COMPACT RETRACTIONS AND SCHAUDER DECOMPOSITIONS IN BANACH SPACES

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ABSTRACT. In our note we show the very close connection between the existence of a Finite Dimensional Decomposition (FDD for short) for a separable Banach space X and the existence of a Lipschitz retraction of X onto a small (in a certain precise sense) generating convex and compact subset K of X.

In one direction, if X admits an FDD then we construct a Lipschitz retraction onto a small generating convex and compact set K. On the other hand, we prove that if X admits a small generating compact Lipschitz retract then X has the π -property. We note that it is still unknown if the π -property is isomorphically equivalent to the existence of an FDD.

For dual Banach spaces this is true, so our results lead in particular to a characterization of the FDD property for dual Banach spaces Xin terms of the existence of Lipschitz retractions onto small generating convex and compact subsets of X.

It is conceivable that our results will find applications in the area of Lipschitz isomorphisms of Banach spaces.

Our arguments make critical use of the Lipschitzization of coarse Lipschitz mappings due to J. Bourgain, and of an unpublished complementability result of V. Milman.

We give an example of a small generating convex compact set which is not a Lipschitz retract of C[0, 1], although it is contained in a small convex Lipschitz retract and contains another one.

In the last part of our note we characterize isomorphically Hilbertian spaces as those Banach spaces X for which every convex and compact subset is a Lipschitz retract of X. Finally, we prove that a convex and compact set K in any Banach space with a Uniformly Rotund in Every Direction norm is a uniform retract, of every bounded set containing it, via the nearest point map.

1. INTRODUCTION

The study of retractions is a large area of research in topology and nonlinear analysis with many applications. In the uniform or Lipschitz setting

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the authoritative monograph [BL00] contains many fundamental results and the general point of view.

The results in our paper were originally motivated by the natural question asked by Godefroy and Ozawa in [GO14] (and then subsequently in [God15a], [God15b], [God20], and [GLP19]) whether every separable Banach space admits a generating convex and compact Lipschitz retract (GCCR, for short). By a generating set in a Banach space X we mean the set C such that the closed linear span of C is the whole space X. We will say that C is a λ -GCCR if it is a GCCR and there exists a λ -Lipschitz retraction from X onto C. A positive answer to this problem would immediately imply that every separable Banach space is Lipschitz approximable, solving an important open problem of Kalton. Using a rather short, but ingenious argument, Kalton [Kal12] showed that indeed if X has a separable dual (or it is itself a separable dual) then X is approximable. But his compact identity-approximating mappings are far from being retractions. In [GO14] it is noted that the retractions can be constructed rather easily if X has an unconditional basis, but the same approach fails even for spaces with a Schauder basis.

Our first result (Theorem 3.3) is that a Banach space X with an FDD admits a GCCR. Moreover, if X has a monotone FDD, then it has a $(5 + \epsilon)$ -GCCR, for every $\epsilon > 0$. If X has a monotone Schauder basis, then it has a $(1 + \epsilon)$ -GCCR, for every $\epsilon > 0$. Our GCCR is roughly speaking a diamond shaped set, and the retraction mapping is somehow aligned with the canonical projections but it is certainly not the nearest point map.

The compact sets used in our arguments are in some sense small. Later on, for the purposes of proving results going in the opposite direction, we proceed by formally defining the quantitative concept of a small set, which is intuitively a generating compact subset of X that is contained in sufficiently small neighbourhoods of its finite dimensional sections. The smallness condition restricts the asymptotic behaviour of the compact set, in a certain sense, but otherwise it leaves a complete freedom as regards its possible shape. It is curious that our results in both directions lead to small compacts of roughly the same size.

Our second, and perhaps the main result of the note (Theorem 4.7), is that if a Banach space admits a small Lipschitz retract (in particular, a small GCCR) then it has the π -property. Note that a Banach space with the metric π -property has an FDD [Joh70], see also [Cas01] Thm. 6.4, and it is still open if the π -property implies metric π -property (and hence FDD) under an equivalent renorming. So our previous results combined together are possibly a characterization of the existence of an FDD property for the Banach space X in terms of the existence of Lipschitz retractions onto small GCCR. In the case of dual spaces the characterization holds true, thanks to Theorem 1.3 in [JRZ71].

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The proof is based on several deep ingredients. Of course, one would like to use the (Gateaux) differentiation theory for Lipschitz mappings in separable Banach spaces in order to pass from the compact Lipschitz retraction to its linearization with a small range. However, to do this directly using the abundance of points of Gateaux differentiability does not seem possible. Instead, our proof takes a detour, and produces the finite rank linear projections (needed for the π -property) indirectly, only proving their existence. We first pass from the small retract K to a finite dimensional subspace E_n , which contains the bulk of the points of K, in a certain sense. Of course, we do not immediately have in our hands a good Lipschitz retraction from Xonto E_n , but using the "Lipschitzization" of coarse Lipschitz mappings due to Bourgain [Bou87] (in the formulation of Begun [Beg99]) we obtain a good Lipschitz almost retraction to E_n from finite dimensional subspaces G of X, of controlled dimension, which contain E_n . For the next step we use the deep and yet unpublished result, due to Vitali Milman, communicated to us by Bill Johnson. Namely, the projection constant $\lambda(E_n, X)$ of a *n*-dimensional subspace E_n of X is witnessed by $\lambda(E_n, G)$ for a certain finite dimensional subspace G of X, of dimension roughly 3^n . This unexpected fact makes our subsequent argument quantitatively independent of the Banach space X. At this point we may use the differentiation theory and averaging in the finite dimensional setting, applied to the finite dimensional approximate version of our retraction (crucially using the smallness assumption for K) to produce a good linear projection from G to E_n . But this implies that $\lambda(E_n, X)$ cannot be too large, which eventually yields the π -property for X.

Since the π -property is equivalent to the FDD property for dual Banach spaces [**JRZ71**], as a corollary to our above results we obtain a retractional characterization of an FDD in the class of dual Banach spaces.

We remark that our techniques above are applicable to small compacts only, and we do not know if analogous results hold for general convex compact subsets of X.

Using the same approach, we also give a variant of the result of Godefroy and Kalton in [GK03], who characterized the BAP property by means of a sequence of finite rank Lipschitz mappings pointwise convergent to the identity. Namely, using the finite rank Lipschitz retractions we characterize the π -property.

We also note that if there is a Lipschitz retraction of X onto a convex compact set K, and Y is a closed linear subspace of X spanned by K, which is linearly complemented in its bidual Y^{**} , then Y is a complemented subspace of X. This means that a natural way of getting the Lipschitz retraction from X onto K is to simply compose the linear projection from X onto Y with a Lipschitz retraction from Y onto K.

In the next section, we give an example of a small compact convex set K in the space C[0, 1], which is contained in a small GCCR, and contains another small GCCR, but such that K is not a GCCR for the space C[0, 1]. This example underlines the subtleness of the existence of Lipschitz retractions.

In our final section we first give a new characterization of isomorphically Hilbertian spaces as those Banach spaces for which every convex and compact subset is a Lipschitz retract.

Then we proceed to the problem of uniformly continuous retractions onto convex compact sets. Our main result is that if a Banach space is equipped with a URED norm, then every convex and compact subset is a uniformly continuous retract, from any bounded superset, with respect to the nearest point map. We recall that every separable Banach space admits an equivalent URED renorming. This result implies, in particular, that every convex and compact set is an absolute uniform retract, a fact recently established in [CCW21]. The case of a general (nonseparable) space X is also treated.

2. Preliminaries

Our notation and terminology is standard. We work in the setting of real Banach spaces throughout this note. For the general concepts and results of Banach space theory we refer to [Fab+11]. The background on the various forms of the approximation property can be found in [Cas01]. For the theory of Lipschitz and uniform retracts we refer to the first two chapters in the monograph [BL00].

Let us now pass to some definitions and results which are used heavily in our note. We start with the formulation of several classical concepts related to the approximation property, in the growing generality.

A Schauder basis for a real Banach space X is a sequence $(x_n) \subset X$ with the property that for every $x \in X$, there exists a unique sequence $(\alpha_n) \subset \mathbb{R}$ such that

$$\left| \left| x - \sum_{i=1}^{n} \alpha_i x_i \right| \right| \xrightarrow{n \to \infty} 0.$$

In this case we say that X has a Schauder basis and we call the projections $P_n(x) = \sum_{i=1}^n \alpha_i x_i$ the natural projections of the Schauder basis. If (x_n) is a Schauder basis in a Banach space X, then we denote by (x_n^*) the coordinate functionals, which form a basic sequence in the dual space X^* .

Definition 2.1. A sequence (E_n) of finite dimensional subspaces of a Banach space X is called the finite dimensional decomposition (FDD for short) if for every $x \in X$ there is a unique sequence $x_n \in E_n$ so that

$$\left| \left| x - \sum_{i=1}^{n} x_i \right| \right| \xrightarrow{n \to \infty} 0.$$

In this case we say that X has an FDD and we call the projections $P_n(x) = \sum_{i=1}^n x_i$ the natural projections of the FDD.

If X is a Banach space with a Schauder basis (resp. FDD) then the natural projections of the basis (resp. FDD) are uniformly bounded. Moreover, X

can be equivalently renormed so that this uniform bound is 1. In this case we say that the Schauder basis (resp. FDD) is monotone.

Definition 2.2. A Banach space X is said to have the π -property if there is a uniformly bounded net of finite rank projections (S_{α}) on X converging strongly to the identity on X. If this uniform bound is $\lambda \geq 1$ then we say that X has the π_{λ} -property. In the case when $\lambda = 1$ we say that X has the metric π -property.

Proposition 2.3 ([Cas01]). For a separable Banach space X and $\lambda \ge 1$, the following are equivalent:

- X has the π_{λ} -property
- There is a sequence of λ-bounded finite rank projections (S_n) on X pointwise converging to the identity such that

$$S_m S_n = S_n \quad \forall m \ge n$$

As we mentioned above, it seems to be an open problem whether the π -property is equivalent to the existence of an FDD for a separable Banach space.

Definition 2.4. Let X be a Banach space. If there is a uniformly bounded net of finite rank operators (T_{α}) on X tending strongly to the identity on X, then we say that X has the bounded approximation property (BAP for short). If $\lambda \geq 1$ is a uniform bound for the net then we say that X has the λ -bounded approximation property (λ -BAP for short). We refer to the 1-BAP as the metric approximation property (MAP for short).

Definition 2.5. Let X be a Banach space. If there is a net of finite rank operators (T_{α}) on X converging to the identity on X uniformly on compacta, then we say that X has the approximation property (AP for short).

Definition 2.6. A Banach space X is said to have the compact approximation property if for every $\varepsilon > 0$ and every compact set K in X there is a compact operator $T \in \mathcal{L}(X)$ so that $||Tx - x|| \le \varepsilon$ for all $x \in K$.

For an arbitrary Banach space X, the previously defined concepts are ordered from the strongest to the weakest, that is,

Schauder basis \Rightarrow FDD $\Rightarrow \pi$ -property \Rightarrow BAP \Rightarrow AP \Rightarrow CAP.

With the possible exception of FDD $\Rightarrow \pi$ -property, none of the above implications can be reversed ([Sza87],[Rea], [FJ73] and [Wil92]).

Let us pass to the non-linear approximation properties.

