k^q) = \emptyset$ or $J_k^p = J_k^q$), where $\operatorname{int}(I)$ denotes the interior of I.

We proceed by induction on the total number $n+m$ of d-rectangles. If $n+m=2$, then A and B are d-rectangles and we can apply step 1. Suppose that the theorem holds for $n+m-1$, where $n+m \geqslant 3$. Then we can find a hyperplane $P : \{z_i = a_i\}$ such that some $A_r \subset \{z_i \geqslant a_i\}$ and some $A_s \subset \{z_i \leqslant a_i\}$.

If the hyperplane has as equation $P : \{z_1 = a_1\}$, the proof is the same as the classical proof of Hadwiger and Ohmann for the addition of sets in $\mathbb{R}^d$. We include it for the sake of completeness. The sets

$$A^+ = A \cap \{z_1 \geqslant a_1\}, \qquad A^- = A \cap \{z_1 \leqslant a_1\}$$

are unions of d-rectangles whose sum is strictly less than n. We choose a parallel hyperplane $Q : \{z_1 = b_1\}$ verifying that

$$\frac{|B^\pm|}{|B|} = \frac{|A^\pm|}{|A|}, \tag{4.1.12}$$

where B^+ and B^- are the sets given by

$$B^+ = B \cap \{z_1 \geqslant b_1\}, \qquad B^- = B \cap \{z_1 \leqslant b_1\}.$$

Moreover, B^+ and B^- are disjoint unions of d-rectangles whose sum is at most m. We apply the induction hypothesis to the pairs A^+, B^+ and A^-, B^-, and we obtain

$$\begin{aligned} |A^+ * B^+| &\geqslant (|A^+|^{1/d} + |B^+|^{1/d})^d \\ |A^- * B^-| &\geqslant (|A^-|^{1/d} + |B^-|^{1/d})^d. \end{aligned} \tag{4.1.13}$$

On the other hand, $P * Q$ is contained in another vertical plane $\{z_1 = a_1 + b_1\} \subset \mathbb{R}^d$, $A^+ * B^+ \subset (P*Q)^+$, and $A^- * B^- \subset (P*Q)^-$. Therefore $A^+ * B^+$ and $A^- * B^-$ are disjoint sets (up to a null set) in $A * B$. Combining this with (4.1.12) and (4.1.13) we get the inequality

$$\begin{aligned} |A * B| &\geqslant |A^+ * B^+| + |A^- * B^-| \\ &\geqslant (|A^+|^{1/d} + |B^+|^{1/d})^d + (|A^-|^{1/d} + |B^-|^{1/d})^d \\ &= (|A^+| + |A^-|)\left[1 + \left(\frac{|B|}{|A|}\right)^{1/d}\right]^d \\ &= (|A|^{1/d} + |B|^{1/d})^d, \end{aligned}$$

and the theorem is proved for such A and B.

If there is no such hyperplane with equation $P : \{z_1 = a_1\}$ but with equation $P : \{z_2 = a_2\}$, then for any u, v, p, q, $I_1^u = I_1^v = I_1$, $J_1^p = J_1^q = J_1$

and for some $r \neq s$, $\text{int}(I_2^r) \cap \text{int}(I_2^s) = \emptyset$, and we can write

$$A = \bigcup_i I_1 \times I_2^i \times \ldots \times I_d^i = I_1 \times \left(\bigcup_i I_2^i \times \ldots \times I_d^i \right) = I_1 \times \tilde{A}$$

$$B = \bigcup_j J_1 \times J_2^j \times \ldots \times J_d^j = J_1 \times \left(\bigcup_j J_2^j \times \ldots \times J_d^j \right) = J_1 \times \tilde{B}.$$

We have seen in (4.1.10) that

$$|A * B| \geqslant |A *_{z_1', w_1'} B|.$$

Now we repeat the above argument, where now we apply the induction hypothesis to the product $*_{z_1', w_1'}$, thus the sets $A^+ *_{z_1', w_1'} B^+$ and $A^- *_{z_1', w_1'} B^-$ are disjoint (up to a null set). Hence, by (4.1.10) we obtain

$$|A*B| \geqslant |A*_{z_1', w_1'} B| \geqslant |A^+ *_{z_1', w_1'} B^+| + |A^- *_{z_1', w_1'} B^-| \geqslant (|A|^{1/d} + |B|^{1/d}|)^{1/d}$$

and the result is proved.

Repeating this reasoning we have covered the general case where $P : \{z_i = a_i\}$.

Step 3. Let us prove (4.0.1) for A and B are measurable sets such that $A * B$ is measurable. We can suppose that A, B and $A * B$ have finite measure, since otherwise the inequality is trivial. Fix $\varepsilon > 0$ and take an open set O such that $A * B \subset O$ and $|O \setminus A * B| < \varepsilon$. Take open sets $O_A \supset A$ and $O_B \supset B$ such that $|O_A \setminus A| < \varepsilon$ and $|O_B \setminus B| < \varepsilon$. Since $*$ is continuous, we can assume also that $O_A * O_B \subset O$. Now we approximate the open sets O_A and O_B from inside by dyadic d-rectangles, D_A and D_B so that $|O_A \setminus D_A| < \varepsilon$, $|O_B \setminus D_B| < \varepsilon$. Using step 2 for D_A and D_B, we obtain

$$\begin{aligned}(|A * B| + \varepsilon)^{1/d} &\geqslant |O|^{1/d} \geqslant |O_A * O_B|^{1/d} \geqslant |D_A * D_B|^{1/d} \\ &\geqslant |D_A|^{1/d} + |D_B|^{1/d} \geqslant (|A| - 2\varepsilon)^{1/d} + (|B| - 2\varepsilon)^{1/d}.\end{aligned}$$

Taking $\varepsilon \to 0$ we obtain (4.0.1). □

As a particular case, we have the Brunn-Minkowski inequality in nilpotent groups.

Theorem 4.1.3 (Brunn-Minkowski inequality in nilpotent groups). *Let G be a simply connected nilpotent group of topological dimension d with Haar measure μ and let $A, B \subset G$ be measurable sets such that $A \cdot B$ is measurable. Then we have*

$$\mu(A \cdot B)^{1/d} \geqslant \mu(A)^{1/d} + \mu(B)^{1/d}. \tag{4.1.14}$$

Proof. We denote $\mathbb{A} = \log(A)$, $\mathbb{B} = \log(B)$. Using Proposition 2.2.4 and Theorem 4.0.1, we have

$$\begin{aligned}\mu(A \cdot B) &= |\log(A \cdot B)| = |\log(\exp(\mathbb{A}) \cdot \exp(\mathbb{B}))| = |\mathbb{A} * \mathbb{B}| \\ &\geqslant (|\mathbb{A}|^{1/d} + |\mathbb{B}|^{1/d})^d = (\mu(A)^{1/d} + \mu(B)^{1/d})^d.\end{aligned}$$ □

Remark 4.1.4. Since the right-hand side of (4.1.14) is symmetric in A and B, it follows

$$\min\{\mu(A \cdot B), \mu(B \cdot A)\}^{1/d} \geqslant \mu(A)^{1/d} + \mu(B)^{1/d}.$$

An example where $\mu(A \cdot B)$ and $\mu(B \cdot A)$ are different can be found in [108].

4.1.1 A sufficient condition for strict inequality in the Heisenberg group

A set A in the Heisenberg group $\mathbb{H}^1$ of the form $A = A_1 \times A_2$, where A_1 is a measurable set in $\mathbb{R}^2$ and A_2 is a measurable set in $\mathbb{R}$ is called a *generalized cylinder*.

In this subsection we prove in Proposition 4.1.5 that the Brunn-Minkowski inequality (4.1.14) is strict in $\mathbb{H}^1$ for a pair of generalized cylinders A and B such that the volumes of A_1 and B_1 are positive.

Recall that a point a in $\mathbb{R}^d$ is a *density point of* A if

$$\lim_{r \to 0^+} \frac{|A \cap B(a,r)|}{|B(a,r)|} = 1,$$

where $B(a,r)$ is the ball of center a and radius r with the CC distance. The set of density points of a set A will be denoted as A^o. We can always normalize a set by including its density points in the set. The existence of a density point in A implies that the volume of A is positive.

Proposition 4.1.5. *Let $A, B \subset \mathbb{H}^1$ be generalized cylinders such that $A \cdot B$ and $A + B$ are measurable. Suppose that $|A_1| > 0$ and $|B_1| > 0$. Then*

$$|A \cdot B| > |A + B|. \tag{4.1.15}$$

Proof. By Fubini's theorem, we have

$$|A \cdot B| = \int_{A_1 + B_1} h(s_1) ds_1,$$

where $h(s_1) = |\{t + t' + \operatorname{Im}(z\overline{(s_1 - z)}) : t \in A_2,\ t' \in B_2,\ z \in K(s_1)\}|_1$ and $K(s_1) = I \cap (s_1 - J)$. Denoting $s_1 = (s_x, s_y)$, we can see that $\operatorname{Im}(z\overline{(s_1 - z)}) = \operatorname{Im}(z\overline{s_1}) = ys_x - xs_y$. We write

$$I_{s_1} = \{ys_x - xs_y : (x,y) \in K(s_1)\}.$$

By the Brunn-Minkowski inequality in $\mathbb{R}$,

$$\begin{aligned} h(s_1) &= |\{s_2 + \operatorname{Im}(z\overline{s_1}) : s_2 \in A_2 + B_2,\ z \in K(s_1)\}|_1 \\ &= |\{s_2 + a : s_2 \in A_2 + B_2,\ a \in I_{s_1}\}|_1 \\ &\geqslant |A_2 + B_2|_1 + |I_{s_1}|_1. \end{aligned}$$

We assert that if $|K(s_1)|_2 > 0$, then $|I_{s_1}|_1 > 0$. To see that, we can take the diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto \left(ys_x - xs_y, \frac{x}{2s_x} - \frac{y}{2s_y}\right)$. Then $|\mathrm{Jac}(\phi)| = 1$ and applying the change of variables formula to ϕ^{-1}, we have

$$0 < |K(s_1)|_2 = \int_{\mathbb{R}^2} \chi_{K(s_1)}(z)dz = \int_{\mathbb{R}^2} \chi_{\phi(K(s_1))}(z)dz = |\phi(K(s_1))|_2.$$

Now we use that, for any set $O \subseteq \mathbb{R}^2$ with $|O|_2 > 0$, it holds that $|\pi_1(O)|_1 > 0$ where $\pi_1(x, y) = x$, since $|\pi_1(O)|_1 = 0$ implies $|O|_2 \leqslant |\pi_1(O) \times \mathbb{R}|_2 = 0$. Hence

$$|I_{s_1}|_1 = |\pi_1(\phi(K(s_1)))|_1 > 0.$$

To complete the proof it remains to show that $\{s_1 \in A_1 + B_1 : |K(s_1)|_2 > 0\}$ has positive measure. Let $a \in A_1^o$, $b \in B_1^o$ and $s_1 = a + b \in A_1^o + B_1^o$. Then $a = s_1 - b$ is a density point in $s_1 - B_1$ and therefore a is a density point in $A_1 \cap (s_1 - B_1) = K(s_1)$ which implies that $|K(s_1)|_2 > 0$. Finally $A_1^o + B_1^o \subseteq A_1 + B_1$ has positive measure since $|A_1^o + B_1^o|_2 \geqslant |A_1^o|_2 = |A_1|_2 > 0$, and

$$|\{s_1 \in A_1 + B_1 : |K(s_1)|_2 > 0\}|_2 \geqslant |\{s_1 \in A_1^o + B_1^o : |K(s_1)|_2 > 0\}|_2 > 0. \quad \square$$

Remark 4.1.6. In order to characterize the equality in (4.1.14) for generalized cylinders, we can distinguish several cases. If A and B lie in parallel vertical hyperplanes, then $|A \cdot B| = 0$ and we have equality in (4.1.14). If A and B are convex and homothetic then either $|A_1|_2 > 0$ and B_1 is a point and the equality holds, or $|A_1|_2 > 0$ and $|B_1|_2 > 0$, and therefore, by Proposition 4.1.5 jointly with the (Euclidean) Brunn-Minkowski inequality, equality does not hold in (4.1.14). The same argument works if A and B lie in horizontal hyperplanes with $|A_1|_2 > 0$ and $|B_1|_2 > 0$. The case in which A and B lie in horizontal hyperplanes with $|A_1|_2 = 0$ is not known in general.

4.2 Consequences

Another equivalent version of the Brunn-Minkowski inequality in Euclidean space is the Prékopa-Leindler inequality. Now we show how the proof of the Prékopa-Leindler inequality from the Brunn-Minkowski inequality can be adapted to the case of nilpotent groups.

Theorem 4.2.1 (Prékopa-Leindler inequality in nilpotent groups)**.** *Let G be a simply connected nilpotent group of topological dimension d with Haar measure μ. Let $f, g, h : G \to \mathbb{R}_0^+$ be measurable functions and $0 < \alpha < 1$ verifying*

$$h(a \cdot b) \geqslant f(a)^{1-\alpha} g(b)^{\alpha} \quad \forall a, b \in G. \tag{4.2.1}$$

Then

$$\int_G h d\mu \geqslant \frac{1}{(1-\alpha)^{d(1-\alpha)}\alpha^{d\alpha}} \left(\int_G f d\mu\right)^{1-\alpha} \left(\int_G g d\mu\right)^{\alpha}. \tag{4.2.2}$$

Proof. We proceed by induction on d.

Let $d = 1$ and $a \cdot b \in \{f > \lambda\} \cdot \{g > \lambda\}$. Then we have $h(a \cdot b) \geqslant f(a)^{1-\alpha} g(b)^{\alpha} > \lambda$, and as a consequence

$$\{h > \lambda\} \supset \{f > \lambda\} \cdot \{g > \lambda\}.$$

Now we can apply Theorem 4.1.3 to get

$$\mu(\{h > \lambda\}) \geqslant \mu(\{f > \lambda\}) + \mu(\{g > \lambda\}).$$

Integrating in λ and using Cavalieri's Principle,

$$\int_G h d\mu = \int_0^{\infty} \mu(\{h > \lambda\}) d\lambda \geqslant \\ \int_0^{\infty} \big(\mu(\{f > \lambda\}) + \mu(\{g > \lambda\})\big) d\lambda = \int_G f d\mu + \int_G g d\mu. \tag{4.2.3}$$

Now we use the weighted inequality between the geometric and arithmetic means,

$$\int_G f d\mu + \int_G g d\mu \geqslant \left(\frac{\int_G f d\mu}{1-\alpha}\right)^{1-\alpha} \left(\frac{\int_G g d\mu}{\alpha}\right)^{\alpha}. \tag{4.2.4}$$

From (4.2.3) and (4.2.4) we have (4.2.2).

Suppose that Theorem 4.2.1 holds for $d-1$. We shall prove (4.2.4) for the functions f, g, h composed with exp and use Proposition 2.2.4. Let $z' = (z_1, \ldots, z_{d-1})$, $w' = (w_1, \ldots, w_{d-1}) \in \mathbb{R}^{d-1}$. By (2.2.1), we can write $(z', z_d) * (w', w_d) = (z' *' w', z_d + w_d + P_d(z', w'))$. Recall that $\mathbb{R}^d$ is isomorphic to $\mathfrak{g}$ once we fix the strong Malcev basis $\{X_1, \ldots, X_d\}$, and X_d spans an ideal $\mathbb{H}_1$ in $\mathfrak{g}$. Thus $\mathfrak{g}/\mathbb{H}_1 \cong (\mathbb{R}^{d-1}, *')$ is a nilpotent group. Now we define the functions $\tilde{f}, \tilde{g}, \tilde{h} : \mathbb{R} \to \mathbb{R}_0^+$ by

$$\tilde{f}(z_d) = (f \circ \exp)(z', z_d), \\ \tilde{g}(w_d) = (g \circ \exp)(w', w_d), \\ \tilde{h}(t) = (h \circ \exp)(z' *' w', t + P_d(z', w')).$$

Let us see that these functions verify (4.2.1):

$$\begin{aligned}\tilde{h}(z_d + w_d) &= (h \circ \exp)((z', z_d) * (w', w_d)) = h(\exp(z', z_d) \cdot \exp(w', w_d)) \\ &\geqslant (f \circ \exp)^{1-\alpha}(z', z_d)(g \circ \exp)^{\alpha}(w', w_d) = \tilde{f}^{1-\alpha}(z_d)\tilde{g}^{\alpha}(w_d). \quad (4.2.5)\end{aligned}$$

By induction hypothesis,

$$\int_{\mathbb{R}} \tilde{h}(t)dt \geqslant \frac{1}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}} \left(\int_{\mathbb{R}} \tilde{f}(z_d)dz_d\right)^{1-\alpha} \left(\int_{\mathbb{R}} \tilde{g}(w_d)dw_d\right)^{\alpha}. \quad (4.2.6)$$

By the invariance of the 1-dimensional Lebesgue measure by translations we get

$$\int_{\mathbb{R}} (h \circ \exp)(z' *' w', t)dt = \int_{\mathbb{R}} \tilde{h}(t)dt. \quad (4.2.7)$$

Inequality (4.2.5) is valid for any $z', w' \in \mathbb{R}^{d-1}$, and we can define the functions $F, G, H : \mathbb{R}^{d-1} \to \mathbb{R}_0^+$ given by

$$\begin{aligned} F(z') &= \frac{1}{(1-\alpha)} \int_{\mathbb{R}} \tilde{f}(z_d)dz_d \\ G(w') &= \frac{1}{\alpha} \int_{\mathbb{R}} \tilde{g}(w_d)dw_d \\ H(z') &= \int_{\mathbb{R}} (h \circ \exp)(z', t)dt. \end{aligned} \quad (4.2.8)$$

Applying (4.2.7) we can rewrite (4.2.6) as

$$H(z' *' w') = \int_{\mathbb{R}} \tilde{h}(t)dt \geqslant F(z')^{1-\alpha}G(w')^{\alpha} \quad \forall z', w' \in \mathbb{R}^{d-1},$$

and again by the induction hypothesis, we get

$$\int_{\mathbb{R}^{d-1}} H(z')dz' \geqslant \frac{1}{(1-\alpha)^{(d-1)(1-\alpha)}\alpha^{(d-1)\alpha}} \left(\int_{\mathbb{R}^{d-1}} F(z')dz'\right)^{1-\alpha} \left(\int_{\mathbb{R}^{d-1}} G(w')dw'\right)^{\alpha}.$$

The result now follows from Fubini's theorem. □

The Prékopa-Leindler inequality in $\mathbb{R}^d$ is usually stated using $h((1-\alpha)x + \alpha y)$ instead of $h(x+y)$ in order to eliminate the factor $((1-\alpha)^{d(1-\alpha)}\alpha^{d\alpha})^{-1}$. This can be done when dilations are defined, and in this case, this inequality take a more pleasant expression.

Corollary 4.2.2. *Let G be a stratifiable group of topological dimension d with Haar measure μ and homogeneous dimension Q. Let $f, g, h : G \to \mathbb{R}_0^+$ be measurable functions, and $0 < \alpha < 1$ verifying*

$$h(\delta_{(1-\alpha)}a \cdot \delta_{\alpha}b) \geqslant f(a)^{1-\alpha}g(b)^{\alpha} \quad \forall a, b \in G.$$

Then

$$\int_G h d\mu \geqslant (1-\alpha)^{(Q-d)(1-\alpha)} \alpha^{(Q-d)\alpha} \left(\int_G f d\mu \right)^{1-\alpha} \left(\int_G g d\mu \right)^{\alpha}.$$

Proof. For the sake of simplicity, $\delta_\lambda(a)$ will be just written as λa for any $\lambda > 0$ and $a \in G$. We denote $a' = (1-\alpha)a$, $b' = \alpha b$, $f_{1-\alpha}(a) = f(\frac{a}{1-\alpha})$ and $g_\alpha(a) = g(\frac{a}{\alpha})$. Then we have

$$h(a' \cdot b') \geqslant f(a)^{1-\alpha} g(b)^{\alpha} = f\left(\frac{a'}{1-\alpha}\right)^{1-\alpha} g\left(\frac{b'}{\alpha}\right)^{\alpha} = f_{1-\alpha}(a')^{1-\alpha} g_\alpha(b')^{\alpha}.$$

By Theorem 4.2.1, we have

$$\int_G h d\mu \geqslant \frac{1}{(1-\alpha)^{d(1-\alpha)} \alpha^{d\alpha}} \left(\int_G f_{1-\alpha} d\mu \right)^{1-\alpha} \left(\int_G g_\alpha d\mu \right)^{\alpha}.$$

Using now Proposition 2.2.11,

$$\int_G f_{1-\alpha}(a) d\mu(a) = \int_G f\left(\frac{a}{1-\alpha}\right) d\mu(a) = (1-\alpha)^Q \int_G f(a') d\mu(a'),$$

and after using also Proposition 2.2.11 for the integral of g_α, we obtain

$$\int_G h d\mu \geqslant (1-\alpha)^{(Q-d)(1-\alpha)} \alpha^{(Q-d)\alpha} \left(\int_G f d\mu \right)^{1-\alpha} \left(\int_G g d\mu \right)^{\alpha}. \qquad \square$$

As we can find in [156], there are several equivalent statements for the Brunn-Minkowski inequality in Euclidean space. Similarly, we have the following result.

Corollary 4.2.3 (Multiplicative Brunn-Minkowski inequalities in Carnot groups)**.** *Let G be a Carnot group of topological dimension d with Haar measure μ and homogeneous dimension Q. Let $A, B \subset G$ be measurable sets such that $A \cdot B$ is measurable, and $0 < \alpha < 1$. Then*

$$\mu(\delta_{(1-\alpha)}A \cdot \delta_\alpha B)^{1/d} \geqslant (1-\alpha)^{Q/d} \mu(A)^{1/d} + \alpha^{Q/d} \mu(B)^{1/d}.$$
$$\mu(\delta_{(1-\alpha)}A \cdot \delta_\alpha B) \geqslant (1-\alpha)^{(Q-d)(1-\alpha)} \alpha^{(Q-d)\alpha} \mu(A)^{1-\alpha} \mu(B)^{\alpha}.$$

Proof. We use Theorem 4.1.3 with the sets $\delta_{(1-\alpha)}A$ and $\delta_\alpha B$, and from Proposition 2.2.11 we get the first inequality.

For the second one, we take $f = \chi_A$, $g = \chi_B$ and $h = \chi_{\delta_{(1-\alpha)}A \cdot \delta_\alpha B}$ and apply Corollary 4.2.2, obtaining the result. $\square$

Notes

Notes of § 4.1 1. Theorem 4.0.1 is an extension of the result obtained by Leonardi and Masnou [108] in Heisenberg groups. They prove the theorem first for the case where A and B are cubes in $\mathbb{R}^{2n+1}$ of the form $A_1 \times A_2$ where A_1 is a dyadic cube in $\mathbb{R}^{2n}$ and A_2 is a measurable set in $\mathbb{R}$, then when A and B are unions of a finite number of cubes, using then an approximation argument. This has the crucial property that either exists a vertical hyperplane that separates cubes or the union is a cube itself. We call a hyperplane vertical when is also a hyperplane after left multiplication. Then we can consider only vertical hyperplanes to separate cubes. In $\mathbb{R}^d$ with a product of the form (*) this property is not true, since the union of the cubes takes the form

$$\bigcup_i I_1 \times \ldots \times I_{n_1} \times I^i_{n_1+1} \times \ldots \times I^i_d = I_1 \times \ldots \times I_{n_1} \times \left(\bigcup_i I^i_{n_1+1} \times \ldots \times I^i_d \right).$$

This set is not of the form $A_1 \times A_2$ and the argument fails. While the proof of Leonardi and Masnou only works in Heisenberg groups, this arguments can be seen as the first step of an induction argument developed in this chapter.

The exponent of (4.0.1) cannot be improved to Q^{-1}. Indeed, if we replace the exponent d^{-1} by Q^{-1}, Monti [124] proved that geodesic balls would be isoperimetric sets among sufficiently regular sets, which are not. By sufficiently regular sets we refer to sets whose Minkowski content coincides with its perimeter, a property that holds whenever the boundary is C^2, see [127].

Juillet [102] developed a method to disprove Brunn-Minkowski inequalities, first connected to the geodesic Brunn-Minkowski inequality and the measure contraction property of metric measure spaces. He used this method to disprove the Brunn-Minkowski inequality in $\mathbb{H}^n$ with any exponent between $(2n+1)^{-1}$ and $(2n+2)^{-1}$.

After [141] was completed, the author was informed that Theorem 4.0.1 was also proven by Bobkov [16] in 2011, where he used Knothe's map to get the Brunn-Minkowski inequality for convex sets and obtained the general result after proving the equivalent analytic version of the theorem, the Prékopa-Leindler inequality.

Notes of § 4.2 1. Terence Tao [160; 161] posted an entry in his blog in 2011 explaining how to produce a Prékopa-Leindler inequality in any nilpotent Lie group of topological dimension d, which provides a natural way to prove the multiplicative Brunn-Minkowski inequality with exponent d^{-1}. We remark that the path developed in this chapter is the opposite and is equivalent.

Chapter 5

Pansu-Wulff shapes in $\mathbb{H}^1$

This chapter gather the results obtained with Ritoré in [142].

We consider a convex set K with boundary of class C^2_+ (i.e., so that ∂K is of class C^2 and ∂K has positive geodesic curvature everywhere) and an open region $\Omega \subseteq \mathbb{H}^1$ with boundary $\partial\Omega = S$ of class C^2. We compute the first variation of the area functional associated to a vector field U with compact support in the regular part of S to get

$$A'_K(0) = \int_S u \left(\operatorname{div}_S \eta_K \right) dS.$$

In this formula $u = \langle U, N \rangle$ is the normal component of the variation and $\operatorname{div}_S \eta_K$ is the divergence on S of the vector field $\eta_K = \pi_K(\nu_h)$, where $\nu_h = N_h/|N_h|$ is the horizontal unit normal and π_K is the projection map defined in Subsection 2.1.1.

The function $H_K = \operatorname{div}_S \eta_K$ is called the *mean curvature* of S. Further calculations imply that H_K is equal to $\langle D_Z \eta_K, Z \rangle$, where $Z = -J(\nu_h)$ is the horizontal direction on the regular part of S. Hence the mean curvature function is localized on the horizontal curves of S. It is not difficult to check that a horizontal curve in a surface with mean curvature H_K must satisfy a differential equation depending on H_K. Hence we can reconstruct the regular part of a surface with prescribed mean curvature by taking solutions of this differential equation. Furthermore, we might be able classify surfaces with prescribed mean curvature by classifying solutions of this ordinary differential equation and by looking at the interaction of these curves with the singular set S_0 of S composed of the points where the tangent plane is horizontal, as was done in [151] for the standard sub-Riemannian perimeter.

Key observations are that horizontal straight lines are solutions of the differential equation for $H_K = 0$ and that horizontal liftings of the curve $|\cdot|_K = 1$ are solutions for $H_K = 1$. The strict convexity of ∂K together with the invariance of the equation by left-translations and dilations imply that all solutions are of this type.

Hence, given a convex body $K \subset \mathbb{R}^2$ containing 0 in its interior and its associated left–invariant norm $|\cdot|_K$, we consider the set $\mathbb{B}_K$ obtained as the ball enclosed by the horizontal liftings of all translations of the curve ∂K containing 0. It is not difficult to prove that this way we obtain a topological sphere $\mathbb{S}_K$ with two poles on the same vertical line, that is the union of two graphs. Moreover the boundary of $\mathbb{B}_K$ is C^2 outside the poles (indeed C^ℓ if the boundary of K is of class C^ℓ, $\ell \geqslant 2$) and of regularity C^2 around the poles. When $K = D$, these sets were build by P. Pansu [137] and are frequently referred to as Pansu spheres. We remark that Pansu spheres' $\mathbb{B}_D$ are of class C^2 but not C^3 near the singular points, see Proposition 3.15 in [46] and Example 3.3 in [151].

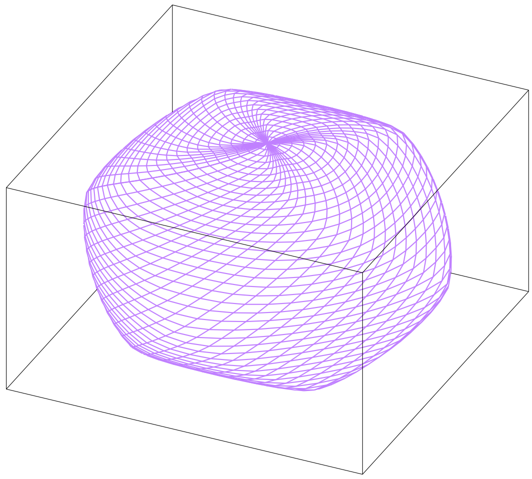

Figure 5.1: The set $\mathbb{B}_K$ when K is the unit ball of the r-norm $||(x,y)||_r = \big(|x|^r + |y|^r\big)^{1/r}$, $r = 1.5$

We observe that these objects have constant mean curvature. Hence they are critical points of the sub-Finsler area functional under a volume constraint. Further evidence that they have stronger minimization properties is given in Section 5.5, where it is proven that, under a geometric condition, a set of finite perimeter E with volume equal to the volume of $\mathbb{B}_K$ has perimeter larger than or equal to the one of the ball $\mathbb{B}_K$. A slightly weaker result for the Euclidean norm was proven in [147]. Moreover, at the end of Section 5.5 it is proven that

This chapter is organized into several sections. In the next one we fix notation and give some background, focusing specially in properties of the sub-Finsler perimeter. In Section 5.2 we compute the first variation of perimeter for surfaces of class C^2 and prove the property that the regular part of the surface is foliated by horizontal liftings of translations of homothetic expansions of ∂K. In Section 5.3 we define the Pansu-Wulff shapes and compute some examples and prove regularity properties of these objects. In

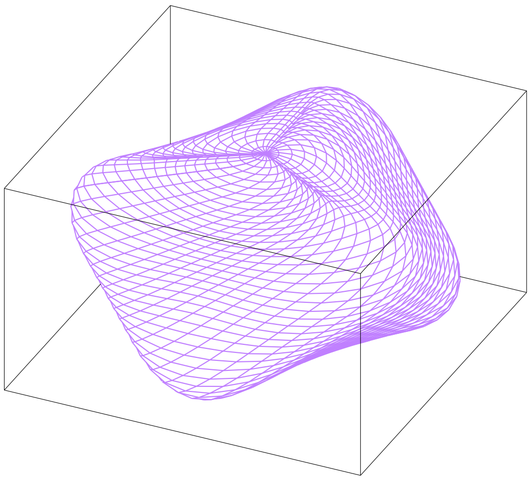

Figure 5.2: The set $\mathbb{B}_K$ when K is a smooth approximation of the triangular norm

Section 5.4 we study some geometric properties of the Pansu-Wulff shapes and in Section 5.5 we obtain a minimization property of these Pansu-Wulff shapes. Finally, in Subsection 5.5.1 we prove that Pansu-Wulff shapes are the only minimizers of perimeter under a slightly stronger geometric condition. This property indicates that these shapes are good candidates to be solutions of the sub-Finsler isoperimetric problem in $\mathbb{H}^1$.

5.1 The pseudo-hermitian connection

The *pseudo-hermitian* connection ∇ on $\mathbb{H}^1$ is the only affine connection satisfying the following properties:

1. ∇ is a metric connection,
2. $\mathrm{Tor}(U, V) = 2\,\langle J(U), V\rangle T$ for all vector fields U, V.

The existence of the pseudo-hermitian connection can be easily obtained adapting the proof of existence of the Levi-Civita connection, see Theorem 3.6 in [57].

We shall use the following relation between the pseudo-hermitian and the Levi-Civita connections.

Lemma 5.1.1. *Let U, V and W be vector fields where V and W are horizontal. Then the following equation holds*

$$\langle \nabla_U V, W\rangle = \langle D_U V, W\rangle + \langle J(W), V\rangle\langle T, U\rangle. \tag{5.1.1}$$

In particular

$$\nabla_U V = D_U V - \langle T, U\rangle J(V). \tag{5.1.2}$$

Proof. By Koszul formula, see § 3 in [57]. The terms in the first two lines are equal to $\langle D_U V, W\rangle$. The last three terms can be computed using the expression for the torsion to get

$$\langle J(W), V\rangle\langle T, U\rangle.$$

This proves (5.1.1). □

Using Koszul formula it can be easily seen that $\nabla X = \nabla Y = 0$.

Corollary 5.1.2. *Let $\gamma : I \to S$ be a curve on $\mathbb{H}^1$ and let ∇/ds, D/ds be the covariant derivatives induced by the pseudo-hermitian connection and the Levi-Civita connection in γ, respectively. Let V be a vector field along γ. Then we have*

$$\frac{\nabla}{ds}V = \frac{D}{ds}V - \langle\dot{\gamma}, T\rangle J(V). \tag{5.1.3}$$

In particular, if γ is a horizontal curve, the covariant derivatives coincide.

5.2 First variation of sub-Finsler area

In this section we fix a convex body $K \subset \mathbb{R}^2$ containing 0 in its interior with C^2_+ boundary and consider its associated left-invariant norm $|\cdot|_K$ in $\mathbb{H}^1$. Since the convex body is fixed, we drop the subscript along this section.

Let S be an oriented C^2 surface immersed in $\mathbb{H}^1$. Let U be a C^2 vector field with compact support on S, normal component $u = \langle U, N\rangle$ and associated one-parameter group of diffeomorphisms $\{\varphi_s\}_{s\in\mathbb{R}}$. In this subsection we compute the first variation of the sub-Finsler area $A(s) = A(\varphi_s(S))$. More precisely

Theorem 5.2.1. *Let S be an oriented C^2 surface immersed in $(\mathbb{H}^1, K)$. Let U be a C^2 vector field with compact support on S, normal component $u = \langle U, N\rangle$ and $\{\varphi_s\}_{s\in\mathbb{R}}$ the associated one-parameter group of diffeomorphisms. Let $\eta = \pi(\nu_h)$. Then we have*

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S \big(u \operatorname{div}_S \eta - 2u\langle N, T\rangle\langle J(N_h), \eta\rangle\big)\, dS \\ - \int_S \operatorname{div}_S\big(u\eta^\top\big)\, dS, \end{aligned} \tag{5.2.1}$$

where div_S *is the Riemannian divergence in S, and the superscript $\top$ indicates the tangent projection to S.*

In the proof of Theorem 5.2.1 we shall make use of the following Lemma and its consequences.

