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On the mathematical theory of behavioral swarms emerging collective dynamics

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This paper considers a system consisting of a number of interacting living entities whose state at the microscopic scale is heterogeneously distributed among the said entities. This state includes, in addition, the classical mechanical variables, such as position and velocity, also a behavioral variable which is modified by interactions. It is shown how

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the pioneering ideas proposed in Bellomo *et al.* [Towards a mathematical theory of behavioral swarms, *ESAIM Control Optim. Calc. Var.* **26** (2020) 125] can be developed towards modeling behavioral swarms within a quest towards a mathematical theory of living systems. The first part of the paper presents a qualitative analysis of the emerging behaviors predicted by the model in aforementioned work. Some simulations follow to depict the said emerging behaviors. The last part of the paper is devoted to derive a new, more general theory in view of applications to model living systems.

Keywords: Active particles; complexity; emerging behaviors; living systems; swarms.

AMS Subject Classification 2020: 82D99, 91D10

1. Plan of the Paper

This paper is motivated by a quest towards a mathematics for living systems through an approach which develops the mathematical theory of swarms to model the dynamics of large systems of interacting living entities. We are aware that we are facing a goal that is too ambitious to actually be achieved. On the other hand, we believe that some very preliminary efforts deserve to be developed and that the achievement of, even if partial, results would be ultimately worth to be chased.

A strategy for pursuing this goal is available in Ref. 8 and references therein. This strategy lies in the derivation of a mathematical structure capturing the complexity features of living systems made up of several interacting entities and in specializing this framework towards the modeling of specific systems by means of the mathematical description of the interactions involving the said entities. This approach should be developed within a multiscale framework[33] according to a unified vision of all physical systems at all representation and modeling scales,[26] see also Refs. 15, 20 and the application.[16] The survey[8] reports about all technical aspects of the so-called *kinetic theory of active particles* which has promoted several applications, just mentioning a few of them, in biology,[14] economics,[9, 30] crowd dynamics.[10, 11]

This approach is somewhat based on the methods of statistical physics,[2, 19, 32] but it differs from classical kinetic theories mainly in that the state at the microscopic scale of the interacting entities includes behavioral variables in addition to the classical ones, such as position and velocity. Borrowing some definitions of active particle methods, the aforementioned entities can be viewed as *active particles*, shortly *a-particles*, while the behavioral variable can be called *activity*. The key feature of the method, called *kinetic theory of active particles*,[8] is the modeling of interactions which, beyond binary interactions, are nonlocal and nonlinearly additive.

However, the method requires the continuity of the distribution function, which can be valid only for very large number of a-particles. Therefore, it is worth developing a critical revisiting to account for small numbers of interacting entities. Because of these criticisms, some authors have used systems with discrete states, for instance,[9] which certainly contributes to a technical improvement of the approach, but this matter needs, according to the authors' opinion, further understanding. As an alternative, a recent paper[13] has proposed a new approach to the modeling swarms with active heterogeneous internal variables by following the idea of swarms with internal thermodynamical variables, see Refs. 24, 25 and for further developments.[22, 23] The search for a mathematical theory of behavioral swarms is also motivated by a recent literature on the applications of swarm theory to model social and economic systems.[1, 4, 5, 30]

Our paper aims at developing a conceptual alternative to Ref. 8 within the framework of a quest to a mathematical theory of living systems.[8] In more details, we consider a system constituted by a number N of interacting living entities whose state at the microscopic scale is heterogeneously distributed among the said entities. This micro-state includes, in addition the classical mechanical variables, for instance, localization and velocity, also a *behavioral variable*.

We consider a system in absence of birth–death dynamics. Then, the number of interacting entities is a constant of the system. However, some reasonings are developed to include in the approach also proliferative–destructive interactions which make N depending on time, however finite. Indeed, this is the first step to show how the pioneering ideas proposed in Ref. 13 can be developed towards a mathematical theory of behavioral swarms. The plan of this paper is as follows.

Section 2 presents class of behavioral swarms consistent with the general framework proposes in Ref. 13. The main feature of these models is that the micro-scale state of each a-particles is defined by position, velocity and a behavioral variable called *activity* which is modified by interactions and, in turn, modifies the mechanical dynamics. The derivation is based on a pseudo-Newtonian framework, where the dynamics is induced for each a-particle, through interactions, by the surrounding a-particles. We consider both first and second order, where the dynamics corresponds, respectively, to velocity and acceleration. Models undergoing topological interactions are also considered, where this term is used to indicate that a-particles interact with a fixed number of individual entities within their sensitivity domain.[7]

Section 3 develops a qualitative analysis of the first-order models[13] mainly focusing on the trend to the asymptotic emerging behavior of the system with special attention to flocking dynamics. First, a reformulation about consensus of activity variables is proposed, then the cases of constant and time-varying activity variables are studied.

Section 4 develops the same qualitative analysis referring now to second-order models, where acceleration is induced by interactions and is referred to a pseudo-inertia. As in Sec. 3, the key feature of the qualitative study consists in understanding how the activity variable modifies the emerging collective behaviors.

Section 5 presents some sample simulations that provide a quantitative picture of the asymptotic behavior of the system based on the proofs delivered in Secs. 3 and 4. First, some specific case studies are selected corresponding to well-defined theorems in the preceding sections, then simulations show both the asymptotic trend and the emerging patterns.

Section 6 is devoted to research perspectives focusing on modeling topics by showing how the mathematical theory can lead to the derivation of models for well-defined systems. The dynamics of human crowds is selected as a specific example. Then, some perspective ideas are proposed towards further development of the mathematical theory.

2. On the Derivation of Planar Behavioral Swarms

In this section, we present a new class of planar swarm models consistent with the mathematical structures proposed in Sec. 2 of Ref. 13, but more general than those used in the simulations in Sec. 4 of the same paper. The model accounts for the influence, over the collective motion, of a heterogeneously distributed activity variable corresponding to a specific social state. The activity variable is modified by the interaction dynamics. The derivation accounts for the so-called topological interactions introduced in Ref. 7, see also the analytic formulation.[12] Actually, these simulations have empirically shown some interesting emerging behaviors that motivate the analytical study proposed in our paper.

The contents of this section are focused on the derivation of first- and second-order models which will be studied in the following sections. We also discuss, how this class of models has some interest in specific applications, for instance to depict the dynamics which appears in crowds.[3, 10, 11]

2.1. *Presentation of modeling spirit*

The derivation of models is here developed with the aim of showing, by a number of simple case studies, how the dynamics of the activity variable modifies the patterns of the flow dynamics. Specific models can be derived within the following framework:

(1) The state of the system is given by the whole set of directions (heading angles) θ_i, rotation speeds σ_i and activities u_i of all a-particles with $i \in [N] := \{1, \ldots, N\}$. Each ith a-particle in the swarm, shortly i-particle, moves with the same speed $v = v_0 = 1$. The activity u_i is supposed to correspond to the level of a specific behavioral variable with $u_i \in [0, 1]$, where $u_i = 0$ and $u_i = 1$ define, respectively, the minimal and maximal admissible level. An example of behavioral variable is the stress which, in living systems, can be generated by the perception of stress.

(2) Each a-particle has a visibility angle $2\theta_v$ which is symmetric with respect to the velocity direction. Hence, it has a visibility domain $\Omega_i^v = [\theta_i - \theta_v, \theta_i + \theta_v]$ supposed to be a circular sector with radius R. Each a-particle interacts only with other particles in Ω_i^v or even with a fixed "small" number of entities within a sensitivity domain $\Omega_i^s \subseteq \Omega_i^v$ corresponding to a sector with radius R_s. This sensitivity radius is finite and depends on the number of particles selected for the interaction.[8]

(3) Each i-particle, has an individual movement with direction $\boldsymbol{\nu}_i$, corresponding to θ_i, but all of them share a common trend $\boldsymbol{\nu}^\nu$, corresponding to θ^ν, while each ith a-particle is subject to an attraction towards the mean velocity direction

φ_i^v (or φ_i^s) of the particles in Ω_i^v (or Ω_i^s). This attraction depends also on the behavioral state of a-particles in the said domains.

(4) The decision process by which a-particles modify their motion according to the following behavioral sequence: first the a-particle modifies the activity and subsequently the direction of motion.

(5) The dynamics of the social state depends on the specific features of the specific behavioral variable under consideration. A simple case corresponds to a consensus dynamics with respect to the a-particles in Ω_i or Ω_i^s.

Remark 2.1. We consider the case $\Omega_i^s \subseteq \Omega_i^v$ although in some special case it might even be $\Omega_i^v \subseteq \Omega_i^s$, see Ref. 8 to discuss this critical case. Figure 1 shows the case $\Omega_i^s \subseteq \Omega_i^v$. If we consider the case $\Omega_i^s = \Omega_i^v$, notations can be simplified by considering interactions in the generic domain Ω_i for both Ω_i^v and Ω_i^s. An additional case, often considered in literature, as well as in the following section, corresponds to interactions of the ith particles with all particles, i.e. Ω_i includes all particles.

Let us now transfer these simple rules into interactions models, to be inserted into the general structures proposed in Sec. 2 in Ref. 13, where some technical cases are here implemented.

The assumption that the same unit speed is shared by all particles defines the velocity of the i-particle yields:

$$\boldsymbol{v}_i = \cos\theta_i \boldsymbol{i} + \sin\theta_i \boldsymbol{j} \Rightarrow \frac{d\boldsymbol{v}_i}{dt} = (-\sin\theta_i \boldsymbol{i} + \cos\theta_i \boldsymbol{j})\sigma_i,$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ denote the unit vectors of an orthogonal frame and $\theta_i \in [0, 2\pi)$ denotes the velocity direction (heading angle).

In addition, the following physical quantities can be defined:

- The mechanical state of each particle is defined by the flight direction θ_i and rotational speed σ_i, i.e. the time derivative of θ_i. In addition, the following notations

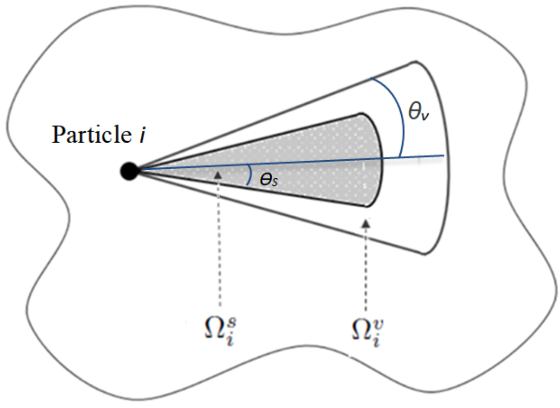


Fig. 1. (Color online) Sensitivity and interaction domain.

are used:

$$\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N), \quad \Theta = (\theta_1, \ldots, \theta_N) \quad \text{and} \quad U = (u_1, \ldots, u_N),$$

which define a spatial, angular and activity configuration vectors, respectively.

