Turkish Journal of Mathematics

Volume 46 | Number 8 Article 13

1-1-2022

Translating solitons of translation and homothetical types

MUHİTTİN EVREN AYDIN

RAFAEL LOPEZ

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 3245 – 3259
© TÜBİTAK
doi:10.55730/1300-0098.3331

Translating solitons of translation and homothetical types

Muhittin Evren AYDIN[1,*], **Rafael LÓPEZ**[2]
[1]Department of Mathematics, Faculty of Science, Fırat University, Elazığ, Turkey
[2]Department of Geometry and Topology, University of Granada, Granada, Spain

Received: 22.12.2021 • **Accepted/Published Online:** 20.09.2022 • **Final Version:** 09.11.2022

Abstract: We prove that if a translating soliton can be expressed as the sum of two curves and one of these curves is planar, then the other curve is also planar and consequently the surface must be a plane or a grim reaper. We also investigate translating solitons that can be locally written as the product of two functions of one variable. We extend the results in Lorentz-Minkowski space.

Key words: Mean curvature flow, translating soliton, surfaces of translation, homothetical surface

1. Introduction

Let $\vec{v}\in\mathbb{R}^3$ be nonzero vector. A *translating soliton* in Euclidean 3-dimensional space $\mathbb{R}^3$ with respect to $\vec{v}$, called the *velocity* of the flow, is a surface M whose mean curvature H satisfies

$$H(p)=\langle N(p),\vec{v}\rangle, \tag{1.1}$$

for all $p\in M$, where N is the unit normal vector field on M. Translating solitons appear in the theory of the mean curvature flow of Huisken and Ilmanen as the solutions of the flow when M evolves purely by translations along the direction $\vec{v}$ ([9, 10]). In particular, $M+t\vec{v}$, $t\in\mathbb{R}$, satisfies that fixed t, the normal component of the velocity vector $\vec{v}$ at each point is equal to the mean curvature at that point. In nonparametric way $z=u(x,y)$, Equation (1.1) is

$$(1+u_y^2)u_{xx}-2u_xu_yu_{xy}+(1+u_x^2)u_{yy}=2(1+u_x^2+u_y^2)(-v_1u_x-v_2u_y+v_3), \tag{1.2}$$

where the subindices indicate the corresponding partial differentiation and $\vec{v}=(v_1,v_2,v_3)$. This equation is a quasilinear elliptic PDE, so the solvability is not assured. Some results of the solvability of the Dirichlet problem can be proved by assuming convexity in the initial data ([14]). A way to reduce the complexity of (1.2) is assuming some type of symmetry on the surface which makes that (1.2) converts into an ordinary differential equation, where classical theory ensures the local existence of solutions. Following this strategy, we can assume that the surface is invariant under a uniparametric group of translations (cylindrical surfaces) or rotations (surfaces of revolution). Both families of surfaces are classified and play a remarkable role in the theory of translating solitons. We now describe both examples. Let (x,y,z) be the canonical coordinates of $\mathbb{R}^3$.

*Correspondence: meaydin@firat.edu.tr
2010 *AMS Mathematics Subject Classification:* 53A10, 53C21, 53C42

1. Cylindrical surfaces. The translating solitons are planes parallel to the velocity vector $\vec{v}$ if the rulings are parallel to $\vec{v}$ or grim reapers otherwise. See a detailed discussion in [14]. For grim reapers, we can assume without loss of generality that $\vec{v}=(0,0,1)$. Let w be the direction of the rulings. After a rotation about the z-axis, let $w=\cos\theta e_1+\sin\theta e_3$, where $\{e_1,e_2,e_3\}$ is the canonical basis of $\mathbb{R}^3$, $e_3=\vec{v}$ and $\cos\theta\neq 0$. The generating curve is included in the plane spanned by $\{e_2,e\}$, where $e=-\sin\theta e_1+\cos\theta e_3$. If we write this curve as $\beta(s)=se_2+u(s)e$, then u satisfies $u''=\cos\theta(1+u'^2)$. This equation can be completely integrated. For example, if $\theta=0$, $u(s)=-\log(\cos(s+a))+b$, $a,b\in\mathbb{R}$. We point out that if a translating soliton is a ruled surface, then it must be cylindrical, hence a plane or a grim reaper ([8]).
2. Surfaces of revolution. The rotation axis is not arbitrary and must be parallel to the velocity vector $\vec{v}$. There are two types of rotational translating solitons depending on whether or not the surface meets the rotation axis ([1, 4]). In the first case, the surface is known in the literature as the bowl soliton and in the second one, the surfaces have a winglike shape.

Another way to address Equation (1.2) is by the technique of separation of variables. We have two possibilities, $u(x,y)=f(x)+g(y)$ and $u(x,y)=f(x)g(y)$, where f and g are smooth functions of one variable. In both cases, Equation (1.2) is an ODE where the unknowns are the functions $f=f(x)$ and $g=g(y)$. If $z=f(x)+g(y)$, the translating soliton equation (1.2) is now

$$(1+g'^2)f''+(1+f'^2)g''=(1+f'^2+g'^2)(-v_1f'-v_2g'+v_3), \tag{1.3}$$

where $'$ indicates the derivative with respect to the corresponding variable. In [13], the second author proved that if $\vec{v}=(0,0,1)$, grim reapers are the only solutions of (1.3). Let us observe that the planes parallel to $\vec{v}$ are not graphs on the xy-plane. A surface that is the graph of $z=f(x)+g(y)$ can be expressed as the sum of two planar curves $\alpha(x)+\beta(y)$, where $\alpha(x)=(x,0,f(x))$ and $\beta(y)=(0,y,g(y))$. Let us observe that both curves are contained in orthogonal planes. More generally, a surface is said to be a *translation surface* if it is the sum of two curves called *generating curves*. The name of the translation surface is due to the fact that surface can be viewed from the kinematic viewpoint as the translation of the curve α (of β) by means of translations through β (or α, respectively). Thus the result in [13] is only a partial answer to the following.