Definition 2.7. A map T from a metric space M into another metric space N is said to be Lipschitz if there exists some $\lambda > 0$ such that

$$d(T(x), T(y)) \le \lambda d(x, y) \quad \forall x, y \in M.$$

We say that λ is the Lipschitz constant for T and we call the infimum of all Lipschitz constants for T the Lipschitz norm of T, that is,

 $||T||_{Lip} = \inf \left\{ \lambda > 0, \text{ Lipschitz constant for } T \right\} = \sup_{x,y \in M, x \neq y} \frac{d(T(x), T(y))}{d(x, y)}.$

If $\lambda > 0$ is a Lipschitz constant for T then we say that T is λ -Lipschitz.

Definition 2.8. Let X be a Banach space. If there is a net of finite rank Lipschitz maps (T_{α}) on X, whose Lipschitz norms are uniformly bounded, converging uniformly on compact to the identity on X, then we say that X has the Lipschitz bounded approximation property. If this net is bounded by $\lambda \geq 1$ then we say that X has the λ -Lipschitz bounded approximation property.

Theorem 2.9 ([GK03] Theorem 5.3). Let X be an arbitrary Banach space. Then the following conditions are equivalent:

- X has the λ-BAP.
- The Lipschitz free space $\mathcal{F}(X)$ has the λ -BAP.
- X has the λ -Lipschitz bounded approximation property.

Definition 2.10. We say that a complete metric space M is approximable whenever there is a subadditive map $\omega : [0, \infty) \to [0, \infty)$ with $\lim_{t\to 0} \omega(t) = 0$ so that for every finite set $E \subset M$ and every $\varepsilon > 0$ we can find a uniformly continuous map $\psi : M \to M$ such that $d(x, \psi(x)) < \varepsilon$ for every $x \in E$, $\psi(M)$ is relatively compact and the modulus of continuity of ψ is bounded by ω .

If $\omega(t) = Lt$ for some L > 0 then we say that M is Lipschitz approximable.

If M is separable then it is easy to see that M is approximable (resp. Lipschitz approximable) if and only if there is an equi-uniformly continuous (resp. equi-Lipschitz) sequence of maps $\psi_n : M \to M$ with relatively compact range such that $\lim_{n \to \infty} d(x, \psi_n(x)) = 0$ for every $x \in X$.

These concepts were introduced and studied in several papers, [Kal04], [Kal12] and [GLZ14].

Kalton proved in [Kal12] that every Banach space with a separable dual (or a separable dual space itself) is approximable. It is still an open problem if every separable Banach space is approximable. Godefroy, on the other hand, observed in [God20] that the compact approximation property implies that the space is Lipschitz approximable, showing that the Lipschitz approximability is a strictly weaker property than the AP.

In the next sections, we are going to make a repeated use of the following concepts.

Definition 2.11. Let M be a metric space and $N \subset M$. A retraction from M onto N is a map $R: M \to N$ such that $R|_N = Id_N$. In this case we say that N is a retract of X. If R is Lipschitz (resp. uniformly continuous) then

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we say that R is a Lipschitz (resp. uniformly continuous) retraction onto N and N is a Lipschitz (resp. uniformly continuous) retract of M.

If N is a Lipschitz (resp. uniformly continuous) retract of M for every metric space M containing it, we say that N is an absolute Lipschitz (resp. uniformly continuous) retract.

We will say that a subset K of a Banach space X generates X whenever the closed linear span of K is equal to X. A Lipschitz retract K of a Banach space X that is convex, compact, and generates X is going to be called a generating convex compact retract (GCCR for short) of X.

Note that the existence of a GCCR implies that the space X is Lipschitz approximable.

Definition 2.12. Let M be a metric space, $N \subset M$. We say that $R: M \to M$ N is a proximity map (or a nearest point map) onto N if

$$d(R(x), x) = \inf_{y \in N} d(y, x) \quad \forall x \in M.$$

A proximity map may not exist in some situations and if it exists it may not be unique. If M is a uniformly convex Banach space then this map is known to be unique and uniformly continuous whenever N is a closed convex subset [Bjo79].

Definition 2.13. A normed space X is said to be uniformly rotund in the direction $z \in X$ if whenever (x_n) and (y_n) are two sequences in X such that

- (1) $||x_n|| = ||y_n|| = 1$ for every $n \in \mathbb{N}$,
- (2) $\lim_{n \to \infty} \left| \left| \frac{x_n + y_n}{2} \right| \right| = 1,$ (3) There is a sequence of real numbers (r_n) such that $x_n y_n = r_n z$ for every $n \in \mathbb{N}$,

then $\lim_{n \to \infty} ||x_n - y_n|| = 0.$

If X is uniformly rotund in the direction z for every $z \in S_X$ then we say that X is uniformly rotund in every direction (URED for short).

3. Compact Lipschitz retracts

The problem whether every separable Banach space has a GCCR was posed in [GO14]. In fact, having a GCCR for a Banach space X gives a lot of information. For instance, it implies that X is Lipschitz approximable and that X has the BAP if and only if $\mathcal{F}(K)$ has the BAP by Theorem 2.9. Our goal in this section is a construction of a certain diamond shaped convex and compact set K, which generates the Banach space X, together with a Lipschitz retraction from X onto K. Our construction is performed under the assumption that X has a FDD so in particular it gives a positive answer to the question asked in [GLP19], whether Pelczynski's space has a GCCR. In order to make the construction work, we are forced to make Kin some sense small. The method breaks down if we try to replace the FDD by a weaker concept, e.g. a Markushevich basis or so. As we will find out in the subsequent parts of our note, there is a good reason for that. Namely, the kind of retractions we are using here imply that X has the π -property (which is possibly equivalent to having an FDD).

Our approach is to define some Lipschitz retractions onto increasing finitedimensional sections of K, so that then we proceed by taking a limit of these mappings to define the final retraction onto K. All these initial retractions will be defined as compositions of little perturbations of the natural projections of the FDD. To this end, we first prove some Lemmas stating the Lipschitz behaviour of the perturbed projections.

Let us proceed with the construction. Let $(X, || \cdot ||)$ be a Banach space with a monotone FDD $(X_n)_{n \in \mathbb{N}}$ whose natural projections are $(P_n)_{n \in \mathbb{N}}$. For every $x \in X$ and $i \in \mathbb{N}$ we are going to denote $x_i = (P_i - P_{i-1})(x)$ where $P_0 \equiv 0$. Let $(r_n)_{n \in \mathbb{N}}$ be an arbitrary decreasing sequence of positive real numbers. For every $m \in \mathbb{N}$ we define the function $f_m : X \to \mathbb{R}$, by setting $f_1 : X \to \{r_1\}$ the constant function, and

$$f_m(x) = r_m \left(1 - \sum_{i=1}^{m-1} \frac{||x_i||}{r_i} \right), \ m \ge 2.$$

We will define for $n \in \mathbb{N}$ the following subsets of X

$$K = \overline{\operatorname{co}}\left(\bigcup_{k \in \mathbb{N}} r_k B_{X_k}\right) \quad , \quad K_n = \overline{\operatorname{co}}\left(\bigcup_{k=1}^n r_k B_{X_k}\right).$$

Finally, given $n, m \in \mathbb{N}$ such that $m \leq n$ we consider $E_{n,m} = P_m(X) \cup K_n$, and $F_{n,m} : E_{n,m} \to E_{n,m-1}$ given by

$$F_{n,m}(x) = \begin{cases} x , & \text{if } \sum_{i=1}^{m} \frac{||x_i||}{r_i} \leq 1, \\ P_{m-1}(x) , & \text{if } \sum_{i=1}^{m-1} \frac{||x_i||}{r_i} \geq 1, \\ P_{m-1}(x) + \frac{x_m}{||x_m||} f_m(x) , & \text{if } \sum_{i=1}^{m-1} \frac{||x_i||}{r_i} < 1 < \sum_{i=1}^{m} \frac{||x_i||}{r_i} \end{cases}$$

where, if m = 1, we consider $\sum_{i=1}^{0} \frac{||x_i||}{r_i} = 0$.

One may see every $F_{n,m}$ as a perturbation of P_{m-1} restricted to $E_{n,m}$. The next Lemma 3.1 states that it is possible to approximate the Lipschitz behaviour of $F_{n,m}$ by $F_{n-1,m}$.

Lemma 3.1. There exists a sequence $(A_m)_{m \in \mathbb{N}} \subset \mathbb{R}^+$ such that

$$f_m$$
 is $\frac{r_m A_{m-1}}{r_{m-1}}$ -Lipschitz $\forall m > 1$,

and if $m, n \in \mathbb{N}$ such that $m \leq n-1$ then,

$$\begin{aligned} ||F_{n,m}(x) - F_{n,m}(y)|| &\leq \frac{r_n A_n}{r_{n-1}} ||x - y|| + ||F_{n-1,m} \circ P_{n-1}(x) - F_{n-1,m} \circ P_{n-1}(y)|| \\ for \ every \ x, y \in E_{n,m}. \end{aligned}$$

Proof. Just consider $A_m = 2m$. Then,

$$||P_m(x)||_{\ell_1(\{X_i\}_{i=1}^m)} \le A_m ||P_m(x)|| \quad \forall x \in X,$$

so we have that

$$|f_m(x) - f_m(y)| = \left| r_m \left(\sum_{i=1}^{m-1} \frac{||x_i|| - ||y_i||}{r_i} \right) \right| \le \frac{r_m}{r_{m-1}} \sum_{i=1}^{m-1} ||x_i - y_i||$$
$$\le \frac{r_m A_{m-1}}{r_{m-1}} ||P_{m-1}(x - y)|| \le \frac{r_m A_{m-1}}{r_{m-1}} ||x - y||.$$

Now, to prove the second part of the lemma, let us define $G_{n-1,m}: E_{n,m} \to E_{n-1,m-1}$ by

$$G_{n-1,m}(x) = F_{n-1,m}(P_{n-1}(x)) \quad \forall x \in E_{n,m}.$$

Taking into account the previous definitions, it is immediate that

$$G_{n-1,m}(x) = F_{n,m}(P_{n-1}(x)) \quad \forall x \in E_{n,m}$$

Hence, it holds that $g_n := F_{n,m} - G_{n-1,m} = P_n - P_{n-1}$. Now, if $x \neq y \in E_{n,m}$ are such that $\sum_{i=1}^m \frac{||x_i||}{r_i} \leq 1$ and $\sum_{i=1}^m \frac{||y_i||}{r_i} \leq 1$ then $\frac{||F_{n,m}(x) - F_{n,m}(y)||}{||x - y||} = 1.$

If that is not the case, then we may assume that $\sum_{i=1}^{m} \frac{||y_i||}{r_i} > 1$, so $||y_n|| = 0$ and $F_{n,m}(y) = F_{n,m}(P_{n-1}(y)) = G_{n-1,m}(y)$. It turns out that

$$\frac{||F_{n,m}(x) - F_{n,m}(y)||}{||x - y||} \le \frac{||g_n(x)|| + ||G_{n-1,m}(x) - G_{n-1,m}(y)||}{||x - y||}$$