Lemma 5.2.2. *Let $\gamma : I \to \mathbb{H}^1$ be a C^1 curve, where $I \subset \mathbb{R}$ is an open interval, and V a horizontal vector field along γ. We have*

$$\frac{d}{ds}|V|_* = \langle \frac{D}{ds}V, \pi(V)\rangle + \langle \gamma', T_\gamma\rangle\langle V, J(\pi(V))\rangle. \tag{5.2.2}$$

Proof. We fix $s_0 \in I$ and let $p = \gamma(s_0)$. Assume that $\pi(V(s_0)) = aX_p + bY_p$, for some $a, b \in \mathbb{R}$. Take the vector field $W(s) := aX_{\gamma(s)} + bY_{\gamma(s)}$ along γ. It coincides with $\pi(V(s_0))$ when $s = s_0$, and it is the restriction to γ of the left-invariant vector field $aX + bY$. In particular, $|(aX + bY)_{\gamma(s)}|_{\gamma(s)} = 1$ for all $s \in I$. Hence

$$|V(s)|_* \geqslant \langle V(s), (aX + bY)_{\gamma(s)}\rangle \quad \text{for all } s \in I,$$

and, since equality holds in the above inequality when $s = s_0$, we have

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=s_0} |V(s)|_* &= \frac{d}{ds}\bigg|_{s=s_0} \langle V(s), (aX + bY)_{\gamma(s)}\rangle \\ &= \langle \frac{\nabla}{ds}\bigg|_{s=s_0} V(s), \pi(V(s_0))\rangle \end{aligned}$$

since

$$\frac{\nabla}{ds}\bigg|_{s=s_0} (aX + bY)_{\gamma(s)} = a\nabla_{\gamma'(s_0)}X + b\nabla_{\gamma'(s_0)}Y = 0.$$

The result follows from the relation between the covariant derivatives given in Equation (5.1.3). □

Remark 5.2.3. In the proof of Lemma 5.2.2 we have obtained the equality

$$\frac{d}{ds}|V|_* = \langle \frac{\nabla}{ds}V, \pi(V)\rangle$$

for a horizontal vector field V along a curve γ. Since ∇ is a metric connection, we also have

$$\frac{d}{ds}|V|_* = \langle \frac{\nabla}{ds}V, \pi(V)\rangle + \langle V, \frac{\nabla}{ds}\pi(V)\rangle.$$

Hence we get

$$\langle V, \frac{\nabla}{ds}\pi(V)\rangle = 0 \tag{5.2.3}$$

for a horizontal vector field V along γ, where ∇/ds is the covariant derivative induced by the pseudo-hermitian connection on γ. Taking into account the relation between the Levi-Civita and pseudo-hermitian connections we deduce from (5.2.3) and (5.1.3)

$$\langle V, \frac{D}{ds}\pi(V) - \langle \dot\gamma, T_\gamma\rangle J(\pi(V))\rangle = 0. \tag{5.2.4}$$

The following is an easy consequence of Lemma 5.2.2

Corollary 5.2.4. *Let F be a vector field tangent to S and γ an integral curve of F. We have*

$$\langle \frac{D}{ds}\eta_\gamma, \nu_h\rangle = -\langle F, T\rangle\langle \eta, J(\nu_h)\rangle. \tag{5.2.5}$$

In particular, if F is horizontal,

$$\langle \frac{D}{ds}\eta_\gamma, \nu_h\rangle = 0. \tag{5.2.6}$$

Proof. We take $V = \nu_h$ and we get (5.2.5) from equation (5.2.4). □

Proof of Theorem 5.2.1. Standard variational arguments, see the proof of Lemma 4.3 in [151], yield

$$A'(0) = \frac{d}{ds}\Big|_{s=0} A(\varphi_s(S)) = \int_S \left(\frac{d}{ds}\Big|_{s=0} |(N_s)_h|_* + |N_h|_* \operatorname{div}_S U\right) dS,$$

where N_s is a smooth choice of unit normal to $\varphi_s(S)$ for small s. We fix a point $p \in S$ and consider the curve $\gamma(s) = \varphi_s(p)$. Lemma 5.2.2 now implies

$$\frac{d}{ds}\Big|_{s=0} |(N_s)_h|_* = \langle \frac{D}{ds}\Big|_{s=0} (N_s)_h, \eta_p\rangle + \langle U_p, T_p\rangle\langle (N_h)_p, J(\eta_p)\rangle,$$

By the definition of $(N_s)_h$ we also have

$$\frac{D}{ds}\Big|_{s=0} (N_s)_h = \frac{D}{ds}\Big|_{s=0} \big(N_s - \langle N_s, T\rangle T\big),$$

where N_s is the Riemannian unit normal to $\varphi_s(S)$. A well-known lemma in Riemannian geometry implies

$$\frac{D}{ds}\Big|_{s=0} N_s = -(\nabla_S u)(p) - A_S(U_p^\top),$$

where A_S is the Weingarten endomorphism of S. Since $\frac{D}{ds}\big|_{s=0} T = J(U_p)$ and η is horizontal, calling

$$B(U) = -\langle N, T\rangle\langle J(U), \eta\rangle + \langle U, T\rangle\langle N_h, J(\eta)\rangle,$$

we get

$$\begin{aligned}\tfrac{D}{ds}\big|_{s=0} |(N_s)_h|_* &= \big(\langle -\nabla_S u - A_S(U^\top), \eta\rangle\big)_p + B(U_p)\\ &= -\langle \nabla_S u, \eta\rangle_p + B(U_p^\perp) + \big(-\langle A_S(U^\perp), \eta\rangle_p + B(U_p^\top)\big)\\ &= \big(-\langle \nabla_S u, \eta\rangle - 2u\langle N, T\rangle\langle J(N_h), \eta\rangle\big)_p + U_p^\top(|N_h|_*).\end{aligned}$$

Observe that

$$\begin{aligned}-\langle\nabla_S u,\eta\rangle &= u\operatorname{div}_S\eta-\operatorname{div}_S(u\eta)\\ &= u\operatorname{div}_S\eta-\operatorname{div}_S(u\eta^\top)-\operatorname{div}_S(u\langle N,\eta\rangle N)\\ &= u\operatorname{div}_S\eta-\operatorname{div}_S(u\eta^\top)-u|N_h|_*\operatorname{div}_S N.\end{aligned}$$

Hence we get

$$\begin{aligned}A'(0)=\int_S\big(u\operatorname{div}_S\eta-2u\langle N,T\rangle\langle J(N_h),\eta\rangle\big)\,dS\\ +\int_S\operatorname{div}_S\big(|N_h|_*U^\top-u\eta^\top\big)\,dS.\end{aligned}$$

From here we obtain formula (5.2.1) since the integral $\int_S|N_h|_*U^\top dS$ is equal to 0 by the divergence theorem for Lipschitz vector fields. □

Now we simplify the first term appearing in the first variation formula (5.2.1).

Lemma 5.2.5. *Let S be a C^2 surface immersed in $(\mathbb{H}^1,K)$ with unit normal N horizontal unit normal ν_h. Let $Z=J(\nu_h)$. Then we have*

$$\operatorname{div}_S\eta-2\langle N,T\rangle\langle J(N_h),\eta\rangle=\langle D_Z\eta,Z\rangle. \tag{5.2.7}$$

Proof. Let us consider the orthonormal basis in $S\setminus S_0$ given by the vector fields $Z=-J(\nu_h)$ and $E=\langle N,T\rangle\nu_h-|N_h|T=a\nu_h+bT$. Using equation (5.2.5) with $F=E$, we get

$$\begin{aligned}\langle D_E\eta,E\rangle &= a\langle D_E\eta,\nu_h\rangle+b\langle D_E\eta,T\rangle\\ &= -a\langle E,T\rangle\langle\eta,J(\nu_h)\rangle+b\big(E(\langle\eta,T\rangle)-\langle\eta,D_ET\rangle\big)\\ &= -ab\langle\eta,J(\nu_h)\rangle-ab\langle\eta,J(\nu_h)\rangle\\ &= -2ab\langle\eta,J(\nu_h)\rangle,\end{aligned}$$

as $D_ET=J(E)=aJ(\nu_h)=-aZ$. From $ab=-\langle N,T\rangle|N_h|$ we obtain

$$\langle D_E\eta,E\rangle=2\langle N,T\rangle\langle\eta,J(N_h)\rangle.$$

Taking into account this equation and that $\operatorname{div}_S\eta=\langle D_Z\eta,S\rangle+\langle D_E\eta,E\rangle$, we obtain equation (5.2.7). □

Definition 5.2.6. Given an oriented surface S immersed in $(\mathbb{H}^1,K)$ endowed with a smooth strictly convex left-invariant norm $|\cdot|_K$, its mean curvature is the function

$$H=\langle D_Z\eta_K,Z\rangle, \tag{5.2.8}$$

defined on $S\setminus S_0$.

Corollary 5.2.7. *Let S be an oriented C^2 surface immersed in $(\mathbb{H}^1, K)$. Let U be a C^2 vector field with compact support on $S \setminus S_0$, normal component $u = \langle U, N\rangle$ and associated one-parameter group of diffeomorphisms $\{\varphi_s\}_{s\in\mathbb{R}}$. Then*

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S uH\, dS,$$

where H is the mean curvature of S defined in (5.2.8).

By equation (5.2.8), a unit speed horizontal curve Γ contained in the regular part of a surface S satisfy the equation

$$\langle \frac{D}{ds}\pi(J(\dot\Gamma)), \dot\Gamma\rangle = H, \tag{5.2.9}$$

where D/ds is the covariant derivative along Γ. Uniqueness of curves Γ satisfying (5.2.9) with given initial conditions $\Gamma(0), \dot\Gamma(0)$ cannot be obtained from (5.2.9). In the next result we prove that the horizontal components of Γ satisfy indeed an ordinary differential equation, thus providing uniqueness with given initial conditions.

Corollary 5.2.8. *Let S be a C^2 oriented surface immersed in $(\mathbb{H}^1, K)$ with mean curvature H. Let $\Gamma : I \to S \setminus S_0$ be a horizontal curve in the regular part of S parameterized by arc-length with $\Gamma(s) = (x_1(s), x_2(s), t(s))$. Then $\gamma(s) = (x_1, x_2)$ satisfies a differential equation of the form*

$$\ddot\gamma = F(\dot\gamma), \tag{5.2.10}$$

where $F(\dot\gamma) = H\,[A(\dot\gamma)](\dot\gamma)$ and A is a nonsingular C^1 matrix of order 2.

Proof. Let $\frac{D}{ds}$ be the covariant derivative along the curve Γ. Since Γ is horizontal and parameterized by arc-length, the vector field $\frac{D}{ds}\dot\Gamma$ along Γ is proportional to $J(\dot\Gamma)$. Then there exists a function $\lambda : I \to \mathbb{R}$ such that

$$\tfrac{D}{ds}\dot\Gamma = \lambda J(\dot\Gamma).$$

Taking scalar product with $\eta = \pi(J(\dot\Gamma))$ we get

$$\lambda = \frac{\langle \frac{D}{ds}\dot\Gamma, \pi(J(\dot\Gamma))\rangle}{|J(\dot\Gamma)|_*} = \frac{\frac{d}{ds}\langle \dot\Gamma, \pi(J(\dot\Gamma))\rangle - H}{|J(\dot\Gamma)|_*}.$$

Hence we have

$$|J(\dot\Gamma)|_* \tfrac{D}{ds}\dot\Gamma - \dot f\, J(\dot\Gamma) = -HJ(\dot\Gamma), \tag{5.2.11}$$

where $f = \langle \dot\Gamma, \pi(J(\dot\Gamma))\rangle$. Since $\dot\Gamma = \dot x_1 X + \dot x_2 Y$, $\frac{D}{ds}\dot\Gamma = \ddot x_1 X + \ddot x_2 Y$, and $J(\dot\Gamma) = -\dot x_2 X + \dot x_1 Y$, equation (5.2.11) is equivalent to the system

$$\begin{aligned} |J(\dot\Gamma)|_*\, \ddot x_1 + \dot f \dot x_2 &= H\dot x_2, \\ |J(\dot\Gamma)|_*\, \ddot x_2 - \dot f \dot x_1 &= -H\dot x_1. \end{aligned} \tag{5.2.12}$$

Let us compute $\dot f = df/ds$. Writing $\pi(aX+bY) = \pi_1(a,b)X + \pi_2(a,b)Y$ we have

$$f = \langle \dot\Gamma, \pi(J(\dot\Gamma))\rangle = \dot x_1 \pi_1(-\dot x_2, \dot x_1) + \dot x_2 \pi_2(-\dot x_2, \dot x_1)$$

and so:

$$\dot f = \left(\pi_1 + \dot x_1 \frac{\partial \pi_1}{\partial x_2} + \dot x_2 \frac{\partial \pi_2}{\partial x_2}\right) \ddot x_1 + \left(\pi_2 - \dot x_1 \frac{\partial \pi_1}{\partial x_1} - \dot x_2 \frac{\partial \pi_2}{\partial x_1}\right) \ddot x_2 = g \ddot x_1 + h \ddot x_2,$$

where the functions π_1, π_2 are evaluated at $(-\dot x_2, \dot x_1)$. Hence equation (5.2.12) is equivalent to

$$\begin{pmatrix} |J(\dot\Gamma)|_* + g\dot x_1 & h\dot x_2 \\ -g\dot x_1 & |J(\dot\Gamma)|_* - h\dot x_1 \end{pmatrix} \begin{pmatrix} \ddot x_1 \\ \ddot x_2 \end{pmatrix} = H \begin{pmatrix} \dot x_2 \\ -\dot x_1 \end{pmatrix} \tag{5.2.13}$$

The determinant of the square matrix in (5.2.13) is equal to

$$|J(\dot\Gamma)|_* \big(|J(\dot\Gamma)|_* + (g\dot x_1 - h\dot x_1)\big).$$

Since

$$g\dot x_1 - h\dot x_2 = \big(\pi_1 \dot x_2 - \pi_2 \dot x_1\big) + \sum_{i,j=1}^{2} \dot x_i \dot x_j \frac{\partial \pi_i}{\partial x_j} = -|J(\dot\Gamma)|_* + \sum_{i,j=1}^{2} \dot x_i \dot x_j \frac{\partial \pi_i}{\partial x_j}$$

we get that the determinant is equal to

$$|J(\dot\Gamma)|_* \sum_{i,j=1}^{2} \dot x_i \dot x_j \frac{\partial \pi_i}{\partial x_j}$$

and we write

$$\sum_{i,j=1}^{2} \dot x_i \dot x_j \frac{\partial \pi_i}{\partial x_j} = \begin{pmatrix} \dot x_1 & \dot x_2 \end{pmatrix} \begin{pmatrix} \partial\pi_1/\partial x_1 & \partial\pi_1/\partial x_2 \\ \partial\pi_2/\partial x_1 & \partial\pi_2/\partial x_2 \end{pmatrix} \begin{pmatrix} \dot x_1 \\ \dot x_2 \end{pmatrix}.$$

Since the kernel of $\big(\partial\pi_i/\partial x_j\big)_{ij}$ is generated by $(-\dot x_2, \dot x_1)$, we have

$$\begin{pmatrix} \partial\pi_1/\partial x_1 & \partial\pi_1/\partial x_2 \\ \partial\pi_2/\partial x_1 & \partial\pi_2/\partial x_2 \end{pmatrix} \begin{pmatrix} \dot x_1 \\ \dot x_2 \end{pmatrix} \neq 0,$$

and, since the image of $\big(\partial\pi_i/\partial x_j\big)_{ij}$ is generated by $(\dot x_1, \dot x_2)$, we get

$$\begin{pmatrix} \dot x_1 & \dot x_2 \end{pmatrix} \begin{pmatrix} \partial\pi_1/\partial x_1 & \partial\pi_1/\partial x_2 \\ \partial\pi_2/\partial x_1 & \partial\pi_2/\partial x_2 \end{pmatrix} \begin{pmatrix} \dot x_1 \\ \dot x_2 \end{pmatrix} \neq 0.$$

So we can invert the matrix in (5.2.13) to get (5.2.10). □

Remark 5.2.9. It is not difficult to prove that

$$\tfrac{D}{ds}\pi(J(\dot\Gamma)) = H\dot\Gamma - |J(\dot\Gamma)|_* T.$$

Indeed it is only necessary to show that $\langle \frac{D}{ds}\pi(J(\dot\Gamma)), J(\dot\Gamma)\rangle = 0$, which follows from (5.2.6) using that $J(\dot\Gamma) = \nu_h$. Observe that the above equation is equivalent to

$$\big[\tfrac{D}{ds}\pi(J(\dot\Gamma))\big]_h = H\dot\Gamma.$$

Writing $\dot\Gamma = \dot x X + \dot y Y$, we have

$$\begin{pmatrix} \partial\pi_1/\partial x_1 & \partial\pi_1/\partial x_2 \\ \partial\pi_2/\partial x_1 & \partial\pi_2/\partial x_2 \end{pmatrix} \begin{pmatrix} -\ddot y \\ \ddot x \end{pmatrix} = H \begin{pmatrix} \dot x \\ \dot y \end{pmatrix}.$$

However, since the determinant of the square matrix is 0 we cannot invert it to obtain an ordinary differential equation for $(\ddot x, \ddot y)$.

Lemma 5.2.10. *Let K be a C^2_+ convex body in $\mathbb{R}^2$ with $0 \in \operatorname{int}(K)$. Let $\gamma : I \to \mathbb{R}^2$ be a unit speed clockwise parameterization of a translation of the unit sphere of ∂K by a vector $v \in \mathbb{R}^2$. Let Γ be a horizontal lifting of z. Then Γ satisfies the equation*

$$1 = \langle \tfrac{D}{ds}\pi(J(\dot\Gamma)), \dot\Gamma\rangle. \qquad (5.2.14)$$

Proof. We have $\pi(J(\dot\Gamma)) = \pi_1(J(\dot\gamma))X + \pi_2(J(\dot\gamma))Y$. Since $J(\dot\gamma)$ is the outer normal to the unit sphere at $\gamma - v$ we have $\gamma - v = \big(\pi_1(J(\dot\Gamma)), \pi_2(J(\dot\Gamma))\big)$. Hence $\frac{D}{ds}\pi(J(\dot\Gamma)) = \dot x X + \dot y Y$ and we get (5.2.14). □

Lemma 5.2.11. *Let K be a C^2_+ convex body in $\mathbb{R}^2$ with $0 \in \operatorname{int}(K)$ and Γ a horizontal curve parameterized by arc-length satisfying the equation*

$$\langle \tfrac{D}{ds}\pi(J(\dot\Gamma)), \dot\Gamma\rangle = H,$$

with $H \in \mathbb{R}$. Then $\sigma(s) = h_\lambda(\Gamma(s/\lambda))$ is parameterized by arc-length and $\langle \frac{D}{ds}\pi(J(\dot\sigma)), \dot\sigma\rangle = H/\lambda$.

Proof. We have $\dot\sigma(s) = \dot\Gamma(s/\lambda)$ and $J(\dot\sigma(s)) = J(\dot\Gamma(s/\lambda))$. □

Remark 5.2.12. Horizontal straight lines are solutions of

$$\langle \tfrac{D}{ds}\pi(J(\dot\Gamma)), \dot\Gamma\rangle = 0$$

since $\dot\Gamma$ is the restriction of a left-invariant vector field in $\mathbb{H}^1$ and so they are $J(\dot\Gamma)$ and $\pi(J(\dot\Gamma))$.

Theorem 5.2.13. *Let K be a C^2_+ convex body in $\mathbb{R}^2$ with $0 \in \operatorname{int}(K)$. Let Γ be a horizontal curve satisfying the equation*

$$\langle \tfrac{D}{ds}\pi(J(\dot\Gamma)), \dot\Gamma\rangle = H, \tag{5.2.15}$$

for some $H \geqslant 0$. Then Γ is either a horizontal straight line if $H = 0$ or the horizontal lifting of a dilation and traslation of a unit speed clockwise parameterization of ∂K in case $H > 0$.

Proof. Horizontal straight lines and horizontal liftings of translations and dilations of ∂K satisfy equation (5.2.15). Uniqueness follow since the projection to $t = 0$ satisfy equation (5.2.10) and, by using translations and dilations, we can obtain any prescribed initial condition. □

Remark 5.2.14. The result in Theorem 5.2.13 includes that constant mean curvature surfaces for the sub-Riemannian area in the Heisenberg group are foliated by geodesics.

To finish this section we prove the following result, that holds trivially for variations supported in the regular part of S.

Proposition 5.2.15. *Let S be a compact C^2 oriented surface in $(\mathbb{H}^1, K)$ enclosing a region E. Assume that S has constant mean curvature H and a finite number of singular points. Then*

1. *S is a critical point of the sub-Finsler area for any volume-preserving variation.*
2. *S is a critical point of the functional $A - H\,|\cdot|$.*

Proof. It is only necessary to prove that if U is a smooth vector field with compact support in $\mathbb{H}^1$ and $\{\varphi_s\}_{s\in\mathbb{R}}$ is its associated flow, then

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S Hu\,dS.$$

From formula (5.2.1) this is equivalent to proving that

$$\int_S \operatorname{div}_S\left(u\eta^\top\right) dS = 0.$$

To compute the integral $\int_S u\eta^\top dS$ we consider the finite number of singular points $p_1, \ldots, p_n$, and take small disjoint balls $B_i(p_i)$ centered at the points p_i. For $\varepsilon > 0$ small enough so that the balls $B_\varepsilon(p_i)$ are contained in B_i we have

$$\int_{S\setminus\bigcup_{i=1}^n B_\varepsilon(p_i)} \operatorname{div} u\eta^\top\, dS = \sum_{i=1}^n \int_{\partial B_\varepsilon(p_i)} \langle \xi_i, u\eta^\top\rangle\, d(\partial B_\varepsilon(p_i)),$$

where ξ_i is the unit inner normal to $\partial B_\varepsilon(p_i)$. Since $u\eta^\top$ is bounded and the lengths of $\partial B_\varepsilon(p_i)$ go to 0 when $\varepsilon \to 0$ we have

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{n}\int_{\partial B_\varepsilon(p_i)}\langle \xi_i, u\eta^\top\rangle\, d(\partial B_\varepsilon(p_i)) = 0.$$

Since the modulus of

$$\begin{aligned}\operatorname{div}_S(u\eta^\top) &= \langle \nabla_S u, \eta^\top\rangle + u\operatorname{div}_S\eta^\top\\ &= \langle \nabla_S u, \eta^\top\rangle + u\,(\operatorname{div}_S\eta - \langle \eta^\top, N\rangle \operatorname{div}_S N)\end{aligned}$$

is uniformly bounded, the dominated convergence theorem implies

$$\begin{aligned}\int_S \operatorname{div}_S u\eta^\top dS &= \lim_{\varepsilon\to 0}\int_{S\setminus\bigcup_{i=1}^n B_\varepsilon(p_i)} \operatorname{div} u\eta^\top\, dS\\ &= \lim_{\varepsilon\to 0}\sum_{i=1}^{n}\int_{\partial B_\varepsilon(p_i)}\langle \xi_i, u\eta^\top\rangle\, d(\partial B_\varepsilon(p_i)) = 0.\end{aligned}$$ □

Corollary 5.2.16 (Minkowski formula)**.** *Let S be a compact C^2 oriented surface in $(\mathbb{H}^1, K)$ enclosing a region E. Assume that S has constant mean curvature H and a finite number of singular points. Then*

$$3A(S) - 4H|E| = 0. \tag{5.2.16}$$

Proof. We consider the vector field $W = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\frac{\partial}{\partial t}$ and its associated flow $\varphi_s((x,y,t)) = (e^s x, e^s y, e^{2s}t)$. Since

$$\left.\frac{d}{ds}\right|_{s=0} A(\varphi_s(S)) = 3A(S), \quad \left.\frac{d}{ds}\right|_{s=0} |\varphi_s(E)| = 4|E|,$$

Proposition 5.2.15 implies

$$0 = \left.\frac{d}{ds}\right|_{s=0} A(\varphi_s(S)) - H\left.\frac{d}{ds}\right|_{s=0} |\varphi_s(E)| = 3A(S) - 4H|E|.$$ □

5.3 Pansu-Wulff spheres and examples

We consider a convex body $K \subset \mathbb{R}^2$ containing 0 in its interior and the associated norm $|\cdot|_K$ in $\mathbb{H}^1$.

Definition 5.3.1. Consider a clockwise-oriented L-periodic parameterization $\gamma : \mathbb{R} \to \mathbb{R}^2$ of ∂K. For fixed $v \in \mathbb{R}$ take the translated curve $u \mapsto \gamma(u+v) - \gamma(v)$ and its horizontal lifting $\Gamma_v : \mathbb{R} \to \mathbb{H}^1$ with initial point $(0,0,0)$ at $u = 0$.

The set $\mathbb{S}_K$ is defined as

$$\mathbb{S}_K = \bigcup_{v\in[0,L)} \Gamma_v([0,L]). \tag{5.3.1}$$

We shall refer to $\mathbb{S}_K$ as the *Pansu-Wulff sphere* associated to the left-invariant norm $|\cdot|_K$.

When $K = D$, the closed unit disk centered at the origin in $\mathbb{R}^2$, the Pansu-Wulff sphere $\mathbb{S}_D$ is Pansu's sphere, see [136; 137].

Remark 5.3.2. In the construction of the Pansu-Wulff sphere we are not assuming any regularity on the boundary of K. Since ∂K is a locally Lipschitz curve, its horizontal lifting is well defined.

Remark 5.3.3. The set $\mathbb{S}_K$ is union of curves leaving from $(0,0,0)$ that meet again at the point $(0,0,2|K|)$. Since γ is L-periodic, the construction is L-periodic in v and so $\mathbb{S}_K$ is the image of a continuous map from a sphere to $\mathbb{H}^1$.

Example 5.3.4. Given the Euclidean norm $|\cdot|$ in $\mathbb{R}^2$ and $a = (a_1, a_2)$, where $a_1, a_2 > 0$, we define the norm:

$$||(x_1,x_2)||_a = |(\tfrac{x_1}{a_1}, \tfrac{x_2}{a_2})|.$$

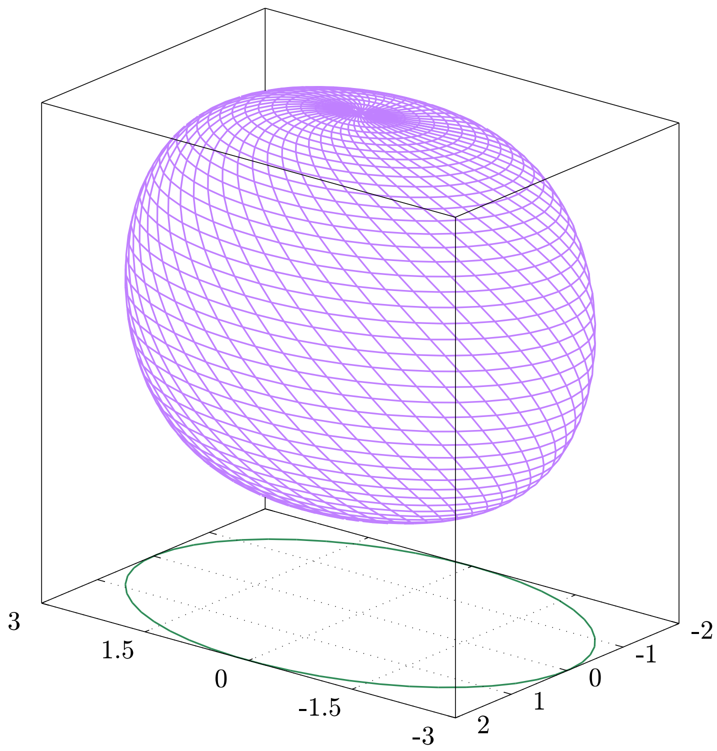

Figure 5.3: The Pansu-Wulff sphere associated to the norm $||\cdot||_a$ with $a = (1, 1.5)$. Observe that the projection to the horizontal plane $t = 0$ is an ellipse with semiaxes of lengths 2 and 3.

The unit ball K_a for this norm is an ellipsoid with axes of length a_1 and a_2. We parameterize clockwise ∂K by

$$\gamma(s) = (a_1 \sin(s), a_2 \cos(s)), \quad s \in \mathbb{R}.$$

This parameterization is injective of period 2π. The translation of this curve to the origin by the point $-\gamma(v)$ is given by the curve

$$\Lambda_v(u) = \gamma(u+v) - \gamma(v).$$

The horizontal lifting of Λ_v is given by $(\Lambda_v(u), t_v(u))$, where

$$t_v(u) = \int_0^u \big[\Lambda_v(\xi) \cdot J(\dot{\Lambda}_v(\xi))\big]\, d\xi.$$

Since

$$\Lambda_v(\xi) \cdot J(\dot{\Lambda}_v(\xi)) = (\gamma(\xi+v) - \gamma(v)) \cdot J(\dot{\gamma}(\xi+v)),$$

we get

$$t_v(u) = a_1 a_2 \big(u + \sin(v)\cos(u+v) - \cos(v)\sin(u+v)\big).$$

Hence a parameterization of $\mathbb{S}_{K_a}$ is given by

$$\begin{aligned} x(u,v) &= a_1\big(\sin(u+v) - \sin(v)\big) \\ y(u,v) &= a_2\big(\cos(u+v) - \sin(v)\big), \\ t(u,v) &= a_1 a_2\big(u + \sin(v)\cos(u+v) - \cos(v)\sin(u+v)\big). \end{aligned}$$

Example 5.3.5. Given any convex set K containing 0 in its interior, we can parameterize its Lipschitz boundary ∂K as

$$\gamma(s) = \big(x(s), y(s)\big) = r(s)\,\big(\sin(s), \cos(s)\big), \quad s \in \mathbb{R}.$$

where $r(s) = \rho(\sin(s), \cos(s))$ and ρ is the radial function of K defined as $\rho(u) = \sup\{\lambda \geqslant 0 : \lambda u \in K\}$ for any vector u of modulus 1 in $\mathbb{R}^2$.

A horizontal lifting of the curve γ passing through the point $(\gamma(0), 0)$ can be obtained computing

$$t(s) = \int_0^s \gamma(\xi) \cdot J(\dot{\gamma}(\xi))\, d\xi = \int_0^s r^2(\xi)\, d\xi,$$

since $J(\dot{\gamma}(s)) = r(s)\,(\sin(s), \cos(s)) + \dot{r}(s)\,(-\cos(s), \sin(s))$. Hence the curve

$$\Gamma(s) = \big(x(s), y(s), t(s)\big) = \Big(\gamma(s), \int_0^s r^2(\xi)\, d\xi\Big)$$

is a horizontal lifting of the curve γ.

Now we translate all these curves to pass through the origin of $\mathbb{H}^1$. This way we get the parameterization Φ_K of $\mathbb{S}_K$ given by

$$(u,v) \mapsto \ell_{-\Gamma(v)}(\Gamma(u+v))$$

for $(u,v) \in [0, 2\pi]^2$. Since

$$\ell_{(x_0,y_0,t_0)}(x,y,t) = \big(x + x_0, y + y_0, t + t_0 + (x y_0 - x_0 y)\big),$$

computing the left-translation using the expression for Γ obtained before we get

$$\begin{aligned} x(u,v) &= r(u+v)\sin(u+v) - r(v)\sin(v), \\ y(u,v) &= r(u+v)\cos(u+v) - r(v)\cos(v), \\ t(u,v) &= r(v)r(u+v)\big(\sin(v)\cos(u+v) - \cos(v)\sin(u+v)\big) \\ &\quad + \textstyle\int_v^{u+v} r^2(\xi)\,d\xi. \end{aligned} \tag{5.3.2}$$

The parameterization given by equations (5.3.2) is useful to obtain regularity properties of $\mathbb{S}_K$. If ∂K is of class C^ℓ, $\ell \geqslant 0$, its radial function $r(s) = (x(s)^2 + y(s)^2)^{1/2}$ is of class C^ℓ and hence the parameterization Φ_K is an immersion of class C^ℓ for $0 < u < 2\pi$.

Example 5.3.6. Let $\ell > 1$. We consider the ℓ-norm in $\mathbb{R}^2$ defined as

$$||(x_1, x_2)||_\ell = \big(|x_1|^\ell + |x_2|^\ell\big)^{1/\ell}.$$

Denote by K_ℓ the unit ball for this ℓ-norm. We can parametrize the unit circle $||\cdot||_\ell = 1$ using (5.3.2). In this case

$$\rho(x,y) = \frac{1}{\big(|x|^\ell + |y|^\ell\big)^{1/\ell}}, \qquad |(x,y)| = 1.$$

By the previous example, the Pansu-Wulff sphere $\mathbb{S}_{K_\ell}$ is parameterized by equations (5.3.2).

Remark 5.3.7. Assume we have a sequence of of convex sets (K_i) converging in Hausdorff distance to a limit convex set K. Then the radial functions r_{K_i} uniformly converge to the radial function r of the limit set K. Hence equations (5.3.2) imply that the Pansu-Wulff spheres $\mathbb{S}_{K_i}$ converge in Hausdorff distance to a ball bounded by the horizontal liftings of translations of a parameterization γ of ∂K.

Since $\lim_{\ell\to 1} ||\cdot||_\ell = ||\cdot||_1$ and $\lim_{\ell\to\infty} ||\cdot||_\ell = |\cdot|_\infty$, we can use the previous argument to show that the Pansu-Wulff spheres $\mathbb{S}_{K_\ell}$ converge to the two spheres $\mathbb{S}_1$ and $\mathbb{S}_\infty$. Under these conditions, it is not difficult to check that the corresponding perimeters converge to the limit perimeter.