Problem 1. Classify all translating solitons that are translation surfaces.

This problem has its analogy in the classical theory of minimal surfaces of $\mathbb{R}^3$. Scherk proved that besides the planes, the only minimal surface that can be expressed as $z=f(x)+g(y)$ is

$$u(x,y)=\frac{1}{c}\log\left|\frac{\cos(cy)}{\cos(cx)}\right|,$$

where $c\neq 0$ ([17]). More recently, Dillen *et al.* proved that if one of the generating curves of a minimal surface of translation type is planar, then the other generating curve is also planar ([5]) and the surface belongs to a family of minimal surfaces discovered by Scherk ([16]). Surprisingly, very recently the second author together Hasanis and Perdomo discovered many minimal surfaces of translation type where both generating curves are not planar ([7, 15]).

In this paper, we follow the same approach for translating solitons. However, the presence of the vector $\vec{v}$ in Equation (1.1) makes a great difference because $\vec{v}$ is an arbitrary vector in relation with the spatial coordinates (x, y, z) of $\mathbb{R}^3$. We give a partial answer to Problem 1 assuming that one of the generating curves is planar and proving that the surface is a plane or a grim reaper (Theorem 2.2). As a previous step, we prove this result in case the two generating curves are planar curves but not necessarily contained in orthogonal planes (Theorem 2.1). Both results are analogous to the minimal surfaces obtained in [5]. The goal of both theorems is that we do not presuppose any relation between the velocity vector $\vec{v}$ and the surface. More precisely, the notion of translation surface is affine because we use the sum of vectors of $\mathbb{R}^3$. However, the velocity vector $\vec{v}$ in the translating soliton equation (1.1) is assumed in all its generality without any relation with the coordinates of $\mathbb{R}^3$. This should be pointed out because one may be tempted to fix $\vec{v}$ since Equation (1.1) is invariant after a rigid motion. However, such rigid motion also changes the spatial coordinates of $\mathbb{R}^3$. This seems to be subtle, but if one assumes that the surface is $z = f(x)+g(y)$, then the vector $\vec{v}$ must be arbitrary. All this complicates the demonstrations, which are not straightforward.

The second case of separation of variables that we investigate for the translating soliton equation is $z = f(x)g(y)$. Then (1.2) is

$$(1 + f^2g'^2)gf'' - 2fgf'^2g'^2 + (1 + g^2f'^2)fg'' - 2(1 + f'^2g^2 + f^2g'^2)(-v_1f'g - v_2fg' + v_3) = 0. \tag{1.4}$$

Let us observe the symmetry of (1.4) in terms of f or g, hence any discussion on one of both functions also holds for the other one. As far as the authors know, the first approach to this kind of surface in relation to the study of the curvature of surfaces appeared in [18, 19], where the authors coined this type of surface as *homothetical surfaces* (see also [6, 12]). We have the analogous question.

Problem 2. Classify all translating solitons of homothetical type.

It was proved in [13], and in the particular case $\vec{v} = (0, 0, 1)$, that the only homothetical translating solitons are grim reapers. Grim reapers appear when one of the functions f or g are constant. Indeed, if say $f(x) = a$, $a \in \mathbb{R}$, then the parametrization of the surface is $X(x, y) = (x, y, ag(y))$ deducing that the surface is cylindrical and the rulings are parallel to the vector e_1 of the canonical basis. In contrast to Equation (1.3), now Equation (1.4) is more difficult to work. The result that we prove is assuming that $\vec{v}$ is one of the canonical directions of $\mathbb{R}^3$ and proving that the surface is a plane or a grim reaper (Theorem 3.1). Again we can make the same observation as before and although this seems elementary analysis and would yield no nontrivial solutions besides cylindrical surfaces, one can expect the existence of new examples. For instance, in the family of minimal surfaces, the plane and the helicoid (which is not cylindrical but ruled) are the only homothetical surfaces ([18]). But if one replaces $z = f(x)g(y)$ by $h(z) = f(x)g(y)$, then there are many minimal surfaces ([16]).

Finally, in Section 4 we extend all the above results for translating solitons in Lorentz-Minkowski space $\mathbb{R}^3_1$. Since the underlying affine space for $\mathbb{R}^3_1$ coincides with the Euclidean space, the concepts of translation surfaces and homothetical surfaces are equally valid in the Lorentzian setting. The results are analogous to that of Euclidean space.

2. Translating solitons of translation type

Consider a translation surface where the generating curves are planar curves. If the planes containing the generating curves are orthogonal and $\vec{v}$ is parallel to both planes, the second author proved that the only translating solitons are grim reapers whose rulings are parallel to one of the above planes ([14]). We now investigate Problem 1 in case that $\vec{v}$ is arbitrary and the generating curves are planar but not necessarily contained in orthogonal planes.

Theorem 2.1 *Planes and grim reapers are the only translating solitons that are the sum of two planar curves.*

Proof If the planes containing the curves are parallel then the sum of the two curves is (part of) a plane. Suppose now that both planes are not parallel. After renaming coordinates, we will assume that the z-axis is the common straight line of the two planes, one of the generating curves is included in the plane of equation $x = 0$ and the other in the plane $cx + y = 0$, $c \in \mathbb{R}$. The cosine of angle between the two planes is $c/\sqrt{1+c^2}$ and if $c = 0$ then both planes become perpendicular. The first curve parametrizes as $\beta(y) = (0, y, g(y))$ and the second one by $\alpha(x) = (x, -cx, f(x))$, where f and g are two smooth functions defined in intervals I and J, respectively. Thus a parametrization of the surface is

$$X(x, y) = \alpha(x) + \beta(y) = (x, y - cx, f(x) + g(y)).$$

Notice that if we name $\tilde{y} = y - cx$, then the surface is $z = f(x) + g(\tilde{y} + cx)$. These surfaces are known in the literature as affine translation surfaces ([11]).