It is enough to prove that $\frac{||g_n(x)||}{||x-y||} \leq \frac{r_n A_n}{r_{n-1}}$. If $\sum_{i=1}^n \frac{||x_i||}{r_i} > 1$ then $||x_n|| = 0$ and it is obviously true. Otherwise,

$$\frac{||x_n||}{r_n} \le 1 - \sum_{i=1}^{n-1} \frac{||x_i||}{r_i} \le \sum_{i=1}^{n-1} \frac{||y_i|| - ||x_i||}{r_i} \le \sum_{i=1}^{n-1} \frac{||x_i - y_i||}{r_i} \le \frac{\sum_{i=1}^{n-1} ||x_i - y_i||}{r_{n-1}}$$

So it is true that

$$||x - y||_{\ell_1(\{X_i\}_{i=1}^n)} \ge ||x_n|| \frac{r_{n-1}}{r_n}.$$

Finally,

$$\frac{||g_n(x)||}{||x-y||} \le \frac{||x_n||A_n}{||x-y||_{\ell_1(\{X_i\}_{i=1}^n)}} \le \frac{||x_n||A_n}{||x_n||\frac{r_{n-1}}{r_n}} \le \frac{r_nA_n}{r_{n-1}}.$$

Proposition 3.2. For every $j \in \mathbb{N}$ and $x, y \in P_j(X)$ the following implications hold true: (3.1)

$$if \sum_{i=1}^{j-1} \frac{||x_i||}{r_i}, \sum_{i=1}^{j-1} \frac{||y_i||}{r_i} \le 1 \Rightarrow ||F_{j,j}(x) - F_{j,j}(y)|| \le \left(5 + \frac{r_j A_{j-1}}{r_{j-1}}\right) ||x - y||,$$

(3.2) if
$$\sum_{i=1}^{j-1} \frac{||y_i||}{r_i} > 1 \Rightarrow ||F_{j,j}(x) - F_{j,j}(y)|| \le \left(1 + \frac{r_j A_{j-1}}{r_{j-1}}\right) ||x - y||.$$

Proof. First we prove (3.1) case by case:

If $\sum_{i=1}^{j} \frac{||x_i||}{r_i} > 1$ and $\sum_{i=1}^{j} \frac{||y_i||}{r_i} > 1$, then from the definition of f_j we know that $f_j(y) < ||y_j||$ so

$$\begin{aligned} &\frac{\left|\left|x_{j}\right|\left|y_{j}\right|\left|f_{j}(x)-y_{j}\right|\left|x_{j}\right|\right|\left|f_{j}(y)\right|\right|}{\left|\left|x_{j}\right|\right|\left|\left|y_{j}\right|\right|\right|} \\ &\leq \frac{\left|\left|x_{j}\right|\left|y_{j}\right|\right|f_{j}(x)-x_{j}f_{j}(y)\right|\left|x_{j}\right|\right|\left|+\left|\left|x_{j}f_{j}(y)\right|\left|x_{j}\right|\right|-y_{j}\right|\left|x_{j}\right|\left|f_{j}(y)\right|\right|}{\left|\left|x_{j}\right|\right|\left|\left|y_{j}\right|\right|\right|} \\ &= \frac{\left|f_{j}(x)\right|\left|y_{j}\right|\right|-f_{j}(y)\left|\left|x_{j}\right|\right|\right|}{\left|\left|y_{j}\right|\right|} + \frac{\left|\left|x_{j}-y_{j}\right|\right|f_{j}(y)\right|}{\left|\left|y_{j}\right|\right|} \\ &\leq \frac{\left|f_{j}(x)\right|\left|y_{j}\right|\right|-f_{j}(y)\left|\left|y_{j}\right|\right|+\left|f_{j}(y)\right|\left|y_{j}\right|\right|-f_{j}(y)\left|\left|x_{j}\right|\right|\right|}{\left|\left|y_{j}\right|\right|} \\ &= \left|f_{j}(x)-f_{j}(y)\right| + \frac{f_{j}(y)\left|\left|x_{j}-y_{j}\right|\right|}{\left|\left|y_{j}\right|\right|} + \left|\left|x_{j}-y_{j}\right|\right| \\ &\leq \frac{r_{j}A_{j-1}}{r_{j-1}}\left|\left|x-y\right|\right|+2\left|\left|x_{j}-y_{j}\right|\right| \leq \left(4 + \frac{r_{j}A_{j-1}}{r_{j-1}}\right)\left|\left|x-y\right|\right|. \end{aligned}$$

Hence, we have that

$$\begin{aligned} ||F_{j,j}(x) - F_{j,j}(y)|| \\ &\leq ||P_{j-1}(x-y)|| + ||(F_{j,j} - P_{j-1})(x) - (F_{j,j} - P_{j-1})(y)|| \\ &= ||P_{j-1}(x-y)|| + \left| \left| \frac{x_j}{||x_j||} f_j(x) - \frac{y_j}{||y_j||} f_j(y) \right| \right| \\ &= ||P_{j-1}(x-y)|| + \frac{||x_j||y_j||f_j(x) - y_j||x_j||f_j(y)||}{||x_j|| ||y_j||} \\ &\leq \left(5 + \frac{r_j A_{j-1}}{r_{j-1}}\right) ||x-y||. \end{aligned}$$

If $\sum_{i=1}^{j} \frac{||x_i||}{r_i} \leq 1$ and $\sum_{i=1}^{j} \frac{||y_i||}{r_i} > 1$, then from the definition of f_j it follows that $||x_j|| \leq f_j(x)$ and $||y_j|| - f_j(y) > 0$ so

$$\frac{\left|\left|x_{j}\right|\left|y_{j}\right|\right| - y_{j}f_{j}(y)\right|\right|}{\left|\left|y_{j}\right|\right|} \leq \frac{\left|\left|x_{j}\right|\left|y_{j}\right|\right| - y_{j}\left|\left|y_{j}\right|\right|\right| + \left|\left|y_{j}\right|\left|y_{j}\right|\right| - y_{j}f_{j}(y)\right|\right|}{\left|\left|y_{j}\right|\right|} \\
= \left|\left|x_{j} - y_{j}\right|\right| + \left(\left|\left|y_{j}\right|\right| - f_{j}(y)\right) \\
= \left|\left|x_{j} - y_{j}\right|\right| + \left(\left|\left|y_{j}\right|\right| - \left|\left|x_{j}\right|\right|\right) + \left(\left|\left|x_{j}\right|\right| - f_{j}(y)\right) \\
\leq 2\left|\left|x_{j} - y_{j}\right|\right| + \left(f_{j}(x) - f_{j}(y)\right) \\
\leq \left(4 + \frac{r_{j}A_{j-1}}{r_{j-1}}\right)\left|\left|x - y\right|\right|.$$

Hence, we deduce that

$$\begin{split} ||F_{j,j}(x) - F_{j,j}(y)|| \\ &\leq ||P_{j-1}(x-y)|| + ||(F_{j,j} - P_{j-1})(x) - (F_{j,j} - P_{j-1})(y)|| \\ &= ||P_{j-1}(x-y)|| + \left| \left| x_j - \frac{y_j}{||y_j||} f_j(y) \right| \right| \\ &= ||P_{j-1}(x-y)|| + \frac{||x_j||y_j|| - y_j f_j(y)||}{||y_j||} \\ &\leq \left(5 + \frac{r_j A_{j-1}}{r_{j-1}} \right) ||x-y||. \end{split}$$

The case when $\sum_{i=1}^{j} \frac{||x_i||}{r_i} \leq 1$ and $\sum_{i=1}^{j} \frac{||y_i||}{r_i} \leq 1$ is trivially true because $F_{j,j}$ acts as the identity, so we have proven (3.1).

Finally, we prove (3.2) distinguishing between 3 different cases:

$$\begin{aligned} \text{If} & \sum_{i=1}^{j-1} \frac{||x_i||}{r_i} \ge 1 \text{ then } F_{j,j}(x) - F_{j,j}(y) = P_{j-1}(x-y) \text{ so it is straightforward} \\ \text{If} & \sum_{i=1}^{j-1} \frac{||x_i||}{r_i} < 1 < \sum_{i=1}^{j} \frac{||x_i||}{r_i} \text{ then, as } 1 \le \sum_{i=1}^{j-1} \frac{||y_i||}{r_i}, \text{ it holds that} \\ |f_j(x)| &= r_j \left(1 - \sum_{i=1}^{j-1} \frac{||x_i||}{r_i} \right) \\ &\le r_j \left(\sum_{i=1}^{j-1} \frac{||y_i|| - ||x_i||}{r_i} \right) = |f_j(x) - f_j(y)| \le \frac{r_j A_{j-1}}{r_j} ||x-y||. \end{aligned}$$

Hence,

$$||F_{j,j}(x) - F_{j,j}(y)|| = \left| \left| P_{j-1}(x-y) + \frac{x_j}{||x_j||} f_j(x) \right| \right|$$
$$\leq \left(1 + \frac{r_j A_{j-1}}{r_{j-1}} \right) ||x-y||.$$

Finally, if $\sum_{i=1}^{j} \frac{||x_i||}{r_i} \le 1$, then

$$\begin{aligned} ||x_j|| &\leq r_j \left(1 - \sum_{i=1}^{j-1} \frac{||x_i||}{r_i} \right) \leq r_j \left(\sum_{i=1}^{j-1} \frac{||y_i|| - ||x_i||}{r_i} \right) = |f_j(x) - f_j(y)| \\ &\leq \frac{r_j A_{j-1}}{r_{j-1}} ||x - y||, \end{aligned}$$

and so,

$$||F_{j,j}(x) - F_{j,j}(y)|| = ||P_{j-1}(x-y) + x_j||$$

$$\leq \left(1 + \frac{r_j A_{j-1}}{r_{j-1}}\right) ||x-y||$$

Theorem 3.3. There is a sequence $(q_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ such that for every Banach space X with a monotone FDD (X_n) , and every decreasing sequence $(r_n) \subset \mathbb{R}^+$ satisfying $\frac{r_n}{r_{n-1}} \leq q_n$, the set

$$K = \overline{\operatorname{co}}\left(\bigcup_{n \in \mathbb{N}} r_n B_{X_n}\right)$$

is a compact Lipschitz retract of X.

Proof. Consider some $\delta > 0$ and $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ such that

$$\prod_{n \in \mathbb{N}} (1 + \delta_n) \le 1 + \delta_n$$

and consider a sequence $(a_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ such that the sequence given by

$$(\alpha_n)_{n\in\mathbb{N}} := \left(\sum_{k=n+1}^{\infty} a_k A_k\right)_{n\in\mathbb{N}}$$

verifies that

$$\alpha_n \le \frac{\delta_n}{2} \qquad \forall n \in \mathbb{N}.$$

Now we set

$$q_n = \min\left\{a_n, \frac{\delta_n}{2A_{n-1}}\right\}.$$

Suppose that (r_n) is a sequence as in the statement of the Theorem. Given any $n \in \mathbb{N}$ we define the following retraction

$$F_n = F_{n,1} \circ \cdots \circ F_{n,n} \circ P_n : X \to K_n.$$

For a given $x \in X$ and $n \in \mathbb{N}$ we set $\widetilde{m} = \max\left\{k \in \{1, \dots, n+1\} : \sum_{i=1}^{k-1} \frac{||x_i||}{r_i} \le 1\right\}$ so that it is possible to compute $F_n(x)$ as

$$F_n(x) = \begin{cases} P_{\widetilde{m}-1}(x) + \frac{x_{\widetilde{m}}}{\|x_{\widetilde{m}}\|} f_{\widetilde{m}}(x), & \text{if } \widetilde{m} \le n, \\ P_n(x) & \text{if } \widetilde{m} = n+1 \end{cases}$$

Now, if $x, y \in P_n(X)$, we claim that $||F_n(x) - F_n(y)|| \le 5(1 + \delta)$. Indeed, let us consider

$$m = \max\left\{k \in \{1, \dots, n+1\} : \sum_{i=1}^{k-1} \frac{||x_i||}{r_i} \le 1, \sum_{i=1}^{k-1} \frac{||y_i||}{r_i} \le 1\right\}.$$

If m = n + 1 then $F_n(x) - F_n(y) = x - y$.