- The rate η_{ij} of interaction between the i- and j-particles can be supposed a constant quantity, i.e. $\eta_{ij} \equiv \eta_0$. In general, η_{ij} might be proportional to the distance between the interacting entities:

$$g_{ij} = g_{ij}(\boldsymbol{x}) = \frac{\phi_{ij}(\boldsymbol{x})}{\sum_{k\in\Omega_i} \phi_{ik}(\boldsymbol{x})} \qquad \text{with } \phi_{ij}(\boldsymbol{x}) := \phi(||\boldsymbol{x}_j - \boldsymbol{x}_i||),$$

where $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ is a positive, Lipschitz continuous and non-increasing weight function. An explicit example is as follows:

$$\phi_{ij}(\boldsymbol{x}) = \exp(-\alpha\|\boldsymbol{x}_j - \boldsymbol{x}_i\|) \quad \text{or} \quad \phi_{ij}(\boldsymbol{x}) = \frac{1}{(1 + \|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)^{\delta/2}} \qquad \delta \geq 0.$$

- The attraction direction φ_i^s by the a-particles in Ω_i is given by the weighted sum of θ_j's by g_{ij}.

$$\varphi_i^s = \varphi_i^s(\boldsymbol{x}, \Theta) = \sum_{j\in\Omega_i} g_{ij}(\boldsymbol{x})\theta_j.$$

- The direction ω_i, which effectively attracts the movement direction of the i-particle, is defined by the convex combination of θ^ν and φ_i^s weighted by the activity

$$\omega_i = \omega_i(\boldsymbol{x}, \Theta, U; \theta^\nu) := u_i\,\theta^\nu + (1 - u_i)\varphi_i^s(\boldsymbol{x}, \Theta),$$

where θ^ν is the angle related to commonly preferred direction, while ω_i ought to be referred either to the visibility or the sensitivity domain as indicated in Remark 2.1.

- The dynamics by the consensus of the i-particle to the j-particles in Ω_i depends on a parameter β as follows:

$$\frac{du_i}{dt} = \beta \sum_{j\in\Omega_i} \psi_{ij}(u_j - u_i), \tag{2.1}$$

where $\psi_{ij} = \psi(\|\boldsymbol{x}_j - \boldsymbol{x}_i\|)$ is a positive, analytic and non-increasing function in its argument. When we go beyond the consensus dynamics, then different models of interaction can be considered keeping, however, the form as in (2.1).

2.2. *First-order model*

Consider a class of first-order models where the rotational speed is phenomenologically modeled by a first-order differential equation describing the alignment attraction of θ_i to ω_i:

$$\frac{d\theta_i}{dt} = \gamma\big(\omega_i(\boldsymbol{x}, \Theta, U; \theta^\nu) - \theta_i\big),$$

where γ is a parameter measuring the inverse of relaxation coefficient.

Let us specialize the components of $\boldsymbol{x}$ by $\boldsymbol{x} = (x, y)$ and transfer the aforementioned assumptions into the mathematical structure to get

$$\begin{cases} \dfrac{du_i}{dt} = \beta \displaystyle\sum_{j\in\Omega_i} \psi_{ij}(u_j - u_i), \\ \dfrac{d\boldsymbol{x}_i}{dt} = (\cos\theta_i, \sin\theta_i), \\ \dfrac{d\theta_i}{dt} = \gamma\left(u_i\,\theta^\nu + \dfrac{1-u_i}{\sum_{k\in\Omega_i}\phi_{ik}(\boldsymbol{x})} \displaystyle\sum_{j\in\Omega_i} \phi_{ij}(\boldsymbol{x})\theta_j - \theta_i \right). \end{cases} \tag{2.2}$$

In the sequel, we consider some special case of the dynamics as well as some simplification. Suppose that all the activity variables have unity initially:

$$u_i(0) = 1 \quad \text{for all } i \in [N].$$

Then, it is easy to see that

$$u_i(t) = 1, \quad t > 0, \quad i \in [N].$$

Hence, system (2.2) becomes

$$\begin{cases} \dfrac{d\boldsymbol{x}_i}{dt} = (\cos\theta_i, \sin\theta_i), \\ \dfrac{d\theta_i}{dt} = \gamma(\theta^\nu - \theta_i). \end{cases} \tag{2.3}$$

By direct calculation, one has an alignment: for $t \geq 0$,

$$|\theta_i(t) - \theta^\nu| = |\theta_i^0 - \theta^\nu| e^{-\gamma t}$$

and

$$\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| \leq \|\boldsymbol{x}_i^0 - \boldsymbol{x}_j^0\| + \frac{\sqrt{2}}{\gamma}(1 - e^{-\gamma t}).$$

2.3. *Second-order models*

Consider second-order models in which the rotational dynamic is modeled by an acceleration term involving the rotational speed σ_i, i.e. the time derivative of θ_i. In more details, we consider the following specific acceleration model:

$$\frac{d\sigma_i}{dt} = \gamma_a(\omega_i(\boldsymbol{x}, \Theta, U; \theta^\nu) - \theta_i) - \gamma_b \frac{d\theta_i}{dt}$$

corresponding to an acceleration induced by the chased flight directions reduced by a viscous action. Here, γ_a and γ_b are positive parameters. This phenomenological assumption yields

$$\begin{cases} \dfrac{du_i}{dt} = \beta \displaystyle\sum_{j\in\Omega_i} \psi_{ij}(u_j - u_i), \\ \dfrac{d\boldsymbol{x}_i}{dt} = (\cos\theta_i, \sin\theta_i), \\ \dfrac{d\theta_i}{dt} = \sigma_i, \\ \dfrac{d\sigma_i}{dt} = \gamma_a \left(u_i\, \theta^\nu + \dfrac{1-u_i}{\sum_{k\in\Omega_i} \phi_{ik}(\boldsymbol{x})} \displaystyle\sum_{j\in\Omega_i} \phi_{ij}(\boldsymbol{x})\theta_j - \theta_i \right) - \gamma_b \dfrac{d\theta_i}{dt}. \end{cases} \tag{2.4}$$

The same case studies, i.e. constant activity and attractions to the high or low values of the activity can be treated by technical calculations analogous to those in Sec. 2.1.

2.4. *Topological interactions*

The concept of topological interactions was introduced in Ref. 7 according to a conjecture that interactions involve only a fixed number m of the n i-particles within the visibility domain Ω_i^v are involved in the interaction. This conjecture defines the sensitivity domain Ω_s. In agreement with Remark 2.1, $m < n$ and $R_s < R_v$. In addition, the spatial dependence of the activity can be neglected in the modeling of interactions as m is of a smaller order with respect to n, do that R_s is also small with respect to the visibility radius R_v. In this case, the model is simply written as follows:

$$\begin{cases} \dfrac{du_i}{dt} = \beta \displaystyle\sum_{j\in\Omega_i^s} (u_j - u_i), \\ \dfrac{d\boldsymbol{x}_i}{dt} = (\cos\theta_i, \sin\theta_i), \\ \dfrac{d\theta_i}{dt} = \gamma \left(u_i\theta^\nu + (1-u_i) \displaystyle\sum_{j\in\Omega_i^s} (\theta_j - \theta_i) \right) \end{cases} \tag{2.5}$$

while analogous calculations can be applied to the structures corresponding to the case of a constant activity.

2.5. *Further remarks*

The different classes of models reported in this section can be viewed as particular cases of the mathematical approach in Ref. 13. A key topic to be investigated consists in understanding how far these models can be particularized in well-defined applications and, looking forward, further developments of the mathematical framework can be further developed within a continuing quest towards a mathematical theory of dynamical systems which aims at modeling the complex behavior of living systems.

These two topics are treated in Sec. 6 specifically with focus on the first topic, by referring to the modeling of human crowds in crisis situations, for instance evacuation under stress conditions[10] or under contagion risk.[27–29] Then, the second topic is treated still by looking at real-world applications and by taking advantage of the qualitative analysis developed in the following sections. In view of this qualitative analysis we report, for sake of completeness, a technical result related to the Gronwall lemma in Ref. 21.

Lemma 2.1. (Ref. 21) (i) *Let* $y : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ *be a differentiable function satisfying*

$$y' \leq -\alpha y + f, \quad t > 0, \quad y(0) = y_0,$$

where α *is a positive constant and* $f : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ *is a continuous function decaying to zero as its argument goes to infinity. Then* y *satisfies*

$$y(t) \leq \frac{1}{\alpha} \max_{s\in[t/2,t]} |f(s)| + y_0 e^{-\alpha t} + \frac{\|f\|_{L^\infty}}{\alpha} e^{-\frac{\alpha t}{2}}, \quad t \geq 0.$$

(ii) *Let* $y : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ *be a differentiable function satisfying*

$$y' \leq -py + q,$$

where p *and* q *are non-negative integrable functions. Then* y *satisfies*

$$y(t) \leq y_0 e^{-\int_0^t p(\tau)d\tau} + e^{-\int_{\frac{t}{2}}^t p(\tau)d\tau} \int_0^{\frac{t}{2}} q(\tau)d\tau + q\left(\frac{t}{2}\right) \int_{\frac{t}{2}}^t e^{-\int_s^t p(\tau)d\tau} ds, \quad t \geq 0.$$

3. Emergent Dynamics of the First-Order Flocking Model

This section presents an analytic study of qualitative behaviors, mainly focused on alignment and flocking dynamics, of the dynamics of the class of first-order models presented in Sec. 2.2 for different types of activity and space interactions. First, in Sec. 3.2 we consider all-to-all couplings, where the interaction domain Ω_i becomes

$$\Omega_i := \{1, \ldots, N\}, \quad i \in [N]. \tag{3.1}$$

Then, we study some aspects of the dynamics in the case of nearby interactions in Sec. 3.3. We refer to the concepts proposed in Remark 2.1. Before we move on, we define several time-dependent indices and diameter variables as follows:

$$M_{1,t} := \operatorname*{argmax}_{i\in[N]} u_i(t), \quad m_{1,t} := \operatorname*{argmin}_{i\in[N]} u_i(t),$$
$$M_{2,t} := \operatorname*{argmax}_{i\in[N]} \theta_i(t), \quad m_{2,t} := \operatorname*{argmin}_{i\in[N]} \theta_i^t,$$
$$M_{3,t} := \operatorname*{argmax}_{i\in[N]} |\tilde{\theta}_i(t)|,$$

$$u_{M_{1,t}}(t) := \max_{1\le i\le N} u_i(t), \quad u_{m_{1,t}}(t) := \min_{1\le i\le N} u_i(t),$$
$$\mathcal{D}(U(t)) := u_{M_{1,t}}(t) - u_{m_{1,t}}(t), \quad \mathcal{D}(\Theta) := \theta_{M_{2,t}}(t) - \theta_{m_{2,t}}(t).$$

For notational simplicity, we suppress t dependence in the above indices, e.g.

$$u_{M_1} := u_{M_{1,t}}, \quad u_{m_1} := u_{m_{1,t}}.$$

Note that these extremal variables, e.g. $u_{M_1}(t)$, are Lipschitz continuous, so they are differentiable a.e. in t.