In case that f or g is a linear function, then the surface is cylindrical and the surface must be a plane or a grim reaper, proving the result. Now we discard this case. Then there are $x_0 \in I$ and $y_0 \in J$ such that $f''(x_0) \neq 0$ and $g''(y_0) \neq 0$. Then $f'' \neq 0$ and $g'' \neq 0$ in some subintervals around x_0 and y_0 respectively, which we can assume to be I and J. In both intervals, there are points where $f' \neq 0$ and $g' \neq 0$, otherwise f or g would be constant functions. Abusing of notation, suppose $f'f''(x_0) \neq 0$ and $g'g''(y_0) \neq 0$ and analogously, $f'f'' \neq 0$ in I and $g'g'' \neq 0$ in J. If $\vec{v} = (v_1, v_2, v_3)$, Equation (1.2) is writen as

$$\left(1 + g'^2\right) f'' + \left(1 + c^2 + f'^2\right) g'' = 2\left(-v_1\left(f' + cg'\right) - v_2 g' + v_3\right)\left(1 + g'^2 + \left(f' + cg'\right)^2\right).$$

Divided by $(1 + g'^2)(1 + c^2 + f'^2)$,

$$\frac{f''}{1 + c^2 + f'^2} + \frac{g''}{1 + g'^2} = 2\left(-v_1\left(f' + cg'\right) - v_2 g' + v_3\right)\frac{1 + g'^2 + \left(f' + cg'\right)^2}{(1 + g'^2)(1 + c^2 + f'^2)}.$$

Because the left-hand side is the sum of a function on the variable x and a function on the variable y, when we differentiate with respect to x and next with respect to y, these terms are zero. The corresponding differentiations on the right-hand side give the expression

$$\left(\sum_{n=0}^{4} P_n(y) {f'}^n\right) \frac{f'' g''}{(1 + g'^2)^2(1 + c^2 + f'^2)^2} = 0,$$

where P_n are functions on the variable y. Thus $\sum_{n=0}^{4} P_n(y) f'(x)^n = 0$ in $I \times J$. Since this is a polynomial of the function $f' = f'(x)$, all coefficients P_n must vanish in J. The computation of P_4 yields $P_4 = -v_1 g'$, deducing $v_1 = 0$ because $g' \neq 0$. Taking into account that $v_1 = 0$, the computation of P_2 gives

$$P_2 = c(-v_3 {g'}^2 - 2v_2 g' + v_3).$$

We discuss two cases:

1. Case $c=0$. Then all P_n are trivially 0 except P_1, which is $P_1=-g'(v_2g'^3+3v_2g'-2v_3)$. From $P_1=0$ and because $g'\neq 0$, we have $v_2g'^3+3v_2g'-2v_3=0$. Since $g''\neq 0$, the functions $\{1,g',g'^3\}$ are linearly independent, concluding $v_2=v_3=0$, so $\vec{v}=0$ obtaining a contradiction.

2. Case $c\neq 0$. Then $-v_3g'^2-2v_2g'+v_3=0$. Thus $v_2=v_3=0$ again, which is contradictory.

□

We point out that in [20] the authors obtained a partial result of Theorem 2.1 in case that $\vec{v}$ is one vector of the canonical basis.

Our next progress in Problem 1 is considering that one of the generating curves is nonplanar.

Theorem 2.2 *Planes and grim reapers are the only translating solitons that are the sum of two curves and where one of the generating curves is planar.*

Proof Suppose that the surface is parametrized by $X(s,t)=\alpha(s)+\beta(t)$, where β is a planar curve. Without loss of generality, we assume that β is contained in the plane Π of equation $x=0$ and that β parametrizes as $\beta(y)=(0,y,g(y))$, where g is a smooth function defined in an interval J. The proof of theorem is by contradiction so by Theorem 2.1, we suppose that the curve α is not planar. Since α is a space curve, then α is a graph on one of the coordinates axes. We can assume that this axis is the x-axis because otherwise, the curve α would be contained in a plane parallel to Π and the sum of α and β would be (part of) a plane. Definitively, α can be expressed as $\alpha(x)=(x,f(x),h(x))$, where f and h are two smooth functions defined in an interval $I\subset\mathbb{R}$. If we parametrize the surface by $X(x,y)=(x,y+f(x),h(x)+g(y))$, the unit normal vector field is

$$N=\frac{1}{\sqrt{1+g'^2+(f'g'-h')^2}}(f'g'-h',-g',1)$$

and the mean curvature H is

$$H=\frac{(h''-f''g')\left(1+g'^2\right)+\left(1+f'^2+h'^2\right)g''}{2\left(1+g'^2+(f'g'-h')^2\right)^{3/2}}.$$

Let $\vec{v}=(v_1,v_2,v_3)$. The translating soliton equation (1.2) is

$$(h''-f''g')\left(1+g'^2\right)+\left(1+f'^2+h'^2\right)g''=2\left(v_1(f'g'-h')-v_2g'+v_3\right)\left(1+g'^2+(f'g'-h')^2\right). \tag{2.1}$$

If f or h are linear functions, then the generating curve α is planar, which is not possible. Therefore, with a similar argument as in the beginning of the proof of Theorem 2.1, we can assume that in some subintervals of I and J, we have $f'f''h'h''\neq 0$ and $g'g''\neq 0$. Without loss of generality we will assume that these subintervals are I and J again. Our arguments will use the next two claims.

Claim 1. If there are $a,b,c\in\mathbb{R}$ such that $a+bf'(x)^2+ch'(x)^2=0$ for all $x\in I$, then either $abc\neq 0$ or $a=b=c=0$. In the first case, we conclude that $h'(x)=\pm\sqrt{m_0+m_1f'(x)^2}$, where $m_0,m_1\neq 0$, $m_0,m_1\in\mathbb{R}$.

The proof of the claim is as follows. According to the value of the constant b, we have two cases. If $b=0$, then $a+ch'^2=0$. In case that $c=0$, then $a=0$ and the claim is proved. If $c\neq 0$, we deduce that $h'h''=0$, which is not possible. The other case is $b\neq 0$. With a similar argument, we deduce $c\neq 0$.