If m = n then $F_n(x) = F_{n,n}(x)$ and $F_n(y) = F_{n,n}(y)$ and we use Proposition 3.2 to finish this case.

If m = n - 1 then $F_n(x) = F_{n,n-1}(F_{n,n}(x))$ and $F_n(y) = F_{n,n-1}(F_{n,n}(y))$, and we know from the definition of m that x or y verifies (3.2) of Proposition 3.2 for j = n. Hence, using Lemma 3.1 together with Proposition 3.2 we get that

$$\begin{split} ||F_{n}(x) - F_{n}(y)|| &= ||F_{n,n-1}(F_{n,n}(x)) - F_{n,n-1}(F_{n,n}(y))|| \\ &\leq \frac{r_{n}A_{n}}{r_{n-1}}||F_{n,n}(x) - F_{n,n}(y)|| \\ &+ ||(F_{n-1,n-1} \circ P_{n-1})(F_{n,n}(x)) - (F_{n-1,n-1} \circ P_{n-1})(F_{n,n}(y))| \\ &\leq \left(5 + \frac{r_{n-1}A_{n-2}}{r_{n-2}} + \frac{r_{n}A_{n}}{r_{n-1}}\right)||F_{n,n}(x) - F_{n,n}(y)|| \\ &\leq \left(5 + \frac{r_{n-1}A_{n-2}}{r_{n-2}} + \frac{r_{n}A_{n}}{r_{n-1}}\right)\left(1 + \frac{r_{n}A_{n-1}}{r_{n-1}}\right)||x - y|| \\ &\leq \left(5 + \frac{\delta_{n-1}}{2} + \alpha_{n-1}\right)\left(1 + \frac{\delta_{n}}{2}\right)||x - y|| \leq 5(1 + \delta)||x - y||. \end{split}$$

Otherwise, if $m \leq n-2$ then

$$F_n(x) = F_{n,m} \circ \cdots \circ F_{n,n}(x)$$
 and $F_n(y) = F_{n,m} \circ \cdots \circ F_{n,n}(y)$.

Also, from the definition of m we get that, if $n \ge p \ge m+2$, then the point $F_{n,p} \circ \cdots \circ F_{n,n}(x)$ or the point $F_{n,p} \circ \cdots \circ F_{n,n}(y)$ satisfies (3.2) of Proposition 3.2 for j = p - 1. Also the point x or the point y satisfies (3.2) for j = n. Having this in mind and using the same argument as in the previous step,

we check that

$$\begin{split} ||F_{n}(x) - F_{n}(y)|| &= ||F_{n,m} \circ \cdots \circ F_{n,n}(x) - F_{n,m} \circ \cdots \circ F_{n,n}(y)|| \\ &\leq \left(5 + \frac{r_{m}A_{m-1}}{r_{m-1}} + \sum_{k=m+1}^{n} \frac{r_{k}A_{k}}{r_{k-1}}\right) ||F_{n,m+1} \circ \cdots \circ F_{n,n}(x) - F_{n,m+1} \circ \cdots \circ F_{n,n}(y)|| \\ &\leq \left(5 + \frac{r_{m}A_{m-1}}{r_{m-1}} + \sum_{k=m+1}^{n} \frac{r_{k}A_{k}}{r_{k-1}}\right) \left(1 + \frac{r_{m+1}A_{m}}{r_{m}} + \sum_{k=m+2}^{n} \frac{r_{k}A_{k}}{r_{k-1}}\right) \\ &\cdot ||F_{n,m+2} \circ \cdots \circ F_{n,n}(x) - F_{n,m+2} \circ \cdots \circ F_{n,n}(y)|| \\ & \leq \left(5 + \frac{r_{m}A_{m-1}}{r_{m-1}} + \sum_{k=m+1}^{n} \frac{r_{k}A_{k}}{r_{k-1}}\right) \left(\prod_{j=m+1}^{n-1} \left(1 + \frac{r_{j}A_{j-1}}{r_{j-1}} + \sum_{k=j+1}^{n} \frac{r_{k}A_{k}}{r_{k-1}}\right)\right) \\ &\cdot ||F_{n,n}(x) - F_{n,n}(y)|| \\ &\leq 5 \left(\prod_{j=m}^{n-1} \left(1 + \frac{r_{j}A_{j-1}}{r_{j-1}} + \sum_{k=j+1}^{n} \frac{r_{k}A_{k}}{r_{k-1}}\right)\right) \left(1 + \frac{r_{n}A_{n-1}}{r_{n-1}}\right) ||x - y|| \\ &\leq 5 \left(\prod_{j=m}^{n-1} \left(1 + \frac{\delta_{j}}{2} + \alpha_{j}\right)\right) \left(1 + \frac{\delta_{n}}{2}\right) ||x - y|| \leq 5 \prod_{j=m}^{n} (1 + \delta_{j})||x - y|| \\ &\leq 5(1 + \delta)||x - y||. \end{split}$$

It is easy to see that $\forall x \in P_n(X)$, if k > n then $F_k(x) = F_n(x)$ so we may define the following map

$$F: \bigcup_{n \in \mathbb{N}} P_n(X) \to \operatorname{co}\left(\bigcup_{n \in \mathbb{N}} r_n B_{X_n}\right),$$
$$F(x) = \lim_{n \to \infty} F_n(x),$$

which is a $5(1+\delta)$ -Lipschitz retraction. Considering now $R: X \to K$ as the extension of F to the whole X, we are done.

Remark 3.4. We may actually choose $q_n = \frac{1}{n2^{n+1}}$. This arises from choosing $\delta_n = 2^{-n+1}$ and $a_k = \frac{1}{k2^{k+1}}$, so that $\alpha_n = 2^{-n}$.

Notice that the restriction of the previous retraction R to one of the blocks of the FDD is the radial projection. Hence, it is not possible to obtain an estimate for the Lipschitz norm of R better than 2 for general FDD spaces. Next, we are going to treat the special case when the blocks of the FDD are of dimension 1, that is when X has a Schauder basis, which leads to a much better estimate on the Lipschitz norm of the retraction.

From now on, $\dim X_n = 1$ for every $n \in \mathbb{N}$ and $(e_n, e_n^*)_{n \in \mathbb{N}}$ is a monotone Schauder basis in X such that $||e_n|| = 1$ and $\langle e_n \rangle = X_n$. From now on $r_n \in \mathbb{R}^+$ is a decreasing sequence, (P_n) is the sequence of projections of the basis and we keep denoting $x_i = (P_i - P_{i-1})(x)$ for every $x \in X$. We

are going to keep the previous definition for $F_{n,m}$ and f_m , as well as for $E_{n,m}$. The main difference between this case and the general one is stated in Proposition 3.5.

Proposition 3.5.

$$F_{m,m}$$
 is $\left(1 + \frac{r_m A_{m-1}}{r_{m-1}}\right)$ -Lipschitz $\forall m \in \mathbb{N}$.

Proof. We are going to prove it case by case. We have already proved in Proposition 3.2 the case when $x, y \in E_{m,m} = P_m(X)$ are such that $\sum_{i=1}^{m-1} \frac{||y_i||}{r_i} > 1$ or $\sum_{i=1}^{m-1} \frac{||x_i||}{r_i} > 1$, so let us assume throughout all the proof that $\sum_{i=1}^{m-1} \frac{||x_i||}{r_i} \le 1$ and $\sum_{i=1}^{m-1} \frac{||y_i||}{r_i} \le 1$. If $\sum_{i=1}^{m} \frac{||x_i||}{r_i} \le 1$ and $\sum_{i=1}^{m} \frac{||y_i||}{r_i} \le 1$ then $F_{m,m}(x) - F_{m,m}(y) = x - y$ so it is straightforward that $||F_{m,m}(x) - F_{m,m}(y)|| = ||x - y||$.

If $\sum_{i=1}^{m} \frac{||x_i||}{r_i} \leq 1$ and $\sum_{i=1}^{m} \frac{||y_i||}{r_i} > 1$, then we split this case into 2 different subcases. For these subcases we are going to set

$$t = \frac{e_m^*(x) - \frac{e_m^*(y)}{|e_m^*(y)|} f_m(y)}{e_m^*(x) - e_m^*(y)}$$

whenever $e_m^*(x) \neq e_m^*(y)$ and $t = \infty$ otherwise.

Subcase $t \notin [0, 1]$.

In this subcase $|e_m^*(y)| = ||y_m|| > f_m(y) \ge 0$ so then $e_m^*(x) \ne 0$ because otherwise $t = \frac{f_m(y)}{|e_m^*(y)|} \in [0, 1]$. We claim that in this subcase $\frac{e_m^*(x)}{|e_m^*(x)|} = \frac{e_m^*(y)}{|e_m^*(y)|}$. Indeed, if not, multiplying and dividing t by $\frac{e_m^*(x)}{|e_m^*(x)|} = \frac{|e_m^*(y)|}{|e_m^*(x)|} = -\frac{|e_m^*(y)|}{|e_m^*(y)|}$, it is easy to see that

$$t = \frac{|e_m^*(x)| + f_m(y)}{|e_m^*(x)| + |e_m^*(y)|} \in [0, 1],$$

which is not possible. This means that $\frac{e_m^*(x)}{|e_m^*(x)|} = \frac{e_m^*(y)}{|e_m^*(y)|}$ so now it can be easily seen that

$$t = \frac{|e_m^*(x)| - f_m(y)}{|e_m^*(x)| - |e_m^*(y)|}$$

As $|e_m^*(x)| - f_m(y) \ge |e_m^*(x)| - |e_m^*(y)|$ we claim that $|e_m^*(x)| - f_m(y) \ge 0$. In fact, we check case by case and obtain the following scheme

$$\begin{cases} \text{if } t = \infty \quad \Rightarrow \ |e_m^*(x)| - |e_m^*(y)| = 0 \Rightarrow \ |e_m^*(x)| - f_m(y) \ge 0, \\ \text{if } t < 0 \quad \Rightarrow \ |e_m^*(x)| - |e_m^*(y)| < 0 \Rightarrow \ |e_m^*(x)| - f_m(y) > 0, \\ \text{if } t > 1 \quad \Rightarrow \ |e_m^*(x)| - |e_m^*(y)| > 0 \Rightarrow \ |e_m^*(x)| - f_m(y) > 0. \end{cases}$$

Then,

$$\left| e_m^*(x) - \frac{e_m^*(y)}{|e_m^*(y)|} f_m(y) \right| = |e_m^*(x)| - f_m(y) \le |f_m(x) - f_m(y)| \le \frac{r_m A_{m-1}}{r_{m-1}} ||x - y||,$$
and we have