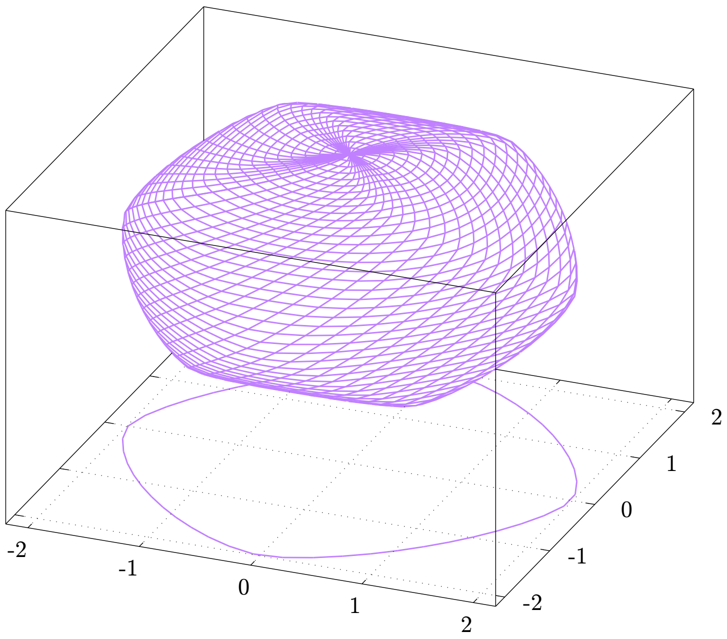

Figure 5.4: The Pansu-Wulff sphere $\mathbb{S}_{K_\ell}$ for the ℓ-norm, $\ell = 1.5$. The horizontal curve is the projection of the equator to the plane $t = 0$. We observe that the Pansu-Wulff sphere projects to the set $|\cdot|_\ell \leqslant 2$ in the $t = 0$ plane.

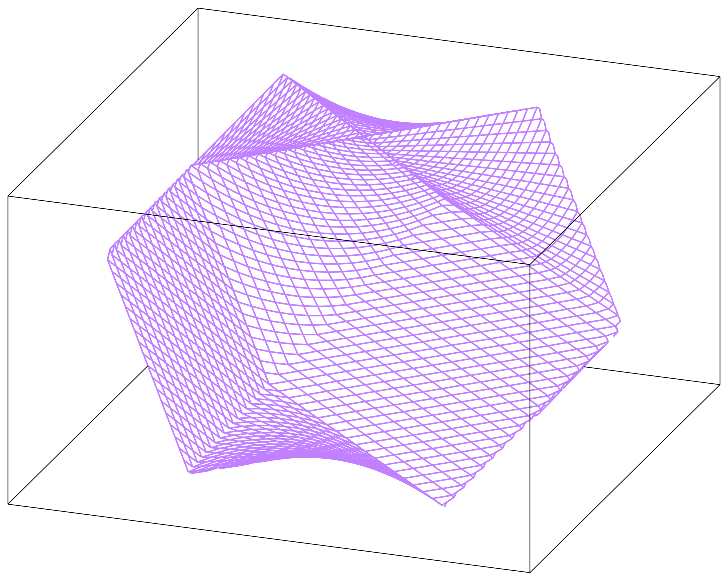

Figure 5.5: The sphere $\mathbb{S}_1$ obtained as Hausdorff limit of the Pansu-Wulff spheres $\mathbb{S}_{K_r}$ of the ℓ-norm when ℓ converges to 1.

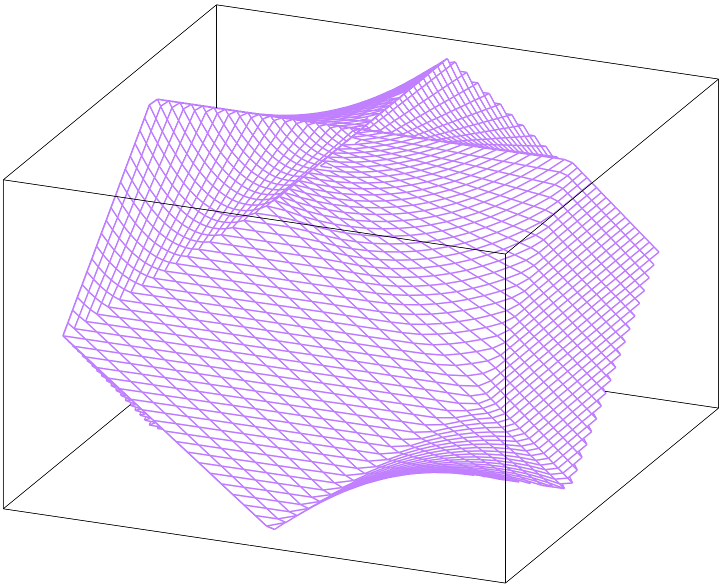

Figure 5.6: The sphere $\mathbb{S}_\infty$ obtained as Hausdorff limit of the Pansu-Wulff spheres $\mathbb{S}_{K_r}$ of the ℓ-norm when ℓ converges to ∞

Example 5.3.8. Let us consider the equilateral triangle T in the plane $\mathbb{R}^2$ defined as the convex envelope of the points $a_1 = (0,1)$, $a_2 = (\sqrt{3}/2, -1/2)$, $a_3 = (-\sqrt{3}/1, -1/2)$. We can define a norm $|\cdot|_T$ by the equality

$$|x|_T = \max\big\{\langle x, a_i\rangle : i = 1,2,3\big\}, \quad x \in \mathbb{R}^2.$$

The unit ball of the norm $|\cdot|_T$ is the triangle T. It is neither smooth nor strictly convex. However we may consider the approximating norms

$$|x|_{T,\ell} = \left(\sum_{i=1}^{3} \max\{\langle x, a_i\rangle, 0\}^\ell\right)^{1/\ell}.$$

These norms are smooth and strictly convex and $\lim_{\ell\to\infty} |x|_{T,\ell} = |x|_T$. Hence the Pansu-Wulff spheres $\mathbb{S}_{K_{T,\ell}}$ converge in Hausdorff distance when $\ell \to \infty$ to the sphere $\mathbb{S}_T$ obtained by traslating ∂T to touch the origin and lifting the obtained curves as horizontal ones to $\mathbb{H}^1$.

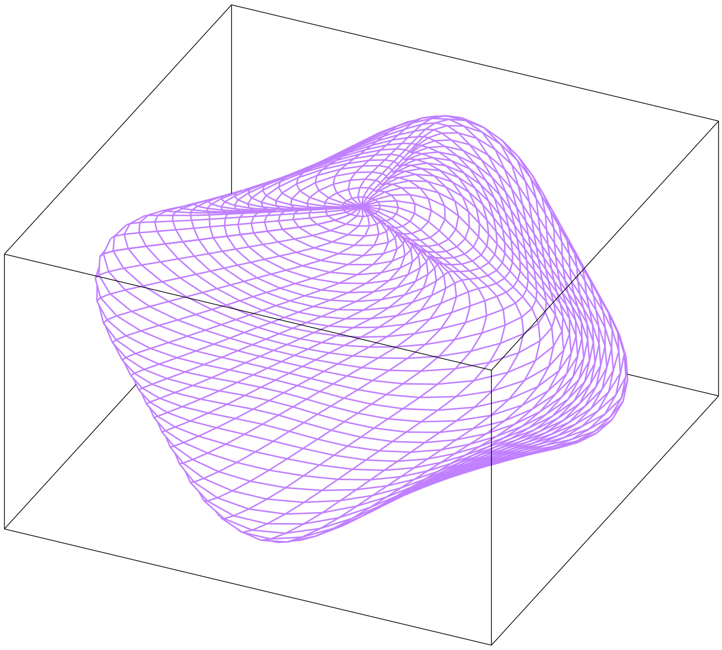

Figure 5.7: The Pansu-Wulff sphere $\mathbb{S}_{T,\ell}$ for the norm $||\cdot||_{T,\ell}$, with $r = 2$.

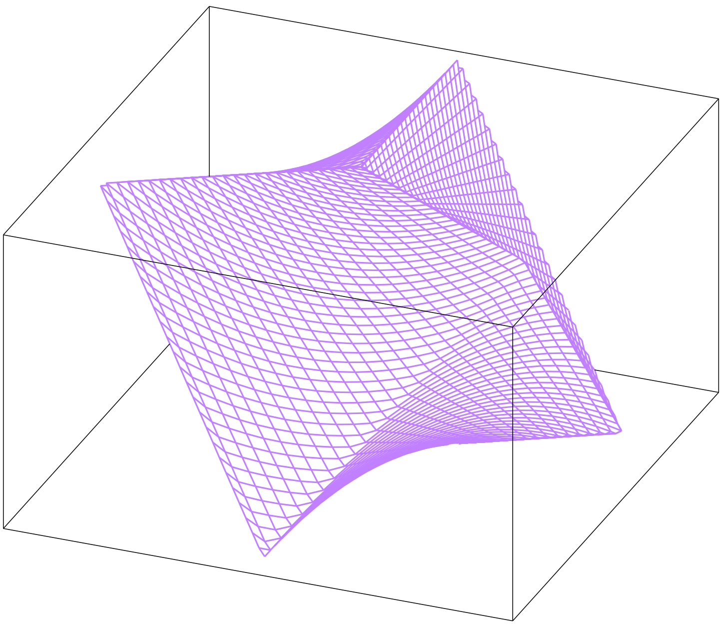

Figure 5.8: The sphere $\mathbb{S}_T$ obtained as limit of the Pansu-Wulff spheres $\mathbb{S}_{T,\ell}$ when $r \to \infty$.

5.4 Geometric properties of the Pansu-Wulff spheres

In this section we show several geometric properties of the Pansu-Wulff spheres $\mathbb{S}_K$ associated with a left-invariant norm $|\cdot|_K$. We start by looking at the projection of the sphere to the $t = 0$ plane. This projection is determined by the geometry of the convex set K.

Given a convex body $K \subset \mathbb{R}^n$, the *difference body* of K is the set

$$\mathrm{D}K = K - K = \{x - y : x, y \in K\}.$$

The difference body $\mathrm{D}K$ is a *centrally symmetric* convex body. This means that $-x \in \mathrm{D}K$ whenever $x \in \mathrm{D}K$. If h_K is the support function of K then the support function of $\mathrm{D}K$ is given by

$$h_{\mathrm{D}K}(u) = h_K(u) + h_K(-u),$$

see [156, p. 140]. This is the width of K in the direction of u.

Lemma 5.4.1. *Let $K \subset \mathbb{R}^n$ be a convex body with $0 \in \mathrm{int}(K)$. We consider the set*

$$K_0 = \bigcup_{p\in\partial K} (-p + K). \tag{5.4.1}$$

Then we have

1. $0 \in K_0$.
2. *K_0 is a convex body.*
3. *K_0 is the difference body of K. In particular, K_0 is centrally symmetric.*
4. *If K is centrally symmetric then $K_0 = 2K$.*
5. *We have*
$$\bigcup_{p\in\partial K} (-p + K) = \bigcup_{p\in\partial K} (-p + \partial K).$$

Proof. To prove 1 take into account that $0 = -p + p \in -p + K \subset K_0$ for any $p \in \partial K$.

To prove 2, we take $p_1, p_2 \in \partial K$, $q_1, q_2 \in K$ and $\lambda \in [0, 1]$. Then

$$\lambda(-p_1 + q_1) + (1 - \lambda)(-p_2 + q_2) = -p_\lambda + q_\lambda,$$

where

$$p_\lambda = \lambda p_1 + (1 - \lambda)p_2, \qquad q_\lambda = \lambda q_1 + (1 - \lambda)q_2.$$

If $p_\lambda = q_\lambda$ then $-p_\lambda + q_\lambda = 0 \in K_0$. Otherwise the segment $[p_\lambda, q_\lambda]$ is not trivial and contained in K. Let $\mu_0 \geqslant 1$ such that $q_\lambda + \mu_0(p_\lambda - q_\lambda) \in \partial K$. The value μ_0 is computed as the supremum of the set $\{\mu \geqslant 0 : q_\lambda + \mu(p_\lambda - q_\lambda) \in K\}$. We have

$$-p_\lambda + q_\lambda = -(q_\lambda + \mu_0(p_\lambda - q_\lambda)) + (q_\lambda + (\mu_0 - 1)(p_\lambda - q_\lambda)).$$

The point $q_\lambda + \mu_0(p_\lambda - q_\lambda)$ belongs to ∂K by the choice of μ_0 and the point $q_\lambda + (\mu_0 - 1)(p_\lambda - q_\lambda)$ belongs to K since $0 \leqslant \mu_0 - 1 \leqslant \mu_0$. Hence $-p_\lambda + q_\lambda \in K_0$ and so K_0 is convex.

To prove 3, we take a vector v with $\langle v, v\rangle = 1$. Let $q \in \partial K_0$ such that

$$h_{K_0}(v) = \langle q, v\rangle \geqslant \langle z, v\rangle \quad \forall\ z \in K_0. \tag{5.4.2}$$

By the definition of K_0, there exists $p \in \partial K$ such that $q \in -p + K$. We claim that $q \in -p + \partial K$: otherwise $p + q \in \operatorname{int}(K)$ and there exists $\varepsilon > 0$ such that $p + q + \varepsilon v \in K$. So we have

$$\langle -p + (p + q + \varepsilon v), v\rangle = \langle q + \varepsilon v, v\rangle = \langle q, v\rangle + \varepsilon > \langle q, v\rangle.$$

Since $p + q + \varepsilon v \in K$ this yields a contradiction. Hence $q \in -p + \partial K = \partial(-p + K)$ for some $p \in \partial K$.

Since $-p + K \subset K_0$ and q is a boundary point for both sets, we deduce that v is a normal vector to $-p+K$ at q. As $h_{-p+K}(v) = -\langle p, v\rangle + h_K(v)$,we have

$$h_{K_0}(v) = h_{-p+K}(v) = h_K(v) + \langle p, -v\rangle.$$

It remains to prove that $h_K(-v) = \langle p, -v\rangle$. Assume by contradiction that $\langle p, -v\rangle < h_K(-v) = \langle x, -v\rangle$ for some $x \in \partial K$. Then we have

$$\langle -x + (p + q), v\rangle = \langle -x + p, v\rangle + \langle q, v\rangle > \langle q, v\rangle,$$

that cannot hold by (5.4.2) since $p+q \in K$ and so $-x+p+q \in -x+K \subset K_0$.

To prove 4, we note that $h_K(v) = h_K(-v)$ when K is centrally symmmetric and, by 3, $h_{K_0} = 2h_K$. Hence $K = 2K_0$.

Finally, to prove 5 we notice that $\bigcup_{p\in\partial K}(-p + K) \supset \bigcup_{p\in\partial K}(-p + \partial K)$. To prove the remaining inclusion we take $p \in \partial K$ and $u \in K$ such that $q = -p + u \in \bigcup_{p\in\partial K}(-p + K)$. Then Lemma 5.4.2 allows us to find $p_1, u_1 \in \partial K$ such that $q = -p + u = -p_1 + u_1$. Hence $q \in \bigcup_{p\in\partial K}(-p + \partial K)$. □

Lemma 5.4.2. *Let $K \subset \mathbb{R}$ be a convex body, and $a, b \in K$. Then there exist $p, q \in \partial K$ such that $b - a = q - p$.*

Proof. If $a = b$ or $a, b \in \partial K$ the result follows trivially. Henceforth we assume $a \neq b$ and that at least a or b is an interior point of K. We pick a point $c \in K$ out of the line ab. Let P be the plane containing a, b, c and $W = K \cap P$. The set W is a convex body in P and the boundary of W in P is contained in ∂K. We take orthogonal coordinates (x, y) in P so that $(b-a)$ points into the positive direction of the y-axis. Let I be the orthogonal projection in P of W onto the x-axis.

Given $x \in I$, define the set $W(x)$ as $\{y \in \mathbb{R} : (x,y) \in W\}$. A simple application of Kuratowski criterion, see Theorem 1.8.8 in [156], implies that $W(x_i)$ converges to $W(x)$ in Hausdorff distance when x_i converges to x. Hence the function $x \in I \mapsto |W(x)|$ is continuous and takes a value larger than $|b-a|$ at the projection of a, b over the x-axis. If $|W(x)| = |b-a|$ for some $x \in I$, we take as p, q the extreme points of the interval $W(x)$ chosen so that $q - p = b - a$ to conclude the proof. Otherwise, we would have $|W(x_0)| > |b-a|$ at an extreme point x_0 of I. We may choose two points $p, q \in W(x_0)$ such that the length $|[p,q]|_1 = |b-a|$ and $q - p = b - a$. Since $W(x_0)$ is contained in the boundary of W in P, it is contained in ∂K and so $p, q \in \partial K$. □

Now we refine the results in Lemma 5.4.1 when K is strictly convex and has boundary of class C^ℓ_+, $\ell \geqslant 2$. We say that a convex body K is of class C^ℓ_+, $\ell \geqslant 1$, when ∂K is of class C^ℓ and its normal map $\mathcal{N}_K : \partial K \to \mathbb{S}^1$ is a diffeomorphism of class $C^{\ell-1}$.

Corollary 5.4.3. *Let $K \subset \mathbb{R}^2$ be a convex body with $0 \in \text{int}(K)$. Then*

1. *If $K \subset \mathbb{R}^2$ is strictly convex, then K_0 is strictly convex.*
2. *If K is of class C^ℓ_+, $\ell \geqslant 2$, then K_0 is of class C^ℓ_+.*

Proof. To prove that K_0 is strictly convex, we take two different points $x_1 - x_2, y_1 - y_2 \in \partial K_0$, with $x_i, y_i \in K$, $i = 1, 2$. Then the four points belong to the boundary of K. For any $\lambda \in (0,1)$, we write the convex combination $\lambda(x_1 - x_2) + (1-\lambda)(y_1 - y_2)$ as

$$x_\lambda - y_\lambda = (\lambda x_1 + (1-\lambda)y_1) - (\lambda x_2 + (1-\lambda)y_2).$$

Since $x_1 \neq y_1$ or $x_2 \neq y_2$, the strict convexity of K implies that x_λ or y_λ is an interior point of K. Then $x_\lambda - y_\lambda$ is an interior point of K_0. Since $\lambda \in (0,1)$ and the boundary points are arbitrary, the set K_0 is strictly convex.

To prove the boundary regularity of K_0 we follow Schneider's arguments [156, p. 115] and observe that the support function h_K of K is defined, when $u \neq 0$, by

$$h_K(u) = \langle u, \mathcal{N}_K^{-1}(u) \rangle,$$

where $\mathcal{N}_K : \partial K \to \mathbb{S}^1$ is the Gauss map, a diffeomorphism of class $C^{\ell-1}$ since K is of class C^ℓ_+. By Corollary 7.1.3 in [156]

$$\nabla h_K(u) = \mathcal{N}_K^{-1}\left(\frac{u}{|u|}\right), \tag{5.4.3}$$

and so h_K is of class C^ℓ. This implies that the support function of K_0, $h_{K_0}(u) = h_K(u) + h_K(-u)$, is of class C^ℓ. Hence the polar body K_0^* of K_0 has boundary of class C^ℓ. The Gauss map $\mathcal{N}_{K_0^*}$ of K_0^* can be described as

$$\mathcal{N}_{K_0^*} : \rho(K_0^*, u)u \mapsto \frac{\mathcal{N}_K^{-1}(u)}{|\mathcal{N}_K^{-1}(u)|},$$

where $\rho(K_0^*, \cdot) = h_K^{-1}(\cdot)$ is the radial function of K_0^*, of class $C^{\ell-1}$. Hence $\mathcal{N}_{K_0^*}$ is a diffeomorphism of class $C^{\ell-1}$ and so K_0^* is of class C_+^ℓ. Now the support function of K_0^* is of class C_+^ℓ and we reason in the same way interchanging the roles of K_0^* and K_0 to get the result. □

Remark 5.4.4. If $K \subset \mathbb{R}^2$ is a centrally symmetric convex body, for any $p \in \partial K$, the line passing through p and $-p$ divides K into two regions of equal area. Hence the line through 0 and $-2p$ divides $-p + K$ into two regions of the same area. When p moves along ∂K, the point $-2p$ parametrizes $\partial(2K)$.

Let K be a convex set of class C_+^ℓ, $\ell \geqslant 2$, $C = \partial K$ and $\gamma : \mathbb{R} \to \mathbb{R}^2$ an L-periodic clockwise arc-length parameterization of C, with $L = \text{length}(C)$. The set $K_0 = \bigcup_{p \in C}(-p + K)$ has smooth boundary C_0. For any $v \in \mathbb{R}$, we denote by $\gamma_v(u) = \gamma(u + v) - \gamma(v)$. Let $\Gamma_v = (\gamma_v, t_v)$ be the horizontal lifting of γ_v with $t_v(0) = 0$. If we call $\Omega_v(u)$ the planar region delimited by the segment $[0, \gamma_v(u)]$ and the restriction of γ_v to $[0, u]$ then a standard application of the Divergence Theorem to the vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ implies

$$t_v(u) = \int_0^u \langle \gamma_v, J(\dot{\gamma}_v)\rangle(\xi)\, d\xi = 2\,|\Omega_v(u)|.$$

Our next goal is to prove that $\mathbb{S}_K$ is the union of two graphs defined in K_0 of class C^2 and coinciding on ∂K_0.

Theorem 5.4.5. *Let $K \subset \mathbb{R}^2$ be a convex body with C_+^ℓ boundary, $\ell \geqslant 2$. Then*

1. *$\mathbb{S}_K$ is of class C^ℓ outside the poles.*
2. *There exist two functions $g_1, g_2 : K_0 \to \mathbb{R}$ of class C^ℓ on $\text{int}(K_0)$ such that*

   $$\mathbb{S}_K = \text{graph}(g_1) \cup \text{graph}(g_2),$$

   with $g_1 > g_2$ on $\text{int}(K_0)$ and $g_1 = g_2$ on C_0. This imples that $\mathbb{S}_K$ is an embedded surface.

   Moreover, if K is centrally symmetric then $g_1 + g_2 = 2|K|$ and hence $\mathbb{S}_K$ is symmetric with respect to the horizontal Euclidean plane $t = |K|$.

Definition 5.4.6. The domain delimited by the embedded sphere $\mathbb{S}_K$ is a ball $\mathbb{B}_K$ that we call the *Pansu-Wulff shape* of $|\cdot|_K$.

Proof of Theorem 5.4.5. That $\mathbb{S}_K$ is C^ℓ outside the singular set follows from the parameterization (5.3.2) since the function $r(s)$ is of class C^ℓ. This proves 1.

We break the proof of 2 into several steps. Recall that $C = \partial K$ and $C_0 = \partial K_0$.

Step 1. Given $x \in K_0 \backslash \{0\}$, we claim that $x \in C - p$ for some $p \in C$ if and only if the segment $[p, p+x]$ is contained in K and $p, p+x \in C$. This means that the number of curves $C - p$, with $p \in C$, passing through $x \neq 0$ coincides with the number of segments parallel to x of length $|x|$ and boundary points in C. This step is trivial.

Step 2. Given $x \in K_0 \setminus \{0\}$, the number of segments $[p, p+x]$ contained in K with $p, p+x \in C$ is either 1 or 2. The first case corresponds to maximal length and happens if and only if x belongs to C_0.

To prove this we consider $v = x/|x|$ and a line L orthogonal to v. For any z in L we consider the intersection $I_z = L_z \cap K$, where L_z is the line passing through z with direction v. The set $J = \{z \in L : I_z \neq \emptyset\}$ is a non-trivial segment in L. The strict convexity of K implies that the map $F : J \to \mathbb{R}$ defined by $F(z) = |I_z|$ is strictly concave. Since F vanishes at the extreme points of J, it has just one maximum point $z_0 \in \mathrm{int}(J)$ and each value in the interval $(0, F(z_0))$ is taken by two different points in J. The observation that there is a bijective correspondence between the segments $[p, p+x]$ contained in K with $p, p+x \in C$ and the points $z \in L$ with $F(z) = |x|$ proves the first part of the claim.

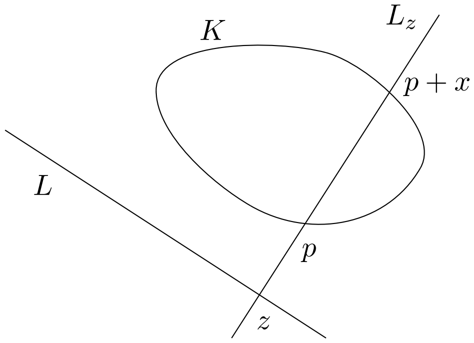

Figure 5.9: Construction of the map F

To prove the second part of the claim we fix some $x \in K_0$. We take $p \in C$ such that the segment $[p, p+x]$ is contained in K and $p, p+x \in C$. Assume first that $x \in C_0$. If there were a larger segment $[q, q+\mu x]$ contained in K with $q, q+\mu x \in C$ and $\mu > 1$ then we would have $\mu x \in C - q \subset K_0$, a contradiction. Hence the length of $[p, p+x]$ is is the largest possible in the direction of x. Assume now that the length of $[p, p+x]$ yields the maximum of length of intervals contained in K in the direction of x. If $x \notin C_0$ then

x is an interior point of K_0 and, since $0 \in \operatorname{int}(K_0)$, there would exist $\lambda > 1$ such that $\lambda x \in K_0$. Hence there is some $q \in C$ such that $\lambda x \in C - p$ and the segment $[q, q + \lambda x] \subset C$ and has length larger than $|x|$, a contradiction that proves that $x \in C_0$.

Step 3. Given any point $x \in \operatorname{int}(K_0)$, there are exactly two points in $\mathbb{S}_K$ at heights $g_1(x) > g_2(x)$. In case K is centrally symmetric then $g_1(x) + g_2(x) = 2|K|$.

By the previous steps, there are exactly two points $p, q \in C$ so that $p + x, q + x \in C$ and the segments $[p, p + x], [q, q + x]$ are contained in K. We may assume that $p, p + x, q + x, q$ are ordered clockwise along C. The heights of the points in $\mathbb{S}_K$ projecting over x are given by twice the areas of the sets A and B, where A is determined by the portion of C from p to $p+x$ and the segment $[p + x, p]$, and B is determined by the portion of C from q to $q + x$ and the segment $[q + x, q]$. Since A is properly contained in B we have $g_2(x) = 2|A| < 2|B| = g_1(x)$.

In case K is centrally symmetric, the central symmetry maps $p + x$ to q and $q + x$ to p since $[p, p + x]$ and $[q, q + x]$ are the only segments in K of length $|x|$ with boundary points on C. Hence $|A| + |B| = |K|$ and so $g_1(x) + g_2(x) = 2|K|$.

Step 4. The functions g_1, g_2 are of class C^ℓ in $\operatorname{int}(K_0) \setminus \{0\}$.

This follows from the implicit function theorem since $\mathbb{S}_K$ is C^ℓ outside the poles. □

Theorem 5.4.7. *Let $K \subset \mathbb{R}^2$ be a convex body of class C^2_+. Then $\mathbb{S}_K$ is of class C^2 around the poles.*

Proof. We consider a horizontal lifting $\Gamma = (x, y, t)$ of a clockwise arc-length parametrization γ of ∂K. Then a parameterization of $\mathbb{S}_K$ is given by $(\mathbf{x}, \mathbf{y}, \mathbf{t})(u, v) = \ell_{-\Gamma(v)}(\Gamma(u + v))$. This means

$$\begin{aligned} \mathbf{x}(u, v) &= x(u + v) - x(v), \\ \mathbf{y}(u, v) &= y(u + v) - y(v), \\ \mathbf{t}(u, v) &= t(u + v) - t(v) - x(u + v)y(v) + y(u + v)x(v). \end{aligned} \tag{5.4.4}$$

The tangent vectors $\partial/\partial u, \partial/\partial u$ are the image of $(1, 0)$ and $(0, 1)$ under the parameterization and are given by

$$\begin{aligned} \frac{\partial}{\partial u} &= \dot{x}(u + v)\, X + \dot{y}(u + v)\, Y. \\ \frac{\partial}{\partial v} &= \big(\dot{x}(u + v) - \dot{x}(v)\big)\, X + \big(\dot{y}(u + v) - \dot{y}(v)\big)\, Y + h(u, v)\, T, \end{aligned}$$

where

$$h(u, v) = 2\big(\dot{x}(v)(y(u + v) - y(v)) - \dot{y}(v)(x(u + v) - x(v))\big). \tag{5.4.5}$$

Geometrically, $h(u,v)$ is the scalar product of the position vector $(x(u+v)-x(v), y(u+v)-y(v))$ with $J((\dot{x},\dot{y}))$, that is always negative for $u>0$. A Riemannian unit normal vector N can be easily computed from the expressions of $\partial/\partial u$ and $\partial/\partial v$ and is given by

$$N = \frac{h\big(\dot{y}(u+v)X - \dot{x}(u+v)Y\big) + gT}{\big(h^2+g^2\big)^{1/2}}, \tag{5.4.6}$$

where

$$g(u,v) = \dot{x}(v)\dot{y}(u+v) - \dot{y}(v)\dot{x}(u+v). \tag{5.4.7}$$

We have

$$|N_h| = \frac{|h|}{\big(h^2+g^2\big)^{1/2}}, \qquad \langle N, T\rangle = \frac{g}{\big(h^2+g^2\big)^{1/2}}$$

Let us see that $\mathbb{S}_K$ is a C^2 surface near the south pole $(0,0,0)$. The arguments for the north pole of are similar. To see that $\mathbb{S}_K$ is C^1 near the south pole, it is enough to check that N extends continuously to $u=0$. Let us see that

$$\lim_{(u,v)\to(0,v_0)} N(u,v) = -T. \tag{5.4.8}$$

Since $g<0$, from the expression (5.4.6) it is enough to prove that

$$\lim_{(u,v)\to(0,v_0)} \frac{h}{g}(u,v) = 0. \tag{5.4.9}$$

Since x and y are functions of class C^2, we use Taylor expansions around v to get

$$\begin{aligned} x(u+v) &= x(v) + \dot{x}(v)u + R(u,v)u, \quad & y(u+v) &= y(v) + \dot{y}(v)u + R(u,v)u, \\ \dot{x}(u+v) &= \dot{x}(v) + \ddot{x}(v)u + R(u,v)u, \quad & \dot{y}(u+v) &= \dot{y}(v) + \ddot{y}(v)u + R(u,v)u. \end{aligned}$$

In the above equations R denotes a continuous functions of (u,v) (depending on the equation) that converges to 0 when $u\to 0$ independently of v. This follows from the integral expression for the reminder in Taylor's expansion. Then we have

$$\begin{aligned} \lim_{(u,v)\to(0,v_0)} \frac{h}{g}(u,v) &= \lim_{(u,v)\to(0,v_0)} \frac{R(u,v)u}{-\kappa(v)u + R(u,v)u} \\ &= \lim_{(u,v)\to(0,v_0)} \frac{R(u,v)}{-\kappa(v) + R(u,v)} = 0, \end{aligned}$$

where

$$\kappa(v) = \big(\dot{y}\ddot{x} - \dot{x}\ddot{y}\big)(v)$$

is the (positive) geodesic curvature of γ. This proves (5.4.9) and so $\mathbb{S}_K$ is of class C^1 around $(0,0,0)$.