If $a = 0$, then $bf'(x)^2 + ch'(x)^2 = 0$ for all $x \in I$, in particular, $bc < 0$. Then $h'(x) = \pm\sqrt{-b/c}f'(x)$ so $h(x) = \pm\sqrt{-b/c}f(x) + m$, $m \in \mathbb{R}$. Thus $\alpha(x) = (x, f(x), \pm\sqrt{-b/c}f(x) + m)$ concluding that α is planar, which is contradictory. Thus $a \neq 0$. Hence $h'^2 = -a/c - b/cf'^2$ and the result follows by taking $m_0 = -a/c$ and $m_1 = -b/c$.

Claim 2. Suppose $h' = \pm\sqrt{m_0 + m_1 f'^2}$, where $m_0, m_1 \neq 0$. Then the functions $\{1, f'^2, f'h'\}$ are linearly independent.

The proof is the following. Since $f'' \neq 0$, let us introduce $s = f'$. Then the Wronskian of the set $\{1, s^2, \pm s\sqrt{m_0 + m_1 s^2}\}$ is $\mp\frac{2m_0^2}{(m_0+m_1s^2)^{3/2}}$, and this proves the claim.

We come back to the proof of the theorem. Dividing (2.1) by $Q = 1 + f'^2 + h'^2$ and differentiating with respect to x, we obtain a polynomial equation on g'

$$\sum_{n=0}^{3} P_n(x) g'^n = 0,$$

where

$$\begin{aligned}
P_0(x) &= \left(\frac{h'' - 2(v_3 - v_1 h')(1 + h'^2)}{Q}\right)' \\
P_1(x) &= \left(\frac{-f'' - 2f'\left(3v_1 h'^2 - 2v_3 h' + v_1\right) + 2v_2 h'^2 + 2v_2}{Q}\right)' \\
P_2(x) &= \left(\frac{h'' - 2f'^2\left(v_3 - 3v_1 h'\right) - 4v_2 f'h' + 2v_1 h' - 2v_3}{Q}\right)' \\
P_3(x) &= \left(\frac{-f'' + 2(v_2 - v_1 f')(1 + f'^2)}{Q}\right)'.
\end{aligned}$$

Thus there are real constants $p_n \in \mathbb{R}$, $0 \leq n \leq 3$, such that

$$\begin{aligned}
h'' - 2(v_3 - v_1 h')(1 + h'^2) &= p_0 Q \\
-f'' - 2f'\left(3v_1 h'^2 - 2v_3 h' + v_1\right) + 2v_2 h'^2 + 2v_2 &= p_1 Q \\
h'' - 2f'^2\left(v_3 - 3v_1 h'\right) - 4v_2 f'h' + 2v_1 h' - 2v_3 &= p_2 Q \\
-f'' + 2(v_2 - v_1 f')(1 + f'^2) &= p_3 Q.
\end{aligned} \tag{2.2}$$

In order to simplify the notation, set

$$c_1 = p_0 - p_2, \quad c_2 = p_1 - p_3.$$

We distinguish two cases.

1. Case $v_3 = 0$. There are two subcases.

(a) Subcase $v_1 = 0$. After a dilation of $\mathbb{R}^3$, we can assume $\vec{v} = (0,1,0)$. Equations (2.2) are

$$\begin{aligned} h'' &= p_0 Q, &(2.3)\\ -f'' + 2(1+h'^2) &= p_1 Q, &(2.4)\\ h'' - 4f'h' &= p_2 Q, &(2.5)\\ -f'' + 2(1+f'^2) &= p_3 Q. &(2.6) \end{aligned}$$

Combining (2.4) and (2.6) and using the value of Q, we have

$$c_1 + (c_1+2)f'^2 + (c_1-2)h'^2 = 0.$$

Because the three coefficients are distinct from 0, Claim 1 implies that $h' = \pm\sqrt{m_0 + m_1 f'^2}$, where $m_0, m_1 \neq 0$. Using now (2.3) and (2.5), $4f'h' = c_1 Q$, or equivalently,

$$c_2 + m_0 c_2 + (c_2 + c_2 m_1) f'^2 \mp 4f'h' = 0.$$

From Claim 2, the functions $\{1, f'^2, f'h'\}$ are linearly independent, hence the coefficients must vanish, obtaining a contradiction.

(b) Subcase $v_1 \neq 0$. Since $\vec{v} = (v_1, v_2, 0)$, after a dilation of $\mathbb{R}^3$, we can assume that $\vec{v} = (1, v_2, 0)$. Now (2.2) is

$$\begin{aligned} h'' + 2h'(1+h'^2) &= p_0 Q, &(2.7)\\ -f'' + 2v_2(1+h'^2) - 2f'(1+3h'^2) &= p_1 Q, &(2.8)\\ h'' + (2 - 4v_2 f' + 6f'^2)h' &= p_2 Q, &(2.9)\\ -f'' + 2(v_2 - f')(1+f'^2) &= p_3 Q. &(2.10) \end{aligned}$$

Combining (2.7) and (2.9),

$$2h'(2v_2 f' - 3f'^2 + h'^2) = c_1 Q = c_1(1 + f'^2 + h'^2). \tag{2.11}$$

Hence we can get the expression

$$h'^2 = \frac{-c_1(1+f'^2) + 4v_2 f'h' - 6h'f'^2}{c_1 - 2h'}.$$

From (2.8) and (2.10), we have

$$-2v_2 f'^2 + 2f'^3 + (2v_2 - 6f')h'^2 = c_2 Q = c_2(1 + f'^2 + h'^2).$$

Substituting the above value of h'^2,

$$(-4c_1 f'^3 + 2c_1 v_2 f'^2 - 3c_1 f' + c_1 v_2) + h'(2c_2 v_2 f' - 4c_2 f'^2 - c_2 - 4v_2^2 f' + 16 v_2 f'^2 - 16 f'^3) = 0.$$

In this polynomial equation on h' of degree ≤ 1, if the coefficient of h' is 0, then this is a polynomial on f' and the leading coefficient of f'^3 is not 0, which is not possible. Thus the coefficient of h' is not 0, obtaining

$$h' = -\frac{2c_1 v_2 f'^2 - 4c_1 f'^3 - 3c_1 f' + c_1 v_2}{2c_2 v_2 f' - 4c_2 f'^2 - c_2 - 4v_2^2 f' + 16 v_2 f'^2 - 16 f'^3}.$$

Substituting into (2.11), and after some manipulations, we have an expression of type

$$\sum_{n=0}^{9} A_n f'(x)^n = 0,$$

where A_n are real constants. Thus all coefficients A_n must vanish. However, the computation of A_9 gives $A_9 = -512$. This completes the proof for the case $v_3 = 0$.