 $||F_{m,m}(x) - F_{m,m}(y)|| = \left| \left| P_{m-1}(x-y) + \left(x_m - \frac{y_m}{||y_m||} f_m(y) \right) \right| \right|$ $\leq ||x-y|| + \left| e_m^*(x) - \frac{e_m^*(y)}{|e_m^*(y)|} f_m(y) \right|$ $\leq ||x-y|| + \frac{r_m A_{m-1}}{r_{m-1}} ||x-y||.$

The subcase $t \in [0, 1]$ is simpler. Due to the convexity of the norm and the fact that $||P_{m-1}|| = 1$ we have

$$||F_{m,m}(x) - F_{m,m}(y)|| = ||P_{m-1}(x-y) + t(x_m - y_m)|| \le ||x-y||,$$

so we are finally done with both subcases. Now, if $\sum_{i=1}^{m} \frac{||x_i||}{r_i} > 1$ and $\sum_{i=1}^{m} \frac{||y_i||}{r_i} > 1$ then $||x_m|| > 0$ and $||y_m|| > 0$ and we are also splitting this case into 2 subcases for technical reasons:

Subcase
$$\frac{x_m}{||x_m||} = \frac{y_m}{||y_m||}$$
. Here,

$$\left| \left| \frac{x_m}{||x_m||} f_m(x) - \frac{y_m}{||y_m||} f_m(y) \right| \right| = |f_m(x) - f_m(y)| \le \frac{r_m A_{m-1}}{r_{m-1}} ||x - y||,$$
there

so then

$$||F_{m,m}(x) - F_{m,m}(y)|| = \left| \left| P_{m-1}(x-y) + \left(\frac{x_m}{||x_m||} f_m(x) - \frac{y_m}{||y_m||} f_m(y) \right) \right| \right|$$

$$\leq \left(1 + \frac{r_m A_{m-1}}{r_{m-1}} \right) ||x-y||.$$

Finally, for the subcase when $\frac{x_m}{||x_m||} = -\frac{y_m}{||y_m||}$ we consider

$$t = \frac{f_m(x) + f_m(y)}{||x_m|| + ||y_m||} \in [0, 1]$$

and again by convexity,

$$||F_{m,m}(x) - F_{m,m}(y)|| = ||P_{m-1}(x-y) + t(x_m - y_m)|| \le ||x-y||.$$

Theorem 3.6. For every $\delta > 0$ there exists a sequence $(q_n) \subset \mathbb{R}^+$ such that for every Banach space X with a monotone Schauder basis (e_n) and every decreasing sequence $(r_n) \subset \mathbb{R}^+$ satisfying $\frac{r_n}{r_{n-1}} \leq q_n$, the set

$$K = \overline{\operatorname{co}}\left(\bigcup_{k \in \mathbb{N}} r_k B_{\langle e_k \rangle}\right)$$

is a $(1+\delta)$ -Lipschitz retract of X.

Proof. Consider $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ such that

$$\prod_{n \in \mathbb{N}} (1 + \delta_n) \le 1 + \delta,$$

and consider a sequence $(a_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ such that the sequence given by

$$(\alpha_n)_{n \in \mathbb{N}} := \left(\sum_{k=n+1}^{\infty} a_k A_k\right)_{n \in \mathbb{N}}$$

verifies that

$$\alpha_n \le \frac{\delta_n}{2} \qquad \forall n \in \mathbb{N}.$$

Now we set

$$q_n = \min\left\{a_n, \frac{\delta_n}{2A_{n-1}}\right\}.$$

Suppose that (r_n) is as in the statement of the Theorem. We fix n > 1, so that making use of Lemma 3.1 and Proposition 3.5, the following holds

If
$$m = 1$$
 $\Rightarrow ||F_{n,1}||_{Lip} \le 1 + \sum_{k=2}^{n} \frac{r_k A_k}{r_{k-1}}.$
If $m \in \{2, \dots, n-1\}$ $\Rightarrow ||F_{n,m}||_{Lip} \le 1 + \frac{r_m A_{m-1}}{r_{m-1}} + \sum_{k=m+1}^{n} \frac{r_k A_k}{r_{k-1}}$
If $m = n$ $\Rightarrow ||F_{n,n}||_{Lip} \le 1 + \frac{r_n A_{n-1}}{r_{n-1}}.$

Let us consider now the composed retraction

$$F_n = F_{n,1} \circ \cdots \circ F_{n,n} \circ P_n : X \to K_n.$$

As in Theorem 3.3 it is enough to show that $||F_n||_{Lip} \leq 1 + \delta$:

$$\begin{split} ||F_{n}||_{Lip} &\leq \prod_{m=1}^{n} ||F_{n,m}||_{Lip} \\ &\leq \left(1 + \sum_{k=2}^{n} \frac{r_{k}A_{k}}{r_{k-1}}\right) \left(\prod_{m=2}^{n-1} \left(1 + \frac{r_{m}A_{m-1}}{r_{m-1}} + \sum_{k=m+1}^{n} \frac{r_{k}A_{k}}{r_{k-1}}\right)\right) \\ &\quad \cdot \left(1 + \frac{r_{n}A_{n-1}}{r_{n-1}}\right) \\ &\leq (1 + \alpha_{1}) \left(\prod_{m=2}^{n-1} \left(1 + \frac{\delta_{m}}{2} + \alpha_{m}\right)\right) \left(1 + \frac{\delta_{n}}{2}\right) \leq \prod_{m=1}^{n} (1 + \delta_{m}) \leq 1 + \delta. \end{split}$$

4. π -property and compact Lipschitz retractions

We pass to the results concerning the necessary conditions on the Banach space X so that X admits a GCCR $K \subset X$. Our methods require a certain quantitative "smallness" condition to be satisfied for K. Under such assumption we show that X must have the π -property. In fact, our argument makes no use of the convexity of K. The crucial condition is smallness. Our proof uses three main ingredients. The unpublished Milman lemma (communicated to us, with proof, by Bill Johnson) concerning the projection constant of a finite dimensional subspace of a Banach space, the finite dimensional "Lipschitzization" of coarse Lipschitz maps due to Bourgain (and streamlined by Begun), and the averaging of derivatives for finite dimensional Lipschitz maps. We start with a well-known fact.

Given $r \in \mathbb{R}^+$, we are going to denote $[r] = \max\{n \in \mathbb{N} \cup \{0\} : n \leq r\}$.

Lemma 4.1. For every $n \in \mathbb{N}$ and $\varepsilon \in (0, 1)$, if $(E, || \cdot ||)$ is a Banach space of dimension n, then there exists a renorming $| \cdot |$ of E such that $(E, | \cdot |)$ embeds isometrically in ℓ_{∞}^{N} where $N = \left[(1 + 2/\varepsilon)^{n}\right]$ and

$$|x| \le ||x|| \le \frac{|x|}{1-\varepsilon}.$$

Proof. By Lemma 2.6 of [MS86], we know that there exists an ε -net in S_{X^*} consisting of N points, namely $\{x_1^*, \ldots, x_N^*\}$. Just consider the norm $|x| = \max_{i \in \{1, \ldots, N\}} x_i^*(x)$.

If X is a Banach space and $E \subset X$ is a subspace, then the projection constant of E in X is defined as

$$\lambda(E, X) = \inf \{ ||P|| : P : X \to E, P|_E = Id_E \}.$$

Lemma 4.2 (Vitali Milman-unpublished). Let X be a Banach space. For every $\varepsilon \in (0,1)$ and a subspace $E \subset X$ of dimension $\dim(E) = n$, there is another subspace $G_E \subset X$ containing E such that $\dim(G_E) \leq (1+\frac{2}{\varepsilon})^n$ and

$$\lambda(E, X) \le \frac{2}{1-\varepsilon}\lambda(E, G_E).$$

Proof. (communicated to us by Bill Johnson) We follow the trace duality arguments set up in [Joh+79]. Pick $\varepsilon \in (0, 1)$. For a given Banach space Y with a finite dimensional subspace $E \subset Y$ we define a pair of norms on the space of all linear operators $\mathcal{L}(E)$

$$||T||_{Y} = \inf \left\{ ||\widetilde{T}|| : \widetilde{T} \in \mathcal{L}(Y, E) , \widetilde{T}\Big|_{E} = T \right\},$$
$$||T||_{\Lambda Y} = ||i_{E}T||_{\Lambda},$$

where $i_E : E \to Y$ is the inclusion map and $|| \cdot ||_{\Lambda}$ refers to the nuclear norm in $\mathcal{L}(E, Y)$, that is,

$$||T||_{\Lambda Y} = \inf \bigg\{ \sum_{i=1}^{n} ||x_i^*|| \cdot ||y_i|| : n \in \mathbb{N}, T = \sum_{i=1}^{n} x_i^* \otimes y_i, x_i^* \in E^*, y_i \in Y \bigg\}.$$

We know from [Joh+79] pg. 377 that both norms are in trace duality. More precisely, we have a dual pairing $\langle \mathcal{L}(E), \mathcal{L}(E) \rangle$ given by $\langle T, S \rangle = tr(ST)$ for every $T, S \in \mathcal{L}(E)$, such that

$$(\mathcal{L}(E), ||\cdot||_Y)^* = (\mathcal{L}(E), ||\cdot||_{\Lambda Y}).$$

Thanks to this interpretation, we can compute

t

$$\lambda(E,Y) = ||Id_E||_Y = \sup_{||T||_{\Lambda Y}=1} tr(TId_E) = \sup_{T \in \mathcal{L}(E)} tr\left(\frac{T}{||T||_{\Lambda Y}}\right)$$
$$= \sup_{T \in \mathcal{L}(E)} \frac{1}{\left|\left|\frac{T}{tr(T)}\right|\right|_{\Lambda Y}} = \sup_{tr(T)=1} \frac{1}{||T||_{\Lambda Y}} = \frac{1}{\inf_{tr(T)=1} ||T||_{\Lambda Y}}.$$

Returning to the situation of our theorem, E is now a subspace of X. Let us first take $\mu \in (0, 1/2)$. Now we take, for $\delta = \frac{(1-\varepsilon)(1-\mu)-(1-\varepsilon)1/2}{\lambda(E,X)} > 0$, a trace one operator $S \in \mathcal{L}(E)$ such that

$$\inf_{r(T)=1} ||T||_{\Lambda X} \ge ||S||_{\Lambda X} - \delta.$$

We also take the norm $|\cdot|$ given by Lemma 4.1 so that $(E, |\cdot|)$ is isometrically a subspace of $\ell_{\infty}^{\varphi(n)}$ where $\varphi(n) = \left[\left(1 + \frac{2}{\varepsilon}\right)^n\right]$ and, denoting $|\cdot|_{\Lambda Y}$ the nuclear norm taking $(E, |\cdot|)$ as the domain of the operators instead of $(E, ||\cdot||)$, we have for every superspace $Y \supset E$ that

$$||S||_{\Lambda Y} \le |S|_{\Lambda Y} \le \frac{||S||_{\Lambda Y}}{1 - \varepsilon}$$