3.1. *Structure and reformulation of system*

Here, we define the global flocking in active swarm particle model.

Definition 3.1. Let $\mathcal{P} := \{(u_i, \boldsymbol{x}_i, \theta_i)\}$ be a time-dependent state for a-particle system. Then, the ensemble $\mathcal{P}$ exhibits a global (asymptotic) flocking if the following three conditions hold:

(1) Activity variables tend to zero asymptotically:

$$\lim_{t\to\infty} \max_{1\le i,j\le N} |u_i(t) - u_j(t)| = 0.$$

(2) Relative positions are uniformly bounded:

$$\sup_{0\le t<\infty} \max_{1\le i,j\le N} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| < \infty.$$

(3) Relative heading angles tend to zero asymptotically:

$$\lim_{t\to\infty} \max_{1\le i,j\le N} |\theta_i(t) - \theta_j(t)| = 0.$$

Remark 3.1. For the generalized state $\tilde{\mathcal{P}} := \{(u_i, \boldsymbol{x}_i, \theta_i, \sigma_i)\}$, a global flocking can be defined similarly by adding an extra condition:

$$\lim_{t\to\infty} \max_{1\le i,j\le N} |\sigma_i(t) - \sigma_j(t)| = 0.$$

By the definition of the dynamics of activity particles $\boldsymbol{x}_i$, the relative positions can be controlled by the relative heading angles, which makes flocking estimates more simpler.

Lemma 3.1. *Let* $\{(u_i, \boldsymbol{x}_i, \theta_i)\}$ *be a solution to the first-order a-particle system* (2.2). *Then, one has*

$$\frac{d}{dt}\mathcal{D}(\boldsymbol{x}(t)) \le \sqrt{2}\mathcal{D}(\Theta(t)), \quad a.e.\ t \ge 0,$$

where $\mathcal{D}(\boldsymbol{x}(t)) := \max_{1\le i,j\le N} |\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)|$.

Proof. By the mean-value theorem, one has

$$|\cos\theta_i - \cos\theta_j| \le |\theta_i - \theta_j|$$

and

$$|\sin\theta_i - \sin\theta_j| \le |\theta_i - \theta_j|.$$

This yields

$$\left|\frac{d}{dt}(\boldsymbol{x}_i - \boldsymbol{x}_j)\right| = |(\cos\theta_i - \cos\theta_j, \sin\theta_i - \sin\theta_j)| \le \sqrt{2}|\theta_i - \theta_j|.$$

Hence, one has

$$\left|\frac{d}{dt}\mathcal{D}(\boldsymbol{x})\right| \le \sqrt{2}\mathcal{D}(\Theta). \qquad \square$$

Remark 3.2. Lemma 3.1 implies that the exponential decay of $\mathcal{D}(\Theta)$ guarantees the uniform boundedness of $D(\boldsymbol{x})$, which satisfies the second condition for flocking.

On the other hand, note that the third equation of (2.2) can be rewritten as follows:

$$\begin{aligned}
\frac{d\theta_i}{dt} &= \gamma\left(u_i\theta^\nu + \frac{1-u_i}{\sum_{k\in\Omega_i}\phi_{ik}(\boldsymbol{x})}\sum_{j\in\Omega_i}\phi_{ij}(\boldsymbol{x})\theta_j - \theta_i\right)\\
&= \gamma\left(u_i\theta^\nu + \frac{1-u_i}{\sum_{k\in\Omega_i}\phi_{ik}(\boldsymbol{x})}\sum_{j\in\Omega_i}\phi_{ij}(\boldsymbol{x})(\tilde\theta_j + \theta^\nu) - (\tilde\theta_i + \theta^\nu)\right)\\
&= \gamma\Bigg(u_i\theta^\nu + (1-u_i)\theta^\nu + \frac{1-u_i}{\sum_{k\in\Omega_i}\phi_{ik}(\boldsymbol{x})}\\
&\qquad \times\sum_{j\in\Omega_i}\phi_{ij}(\boldsymbol{x})\tilde\theta_j - u_i\tilde\theta_i + (1-u_i)\tilde\theta_i - \theta^\nu\Bigg)\\
&= \gamma\left(-u_i\tilde\theta_i + \frac{1-u_i}{\sum_{k\in\Omega_i}\phi_{ik}(\boldsymbol{x})}\sum_{j\in\Omega_i}\phi_{ij}(\boldsymbol{x})(\tilde\theta_j - \tilde\theta_i)\right).
\end{aligned}$$

Motivated by this, we set

$$\tilde\theta_i := \theta_i - \theta^\nu, \quad i \in [N].$$

Then, system (2.2) becomes

$$\begin{cases}
\dfrac{du_i}{dt} = \beta\displaystyle\sum_{j\in\Omega_i}\psi_{ij}(u_j - u_i),\\
\dfrac{d\boldsymbol{x}_i}{dt} = (\cos\theta_i, \sin\theta_i),\\
\dfrac{d\tilde\theta_i}{dt} = \gamma\left(-u_i\tilde\theta_i + \dfrac{(1-u_i)}{\sum_{k\in\Omega_i}\phi_{ik}(\boldsymbol{x})}\displaystyle\sum_{j\in\Omega_i}\phi_{ij}(\boldsymbol{x})(\tilde\theta_j - \tilde\theta_i)\right),
\end{cases} \tag{3.2}$$

and this form of the system is useful for the easy computation in flocking analysis. So, we will usually deal with this form.

3.2. *All-to-all interactions*

In this section, we consider system (3.2) in the case of all-to-all coupling (3.1):

$$\begin{cases} \dfrac{du_i}{dt} = \dfrac{\tilde{\beta}}{N}\displaystyle\sum_{j=1}^{N}\psi_{ij}(u_j - u_i), \\ \dfrac{d\boldsymbol{x}_i}{dt} = (\cos\theta_i, \sin\theta_i), \\ \dfrac{d\tilde{\theta}_i}{dt} = \gamma\left(-u_i\,\tilde{\theta}_i + \dfrac{1-u_i}{\sum_{k=1}^{N}\phi_{ik}(\boldsymbol{x})}\displaystyle\sum_{j=1}^{N}\phi_{ij}(\boldsymbol{x})(\tilde{\theta}_j - \tilde{\theta}_i)\right), \end{cases} \tag{3.3}$$

where $\beta = \frac{\tilde{\beta}}{N}$. One of the benefits for all-to-all coupling is that the whole activity variables make the consensus, exponentially fast, at the value which can be pre-computed at the beginning.

Lemma 3.2. *Let* $\{(u_i, \boldsymbol{x}_i, \theta_i)\}$ *be a solution to* (3.3). *Then, one has*

$$\sum_{i=1}^{N} u_i(t) = \sum_{i=1}^{N} u_i^0, \quad t \geq 0.$$

Proof. The sum of $(3.3)_1$ overall i yields

$$\frac{d}{dt}\sum_{i=1}^{N} u_i = \frac{\tilde{\beta}}{N}\sum_{i,j=1}^{N}\psi_{ij}(u_j - u_i) = -\frac{\tilde{\beta}}{N}\sum_{i,j=1}^{N}\psi_{ij}(u_j - u_i) = 0,$$

where we used the skew-symmetry of $\psi_{ij}(u_j - u_i)$ under the index exchange transformation $(i, j) \leftrightarrow (j, i)$. □

Remark 3.3. If all u_i tend to the same constant u_∞, then u_∞ must be equal to $\frac{1}{N}\sum_{i=1}^{N} u_i^0$.

Now, we are ready to study the consensus of activity variables using a diameter functional.

Proposition 3.1. *Suppose that* ψ *has a positive lower bound*

$$\inf_{0\leq r<\infty}\psi(r) \geq \psi_* > 0, \tag{3.4}$$

and let $\{(u_i, \boldsymbol{x}_i, \tilde{\theta}_i)\}$ *be a solution to* (3.3). *Then, one has*

$$\mathcal{D}(U(t)) \leq \mathcal{D}(U^0)e^{-\tilde{\beta}\psi_* t}, \quad t \geq 0,$$

where $\mathcal{D}(U^0) := \mathcal{D}(U(0))$.

Proof. Note that since the right-hand side of $(3.3)_1$ is analytic, u_i is also analytic. So for each pair (i, j), the zero set of $u_i - u_j$ must be finite in any finite-time interval. Therefore, for given $t \in (0, \infty)$, we can decompose the time interval $[0, t)$ into a finite union of subintervals $\cup_{i=1}^{n}[t_{i-1}, t_i)$ such that

$$0 = t_0 < t_1 < \cdots < t_n = t$$

and extremal indices M_t and m_t are constant on each subinterval $[t_{i-1}, t_i)$.

For each subinterval $[t_{i-1}, t_i)$, it follows from $(3.3)_1$ that for $t \in [t_{i-1}, t_i)$,

$$\frac{du_{M_1}}{dt} \leq \frac{\tilde{\beta}}{N}\psi(\mathcal{D}(\boldsymbol{x}))\sum_{j=1}^{N}(u_j - u_{M_1}), \quad \frac{du_{m_1}}{dt} \geq \frac{\tilde{\beta}}{N}\psi(D(\boldsymbol{x}))\sum_{j=1}^{N}(u_j - u_{m_1}).$$

This yields

$$\frac{d}{dt}\mathcal{D}(U) \leq -\tilde{\beta}\psi(\mathcal{D}(\boldsymbol{x}))\mathcal{D}(U) \leq -\tilde{\beta}\psi_*\mathcal{D}(U), \quad t \in [t_{i-1}, t_i).$$

Then, we use the continuity of $\mathcal{D}(U)$ across the time $t = t_i$ and we apply Gronwall's lemma in each subinterval $[t_{i-1}, t_i)$ to derive the desired exponential convergence to zero. □

Remark 3.4. The result of this lemma yields the exponential convergence of u_i to $u_c^0 := \frac{1}{N}\sum_{i=1}^{N} u_i(0)$ and the condition (3.4) for ψ can be relaxed to include zero in the range of ψ.

Now, we consider the asymptotic behavior of activity particles in terms of preferred direction.

Theorem 3.1. *Let* $\{(u_i, \boldsymbol{x}_i, \theta_i)\}$ *be a solution to* (3.3). *Then, the distances between directions and preferred direction have exponential decay estimate*:

$$|\theta_i(t) - \theta^\nu| = |\tilde{\theta}_i| \leq |\tilde{\theta}_{M_{3.0}}(0)|e^{-\gamma \underline{u} t},$$

where $\underline{u}$ *is a positive constant defined by*

$$\underline{u} := \min_{1 \leq i \leq N} u_i(0).$$

Moreover, if $\underline{u} > 0$, *every activity particle keeps in some boundary of the particle which only follows the preferred direction, i.e.*

$$\|\boldsymbol{x}_i(t) - \boldsymbol{x}_i^0 - \boldsymbol{v}^\nu t\| \leq \frac{\sqrt{2}|\tilde{\theta}_{M_{3,0}}(0)|}{\gamma \underline{u}},$$

where $\boldsymbol{v}^\nu := (\cos\theta^\nu, \sin\theta^\nu)$.