To prove that $\mathbb{S}_K$ is of class C^2 around the origin it is enough to show that the Riemannian second fundamental form of $\mathbb{S}_K$ converges to 0 when $(u,v)\to(0,v_0)$. We first compute

$$\lim_{(u,v)\to(0,v_0)} D_{\partial/\partial u}N.$$

Since

$$\begin{aligned} D_{\partial/\partial u}N &= \frac{\partial}{\partial u}\bigg(\frac{h\dot y(u+v)}{\sqrt{h^2+g^2}}\bigg)X - \frac{\partial}{\partial u}\bigg(\frac{h\dot x(u+v)}{\sqrt{h^2+g^2}}\bigg)Y + \frac{g}{\sqrt{h^2+g^2}}J(\tfrac{\partial}{\partial u}) \\ &+ \bigg(\frac{\partial}{\partial u}\bigg(\frac{g}{\sqrt{h^2+g^2}}\bigg) + \frac{h}{\sqrt{h^2+g^2}}\bigg)T. \end{aligned} \tag{5.4.10}$$

A direct computation taking into account $\frac{\partial h}{\partial u}=2g$ yields

$$\frac{\partial}{\partial u}\bigg(\frac{h}{\sqrt{h^2+g^2}}\bigg) = \frac{2g^3-gh\frac{\partial g}{\partial u}}{(h^2+g^2)^{3/2}}, \qquad \frac{\partial}{\partial u}\bigg(\frac{g}{\sqrt{h^2+g^2}}\bigg) = \frac{h^2\frac{\partial g}{\partial u}-2g^2h}{(h^2+g^2)^{3/2}}.$$

It is straightforward to check from the Taylor expressions that

$$\lim_{(u,v)\to(0,v_0)} \frac{h}{g^2}(u,v) = \lim_{(u,v)\to(0,v_0)} \frac{-\kappa(v_0)u^2+R(u,v)u^2}{\kappa(v_0)^2u^2+R(u,v)u^2} = \frac{-1}{\kappa(v_0)}.$$

Then we immediately get, dividing by $-g^3$,

$$\lim_{(u,v)\to(0,v_0)} \frac{\partial}{\partial u}\bigg(\frac{h}{\sqrt{h^2+g^2}}\bigg) = \lim_{(u,v)\to(0,v_0)} \frac{-2+\frac{h}{g^2}\frac{\partial g}{\partial u}}{((\frac{h}{g})^2+1)^{3/2}} = -1$$

and

$$\lim_{(u,v)\to(0,v_0)} \frac{\partial}{\partial u}\bigg(\frac{g}{\sqrt{h^2+g^2}}\bigg) = \lim_{(u,v)\to(0,v_0)} \frac{-\frac{h}{g}\frac{h}{g^2}\frac{\partial g}{\partial u}+2\frac{h}{g}}{((\frac{h}{g})^2+1)^{3/2}} = 0.$$

Taking limits in (5.4.10) we get

$$\lim_{(u,v)\to(0,v_0)} D_{\partial/\partial u}N = J(\tfrac{\partial}{\partial u}) - J(\tfrac{\partial}{\partial u}) + 0 = 0.$$

We complete $\frac{\partial}{\partial v}$ to an orthonormal basis of the tangent plane by adding the vector

$$E = \frac{\frac{\partial}{\partial v} - \langle\frac{\partial}{\partial u},\frac{\partial}{\partial v}\rangle\frac{\partial}{\partial u}}{(1-\langle\frac{\partial}{\partial u},\frac{\partial}{\partial v}\rangle^2)^{1/2}}.$$

Since $\lim_{(u,v)\to(0,v_0)} \frac{\partial}{\partial v}=0$, we have

$$\begin{aligned} \lim_{(u,v)\to(0,v_0)} D_E N &= \lim_{(u,v)\to(0,v_0)} D_{\partial/\partial v}N \\ &= \lim_{(u,v)\to(0,v_0)} \bigg(-\frac{\partial}{\partial v}\bigg(\frac{h}{\sqrt{h^2+g^2}}\bigg)J(\tfrac{\partial}{\partial u}) + \frac{\partial}{\partial v}\bigg(\frac{g}{(h^2+g^2)^{1/2}}\bigg)\bigg). \end{aligned}$$

A computation shows that

$$\frac{\partial}{\partial v}\left(\frac{h}{\sqrt{h^2+g^2}}\right) = \frac{g^2\frac{\partial h}{\partial v} - gh\frac{\partial g}{\partial v}}{(h^2+g^2)^{3/2}}, \qquad \frac{\partial}{\partial v}\left(\frac{g}{\sqrt{h^2+g^2}}\right) = \frac{h^2\frac{\partial g}{\partial v} - gh\frac{\partial h}{\partial v}}{(h^2+g^2)^{3/2}}.$$

We trivially have

$$\lim_{(u,v)\to(0,v_0)} \frac{\partial h}{\partial v}(u,v) = \lim_{(u,v)\to(0,v_0)} \frac{\partial g}{\partial v}(u,v) = 0.$$

Hence

$$\lim_{(u,v)\to(0,v_0)} \frac{\partial}{\partial v}\left(\frac{g}{\sqrt{h^2+g^2}}\right) = \lim_{(u,v)\to(0,v_0)} \frac{-\frac{h}{g}\frac{h}{g^2}\frac{\partial g}{\partial v} + \frac{h}{g^2}\frac{\partial h}{\partial v}}{((\frac{h}{g})^2+1)^{3/2}} = 0.$$

On the other hand

$$\lim_{(u,v)\to(0,v_0)} \frac{\partial}{\partial v}\left(\frac{h}{\sqrt{h^2+g^2}}\right) = \lim_{(u,v)\to(0,v_0)} \frac{-\frac{1}{g}\frac{\partial h}{\partial v} + \frac{h}{g^2}\frac{\partial g}{\partial v}}{(h^2+g^2)^{3/2}} = 0.$$

This equality holds from the Taylor expansions since

$$\lim_{(u,v)\to(0,v_0)} \frac{1}{g}\frac{\partial h}{\partial v}(u,v) = \lim_{(u,v)\to(0,v_0)} \frac{R(u,v)u}{-\kappa(v)u + R(u,v)u} = 0.$$

So we conclude that $\lim_{(u,v)\to(0,v_0)} D_E N = 0$. □

5.5 Minimization property of the Pansu-Wulff shapes

We prove in this section a minimization property satisfied by the balls $\mathbb{B}_K$. Let K be a convex body containing 0 in its interior. We assume that K is of class C^ℓ_+, with $\ell \geqslant 2$.

Definition 5.5.1. Given $\mathbb{S}_K$, we let $g : K_0 \to \mathbb{R}$ be the function $g(x) = (g_1(x) + g_2(x))/2$, where g_1 and g_2 are the functions obtained in Theorem 5.4.5.

We also introduce the notation $\mathbb{S}_K^+ := \mathbb{S}_K \cap \{(x,t) : t \geqslant g(x)\}$, $\mathbb{S}_K^- := \mathbb{S} \cap \{(x,t) : t \leqslant g(x)\}$ and $D_0 = \{(x, g(x)) : x \in K_0\}$.

Theorem 5.5.2. *Let $K \subset \mathbb{R}^2$ be convex body of class C^ℓ_+, with $\ell \geqslant 2$ and $0 \in \operatorname{int}(K)$. Let $r > 0$ and $h : rK_0 \to \mathbb{R}$ a C^0 function. Consider a subset $E \subset \mathbb{H}^1$ with finite volume and finite K-perimeter such that*

$$\operatorname{graph}(h) \subseteq E \subset rK_0 \times \mathbb{R}.$$

Then

$$|\partial E|_K \geqslant |\partial \mathbb{B}_E|_K, \tag{5.5.1}$$

where $\mathbb{B}_E$ is the Wulff shape in $(\mathbb{H}^1, K)$ with $|E| = |\mathbb{B}_E|$.

Proof. Let $g_r : rK_0 \to \mathbb{R}$ the function defined by $g_r(x) = r^2 g(\frac{1}{r}x)$, where g is the function in Definition 5.5.1. Let D be the graph of g_r. We know that D divides the Wulff shape $r\mathbb{S}_K$ into two parts $r\mathbb{S}_K^+$ and $r\mathbb{S}_K^-$. Let W^+ and W^- the vector fields in $rK_0 \times \mathbb{R} \setminus L$ defined by translating vertically the vector fields

$$\pi_K(\nu_0)\big|_{r\mathbb{S}_K^+}, \quad \pi_K(\nu_0)\big|_{r\mathbb{S}_K^-},$$

respectively. Here ν_0 is the horizontal unit normal to $\mathbb{S}_K$.

As a first step in the proof we are going to show that if $F \subset rK_0 \times \mathbb{R}$ is a set of finite volume and K-perimeter so that rel int$(D) \subset$ int(F), then the inequality

$$\tfrac{1}{r}|F| \leqslant \int_D \langle W^+ - W^-, N_D \rangle dD + |\partial F|_K \tag{5.5.2}$$

holds, where N_D is the Riemannian normal pointing down and dD is the Riemannian measure of D. Equality holds in (5.5.2) if and only if $W^+ = \pi_K(\nu_h)$ $|\partial_K F|$-a.e. on $F^+ = F \cap \{t \geqslant g_r\}$ and $W^- = \pi_K(\nu_h)$ $|\partial_K F|$-a.e. on $F^- = F \cap \{t \leqslant g_r\}$. Here ν_h is the horizontal unit normal to F.

To prove (5.5.2) we consider two families of functions. For $0 < \varepsilon < 1$ we consider smooth functions φ_ε, depending on the Riemannian distance to the vertical axis $L = \{x = y = 0\}$, so that $0 \leqslant \varphi_\varepsilon \leqslant 1$ and

$$\begin{aligned} \varphi_\varepsilon(p) &= 0, & d(p,L) &\leqslant \varepsilon^2, \\ \varphi_\varepsilon(p) &= 1, & d(p,L) &\geqslant \varepsilon, \\ |\nabla\varphi_\varepsilon(p)| &\leqslant 2/\varepsilon, & \varepsilon^2 \leqslant d(p,L) &\leqslant \varepsilon. \end{aligned}$$

Again for $0 < \varepsilon < 1$ we consider smooth functions ψ_ε, depending on the Riemannian distance to the Euclidean hyperplane $\Pi_0 = \{t = 0\}$, so that $0 \leqslant \psi_\varepsilon \leqslant 1$ and

$$\begin{aligned} \psi_\varepsilon(p) &= 1, & d(p,\Pi_0) &\leqslant \varepsilon^{-1/2}, \\ \psi_\varepsilon(p) &= 0, & d(p,\Pi_0) &\geqslant \varepsilon^{-1/2} + 1, \\ |\nabla\psi_\varepsilon(p)| &\leqslant 2, & \varepsilon^{-1/2} \leqslant d(p,\Pi_0) &\leqslant \varepsilon^{-1/2} + 1. \end{aligned}$$

For any $\varepsilon > 0$, the vector field $\varphi_\epsilon\psi_\varepsilon W$ has compact support.

It is easy to prove that F^+ and F^- have finite K-perimeter. Since F^+ has also finite (sub-Riemannian) perimeter, applying the Divergence Theorem to F^+ and the horizontal vector field $\varphi_\epsilon\psi_\varepsilon W^+$, we have

$$\begin{aligned} \int_{F^+} \operatorname{div}(\varphi_\epsilon\psi_\varepsilon W^+) d\mathbb{H}^1 = &\int_D \langle \varphi_\epsilon\psi_\varepsilon W^+, N_D \rangle dD \\ &+ \int_{\{t > g_r\}} \langle \varphi_\epsilon\psi_\varepsilon W^+, \nu_h \rangle d|\partial F|. \end{aligned} \tag{5.5.3}$$

Where N_D is the Riemannian unit normal to D pointing into F^-, dD is the Riemannian area element on D, and ν_h is the outer horizontal unit normal to F.

We take limits in the left hand side of Equation (5.5.3) when $\varepsilon \to 0$. We write

$$\int_{F^+} \operatorname{div}(\varphi_\epsilon \psi_\varepsilon W^+) d\mathbb{H}^1 = \int_{F^+} \varphi_\epsilon \psi_\varepsilon \operatorname{div} W^+ d\mathbb{H}^1 + \int_{F^+} \langle \nabla(\varphi_\epsilon \psi_\varepsilon), W^+ \rangle d\mathbb{H}^1. \tag{5.5.4}$$

Since $\langle \varphi_\varepsilon \nabla \psi_\varepsilon, W^+ \rangle$ is bounded and converges pointwise to 0, and

$$\int_{F^+} \langle \psi_\varepsilon \nabla \varphi_\varepsilon, W^+ \rangle \leqslant \int_{\{(x,t):\varepsilon^2<|x|<\varepsilon,\ 0<t<\varepsilon^{-1/2}+1\}} \psi_\varepsilon |\nabla \varphi_\epsilon| d\mathbb{H}^1,$$

we have

$$\lim_{\varepsilon\to 0} \int_{F^+} \langle \nabla(\varphi_\epsilon \psi_\varepsilon), W^+ \rangle d\mathbb{H}^1 = 0. \tag{5.5.5}$$

On the other hand, $\operatorname{div} W^+ = \frac{1}{r}$, the mean curvature of $r\mathbb{B}_K$. We consider the orthonormal vectors $Z = -J(\nu_h)$, $E = \langle N, T\rangle\, \nu_h - |\nu_h|\, T$ and N, globally defined on $(rK_0 \times \mathbb{R}) \setminus L$ by vertical translations. We know from Lemma 5.2.5 that

$$\langle D_Z W^+, Z\rangle = \tfrac{1}{r}, \qquad \langle D_E W^+, E\rangle = 2\,\langle N, T\rangle |N_h|\, \langle W^+, J(\nu_h)\rangle.$$

It remains to compute $\langle D_N W^+, N\rangle$. We express $N = \lambda E + \mu T$ as a linear combination of E and T, where $\lambda = |N_h|/\langle N, T\rangle$, $\mu = 1/\langle N, T\rangle$. Observe that $\langle N, T\rangle \neq 0$ on $\operatorname{int}(K_0)$ since $r\mathbb{S}_K^+$ is a t-graph. So we have

$$\begin{aligned}
\langle D_N W^+, N\rangle &= \lambda \langle D_E W^+, N\rangle + \mu \langle D_T W^+, N\rangle \\
&= \lambda^2 \langle D_E W^+, E\rangle + \lambda\mu \langle D_E W^+, T\rangle + \mu \langle J(W^+), N_h\rangle \\
&= \lambda^2 \langle D_E W^+, E\rangle - \lambda\mu \langle N, T\rangle \langle W^+, J(\nu_h)\rangle - \mu |N_h| \langle W^+, J(\nu_h)\rangle \\
&= \left(\frac{|N_h|}{\langle N, T\rangle}\right)^2 \langle D_E W^+, E\rangle - \frac{1}{\langle N, T\rangle^2}\, \langle D_E W^+, E\rangle \\
&= \langle D_E W^+, E\rangle,
\end{aligned}$$

where we have used that $D_T W^+ = J(W^+)$ since W^+ is a linear combination of W^+, Y multiplied by functions that do not depend on t. Hence

$$\operatorname{div} W^+ = \langle D_Z W^+, Z\rangle + \langle D_E W^+, E\rangle + \langle D_N W^+, N\rangle = \frac{1}{r}$$

on $\operatorname{int}(K_0)$. Since $\varphi_\epsilon \psi_\varepsilon \operatorname{div} W^+$ is uniformly bounded, F^+ has finite volume and $\lim_{\varepsilon\to 0} \varphi_\epsilon \psi_\varepsilon = 1$, we can apply Lebesgue's Dominated Convergence Theorem to get

$$\lim_{\varepsilon\to 0} \int_{F^+} \varphi_\epsilon \psi_\varepsilon \operatorname{div} W^+ d\mathbb{H}^1 = \tfrac{1}{r} |F^+|. \tag{5.5.6}$$

So we get from (5.5.4), (5.5.5) and (5.5.6)

$$\lim_{\varepsilon\to 0} \int_{F^+} \operatorname{div}(\varphi_\varepsilon \psi_\varepsilon W^+)\, d\mathbb{H}^1 = \tfrac{1}{r}\, |F^+|. \tag{5.5.7}$$

Now we treat the remainings terms in (5.5.3). Using the representation of perimeter obtained in (2.3.4) for sets of finite K-perimeter sets we have

$$\int_{\{t>g_r\}} \langle W^+, \nu_h\rangle d|\partial F| \leqslant \int_{\{t>g_r\}} |\nu_h|_* d|\partial F| = |\partial F^+|_K, \tag{5.5.8}$$

with equality if and only if $W^+ = \pi(\nu_h)$ $|\partial F|$-a.e. on $\{t > g_r\}$. From equations (5.5.7) and (5.5.8), taking limits in Equation (5.5.3) when $\varepsilon \to 0$,

$$\tfrac{1}{r}|F^+| \leqslant \int_D \langle W^+, N_D\rangle dD + |\partial F^+|_K, \tag{5.5.9}$$

with equality if and only if $W^+ = \pi(\nu_h)$ $|\partial F|$-a.e. on $\partial F \cap \{t > g_r\}$.

We consider now the foliation of $rK_0 \times \mathbb{R}$ by vertical translations of $r\mathbb{S}_K^-$. Reasoning as in the previous case we get

$$\tfrac{1}{r}|F^-| \leqslant -\int_D \langle W^-, N_D\rangle dD + |\partial F^-|_K. \tag{5.5.10}$$

with equality if and only if $W^- = \pi(\nu_h)$ $|\partial F|$-a.e. on $\partial\{t < g_r\}$. Hence, adding (5.5.9) and (5.5.10), and taking into account $|\partial F|_K(\mathbb{H}^1 \setminus D) = |\partial F|_K$ and that $F \cap D$ does not contribute to the volume of F, we get

$$\tfrac{1}{r}|F| \leqslant \int_D \langle W^+ - W^-, N_D\rangle dD + |\partial F|_K,$$

and so (5.5.2) holds, with equality if and only if equalities (5.5.9) and (5.5.10) hold. This completes the first part of the proof.

Recall that $h : rK_0 \to \mathbb{R}$ is a function so that $D = \operatorname{graph}(h) \subset E$. We take two values $t_m < t_M$ such that

$$h + t_m < g_r < h + t_M.$$

We apply inequality (5.5.2) to the set $B = B^- \cup B^0 \cup B^+$, where

- $B^0 = \{(x,t) : x \in rK_0, |t - g_r| \leqslant (t_M - t_m)/2\}$,
- $B^+ = r\mathbb{B}_K^+ + (0, (t_M - t_m)/2)$,
- $B^- = r\mathbb{B}_K^- - (0, (t_M - t_m)/2)$.

By construction, $D = \operatorname{graph}(g_r) \subset B^0$. Since the lateral boundary of B^0 is contained in $\partial(rK_0 \times \mathbb{R})$ and the outer unit normal to $\partial(rK_0 \times \mathbb{R})$ coincides with W^+ and W^-, the lateral K-boundary area of B^0 is equal to

$$(t_M - t_m)\int_{\partial(rK_0)} |\nu_0|_* d(\partial(rK_0)), \tag{5.5.11}$$

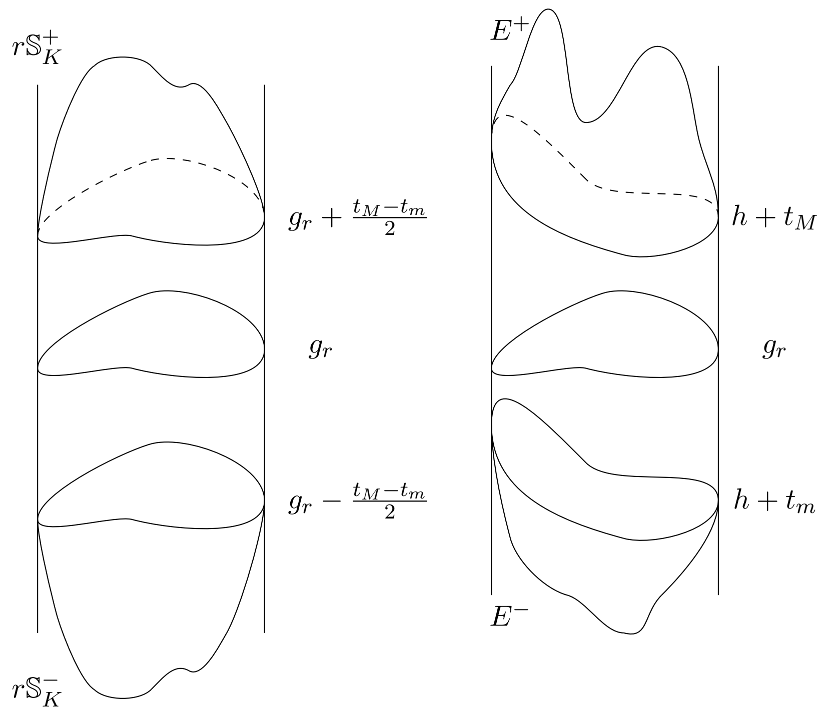

Figure 5.10: Geometric construction in the proof of Theorem 5.5.2

where $d(\partial(rK_0))$ is the Riemannian length element of the C^1 curve $\partial(rK_0)$. Hence we get

$$|\partial B|_K = (t_M - t_m) \int_{\partial(rK_0)} |\nu_0|_* d(\partial(rK_0)) + |\partial(r\mathbb{B}_K)|_K.$$

On the other hand, since

$$|B| = |r\mathbb{B}_K| + |rK_0|(t_M - t_m),$$

we obtain

$$\begin{aligned} \tfrac{1}{r}(|r\mathbb{B}_K| + |rK_0|(t_M - t_m)) &= \int_D \langle W^+ - W^-, N_D \rangle dD \\ &+ (t_M - t_m) \int_{\partial(rK_0)} |\nu_0|_* d\partial(rK_0) + |\partial(r\mathbb{B}_K)|_K. \end{aligned} \tag{5.5.12}$$

Now we apply (5.5.2) to the set $\mathbb{E}$ consisting on the union of $E^+ = E \cap \{t \geqslant h\}$ translated by the vector $(0, t_M)$, $E^- = E \cap \{t \leqslant h\}$ translated by the vector $(0, t_m)$ and the vertical filling in between the two sets. We reason as before to get

$$\begin{aligned}\tfrac{1}{r}(|E|+|rK_0|(t_M-t_m)) \leqslant \int_D \langle W^+ - W^-, N_D\rangle dD \\ + (t_M - t_m)\int_{\partial D} |\nu_0|_* d\partial D_0 + |\partial E|_K.\end{aligned} \tag{5.5.13}$$

From (5.5.12) and (5.5.13) we get

$$|\partial E|_K \geqslant |\partial(r\mathbb{B}_K)|_K + \tfrac{1}{r}(|E| - |r\mathbb{B}|).$$

Let $f(\rho) = |\partial(\rho\mathbb{B}_K)|_K + \frac{1}{\rho}(|E| - |\rho\mathbb{B}|)$. Since $\rho\mathbb{B}_K$ has mean curvature $\frac{1}{\rho}$, Theorem 5.2.15 guarantees that the Wulff shape $\rho\mathbb{B}_K$ is a critical point of $A - \frac{1}{\rho}|\cdot|$ for any variation. Therefore $|\partial(\rho\mathbb{B}_K)|'_K - \frac{1}{\rho}|\rho\mathbb{B}_K|' = 0$ where primes indicates the derivative with respect to ρ. Hence we have

$$f'(\rho) = -\tfrac{1}{\rho^2}(|E| - |\rho\mathbb{B}_K|).$$

So the only critical point of f corresponds to the value ρ_0 so that $|\rho_0\mathbb{B}_K| = |E|$. Since the function $\rho \mapsto |\rho B_K|$ is strictly increasing and takes its values in $(0, +\infty)$, we obtain that $f(\rho)$ is a convex function with a unique minimum at ρ_0. Hence we obtain

$$|\partial E|_K \geqslant f(r) \geqslant f(\rho_0) = |\partial(r_0\mathbb{B}_K)|_K, \tag{5.5.14}$$

which implies (5.5.1). □

5.5.1 A uniqueness result in $rK_0 \times \mathbb{R}$

We consider a horizontal lifting $\Gamma = (x, y, t)$ of a clockwise arc-length parametrization γ of ∂K. Then a parameterization of $\mathbb{S}_K$ is given by $(\mathbf{x}, \mathbf{y}, \mathbf{t})(u, v) = \ell_{-\Gamma(v)}(\Gamma(u+v))$. This means

$$\begin{aligned}\mathbf{x}(u,v) &= x(u+v) - x(v),\\ \mathbf{y}(u,v) &= y(u+v) - y(v),\\ \mathbf{t}(u,v) &= t(u+v) - t(v) - x(u+v)y(v) + y(u+v)x(v).\end{aligned} \tag{5.5.15}$$

As can be seen in the proof of Theorem 5.4.5, there is a unique value $v'(v) \equiv v'$ such that

$$\gamma_v(v') \in \partial K_0, \tag{5.5.16}$$

where $\gamma_v(u) = \gamma(v+u) - \gamma(v)$. We shall use the notation

$$\mathbb{S}_K^- = \bigcup_{v\in[0,L)} \Gamma_v(0, v'], \qquad \mathbb{S}_K^+ = \bigcup_{v\in[0,L)} \Gamma_v[v', L),$$

where Γ_v is the horizontal lifting of $\gamma_v(u)$ and L is the length of γ. We write ν_0 to denote the horizontal unit normal vector to $\mathbb{S}_K$. It is easy to see that

$$|\nu_0(u,v)|_* = |J(\dot\gamma_v(u))|_* = |J(\dot\gamma(u+v))|_* = |\gamma(u+v)|. \tag{5.5.17}$$

Lemma 5.5.3. *Let β an arc-length parametrization of ∂K_0. The map $\psi : [0, L] \to \partial K_0$ given by*

$$\psi(v) = \gamma_v(v')$$

is bijective, C^1 and $(\beta^{-1} \circ \psi)' \geqslant 1$.

Proof. Fixed v_0, the curve $\gamma(v_0') - \gamma(\cdot)$ is in K_0 and equal to ψ at v_0, and we can take s_0 such that

$$\beta(s_0) = \psi(v_0), \qquad \dot\beta(s_0) = -\dot\gamma(v_0).$$

In particular, we have

$$\psi(v_0) = \beta\big(\dot\beta^{-1}(-\dot\gamma(v_0)\big)$$

and $\psi = \beta \circ \dot\beta^{-1} \circ -\dot\gamma$. Hence ψ is bijective. Since ∂K_0 has less curvature than ∂K and $\dot\beta(s_0) = -\dot\gamma(v_0)$, it follows that

$$\dot\beta^{-1}(-\dot\gamma(v_0 + t)) \geqslant -\dot\gamma^{-1}(-\dot\gamma(v_0 + t)) = v_0 + t.$$

Hence we get

$$(\beta^{-1} \circ \psi)(v_0 + t) - (\beta^{-1} \circ \psi)(v_0) \geqslant t,$$

and it follows that $(\beta^{-1} \circ \psi)' \geqslant 1$. □

Given $f \in C^0(\partial K_0)$, we shall abuse the notation and write $f \equiv f \circ \psi$ when the domain is $[0, L)$. We define the sets

$$\begin{aligned} \mathbb{S}_f^- &= \bigcup_{v \in [0,L)} \Gamma_v(0, v'] + (0, 0, f(v)) \\ \mathbb{S}_f^+ &= \bigcup_{v \in [0,L)} \Gamma_v[v', L) + (0, 0, f(v)). \end{aligned} \tag{5.5.18}$$

Lemma 5.5.4. *Let f and g be Euclidean Lipschitz functions in ∂K_0. Then*

$$|\mathbb{S}_f^-|_K + |\mathbb{S}_g^+|_K - |\partial\mathbb{B}_K|_K \geqslant \int_0^L (f'(v) - g'(v))\Big(\int_0^{v'(v)} |\gamma(v + u)| du\Big) dv, \tag{5.5.19}$$

where $v'(v)$ is defined in (5.5.16)*. Moreover, equality holds in* (5.5.19) *if and only if $f' = g' = 0$ almost everywhere.*

Proof. Let Φ is the parametrization of $\mathbb{S}_K$ given in (5.5.15). The map $\Phi_f(u, v) = \Phi(u, v) + (0, 0, f(v))$ is a parametrization of $\mathbb{S}_f$. By the representation of the sub-Finsler perimeter for Euclidean Lipschitz surfaces (2.3.10), we have

$$\begin{aligned}
|\mathbb{S}_f^-|_K &= \int_0^L \int_0^{v'} |\partial_u \Phi_f \times \partial_v \Phi_f| \Big| \frac{(\partial_u \Phi_f \times \partial_v \Phi_f)_h}{|\partial_u \Phi_f \times \partial_v \Phi_f|} \Big|_* dudv \\
&= \int_0^L \int_0^{v'} |(\partial_u \Phi_f \times \partial_v \Phi_f)_h|_* dudv \\
&= \int_0^L \int_0^{v'} |(\partial_u \Phi \times \partial_v \Phi + \partial_u \Phi \times (0,0,f'))_h|_* dudv.
\end{aligned}$$

Let $\tau_0 = (\partial_u \Phi \times \partial_v \Phi)_h$ and $\tau_{f'} = \tau_0 + (\partial_u \Phi(u,v) \times f'(v))_h$. By definition of Φ, it is clear that

$$(\partial_u \Phi(u,v) \times f'(v))_h = f'(v) J(\dot\gamma(u+v)).$$

Since $\tau_0/|\partial_u \Phi \times \partial_v \Phi|$ and $J(\dot\gamma(u+v))$ are the horizontal part of the normal to $\mathbb{S}_K$ and the horizontal unit normal to $\mathbb{S}_K$ respectively, they are the same vector but for a multiplicative constant. Hence $\pi(\tau_0(u,v)) = \pi(J(\dot\gamma(u+v)))$. Using (5.5.17) and the definition of π, we conclude that

$$\begin{aligned}
|\mathbb{S}_f^-|_K =& \int_0^L \int_0^{v'} \langle \tau_{f'}(u,v), \pi(\tau_{f'}(u,v)) \rangle dudv \\
\geqslant& \int_0^L \int_0^{v'} \langle \tau_{f'}(u,v), \pi(\tau_0(u,v)) \rangle dudv \\
=& \int_0^L \int_0^{v'} \langle \tau_0(u,v), \pi(\tau_0(u,v)) \rangle dudv \\
&+ \int_0^L \int_0^{v'} f'(v) \langle J(\dot\gamma(u+v)), \pi(\tau_0(u,v)) \rangle dudv \\
=& |\mathbb{S}_K^-|_K + \int_0^L f'(v) \int_0^{v'} |\gamma(v+u)| dudv.
\end{aligned} \tag{5.5.20}$$

Reasoning similarly for $|\mathbb{S}_g^+|_K$ and adding we get

$$\begin{aligned}
&|\mathbb{S}_f^-|_K + |\mathbb{S}_g^+|_K - |\partial \mathbb{B}_K|_K \\
&\geqslant \int_0^L f'(v) \int_0^{v'} |\gamma(v+u)| dudv + \int_0^L g'(v) \int_{v'}^L |\gamma(v+u)| dudv \\
&= \int_0^L (f'(v) - g'(v)) \int_0^{v'} |\gamma(v+u)| dudv + \int_0^L g'(v) dv \int_0^L |\gamma(v+u)| du \\
&= \int_0^L (f'(v) - g'(v)) \int_0^{v'} |\gamma(v+u)| dudv.
\end{aligned}$$

where we used that $\int_0^L |\gamma(v+u)| du$ is independent of v and h is L-periodic.

Equality holds in (5.5.20) if and only if $\pi(\tau_{f'}) = \pi(\tau_0)$ almost everywhere. We know that $\tau_{f'}(u,v) = \lambda(u,v)\tau_0(u,v)$ for some λ. Let v_0 and $\varepsilon > 0$

such that $f'(v_0)$ exists and $f'(v_0) = -\varepsilon$. Notice that $\Phi(u, v_0)$ tends to a singular point as u tends to L. Hence $\lim_{u\to L} \tau_0(u, v_0) = 0$. Let u_0 and $\Omega \subset [0, L[\times[0, L[$ open such that, for any u and v in Ω it holds

$$\begin{aligned}
&|\tau_0(u_0, v_0)| < \varepsilon/4 \\
&|\tau_0(u, v) - \tau_0(u_0, v_0)| < \varepsilon/4 \\
&|f'(v) - f'(v_0)| < \varepsilon/2.
\end{aligned}$$

Then $|\tau_0(u, v)| < \varepsilon/2 < |f'(v)||J(\dot\gamma(u + v))|$ and $\tau_{f'}(u, v) = \lambda(u, v)\tau_0(u, v)$ for $\lambda(u, v) < 0$. It follows that $\pi(\tau_{f'}) \neq \pi(\tau_0)$ in Ω. □

The vector fields U^+ and U^- in $(rK_0 \times \mathbb{R}) \setminus L$ are defined by vertical translations of

$$J\big(\nu_0|_{\mathbb{S}^+_{\rho_0}}\big), \qquad J\big(\nu_0|_{\mathbb{S}^-_{\rho_0}}\big).$$

For any (x, y, t) in the interior of $(rK_0 \times \mathbb{R}) \setminus L$ we can write locally U^+ as

$$U^+ = r\big(\dot{\mathbf{x}}(\mathbf{u} + \mathbf{v})X + \dot{\mathbf{y}}(\mathbf{u} + \mathbf{v})Y\big),$$

where $(\mathbf{u}, \mathbf{v}) = (\mathbf{x}, \mathbf{y})^{-1}$, $\mathbf{u} > \mathbf{v}'$ is well defined by Theorem 5.4.5. Therefore there exists a unique integral curve of U^+ passing through (x, y, t). Given $q \in \partial(rK_0) \times \mathbb{R}$, we consider the vertical translation containing q of the horizontal lifting of a parametrization of $\partial(rK_0)$. Such a curve is denoted by Ψ_q and assume that $\Psi_q(0) = q$, and is also an integral curve of U^+. Hence an integral curve of U^+ in $\partial(rK_0) \times \mathbb{R}$ can move along $\partial(rK_0) \times \mathbb{R}$ or move into $\mathrm{int}(rK_0) \times \mathbb{R}$.

Theorem 5.5.5. *Let $r > 0$ and $h : rK_0 \to \mathbb{R}$ a C^0 function. Consider a subset $E \subset \mathbb{H}^1$ with finite volume and K-perimeter such that*

$$\mathrm{graph}(h) \subseteq E \subset rK_0 \times \mathbb{R}.$$

Moreover, assume that there exists $h_-, h_+ : \partial rK_0 \to \mathbb{R}$ Euclidean Lipschitz functions with $h_- \leqslant h_+$ such that

i) $\mathrm{graph}(h_\pm) \subset \partial E$,

ii) $C := \{(z, t) : z \in \partial K_0, \quad h_-(z) \leqslant t \leqslant h_+(z)\} \subset \partial E$,

iii) There exists $\{t_n\} \searrow 0$ such that for any $q \in \mathrm{graph}(h_+)$ and $p \in \mathrm{graph}(h_-)$,

$$\Psi_q(t_n) \notin \partial E, \qquad \Psi_p(-t_n) \notin \partial E,$$

where Ψ_q is the vertical translation of the horizontal lifting of a clockwise parametrization of $\partial(rK_0) \times \mathbb{R}$ passing through q at 0. Then, equality holds in (5.5.1) if and only if the sets E and $\mathbb{B}_E$ coincide.

Proof. We proof of Theorem 5.5.5 is divided into several steps. We shall use the notation used in the proof of Theorem 5.5.2.