It is well-known (Proposition 47.6 in [Tre06]) that $i_E S$ admits an extension $\widetilde{S} : \ell_{\infty}^{\varphi(n)} \to X$ almost preserving the nuclear norm, that is $|S|_{\Lambda X} \ge (1-\mu)||\widetilde{S}||_{\Lambda}$. By Proposition 8.7 from [TJ89] we know that there exist $x_1, \ldots, x_{\varphi(n)} \in X$ such that $\widetilde{S} = \sum_{i=1}^{\varphi(n)} e_i^* \otimes x_i$ and

$$||\widetilde{S}||_{\Lambda} = \sum_{i=1}^{\varphi(n)} ||x_i||,$$

where $e_i^* \in (\ell_{\infty}^{\varphi(n)})^*$ are the coordinate functionals. Just considering $G_E = [x_i]_{i=1}^{\varphi(n)}$, we can see that

$$\begin{aligned} ||S||_{\Lambda X} &\geq (1-\varepsilon)|S|_{\Lambda X} \geq (1-\varepsilon)(1-\mu)||S||_{\Lambda} \\ &\geq (1-\varepsilon)(1-\mu)|S|_{\Lambda G_E} \geq ||S||_{\Lambda G_E}(1-\varepsilon)(1-\mu). \end{aligned}$$

Finally, taking into acount that $\lambda(E, G_E) \leq \lambda(E, X)$, we finish the proof because

$$\lambda(E,X) \leq \frac{1}{||S||_{\Lambda X} - \delta} \leq \frac{1}{(1-\varepsilon)(1-\mu)||S||_{\Lambda G_E} - \delta}$$
$$\leq \frac{1}{(1-\varepsilon)(1-\mu)\inf_{tr(T)=1}||T||_{\Lambda G_E} - \delta}$$
$$= \lambda(E,G_E)\frac{1}{(1-\varepsilon)(1-\mu) - \delta\lambda(E,G_E)} \leq \frac{2}{1-\varepsilon}\lambda(E,G_E)$$

Definition 4.3. Given a separable Banach space X, we will say that $\beta = (e_n) \subset X$ is a fundamental sequence if $[e_n] = X$ and we will denote $E_n^\beta = [e_i]_{i=1}^n$.

Definition 4.4. Let X be a separable Banach space, $\beta = (e_n)$ a fundamental sequence and $K \subset X$ a bounded subset. We will define the following concepts:

• The sequence of inner radii (r_n^β) given by

$$r_n^\beta = \sup\left\{r \ge 0 \ : \ B_{E_n^\beta}(x,r) \subset K \cap E_n^\beta \ , \ x \in X\right\} \quad \forall n \in \mathbb{N}.$$

• The sequence of heights (h_n^β) given by

$$h_n^\beta = \sup \left\{ d(x, E_n^\beta) \ : \ x \in K \right\} \quad \forall n \in \mathbb{N}.$$

Definition 4.5. We say that a bounded subset K of a separable Banach space X is small if there exist an $\varepsilon \in (0, 1)$, a fundamental sequence $\beta = (e_n)$ in X and a strictly increasing $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$0 < \frac{h_{\sigma(n)}^{\beta}}{r_{\sigma(n)}^{\beta}} \le \frac{1}{2\sigma(n)^2 \left(\left(1 + \frac{2}{\varepsilon}\right)^{\sigma(n)} + 2 \right)} \quad \forall n \in \mathbb{N}.$$

Note that such sets are necessarily compact and generate X.

If $\beta = (e_n)$ is a monotone Schauder basis and $X_i = \langle e_i \rangle$, then we know from Theorem 3.3 and Remark 3.4 that for every sequence $(r_n) \subset \mathbb{R}^+$ such that

$$\frac{r_n}{r_{n-1}} \le q_n = \frac{1}{n2^{n+1}},$$

the compact $K = \operatorname{co}\left(\bigcup_{k \in \mathbb{N}} r_k B_{X_k}\right)$ is a GCCR. In this case it is easily seen that there is a C > 0 independent of $n \in \mathbb{N}$ and X with

$$\frac{h_n^\beta}{r_n^\beta} \le C \frac{q_n}{n} = \frac{C}{n^{2}2^{n+1}},$$

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where the right hand side of the inequality is a very similar sequence to the one given in the definition of smallness.

More generally, it is easy to check using Remark 3.4 that the following result holds.

Proposition 4.6. If a separable Banach space X has an FDD then X admits a small GCCR.

We now pass to the promised opposite implication. Note that the convexity assumption on the generating compact K is not needed.

Theorem 4.7. Let X be a separable Banach space. If there exists a Lipschitz retraction from X onto a small compact subset, then X has the π -property.

Proof. Assume that there is a Lipschitz retraction from X onto a small compact K. Take $\varepsilon \in (0,1)$, $\beta = (e_n)$ a fundamental sequence of X and $\sigma : \mathbb{N} \to \mathbb{N}$ strictly increasing for which the inequality of Definition 4.5 holds true, and let $\varphi(n) = \left[\left(1 + \frac{2}{\varepsilon}\right)^{\sigma(n)}\right]$, $E_n = E_{\sigma(n)}^{\beta}$, $h_n = h_{\sigma(n)}^{\beta}$ and $r_n = r_{\sigma(n)}^{\beta}$. Lemma 4.2 guarantees that for every $n \in \mathbb{N}$ there is a finite dimensional subspace $G_n \subset X$ of dimension $\dim(G_n) = \varphi(n)$ such that for every projection $P : G_n \to E_n$, the inequality $||P|| \ge (1 - \varepsilon)\frac{\lambda(E_n, X)}{2}$ holds. Assume that $R : X \to K$ is the Lipschitz retraction, then taking $C_n : K \to E_n$ a nearest point map (it may not be unique), we define $\widetilde{R}_n = (C_n \circ R)|_{G_n} : G_n \to E_n$ for every $n \in \mathbb{N}$. Now,

$$||\widetilde{R}_n(x) - \widetilde{R}_n(y)|| \le ||R|| \left(||x - y|| + \frac{2h_n}{||R||} \right) \quad \forall x, y \in G_n,$$

so by the Proposition of [Beg99], for every $\tau > 0$, there is a Lipschitz mapping $R_{n,\tau}: G_n \to E_n$

such that

$$||R_{n,\tau}||_{Lip} \le ||R|| \left(1 + \frac{\varphi(n)h_n}{||R||\tau}\right),$$
$$||R_{n,\tau}(x) - \widetilde{R}_n(x)|| \le ||R|| \left(\tau + \frac{2h_n}{||R||}\right) \quad \forall x \in G_n$$

For the rest of the argument we fix $x_n \in K_n := K \cap E_n$ such that $B_{E_n}(x_n, r_n) \subset K_n$ (it exists by the definition of the inner radius and the compactness). Now we choose $\tau_n = \frac{\varphi(n)h_n}{||R||}$, and define $R_n : G_n \to E_n$ by $R_n(x) = R_{n,\tau_n}(x+x_n) - x_n$. If $x + x_n \in K_n$ then $\widetilde{R}_n(x+x_n) = x + x_n$ acts as an identity. Hence we have that for every $x \in K_n + \{-x_n\}$

$$||R_n(x) - x|| = ||R_{n,\tau_n}(x + x_n) - \widetilde{R}_n(x + x_n)|| \le h_n(\varphi(n) + 2) =: \rho_n.$$

Now, let $(a_i, a_i^*)_{i=1}^{\varphi(n)}$ be a normalized linear basis for G_n with projections $(S_i)_{i=1}^{\varphi(n)}$ such that $(a_i, a_i^*)_{i=1}^{\sigma(n)}$ is an Auerbach basis for E_n . Then,

$$B_n = r_n \operatorname{co} \left(\{ \pm a_i , i = 1, \dots, \sigma(n) \} \right) \subset K_n + \{ -x_n \}.$$

Fix a sequence (δ_k) of positive numbers converging to zero. Now we define for every $k \in \mathbb{N}$ the compact

$$B_{n,k} = B_n + \delta_k \sum_{i=\sigma(n)+1}^{\varphi(n)} [-a_i, a_i] \subset G_n.$$

The geometrical shape of this set can be described as a r_n -multiple of the unit ball of $\ell_1^{\sigma(n)}$, a sort of a base of a hypercylinder, located in E_n times a hypercube of side length $2\delta_k$ sticking into the remaining dimensions of G_n . Letting δ_k go to zero of course means that this set gets squashed down to its base in E_n . For any $i \in \{1, \ldots, \sigma(n)\}$ we denote $(B_{n,k})_i = (Id - a_i^* \otimes a_i)(B_{n,k})$. This set is just a one-codimensional section (or a projection of rank $\varphi(n)-1$) of $B_{n,k}$ which reduces the base by one coordinate.

In order to recover the shape $B_{n,k}$ from its section $(B_{n,k})_i$ we pass from any point $x^i \in (B_{n,k})_i$ to boundary point of $B_{n,k}$ which got projected onto it. The newly acquired coordinate vector will then be denoted by $x_i(x^i)$ and given by the formula

$$x_{i}(x^{i}) = \left(r_{n} - \sum_{\substack{j=1\\ j \neq i}}^{\sigma(n)} a_{j}^{*}(x^{i})\right) a_{i}, \ x^{i} \in (B_{n,k})_{i}.$$

For convenience in our computations, we also introduce the quantity

$$z_i(x^i) = R_n(x^i + x_i(x^i)) - R_n(x^i - x_i(x^i)) - 2x_i(x^i).$$

As $S_{\sigma(n)}$ is a linear projection, we know that $-S_{\sigma(n)}(x^i + x_i(x^i)) + S_{\sigma(n)}(x^i - x_i(x^i)) = -2x_i(x^i)$. Using this and a triangle inequality with four terms we have that

$$\begin{aligned} |z_{i}(x^{i})|| &\leq \left| \left| R_{n} \left(x^{i} + x_{i}(x^{i}) \right) - R_{n} \left(S_{\sigma(n)} \left(x^{i} + x_{i}(x^{i}) \right) \right) \right| \right| \\ &+ \left| \left| R_{n} \left(S_{\sigma(n)} \left(x^{i} + x_{i}(x^{i}) \right) \right) - S_{\sigma(n)} \left(x^{i} + x_{i}(x^{i}) \right) \right| \right| \\ &+ \left| \left| R_{n} \left(S_{\sigma(n)} \left(x^{i} - x_{i}(x^{i}) \right) \right) - R_{n} \left(x^{i} - x_{i}(x^{i}) \right) \right) \right| \right| \\ &+ \left| \left| S_{\sigma(n)} \left(x^{i} - x_{i}(x^{i}) \right) - R_{n} \left(S_{\sigma(n)} \left(x^{i} - x_{i}(x^{i}) \right) \right) \right| \right| \\ &\leq 2 ||R_{n}||\varphi(n)\delta_{k} + 2\rho_{n}. \end{aligned}$$

For a set $J \subset \{1, \ldots, \varphi(n)\}$ with $\#J = m \ge 1$, we define the measure in $[a_i]_{i \in J}$ as

$$\lambda_J^m(A) = \lambda_m \left(\prod_{i \in J} \left(a_i^*(A) \right) \right) \quad \forall A \in \mathcal{M}_J^m,$$

where λ_m is the Lebesgue measure in \mathbb{R}^m and $\mathcal{M}_J^m = \left\{ A \subset [a_i]_{i \in J} : \prod_{i \in J} \left(a_i^*(A)\right) \text{ is Lebesgue measurable subset of } \mathbb{R}^m \right\}$. If $J = \{1, \dots, \varphi(n)\}$ we denote $\lambda^{\varphi(n)} = \lambda_J^{\varphi(n)}$ and for every $i \in \{1, \dots, \varphi(n)\}$, if $J = \{1, \dots, \varphi(n)\} \setminus \mathbb{R}^n$