Proof. If follows from (3.3) that

$$\frac{d\tilde{\theta}_{M_3}}{dt} = -\gamma u_{M_3}\tilde{\theta}_{M_3} + \gamma(1 - u_{M_3})\sum_{j=1}^{N}\frac{\phi_{M_3 j}(\boldsymbol{x})}{\sum_{k=1}^{N}\phi_{M_3 k}(\boldsymbol{x})}(\tilde{\theta}_j - \tilde{\theta}_{M_3}). \tag{3.5}$$

We multiply $\mathrm{sgn}(\tilde{\theta}_{M_3})$ to the above equation to find

$$\frac{d|\tilde{\theta}_{M_3}|}{dt} \leq -\gamma u_{M_3}|\tilde{\theta}_{M_3}|, \tag{3.6}$$

where we used the relation

$$(\tilde{\theta}_j - \tilde{\theta}_{M_3})\mathrm{sgn}(\tilde{\theta}_{M_3}) \leq 0, \quad \forall j \in [N].$$

Since $\mathrm{sgn}(\tilde{\theta}_{M_3})$ is discontinuous at the instant in which $\tilde{\theta}_{M_3} = 0$, the derivation of (3.6) from (3.5) needs a justification. In fact, this can be done rigorously using the smooth approximations of $\mathrm{sgn}(\tilde{\theta}_{M_3})$ following the modification procedure (see Ref. 17 for detailed arguments). This yields

$$|\tilde{\theta}_{M_3}(t)| \leq |\tilde{\theta}_{M_3^0}(0)|e^{-\gamma \underline{u} t}, \quad t \geq 0,$$

which implies the first assertion. Next, one has

$$\begin{aligned} \|\boldsymbol{v}_i(t) - \boldsymbol{v}^\nu\| &\leq |\cos\theta_i - \cos\theta^\nu| + |\sin\theta_i - \sin\theta^\nu| \\ &\leq \sqrt{2}|\theta_i(t) - \theta^\nu| \leq \sqrt{2}|\tilde{\theta}_{M_3^0}(0)|e^{-\gamma\underline{u}t}. \end{aligned}$$

Therefore, one has

$$\begin{aligned} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_i^0 - \boldsymbol{v}^\nu t\| &= \left|\int_0^t \left(\frac{d\boldsymbol{x}_i(s)}{ds} - \boldsymbol{v}^\nu\right) ds\right| \leq \int_0^t \|\boldsymbol{v}_i(s) - \boldsymbol{v}^\nu\| ds \\ &\leq \sqrt{2}|\tilde{\theta}_{M_3^0}(0)| \int_0^t e^{-\gamma\underline{u}s} ds = \frac{\sqrt{2}|\tilde{\theta}_{M_3^0}(0)|}{\gamma\underline{u}}(1 - e^{-\gamma\underline{u}t}), \end{aligned}$$

which shows the second assertion. □

Remark 3.5. Theorem 3.1 guarantees the exponential decay of $D(\Theta)$ and uniform boundedness of $D(\boldsymbol{x})$. With Proposition 3.1, we can have the flocking estimate.

Although we already have the flocking estimate for (3.3), we would like to show the improved decay rate of $D(\Theta)$ for the special case, *constant activity variables*. If the activity variables are the same constant, of which situation is almost like when the consensus of them is reached, the decay rate more speeds up.

Consider constant activity variables, i.e. $u_i \equiv u$ for all i. This is actually just the case of (3.3) with the specific initial data $u_i(0) = u_j(0)$ for all $i \neq j$. The system (3.3) can be simplified as follows:

$$\begin{cases} \dfrac{d\boldsymbol{x}_i}{dt} = (\cos\theta_i, \sin\theta_i), \\ \dfrac{d\tilde{\theta}_i}{dt} = \gamma\left(-u\tilde{\theta}_i + (1-u)\displaystyle\sum_{j=1}^N \frac{\phi_{ij}(\boldsymbol{x})}{\sum_{k=1}^N \phi_{ik}(\boldsymbol{x})}(\tilde{\theta}_j - \tilde{\theta}_i)\right). \end{cases} \tag{3.7}$$

Theorem 3.2. *Suppose ϕ has an upper bound ϕ_∞ and let $\{\boldsymbol{x}_i, \tilde{\theta}_i\}$ be a solution to* (3.7). *Then, the following flocking estimates hold*:

$$\text{(i)} \qquad \sup_{0\le t<\infty} \mathcal{D}(\boldsymbol{x}(t)) \le \mathcal{D}^\infty \quad \textit{for some constant } \mathcal{D}^\infty > 0,$$

$$\text{(ii)} \qquad \mathcal{D}(\tilde{\Theta}(t)) \le \mathcal{D}(\tilde{\Theta}^0)\exp\left[-\left(\gamma u + \frac{\gamma(1-u)}{\phi_\infty}\phi(\mathcal{D}^\infty)\right)t\right], \quad t \ge 0.$$

Proof. Using the relation

$$\sum_{j=1}^N \frac{\phi_{ij}(\boldsymbol{x})}{\sum_{k=1}^N \phi_{ik}(\boldsymbol{x})} \ge \frac{\phi(\mathcal{D}(\boldsymbol{x}))}{N\phi_\infty}$$

one can derive

$$\frac{d}{dt}\mathcal{D}(\tilde{\Theta}(t)) \le -\gamma u \mathcal{D}(\tilde{\Theta}(t)) - \gamma(1-u)\frac{\phi(\mathcal{D}(\boldsymbol{x}))}{\phi_\infty}\mathcal{D}(\tilde{\Theta}(t)). \tag{3.8}$$

By ignoring the last negative term in (3.8) temporarily, we know that $\mathcal{D}(\Theta)$ decreases exponentially fast by Gronwall's lemma, which means there exists an upper bound $\mathcal{D}^\infty$ of $\mathcal{D}(\boldsymbol{x})$. Then, putting $\mathcal{D}^\infty$ instead of $\mathcal{D}(\boldsymbol{x})$ in (3.8) yields the desired result. □

3.3. *Networks with topological distance*

In this section, we consider the first-order model where a-particles undergo the so-called *topological interactions* which, as we have seen in Sec. 2.4, are taken into account by the terms ϕ, ϕ_{ij} that can be defined as *network functions*. These denote the quantity of influence from the particle j to i within the sensitivity domain Ω_s. The model is the so-called *the closest neighbors model*, following the convention of the name in Ref. 18, where every particle is affected only by a fixed number q closest neighbors. In terms of differential equations, it can be seen taking the advantage of the meaning of $\Omega_i = \Omega_i^s$. Here, $\Omega_i = \Omega_i(t)$ depends on time and is chosen to contain only the closest q particles. (Ties are dealt by selecting the particles with smaller index.) That is, $j \in \Omega_i(t)$ if and only if

$$\#\{l \ne i, j : \|\boldsymbol{x}_i - \boldsymbol{x}_l\| < \|\boldsymbol{x}_i - \boldsymbol{x}_j\|\} + \#\{i \ne l < j : \|\boldsymbol{x}_i - \boldsymbol{x}_l\| = \|\boldsymbol{x}_i - \boldsymbol{x}_j\|\} < q.$$

We assume $i \in \Omega_i$. For a technical reason, we only consider the case of constant activity variables, i.e. $u_i \equiv u$ with constant weight function ϕ.

Consider system (2.5) with $u_i \equiv u \in [0,1)$ for all i. Then, (2.5) is simplified as follows:

$$\begin{cases} \dfrac{d\boldsymbol{x}_i}{dt} = (\cos\theta_i, \sin\theta_i), \\[2ex] \dfrac{d\theta_i}{dt} = \gamma\left(u\theta^\nu + (1-u)\displaystyle\sum_{j\in\Omega_i}(\theta_j - \theta_i)\right). \end{cases} \tag{3.9}$$

In order to use the digraph theory, we adopt a matrix representation for Ω_i:

$$a_{ij} = \begin{cases} 1 & \text{if} \quad j \in \Omega_i, \\ 0 & \text{otherwise.} \end{cases}$$

Now, the system can be rewritten as

$$\begin{cases} \dfrac{d\boldsymbol{x}_i}{dt} = (\cos\theta_i, \sin\theta_i), \\ \dfrac{d\theta_i}{dt} = \gamma\left(u\theta^\nu + (1-u)\displaystyle\sum_{j=1}^N a_{ij}(\theta_j - \theta_i)\right). \end{cases} \tag{3.10}$$

Throughout this section, the representation Ω_i and a_{ij} will be used interchangeably. When we deal with the topological interactions, it is useful to use some linear algebra knowledge related to graph. Here, we recall some definitions related to graph and provide the useful lemma.

Definition 3.2. Let $A = (a_{ij})_{N\times N}$ be a non-negative matrix.

(1) The matrix A is *scrambling* if for any pair i, j of indexes there exists l such that $a_{il}, a_{jl} > 0$.
(2) The *ergodicity coefficient* of A is

$$\mu(A) := \min_{i,j} \sum_{l=1}^N \min\{a_{il}, a_{jl}\}.$$

Remark 3.6. (1) A matrix A is scrambling if and only if $\mu(A) > 0$.
(2) If A is scrambling, $\mu(A) \geq \min\{a_{ij} : a_{ij} > 0\}$.

Lemma 3.3. (Ref. 18) *Let $A = (a_{ij})_{N\times N}$ be a non-negative matrix such that each row sum is equal to n. For any vector $\boldsymbol{v} = (v_1, \ldots, v_N) \in \mathbb{R}^N$, we set $w = (w_1, \ldots, w_N) \in \mathbb{R}^n$ such that $x = Av$. Then, we have*

$$\max_{i,j} |w_i - w_j| \leq (n - \mu(A)) \max_{l,m} |v_l - v_m|.$$

The key idea to prove our result is based on Gronwall's lemma. Hence, we start by obtaining the needed relation.

Lemma 3.4. *The following estimates hold: for a.e. $t \geq 0$,*

$$\frac{d}{dt}\|\boldsymbol{x}_i - \boldsymbol{x}_j\| \leq 2\left|\sin\left(\frac{\theta_i - \theta_j}{2}\right)\right|, \quad \frac{d}{dt}\mathcal{D}(\Theta) \leq -\gamma(1-u)\mu(t)\mathcal{D}(\Theta),$$

where $\mu(t) := \mu(A(t))$ represents the ergodicity coefficient of $A(t)$.