Step 1. Equality holds in (5.5.14) and by convexity $r = \rho_0$. Moreover, equality also holds in (5.5.13) and $\pi_K(\nu_h)$ coincides with W^+ in $(\partial E \cap \{t \geqslant g_r\}) \setminus L$, and with W^- in $(\partial E \cap \{t \leqslant g_r\}) \setminus L$, where ν_h is the horizontal unit normal to E. Hence $\nu_h = \lambda\nu_0$ with $\lambda > 0$ by the strict convexity of K. Since ν_0 and ν_h has unit norm we get $\nu_h = \nu_0$. By [129, Theorem 1.2], $\partial E \setminus L$ is a $\mathbb{H}$-regular hypersurface and by [147, Lemma 2.5] the integral curves of $J(\nu_h)$ starting from points in $(\partial E \cap \{t \geqslant g_r\}) \setminus L$ are contained in $\partial E \setminus L$. Such a curve is also an integral curve of U^+ or U^-. Thus, by hypothesis, the sets $r\mathbb{S}^+_{h_{r,+}}$, $r\mathbb{S}^+_{h_{r,-}}$ and C are contained in ∂E, where $h_{r,\pm}(x) = r^2 h_\pm(\frac{x}{r})$. Hence

$$|\partial E|_K \geqslant |r\mathbb{S}^+_{h_{r,+}}|_K + |r\mathbb{S}^-_{h_{r,-}}|_K + |C|_K. \tag{5.5.21}$$

Step 2. Let us denote $\varphi(v) = \int_0^{v'} |\gamma(v+u)| du$ and $\phi = h_{r,+} - h_{r,-}$. Thanks to Lemma 5.5.4 and the homogeneity of the K-perimeter, we get

$$|r\mathbb{S}^+_{h_{r,+}}|_K + |r\mathbb{S}^-_{h_{r,-}}|_K - |\partial\mathbb{B}_{\rho_0}|_K \geqslant r^3 \int_0^L \phi'\varphi dv. \tag{5.5.22}$$

From the expression for the lateral perimeter (5.5.11), Lemma 5.5.3 and (5.5.17), we have

$$|C|_K = \int_{\partial K_0} (h_+ - h_-)|\nu_0|_* d(\partial r K_0) \geqslant r^3 \int_0^L \phi(v)|\gamma(v'+u)| dv. \tag{5.5.23}$$

Moreover, taking the derivative of φ we get $|\gamma(v'+u)| = \varphi'(v) + |\gamma(v)|$. Hence we obtain

$$\begin{aligned}\int_0^L \phi'\varphi dv + \int_0^L \phi(v)|\gamma(v'+u)| dv &\geqslant \int_0^L \phi'\varphi dv + \int_0^L \phi\varphi' dv + \int_0^L \phi|\gamma| dv \\ &= [\phi\varphi]_0^L + \int_0^L \phi|\gamma| dv = \int_0^L \phi|\gamma| dv. \end{aligned} \tag{5.5.24}$$

Substituting (5.5.22) and (5.5.23) in (5.5.21) and using (5.5.24), we get

$$|\partial E|_K \geqslant |\partial\mathbb{B}_{\rho_0}|_K + r^3 \int_0^L \phi(v)|\gamma(v)| dv \geqslant |\partial\mathbb{B}_{\rho_0}|_K. \tag{5.5.25}$$

If equality holds then we have equality in (5.5.25) and in (5.5.19) from where we obtain that $h_+ = h_-$ and h_+ and h_- are constant. Hence $r\mathbb{S}^+_{h_{r,+}} \cup r\mathbb{S}^+_{h_{r,-}}$ is a translation of $r\mathbb{S}_K \setminus r\mathbb{S}^0_K$ and $r\mathbb{S}_K \setminus r\mathbb{S}^0_K \subseteq \partial E$.

Step 3. We claim that $\mathbb{B}_{\rho_0} \subseteq \bar{E}$. To prove this we shall show that $\mathbb{B}_{\rho_0} \setminus L \subseteq \bar{E}$ reasoning by contradiction. If $\mathbb{B}_{\rho_0} \setminus L \not\subseteq \bar{E}$, since $r\mathbb{S}_K = \partial\mathbb{B}_{\rho_0} \subseteq \partial E$, there is a point p in the interior of $\mathbb{B}_{\rho_0} \setminus L$ so that $p \notin \bar{E}$.

The Euclidean orthonormal projection x of p over $t = 0$ lies in $D \subseteq E$. Hence there is a point q in the segment $[p, x] \subseteq \mathbb{B}_{\rho_0}$ that belongs to $\partial E \setminus L$. As $\partial E \setminus L$ is $\mathbb{H}$-regular, the perimeter of ∂E in a small ball contained in the interior of $\mathbb{B}_{\rho_0} \setminus L$ and centered at q is positive, and so, $|\partial E| > |\partial \mathbb{B}_{\rho_0}|$, which contradicts our assumption that equality holds in (5.5.1). This implies $\mathbb{B}_{\rho_0} \subseteq \bar{E}$. As $|\mathbb{B}_{\rho_0}| = |E|$ we obtain $\mathbb{B}_{\rho_0} = \bar{E}$ by the normalization of E. □

Remark 5.5.6. Hypothesis *i*), *ii*) and *iii*) in Theorem 5.5.5 holds as soon as the intersection of ∂E with $\partial r K_0 \times \mathbb{R}$ is an Euclidean Lipschitz curve.

Notes

Notes of § 5.2 1. In [154], the author obtained an expression of the mean curvature of a C^2 surface in terms of a parametrization when $\mathbb{H}^1$ is endowed with the left-invariant norm $|\cdot|_\infty$, and defined a notion of distributional mean curvature for polygonal norms.

The result in Theorem 5.2.13 includes that constant mean curvature surfaces for the sub-Riemannian area in the Heisenberg group are foliated by geodesics. This result can be found, with slight variations, in [33; 36; 34; 79; 78].

Notes of § 5.5 1. Theorem 5.5.2 differs from Theorem 3.1 in [147] in two aspects when restricted to the sub-Riemannian case. On the one hand, in Theorem 5.5.2 we consider sets with a membrane not necessarily the same as the one of the Pansu-Wulff shape, but any continuous function over the cylinder. But on the other hand, we do not characterize those sets for which equality holds in (5.5.1).

Chapter 6

Area-minimizing t-graphs with low-regularity in $\mathbb{H}^1$

This chapter is devoted to the results obtained with Giovannardi and Ritoré given in [87].

We consider $(\mathbb{H}^1, K)$ and provide examples of entire area-minimizing horizontal graphs which are locally Lipschitz in Euclidean sense. A large number of them fail to have further regularity properties. The examples are obtained prescribing as singular set a horizontal line or a finite union of horizontal half-lines extending from a given point. Of particular interest are the conical examples invariant by the non-isotropic dilations of $\mathbb{H}^1$. In the sub-Riemannian case these examples were investigated in [91] and [146].

The chapter is organized the following way. In Section 6.1 we obtain a necessary and sufficient condition, based on Theorem 5.2.1, for a surface to be a critical point of the sub-Finsler area. We assume that the surface is piecewise C^2, composed of pieces meeting in a C^1 way along C^1 curves. This condition will allow us to construct area-minimizing examples in Proposition 6.2.3 of Section 6.2, and examples with low regularity in Proposition 6.2.4. The same construction, keeping fixed the angle at one side of the singular line, provides examples of area-minimizing cones, see Corollary 6.2.5. Finally, in Section 6.3 we provide examples of area-minimizing cones in the spirit of [91]. These examples are obtained in Theorem 6.3.2 from circular sectors of the area-minimizing cones with one singular line obtained in Corollary 6.2.5.

6.1 The first variation formula and a stationary condition

In this section we present some consequences of the first variation formula. We consider $(\mathbb{H}^1, K)$, where K is of class C^2_+ with $0 \in \mathrm{int}(K)$.

Using Theorem 5.2.1 we can prove the following necessary and sufficient condition for a surface S to be A_K-stationary. When a surface S of class C^1 is divided into two parts S^+, S^- by a singular curve S_0 so that S^+, S^- are of class C^2 up to the boundary, the tangent vectors Z^+, Z^- can be chosen so that they parameterize the characteristic curves as curves leaving from S_0, see Corollary 3.6 in [33]. In this case $\eta^+ = \pi(\nu_h) = \pi(J(Z^+))$ and $\eta^- = \pi(J(Z^-))$.

Corollary 6.1.1. *Let S be an oriented surface of class C^1 such that the singular set S_0 is a C^1 curve. Assume that $S \setminus S_0$ is the union of two surfaces S^+, S^- of class C^2 meeting along S_0. Let η^+, η^- the restrictions of η to S^+ and S^-, respectively. Then S is area-stationary if and only if*

1. *$H_K = 0$, and*
2. *$\eta^+ - \eta^-$ is tangent to S_0.*

In particular, condition $H_K = 0$ implies that $S \setminus S_0$ is foliated by horizontal straight lines.

Proof. We may apply the divergence theorem to the second term in (5.2.1) to get

$$\frac{d}{ds}\bigg|_{s=0} A_K(\varphi_s(S)) = \int_{S \setminus S_0} H_K u \, dS - \int_{S_0} u \, \langle \xi, (\eta^+ - \eta^-)^\top \rangle \, dS,$$

where ξ is the outer unit normal to S^+ along S_0. Hence the stationary condition is equivalent to $H = 0$ on $S \setminus S_0$ and $\langle \xi, \eta^+ - \eta^- \rangle = 0$. The latter condition is equivalent to that $\eta^+ - \eta^-$ be tangent to S_0.

That $H_K = 0$ implies that $S \setminus S_0$ is foliated by horizontal straight lines was proven in Theorem 3.14 in [142]. □

Since $\nu^+ = J(Z^+), \nu^- = J(Z^-)$, where Z^+ and Z^- are the extensions of the horizontal tangent vectors in S^+, S^-, we have that the second condition in Corollary 6.1.1 is equivalent to

$$\pi(J(Z^+)) - \pi(J(Z^-)) \text{ is tangent to } S_0. \tag{6.1.1}$$

So a natural question is, given a C^2_+ convex body K containing 0 in its interior, and a unit vector $v \in \mathbb{S}^1$, can we find a pair of unit vectors Z^+, Z^- such that (6.1.1) is satisfied? If such vectors exist, how many pairs can we get? The answer follows from next Lemma.

Lemma 6.1.2. *Let K be a convex body of class C^2_+ such that $0 \in \text{int}(K)$. Given $v \in \mathbb{R}^2 \setminus \{0\}$, let $L \subset \mathbb{R}^2$ be the vector line generated by v. Then, for any $u \in \partial K$, we have the following possibilities*

1. *The only $w \in \partial K$ such that $w - u \in L$ is $w = u$, or*

2. There is only one $w \in \partial K$, $w \neq u$ such that $w - u \in L$.

The first case happens if and only if L is parallel to the support line of K at u.

Proof. Let T be the translation in $\mathbb{R}^2$ of vector u. Then $T(L)$ is a line that meets ∂K at u. The line $T(L)$ intersects ∂K only once when L is the supporting line of $T(K)$ at 0; otherwise L intersects ∂K at another point $w \neq u$ so that $w - u \in L$. □

Remark 6.1.3. We use Lemma 6.1.2 to understand the behavior of characteristic curves meeting at a singular point $p \in S_0$. Let Z^+, Z^- be the tangent vectors to the characteristic lines starting from p. Let ν^+, ν^- be the vectors $J(Z^+), J(Z^-)$, and L the line generated by the tangent vector to S_0 at p. The condition that S is stationary implies that $\eta^+ - \eta^- \in L$. If $w = \eta^+$ and $u = \eta^-$ are equal then $\nu^+ = \nu^-$ are orthogonal to L, which implies that Z^+, Z^- lie in L. This is not possible since characteristic lines meet tranversaly the singular line, again by Corollary 3.6 in [33].

Hence $\eta^+ \neq \eta^-$ and η^+ is uniquely determined from η^- by Lemma 6.1.2. Obviously the roles of η^+ and η^- are interchangeable.

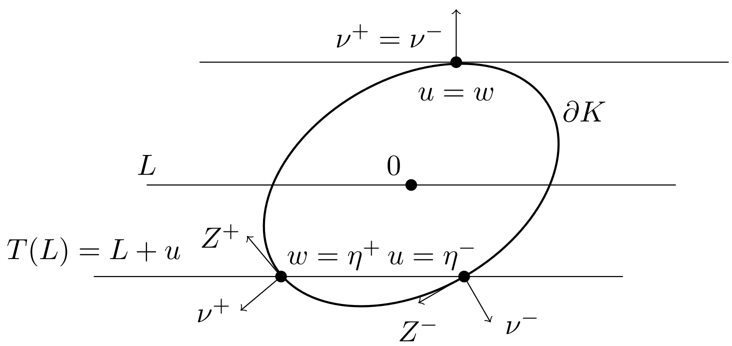

Figure 6.1: Geometric construction to obtain $w = \eta^+$ from $u = \eta^-$ so that the stationary condition is satisfied. The case $\nu^+ = \nu^-$ cannot hold.

6.2 Examples of entire K-perimeter minimizing horizontal graphs with one singular line

Remark 6.1.3 implies that Z^- can be uniquely determined from Z^+ when S is a stationary surface. Let us see that this result can be refined to provide a smooth dependence of the oriented angle $\angle(v, Z^-)$ in terms of $\angle(v, Z^+)$. We use complex notation for horizontal vectors assuming that the horizontal distribution is positively oriented by $v, J(v)$ for any $v \in \mathcal{H} \setminus \{0\}$.

Lemma 6.2.1. *Let K be a convex body of class C^2_+ with $0 \in \operatorname{int}(K)$. Consider a unit vector $v \in \mathbb{R}^2$ and let $L \subset \mathbb{R}^2$ be the vector line generated by v. Then, for any $\alpha \in (0,\pi)$ there exists a unique $\beta \in (\pi, 2\pi)$ such that if $Z^+ = ve^{i\alpha}$, $Z^- = ve^{i\beta}$, then $\pi(J(Z^+)) - \pi(J(Z^+))$ belongs to L.*

Moreover the function $\beta : (0,\pi) \to (\pi, 2\pi)$ is of class C^1 with negative derivative.

Proof. We change coordinates so that L is the line $y = 0$. We observe that $Z^+ = ve^{i\alpha}$ implies that $J(Z^+) = ve^{i(\alpha+\pi/2)}$. We define $(x,y) : \mathbb{S}^1 \to \partial K$ by

$$(x(\alpha), y(\alpha)) = \mathcal{N}_K^{-1}(ve^{i(\alpha+\pi/2)}),$$

where $\mathcal{N}_K : \partial K \to \mathbb{S}^1$ is the (outer) Gauss map of ∂K. The functions x, y are C^1 since $\mathcal{N}_K$ is C^1. The point $(x(\alpha), y(\alpha))$ is the only one in ∂K such that the clockwise oriented tangent vector to ∂K makes an angle α with the positive direction of the line L. A line parallel to L meets ∂K at a single point only when $\alpha + \pi/2 = \pi/2$ or $\alpha + \pi/2 = 3\pi/2$. Hence, for $\alpha \in (0,\pi)$, there is a unique $\beta \in (\pi, 2\pi)$ such that

$$(x(\beta), y(\beta)) - (x(\alpha), y(\alpha)) \in L.$$

Observe that, for $\alpha \in (0,\pi)$, we have $dy/d\alpha > 0$ and, for $\beta \in (\pi, 2\pi)$, we get $dy/d\beta < 0$. we can use the implicit function theorem (applied to $y(\beta) - y(\alpha)$) to conclude that β is a C^1 function of α. Moreover

$$\frac{d\beta}{d\alpha} = \frac{dy/d\alpha}{dy/d\beta} < 0. \qquad \square$$

Now we give the main construction in this section.

We fix a vector $v \in \mathbb{R}^2 \setminus \{0\}$ and the line $L_v = \{\lambda v : \lambda \in \mathbb{R}\}$. For every $\lambda \in \mathbb{R}$, we consider two half-lines, $r^+_\lambda, r^-_\lambda \subset \mathbb{R}^2$, extending from the point $p = \lambda v \in L_v$ with angles $\alpha(\lambda)$ and $\beta(\lambda)$ respectively. Here $\alpha : \mathbb{R} \to (0,\pi)$ is a non-increasing function and $\beta(\lambda)$ is the composition of $\alpha(\lambda)$ with the function obtained in Lemma 6.2.1. Hence $\beta(\lambda)$ is a non-decreasing function. The line L_v can be lifted to the horizontal straight line $R_v = L_v \times \{0\} \subset \mathbb{H}^1$ passing through the point $(0,0,0)$, and the half-lines $r^\pm_\lambda$ can be lifted to horizontal half-lines $R^\pm_\lambda$ starting from the point $(\lambda v, 0)$ in the line R_v.

The surface obtained as the union of the half-lines R^+_λ and R^-_λ, for $\lambda \in \mathbb{R}$, is denoted by $\Sigma_{v,\alpha}$. Since any $R^\pm_\lambda$ is a graph over $r^\pm_\lambda$ and $\bigcup_{\lambda\in\mathbb{R}}(r^+_\lambda \cup r^-_\lambda)$ covers the xy-plane, we can write the surface $\Sigma_{v,\alpha}$ as the graph of a continuous function $u_\alpha : \mathbb{R}^2 \to \mathbb{R}$. Writing $v = e^{i\alpha_0}$, the surface $\Sigma_{v,\alpha}$ can be parametrized by $\Psi : \mathbb{R}^2 \to \mathbb{R}^3$ as follows

$$\Psi(\lambda,\mu) = \begin{cases} \big(\lambda e^{i\alpha_0} + \mu e^{i(\alpha_0+\alpha(\lambda))}, -\mu\lambda \sin\alpha(\lambda)\big), & \mu \geqslant 0, \\ \big(\lambda e^{i\alpha_0} + |\mu| e^{i(\alpha_0+\beta(\lambda))}, -|\mu|\lambda \sin\beta(\lambda)\big), & \mu \leqslant 0. \end{cases} \tag{6.2.1}$$

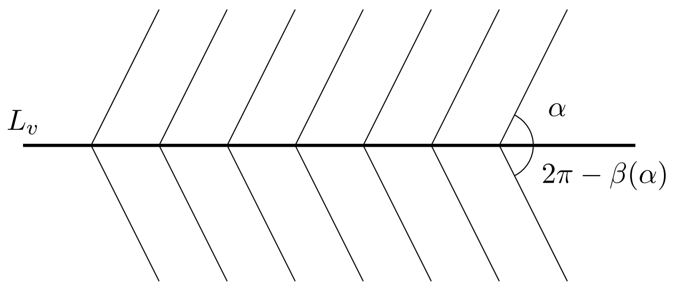

Figure 6.2: The planar configuration to obtain the surface $\Sigma_{v,\alpha}$. Here α is a constant function and K is the unit disc D. Such surfaces were called *herringbone surfaces* by Young [166] as they are the union of horizontal rays that branch out of a horizontal line.

Example 6.2.2. An special example to be considered is the sub-Riemannian cone Σ_α, where $\alpha \in (0, \pi)$. The projection of Σ_α to the horizontal plane $t = 0$ is composed of the line $y = 0$ and the half-lines starting from points in $y = 0$ with angles α and $-\alpha$. This cone can be parametrized, for $s \in \mathbb{R}, t \geqslant 0$, by

$$(u, v) \mapsto (u + v \cos \alpha, v \sin \alpha, -uv \sin \alpha)$$

when $y \geqslant 0$, and by

$$(u + v \cos \alpha, -v \sin \alpha, uv \sin \alpha)$$

when $y \leqslant 0$. A straigthforward computation implies that Σ_α is the t-graph of the function

$$u_\alpha(x, y) = -xy + \cot \alpha \, y|y|. \tag{6.2.2}$$

Observe that

$$\lim_{\alpha \to 0} u_\alpha(x, y) = \begin{cases} +\infty, & y > 0, \\ 0, & y = 0, \\ -\infty, & y < 0, \end{cases} \tag{6.2.3}$$

so that the subgraph of Σ_α converges pointwise locally when $\alpha \to 0$ to a vertical half-space.

The following lemma provides some properties of u_α when $\alpha(\lambda)$ is a smooth function of λ.

Proposition 6.2.3. *Let $\alpha \in C^k(\mathbb{R})$, $k \geqslant 2$, be a non-decreasing function. Then*

i) u_α is a C^k function in $\mathbb{R}^2 \setminus L_v$,

ii) u_α is merely $C^{1,1}$ near L_v when $\beta \neq \alpha + \pi$.

iii) u_α is C^∞ in any open set I of values of λ when $\beta = \alpha + \pi$ on I.

iv) $\Sigma_{v,\alpha}$ *is K-perimeter-minimizing when* $\beta = \beta(\alpha)$.

v) The projection of the singular set of $\Sigma_{v,\alpha}$ *to the xy-plane is* L_v.

Proof. i), ii), iii) and v) are proven in Lemma 3.1 in [146].

We prove iv) by a calibration argument. We shall drop the subscript α to simplify the notation. Let E be the subgraph of u and $F \subseteq \mathbb{H}^1$ such that $F = E$ outside a Euclidean ball centered at the origin. Let $P = \{(z,t) : \langle z, v\rangle = 0\}$, $P^1 = \{(z,t) : \langle z, v\rangle > 0\}$ and $P^2 = \{(z,t) : \langle z, v\rangle < 0\}$. We define two vector fields U^1, U^2 on P^1, P^2 respectively by vertical translations of the vectors $\pi(\nu_E)|_{P^1} = \eta^+$ and $\pi(\nu_E)|_{P^2} = \eta^-$. They are C^2 in the interior of the halfspaces and extend continuously to the boundary plane P. As $\operatorname{div}(U^j)_{(z,t)}$ coincides with the sub-Finsler mean curvature of the translation of $\Sigma_{v,\alpha}$ passing through (z,t) as defined in (5.2.8), and this surfaces are foliated by horizontal straight lines in the interior of the halfspaces, by Theorem 5.2.13 we get

$$\operatorname{div} U^j = 0 \quad j = 1, 2.$$

Here $\operatorname{div} U$ is the Riemannian divergence of the vector field U. We apply the divergence theorem (Theorem 2.1 in [146]) to get

$$0 = \int_{F\cap P^j\cap B} \operatorname{div} U^j = \int_F \langle U^j, \nu_{P^j\cap B}\rangle |\partial(P^j \cap B)| + \int_{P^j\cap B} \langle U^j, \nu_F\rangle |\partial F|.$$

Let $C = P \cap \bar{B}$. Then, for every $p \in C$, we have $\nu_{P^1\cap B} = J(v)$ is a normal vector to the plane P and $\nu_{P^2\cap B} = -J(v)$, $U^1 = \eta^+$ and $U^2 = \eta^-$. Hence, by Lemma 6.2.1, we get

$$\langle U^1, \nu_{P^1\cap B}\rangle + \langle U^2, \nu_{P^2\cap B}\rangle = \langle \eta^+ - \eta^-, J(v)\rangle = 0 \quad p \in C.$$

Adding the above integrals we obtain

$$0 = \sum_{j=1,2} \int_F \langle U^j, \nu_B\rangle d|\partial B| + \int_{B\cap \operatorname{int}(P^j)} \langle U^j, \nu_F\rangle d|\partial F|. \tag{6.2.4}$$

From the Cauchy-Schwarz inequality and the fact that $|\partial F|$ is a positive measure, we get that

$$\sum_{j=1,2} \int_{B\cap P^j} \langle U^j, \nu_F\rangle d|\partial F| \leqslant P_K(F, B). \tag{6.2.5}$$

In particular, if we apply the same reasoning to E, equality holds and

$$0 = \sum_{j=1,2} \int_E \langle U^j, \nu_B\rangle d|\partial B| + P_K(E, B). \tag{6.2.6}$$

From (6.2.4), (6.2.5), (6.2.6) and the fact that $F = E$ in the boundary of B, we get

$$P_K(E, B) \leqslant P_K(F, B).$$

□

The general properties of $\Sigma_{v,\alpha}$ when α is only continuous are given in the following proposition.

Proposition 6.2.4. *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a continuous and non-decreasing function. Then*

i) u_α is locally Lipschitz in Euclidean sense,

ii) E_α is a set of locally finite perimeter in $\mathbb{H}^1$, and

iii) $\Sigma_{v,\alpha}$ is K-perimeter-minimizing in $\mathbb{H}^1$.

Proof. i) and ii) are proven in [146], Proposition 3.2. Let

$$\alpha_\varepsilon(x) = \int_{\mathbb{R}} \alpha(y)\delta_\varepsilon(x-y)dy$$

the usual convolution, where δ is a Dirac function and $\delta_\varepsilon = \frac{\delta(x/\varepsilon)}{\varepsilon}$. Then α_ε is a C^∞ non-decreasing function and α_ε converges uniformly to α on compact sets of $\mathbb{R}$. By Lemma 6.2.1, $\beta_\varepsilon = \beta(\alpha_\varepsilon)$ is a C^1 non-decreasing function. Since β is C^1 with respect to α it follows the uniform convergence on compact sets of β_ε to a function $\bar{\beta}$.

Taking $F \subset \mathbb{H}^1$ so that $F = E$ outside a Euclidean ball centered at the origin. We follow the arguents of the proof of iv) in Proposition 6.2.3 and define vector fields $\operatorname{div}(U_\varepsilon^j)$ translating vertically $\pi(\nu_{E_\varepsilon})$, where E_ε is the subgraph of $\Sigma_{\alpha_\varepsilon}$, to obtain by the divergence theorem

$$\sum_{j=1,2} \int_{B\cap \operatorname{int}(P^i)} \langle U_\varepsilon^j, \nu_{E_\varepsilon}\rangle |\partial E_\varepsilon| = \sum_{j=1,2} \int_{B\cap \operatorname{int}(P^i)} \langle U_\varepsilon^j, \nu_F\rangle |\partial F|,$$

the left hand side is the K-perimeter of E_ε, while the right hand side is trivially bounded by the K-perimeter of F. Therefore

$$P_K(E_\varepsilon, B) \leqslant P_K(F, B).$$

Since E_ε converges uniformly in compact sets to E, we obtain the result. □

We study now with some detail the case when $\Sigma_{v,\alpha}$ is a C^∞ surface.

Corollary 6.2.5. *When α is constant, the surface $\Sigma_{v,\alpha}$ is a K-perimeter-minimizing cone in $\mathbb{H}^1$ of class $C^{1,1}$. The singular set is a horizontal straight line and the regular part of $\Sigma_{v,\alpha}$ is a C^∞ surface.*

The following lemma extends the already known notion that in the sub-Riemannian setting the surfaces $\Sigma_{v,\pi/2}$ are C^∞.

Lemma 6.2.6. *Let $v \in \mathbb{R}^2 \setminus \{0\}$ and $\alpha \in (0,\pi)$ be fixed. If K is centrally symmetric with respect to $O = \frac{1}{2}\eta^+ + \frac{1}{2}\eta^-$ then $\beta(\alpha) = \alpha + \pi$, where $\eta^+ = \pi(J(ve^{i\alpha}))$ and $\eta^- = \pi(J(ve^{i\beta}))$.*

Proof. Let K be centrally symmetric with respect to O. Then η^- is the symmetric point of η^+. On the other hand, the convex body $K - O$ is symmetric with respect to the origin. Then the dual norm is even and, in particular, $\pi_{K-O}(-\nu^+) = -\pi_{K-O}(\nu^+)$. Now, since a translation takes symmetric points on $K - O$ with respect to the origin to symmetric points of K with respect to O, we get $\nu^- = -\nu^+$, that is, $\beta(\alpha) = \alpha + \pi$. □

The existence of a convex body K of class C^2_+ such that $0 \in \mathrm{int}(K)$ for which $\Sigma_{v,\alpha}$ is C^∞ is studied in Corollary 6.2.7 and Proposition 6.2.8.

Corollary 6.2.7. *Let $v \in \mathbb{R}^2 \setminus \{0\}$ and $\alpha \in (0, \pi)$ be fixed. Then there exists a convex body K of class C^2_+ with $0 \in \mathrm{int}(K)$ such that $\Sigma_{v,\alpha}$ is C^∞.*

Proof. To construct the convex body K, fix a point $p \in \{(x, y) : \langle (x, y), ve^{i\alpha}\rangle > 0\}$ and $O \in J(L)+p\cap L$, where L is the vector line generated by v. Then any K of class C^2_+ centrally symmetric with respect to O containing the origin such that $p \in \partial K$ and $ve^{i\alpha} \perp T_p\partial K$ satisfies the hypothesis of Lemma 6.2.6, where $\eta^+ = p$ and η^- is the symmetric of η^+ with respect to O. Thus, by (iii) in Proposition 6.2.3 we get that $\Sigma_{v,\alpha}$ is C^∞. □

Proposition 6.2.8. *Given a convex body K of class C^2_+ with $0 \in \mathrm{int}(K)$, there exists $v \in \mathbb{R}^2$ such that $\Sigma_{v,\pi/2}$ is C^∞.*

Proof. Let p and q be points in K at maximal distance. Then the lines through p and q orthogonal to $q-p$ are support lines to K. Taking $v = q-p$ and setting $p = \eta^+$ we have $q = \eta^-$, while the vectors ν^+ and ν^- are over the line $L(v)$, that is, Z^+ Z^- make angles $\pi/2$ and $3\pi/2$ with $L(v)$. □

For fixed $v \in \mathbb{R}^2$, we define the surface $\Sigma^+_{v,\alpha}$ as the one composed of all the horizontal half-lines R^+_λ and $R^-_\lambda \subseteq \mathbb{R}^2$ extending from the lifting of the point $p = \lambda v \in L_v$, $\lambda \geqslant 0$, to $\mathbb{H}^1$. The surface $\Sigma^+_{v,\alpha}$ has a boundary composed of two horizontal lines and its singular set is the ray $L^+_v = \{\lambda v : \lambda > 0\}$. We present some pictures of such surfaces.

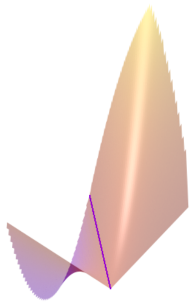

Figure 6.3: The surface $\Sigma^+_{\pi/3,\pi/6}$ associated to the norm $|\cdot|_D$, where D is the unit disk. The singular set corresponds to the purple ray of angle $e^{i\pi/3}$.

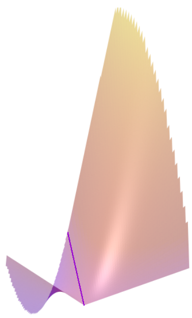

Figure 6.4: The surface $\Sigma^+_{\pi/3,\pi/6}$ associated to the p-norm with $p = 1.5$. The left part of the figure coincides with the left part of Figure 6.3, while the angle β is bigger. Notice that also the height has increased.

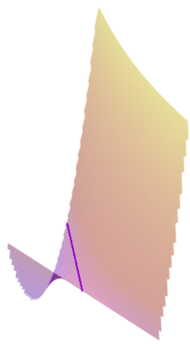

Figure 6.5: The surface $\Sigma^+_{\pi/3,\pi/6}$ with $\beta = \alpha + \pi$. There existence of K is granted by Corollary 6.2.7.

6.3 Area-Minimizing Cones in $\mathbb{H}^1$

We proceed now to construct examples of K-perimeter minimizing cones in $\mathbb{H}^1$ with an arbitrary finite number of horizontal half-lines meeting at the origin. The building blocks for this construction are lifting of circular sectors of the cones considered in Corollary 6.2.5.

We first prove the following result.

Lemma 6.3.1. *Let K be a convex body of class C^2_+ such that $0 \in \operatorname{int}(K)$. Let $u, w \in \mathbb{S}^1$, $\theta = \angle(u, w) > 0$. Then there exists $v \in \mathbb{S}^1$ such that the vector line L_v generated by v splits the sector determined by u and w into two sectors of oriented angles α and β such that $\alpha + \beta = \theta$. Moreover, the stationary condition $\pi(J(u)) - \pi_K(J(w)) \in L_v$ is satisfied.*

Proof. Let $\nu_u = J(u)$, $\nu_w = J(w)$ and $\eta_u = \pi(\nu_u)$, $\eta_w = \pi(\nu_w)$, $\eta_u \neq \eta_w$ since π is a C^1 diffeomorphism. Thus there exists a unique line $\tilde{L}$ passing through η_u and η_w and $L = \tilde{L} - \eta_u$ is a straight line passing though the origin. Notice that $\tilde{L}$ splits ∂K in two connect open components ∂K_1 and ∂K_2. There exist two points $\eta_1 \in \partial K_1$ and $\eta_2 \in \partial K_2$ such that $L + \eta_1$ (resp. $L + \eta_2$) is the support line at η_1 (resp. η_2). Setting $v_1 = N_{\partial K}(\eta_1)$ and $v_2 = N_{\partial K}(\eta_2)$ we gain that v_i for $i = 1, 2$ is perpendicular to L. Without loss of generality we set that $-J(v_1)$ belongs to the portion of plane identified by the θ and $-J(v_2)$ belongs to the portion of plane identified by the $2\pi - \theta$. Then we set $v = -J(v_1)$. Notice that v splits θ in two angles $\beta = \angle(u, v)$, $\alpha = \angle(v, w)$ with $\theta = \alpha + \beta$ and $L = L_v$. □

Now we proceed with the construction inspired by the sub-Riemannian construction in [91]. For $k \geqslant 3$ consider a fixed angle θ_0 and family of positive oriented angles $\theta_1, \ldots, \theta_k$ such that $\theta_1 + \cdots + \theta_k = 2\pi$. Consider the planar vectors $u_0 = (\cos(\theta_0), \sin(\theta_0))$ and

$$u_i = (\cos(\theta_0 + \theta_1 + \cdots + \theta_i), \sin(\theta_0 + \theta_1 + \cdots + \theta_i)), \qquad i = 1, \ldots, k.$$

Observe that $u_k = u_0$. For every $i \in \{1, \ldots, k\}$ consider the vectors u_{i-1}, u_i and apply Lemma 6.3.1 to obtain a family of k vectors v_i in $\mathbb{S}^1$ between u_{i-1} and u_i. We lift the half-lines $L_i = \{\lambda v_i : \lambda \geqslant 0\}$ to horizontal straight lines passing through $(0, 0, 0) \in \mathbb{H}^1$, and we also lift the half-lines

$$\lambda v_i + \{\rho u_{i-1} : \rho \geqslant 0\}, \qquad \lambda v_i + \{\rho u_i : \rho \geqslant 0\},$$

to horizontal straight lines starting from $(\lambda v_i, 0)$. This way we obtain a surface

$$C_K(\theta_0, \theta_1, \ldots, \theta_k)$$

with the following properties

Theorem 6.3.2. *The surface $C_K(\theta_0, \theta_1, \dots, \theta_k)$ is K-perimeter-minimizing cone which is the graph of a C^1 function.*

Proof. $C_K(\theta_0, \theta_1, \dots, \theta_k)$ is a cone by construction. It is an entire graph since it is composed of horizontal lifting of straight half-lines in the xy-plane that covered the whole plane without interesecting themselves transversally. The K-perimeter-minimizing property follows in a similar way to from Proposition 2.4 in [91]. That it is the graph of a C^1 function is proven like in Proposition 3.2(4) in [91]. □

Example 6.3.3. A particular example of area-minimizing cones are those who uses the sub-Riemannian cones C_α restricted to the circular sector with $\theta \in (-\alpha, \alpha)$ as as model piece of the cone. Taking $K = D$, $k \geqslant 3$ and angle$\alpha = \pi/k$, we define

$$C(k) = C_D(\frac{\pi}{k}, \frac{2\pi}{k}, \dots, \frac{2\pi}{k}).$$

Let us denote by u_k the functions in $\mathbb{R}^2$ whose graph is $C(k)$. The behaviour when k tends to infinity of u_k in a disk is analyzed in the folowing Proposition.