2. Case $v_3 \neq 0$. After a dilation, we suppose $\vec{v} = (v_1, v_2, 1)$. We distinguish four subcases.

   (a) Subcase $v_1 = v_2 = 0$. Then (2.2) is

$$h'' - 2\left(1 + h'^2\right) = p_0 Q \tag{2.12}$$

$$-f'' + 4f'h' = p_1 Q \tag{2.13}$$

$$h'' - 2f'^2 - 2 = p_2 Q \tag{2.14}$$

$$-f'' = p_3 Q. \tag{2.15}$$

   We deduce from (2.12) and (2.14) that $c_1 Q = 2f'^2 - 2h'^2$ or equivalently, by $Q = 1 + f'^2 + h'^2$,

$$c_1 + (c_1 - 2)\, f'^2 + (c_1 + 2)\, h'^2 = 0.$$

   Clearly, the coefficients are nonzero and Claim 1 implies $h' = \pm\sqrt{m_0 + m_1 f'^2}$, $m_0, m_1 \neq 0$. Moreover from (2.13) and (2.15), we deduce $c_2 Q = 4f'h'$, and substituting $h'^2 = m_0 + m_1 f'^2$,

$$c_2\,(1 + m_0) + c_2\,(1 + m_1)\, f'^2 \mp 4f'h' = 0,$$

   which gives a contradiction from Claim 2.

   (b) Subcase $v_1 = 0$ and $v_2 \neq 0$. Then (2.2) is

$$h'' - 2\left(1 + h'^2\right) = p_0 Q \tag{2.16}$$

$$-f'' + 4f'h' + 2v_2(1 + h'^2) = p_1 Q \tag{2.17}$$

$$h'' - 2f'^2 - 4v_2 f'h' - 2 = p_2 Q \tag{2.18}$$

$$-f'' + 2v_2\left(1 + f'^2\right) = p_3 Q. \tag{2.19}$$

   It follows from (2.16) and (2.18) that

$$c_1 Q = 2f'^2 - 2h'^2 + 4v_2 f'h'. \tag{2.20}$$

   Similarly from (2.17) and (2.19),

$$c_2 Q = 4f'h' + 2v_2(h'^2 - f'^2). \tag{2.21}$$

   Combining (2.20) and (2.21), we deduce

$$2(1 + v_2^2)(f'^2 - h'^2) = (c_1 - c_2 v_2)\, Q = (c_1 - c_2 v_2)\,(1 + f'^2 + h'^2).$$

Let $c_3 = p_0 - p_2 - v_2(p_1 - p_3)$. This equation is written as

$$c_3 + (c_3 - 2(1+v_2^2))f'^2 + (c_3 + 2(1+v_2^2))h'^2 = 0,$$

where the coefficients of $\{1, f'^2, h'^2\}$ are clearly nonzero. Then Claim 1 implies $h' = \pm\sqrt{m_0 + m_1 f'^2}$, $m_0, m_1 \neq 0$. Coming back to (2.20), we derive

$$4v_2 f'h' = c_1(1+m_0) + 2m_0 + (c_1(1+m_1) + 2m_1 - 2)f'^2.$$

Claim 2 concludes that this subcase is false because $v_2 \neq 0$

(c) Subcase $v_1 \neq 0$ and $v_2 = 0$. Then (2.2) is

$$h'' - 2(1 - v_1 h')\left(1 + h'^2\right) = p_0 Q \tag{2.22}$$
$$-f'' - 2f'\left(3v_1 h'^2 - 2h' + v_1\right) = p_1 Q \tag{2.23}$$
$$h'' - 2f'^2(1 - 3v_1 h') + 2(v_1 h' - 1) = p_2 Q \tag{2.24}$$
$$-f'' - 2v_1 f'\left(1 + f'^2\right) = p_3 Q. \tag{2.25}$$

From (2.22) and (2.24) we derive $f'^2 = A/B$, where

$$A = (2v_1 h' - c_1 - 2)h'^2 - c_1, \quad B = c_1 - 2(1 - 3v_1 h').$$

Let us observe that $B \neq 0$ because $h'' \neq 0$ and $v_1 \neq 0$. Similarly, from (2.23) and (2.25), we have

$$2f'(v_1 f'^2 + 2h'^2 - 3v_1 h'^2) = c_2 Q = c_2\left(1 + f'^2 + h'^2\right).$$

Substituting f' by $\pm\sqrt{A/B}$,

$$c_2\sqrt{B}\left(A + B\left(1 + h'^2\right)\right) = \pm 2\sqrt{A}(Av_1 + Bh'(2 - 3v_1 h')).$$

After squaring both sides, we obtain

$$\sum_{n=0}^{9} A_n h'(x)^n = 0,$$

where A_n are real constants. Being $A_9 = 2^{11}v_1^5 \neq 0$, we arrive to a contradiction.