Proof. Consider the first inequality, for almost every t,

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^2 &= 2\langle \boldsymbol{x}_i-\boldsymbol{x}_j, (\cos\theta_i-\cos\theta_j, \sin\theta_i-\sin\theta_j)\rangle \\ &= 2\|\boldsymbol{x}_i-\boldsymbol{x}_j\|\sqrt{2-2\cos(\theta_i-\theta_j)} = 4\|\boldsymbol{x}_i-\boldsymbol{x}_j\|\cdot\left|\sin\left(\frac{\theta_i-\theta_j}{2}\right)\right|\end{aligned}$$

and

$$\frac{d}{dt}\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^2 = 2\|\boldsymbol{x}_i-\boldsymbol{x}_j\|\frac{d}{dt}\|\boldsymbol{x}_i-\boldsymbol{x}_j\|.$$

Hence, one has

$$\frac{d}{dt}\|\boldsymbol{x}_i-\boldsymbol{x}_j\| \le 2\left|\sin\left(\frac{\theta_i-\theta_j}{2}\right)\right|, \quad \text{a.e. } t\ge 0.$$

Next, we return to the second inequality.

$$\begin{aligned}\frac{d}{dt}|\theta_i-\theta_j|^2 &= 2\langle \theta_i-\theta_j, \dot\theta_i-\dot\theta_j\rangle \\ &= 2\left\langle \theta_i-\theta_j, \gamma(1-u)\left(\sum_{l=1}^N a_{il}(\theta_l-\theta_i) - \sum_{l=1}^N a_{jl}(\theta_l-\theta_j)\right)\right\rangle \\ &= 2\gamma(1-u)\left\langle \theta_i-\theta_j, \left(\sum_{l=1}^N a_{il}\theta_l - \sum_{l=1}^N a_{jl}\theta_l\right) - (q+1)(\theta_i-\theta_j)\right\rangle \\ &\le 2\gamma(1-u)\left|\sum_{l=1}^N a_{il}\theta_l - \sum_{l=1}^N a_{jl}\theta_l\right||\theta_i-\theta_j| - 2\gamma(1-u)(q+1)|\theta_i-\theta_j|^2 \\ &= 2\gamma(1-u)((q+1)-\mu(t))\mathcal{D}(\Theta)|\theta_i-\theta_j| - 2\gamma(1-u)(q+1)|\theta_i-\theta_j|^2,\end{aligned}$$

where we used Lemma 3.3 in the last equality. Since the above relation holds for any i, j, one can derive

$$\begin{aligned}\frac{d}{dt}\mathcal{D}(\Theta)^2 &\le 2\gamma(1-u)((q+1)-\mu(t))\mathcal{D}(\Theta)^2 - 2\gamma(1-u)(q+1)\mathcal{D}(\Theta)^2 \\ &= -2\gamma(1-u)\mu(t)\mathcal{D}(\Theta)^2.\end{aligned}$$

This yields the desired estimate. □

As one can easily expect, the dynamics of system (3.10) heavily depends on fixed number q, which denote the interaction particle number. So, we will show the dynamics result into the two cases, one for more than half and one for less than half.

- **Case A:** $q \ge \frac{1}{2}(N-1)$. First, we consider the case in which the number of particles interacting with each particle is more than half. In this case, since all particles are organically related, we can easily expect for them to exhibit a global asymptotic flocking. Actually, there is no required condition for the asymptotic flocking. The following theorem tells it.

Theorem 3.3. *Assume $u \in [0,1)$ and $q \geq \frac{1}{2}(N-1)$. Then, the relative heading angle goes to zero exponentially fast, and the relative positions are uniformly bounded.*

Proof. Note that $A(t)$ is scrambling for all $t \geq 0$ from $q \geq \frac{1}{2}(N-1)$. So, we have

$$\mu(A) \geq \min\{a_{ij} : a_{ij} > 0\} = 1.$$

From this, we obtain

$$\mathcal{D}(\Theta(t)) \leq \mathcal{D}(\Theta(0))e^{-\gamma(1-u)\int_0^t \mu(s)ds} \leq \mathcal{D}(\Theta(0))e^{-\gamma(1-u)t}, \quad t \geq 0,$$

where the first inequality comes from Lemma 3.4. By Remark 3.2, we have the uniform boundedness of spatial diameter. □

- **Case B:** $q < \frac{1}{2}(N-1)$. In this case, we use the notion of *disturbed subgraph* to prove our goal (Theorem 3.4). The following definitions and notions follow the convention in Ref. 18. Recall that a *digraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V} = \{1, \ldots, k\}$ of vertices and a set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of *arcs*.

Definition 3.3. For a given $\rho > 0$, the disturbed subgraph $\mathcal{H}_\rho$ of the initial digraph $\mathcal{G}(0)$ is defined as $\mathcal{H}_\rho := \mathcal{G}(A_\rho) = (\mathcal{V}, \mathcal{E}_\rho)$ where $A_\rho = (\tilde{a}_{ij})$ to satisfy the followings:

$$\begin{aligned} &(1) \qquad \text{for all } i \leq k, \quad \tilde{a}_{ii} = 1 \quad \text{and} \\ &(2) \qquad \text{for } i \neq j, \quad \tilde{a}_{ij} = 1 \quad \text{if and only if} \\ &\qquad\quad \#\{l \neq j, i : \|\boldsymbol{x}_i(0) - \boldsymbol{x}_l(0)\| < \|\boldsymbol{x}_i(0) - \boldsymbol{x}_j(0)\| + 2\rho\} < q \\ &\qquad\quad \text{and} \quad \tilde{a}_{ij} = 0 \quad \text{otherwise.} \end{aligned}$$

If $\mathcal{H}_\rho$ is a rooted digraph, d_ρ and $\mathcal{R}_\rho$ are defined to be its smallest depth and the set of the roots r of $\mathcal{H}_\rho$ such that $d_r = d_\rho$, respectively. Also, let $\mathbf{r}_\rho$ to be the cardinality of $\mathcal{R}_\rho$ and for each $\alpha > 0$,

$$\mathbf{a}_{\rho\alpha} := \frac{\mathbf{r}_\rho}{e^{(q+1)\alpha}}\left(e^\alpha - \sum_{n=1}^{d_\rho - 1}\frac{\alpha^n}{n!}\right) \quad \text{and} \quad \bar{\mathbf{a}} := \gamma(1-u)\sup_{\rho>0}\sup_{\alpha>0}\frac{\mathbf{a}_{\rho\alpha}\cdot\rho}{\alpha}\cdot$$

Here, we provide a technical lemma which will be used crucially to show the exponential decay of $\mathcal{D}(\Theta)$. This lemma regards θ_i as the velocity of some variable, which we will denote by χ_i, and derive the partial conservation of graph depending on the relations between χ_i. For $i \in [N]$, we define

$$\chi_i(t) := \int_0^t \theta_i(s)ds \quad \text{with } \chi_i(0) = 0.$$

Then, we can consider the system (3.10) as follows:

$$\begin{cases} \dfrac{d\boldsymbol{x}_i}{dt} = (\cos\theta_i, \sin\theta_i), \\[2mm] \dfrac{d\chi_i}{dt} = \theta_i, \\[2mm] \dfrac{d\theta_i}{dt} = \gamma\left(u\theta^\nu + (1-u)\displaystyle\sum_{j=1}^N a_{ij}(\theta_j - \theta_i)\right). \end{cases} \tag{3.11}$$

Note that the emergence of variable χ_i has no effect on our original system (3.10). It is just introduced to use the following lemma.

Lemma 3.5. (Ref. 18) *Let* $\{\boldsymbol{x}_i, \chi_i, \theta_i\}$ *be a solution to system* (3.11) *such that for some* $t > 0$ *and for any* $1 \le i, j \le N$,

$$\| \chi_i(t) - \chi_j(t)| - |\chi_i^0 - \chi_j^0 \| < \rho.$$

Then, $\mathcal{H}_\rho \subseteq \mathcal{G}(A(t))$, *that is,* $A_\rho \le A(t)$.

We can obtain this result by modifying the proof of Lemma 3.2 in Ref. 18, so we omit details. Now, we are ready to state and prove our result.

Theorem 3.4. *Assume* $u \in [0, 1)$, $q < \frac{1}{2}(N-1)$ *and* $\mathcal{D}(\Theta^0) < \bar{\mathbf{a}}$. *Then, the relative heading angles go to zero exponentially fast and the relative positions are uniformly bounded.*

Proof. Since $\mathcal{D}(\Theta^0) < \bar{\mathbf{a}}$, there exist $\rho, \alpha > 0$ such that

$$\mathcal{D}(\Theta^0) \le \gamma(1-u) \cdot \frac{\mathbf{a}_{\rho\alpha} \cdot \rho}{\alpha}. \tag{3.12}$$

From this, we get $\mathbf{a}_{\rho\alpha} > 0$ and $\mathcal{H}_\rho$ is rooted by the definition of $\mathbf{a}_{\rho\alpha}$. Now, we claim the following relation:

$$\mathcal{D}(\Theta(\tilde{\alpha}t)) \le (1 - \mathbf{a}_{\rho\alpha})^t \mathcal{D}(\Theta^0),$$

where $\tilde{\alpha} = \frac{\alpha}{\gamma(1-u)}$. Obviously, it holds for $t = 0$. Assume that it holds for all $0 \le t \le T$ for some $T \ge 0$. Since $\mathcal{D}(\Theta)$ is non-increasing by Lemma 3.4, one can derive that for any $t \in [0, T]$,

$$\mathcal{D}(\Theta(\tau)) \le (1 - \mathbf{a}_{\rho\alpha})^t \mathcal{D}(\Theta^0) \quad \text{for all } \tau \in [\tilde{\alpha}t, \tilde{\alpha}(t+1)].$$

For any $(j, i) \in \mathcal{E}_\rho$ and $\tilde{\alpha}T \le t \le \tilde{\alpha}(T+1)$,

$$\begin{aligned} \| \chi_i(t) - \chi_j(t)| - |\chi_i^0 - \chi_j^0 \| &\le \int_0^t |\theta_i(s) - \theta_j(s)| ds \le \sum_{\tau=0}^T \int_{\tilde{\alpha}\tau}^{\tilde{\alpha}(\tau+1)} \mathcal{D}(\Theta(s)) ds \\ &\le \tilde{\alpha}\mathcal{D}(\Theta^0) \sum_{\tau=0}^T (1 - \mathbf{a}_{\rho\alpha})^\tau \le \frac{\tilde{\alpha}\mathcal{D}(\Theta^0)}{\mathbf{a}_{\rho\alpha}} \le \rho, \end{aligned}$$

where we used (3.12) in the last inequality.