Proposition 6.3.4. *The sequence u_k converge to 0 uniformly on compact subsets of $\mathbb{R}^2$. Moreover, the sub-Riemannian area of u_k converges locally to the sub-Riemannian area of the plane $t = 0$. Moreover the sub-Riemannian area of u_k converges to the one of the plane $t = 0$.*

Proof. Since u_k is obtained by collating of u_α, where $\alpha = \pi/k$, we can estimate the height of u_k by the height of u_α. By (6.2.2), using polar coordinates (r, θ), where $\theta \in [-\alpha, \alpha]$ and $r < r_0$, we get

$$|u_\alpha| \leqslant 2r_0^2 |\sin(\pi/k)|$$

on $D(r_0) = \overline{B}(0, r_0)$. The claim follows since $\lim_{k\to\infty} \sin(\pi/k) = 0$.

The sub-Riemannian area of the graph of u_k over $D(r_0)$ is given by

$$A_D(u_k, r_0) = \int_{D(r_0)} |\nabla u_k + (-y, x)| dxdy.$$

Since the sub-Riemannian perimeter is rotationally invariant, we can decompose the above integral as k times the area of the cone C_α in the circular sector with $\theta \in (-\alpha, \alpha)$ and $r < r_0$. By (6.2.2), it is immediate that

$$|\nabla u_k(x, y) + (-y, x)| = 2|y| \sin^{-1}(\alpha).$$

A direct computation shows that

$$A_D(u_k, r_0) = \frac{4\pi r_0^3}{3} \frac{1 - \cos \pi/k}{(\pi/k) \sin \pi/k}.$$

Then $A_D(u_k, r_0)$ tends to $\frac{2\pi r_0^3}{3}$ as $k \to +\infty$. □

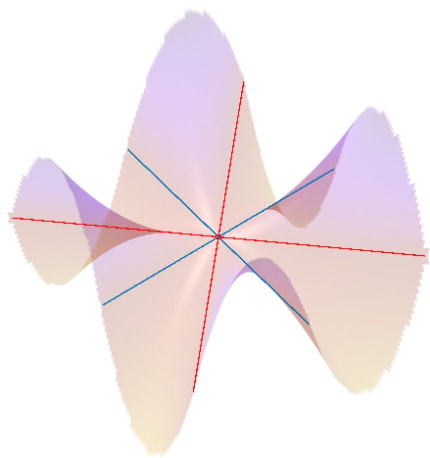

Figure 6.6: The cone $C(4)$. The singular set is composed of the red rays of angle $0, \pi/2, \pi, (3\pi)/2$, while the rays of angles $\pi/4, (3\pi)/4, (5\pi)/4), (7\pi)/4$, where two pieces of the construction meet, are depicted in cyan.

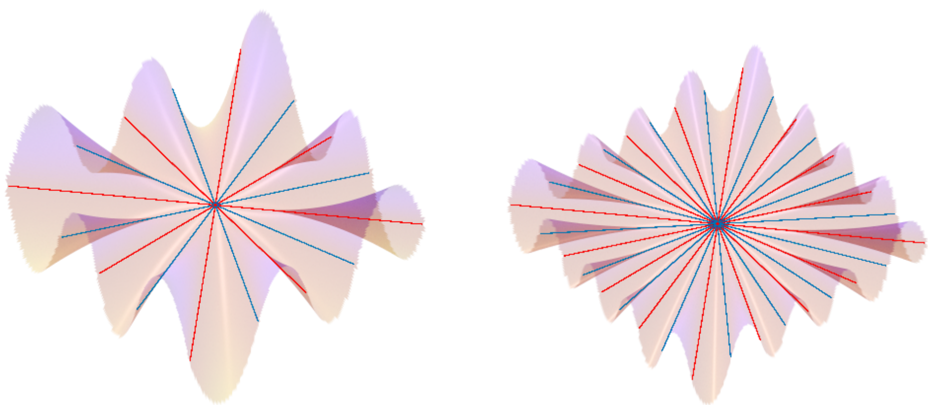

Figure 6.7: The cones $C(8)$ and $C(16)$. They are depicted at the same in this Figure and the previous one. As the number of angles increases, the cone produces more oscilations of smaller height.

Chapter 7

The prescribed mean curvature equation for t-graphs in $\mathbb{H}^n$

This chapter exposes the some of the results obtained during the stay at Università di Trento in colaboration with Giovannardi, Pinamonti and Verzellesi.

The aim of this chapter is to study the prescribed mean curvature equation for t-graphs in the sub-Finsler Heisenberg group $(\mathbb{H}^n, K_0)$. Let $\Omega \subset \mathbb{R}^{2n}$ be a bounded open set, $H \in L^\infty(\Omega)$, $F \in L^1(\Omega, \mathbb{R}^{2n})$ and $u \in W^{1,1}(\Omega)$. We consider the functional

$$\mathcal{I}(u) = \int_\Omega |\nabla u + F|_{K_0,*}\, dxdy + \int_\Omega Hu\, dxdy, \tag{7.0.1}$$

where $|\cdot|_{K_0,*}$ denotes the dual norm of $|\cdot|_{K_0}$. In particular, when $F(x,y) = (-y,x)$ the first term in (7.0.1) coincides with the sub-Finsler area of the t-graph of u, see [142; 69]. Moreover, if K_0 is the Euclidean unit ball centered at the origin and $H = 0$ then (7.0.1) boils down to the classical area functional for t-graphs in Heisenberg group, see [33; 99] and references therein. We say that the graph of u has prescribed K_0-mean curvature H in Ω if u is a minimizer of $\mathcal{I}$. Indeed, the Euler-Lagrange equation associated to $\mathcal{I}$ out of the singular set Ω_0, i.e. the set of points where $\nabla u + F$ vanishes, is given by

$$\operatorname{div}(\pi_{K_0}(\nabla u + F)) = H, \tag{7.0.2}$$

where π_{K_0} is a suitable 0-homogeneous function defined in (2.1.5). When we fix a boundary datum $\varphi \in W^{1,1}(\Omega)$, a solution to the *Dirichlet problem* for the prescribed K_0-mean curvature equation is a minimizer u of $\mathcal{I}$ such that $u - \varphi$ belongs to the Sobolev space $W_0^{1,1}(\Omega)$. Our main result is Theorem 7.4.1, where we prove, under suitable regularity assumptions on the data, that there exists a Lipschitz solution to the Dirichlet problem for the prescribed K_0-mean curvature equation when H is *constant* and satisfies

$$|H| < H_{K_0,\partial\Omega}(z_0) \tag{7.0.3}$$

for each $z_0 = (x_0, y_0) \in \partial\Omega$, where $H_{K_0,\partial\Omega}$ denotes the Finsler mean curvature of the boundary $\partial\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$. Notice that the mean curvature of the graph of u is computed with respect to the downward pointing unit normal and the Finsler mean curvature of $\partial\Omega$ is computed with respect to the inner unit normal. The upper bound (7.0.3) of H in terms of the Finsler mean curvature of the boundary is the Finsler analogous of the standard assumption for the solution to the Dirichlet problem for the classical mean curvature equation in the Euclidean setting as stated in [158], [84] or [83] (see also [85, Theorem 16.11]). The approach of the present chapter, based on the Schauder fixed-point theory, follows the scheme developed in [35] and extends its results both to the case of prescribed constant mean curvature $H \neq 0$ and to the sub-Finsler setting. In Theorem 7.4.1 we can not expect better regularity than Lipschitz, as exhibit in Chapter 6.

Since equation (7.0.2) is sub-elliptic degenerate and it is singular next the singular set, inspired by [35; 138], we introduce a family of elliptic approximating equations given by

$$\operatorname{div}\left(\pi_{K_0}(\nabla u + F)\frac{|\nabla u+F|_*^2}{(\varepsilon^3+|\nabla u+F|_*^3)^{\frac{2}{3}}}\right) = H. \tag{7.0.4}$$

for each $0 < \varepsilon < 1$. To obtain this family of equations we consider a $2n+1$ dimensional convex body K_ε containing the origin in its interior, that converges in the Hausdorff sense to the $2n$ dimensional convex body K_0 as $\varepsilon \to 0$. The choice of the convex body K_ε is not arbitrary. Indeed, we need a specific shape in order to obtain an approximating equation well defined in the classical sense in the singular set. For $0 < \varepsilon < 1$, the convex body K_ε defines a Finsler norm on $T\mathbb{H}^n$ whose associated Finsler area functional is given by

$$\mathcal{I}_\varepsilon(u) = \int_\Omega \left(\varepsilon^3 + |\nabla u + F|_{K_0,*}^3\right)^{\frac{1}{3}}\, dxdy + \int_\Omega Hu\, dxdy.$$

It is easy to see that the Euler-Lagrange equation associated to this functional is elliptic and avoids singularities. Given a boundary datum $\varphi \in C^{2,\alpha}(\bar{\Omega})$, the solvability of the Dirichlet problem associated to (7.0.4) is reduced by [85, Theorem 13.8] to *a priori* estimates in $C^1(\overline{\Omega})$ of a related family of problems. As usual the *a priori* estimates in $C^1(\overline{\Omega})$ consist of three parts: estimates of the supremum of $|u|$, boundary estimates of the gradient of u and interior estimates of the gradient of u. Both the estimates of the supremum and the boundary estimates of the gradient are obtained by a barriers argument that depends on the Finsler distance from the boundary $\partial\Omega$. Due to technical reasons in the construction of the barriers we need to assume the strict inequality in (7.0.3), avoiding the optimal case when H coincides with $H_{K_0,\partial\Omega}(z_0)$ at a given point $z_0 \in \partial\Omega$. We emphasize that these results hold even if the prescribed curvature H is non-constant and

Lipschitz. When H is non-constant, in order to obtain the estimates of the supremum we assume that there exists $\delta \in (0,1]$ such that

$$\left| \int_\Omega Hv \, dxdy \right| \leqslant (1-\delta) \int_\Omega |\nabla v|_{K_0,*} \, dxdy \tag{7.0.5}$$

for each $v \in C_c^\infty(\Omega)$. Assumption (7.0.5) is a standard sufficient condition for the estimates of the supremum of $|u|$ (see [83] or [85]). Moreover, in analogy with the Euclidean case, we show that the sufficient condition (7.0.5) with $\delta > 0$, which is *a priori* stronger than the necessary condition with $\delta = 0$, is redundant when H is constant. The only crucial step where we need that H is constant is the maximum principle for the gradient of the solution that allows us to reduce the interior estimates of the gradient to its boundary estimates. Finally, once we realize that C^1 estimates are independent of the approximation parameter ε, passing to the limit as ε tends to 0 and using Arzelà-Ascoli Theorem we get the existence of a Lispchitz minimizer for the sub-Finsler Dirichlet problem.

The chapter is organized as follows. In Section 7.1 we introduce some preliminary definitions and results, such as the Minkowski norm, the Finsler geometry of a hypersurface in $\mathbb{R}^{2n}$, the Heisenberg group, the sub-Finsler perimeter and the sub-Finsler functional $\mathcal{I}$. Section 7.2 is dedicated to the Finsler approximation by the K_ε convex body of the sub-Finsler convex body K_0. Section 7.3 deals with the *a priori* estimates for the C^1 norm of the solution to the approximating elliptic equations. In particular, Proposition 7.3.4 deals with the *a priori* estimates of $|u|$ when H is Lispchitz and verifies the integral condition (7.0.5), in Proposition 7.3.6 we deduce the boundary estimates of the gradient when H is Lispchitz, in Proposition 7.3.5 we establish the maximum principle for the gradient for H constant, and finally, in Proposition 7.3.8 we achieve *a priori* estimates of $|u|$ when H is constant. To conclude, Section 7.4 contains the main Theorem 7.4.1.

7.1 Preliminaries

Unless otherwise specified, we let $n, d \in \mathbb{N}$, $n, d \geqslant 1$. Given two open sets $A, B \subseteq \mathbb{R}^d$, we write $A \Subset B$ whenever $\overline{A} \subseteq B$.

7.1.1 Finsler geometry of hypersurfaces in Euclidean space

Let $K \subset \mathbb{R}^d$ be a convex body in C^2_+, $0 \in \operatorname{int} K$ and $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega = \Sigma$ of class C^2. Let N be the inner unit normal to Σ. Then the derivative map $(W_{K,\Sigma})_p = -d_p(\pi_K \circ N) : T_p\Sigma \to T_{\pi_K(N(p))}\partial K$ is called the *K-Weingarten map*. Let $\gamma \subseteq \partial K$ be a differentiable curve with $\gamma(0) = \pi_K(p)$ and $\gamma'(0) \in T_{\pi_K(p)}\partial K$. By definition of π_K, the function

$$f(t) = \langle \gamma(t), p \rangle$$

has a maximum at 0 and therefore $\langle \gamma'(0), p\rangle = f'(0) = 0$, which gives $T_{\pi_K(p)}\partial K = T_p\mathbb{S}^{d-1}$. Moreover it is well known that $(dN)_q$ is an endomorphism of $T_q\Sigma$ and therefore $W_{K,\Sigma;p}$ is an endomorphism of $T_p\Sigma$. We define the *K-mean curvature* of Σ as

$$H_{K,\Sigma} = \mathrm{Trace}(W_{K,\Sigma}) = -\operatorname{div}_\Sigma(\pi_K \circ N),$$

where $\operatorname{div}_\Sigma$ is the divergence in the tangent directions to Σ. We remark that $W_{K,\Sigma}$ is neither necessarily self-adjoint nor symmetric. Let us check that $W_{K,\Sigma}$ is diagonalizable. Indeed, given a parametrization X of Σ, dN has a symmetric matrix representation S in the basis $B = \{\partial_{x_1}X, \dots, \partial_{x_d}X\}$. On the other hand, $\pi_K = \mathcal{N}_K^{-1}$ and, since K is in C^2_+, the matrix A which represents $d(\mathcal{N}_K^{-1})$ with respect to B is positive definite. Therefore, there exists an invertible matrix P such that $A = P^tP$. Notice that the matrices P^tPS and PSP^t has the same spectrum, and equal to the spectrum of $W_{K,\Sigma}$. Since S is symmetric we can apply the criterion of Sylvester to obtain that all the eigenvalues of PSP^t are real. The eigenvalues of $W_{K,\Sigma}$ are called *K-principal curvatures* and the eigenvectors of $W_{K,\Sigma}$ are called *K-principal directions.*

7.1.1.1 Finsler distance from the boundary and the Eikonal equation

In this and the following section we want to rely on some results by [113; 112], and so we assume that K is in C^∞_+, i.e. ∂K is of class C^∞ with strictly positive principal curvatures. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega = \Sigma$ of class $C^{2,\alpha}$, for $0 < \alpha \leqslant 1$, and inner unit normal N. We shall adapt Theorem 4.26 in [130] and the remarks at the end of Section 4.5 in [130] to prove existence of a tubular neighborhood of Σ and compute the K-mean curvature of parallel hypersurfaces. The *interior signed K-distance* to Σ is the function $d_{K,\Sigma} : \mathbb{R}^d \to \mathbb{R}$ given by

$$d_{K,\Sigma}(p) = \begin{cases} \min\{|p-q|_K : q \in \Sigma\} & \text{if } p \in \Omega \\ -\min\{|p-q|_K : q \in \Sigma\} & \text{if } p \notin \Omega. \end{cases}$$

Throughout this chapter, we shall use the notation $B_K(p,r)$ to denote the ball of radius $r > 0$ in $\mathbb{R}^d$ associated to the distance $d_{K,p}$. It is easy to check that $|p|_K = |-p|_{-K}$ for any $p \in \mathbb{R}^d$, so that

$$B_{-K}(p,r) = \{q \in \mathbb{R}^d \, : \, |p-q|_K \leqslant r\} \tag{7.1.1}$$

for any $p \in \mathbb{R}^d$ and $r > 0$.

Consider the map $F : \Sigma \times \mathbb{R} \to \mathbb{R}^d$ given by

$$F(q,t) = q + t(\pi_K \circ N)(q).$$

For any $v \in T_q\Sigma$, we have $(dF)_{(q,t)}(v,0) = v + td(\pi_K \circ N)(q)$ and $(dF)_{(q,t)}(0,1) = (\pi_K \circ N)(q)$. Since K contains the origin,

$$\langle \pi_K(N), N \rangle > 0$$

and dF is invertible at $t = 0$. Thus F is locally a diffeomorphism and, being Σ a compact hypersurface, F is a diffeomorphism in a domain $\Sigma \times (-\delta, \delta)$. The set $F(\Sigma \times (-\delta, \delta))$ is called a *tubular neighborhood* of Σ. Notice that if $p = F(q,t)$, then

$$p - q = t(\pi_K \circ N)(q) \tag{7.1.2}$$

and, taking the K-norm, we obtain that $d_{K,\Sigma}(p) = t$. We know (cf. [113]) that, under our assumptions, there exists $\bar{\delta} > 0$ such that

$$d_{K,\Sigma} \in C^{2,\alpha}(\overline{F(\Sigma \times (-\delta, \delta))}).$$

for any $\delta < \bar{\delta}$. Given $|t| < \delta$, we let

$$\Sigma_t = \{p \in \mathbb{R}^d : p = F(q,t) \text{ for some } q \in \partial K\}. \tag{7.1.3}$$

Proposition 7.1.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega = \Sigma$ of class C^2 and let $F(\Sigma \times (-\delta, \delta))$ be a tubular neighborhood of Σ. The K-mean curvature of Σ_t at $p \in \Sigma_t$ is given by*

$$H_{K,\Sigma_t}(p) = \sum_{i=1}^{d-1} \frac{\kappa_i(q)}{1 - t\kappa_i(q)}, \tag{7.1.4}$$

where $q \in \Sigma$ satisfies $p = F(q,t)$ and $\kappa_1(q), \ldots, \kappa_{d-1}(q)$ are the K-principal curvatures of Σ at q.

Proof. Let $\{e_1, \ldots, e_{d-1}\}$ be a basis of K-principal directions of Σ. Then $(dF)_{(q,t)}(e_i, 0) = (1 - t\kappa_i)e_i$. Therefore a basis of principal directions in Σ_t is $\{\frac{e_1}{1-t\kappa_1}, \ldots, \frac{e_{d-1}}{1-t\kappa_{d-1}}\}$. Since we have

$$-d(\pi_K \circ N)_q \left(\frac{e_i}{1 - t\kappa_i} \right) = \frac{\kappa_i}{1 - t\kappa_i} e_i$$

for each $i = 1, \ldots, d-1$ we obtain the result. □

Remark 7.1.2. From (7.1.4), we obtain that the K-mean curvature is increasing in t. In particular, given $q \in \Sigma$ and $p = F(q,t)$ for $t > 0$, it holds that

$$H_{K,\Sigma_t}(p) \geqslant H_{K,\Sigma}(q). \tag{7.1.5}$$

The following Eikonal equation can be deduced using classical arguments. We include the proof for sake of completeness.

Proposition 7.1.3. *It holds that*

$$|\nabla d_{K,\Sigma}(p)|_{K,*} = 1 \tag{7.1.6}$$

for any p where $d_{K,\partial\Omega}$ is differentiable.

Proof. It is clear that, for any p, p' in $\mathbb{R}^d$, we have

$$d_{K,\Sigma}(p') \leqslant |p' - p|_K + d_{K,\Sigma}(p).$$

Taking $p' = p + tv$ where $t > 0$, we get

$$d_{K,\Sigma}(p + tv) - d_{K,\Sigma}(p) \leqslant |tv|_K.$$

Therefore,

$$\langle v, \nabla d_{K,\Sigma}(p)\rangle \leqslant |v|_K. \tag{7.1.7}$$

Taking $v = \pi_K(\nabla d_{K,\Sigma}(p))$ in (7.1.7), we obtain

$$|\nabla d_{K,\Sigma}(p)|_{K,*} \leqslant 1.$$

On the other hand, let $\gamma(t) = F(q_0, t)$. By (7.1.2) we have that

$$d_{K,\Sigma}(\gamma(t)) = t.$$

Taking derivatives in the previous equation, we obtain

$$\langle \gamma'(t), \nabla d_{K,\Sigma}(\gamma(t))\rangle = 1.$$

Since $\gamma'(t) = (\pi_K \circ N)(q_0)$, we get that $|\gamma'(t)|_K = 1$. Using (2.1.4), we get

$$|\nabla d_{K,\Sigma}(\gamma(t))|_{K,*} \geqslant 1. \qquad \square$$

Given a tubular neighborhood $\mathcal{O}$ of $\partial\Omega$ and $p = F(q,t) \in \Omega$, we denote $N_t(p)$ the inner unit normal to Σ_t at p. Let us explicitly compute $\operatorname{div}(\pi_K \circ N_t)(p)$. Let us recall that, to the 0-homogeneity of π_K, we get that

$$q \cdot D\pi_K(q) = 0$$

for any $q \in \mathbb{R}^d$. In particular, taking $q = N_t$, we obtain

$$N_t \cdot D(\pi_K \circ N_t) = N_t \cdot D\pi_K(N_t) \cdot DN_t = 0,$$

which implies that

$$-\operatorname{div}(\pi_K \circ N_t)(p) = -\operatorname{div}_\Sigma(\pi_K \circ N_t)(p) = H_{K,\Sigma_t}(p) \geqslant H_{K,\partial\Omega}(q). \tag{7.1.8}$$

With the next result we better understand the relationship between the Finsler mean curvature of Σ, the Euclidean curvatures of Σ and the Euclidean principal curvatures of K.

Proposition 7.1.4. *Let K be a convex body in C^2_+, $0 \in \operatorname{int} K$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega = \Sigma$ of class C^2 and let N_q be the inner unit normal to Σ at q. Then we have*

$$H_{K,\Sigma}(q) = -\sum_{i=1}^{d-1} \frac{\langle D_{e_i} N_q, e_i\rangle}{k_i^K(\pi_K(N_q))} \tag{7.1.9}$$

where k_i^K are the Euclidean principal curvatures of ∂K and $e_1, \ldots, e_{d-1}$ is an orthonormal basis of Euclidean principal directions of ∂K.

Proof. We shall drop the subscript for π_K. Let q in Σ and $e_1, \ldots, e_{d-1}$ be an orthonormal basis of $\mathbb{R}^{d-1} = T_{\pi(N_q)}\partial K$ such that

$$(d\mathcal{N}_K)_{\pi(N_q)} e_i = k_i^K(\pi(N_q))e_i.$$

By hypothesis, $k_i^K > 0$ for $i = 1, \ldots, d-1$. Here $\mathcal{N}_K$ denotes the Gauss map of ∂K. Then we have

$$H_{K,\Sigma}(q) = -\operatorname{div}_\Sigma(\pi(N_q)) = -\sum_{i=1}^{d-1}\langle D_{e_i}\pi(N_q), e_i\rangle,$$

where D is the Levi-Civita connection in $\mathbb{R}^d$. We claim that $D_{e_i}\pi(N_q) = d\pi(D_{e_i}N_q)$. Indeed, let $\gamma : (\epsilon, \epsilon) \to \Sigma$ such that $\gamma(0) = q$ and $\dot\gamma(0) = e_i$ for $i = 1, \ldots, d-1$. Then we have

$$\begin{aligned} D_{e_i}\pi(N_q) &= \frac{D}{ds}\Big|_{s=0}\pi(N_{\gamma(s)}) = \sum_{j=1}^{d} \frac{d}{ds}\Big|_{s=0}\pi_j(N_{\gamma(s)})\frac{\partial}{\partial x_j} \\ &= \sum_{j=1}^{d} \nabla\pi_j(N_q)\frac{D}{ds}\Big|_{s=0}N_{\gamma(s)}\frac{\partial}{\partial x_j} = (d\pi)_{N_q}D_{e_i}N_q. \end{aligned}$$

Moreover, since $d\pi$ is a symmetric matrix we gain

$$H_{K,\Sigma}(q) = -\sum_{i=1}^{d-1}\langle(d\pi)_{N_q}D_{e_i}N_q, e_i\rangle = -\sum_{i=1}^{d-1}\langle D_{e_i}N_q, (d\pi)_{N_q}e_i\rangle. \tag{7.1.10}$$

Since $\pi = \mathcal{N}_K^{-1}$ we obtain $d\pi = (d\mathcal{N}_K)^{-1}$ and

$$e_i = d\mathcal{N}_K^{-1} d\mathcal{N}_K(e_i) = d\mathcal{N}_K^{-1}(k_i^K(\pi(N_q))e_i) = k_i^K(\pi(N_q))d\pi(e_i),$$

by linearity. Therefore, we have $d\pi(e_i) = (k_i^K(\pi(N_q)))^{-1}e_i$. Hence, plugging this last equality in (7.1.10) we gain (7.1.9). □

7.1.1.2 The Ridge of the Finsler distance

In the previous section we obtained some regularity and geometric properties of $d_{K,\partial\Omega}$ in a tubular neighborhood of $\partial\Omega$. We shall see that some of these properties holds outside a tubular neighborhood. We fix a convex body $K \in C^\infty_+$ and a bounded domain $\Omega \subseteq \mathbb{R}^d$ with $C^{2,1}$ boundary. For any $p \in \Omega$, we let $D(p) := \{q \in \partial\Omega \,:\, d_{K,\partial\Omega}(p) = |p-q|_K\}$. Since $d_{K,\partial\Omega}$ is continuous, then clearly $D(p) \neq \emptyset$ for any $p \in \Omega$. Accordingly, we define the set

$$\Omega_1 := \{p \in \Omega \,:\, D(p) \text{ is a singleton}\}, \tag{7.1.11}$$

and we define the *Ridge* of Ω by

$$R := \Omega \setminus \operatorname{int}\Omega_1.$$

We know, again thanks to [113], that, under our assumptions on K and Ω,

$$d_{K,\partial\Omega} \in C^{2,1}(\operatorname{int}\Omega_1 \cup \partial\Omega). \tag{7.1.12}$$

Moreover, in [112, Corollary 1.6] it is proven that the Hausdorff dimension of R is at most $d-1$. This fact implies that R has empty interior, so that

$$\partial(\operatorname{int}\Omega_1) = \partial\Omega \cup R. \tag{7.1.13}$$

The following result is inspired partially by [60, Lemma 3.4].

Proposition 7.1.5. *Let $p \in \Omega$, let $q \in D(p)$ and let*

$$(p,q) := \{tp + (1-t)q \,:\, t \in (0,1)\}.$$

Then $(p,q) \subseteq \operatorname{int}\Omega_1$ and

$$D(\gamma) = \{q\} \tag{7.1.14}$$

for any $\gamma \in (p,q)$.

Proof. Let p, q be as in the statement, and fix $\gamma \in (p,q)$. We already know that $D(\gamma) \neq \emptyset$. On the other hand, assume that there exists $q' \neq q$ such that $q' \in D(\gamma)$. Let us notice that p, q, q' cannot lie on the same line. Indeed, if by contradiction this was the case, then the only possibility is that p is a convex combination of γ and q'. But then the strict convexity of K would imply that

$$|\gamma - q'|_K \leqslant |\gamma - q|_K < |p-q|_K \leqslant |p-q'|_K < |\gamma - q'|_K,$$

which is a contradiction. This in particular implies that p, γ, q' do not lie on the same line. Therefore, thanks again to the strict convexity of K, we get that

$$|p-q'|_K < |p-\gamma|_K + |\gamma - q'|_K \leqslant |p-\gamma|_K + |\gamma - q|_K = |p-q|_K,$$

a contradiction with $q \in D(p)$. Hence (7.1.14) is proved. Assume by contradiction that $\gamma \in R$. By Corollary 4.11 in [112], any point of the form $q + \lambda(\gamma - q)$ with $\lambda > 1$ has a point in $\partial\Omega$ closer than q. On the other hand, taking w the midpoint of p and γ, then by (7.1.14) it holds that $D(w) = \{q\}$. A contradiction then follows. □

Let us take a point $p \in \operatorname{int}\Omega_1$, and let $q \in D(p)$. Thanks to Proposition 7.1.5, we know that

$$d_{K,\partial\Omega}(z) = |z - q|_K$$

for any z in (p, q). Recalling that $(p, q) \subseteq \operatorname{int}\Omega_1$, together with (7.1.12), and Proposition 7.1.3 it is easy to see that $\nabla d_{K,\partial\Omega}(z) \neq 0$. Thus, at least locally, the level set $\Sigma_{d_K,\partial\Omega}(p)$ is a well defined C^2 hypersurface. Reasoning as in Section 7.1.1.1 we conclude that

$$- \operatorname{div}(\pi_K \circ N_{d_K,\partial\Omega})(p) \geqslant H_{K_0,\partial\Omega}(q) \tag{7.1.15}$$

for any $p \in \operatorname{int}\Omega_1$, where $q \in D(p)$.

7.1.2 Sub-Finsler area

Let $K_0 \subseteq \mathbb{R}^{2n}$ be a C^2_+ convex body with $0 \in \operatorname{int} K_0$ and let $|\cdot|_{K_0}$, $|\cdot|_{K_0,*}$ and π_{K_0} be the associated left-invariant extensions to $\mathcal{H}_0$ (see § 2.1.2). In the following we shall write $|\cdot|_*$ and π instead of $|\cdot|_{K_0,*}$ and π_{K_0} respectively. Given a horizontal vector field U of class C^1, we define $\pi(U)$ as the C^1 horizontal vector field satisfying

$$|U|_* = \langle U, \pi(U)\rangle.$$

Proceeding as in § 2.1.1, it is easy to see that the projection satisfies

$$\pi\left(\sum_i^n f_i X_i + g_i Y_i\right) = \mathcal{N}_{K_0}^{-1}\left(\frac{(f,g)}{\sqrt{|f|^2 + |g|^2}}\right),$$

where $|f|^2 = \langle f, f\rangle$.

Recall that, by Proposition 2.3.4, if E has C^1 boundary ∂E, then

$$P_{K_0}(E) = \int_{\partial E} |N_h|_* d\sigma =: A_{K_0}(\partial E),$$

where N_h is the projection on the horizontal distribution $\mathcal{H}$ of the Riemannian normal N with respect to the metric g and $d\sigma$ is the Riemannian measure of ∂E.

As a significant example, we consider a bounded open set $\Omega \subseteq \mathbb{R}^{2n}$ and a C^1 function $u : \Omega \to \mathbb{R}$. Let $\mathrm{Gr}(u) = \{(x, y, t) \in \mathbb{H}^n \, : \, u(x, y) - t = 0\}$ be the graph of u. Then we have

$$N_h = \frac{\sum_{i=1}^n (u_{x_i} - y)X_i + (u_{y_i} + x)Y_i}{\sqrt{1 + |\nabla u + F|^2}} \quad \text{and} \quad d\sigma = \sqrt{1 + |\nabla u + F|^2}\, dxdy,$$

where $\nabla u(x, y)$ is the Euclidean gradient of $u(x, y)$ and $F(x, y) = (-y, x)$. Therefore we get

$$A_{K_0}(\mathrm{Gr}(u)) = \int_\Omega |\nabla u + F|_* \, dxdy.$$

7.1.3 The sub-Finsler prescribed mean curvature equation

Inspired by the previous computation and the sub-Riemannian problem studied by [35] we consider the following problem. Let $\Omega \subset \mathbb{R}^{2n}$ be a bounded open set and let $F \in L^1(\Omega, \mathbb{R}^{2n})$, $\varphi \in W^{1,1}(\Omega)$ and $H \in L^\infty(\Omega)$. Then we set

$$\mathcal{I}(u) = \int_\Omega |\nabla u + F|_* \, dxdy + \int_\Omega Hu \, dxdy \tag{7.1.16}$$

for each $u \in W^{1,1}(\Omega)$ such that $u - \varphi \in W_0^{1,1}(\Omega)$. We say that $u \in W^{1,1}(\Omega)$ is a *minimizer* for $\mathcal{I}$ if

$$\mathcal{I}(u) \leqslant \mathcal{I}(v)$$

for all $v \in W^{1,1}(\Omega)$ such that $v - \varphi \in W_0^{1,1}(\Omega)$. In [35, Section 3] the authors investigate the first variation of the functional $\mathcal{I}$ when $|\cdot|_*$ is the Euclidean norm $|\cdot|$, taking into account the bad beaviour of the singular set

$$\Omega_0 = \{(x, y) \in \Omega \, : \, (\nabla u + F)(x, y) = 0\}. \tag{7.1.17}$$

In the next result we derive the Euler-Lagrange equation associated to $\mathcal{I}$ for C^2 minimizers.

Proposition 7.1.6. *Let K_0 be a C^2_+ convex body such that $0 \in \mathrm{int}(K_0)$. Let $u \in C^2(\Omega)$ be a minimizer for $\mathcal{I}$ defined in* (7.1.16)*. Assume that $F \in C^1(\Omega, \mathbb{R}^{2n})$. Let Ω_0 be the singular set defined in* (7.1.17)*. Then u satisfies*

$$\mathrm{div}(\pi(\nabla u + F)) = H \tag{7.1.18}$$

in $\Omega \setminus \Omega_0$.