(d) Subcase $v_1 v_2 \neq 0$. Then (2.2) writes

$$h'' - 2(1 - v_1 h')\left(1 + h'^2\right) = p_0 Q \tag{2.26}$$
$$-f'' - 2f'\left(3v_1 h'^2 - 2h' + v_1\right) + 2v_2\left(1 + h'^2\right) = p_1 Q \tag{2.27}$$
$$h'' - 2f'^2(1 - 3v_1 h') - 4v_2 f'h' + 2(v_1 h' - 1) = p_2 Q \tag{2.28}$$
$$-f'' + 2(v_2 - v_1 f')\left(1 + f'^2\right) = p_3 Q. \tag{2.29}$$

From (2.26) and (2.28),

$$2(v_1 h' - 1)h'^2 + 2f'^2(1 - 3v_1 h') + 4v_2 f'h' = c_1 Q = c_1\left(1 + f'^2 + h'^2\right),$$

or equivalently

$$Af'^2 + Bf' + C = 0,$$

where

$$A = c_1 - 2 + 6v_1 h',$$
$$B = -4v_2 h',$$
$$C = c_1 + (c_1 + 2 - 2v_1 h')\, h'^2.$$

Note that A and B cannot vanish because $v_1 v_2 \neq 0$ and $h'' \neq 0$. Then

$$f' = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

On the other hand, (2.27) and (2.29) imply

$$c_2 Q + 2\left(f'(3v_1 h'^2 - 2h') + f'^2(v_2 - v_1 f') - v_2 h'^2\right) = 0. \tag{2.30}$$

As in the previous subcase, our purpose is to substitute the value of f' in order to obtain a polynomial on h'. Let $D = B^2 - 4AC$. If we write (2.30) again in terms of $\sqrt{D}$, then we have a polynomial equation of type $a + b\sqrt{D} = 0$, hence, $a^2 - b^2 D = 0$. Substituting the value of D as well as of A, B, and C, we obtain the desired polynomial equation on h', namely,

$$\sum_{n=0}^{12} A_n h'^n = 0,$$

where A_n are real constants. Because $A_{12} = 2^{16} 3^3 v_1^8 \neq 0$, we arrive to a contradiction, completing the proof of theorem.

□

3. Translating solitons of homothetical type

Let $u(x,y) = f(x)g(y)$ be a homothetical surface. Suppose that the surface is also a translating soliton with respect to $\vec{v}$. There are three initial cases that can be previously considered.

1. Case that f or g is constant. Then the surface is ruled, so we know that the surface is a grim reaper or a plane parallel to the vector $\vec{v}$.

2. Case that f (or g) is linear. Indeed, if $f(x) = ax + b$ with $a, b \in \mathbb{R}$, $a \neq 0$, then (1.4) is a polynomial equation $\sum_{n=0}^{3} A_n(y)x^n = 0$. In particular all coefficients A_n must vanish. The computation of A_3 yields $2a^3 v_2 g'^3$. Since g is not constant and $a \neq 0$, we deduce $v_2 = 0$. Now the computation of A_2 gives $A_2 = 2a^3(av_1 g)g'^2$. Hence, $v_1 = v_3 = 0$, obtaining a contradiction.

3. Case that f and f' (or g and g') are linearly dependent. Assume that $f' = af$, $a \in \mathbb{R}$, $a \neq 0$. Then $f'' = a^2 f$ and (1.4) is a polynomial equation on f of degree 3, namely, $\sum_{n=0}^{3} A_n(y)f^n = 0$. Then all

coefficients A_n must vanish. In particular, $A_0 = -2v_3$, hence $v_3 = 0$. Now $A_1 = 0$ and A_3 lead to

$$\begin{aligned} A_1 = a\left(a+2v_1\right)g + 2v_2 g' + g'' &= 0 \\ A_3 = \left(av_1 g + v_2 g'\right)\left(ag^2 + g'^2\right) + a^2\left(g^2 g'' - gg'^2\right) &= 0. \end{aligned}$$

The linear combination $A_1 - a^2 g A_3 = 0$ writes $-a^4 g^3 + (2av_1 - a^2)gg'^2 + 2v_2 g'^3 = 0$, or equivalently,

$$-a^3 + (2av_1 - a^2)\left(\frac{g'}{g}\right)^2 + 2v_2\left(\frac{g'}{g}\right)^3 = 0.$$

Since all coefficients are not all zero, there is $b \in \mathbb{R}$ such that $g'/g = b$ with $b \neq 0$. Taking into account that $g'' = b^2 g$ and that $a^2 + b^2 \neq 0$, then $A_1 = 0$ is

$$a^2 + (2(av_1 + bv_2) + b^2)g = 0,$$

obtaining a contradiction.

After this discussion, we can assume that $f'f'' \neq 0$ and $g'g'' \neq 0$ in their domains and let us introduce new variables. So, let $p = p(f) = f'$, as well as, $q = q(g) = g'$. Then $p' = f''/f'$ and $q'' = g''/g'$. Let us observe that $pp'qq' \neq 0$. Now the translating soliton equation (1.4) is

$$(1+f^2q^2)gpp' - 2fgp^2q^2 + (1+g^2p^2)fqq' - 2(1+p^2g^2+f^2q^2)(-v_1 pg - v_2 fq + v_3) = 0. \tag{3.1}$$

We now give a partial result on Problem 2 in case that $\vec{v}$ is one of the canonical basis of $\mathbb{R}^3$.

Theorem 3.1 *Grim reapers are the only translating solitons of homothetical type when $\vec{v}$ is one vector of the canonical basis.*

Proof We know that the case $\vec{v} = (0,0,1)$ was solved in [13]. It remains the case that $\vec{v}$ is $(1,0,0)$ or $(0,1,0)$. By the symmetry of Equation (3.1) with respect to v_1 and v_2, it suffices to consider the case that $\vec{v} = (0,1,0)$. Then

$$(1+f^2q^2)gpp' - 2fgp^2q^2 + (1+g^2p^2)fqq' + 2fq(1+p^2g^2+f^2q^2) = 0. \tag{3.2}$$

We divide by fgp^2q^2,

$$\frac{1}{q^2}\left(\frac{p'}{fp}\right) + \frac{1}{p^2}\left(\frac{q'}{gq}\right) + \frac{1}{q}\left(gq' - q\right) + \frac{1}{p}\left(fp' - p\right) + 2\left(\frac{1}{p^2gq} + \frac{g}{q} + \frac{f^2q}{p^2g}\right) = 0.$$