By Lemma 3.5, we can induce

$$A_\rho \le A(t) \quad \text{for all } \tilde{\alpha}T \le t \le \tilde{\alpha}(T+1).$$

Before we move on further, we use change of the variables for the compact form of system (3.10). Denote

$$\tilde{\theta}_i(t) = \theta_i(t) - \gamma u \theta^\nu t, \quad \forall\, i \in [N].$$

Note that the definition of $\tilde{\theta}_i$ is different from the one defined in Sec. 3.2 (all-to-all coupling). Since it will be used only here, we expect there will be no confusion. In addition, note that system $(3.10)_3$ can be rewritten as

$$\dot{\tilde{\Theta}}(t) = \gamma(1-u)(A(t) - (q+1)I)\tilde{\Theta}(t), \quad \forall\, t \ge 0,$$

where $\tilde{\Theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_N)^T$ and I is an $N \times N$ identity matrix. Then, by a similar argument in the proof of Theorem 1.1 in Ref. 18 and with the fact $\mathcal{D}(\tilde{\Theta}) = \mathcal{D}(\Theta)$, we get

$$\mathcal{D}(\Theta(\tilde{\alpha}(T+1))) \le (1 - \mathbf{a}_{\rho\alpha})\mathcal{D}(\Theta(\tilde{\alpha}T)) \le (1 - \mathbf{a}_{\rho\alpha})^{T+1}\mathcal{D}(\Theta^0).$$

By the mathematical induction, this leads to

$$\mathcal{D}(\Theta(t)) \le (1 - \mathbf{a}_{\rho\alpha})^{\lfloor \frac{t}{\tilde{\alpha}} \rfloor}\mathcal{D}(\Theta^0), \quad t \ge 0.$$

Since $\mathcal{D}(\Theta)$ goes to zero exponentially fast, we obtain the desired results. □

4. Emergent Dynamics of the Second-Order Flocking Model

In this section, we study emergent dynamics of the second-order active model:

$$\begin{cases} \dfrac{du_i}{dt} = \beta \displaystyle\sum_{j=1}^N \psi_{ij}(u_j - u_i), \\ \dfrac{d\boldsymbol{x}_i}{dt} = (\cos(\tilde{\theta}_i + \theta^\nu), \sin(\tilde{\theta}_i + \theta^\nu)), \\ \dfrac{d\tilde{\theta}_i}{dt} = \sigma_i, \\ \dfrac{d\sigma_i}{dt} = \gamma_a \left(-u_i\tilde{\theta}_i + (1-u_i)\displaystyle\sum_{j=1}^N \frac{\phi_{ij}(\boldsymbol{x})}{\sum_{k=1}^N \phi_{ik}(\boldsymbol{x})}(\tilde{\theta}_j - \tilde{\theta}_i) \right) - \gamma_b\sigma_i, \end{cases} \tag{4.1}$$

where we used the relation in (2.4) and $\tilde{\theta}_i = \theta_i - \theta^\nu$. In what follows, we study the emergent dynamics of (4.1) with the constant activity variables, i.e. $u_i \equiv u$ for all

i, and the constant weight function $\phi \equiv 1$. Then, the system becomes as follows:

$$\begin{cases} \dfrac{d\boldsymbol{x}_i}{dt} = \big(\cos(\tilde{\theta}_i + \theta^\nu), \sin(\tilde{\theta}_i + \theta^\nu)\big), \\ \dfrac{d\tilde{\theta}_i}{dt} = \sigma_i, \\ \dfrac{d\sigma_i}{dt} = \gamma_a \left(-u\tilde{\theta}_i + \dfrac{(1-u)}{N} \displaystyle\sum_{j=1}^N \big(\tilde{\theta}_j - \tilde{\theta}_i\big) \right) - \gamma_b \sigma_i. \end{cases} \tag{4.2}$$

4.1. *Zero activity* ($u = 0$)

In this case, system (4.2) becomes

$$\begin{cases} \dfrac{d\tilde{\theta}_i}{dt} = \sigma_i, \\ \dfrac{d\sigma_i}{dt} = \dfrac{\gamma_a}{N} \displaystyle\sum_{j=1}^N (\tilde{\theta}_j - \tilde{\theta}_i) - \gamma_b \sigma_i, \end{cases} \tag{4.3}$$

where we omit the first equation in (4.2) since it does not give any effect to the dynamical system.

For the flocking estimate of (4.3), we introduce macro and micro components corresponding to the averaged values and their fluctuations around average values:

$$\tilde{\theta}_c := \frac{1}{N} \sum_i \tilde{\theta}_i, \quad \sigma_c := \frac{1}{N} \sum_i \sigma_i.$$

Note that

$$\tilde{\theta}_c = \theta_c - \theta^\nu, \quad \text{where } \theta_c := \frac{1}{N} \sum_i \theta_i.$$

So, we have

$$\hat{\theta}_i := \theta_i - \theta_c = \tilde{\theta}_i - \tilde{\theta}_c, \quad \hat{\sigma}_i := \sigma_i - \sigma_c.$$

Then, one has

$$\frac{d}{dt}\tilde{\theta}_c = \sigma_c, \quad \frac{d}{dt}\sigma_c = -\gamma_b \sigma_c, \quad t > 0 \tag{4.4}$$

and

$$\begin{cases} \dfrac{d\hat{\theta}_i}{dt} = \hat{\sigma}_i, \quad t > 0, \\ \dfrac{d\hat{\sigma}_i}{dt} = -\gamma_a \hat{\theta}_i - \gamma_b \hat{\sigma}_i. \end{cases} \tag{4.5}$$

Lemma 4.1. *Let* $\{(\tilde{\theta}_i, \sigma_i)\}$ *be a solution to* (4.3). *Then, macro and micro variables satisfy the following assertions*:

(1) *The macro variables* $(\sigma_c, \tilde{\theta}_c)$ *satisfy*

$$\sigma_c(t) = e^{-\gamma_b t}\sigma_c^0, \quad \tilde{\theta}_c(t) = \tilde{\theta}_c^0 + \frac{\sigma_c^0}{\gamma_b}(1 - e^{-\gamma_b t}), \quad t \geq 0.$$

(2) *The micro variables* $(\hat{\sigma}_i, \hat{\theta}_i)$ *satisfy*

$$\hat{\sigma}_i(t) = \mathcal{O}(1)e^{-|\lambda_{1,+}|t}, \quad \hat{\theta}_i(t) = \mathcal{O}(1)e^{-|\lambda_{1,+}|t},$$

where

$$\lambda_{1,-} := \frac{-\gamma_b - \sqrt{\gamma_b^2 - 4\gamma_a}}{2}, \quad \lambda_{1,+} := \frac{-\gamma_b + \sqrt{\gamma_b^2 - 4\gamma_a}}{2}.$$

Proof. (1) It follows from $(4.4)_2$ that

$$\sigma_c(t) = e^{-\gamma_b t}\sigma_c^0, \quad t \geq 0. \tag{4.6}$$

Now, we use $(4.4)_1$ and (4.6) to get

$$\tilde{\theta}_c(t) - \tilde{\theta}_c^0 = \int_0^t e^{-\gamma_b s}\sigma_c^0 ds = \frac{\sigma_c^0}{\gamma_b}(1 - e^{-\gamma_b t}).$$

(2) From (4.5), we see that $\hat{\theta}_i$ satisfies

$$\frac{d^2\hat{\theta}_i}{dt^2} + \gamma_b\frac{d\hat{\theta}_i}{dt} + \gamma_a\hat{\theta}_i = 0.$$

We set

$$\lambda_{1,-} := \frac{-\gamma_b - \sqrt{\gamma_b^2 - 4\gamma_a}}{2}, \quad \lambda_{1,+} := \frac{-\gamma_b + \sqrt{\gamma_b^2 - 4\gamma_a}}{2}. \tag{4.7}$$

Depending on the relative sizes between γ_a and γ_b, we have the following three cases:

- **Case A.1 ($\gamma_b^2 > 4\gamma_a$).** In this case, $\lambda_{1,-}$ and $\lambda_{1,+}$ are both negative real numbers and for some constants c_1 and c_2, one has

$$\hat{\theta}_i(t) = c_1 e^{\lambda_{1,-}t} + c_2 e^{\lambda_{1,+}t}, \tag{4.8}$$

which gives

$$|\hat{\theta}_i(t)| \leq \mathcal{O}(1)e^{-|\lambda_{1,+}|t} \to 0 \quad \text{as } t \to \infty.$$

- **Case A.2 ($\gamma_b^2 = 4\gamma_a$).** In this case, we have

$$\hat{\theta}_i(t) = (c_1 + c_2 t)e^{-\frac{\gamma_b}{2}t}. \tag{4.9}$$

On the other hand, for all $0 < \varepsilon \ll \gamma_b$, we have

$$|\hat{\theta}_i(t)| \leq \mathcal{O}(1)(1+t)e^{-\frac{\gamma_b}{2}t} \leq \mathcal{O}(1)e^{-\frac{(\gamma_b - \varepsilon)}{2}t} = \mathcal{O}(1)e^{-|\lambda_{1,-}|} \quad \text{as } t \to \infty.$$

- **Case A.3 ($\gamma_b^2 < 4\gamma_a$).** In this case, we have

$$\hat{\theta}_i(t) = e^{-\frac{\gamma_b}{2}t}\left[c_1 \cos\left(\frac{\sqrt{4\gamma_a - \gamma_b^2}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{4\gamma_a - \gamma_b^2}}{2}t\right)\right]. \tag{4.10}$$

This results in

$$|\hat{\theta}_i(t)| \le \mathcal{O}(1)e^{-\frac{\gamma_b}{2}t} \quad \text{as } t \to \infty.$$

Since

$$e^{-\frac{(\gamma_b-\varepsilon)}{2}}t \ll e^{-|\lambda_{1,+}|t}$$

one has the desired exponential decay of $|\hat{\tilde{\theta}}_i(t)|$. The exponential decay of $\hat{\sigma}_i(t)$ follows from (4.5) together with the explicit formulas (4.8)–(4.10). □

Thanks to Lemma 4.1, one has the following emergent estimates.

Theorem 4.1. *Let $\{(\tilde{\theta}_i, \sigma_i)\}$ be a solution to* (4.3). *Then, there exist positive constants C_1 and Λ_1 such that*

$$\left|\theta_i(t) - \tilde{\theta}_c^0 - \frac{\sigma_c^0}{\gamma_b}\right| + |\sigma_i(t)| \le C_1 e^{-\Lambda_1 t}, \quad t \ge 0.$$

4.2. *Nonzero constant activity* $u \in (0,1)$

$$\frac{d\sigma_i}{dt} = \gamma_a\left(-u\tilde{\theta}_i + \frac{(1-u)}{N}\sum_{j=1}^{N}\left(\tilde{\theta}_j - \tilde{\theta}_i\right)\right) - \gamma_b\sigma_i.$$

As before, we consider the dynamics of macro and micro components. First, note that macro and micro components satisfy

$$\begin{cases} \dfrac{d\tilde{\theta}_c}{dt} = \sigma_c, \quad t > 0, \\[2mm] \dfrac{d\sigma_c}{dt} = -\gamma_a u\tilde{\theta}_c - \gamma_b\sigma_c, \end{cases}$$

and $(\hat{\theta}_i, \hat{\sigma})$ still satisfy the relation (4.5). We set

$$\lambda_{2,-} := \frac{-\gamma_b - \sqrt{\gamma_b^2 - 4\gamma_a u}}{2} \quad \text{and} \quad \lambda_{2,+} := \frac{-\gamma_b + \sqrt{\gamma_b^2 - 4\gamma_a u}}{2}.$$

Lemma 4.2. *Let $\{(\boldsymbol{x}_i, \theta_i, \sigma_i)\}$ be a solution to* (4.2). *Then, the following assertions hold*:

(1) *The macro variables satisfy*

$$|\tilde{\theta}_c(t)| \le \mathcal{O}(1)e^{-|\lambda_{2,+}|t}, \quad |\sigma_c(t)| \le \mathcal{O}(1)e^{-|\lambda_{2,+}|t}, \quad t \ge 0.$$

(2) *The micro variables* $(\hat{\sigma}_i, \hat{\tilde{\theta}}_i)$ *satisfy*

$$\hat{\sigma}_i(t) = \mathcal{O}(1)e^{-|\lambda_{1,+}|t}, \quad \hat{\theta}_i(t) = \mathcal{O}(1)e^{-|\lambda_{1,+}|t},$$

where $\lambda_{1,+}$ *is defined in* (4.7).