 $\{i\}$, we denote $\lambda_i^{\varphi(n)-1} = \lambda_J^{\varphi(n)-1}$. Then, we are ready to define the linear operators $P_{n,k}: G_n \to E_n$, for every $k \in \mathbb{N}$, as

$$P_{n,k}(v) = \frac{1}{\lambda^{\varphi(n)}(B_{n,k})} \int_{B_{n,k}} dR_n(x)[v] d\lambda^{\varphi(n)}(x).$$

In [Bra+14] pg. 47 the volumes of ℓ_p^n balls B_p^n have been computed as $|B_p^n| = \frac{2^n \Gamma(\frac{1}{p}+1)^n}{\Gamma(\frac{n}{p}+1)}$. Using this result for p = 1 (for the base part of our set $B_{n,k}$) and the standard properties of Lebesgue measure we obtain the following values for our sets for arbitrary $i \in \{1, \ldots, \sigma(n)\}$

$$\lambda_i^{\varphi(n)-1} \big((B_{n,k})_i \big) = \frac{2^{\varphi(n)-1} r_n^{\sigma(n)-1} \delta_k^{\varphi(n)-\sigma(n)}}{(\sigma(n)-1)!}$$
$$\lambda^{\varphi(n)} \big(B_{n,k} \big) = \frac{2^{\varphi(n)} r_n^{\sigma(n)} \delta_k^{\varphi(n)-\sigma(n)}}{\sigma(n)!},$$

so the quotient is

$$\frac{\lambda_i^{\varphi(n)-1}((B_{n,k})_i)}{\lambda^{\varphi(n)}(B_{n,k})} = \frac{\sigma(n)}{2r_n}.$$

Note that the expression

$$R_n(x^i + x_i(x^i)) - R_n(x^i - x_i(x^i)) = z_i(x^i) + 2x_i(x^i)$$

represents the difference of the values of the operator R_n between the endpoints of a segment cutting through $B_{n,k}$, which passes through the point x^i with direction a_i . As $B_{n,k} = \{u + w \in G_n : u \in (B_{n,k})_i, w \in [-x_i(x^i), x_i(x^i)]\}$, thanks to Fubini's Theorem and the Fundamental Theorem of Calculus applied to the *i*-th coordinate, we can compute for each $i \in \{1, \ldots, \sigma(n)\}$

$$\begin{aligned} ||a_{i} - P_{n,k}(a_{i})|| &= \left| \left| a_{i} - \frac{1}{\lambda^{\varphi(n)}(B_{n,k})} \int_{B_{n,k}} dR_{n}(x)[v] d\lambda^{\varphi(n)}(x) \right| \right| \\ &= \left| \left| a_{i} - \frac{1}{\lambda^{\varphi(n)}(B_{n,k})} \int_{(B_{n,k})_{i}} z_{i}(x^{i}) + 2x_{i}(x^{i}) d\lambda_{i}^{\varphi(n)-1}(x^{i}) \right| \right| \\ &= \left| \left| \frac{1}{\lambda^{\varphi(n)}(B_{n,k})} \int_{(B_{n,k})_{i}} z_{i}(x^{i}) d\lambda_{i}^{\varphi(n)-1}(x^{i}) \right| \right| \\ &\leq \frac{\lambda_{i}^{\varphi(n)-1}((B_{n,k})_{i})}{\lambda^{\varphi(n)}(B_{n,k})} (2||R_{n}||\varphi(n)\delta_{k} + 2\rho_{n}) \\ &= \frac{\sigma(n)||R_{n}||\varphi(n)\delta_{k} + \sigma(n)\rho_{n}}{r_{n}}. \end{aligned}$$

We may assume that $P_{n,k}$ pointwise converge in k and define $P_n(x) = \lim_{k \to \infty} P_{n,k}(x)$ for every $x \in G_n$, which is a linear operator from G_n to E_n

satisfying that

$$||P_n(a_i) - a_i|| \le \frac{\sigma(n)h_n(\varphi(n) + 2)}{r_n} \quad \forall i \in \{1, \dots, \sigma(n)\}.$$

Using the fact that K is small and $||a_i^*|| = 1$, $i \in \{1, \ldots, \sigma(n)\}$, we obtain that for every $x \in E_n$

$$||P_n(x) - x|| = \left| \left| \sum_{i=1}^{\sigma(n)} a_i^*(x) (P_n(a_i) - a_i) \right| \right| \le \frac{h_n \sigma(n)^2 (\varphi(n) + 2)}{r_n} ||x|| \le \frac{1}{2} ||x||.$$

Finally, we construct the projection $\widetilde{P}_n = \left(P_n\Big|_{E_n}\right)^{-1} \circ P_n : G_n \to E_n$ of norm

$$||\widetilde{P}_n|| \le \left| \left| \left(P_n \right|_{E_n} \right)^{-1} \right| \right| \cdot ||P_n|| \le 2 \cdot 2||R|| = 4||R|| \quad \forall n \in \mathbb{N}.$$

This implies that X has the π -property since

$$\lambda(E_n, X) \le \frac{2}{1 - \varepsilon} \lambda(E_n, G_n) \le \frac{2}{1 - \varepsilon} ||\widetilde{P}_n|| \le \frac{8||R||}{1 - \varepsilon} \quad \forall n \in \mathbb{N}.$$

Recall that thanks to Theorem 1.3 in [JRZ71], dual Banach spaces have an FDD if and only if they enjoy the π -property. Hence we have the next characterization.

Corollary 4.8. Let X be a separable dual Banach space. Then X has an FDD if and only if X admits a small GCCR, if and only if X admits a small subset which is a Lipschitz retract of X.

In Section 3 we have found a sequence $(q_n) \subset \mathbb{R}^+$ such that for every sequence $r = (r_n) \subset \mathbb{R}^+$ satisfying that $\frac{r_n}{r_{n-1}} \leq q_n$ there is a λ -Lipschitz retraction $R(r) : X \to K(r)$, where $K(r) = \operatorname{co}\left(\bigcup_{k \in \mathbb{N}} r_k B_{X_k}\right)$ for some FDD $(X_n)_{n \in \mathbb{N}}$. Let $r = (r_n)$ be such a sequence and denote for every $k, m \in \mathbb{N}$ the sequence $r^{k,m} = (r_1, \ldots, r_m, r_{m+1}/k, \ldots, r_n/k, \ldots)$. Taking subsequences, we may assume that for every $x \in X$ and every $m \in \mathbb{N}$ there exists $R_m(x) = \lim_{k \to \infty} R(r^{k,m})(x)$ which define retractions onto increasing finite dimensional compacts. This leads to the π -property for Lipschitz retractions.

Definition 4.9. Let X be a separable Banach space and $\lambda > 0$. We say that X has the Lipschitz π_{λ} -property if there exists an increasing sequence of finite dimensional convex subsets (C_n) of X such that $X = \bigcup_{n \in \mathbb{N}} span(C_n)$ and there exists a λ -Lipschitz retraction $R_n : X \to C_n$ for every $n \in \mathbb{N}$.

Analogously to the result of Godefroy and Kalton on the Lipschitz bounded approximation property (Theorem 2.9), we are going to prove that this new property is nothing else but the well-known π -property. This result is a

direct consequence of the next Theorem, which is mainly based on an ultraproduct technique similar to the result of Lindenstrauss in [Lin64], see Corollary 7.3 of [BL00].

Theorem 4.10. Let X be a Banach space, $\lambda_1, \lambda_2 > 0$ and $Y \subset X$ a subspace λ_1 -complemented in its bidual. If there is a λ_2 -Lipschitz retraction from X onto a convex subset K containing 0 such that $\overline{span}(K) = Y$ then Y is $\lambda_1\lambda_2$ -complemented in X.

Proof. Let $R: X \to K \subset Y$ be such a retraction. Then for every $n \in \mathbb{N}$ we define the λ_2 -Lipschitz retraction $R_n: X \to nK$ given by $R_n(x) = nR(x/n)$, for every $x \in X$. As in the proof of Theorem 7.2 of [BL00] we let \mathcal{U} to be a free ultrafilter on \mathbb{N} , and we put

$$S(x) = \lim_{\mathcal{U}} R_n(x) \quad \forall x \in X.$$

It is standard to check that $S: X \to Y^{**}$ is a λ_2 -Lipschitz mapping, which is identity on Y. Now, if $L: Y^{**} \to Y$ is a bounded linear projection, we just define $\tilde{R}: X \to Y$ by

$$\widehat{R}(x) = L \circ S(x) \quad \forall x \in X$$

Finally \widetilde{R} is a $\lambda_1 \lambda_2$ -Lipschitz retraction from X onto Y so by Corollary 7.3 of [BL00] we are done.

Corollary 4.11. Let X be a separable Banach space and $\lambda > 0$. Then X has the Lipschitz π_{λ} -property if and only if it has the π_{λ} -property.

Proof. It is straightforward from Proposition 2.3 and Theorem 4.10. \blacksquare

5. Compacts without Lipschitz retractions

We proceed by constructing an example of a small convex and compact Kin C[0, 1], which is contained in a small GCCR, contains a small GCCR, and yet there is no Lipschitz retraction onto K. The idea behind the construction can be described as follows. The GCCR constructed at the beginning of our note are well "aligned" with the FDD on X, and in the proof that the smallness condition of GCCR implies the π -property the projections are also aligned with the structure of the compact. So our strategy is to employ badly complemented finite dimensional subspaces (in fact, Hilbert spaces) of C[0, 1] as the sections of the sought compact K. In order to glue the decreasing sequence of these pieces together, we use the \mathcal{L}_{∞} -FDD in C[0, 1]. We start with a standard argument.

Lemma 5.1. For any $\varepsilon > 0$ there is an FDD in C[0,1] with associated projections P_n , and a sequence $(a_n) \subset C[0,1]$, $(1 + \varepsilon)$ -equivalent to the ℓ_2 basis such that

- $\sup_{n \to \infty} d(P_n(C[0,1]), \ell_{\infty}^{d(n)}) < \infty$ where $d(n) = \dim(P_n(C[0,1])).$
- $a_n \in (P_n P_{n-1})(C[0,1])$ for every $n \in \mathbb{N}$.

Proof. By Remark 5.2 of [JRZ71] there is an FDD (X_n) in C[0,1] such that

(5.1)
$$\sup_{n\in\mathbb{N}} d(\tilde{Q}_n(C[0,1]), \ell_\infty^{d(n)}) = d < \infty,$$

where the \tilde{Q}_n are the natural projections of the FDD given by $\tilde{Q}_n(\sum_{i=1}^{\infty} x_i) = \sum_{i=1}^{n} x_i$. Take an ℓ_2 -basis $(b_n)_{n \in \mathbb{N}} \subset C[0, 1]$, which is certainly a w^* -null normalized sequence. For a given $\varepsilon > 0$ apply the standard Bessaga-Pelczynski blocking principle (e.g. [Fab+11] p. 194) to obtain a subsequence $(b_{k(n)})$ of the original ℓ_2 basis, and its perturbed companion (a_n) , $\sum_{n=1}^{\infty} ||a_n - b_{k(n)}|| < \varepsilon$, such that $a_n \in (\tilde{Q}_{\eta(n)} - \tilde{Q}_{\eta(n-1)})(C[0,1])$, for some increasing sequence of indices $(\eta(n))$. To finish the proof it remains to let $((P_n - P_{n-1})(C[0,1]))$ be the desired FDD, where $P_n = \tilde{Q}_{\eta(n)}$.