Differentiating with respect to f and g successively, we obtain

$$\left(\frac{1}{q^2}\right)'\left(\frac{p'}{fp}\right)' + \left(\frac{1}{p^2}\right)'\left(\frac{q'}{gq}\right)' + 2\left[\left(\frac{1}{p^2}\right)'\left(\frac{1}{gq}\right)' + \left(\frac{f^2}{p^2}\right)'\left(\frac{q}{g}\right)'\right] = 0.$$

Notice that $\left(\frac{1}{p^2}\right)'\left(\frac{1}{q^2}\right)' \neq 0$. Dividing by $2\left(\frac{1}{p^2}\right)'\left(\frac{1}{q^2}\right)'$ and next differentiating with respect to f and g successively,

$$\left(\frac{\left(\frac{f^2}{p^2}\right)'}{\left(\frac{1}{p^2}\right)'}\right)'\left(\frac{\left(\frac{q}{g}\right)'}{\left(\frac{1}{q^2}\right)'}\right)' = 0. \tag{3.3}$$

1. Case

$$\left(\frac{f^2}{p^2}\right)' = a\left(\frac{1}{p^2}\right)'$$

for some $a \neq 0$. Integrating we have $p^2 = kf^2 - ak$ for some $k \neq 0$. Differentiating with respect to f, we deduce $pp' = kf$. Substituting into (3.2), we have a polynomial equation $B_1(g)f + B_3(g)f^3 = 0$. Thus $B_1 = B_3 = 0$. The computation of these coefficients yield

$$\begin{aligned} B_1 &= -q\left(ag^2k - 1\right)(q' + 2) + 2agkq^2 + gk, \\ B_3 &= q\left(g^2k(q' + 2) - gkq + 2q^2\right). \end{aligned}$$

Equation $B_3 = 0$ can be solved explicitly. Suppose $k > 0$ (an analogous argument if $k < 0$). Then

$$q(g) = g\sqrt{k}\tan\left(m - \frac{2}{\sqrt{k}}\log(g)\right), \quad m \in \mathbb{R}.$$

Substituting into $B_1 = 0$, we conclude

$$2\left(ag^2k - 1\right)\tan^3\left(m - \frac{2\log(g)}{\sqrt{k}}\right) + \sqrt{k}\left(ag^2k + 1\right)\tan^2\left(m - \frac{2\log(g)}{\sqrt{k}}\right) + \sqrt{k} = 0,$$

obtaining a contradiction.

2. Case

$$\left(\frac{q}{g}\right)' = a\left(\frac{1}{q^2}\right)'$$

for some $a \neq 0$. Integrating,

$$g = \frac{q^3}{a + kq^2},$$

for some constant $k \in \mathbb{R}$. Differentiating with respect to g,

$$q' = \frac{q^2\left(3a + kq^2\right)}{\left(a + kq^2\right)^2}.$$

By substituting these values of g and q in (3.2), we obtain

$$\frac{q}{(a + kq^2)^4}\sum_{n=0}^{10} C_n(x)q^n = 0.$$

Then all coefficients C_n must vanish. However, the computation of C_0 gives $C_0 = 2a^4 f$ which is not possible.

□

4. Extension of the results to the Lorentzian setting

In this last section, we extend the results to the Lorentz-Minkowski 3-space $\mathbb{R}^3_1$. Here $\mathbb{R}^3_1$ is the affine space $\mathbb{R}^3$ endowed with the canonical Lorentzian metric $dx^2+dy^2-dz^2$. Denote $\langle\cdot,\cdot\rangle_L$ and $\times_L$ the Lorentzian inner and cross product, respectively.

We first consider a (nonplanar) nondegenerate cylindrical surface $X(s,t)=\alpha(s)+tw$ where $\alpha=\alpha(s)$ is parametrized by the arc length s and $w\in\mathbb{R}^3_1$. The unit normal vector N is parallel to $\alpha'(s)\times_L w$ and hence (1.1) writes

$$\langle w,w\rangle_L \langle\alpha'\times_L w,\alpha''\rangle_L = 2\epsilon\left(\epsilon_1\langle w,w\rangle_L - \langle\alpha'(s),w\rangle_L^2\right)\langle\alpha'(s)\times_L w,\vec{v}\rangle_L, \tag{4.1}$$

where ϵ is the sign of $\langle\alpha'(s)\times_L w,\alpha'(s)\times_L w\rangle_L$ and $\epsilon_1=\langle\alpha'(s),\alpha'(s)\rangle_L$. In case the rulings are lightlike, the surface is a translating soliton if $\langle\alpha'(s)\times_L w,\vec{v}\rangle_L=0$. In particular, this equation holds if $\vec{v}$ is parallel to w being α is an arbitrary curve.

In all Lorentzian versions of the results, we will conclude that the surface is a cylindrical surface. According to the causal character of the rulings, the description of the translating solitons of $\mathbb{R}^3_1$ of cylindrical type is the following ([2]). After a rigid motion of $\mathbb{R}^3_1$, we can fix w.

1. Spacelike rulings. Let $w=(1,0,0)$. If $X(s,t)=(0,s,u(s))+tw$, then (1.1)

$$u''=\begin{cases} 2(1-u'^2)(v_2u'-v_3), & 1-u'^2>0\\ -2(1-u'^2)(v_2u'-v_3), & 1-u'^2<0.\end{cases}$$

   For example, if $\vec{v}=(0,0,1)$, the rulings are orthogonal to $\vec{v}$ and the integration of both equations give

$$u(s)=\begin{cases} -\frac{1}{2}\log(\cosh(-2s+a))+b, & 1-u'^2>0\\ \frac{1}{2}\log(\sinh(2s+a))+b, & 1-u'^2<0,\end{cases}$$

   where $a,b\in\mathbb{R}$. These two curves appeared in [3].