Proof. (1) Note that

$$\frac{d^2\tilde{\theta}_c}{dt^2} + \gamma_b \frac{d\tilde{\theta}_c}{dt} + \gamma_a u \tilde{\theta}_c = 0.$$

Let us now observe that, depending on the relative sizes between γ_a and γ_b, we have the following three cases:

- **Case B.1** $(\gamma_b^2 > 4\gamma_a u)$**.** In this case, we have

$$\tilde{\theta}_c(t) = c_1 e^{\lambda_{2,-}t} + c_2 e^{\lambda_{2,+}t}.$$

  Since

$$\lambda_{2,-} < \lambda_{2,+} < 0$$

  one has

$$|\tilde{\theta}_c(t)| \le \mathcal{O}(1)e^{-|\lambda_{2,+}|t} \to 0 \quad \text{as } t \to \infty.$$

- **Case B.2** $(\gamma_b^2 = 4\gamma_a u)$**.** In this case, one has

$$\tilde{\theta}_c(t) = (c_1 + c_2 t)e^{-\frac{\gamma_b}{2}t}.$$

  On the other hand, for all $0 < \varepsilon \ll \gamma_b$, we have

$$|\tilde{\theta}_c(t)| \le \mathcal{O}(1)(1+t)e^{-\frac{\gamma_b}{2}t} \le \mathcal{O}(1)e^{-\frac{(\gamma_b-\varepsilon)}{2}t} \quad \text{as } t \to \infty.$$

- **Case B.3** $(\gamma_b^2 < 4\gamma_a u)$**.** In this case, we have

$$\tilde{\theta}_c(t) = e^{-\frac{\gamma_b}{2}t}\left[c_1 \cos\left(\frac{\sqrt{4\gamma_a u - \gamma_b^2}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{4\gamma_a u - \gamma_b^2}}{2}t\right)\right],$$

  which results in

$$|\tilde{\theta}_c(t)| \le \mathcal{O}(1)e^{-\frac{\gamma_b}{2}t} \quad \text{as } t \to \infty.$$

  Since

$$e^{-\frac{(\gamma_b-\varepsilon)}{2}t} \ll e^{-|\lambda_{2,+}|t}$$

  one has the desired exponential decay of $|\tilde{\theta}_c(t)|$.

  Therefore, we have

$$|\tilde{\theta}_c(t)| \le \mathcal{O}(1)e^{-|\lambda_{2,+}|t} \quad \text{as } t \to \infty.$$

  Finally, by $(4.8)_1$, one has

$$|\sigma_c(t)| \le \mathcal{O}(1)e^{-|\lambda_{2,+}|t}, \quad t \ge 0.$$

(2) The estimates for micro variables are exactly the same as in Lemma 4.1. □

Thanks to Lemma 4.2, one has the following emergent estimates.

Theorem 4.2. *Let* $\{(\tilde{\theta}_i, \sigma_i)\}$ *be a solution to* (4.3). *Then, there exist positive constants* C_2 *and* Λ_2 *such that*

$$|\tilde{\theta}_i(t)| + |\sigma_i(t)| \le C_2 e^{-\Lambda_2 t}, \quad t \ge 0.$$

Remark 4.1. Note that the results in Theorem 4.2 tell us

$$(\theta_i(t), \sigma_i(t)) \to (\theta^\nu, 0) \quad \text{exponentially fast.}$$

In contrast, for $u = 0$ in Theorem 4.1, one has

$$(\theta_i(t), \sigma_i(t)) \to \left(\theta^\nu + \tilde{\theta}_c^0 + \frac{\sigma_c^0}{\gamma_b}, 0\right) \quad \text{exponentially fast as } t \to \infty.$$

5. Simulations of Flocking Dynamics

This section presents a few sample simulations concerning first- and second-order models with constant activity variables defined in Eqs. (3.3), one with constant weight function $\phi \equiv 1$ and one with time-dependent weight function $\phi = \phi(t)$, and (4.2), respectively. The aim lies in showing quantitative dynamical behaviors which can enrich the description of the qualitative results delivered in the preceding sections.

The initial conditions of the a-particles, at $t = 0$, are supposed to be randomly placed in a region with dimension $[0, 1] \times [0, 1]$, while the velocity direction in $[0, 2\pi)$ is also supposed randomly distributed, see Fig. 2. The following parameters are adopted for all simulations:

$$N = 200, \quad t \in [0, 20], \quad \theta^\nu = \frac{\pi}{3}. \tag{5.1}$$

The selection of the case studies does not claim completeness, but simply shows, also with tutorial aims, how computing can contribute to enlarge, by quantitative

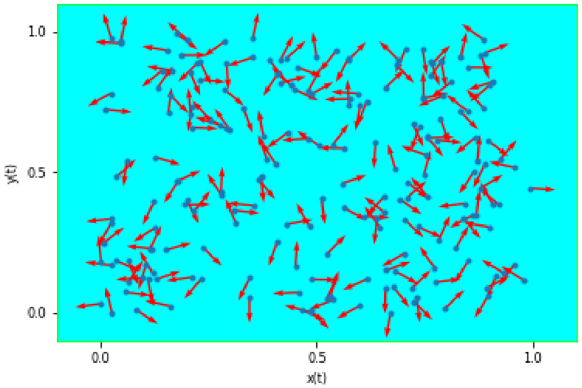


Fig. 2. (Color online) Initial, $t = 0$, positions and velocity directions.

results, the descriptive ability of models. The interested reader can organize different types of simulations referring specifically not to toy models but to real-world case studies.

5.1. *Case study 1: Dynamics of first-order models with constant activity*

This section studies the dynamics of first-order models. In more detail, we consider the first-order model (3.3) with a constant weight function $\phi \equiv 1$ and the analytic result delivered by Theorem 3.1 in the case of constant activity variable and fixed weight function. Initial conditions are visualized in Fig. 2. The dynamical response is studied for the alignment speed $\gamma = 0.3$ and different values $u = 0.1$, $u = 0.4$, $u = 0.7$ of the activity. In practice, γ can be inserted into the time scale.

Simulations in Fig. 3 show an asymptotic global flocking of the dynamics of the diameter functionals:

$$\mathcal{D}^\theta(t) = \max_{1\le i\le N}(|\theta_i(t) - \theta^\nu|) \tag{5.2}$$

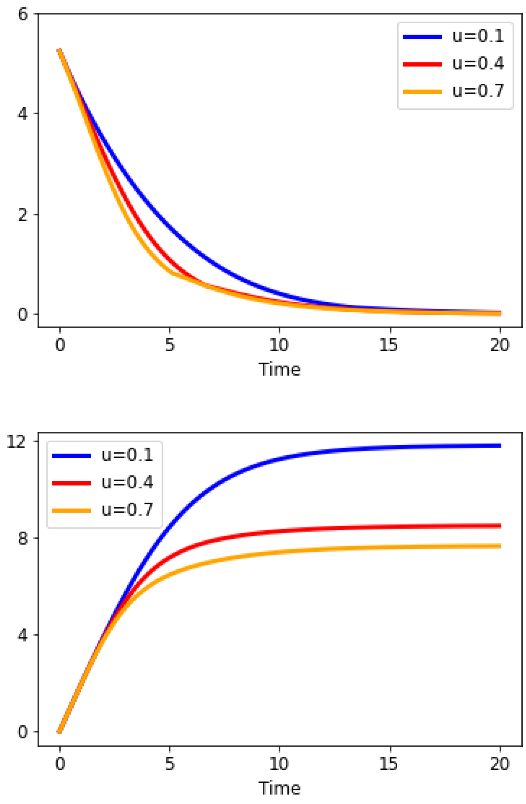


Fig. 3. (Color online) Dynamics, versus time, of the diameters $\mathcal{D}^\theta = \max_{1\le i\le N}(|\theta_i(t) - \theta^\nu|)$ (up) and $\mathcal{D}^{\boldsymbol{x}} = \max_{1\le i\le N}(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_i^0 - \boldsymbol{v}^\nu t\|)$ (down).

and

$$\mathcal{D}^{\boldsymbol{x}}(t) = \max_{1\leq i\leq N}(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_i^0 - \boldsymbol{v}^\nu t\|), \tag{5.3}$$

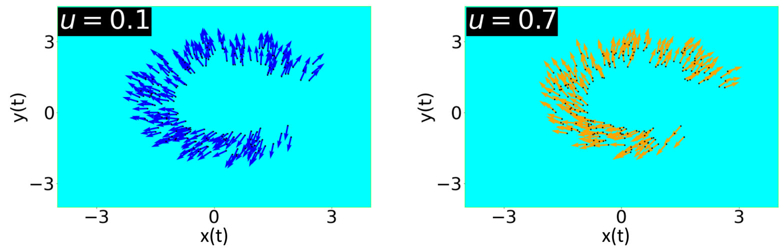
where these two functionals were introduced in Theorem 3.1 as indicators of the "distances" in the alignment and position, respectively.

The up-side of Fig. 3 shows that the alignment decays with an exponential-like profile by a coefficient which increases with u, while the down-side figure shows, for the distance in the positions, an asymptotic trend towards a constant value which increases with decreasing values of the activity.

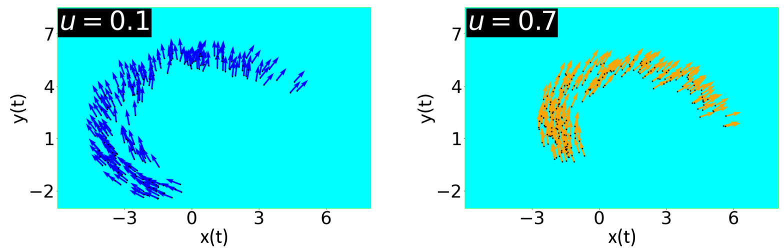
This dynamic is visualized by the flow patterns shown in Fig. 4 corresponding to two times shots $t = 2$ and $t = 5$.

5.2. *Case study 2: Dynamics of first-order models with time-dependent activity variable*

The second case study refers to first-order models specified in Eqs. (3.3) with a time-dependent weight function $\phi = \phi(t)$ and to the qualitative analysis delivered by Theorem 3.1. We consider the parameters (5.1), but initial conditions constituted by two groups, i.e. a first group of 150 particles with an initial activity set to $u = 0.2$, and a second group of 50 particles with a higher level of emotional state $u = 0.8$. The interest to study this type of heterogeneity refers also in showing the dynamics of the activity variable and the consequent role on the overall dynamics. The representation is analogous to that of Fig. 2, then it is not repeated.