Proof. Given $v \in C_c^\infty(\Omega \setminus \Omega_0)$, by [142, Lemma 3.2] the first variation is given by

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\mathcal{I}(u+sv) &= \int_{\Omega\setminus\Omega_0}\frac{d}{ds}\Big|_{s=0}|\nabla(u+sv)+F|_*\ dxdy + \int_{\Omega\setminus\Omega_0} Hv\ dxdy\\ &= \int_{\Omega\setminus\Omega_0}\frac{d}{ds}\Big|_{s=0}|\nabla u+F+s\nabla v|_*\ dxdy + \int_{\Omega\setminus\Omega_0} Hv\ dxdy\\ &= \int_{\Omega\setminus\Omega_0}\langle\nabla v,\pi(\nabla u+F))\rangle\ dxdy + \int_{\Omega\setminus\Omega_0} Hv\ dxdy\\ &= \int_{\Omega\setminus\Omega_0} v\ (H-\operatorname{div}(\pi(\nabla u+F)))\ dxdy. \qquad \square\end{aligned}$$

Remark 7.1.7. When K_0 is the unit disk $D_0\subset\mathbb{R}^{2n}$ centered at 0 of radius 1 we have

$$\pi_{D_0}(\nabla u+F)=\frac{\nabla u+F}{|\nabla u+F|}$$

and (7.1.18) is equivalent to

$$\operatorname{div}\left(\frac{\nabla u+F}{|\nabla u+F|}\right)=H.$$

7.2 The Finsler approximation problem

In this section we develop the Finsler approximation scheme in order to get rid of the singular nature of equation (7.1.18). To this aim, given K_0 a convex body in C^2_+ such that $0\in\operatorname{int}K_0$ and $\varepsilon\in(0,1)$, we denote by K_ε the set

$$K_\varepsilon := \left\{(x,y,t)\in\mathbb{R}^{2n+1} : \left(\frac{|t|}{\varepsilon}\right)^{\frac{3}{2}} + |(x,y)|_{K_0}^{\frac{3}{2}} \leqslant 1\right\}. \qquad (7.2.1)$$

Notice that $K_\varepsilon\subset\mathbb{R}^{2n+1}\equiv T_0\mathbb{H}^n$ (here $T_0\mathbb{H}^n$ denotes the tangent space of $\mathbb{H}^n$ at $p=0$) is a strictly convex body with $0\in\operatorname{int}(K_\varepsilon)$. Moreover ∂K_ε is of class C^1. Indeed it is a level set of the C^1 function

$$g_\varepsilon(x,y,t) := \left(\frac{|t|}{\varepsilon}\right)^{\frac{3}{2}} + |(x,y)|_{K_0}^{\frac{3}{2}},$$

whose gradient never vanishes on ∂K_ε. Hence, the projection π_{K_ε} is well defined and continuous. We shall write $|\cdot|_\varepsilon$, $|\cdot|_{\varepsilon,*}$ and π_ε instead of $|\cdot|_{K_\varepsilon}$, $|\cdot|_{K_\varepsilon,*}$ and π_{K_ε} respectively. The map π^h_ε is defined as the first $2n$ components of π_ε. By abuse of notation, we write $\pi^h_\varepsilon(x,y)=\pi^h_\varepsilon(x,y,-1)$ when there is no confusion.

Proposition 7.2.1. *Let K_0 be a convex body in C^2_+ such that $0\in\operatorname{int}K_0$, and let $K_\varepsilon\subseteq\mathbb{R}^{2n+1}$ be the set defined in* (7.2.1)*. Then the following assertions hold:*

(i) *The map* $\pi^h_\varepsilon : \mathbb{R}^{2n} \setminus \{0\} \to \mathbb{R}^{2n}$ *satisfies*

$$\pi^h_\varepsilon(x,y) = \pi(x,y)\frac{|(x,y)|_*^2}{(\varepsilon^3 + |(x,y)|_*^3)^{\frac{2}{3}}}.$$

(ii) *The map* π^h_ε *can be extended to a* C^1 *map in* $\mathbb{R}^{2n}$ *by setting* $\pi^h_\varepsilon(0,0) = (0,0)$.

(iii) $|(x,y,-1)|_{K_\varepsilon,*} = \left(\varepsilon^3 + |(x,y)|_*^3\right)^{\frac{1}{3}}$.

Proof. Let us prove that

$$\pi_\varepsilon(x,y,-1) = (\alpha\pi(x,y), -\varepsilon(1-\alpha^{3/2}))^{2/3} \tag{7.2.2}$$

for some $0 < \alpha(x,y) < 1$. Given (x,y) in $\mathbb{R}^{2n} \setminus \{0\}$, we denote by t_0 the $2n+1$ coordinate of $\pi_\varepsilon(x,y,-1)$ and we let $K_{t_0} \subset \mathbb{R}^{2n}$ be the convex set defined by

$$K_{t_0} := \{(x',y') \ : \ (x',y',t_0) \in K_\varepsilon\}.$$

Then we have

$$\begin{aligned} K_{t_0} \times \{t_0\} &= \left\{ \left(\frac{|t_0|}{\varepsilon}\right)^{\frac{3}{2}} + |(x',y')|_{K_0}^{\frac{3}{2}} \leqslant 1 \right\} \\ &= \left\{ |(x',y')|_{K_0} \leqslant \left(1 - \left(\frac{|t_0|}{\varepsilon}\right)^{\frac{3}{2}}\right)^{\frac{2}{3}} \right\}. \end{aligned}$$

Hence there exists $\bar\alpha = \bar\alpha(x,y)$ such that $K_{t_0} = \bar\alpha K_0$ and $\pi_{t_0} = \bar\alpha\pi$. On the other hand, since π_ε is the inverse of the Gauss map, we can see that $(x,y,-1)$ is normal to ∂K_ε at $\pi_\varepsilon(x,y,-1)$ and so (x,y) is normal to ∂K_{t_0} at $\pi^h_\varepsilon(x,y)$. Since K_{t_0} is strictly convex, the projection is unique and $\pi^h_\varepsilon = \pi_{t_0}$. Hence (7.2.2) follows. Taking the scalar product of $(x,y,-1)$ with the curve $\beta(s) = (s\pi(x,y), -\varepsilon(1-s^{3/2})^{2/3})$, we get

$$\langle (x,y,-1), \beta(s)\rangle = s|(x,y)|_* + \varepsilon(1-s^{3/2})^{2/3}.$$

Notice that β is in ∂K_ε and $\beta(\alpha)$ is π_ε. Hence in $s=\alpha$ the maximum of the scalar products of $(x,y,-1)$ with an element of K_ε is attained. Thus we can take derivatives in $s=\alpha$ and equal to 0, and get

$$0 = |(x,y)|_* - \varepsilon\frac{\alpha^{\frac{1}{2}}}{(1-\alpha^{3/2})^{\frac{1}{3}}}.$$

Then we obtain

$$\alpha = \frac{|(x,y)|_*^2}{(\varepsilon^3 + |(x,y)|_*^3)^{2/3}}$$

and we get (i). Since $|(x,y,-1)|_{K_\varepsilon,*} = \langle (x,y,-1), \pi_\varepsilon(x,y,-1)\rangle$, a straightforward computation shows (iii). Finally, (ii) follows from (i) and the 2-homogeneity of the map $\pi(\cdot)\,|\cdot|_*^2$. □

Lemma 7.2.2. *Let $u, v \in T_0\mathbb{H}^n$ and $s \in \mathbb{R}$. Then we have*

$$\frac{d}{ds}\Big|_{s=0} |u + sv|_{\varepsilon,*} = \langle v, \pi_\varepsilon(u)\rangle. \tag{7.2.3}$$

Proof. Let $f(s) = |u + sv|_{\varepsilon,*}$ and $g(s) = \langle u + sv, \pi_\varepsilon(u)\rangle$. Notice that $f(s) \geqslant g(s)$ for each $s \in \mathbb{R}$, since by definition $|u+sv|_{\varepsilon,*} \geqslant \langle u+sv, \pi_\varepsilon(u)\rangle$ and $f(0) = |u|_{\varepsilon,*} = \langle u, \pi_\varepsilon(u)\rangle = g(0)$. Therefore, by a standard argument $f'(0) = g'(0)$, and the thesis follows. □

Given $K_0 \subset \mathbb{R}^{2n}$ a convex body in C^2_+ with $0 \in \operatorname{int}(K_0)$, and K_ε defined as in (7.2.1), we extend the reasoning of § 2.1.2 to define a left-invariant norm $|\cdot|_\varepsilon$ on $T\mathbb{H}$ by means of the equality

$$\Big|\sum_{i=1}^n f_i X_i + g_i Y_i + hT\Big|_{\varepsilon,p} = |(f(p), g(p), h(p))|_\varepsilon,$$

for any $p \in \mathbb{H}^n$ with $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$. Again, $|\cdot|_{\varepsilon,*}$ and π_ε can be extended to the tangent bundle in the usual way.

Definition 7.2.3. Given a measurable set $E \subset \mathbb{H}^n$ we say that E has *finite K_ε-perimeter* if

$$P_{K_\varepsilon}(E) = \sup\left\{\int_E \operatorname{div}(U)\ d\mathbb{H}^n, U \in \mathfrak{X}_0(\mathbb{H}^n), |U|_{K_\varepsilon,\infty} \leqslant 1\right\} < +\infty,$$

where $|U|_{K_\varepsilon,\infty} = \sup_{p\in\mathbb{H}^n} |U_p|_\varepsilon$ and $\mathfrak{X}_0(\mathbb{H}^n)$ is the space of C^1 compactly supported vector fields in $\mathbb{H}^n$.

Remark 7.2.4. Notice that we are abusing the notation P_{K_ε} since it is related to the left-invariant basis $\{X_1, Y_1, \ldots, X_n, Y_n, T\}$ of all the tangent bundle of $\mathbb{H}^n$ instead of the horizontal distribution $\mathcal{H}_0$.

Remark 7.2.5. If E has C^1 boundary ∂E, then

$$P_{K_\varepsilon}(E) = \int_{\partial E} |N|_{\varepsilon,*} d\sigma = A_\varepsilon(\partial E),$$

where N is the Riemannian normal with respect to the metric g and $d\sigma$ is the Riemannian measure of ∂E. Indeed by the divergence theorem we have

$$\begin{aligned} P_{K_\varepsilon}(E) &= \sup\left\{\int_E \operatorname{div}(U)\ d\mathbb{H}^n, U \in \mathfrak{X}_0(\mathbb{H}^n), |U|_{K_\varepsilon,\infty} \leqslant 1\right\} \\ &= \sup\left\{\int_{\partial E} \langle U, N\rangle\ d\mathbb{H}^n, U \in \mathfrak{X}_0(\mathbb{H}^n), |U|_{K_\varepsilon,\infty} \leqslant 1\right\} \\ &= \int_{\partial E} |N|_{\varepsilon,*} d\sigma, \end{aligned}$$

where the last equality can be proved proceeding exactly as in [71; 82].

7.2.1 The Finsler prescribed mean curvature equation

We are ready to derive the Finsler prescribed mean curvature equation, essentially in the same way as in the previous section. To this aim, let $\Omega \subset \{t = 0\}$ be a bounded open set and $u : \Omega \to \mathbb{R}$ be a C^2 function. Then we have

$$N = \frac{\sum_{i=1}^{n}(u_{x_i} - y)X_i + (u_{y_i} + x)Y_i - T}{\sqrt{1 + |\nabla u + F|^2}}$$

$$d\sigma = \sqrt{1 + |\nabla u + F|^2}\, dxdy,$$

where $F(x, y) = (-y, x)$. Therefore we get

$$A_{K_\varepsilon}(\mathrm{Gr}(u)) = \int_\Omega |(\nabla u + F, -1)|_{\varepsilon,*}\, dxdy.$$

Therefore, inspired by this computation and thanks to Proposition 7.2.1, given $F \in L^1(\Omega, \mathbb{R}^{2n})$, $\varphi \in W^{1,1}(\Omega)$ and $H \in L^\infty(\Omega)$, we define the approximating Finsler functional $\mathcal{I}_\varepsilon$ by

$$\mathcal{I}_\varepsilon(u) = \int_\Omega \left(\varepsilon^3 + |(\nabla u + F)|_*^3\right)^{\frac{1}{3}}\, dxdy + \int_\Omega Hu\, dxdy, \tag{7.2.4}$$

for any $u \in W^{1,1}(\Omega)$ such that $u - \varphi \in W_0^{1,1}(\Omega)$. Arguing as in the previous section, and thanks to Lemma 7.2.2, we are able to deduce the Euler-Lagrange equation associated to (7.2.4). Indeed, given $v \in C_c^\infty(\Omega)$, by Lemma 7.2.2, the first variation is given by:

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=0} \mathcal{I}_\varepsilon(u + sv) \\
&\qquad = \int_\Omega \frac{d}{ds}\Big|_{s=0} |(\nabla(u + sv) + F, -1)|_{\varepsilon,*}\, dxdy + \int_\Omega Hv\, dxdy \\
&\qquad = \int_\Omega \frac{d}{ds}\Big|_{s=0} |(\nabla u + F, -1) + s(\nabla v, 0)|_{\varepsilon,*}\, dxdy + \int_\Omega Hv\, dxdy \\
&\qquad = \int_\Omega \langle(\nabla v, 0), \pi_\varepsilon((\nabla u + F, -1))\rangle\, dxdy + \int_\Omega Hv\, dxdy \\
&\qquad = \int_\Omega \langle \nabla v, \pi_\varepsilon^h(\nabla u + F)\rangle\, dxdy + \int_\Omega Hv\, dxdy \\
&\qquad = \int_\Omega v(H - \mathrm{div}(\pi_\varepsilon^h(\nabla u + F)))\, dxdy.
\end{aligned}$$

Then the Finsler prescribed mean curvature equation for the graph of u is given by

$$\mathrm{div}(\pi_\varepsilon^h(\nabla u + F)) = H \quad \text{in } \Omega. \tag{7.2.5}$$

7.3 A priori estimates for the Finsler Prescribed Mean Curvature Equation

In this section we want to find classical solutions to the Finsler approximating Dirichlet problem associated to (7.2.5), that is

$$\begin{cases} \operatorname{div}\left(\pi^h_\varepsilon(\nabla u + F)\right) = H & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega, \end{cases} \tag{7.3.1}$$

where $\Omega \subseteq \mathbb{R}^{2n}$ is a bounded domain with $C^{2,\alpha}$ boundary for $0 < \alpha < 1$, K_0 is a convex body in $C^{2,\alpha}_+$ with $0 \in \operatorname{int} K_0$, $H \in Lip(\overline{\Omega})$, $F \in C^{1,\alpha}(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$. To this aim, let us fix some notation. It is easy to see that the map $G : \mathbb{R}^{2n} \setminus \{0\} \to \mathbb{R}^{2n}$ defined by $G(p) = \pi(p)|p|^2_*$ can be extended to a 2-homogeneous and C^1 map setting $G(0) = 0$. Moreover, for any $i = 1, \dots, 2n$

$$D_i(|\cdot|^3_*) = 3G_i(\cdot),$$

where $G = (G_1, \dots, G_{2n})$. Thanks to Proposition 7.2.1, we can write (7.2.5) in the form

$$\operatorname{div}\left(\pi(\nabla u + F)\frac{|\nabla u+F|^2_*}{(\varepsilon^3+|\nabla u+F|^3_*)^{\frac{2}{3}}}\right) = H.$$

An easy computation yields

$$\frac{1}{(\varepsilon^3 + |\nabla u + F|^3_*)^{\frac{5}{3}}}\Big((\varepsilon^3 + |\nabla u + F|^3_*)\operatorname{div}(G(\nabla u + F)) - \\ - 2G(\nabla u + F)(D^2u + DF)G(\nabla u + F)^T\Big) = H.$$

Therefore, we can write (7.2.5) in the familiar form

$$\sum_{i,j=1}^{2n} A^\varepsilon_{i,j}(z, \nabla u; F)D_{i,j}u + B^\varepsilon(z, \nabla u; F) = H,$$

where the coefficients $A^\varepsilon_{i,j}$ and B^ε are defined by

$$A^\varepsilon_{i,j}(z,p;F) := \frac{1}{(\varepsilon^3 + |p + F|^3_*)^{\frac{2}{3}}}D_jG_i(p + F) \\ - \frac{2}{(\varepsilon^3 + |p + F|^3_*)^{\frac{5}{3}}}G_i(p + F)G_j(p + F)$$

and

$$B^\varepsilon(z,p;F) := \frac{1}{(\varepsilon^3 + |p + F|^3_*)^{\frac{2}{3}}}\sum_{i,j=1}^{2n} D_jG_i(p + F)D_iF_j \\ - \frac{2}{(\varepsilon^3 + |p + F|^3_*)^{\frac{5}{3}}}G(p + F)\, DF\, G(p + F)^T.$$

for any $z \in \Omega$ and $p \in \mathbb{R}^{2n}$. Therefore (7.2.5) is a second-order quasi-linear equation. Moreover, thanks to the computations of the previous section and (*iii*) in Proposition 7.2.1, we know that (7.2.5) is the Euler-Lagrange equation associated to the functional

$$u \mapsto \int_\Omega \left(\varepsilon^3 + |\nabla u + F|_*^3\right)^{\frac{1}{3}} + uH\, dz.$$

Since the function $(\varepsilon^3 + |\cdot + F|_*^3)^{1/3}$ is strictly convex we get that (7.2.5) is elliptic. Finally, it is easy to see that the matrix A^ε is symmetric. Therefore we are in position to apply the classical theory for quasi-linear elliptic equation of [85]. In particular, we wish to rely on the following fundamental result, which is a direct consequence of [85, Theorem 13.8] and subsequent remarks.

Proposition 7.3.1. *Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with $C^{2,\alpha}$ boundary, for some $0 < \alpha < 1$, and let $\varphi \in C^{2,\alpha}(\overline{\Omega})$. Let us assume that $A^\varepsilon_{i,j}(\cdot,\cdot;\sigma F), B^\varepsilon(\cdot,\cdot;\sigma F) \in C^\alpha(\overline{\Omega} \times \mathbb{R}^{2n})$ for any $\sigma \in [0,1]$, and that the maps*

$$\sigma \mapsto A^\varepsilon_{i,j}(\cdot,\cdot;\sigma F), \quad \sigma \mapsto B^\varepsilon(\cdot,\cdot;\sigma F)$$

are continuous as maps from $[0,1]$ to $C^\alpha(\overline{\Omega} \times \mathbb{R}^{2n})$. If there exist a constant $M > 0$ such that, for any $\sigma \in [0,1]$, any solution $u \in C^{2,\alpha}(\overline{\Omega})$ to the problem

$$\begin{cases} \operatorname{div}(\pi^h_\varepsilon(\nabla u + \sigma F)) = \sigma H & \text{in } \Omega \\ u = \sigma\varphi & \text{in } \partial\Omega \end{cases} \tag{7.3.2}$$

satisfies

$$\|u\|_{C^1(\overline{\Omega})} \leqslant M,$$

then

$$\begin{cases} \operatorname{div}(\pi^h_\varepsilon(\nabla u + F)) = H & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases} \tag{7.3.3}$$

admits a solution in $C^{2,\alpha}(\overline{\Omega})$.

Our aim is to prove *a priori* estimates for the C^1 norm of solutions to (7.3.2). As a consequence of this procedure we will get C^1 estimates for solutions to (7.3.3) which are uniform in $\varepsilon \in (0,1)$. First of all we need to guarantee the requested regularity for the coefficients of the equation.

Lemma 7.3.2. *Let K_0 be a convex body in $C^{2,\alpha}_+$ with $0 \in \operatorname{int} K_0$. Let $F \in C^{1,\alpha}(\overline{\Omega})$. Then there exists $0 < \beta < 1$ such that $A^\varepsilon_{i,j}(\cdot,\cdot;\sigma F), B^\varepsilon(\cdot,\cdot;\sigma F) \in C^\beta(\overline{\Omega} \times \mathbb{R}^{2n})$ for any $\sigma \in [0,1]$. Moreover, the maps*

$$\sigma \mapsto A^\varepsilon_{i,j}(\cdot,\cdot;\sigma F), \quad \sigma \mapsto B^\varepsilon(\cdot,\cdot;\sigma F)$$

are continuous as maps from $[0,1]$ to $C^\beta(\overline{\Omega} \times \mathbb{R}^{2n})$.

Proof. The second statement follows easily from the definition of the coefficients. Let us prove the first statement. It is clear, thanks to our assumptions on K_0 and F, that $A^{\varepsilon}_{i,j}(\cdot,\cdot,\sigma F)$ and $B^{\varepsilon}(\cdot,\cdot,\sigma F)$ belong to $C^0(\overline{\Omega}\times\mathbb{R}^{2n})$ for any $\sigma\in[0,1]$. Moreover, notice that for any $i,j=1,\ldots,2n$

$$D_j(G_i(p)) = \begin{cases} 2|p|_*\pi_j(p)\pi_i(p) + |p|_*^2 D_i\pi_j(p) & \text{if } p\neq 0 \\ 0 & \text{if } p=0 \end{cases}$$

is $C^{\alpha}(\mathbb{R}^{2n}\setminus 0)$ since ∂K_0 is $C^{2,\alpha}$. Finally, we get

$$\lim_{p\to 0}\frac{|D_jG_i|(p)}{|p|^{\alpha}} = 0.$$

Indeed, we have

$$\begin{aligned}\frac{|D_jG_i|(p)}{|p|^{\alpha}} &= 2\frac{|p|_*}{|p|^{\alpha}}|\pi_j(p)\pi_i(p) + |p|_*^2 D_i\pi_j(p)| \\ &\leqslant 2\frac{|p|_*^{\alpha}}{|p|^{\alpha}}|p|_*^{1-\alpha}\left(|\pi_j(p)\pi_i(p)| + |p|_*|D_i\pi_j(p)|\right) \\ &\leqslant C|p|_*^{1-\alpha}\to 0\end{aligned}$$

as $p\to 0$, since $\frac{|p|_*^{\alpha}}{|p|^{\alpha}}$ is bounded and the last factor in the previous inequality is 0-homogeneous, thus in particular bounded. Then D_jG_i belongs to $C^{\alpha}(\mathbb{R}^{2n})$. Since $A^{\varepsilon}_{i,j}$ and B^{ε} are obtained as composition, sum and product of Hölder functions, the conclusion follows. □

Therefore we are in position to try to apply Proposition 7.3.1. First of all we want to obtain estimates for the C^0 norm of solutions to (7.3.2). In order to do this, inspired by [84], we assume that there exists $\delta\in(0,1]$ such that

$$\left|\int_{\Omega} Hv dz\right| \leqslant (1-\delta)\int_{\Omega}|\nabla v|_* dz \tag{7.3.4}$$

for any non-negative function $v\in C^{\infty}_c(\Omega)$. To justify this assumption, assume that we have a function $u\in C^2(\Omega)$ which solves (7.3.1). Then, multiplying (7.3.1) by a test function $v\in C^{\infty}_c(\Omega)$ and integrating over Ω, we get that

$$\begin{aligned}\left|\int_{\Omega} Hv\, dz\right| = \left|\int_{\Omega} v\,\mathrm{div}(\pi^h_{\varepsilon}(\nabla u+F))dz\right| \leqslant \int_{\Omega}|\langle\pi^h_{\varepsilon}(\nabla u+F),\nabla v\rangle|\, dz \\ \leqslant \int_{\Omega}|\nabla v|_* dx. \quad (7.3.5)\end{aligned}$$

Notice that, as already pointed out in the introduction, (7.3.4) is slightly stronger than (7.3.5). We begin by proving a technical lemma.

Lemma 7.3.3. *Let $\sigma \in [0,1]$ and $\varepsilon \in (0,1)$. Then*

$$\langle p, \pi_\varepsilon^h(p+\sigma F)\rangle \geqslant |p|_* - 1 - |F|_* - |-F|_* \tag{7.3.6}$$

for any $p \in \mathbb{R}^{2n}$ and $z \in \overline{\Omega}$.

Proof. Let us fix $z \in \overline{\Omega}$ and $p \in \mathbb{R}^{2n}$. If $p = 0$ or $p + \sigma F = 0$, then the thesis is trivial. Therefore, assume $p, p + \sigma F \neq 0$. It is clear, recalling Proposition 7.2.1 and using the Cauchy -Schwarz formula (2.1.4), that

$$\begin{aligned}\langle p, \pi_\varepsilon^h(p+\sigma F)\rangle &= \langle p+\sigma F, \pi_\varepsilon^h(p+\sigma F)\rangle - \langle \sigma F, \pi_\varepsilon^h(p+\sigma F)\rangle \\ &\geqslant \frac{|p+\sigma F|_*^3}{(\varepsilon^3 + |p+\sigma F|_*^3)^{\frac{2}{3}}} - \left(\frac{|p+\sigma F|_*^3}{\varepsilon^3 + |p+\sigma F|_*^3}\right)^{\frac{2}{3}} |\sigma F|_* \\ &\geqslant \frac{|p+\sigma F|_*^3}{(\varepsilon^3 + |p+\sigma F|_*^3)^{\frac{2}{3}}} - |F|_*.\end{aligned}$$

Hence, noticing that

$$|p+\sigma F|_* \geqslant |p|_* - |-\sigma F|_* \geqslant |p|_* - |-F|_*$$

by the triangle inequality, it suffices to prove that

$$\frac{|p+\sigma F|_*^3}{(\varepsilon^3 + |p+\sigma F|_*^3)^{\frac{2}{3}}} \geqslant |p+\sigma F|_* - 1. \tag{7.3.7}$$

When $|p+\sigma F|_* \leqslant 1$ (7.3.7) is trivial. Therefore let us assume $|p+\sigma F|_* > 1$. Notice that (7.3.7) is equivalent to

$$|p+\sigma F|_*^{\frac{9}{2}} \geqslant (|p+\sigma F|_* - 1)^{\frac{3}{2}} (\varepsilon^3 + |p+\sigma F|_*^3).$$

Since $a^p - b^p \geqslant (a-b)^p$ when $0 < b < a$ and $p > 1$, it is enough to check that

$$\begin{aligned}|p+\sigma F|_*^{9/2} &\geqslant (|p+\sigma F|_*^{3/2} - 1)(\varepsilon^3 + |p+\sigma F|_*^3) \\ &= \varepsilon^3 |p+\sigma F|_*^{3/2} + |p+\sigma F|_*^{9/2} - \varepsilon^3 - |p+\sigma F|_*^3,\end{aligned}$$

which is clearly true since $|p+\sigma F|_* > 1$ and $\varepsilon < 1$. □

Proposition 7.3.4. *Let $\alpha \in (0,1)$ and K_0 be a convex body in $C_+^{2,\alpha}$ with $0 \in \operatorname{int} K_0$. Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded open set, $\varphi \in C^2(\overline{\Omega})$, $H \in L^\infty(\Omega)$ and $F \in C^0(\overline{\Omega})$. If condition (7.3.4) is satisfied then there exists a constant $C_1 = C_1(n, K_0, \Omega, \varphi, F, \delta) > 0$, independent of $\sigma \in [0,1]$ and $\varepsilon \in (0,1)$, such that, for any solution $u \in C^2(\overline{\Omega})$ to (7.3.2), it holds that*

$$\|u\|_{L^\infty(\Omega)} \leqslant C_1.$$

Proof. Let us notice that (7.3.6), the equivalence between $|\cdot|_*$ and the Euclidean norm and the boundedness of F allow to find constants $a_0, a_2 > 0$, independent of $\sigma \in [0,1]$ and $\varepsilon \in (0,1)$, such that

$$\langle p, \pi_\varepsilon^h(p + \sigma F)\rangle \geqslant a_0|p| - a_2$$

for any $z \in \Omega$ and $p \in \mathbb{R}^{2n}$. This fact, together with the boundedness of H, suggests to relay on [85, Lemma 10.8] to limit ourselves to estimate $\|u\|_{L^1(\Omega)}$. Indeed, it is not difficult to show that [85, Lemma 10.8] remains true when condition (10.23) of [85] allows a positive coefficient multiplying $|p|$. Moreover, its proof can be easily adapted to achieve estimates from above of $\sup_\Omega -u$ in terms of $\|u^-\|_{L^1(\Omega)}$ for any solution of $Qu = 0$ where Q is defined in (10.5) of [85]. In the end it suffices to estimate $\|u^+\|_{L^1(\Omega)}$ and $\|u^-\|_{L^1(\Omega)}$. We only estimate $\|u^+\|_{L^1(\Omega)}$, being the other case analogous. Moreover, up to replacing u by $u - \|\varphi\|_{\infty,\partial\Omega}$, we can assume that $u \leqslant 0$ in $\partial\Omega$. Let us set $v = u^+$. Then it is clear that $v \in W^{1,\infty}(\Omega) \cap W_0^{1,1}(\Omega)$, and moreover ∇v exists in the classical sense for almost every $z \in \Omega$. Therefore, since u is in particular a weak solution to

$$\mathrm{div}(\pi_\varepsilon^h(\nabla u + \sigma F)) = \sigma H,$$

it follows that

$$\int_\Omega \langle \nabla v, \pi_\varepsilon^h(\nabla u + \sigma F)\rangle dz = -\int_\Omega v\sigma H dz. \tag{7.3.8}$$

We claim that

$$\langle \nabla v, \pi_\varepsilon^h(\nabla u + \sigma F)\rangle \geqslant |\nabla v|_* - 1 - |F|_* - |-F|_* \tag{7.3.9}$$

holds in any point where ∇v exists in the classical sense. Indeed, in such points ∇v is either 0 or ∇u. In the first case (7.3.9) is trivial, while in the second case it follows from Lemma 7.3.3. It is well known that, since $v \geqslant 0$ and $v \in W_0^{1,1}(\Omega)$, there exists a sequence of non-negative functions $(v_k)_k \subseteq C_c^\infty(\Omega)$ converging to v strongly in $W_0^{1,1}(\Omega)$. Moreover, thanks to (7.3.4) it holds that

$$\left|\int_\Omega H v_k dz\right| \leqslant (1-\delta)\int_\Omega |\nabla v_k|_* \, dz.$$

Hence, passing to the limit in the previous equation, and recalling that $|\cdot|_*$ is equivalent to the Euclidean norm, we conclude that (7.3.4) holds for v.

Combining this information with (7.3.8) and (7.3.9) we get that

$$\begin{aligned}
0 &= \int_\Omega -\langle \nabla v, \pi_\varepsilon^h(\nabla u + \sigma F)\rangle \, dz - \int_\Omega v\sigma H \, dz \\
&\leqslant \int_\Omega -|\nabla v|_* + 1 + |F|_* + |-F|_* \, dz + \left| \int_\Omega vH \, dz \right| \\
&\leqslant \int_\Omega -|\nabla v|_* + 1 + |F|_* + |-F|_* + (1-\delta)|\nabla v|_* \, dz \\
&\leqslant \int_\Omega 1 + |F|_* + |-F|_* - \delta |\nabla v|_* \, dz,
\end{aligned}$$

which implies

$$\delta \int_\Omega |\nabla v|_* \, dz \leqslant \int_\Omega 1 + |F|_* + |-F|_* \, dz.$$

Thanks to the Poincaré inequality and the equivalence between $|\cdot|_*$ and the Euclidean norm, we conclude that there exists a constant c_1, independent of $\sigma \in [0,1]$ and $\varepsilon \in (0,1)$. Such that

$$\int_\Omega u^+ \, dz \leqslant c_1.$$

Since in the same way we can achieve an estimate for u^-, the thesis follows. □

The next step is to achieve gradient estimates, again in the C^0 norm, for solutions to (7.3.2). As customary in this framework, we want to reduce ourselves to boundary gradient estimates via a suitable maximum principle. To this aim, arguing as in [35], we need to assume the existence of scalar functions $f_1, \ldots, f_{2n} \in C^1(\overline{\Omega})$ such that

$$D_k F_i = D_i f_k \quad \text{for any} \quad i,k = 1, \ldots, 2n. \tag{7.3.10}$$

Thanks to this assumption, the following maximum principle, which is the Finsler counterpart of [35, Proposition 4.3], holds.

Proposition 7.3.5. *Let K_0 be a convex body in $C^{2,\alpha}_+$ for $0 < \alpha < 1$ with $0 \in \operatorname{int} K_0$. Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain. Let $F \in C^1(\Omega, \mathbb{R}^{2n})$ be such that (7.3.10) holds. Let H be a constant. Let $u \in C^2(\overline{\Omega})$ be a solution to (7.3.2). Then*

$$\|\nabla u\|_{\infty,\Omega} \leqslant \|\nabla u\|_{\infty,\partial\Omega} + 2\|f\|_{\infty,\Omega}, \tag{7.3.11}$$

where $f = (f_1, \ldots, f_{2n})$ is as in (7.3.10).