2. Timelike rulings. Let $w=(0,0,1)$. If $X(s,t)=(s,u(s),0)+tw$, then the surface is timelike and Equation (1.1) is

$$u''=2(1+u'^2)(v_1u'-v_2).$$

   If $\vec{v}=(0,1,0)$, the rulings are orthogonal to $\vec{v}$ and the solution is $u(s)=\log(\cos(2s+a))/2+b$, $a,b\in\mathbb{R}$.

3. Lightlike rulings. Then $H=0$, so the translating equation (1.1) is $\langle N,\vec{v}\rangle=0$. Let $w=(1,0,1)$ and $X(s,t)=(u(s),s,-u(s))+tw$. The surface is not degenerated if $u'\neq 0$. Then Equation (1.1) is

$$v_1-2v_2u'-v_3=0.$$

   If $v_2=0$, then $\vec{v}$ is parallel to w and with arbitrary generating curve. Otherwise the function u is linear and $X(s,t)$ is a plane.

Summarizing, the cylindrical translating solitons in $\mathbb{R}^3_1$ are planes (when the rulings are parallel to $\vec{v}$), Lorentzian grim reapers and cylindrical surfaces whose rulings are lightlike and parallel to $\vec{v}$.

As we have pointed out, the extensions of Theorems 2.1, 2.2, and 3.1 to the Lorentzian setting are straightforward and the conclusion is that the surfaces must be cylindrical surfaces.

Theorem 4.1 *A translating soliton in $\mathbb{R}^3_1$ that is the sum of two curves and where one of the generating curve is planar must be a cylindrical surface.*

Theorem 4.2 *A translating soliton in $\mathbb{R}^3_1$ of homothetical type when $\vec{v}$ is one vector of the canonical basis must be a cylindrical surface.*

Acknowledgment

Rafael López has been partially supported by grant no. MTM2017-89677-P, MINECO/AEI/FEDER, UE.

References

[1] Altschuler SJ, Wu LF. Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle. Calculus of Variations and Partial Differential Equations 1994; 2: 101-111. https://doi.org/10.1007/BF01234317

[2] Aydin ME, López R. Ruled translating solitons in Minkowski 3-space. Journal of Geometry and Physics 2021; 170: 104392. doi: 10.1016/j.geomphys.2021.104392

[3] Castro I, Castro-Infantes I, Castro-Infantes J. Curves in the Lorentz-Minkowski plane: elasticae, catenaries and grim-reapers. Open Mathematics 2018; 16: 747-766. https://doi.org/10.1515/math-2018-0069

[4] Clutterbuck J, Schnürer O, Schulze F. Stability of translating solutions to mean curvature flow. Calculus of Variations and Partial Differential Equations 2007; 29: 281- 293. https://doi.org/10.1007/s00526-006-0033-1

[5] Dillen F, Van de Woestyne I, Verstraelen L, Walrave JT. The surface of Scherk in E^3: A special case in the class of minimal surfaces defined as the sum of two curves. Bulletin of the Institute of Mathematics Academia Sinica New Series 1998; 26: 257-267.

[6] Goemans W, Van de Woestyne I. Translation and homothetical lightlike hypersurfaces of semi-Euclidean space. Kuwait Journal of Science & Engineering 2011; 38(2A): 35-42.

[7] Hasanis T, López R. Classification and construction of minimal translation surfaces in Euclidean space. Results in Mathematics 2020; 75(2): 22 pp. https://doi.org/10.1007/s00025-019-1128-2

[8] Hieu DT, Hoang NM. Ruled minimal surfaces in $\mathbb{R}^3$ with density e^z. Pacific Journal of Mathematics 2009; 243(2): 277-285.https://doi.org/10.2140/pjm.2009.243.277

[9] Huisken G, Sinestrari C. Mean curvature flow singularities for mean convex surfaces. Calculus of Variations and Partial Differential Equations 1999; 8: 1-14. https://doi.org/10.1007/s005260050113

[10] Ilmanen T. Elliptic regularization and partial regularity for motion by mean curvature. Memoirs of the American Mathematical Society 1994;108, x+90.

[11] Liu H, Yu Y. Affine translation surfaces in Euclidean 3-space. Proceedings of the Japan Academy, Series A, Mathematical Sciences 2013; 89(9): 111-113. https://doi.org/10.3792/pjaa.89.111

[12] López R, Moruz M. Translation and homothetical surfaces in Euclidean space with constant curvature. Journal of Korean Mathematical Society 2015; 52: 523-535. https://doi.org/10.4134/JKMS.2015.52.3.523

[13] López R. Separation of variables in equations of mean-curvature type. Proceedings of the Royal Society of Edinburgh Section A: Mathematics 2016; 146: 1017-1035. https://doi.org/10.1017/S0308210515000815

[14] López R. The translating soliton equation. In: Minimal Surfaces, Integrable Systems and Visualisation, Springer Proceedings in Mathematics & Statistics; 2021. pp. 187-216. https://doi.org/10.1007/978-3-030-68541-6-11

[15] López R, Perdomo O. Minimal translation surfaces in Euclidean space. The Journal of Geometric Analysis 2017; 27(4): 2926-2937. https://doi.org/10.1007/s12220-017-9788-1

[16] Nitsche JCC. Lectures on Minimal Surfaces. Cambridge University Press, England, 1989.

[17] Scherk H.F. Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen. Journal für die reine und angewandte Mathematik 1835; 13: 185-208. https://doi.org/10.1515/crll.1835.13.185 (in German).

[18] Van de Woestyne I. A new characterization of the helicoids. In: Geometry and topology of submanifolds V, Proceedings of The Conferences on Differential Geometry and Vision & Theory of Submanifolds; Leuven/Brussels, Belgium; 1992. pp. 267-273.

[19] Van de Woestyne I. Minimal homothetical hypersurfaces of a semi-Euclidean space. Results in Mathematics 1995; 27; 333-342. https://doi.org/10.1007/BF03322837

[20] Yoon DW, Yuzbasi ZK. Weighted minimal affine translation surfaces in Euclidean space with density. International Journal of Geometric Methods in Modern Physics 2018; 15: 8 pp. https://doi.org/10.1142/S0219887818501967