(a) $t = 2$



(b) $t = 5$

Fig. 4. (Color online) Flow patterns for different values of u at time $t = 2$ and $t = 5$.

The activity variable is not constant, as it is conditioned by space-depending communication weight functions $\phi_{ij} = \phi_{ij}(\boldsymbol{x})$ and $\psi_{ij} = \psi_{ij}(\boldsymbol{x})$ which model the decay with space of the attractions of the activity and alignment, respectively. In more details, we consider two communication weight functions:

$$\phi_{ij} = \frac{1}{(1 + \|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)^{\delta/2}} \quad \text{with } \delta = 0 \quad \text{or} \quad \delta = 2, \tag{5.4}$$

while, in addition to parameters (5.1), simulations are developed for

$$\psi_{ij} = 1 + \exp(-\|\mathbf{x}_i - \mathbf{x}_j\|), \quad \frac{\beta}{N} = 1.$$

This case study shows the role of the activity variable and of the weight function ϕ_{ij} on the collective dynamics and, as in the case of the first test, we consider the time dynamics of flocking diameters (5.2) and (5.3), as well as patterns of the collective motion for $t = 2$ and $t = 5$. Simulations are shown in Figs. 5 and 6. These simulations show a dynamics somehow analogous to that of case study 1, but the technical difference is that now also the role of the space decay in the interactions is considered.

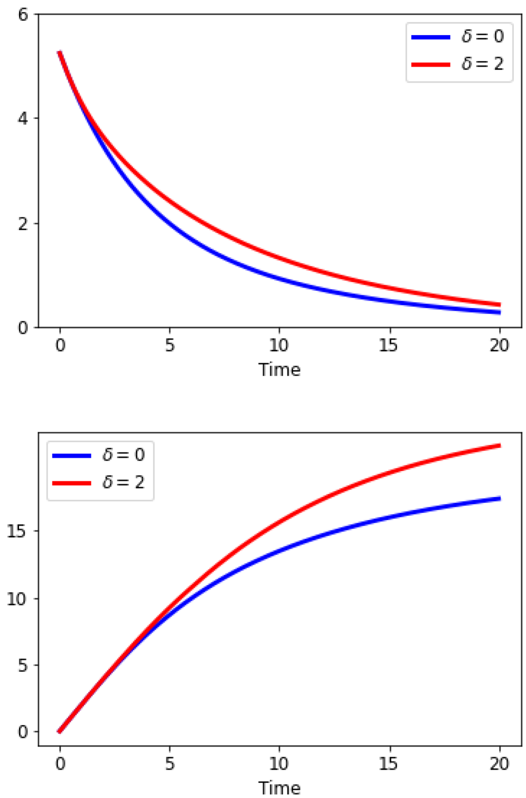
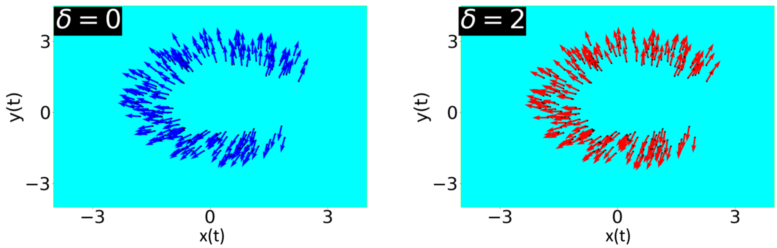
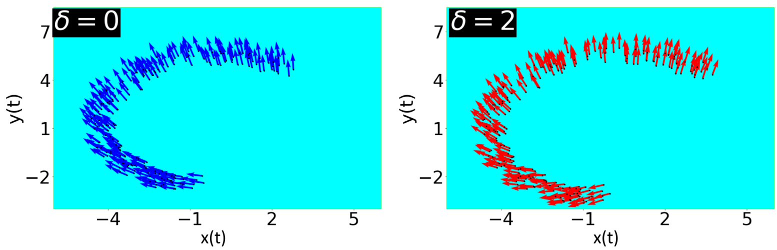


Fig. 5. (Color online) Time evolution of diameters $\max_{1\le i\le N}(|\theta_i(t) - \theta^\nu|)$ (up) and $\max_{1\le i\le N}(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_i^0 - \boldsymbol{v}^\nu t\|)$ (down).



(a) $t = 2$



(b) $t = 5$

Fig. 6. (Color online) Pattern dynamics for $t = 2$ and $t = 5$ with time-varying activity variable.

5.3. *Case study 3: Dynamics of second-order models*

This case study, considers the second-order model (4.2). The test is developed in the case of a constant activity for an initial constant speed: $\sigma_i(0) = 0.5$, for $i \in [N]$. The main objective of the test consists in developing simulations related to the analytic result proved in Theorem 4.2. Simulations are developed for $u = 0.1$ and $u = 0.7$ using parameters (5.1) and the time-varying communication function given by (5.4), with $\delta = 2$, and for $(\gamma_a, \gamma_b) = (0.3, 1)$.

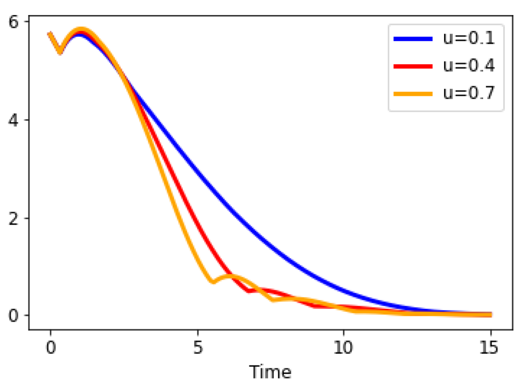
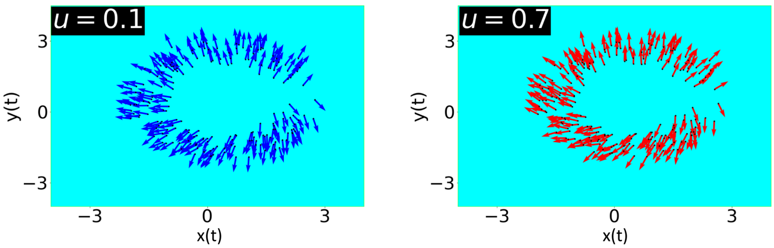
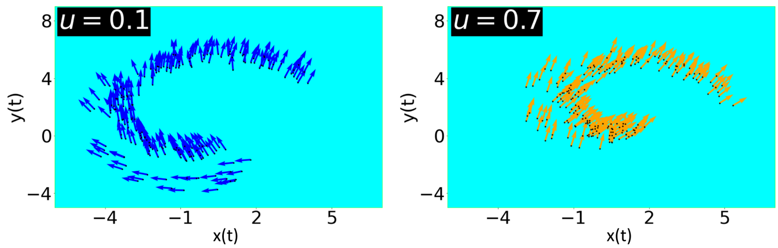


Fig. 7. (Color online) Temporal evolution of $\max_{1\le i\le N}(|\theta_i(t) - \theta^\nu| + |\sigma_i(t)|)$ for second-order model with time-varying communication weight function.



(a) $t = 2$



(b) $t = 5$

Fig. 8. (Color online) Pattern dynamics for $t = 2$ and $t = 5$.

Figure 7 reports the evolution of diameters corresponding to second-order models: $\max_{1\leq i\leq N}(|\theta_i(t) - \theta^\nu| + |\sigma_i(t)|)$, while the pattern formation is shown in Fig. 8 for $t = 2$ and $t = 5$.

5.4. *Critical analysis towards simulation perspectives*

The various simulations presented in this section indicate how a quantitative study of the dynamics of behavioral swarms can be developed. Specifically, the time dynamics of two well-defined metrics, corresponding to angular and spatial distances, have been considered. The study has confirmed the exponential-type trend towards an asymptotic configuration according to the prediction of the qualitative analysis. Additional simulations have shown the patterns by which the swarm moves to the asymptotic configuration.

The overall dynamic quantitatively depends on the parameters of the model, but the qualitative behavior appears to be preserved. Of course, a systematic study of the sensitivity of parameters might be developed, but this objective goes beyond the specific objectives of our paper.

6. A Forward look at Modeling Perspectives

The analytic and computational studies proposed in our paper rely on the use of mathematical structures suitable to account for the behavioral dynamics of the living entities composing the swarm. Models are derived by inserting a detailed description of interactions. It can be shown that these structures can be used in the modeling the dynamics of well-defined living systems consisting in a finite number of living entities. First, we briefly show a possible application the theory of behavioral swarms, and then we focus on a further development of the mathematical structure. As a specific case study, we consider the modeling of human crowds.[11] We can outline how the modeling approach can be developed, at the microscopic scale, consistently with the mathematical frameworks. The approach does not require further developments of the structure, but simply a more specific description of interactions which depends on the strategy that each pedestrian develops by interaction with the other pedestrians as well as with the geometrical and physical features of the venue where the crowd moves.

Let us consider a "swarm" of a finite number of i-pedestrians, with $i \in [N]$. An intuitive model of such strategy is proposed in Ref. 11 as follows:

(1) Each i-pedestrian can develop a walking strategy by decisional hierarchy where interactions first modify the activity and then the motion which depends also on the activity. The strategy, accounts of all pedestrians in her/his individual sensitivity domain Ω_i.
(2) All a-particles are subject to different stimuli, i.e. they tend to the direction from the location $\boldsymbol{x}_i$ of the i-pedestrian to a meeting point or exit; attraction by the main stream computed in Ω_i; attraction towards less congested areas corresponding to the local distribution of density in Ω_i. The choice of the velocity direction corresponds to a weighted selection of these stimuli, depending on the quality of the venue, on the emotional state and on the local density distribution.
(3) The i-pedestrian, once moved to the new velocity direction, perceives the local density in the new visibility domain which differs from the density previously perceived in this domain. If the new density is lower (higher) than the previous one, the i-pedestrian will increase (decrease) the speed.

A challenging perspective lies in further developments of the framework proposed in Ref. 13. It is not an easy task, as causality principles cannot, for evolutionary living systems, rely on a physical background.[31] The mathematical theory proposed in Ref. 8 suggests an alternative to the search of a physical theory to support the derivation of causality principles. The authors mention a *science of living systems*, where the derivation of models is developed within a framework suitable to capture the complexity features of living systems. This strategy leads to a mathematical framework for the derivation of specific model. The complexity features proposed in Ref. 8 are: *Ability to express a strategy*, *Heterogeneity*, *Learning ability*, *Nonlinearity of interactions*, *Darwinian mutations and selection*, see also Ref. 6.

Therefore, it is worth going beyond the present state of the art to understand how the mathematical structures proposed in Ref. 13 can be further developed in order to take into account the aforementioned rationale, by including interactions that are not number conservative. For instance by modeling proliferative and/or destructive events. This development might end up with new mathematical tools, somehow alternative to the kinetic theory for a-particles.

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