Proof. Fix $\sigma \in [0,1]$ and $\varepsilon \in (0,1)$. Let $v \in C^2_c(\Omega)$ and fix $k \in \{1,\dots,2n\}$. Then, multiplying (7.3.2) by $D_k v$, using Proposition 7.2.1, integrating over Ω, integrating by parts and exploiting the properties of F, it holds that

$$\begin{aligned}
0 &= \int_\Omega \left(\operatorname{div} \left(\pi(\nabla u + \sigma F) \frac{|\nabla u+\sigma F|_*^2}{(\varepsilon^3+|\nabla u+\sigma F|_*^3)^{\frac{2}{3}}} \right) - \sigma H \right) D_k v \, dz \\
&= \int_\Omega \operatorname{div} \left(\pi(\nabla u + \sigma F) \frac{|\nabla u+\sigma F|_*^2}{(\varepsilon^3+|\nabla u+\sigma F|_*^3)^{\frac{2}{3}}} \right) D_k v \, dz \\
&= -\sum_{i=1}^{2n} \int_\Omega \left(\pi_i(\nabla u + \sigma F) \frac{|\nabla u+\sigma F|_*^2}{(\varepsilon^3+|\nabla u+\sigma F|_*^3)^{\frac{2}{3}}} \right) D_i D_k v \, dz \\
&= -\sum_{i=1}^{2n} \int_\Omega \left(\pi_i(\nabla u + \sigma F) \frac{|\nabla u+\sigma F|_*^2}{(\varepsilon^3+|\nabla u+\sigma F|_*^3)^{\frac{2}{3}}} \right) D_k D_i v \, dz \\
&= \sum_{i=1}^{2n} \int_\Omega D_k \left(\pi_i(\nabla u + \sigma F) \frac{|\nabla u+\sigma F|_*^2}{(\varepsilon^3+|\nabla u+\sigma F|_*^3)^{\frac{2}{3}}} \right) D_i v \, dz \\
&= \sum_{i,j=1}^{2n} \int_\Omega a_{i,j}^{\varepsilon,\sigma}(x, \nabla u) D_k (D_j u + \sigma F_j) D_i v \, dz \\
&= \sum_{i,j=1}^{2n} \int_\Omega a_{i,j}^{\varepsilon,\sigma}(x, \nabla u) D_j (D_k u + \sigma f_k) D_i v dz,
\end{aligned}$$

where

$$a_{i,j}^{\varepsilon,\sigma}(x,p) = \frac{D_j(G_i)(p+\sigma F)(\varepsilon^3 + |p+\sigma F|_*^3) - 2G_i(p+\sigma F)G_j(p+\sigma F)}{(\varepsilon^3 + |p+\sigma F|_*^3)^{\frac{5}{3}}}.$$

Therefore we proved that

$$\sum_{i,j=1}^{2n} \int_\Omega a_{i,j}^{\varepsilon,\sigma}(x, \nabla u) D_j (D_k u + \sigma f_k) D_i v \, dz = 0 \tag{7.3.12}$$

for any $v \in C^2_c(\Omega)$. Arguing as in [35, Proposition 4.3] it is easy to show that (7.3.12) actually holds for any $v \in C^1_c(\Omega)$. Therefore we proved that $D_k u + \sigma f_k$ is a weak solution to the linear elliptic equation

$$\operatorname{div}(a_{i,j}^{\varepsilon,\sigma} D_j w) = 0.$$

Hence, being $a_{i,j}^{\varepsilon,\sigma}(x, \nabla u)$ bounded over Ω, thanks to [85, Theorem 8.1] with $b_i, c_i, d = 0$ we can conclude that

$$\|\nabla u + \sigma f\|_{\infty,\Omega} \leqslant \|\nabla u + \sigma f\|_{\infty,\partial\Omega},$$

which in particular implies that

$$\|\nabla u\|_{\infty,\Omega} \leqslant \|\nabla u\|_{\infty,\partial\Omega} + 2\|f\|_{\infty,\Omega}. \tag{7.3.13}$$

□

Finally we are left to provide boundary gradient estimates for solutions to (7.3.2). Therefore, inspired by [84], we have to impose some constraints on the values of H depending on the Finsler mean curvature of $\partial\Omega$. More precisely, we require that

$$|H|(z_0) < H_{K_0,\partial\Omega}(z_0) \tag{7.3.14}$$

for any $z_0 \in \partial\Omega$, where $H_{K_0,\partial\Omega}$ is the K_0-mean curvature as defined in § 7.1.1. Here and in the rest of this section we assume that K_0 is a convex body in C^∞_+ such that $0 \in \operatorname{int} K_0$, since we need to apply the results of § 7.1.1.1 and § 7.1.1.2.

Proposition 7.3.6. *Let K_0 be a convex body in C^∞_+ with $0 \in \operatorname{int} K_0$. Let $\Omega \subseteq \mathbb{R}^{2n}$ be an open and bounded set with $C^{2,\alpha}$ boundary, for some $0 < \alpha < 1$. Let $\varphi \in C^2(\overline{\Omega})$, $F \in C^0(\overline{\Omega})$ and $H \in Lip(\Omega)$ satisfying* (7.3.14)*. Finally, assume that there exists a constant $\tilde{C}_1 = \tilde{C}_1(n, K_0, \Omega, \varphi, F, H) > 0$, independent of $\sigma \in [0, 1]$ and $\varepsilon \in (0, 1)$, such that, for any solution $u \in C^2(\overline{\Omega})$ to* (7.3.2)*, it holds that*

$$\|u\|_{\infty,\Omega} \leqslant \tilde{C}_1. \tag{7.3.15}$$

Then, there exists a constant $C_2 = C_2(n, K_0, \Omega, \varphi, F, \tilde{C}_1, H) > 0$, independent of $\sigma \in [0, 1]$ and $\varepsilon \in (0, 1)$, such that, for any solution $u \in C^2(\overline{\Omega})$ to (7.3.2)*, it holds that*

$$\|\nabla u\|_{\infty,\partial\Omega} \leqslant C_2. \tag{7.3.16}$$

Proof. First of all we notice that, being $\partial\Omega$ compact and $H_{K_0,\partial\Omega}$ continuous, (7.3.14) implies the existence of a positive constant C_3 such that

$$|H(z_0)| \leqslant H_{K_0,\partial\Omega}(z_0) - 2C_3 \tag{7.3.17}$$

for any $z_0 \in \partial\Omega$. In order to prove this result we use a barriers argument as in [85, Chapter 14]. Therefore, for any $z_0 \in \partial\Omega$, we have to find a neighborhood $\mathcal{N}$ of z_0 in Ω and two functions $w^+, w^- \in C^2(\mathcal{N})$, called *upper barrier* and *lower barrier* respectively, such that

$$w^+(z_0) = w^-(z_0) = \sigma\varphi(z_0),$$

$$w^-(z) \leqslant u(z) \leqslant w^+(z)$$

for any $z \in \partial\mathcal{N}$,

$$\operatorname{div}(\pi^h_\varepsilon(\nabla w^+ + \sigma F)) \leqslant \sigma H$$

for any $z \in \mathcal{N}$ and

$$\operatorname{div}(\pi^h_\varepsilon(\nabla w^- + \sigma F)) \geqslant \sigma H$$

for any $z \in \mathcal{N}$. In this proof we deal only with the upper barrier, being the other case analogous. In order to find an upper barrier, we consider a tubular neighborhood $\mathcal{O}$ of $\partial\Omega$ and we let $\Gamma_\mu := \{x \in \overline{\Omega} : d_{K_0,\partial\Omega}(x) < \mu\}$, where $\mu > 0$ is small enough to ensure that $\Gamma_\mu \Subset \mathcal{O}$ and $d_{K_0,\partial\Omega}$ denotes the Finsler distance from the boundary. We define $w^+ : \Gamma_\mu \mathbb{R}$ by $w^+(z) := k d_{K_0,\partial\Omega}(z) + \sigma\varphi(z)$, where $k > 0$ has to be chosen. First, thanks to (7.1.12), $w^+ \in C^2(\overline{\Gamma_\mu})$, and for any $z \in \Gamma_\mu$ there exists a unique $z_0 \in \partial\Omega$ such that $d_{K_0,\partial\Omega}(z) = |z - z_0|_{K_0}$. Moreover, it is clear that $w^+(z_0) = \sigma\varphi(z_0)$ for any $z_0 \in \partial\Omega$. Thanks to (7.3.15), if we choose

$$k \geqslant \frac{\tilde{C}_1 + \|\varphi\|_{\infty,\Omega}}{\mu},$$

it follows that $w^+(z) \geqslant u(x)$ for any $z \in \Omega$ with $d_{K_0,\partial\Omega}(z) = \mu$, and so we conclude that $u(z) \leqslant w^+(z)$ for any $z \in \partial\Gamma_\mu$. We are left to show that w^+ is a subsolution to (7.3.2). Therefore it suffices to show that

$$(\varepsilon^3 + |\nabla w^+ + \sigma F|_*^3)^{\frac{5}{3}}(\operatorname{div}(\pi_\varepsilon^h(\nabla w^+ + \sigma F)) - \sigma H) \leqslant 0$$

on Γ_μ. Taking $k > \sup_\Omega |-F|_*$, (7.1.6) ensures that $k\nabla d_{K_0,\partial\Omega}(z) + \sigma F(z) \neq 0$ for any $z \in \Gamma_\mu$ and $\sigma \in [0,1]$. Let us notice that Proposition 7.2.1 and a simple computation imply that

$$\begin{aligned}
(\varepsilon^3 + &|\nabla w^+ + \sigma F|_*^3)^{\frac{5}{3}} \operatorname{div}(\pi_\varepsilon^h(\nabla w^+ + \sigma F)) \\
&= (\varepsilon^3 + |\nabla w^+ + \sigma F|_*^3)^{\frac{5}{3}} \operatorname{div}\left(\frac{\pi(\nabla w^+ + \sigma F)|\nabla w^+ + \sigma F|_*^2}{(\varepsilon^3 + |\nabla w^+ + \sigma F|_*^3)^{\frac{2}{3}}}\right) \\
&= (\varepsilon^3 + |\nabla w^+ + \sigma F|_*^3)\underbrace{\operatorname{div}(\pi(\nabla w^+ + \sigma F)|\nabla w^+ + \sigma F|_*^2)}_{A} \\
&\quad + (\varepsilon^3 + |\nabla w^+ + \sigma F|_*^3)^{\frac{5}{3}} \\
&\qquad \underbrace{|\nabla w^+ + \sigma F|_*^2 \langle \pi(\nabla w^+ + \sigma F), \nabla\left(\varepsilon^3 + |\nabla w^+ + \sigma F|_*^3)^{-\frac{2}{3}}\right)\rangle}_{B}.
\end{aligned}$$

We estimate separately A and B. In the following computations we let $d := d_{K_0,\partial\Omega}$ and $R_\sigma := \sigma\nabla\varphi + \sigma F$. We are going to exploit the fact that, thanks to the homogeneity properties of the equation, the contribution of R_σ as $k \to \infty$ is negligible. Let us notice that by (7.1.6) and (2.1.6) we get

$$\pi(\nabla d_{K_0,\partial\Omega}) \cdot D^2 d_{K_0,\partial\Omega} = 0. \tag{7.3.18}$$

Hence, thanks to (7.1.6), (7.3.18), the 1-homogeneity of $|\cdot|_*$, the 0-homogeneity of π, the -1-homogeneity of $D\pi$ and the properties of $|\cdot|_*$, it holds that

$$
\begin{aligned}
A =&|k\nabla d + R_\sigma|_*^2 \sum_{i=1}^{2n} D_i \left(\pi_i(k\nabla d + R_\sigma)\right) \\
&+ \sum_{i=1}^{2n} \pi_i(k\nabla d + R_\sigma) D_i \left(|k\nabla d + R_\sigma|_*^2\right) \\
=&|k\nabla d + R_\sigma|_*^2 \sum_{i,j=1}^{2n} D_i\pi_j(k\nabla d + R_\sigma)(kD_{ij}d + D_i R_{\sigma,j}) \\
&+ 2|k\nabla d + R_\sigma|_* \pi(k\nabla d + R_\sigma)\cdot(kD^2 d + DR_\sigma)\cdot\pi(k\nabla d + R_\sigma)^T \\
=&k^2 \left|\nabla d + \frac{R_\sigma}{k}\right|_*^2 \sum_{i,j=1}^{2n} D_i\pi_j\left(\nabla d + \frac{R_\sigma}{k}\right)\left(D_{ij}d + \frac{D_i R_{\sigma,j}}{k}\right) \\
&+ 2k^2 \left|\nabla d + \frac{R_\sigma}{k}\right|_* \pi\left(\nabla d + \frac{R_\sigma}{k}\right)\cdot\left(D^2 d + \frac{DR_\sigma}{k}\right)\cdot\pi\left(\nabla d + \frac{R_\sigma}{k}\right)^T \\
=&k^2(1+o(1))(\operatorname{div}(\pi(\nabla d)) + o(1)) \\
&+ 2k^2(1+o(1))(\pi(\nabla d)\cdot D^2 d\cdot\pi(\nabla d)^T + o(1)) \\
=&k^2 \operatorname{div}(\pi(\nabla d)) + o(k^2),
\end{aligned}
$$

which allows to infer that

$$
(\varepsilon^3 + |\nabla w^+ + \sigma F|_*^3)A = k^5 \operatorname{div}(\pi(\nabla d)) + o(k^5)
$$

as $k\to\infty$, where $o(k^2)$ is uniform with respect to $z\in\Gamma_\mu$, $\varepsilon\in(0,1)$ and $\sigma\in[0,1]$. Now, exploiting the same properties as above, we estimate B.

$$
\begin{aligned}
&(\varepsilon^3 + |k\nabla d + R_\sigma|_*^3)^{\frac{5}{3}} B \\
&= -2|k\nabla d + R_\sigma|_*^4 \langle \pi(k\nabla d + R_\sigma), \nabla(|k\nabla d + R_\sigma|_*)\rangle \\
&= -2|k\nabla d + R_\sigma|_*^4 \pi(k\nabla d + R_\sigma)\cdot(kD^2 d + DR_\sigma)\cdot\pi(k\nabla d + R_\sigma)^T \\
&= -2k^5 \left|\nabla d + \frac{R_\sigma}{k}\right|_*^4 \pi\left(\nabla d + \frac{R_\sigma}{k}\right)\cdot\left(D^2 d + \frac{DR_\sigma}{k}\right)\cdot\pi\left(\nabla d + \frac{R_\sigma}{k}\right)^T \\
&= -2k^5(1+o(1))(\pi(\nabla d)\cdot D^2 d\cdot\pi(\nabla d)^T + o(1)) \\
&= -2k^5(1+o(1))o(1) \\
&= o(k^5).
\end{aligned}
$$

as $k\to\infty$ and uniformly with respect to $\varepsilon\in(0,1)$, $\sigma\in[0,1]$ and $z\in\Gamma_\mu$. Finally, it is easy to see that

$$
-(\varepsilon^3 + |\nabla w^+ + \sigma F|_*^3)^{\frac{5}{3}}\sigma H \leqslant (\varepsilon^3 + |\nabla w^+ + \sigma F|_*^3)^{\frac{5}{3}}|H| = k^5|H| + o(k^5)
$$

as $k \to \infty$ and uniformly with respect to $\varepsilon \in (0,1)$, $\sigma \in [0,1]$ and $z \in \Gamma_\mu$. In the end we get that

$$(\varepsilon^3 + |\nabla w^+ + \sigma F|_*^3)^{\frac{5}{3}} (\mathrm{div}(\pi_\varepsilon^h(\nabla w^+ + \sigma F)) - \sigma H) \leqslant k^5 (\mathrm{div}(\pi(\nabla d)) + |H|) + o(k^5)$$

as $k \to \infty$ and uniformly with respect to $\varepsilon \in (0,1)$, $\sigma \in [0,1]$ and $z \in \Gamma_\mu$.

Now, let $z \in \Gamma_\mu$ and let $z_0 \in \partial\Omega$ be such that $d(z) = |z - z_0|_{K_0}$. Thanks to the Lipschitz continuity of H and the equivalence between $|\cdot|_{K_0}$ and the Euclidean norm, there exists a constant C_4 such that

$$|H|(z) = |H|(z_0) + |H|(z) - |H|(z_0) \leqslant |H|(z_0) + C_4 d(z) \leqslant |H|(z_0) + C_4\mu \tag{7.3.19}$$

Hence, thanks to Remark 7.1.4, (7.1.8) and (7.3.17), we conclude that

$$\begin{aligned}
\mathrm{div}(\pi(\nabla d))(z) + |H|(z_0) + C_4\mu &= \mathrm{div}_{\Sigma_{d(z)}}(\pi(\nabla d))(z) + |H|(z_0) + C_4\mu \\
&\leqslant \mathrm{div}_{\partial\Omega}(\pi(\nabla d))(z_0) + |H|(z_0) + C_4\mu \\
&= -H_{K_0,\partial\Omega}(z_0) + |H|(z_0) + C_4\mu \\
&\leqslant -C_3 < 0,
\end{aligned}$$

provided that $\mu \leqslant \frac{C_3}{C_4}$. Hence we found an upper barrier, from which the thesis follows. □

Remark 7.3.7. Assume that $n = 1$, let $\Omega \subset \mathbb{R}^2$ and $K_0 \in C^2_+$ be a convex body of $\mathbb{R}^2$. If (7.3.14) holds then Ω is strictly convex. Indeed, by Proposition 7.1.4 we have

$$0 \leqslant |H| < -\frac{\langle D_{e_1} N_{z_0}, e_1\rangle}{k^{K_0}(\pi(N_{z_0}))} = \frac{k^{\partial\Omega}(z_0)}{k^{K_0}(\pi(N_{z_0}))},$$

where k^{K_0} and $k^{\partial\Omega}$ are the the Euclidean geodesic curvature of ∂K and $\partial\Omega$. Since k^{K_0} is strictly positive we obtain $k^{\partial\Omega}(z_0) > 0$, hence Ω is strictly convex.

To conclude this section, inspired by [158] we want to show that, in the particular case in which H is constant, then we can exploit (7.3.14) in order to obtain uniform estimates of the function, without requiring the validity of (7.3.4). Again, in order to apply the results of § 7.1.1.1 and § 7.1.1.2, we assume that K_0 is a convex body in C^∞_+ such that $0 \in \mathrm{int}\, K_0$ and $\partial\Omega$ belongs to $C^{2,1}$.

We denote by $B_K(p,r)$ the ball of radius $r > 0$ in $\mathbb{R}^d$ associated to the distance $d_{K,p}$ and R the Ridge of Ω as defined in §7.1.1.1 and §7.1.1.2 respectively.

Proposition 7.3.8. *Let K_0 be a convex body in C^∞_+ with $0 \in \mathrm{int}\, K_0$. Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with $C^{2,1}$ boundary, let $\varphi \in C^2(\overline{\Omega})$ and* *let H be a constant which satisfies* (7.3.14)*. There exists a constant $C_1 = C_1(n, K_0, \Omega, \varphi, H, F) > 0$, independent of $\sigma \in [0,1]$ and $\varepsilon \in (0,1)$, such that, for any solution $u \in C^2(\overline{\Omega})$ to* (7.3.2)*, it holds that*

$$\|u\|_{\infty,\Omega} \leqslant C_1.$$

Proof. Let us define the function $v : \operatorname{int}\Omega_1 \mathbb{R}$ by

$$v(z) := \sup_{\partial\Omega} |\varphi| + k d_{K_0,\partial\Omega}(z)$$

for any $z \in \Omega_1$, where $k > 0$ has to be chosen and Ω_1 is the set defined in (7.1.11). We already know (cf. (7.1.12)) that $v \in C^2(\operatorname{int}\Omega_1)$. We repeat the computations of the proof of Proposition 7.3.6, avoiding (7.3.19) thanks to the fact that H is constant, to find $k > 0$, independent of $\varepsilon \in (0,1)$, $\sigma \in [0,1]$ and $z \in \Omega_1$, such that v is a subsolution to (7.3.2) on $\operatorname{int}\Omega_1$. Therefore, arguing as in the proof of [85, Theorem 10.7], it follows that $w := u - v$ is a weak supersolution on $\operatorname{int}\Omega_1$ to a linear elliptic equation of the form

$$\sum_{i,j=1}^{2n} D_i(a_{i,j}(z) D_j w(z)) + \sum_{i=1}^{2n} c_i(z) D_i w(z) = 0.$$

Hence, thanks to [85, Theorem 8.1] and recalling (7.1.13), it follows that

$$\sup_{\Omega_1}(u - v) \leqslant \sup_{\partial\Omega\cup R}((u - v)^+).$$

Noticing that $u - v \leqslant 0$ on $\partial\Omega$ and that $\overline{\operatorname{int}\Omega_1} = \overline{\Omega}$, we obtain that

$$u(z) - v(z) \leqslant \sup_{\Omega}(u - v) = \sup_{\Omega_1}(u - v) \leqslant \sup_{\partial\Omega_1}((u - v)^+) = \sup_{R}((u - v)^+)$$

for any $z \in \Omega$. We are left to show that $\sup_R((u-v)^+) \leqslant 0$. Indeed, assume by contradiction that $\sup_R((u - v)^+) > 0$. Since R is compact, there exists $z_0 \in R$ such that

$$u(z_0) - v(z_0) = \sup_{R}((u - v)^+) = \sup_{R}(u - v).$$

Moreover, z_0 is a maximum point for $u - v$ on $\overline{\Omega}$. Let us fix $y_0 \in \partial\Omega$ such that $d_{K_0,\partial\Omega}(z_0) = |z_0 - y_0|_{K_0}$. Then, thanks to Proposition 7.1.5, it is easy to see that

$$d_{K_0,\partial\Omega}(z) = |z - y_0|_{K_0} \tag{7.3.20}$$

for any z belonging to (y_0, z_0), the segment connecting y_0 and z_0. Let now $\nu := \frac{y_0 - z_0}{|y_0 - z_0|}$. By (7.3.20) it holds that $v(z) < v(z_0)$ for any $z \in (y_0, z_0)$, and moreover

$$D_\nu^+ v(z_0) := \lim_{h\to 0^+} \frac{v(z_0 + h\nu) - v(z_0)}{h} < 0. \tag{7.3.21}$$

Since z_0 is a maximum point of $u - v$, it holds in particular that $D_\nu^+ u(z_0) \leqslant D_\nu^+ v(z_0)$, which implies, together with (7.3.21), that $D_\nu^+ u(z_0) = D_\nu u(z_0) < 0$. This proves that $Du(z_0) \neq 0$. Since then z_0 is a regular point for u, the level set $\{z \in \Omega : u(z) = u(z_0)\}$ is locally a C^2 hypersurface. Therefore there exists a small Euclidean ball B_{eu} such that B_{eu} is tangent to the level set at z_0 and moreover $B_{eu} \subseteq \{z \in \Omega : u(z) \geqslant u(z_0)\}$. Now, since by our assumptions the Finsler balls relative to $-K_0$ are uniformly convex and C^2, there exists $\eta > 0$ and $x_0 \in \Omega$ such that

$$\overline{B_{-K_0}(x_0, \eta)} \subseteq \{z \in \Omega : u(z) \geqslant u(z_0)\} \tag{7.3.22}$$

and $B_{-K_0}(x_0, \eta)$ is tangent to B_{eu} at z_0. Indeed, fix a Finsler ball tangent to B_{eu} at z_0 relative to $-K_0$, say B_F. On one hand, the principal curvatures of ∂B_{eu} at z_0 are fixed. On the other hand, noticing that the principal curvatures of a C^2_+ convex set admit a positive lower bound, we can dilate and translate B_F to make the curvature of B_F as big as we want to ensure that (7.3.22) holds. Notice that

$$d_{K_0,\partial\Omega}(z) \geqslant d_{K_0,\partial\Omega}(z_0) \tag{7.3.23}$$

for any $z \in \overline{B_{-K_0}(x_0, \eta)}$. Indeed, if by contradiction there exists $z \in \overline{B_{-K_0}(x_0, \eta)}$ such that $d_{K_0,\partial\Omega}(z) < d_{K_0,\partial\Omega}(z_0)$, then (7.3.22) would imply

$$u(z) - k d_{K_0,\partial\Omega}(z) \geqslant u(z_0) - k d_{K_0,\partial\Omega}(z) > u(z_0) - k d_{K_0,\partial\Omega}(z_0),$$

a contradiction with the maximality of z_0. Let now $w_0 \in \partial\Omega$ be such that $d_{K,\partial\Omega}(x_0) = |x_0 - w_0|_{K_0}$, and let b_0 be the unique point of intersection between $\partial B_{-K_0}(x_0, \eta)$ and the segment joining w_0 and x_0. Then by (7.1.1), (7.3.20), (7.3.23), the choice of b_0 and the strict convexity of K_0, it holds that

$$\begin{aligned} d_{K_0,\partial\Omega}(x_0) = |x_0 - w_0|_{K_0} &= |x_0 - b_0|_{K_0} + |b_0 - w_0|_{K_0} \\ &= \eta + d_{K_0,\partial\Omega}(b_0) \geqslant \eta + d_{K_0,\partial\Omega}(z_0). \end{aligned}$$

On the other hand, (7.1.1) and the triangle inequality imply

$$d_{K_0,\partial\Omega}(x_0) \leqslant |x_0 - y_0|_{K_0} \leqslant |x_0 - z_0|_{K_0} + |z_0 - y_0|_{K_0} = \eta + d_{K_0,\partial\Omega}(z_0).$$

Putting together the previous inequalities we get that

$$d_{K_0,\partial\Omega}(x_0) = |x_0 - y_0|_{K_0} = |x_0 - z_0|_{K_0} + |z_0 - y_0|_{K_0}, \tag{7.3.24}$$

from which in particular we conclude, exploiting again the strict convexity of K_0, that x_0 lies on (y_0, x_0). Therefore, thanks to this fact, the first equality in (7.3.24) and Proposition 7.1.5, we conclude that $z_0 \in \operatorname{int} \Omega_1$, which is a contradiction. In the end we proved that

$$\sup_\Omega u \leqslant \sup_{\partial\Omega} |\varphi| + k \max_{\overline{\Omega}} d_{K_0,\partial\Omega}.$$

Since the converse estimate can be obtained in a similar way, the thesis is proved. □

7.4 Existence of Lipschitz minimizer for the sub-Finsler functional I

Thanks to the *a priori* estimates of the previous section, together with Proposition 7.3.1 and the uniformity of the estimates with respect to $\varepsilon \in (0,1)$, we are in position both to solve the Finsler Prescribed Mean Curvature Equation and to pass to the limit and find a solution to the Sub-Finsler Prescribed Mean Curvature equation.

Theorem 7.4.1. *Let $K_0 \in C^\infty_+$ be a convex body such that $0 \in \operatorname{int} K_0$. Let $\Omega \subset \mathbb{R}^{2n}$ be a bounded domain with $C^{2,1}$ boundary. Let $\varphi \in C^{2,\alpha}(\overline{\Omega})$, for $0 < \alpha < 1$, and let $F \in C^{1,\alpha}(\overline{\Omega})$ be such that* (7.3.10) *is satisfied. Assume that H is a constant such that* (7.3.14) *holds. Then, for any $\varepsilon \in (0,1)$, the exists a function $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ which solves* (7.3.1)*. Moreover, there exists a constant $M > 0$, independent of $\varepsilon \in (0,1)$, such that any solution u_ε to* (7.3.1) *satisfies*

$$\sup_\Omega |u_\varepsilon| + \sup_\Omega |\nabla u_\varepsilon| \leqslant M. \tag{7.4.1}$$

Finally, there exists a Lipschitz continuous minimizer $u_0 \in Lip(\overline{\Omega})$ for $\mathcal{I}$ with $u_0 = \varphi$ on $\partial\Omega$.

Proof. Let $0 < \varepsilon < 1$. By Proposition 7.3.8, Proposition 7.3.5 and Proposition 7.3.6, there exists a constant $M > 0$ such that, for any $\sigma \in [0,1]$, any solution $u \in C^{2,\alpha}(\overline{\Omega})$ to the problem (7.3.2) satisfies

$$\sup_\Omega |u| + \sup_\Omega |\nabla u| \leqslant M.$$

Then by Proposition 7.3.1 there exists a solution $u_\varepsilon \in C^{2,\alpha}(\bar{\Omega})$ to

$$\begin{cases} \operatorname{div}(\pi^h_\varepsilon(\nabla u + F)) = H & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega. \end{cases}$$

Again by Proposition 7.3.8, Proposition 7.3.5 and Proposition 7.3.6, we have that

$$\sup_\Omega |u_\varepsilon| + \sup_\Omega |\nabla u_\varepsilon| \leqslant M, \tag{7.4.2}$$

where the constant $M > 0$ is uniform in $0 < \varepsilon < 1$. Let $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,1)$ be a sequence such that $\varepsilon_j \to 0$ as $j \to \infty$. Since M is uniform in ε by (7.4.2) we gain that $\sup_\Omega |u_{\varepsilon_j}| \leqslant M$ and that for any $x, y \in \Omega$

$$|u_{\varepsilon_j}(x) - u_{\varepsilon_j}(y)| \leqslant M|x-y|. \tag{7.4.3}$$

Then, by Ascoli-Arzelà theorem there exists $u_0 \in C(\overline{\Omega})$ such that $u_{\varepsilon_j} \to u_0$ uniformly in $\overline{\Omega}$. It is clear that $u = \varphi$ on $\partial\Omega$. Moreover, taking the limit as $j \to 0$ in (7.4.3), we gain that

$$\sup_{x\neq y} \frac{|u_0(x) - u_0(y)|}{|x-y|} \leqslant M,$$

thus u_0 is Lipschitz. We claim that u_0 is a minimizer for $\mathcal{I}$ defined in (7.1.16). Indeed, we have that $\|u_{\varepsilon_j}\|_{W^{1,1}(\Omega)} \leqslant M|\Omega|$, $\|u_0\|_{W^{1,1}(\Omega)} \leqslant M|\Omega|$ and u_{ε_j} converge to u_0 in $L^1(\Omega)$. Moreover, the function $(p,(x,y)) \to |p + F(x,y)|_*$ is positive, continuous and convex in p. Therefore, by [132, Theorem 4.1.2] $\mathcal{I}$, is lower semicontinuous with respect to the strong L^1-topology, from which we have that

$$\mathcal{I}(u_0) \leqslant \liminf_{j\to\infty} \mathcal{I}(u_{\varepsilon_j}). \tag{7.4.4}$$

For each $v \in W^{1,1}(\Omega)$ such that $v - \varphi \in W_0^{1,1}(\Omega)$, it follows that

$$\begin{aligned}
\mathcal{I}(u_{\varepsilon_j}) &= \int_\Omega |\nabla u_{\varepsilon_j} + F|_* \, dz + \int_\Omega H u_{\varepsilon_j} \, dz \\
&\leqslant \int_\Omega (\varepsilon_j^3 + |\nabla u_{\varepsilon_j} + F|_*^3)^{\frac{1}{3}} \, dz + \int_\Omega H u_{\varepsilon_j} \, dz \\
&\leqslant \int_\Omega (\varepsilon_j^3 + |\nabla v + F|_*^3)^{\frac{1}{3}} \, dz + \int_\Omega H v \, dz \\
&\leqslant \varepsilon_j |\Omega| + \int_\Omega |\nabla v + F|_* \, dz + \int_\Omega H v \, dz,
\end{aligned} \tag{7.4.5}$$

where we have used the fact that the Dirichlet solution $u_{\varepsilon_j} \in C^{2\,\alpha}(\bar{\Omega})$ is a minimizer for the functional $v \to \int_\Omega (\varepsilon_j^3 + |\nabla v + F|_*^3)^{\frac{1}{3}} + \int_\Omega H v$ for each $v \in W^{1,1}(\Omega)$ s.t. $v - \varphi \in W_0^{1,1}(\Omega)$. Passing to the liminf in (7.4.5) and taking into account (7.4.4), we obtain $\mathcal{I}(u_0) \leqslant \mathcal{I}(v)$ for each $v \in W^{1,1}(\Omega)$ s.t. $v - \varphi \in W_0^{1,1}(\Omega)$. □

Remark 7.4.2. A deeper look to [113; 112] suggests that it should be possible to prove that the aforementioned results still hold only assuming that K_0 is a convex body in $C_+^{2,\alpha}$ with $0 \in \operatorname{int} K_0$, for some $0 < \alpha < 1$. Accordingly, it is reasonable that in Theorem 7.4.1 the regularity of ∂K_0 can be weakened to $C^{2,\alpha}$, for some $0 < \alpha < 1$.

Appendix A

Alternative proof of Theorem 2.3.3

Given a measure μ, we say that a family of measurable functions H is *PCU-stable* if for every family $\{\alpha_1, \dots, \alpha_n\}$ of $C^1(\mathbb{R}^d)$ that forms a partition of unity at all p in $\mathbb{R}^d$, and every subset $\{v_1, \dots, v_n\}$ in H, then $\sum_i \alpha_i v_i$ is an element of H.

A normal convex integrad is a measurable function $j : \Omega \times \mathbb{R}^k \to \mathbb{R} \cup \{+\infty, -\infty\}$ such that for every p in Ω, $j(p, \cdot)$ is convex and lower semi-continuous.

The family

$$H = \{g \in C^1_c(V; \mathbb{R}^k) : |g|_{K,+\infty} \leqslant 1\}$$

is PCU-stable, since

$$|\sum_{i \in I} \alpha_i(p) g_i(p)| = \sum_{i \in I} \alpha_i(p) |g_i(p)| \leqslant 1.$$

We consider the normal convex functional

$$j(p, z) = -\langle z, \nu_h(p) \rangle.$$

Given p in $\mathbb{R}^d$, let us prove that

$$\inf_{g \in -H} j(p, g(p)) = -|\nu_h(p)|_{-K,*}.$$

It is clear from the definition of the norm that $|g|_K = |-g|_{-K}$. Thus, for $g \in -H$ we have

$$-\langle g, \nu_h \rangle \geqslant -|\nu_h|_{-K,*}.$$

On the other hand, let $\varphi \in C^1_c(\mathbb{R}^k)$ such that $\varphi(p) = 1$ and $0 \leqslant \varphi(q) \leqslant 1$ for every $q \in \mathbb{R}^d$. Thereore the function $\varphi \pi_{-K}(\nu_h)$ is in $C^1_c(\mathbb{R}^d; \mathbb{R}^k)$ and $|\varphi \pi_{-K}(\nu_h)|_{-K,+\infty} \leqslant 1$. Moreover,

$$\langle \varphi(p) \pi_{-K}(\nu_h(p)), \nu_h(p) \rangle = |\nu_h(p)|_{-K,*}.$$

Taking the infimum in H, we get

$$\inf_{g\in H} g(p) \leqslant |\nu_h(p)|_*.$$

Therefore, by Theorem 1 in [20], we get

$$\begin{aligned}\sup_{v\in H}\int_{\mathbb{R}^d}\langle U,\nu_h\rangle d|\partial E| &= \inf_{v\in -H}\int_{\mathbb{R}^d} -\langle U,\nu_h\rangle d|\partial E|\\ &= \int_{\mathbb{R}^d} -|\nu_h|_{-K,*} d|\partial E| = \int_{\mathbb{R}^d} |\nu_h|_{K,*} d|\partial E|.\end{aligned}$$

By (2.3.3) and the definition of perimeter, we get (2.3.5).

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