To simplify the notation, let us put X = C[0,1] and let P_n and (a_n) be given by Lemma 5.1 for some fixed $\varepsilon \in (0,1)$. Now define the function $\sigma : \mathbb{N} \to \mathbb{N}$, so that $\sigma(1) = 1$ and $\sigma(n) = \sigma(n-1) + n$ for every $n \geq 2$. Then, letting $Q_n = P_{\sigma(n)}$ and $X_n = (Q_n - Q_{n-1})(X)$ we have that $Y_n = span(\{a_{\sigma(n-1)+1}, \ldots, a_{\sigma(n)}\})$ is a subspace of X_n , $(1 + \varepsilon)$ -isometric to ℓ_2^n . From now on in this section we are going to denote $E_n = Q_n(X)$.

We define for every $n \in \mathbb{N}$ and $\delta > 0$ the set

$$B_n^{\delta} = \Big\{ x \in X_n : d(x, B_{Y_n}) \le \delta \Big\}.$$

It is clear that B_n^{δ} is a compact convex subset of $(1 + \delta)B_{X_n}$ that generates X_n . We will need the following results to continue with our construction.

Lemma 5.2. Let d be as in (5.1). For every $n \in \mathbb{N}$, if there is a Lipschitz retraction $\varphi : B_{E_n} \to B_{Y_n}$, then there is a Lipschitz retraction $\psi : \ell_{\infty} \to \ell_2^n$ satisfying the following inequality

$$||\psi||_{Lip} \le \frac{2d||\varphi||_{Lip}}{1-\varepsilon}.$$

Proof. By composing φ with the norm 2 radial retraction from E_n onto B_{E_n} we have a retraction $\widehat{\varphi} : E_n \to B_{Y_n}$ with $||\widehat{\varphi}||_{Lip} \leq 2||\varphi||_{Lip}$. Using the technique in the proof of Theorem 4.10, since Y_n is finite dimensional, there exists even a linear projection $\widetilde{\varphi} : E_n \to Y_n$ of norm $||\widetilde{\varphi}|| \leq 2||\varphi||_{Lip}$. Since E_n is *d*-isometric to ℓ_{∞}^M , and Y_n is $(1+\varepsilon)$ isometric to ℓ_2^n the desired estimate follows.

Lemma 5.3 (Lindenstrauss, '64). For every $n \in \mathbb{N}$, if $\psi : \ell_{\infty} \to \ell_2^n$ is a Lipschitz retraction then

$$||\psi||_{Lip} \ge \frac{n^{1/4}}{3}.$$

Proof. Just use Lemma 1.28 of [BL00] with $r = n^{1/4}$ and $\varepsilon = 1$.

Proposition 5.4. There exist sequences $(M_n), (\delta_n) \subset \mathbb{R}^+, M_n \to \infty, \delta_n \to 0$, such that for every $n \in \mathbb{N}$ there is no M_n -Lipschitz retraction from B_{E_n} onto $B_n^{\delta_n}$.

Proof. It suffices to let

$$M_n = \frac{n^{1/4}(1-\varepsilon)}{25d} \quad \forall n \in \mathbb{N}.$$

Now, for a fixed $n \in \mathbb{N}$ we are going to prove the existence of $\delta_n > 0$ satisfying the statement of the Proposition by contradiction. Suppose that for every $\delta > 0$ there exists a retraction $\phi^{\delta} : B_{E_n} \to B_n^{\delta}$ satisfying $||\phi^{\delta}||_{Lip} \leq M_n$. Then we define $N : E_n \to B_{E_n}$ the radial projection, and $C^{\delta} : B_n^{\delta} \to B_{Y_n}$ a nearest point map. The map $\psi^{\delta} = C^{\delta} \circ \phi^{\delta} \circ N : E_n \to B_{Y_n}$ satisfies the following inequality

$$||\psi^{\delta}(x) - \psi^{\delta}(y)|| \le 2M_n \left(||x - y|| + \frac{\delta}{M_n}\right),$$

so we are allowed to use the Proposition of [Beg99]. Then, for every $\tau > 0$ denoting by χ_{τ} the indicator function of τB_{E_n} we have that $\varphi^{\delta,\tau} = \psi^{\delta} * \chi_{\tau}$: $B_{E_n} \to B_{Y_n}$ is a Lipschitz map satisfying

$$\begin{split} ||\varphi^{\delta,\tau}(x) - \psi^{\delta}(x)|| &\leq 2M_n \left(\tau + \frac{\delta}{M_n}\right) \quad \forall x \in B_{E_n}, \\ ||\varphi^{\delta,\tau}||_{Lip} &\leq 2M_n \left(1 + \frac{\delta N_n}{2\tau M_n}\right). \end{split}$$

Let us take a sequence $(\delta_k) \subset \mathbb{R}^+$ decreasing to 0, and put $\tau_k = \frac{\delta_k N_n}{2M_n}$ for every $k \in \mathbb{N}$. Denoting $\varphi_k = \varphi^{\delta_k, \tau_k}$ we have that φ_k pointwise converge to a retraction $\varphi : B_{E_n} \to B_{Y_n}$ with the norm $||\varphi||_{Lip} \leq 4M_n$. By Lemma 5.2 there exists a Lipschitz retraction $\psi : \ell_{\infty} \to \ell_p^n$ such that

$$||\psi||_{Lip} \le \frac{2d||\varphi||_{Lip}}{1-\varepsilon} \le \frac{8dM_n}{1-\varepsilon} = \frac{8}{25}n^{1/4} < \frac{n^{1/4}}{3}.$$

This contradicts Lindenstrauss' Lemma 5.3.

Finally, we are ready to construct nonretractable convex subsets of C[0, 1]. Note that if λ_n below are decreasing sufficiently fast then the resulting set K will be small.

Theorem 5.5. For every sequence $(\lambda_n) \subset \mathbb{R}^+$, the subset of C[0,1] given by

$$K = \overline{\operatorname{co}} \left(\bigcup_{n \in \mathbb{N}} \lambda_n B_n^{\delta_n} \right)$$

is not Lipschitz retractable.

Proof. Suppose there exists a Lipschitz retraction $\phi : C[0,1] \to K$ and let $||\phi||_{Lip} = L$. Then, there is an $n \in \mathbb{N}$ such that $L < \frac{M_n}{2M}$ where M is the constant of the FDD (Q_n) , that is, $||Q_n|| \leq M$ for every $n \in \mathbb{N}$. Now, the retraction $R_n = (Q_n - Q_{n-1}) \circ \phi|_{E_n} : E_n \to \lambda_n B_n^{\delta_n}$ has norm $||R_n||_{Lip} \leq 2ML < M_n$. Finally, the retraction $F_n : E_n \to B_n^{\delta_n}$ given by $F_n(x) = \lambda_n^{-1} R_n(\lambda_n x)$ for every $x \in E_n$ has norm $||F_n||_{Lip} \leq ||R_n||_{Lip} < M_n$ which contradicts Proposition 5.4. ■

6. Nearest point map

The radial projection mapping from a Banach space X onto B_X can be shown to have a Lipschitz constant at most 2, [DW64].

Figueiredo and Karlovitz [FK67] proved that a real normed linear space of dimension 3 or higher is an inner product space if and only if the radial projection onto the unit ball is nonexpansive (see also [Phe58], [Sch65]).

For more general sets C a natural candidate for the retraction mapping is the nearest point map (sometimes also called the proximity map, of which the radial projection is a special case), provided the nearest point is unique. Again, it turns out that such a mapping is nonexpansive for every closed convex set if and only if the space is Hilbertian [Phe58]. In fact, in the papers [Bru74] [FK70], the following characterization is shown. Let X be a strictly convex Banach space with $3 \leq \dim(X)$; then there exists a nonexpansive retraction onto the unit ball if and only if X is a Hilbert space. The restriction to dimension at least 3 is necessary. Karlovitz [Kar72] showed that if X is 2-dimensional then a non-expansive retraction onto any closed convex set always exists and it can be realized as a proximity mapping with respect to a new norm.

Let us begin our last section with a new characterization of the Hilbert space in the spirit of the above results.

Theorem 6.1. Let X be a Banach space. Then X is isomorphic to a Hilbert space if and only if every convex and compact subset $K \subset X$ is a Lipschitz retract of X.

Proof. Of course, it suffices to deal with infinite dimensional spaces X, and in light of the Phelps [Phe58] characterization, it suffices to prove only one implication (passing from retractions to Hilbert space). By inspection of the proof of the Lindenstrauss-Tzafriri characterization of the Hilbert space [LT71], it is clear that a Banach space is isomorphically Hilbert if and only if there exists some $\lambda > 0$ such that every finite-dimensional subspace $E \subset X$ is λ -complemented in X. Proceeding by contradiction, suppose that every convex compact is a Lipschitz retract of X, but the complementation constants $\lambda(E, X)$, where E runs through all finite-dimensional subspaces of X, are not uniformly bounded. It is easy to see that this also means that for any finite-codimensional subspace $Y \subset X$, the complementation constants $\lambda(E, X)$, where $E \subset Y$ is finite dimensional, are not uniformly bounded. We COMPACT RETRACTS

construct inductively a sequence (E_n) of finite -dimensional subspaces of X, such that this sequence forms an FDD with the projection constant at most 2 of its closed span (a subspace of X), so that $\lambda(E_n, X) > n$. Let us describe the inductive step. Having constructed the initial sequence $(E_n)_1^N$ such that the FDD has the projection constant bounded by $2 - \frac{1}{N}$, we use the Mazur technique for constructing basic sequences [Fab+11] p.191. Namely, there exists a finite set of functionals $\{f_i\}_1^M$ from X^* that is $(1 + \frac{1}{2N^2})$ -norming for $E_1 \oplus \cdots \oplus E_N$. Let $Y = \bigcap_1^M \operatorname{Ker} f_i$ be a finite-codimensional subspace of X. Now, it suffices to choose a finite dimensional subspace E_{N+1} of Y such that $\lambda(E_{N+1}, X) > N$. The projection constant for the new FDD is under control.

By the same argument as in the proof of Theorem 4.10 it follows that the Lipschitz norm of any Lipschitz retraction from X onto B_E cannot be less than $\lambda(E, X)$. Let us pass to the construction of K. Fix a sequence (τ_n) of positive numbers such that $\sum \tau_n$ is finite. Finally, set K to be the closed convex hull of $\cup \tau_n B_{E_n}$. Supposing that $\phi : X \to K$ is a λ -Lipschitz retraction, for some $\lambda > 0$, we compose ϕ with the canonical retraction r_n of K onto $\tau_n B_{E_n}$ (which exists due to the structure of K, and has a Lipschitz constant at most 6), for $n > 10\lambda$. Then $r_n \circ \phi : X \to \tau_n B_{E_n}$ is a Lipschitz retraction of Lipschitz norm at most 6λ , hence $\lambda(E_n, X) \leq 6\lambda$ which is a contradiction.

This leaves us with the natural question whether at least uniformly continuous retractions onto convex compact subsets of a general Banach space are always possible. We give a strong positive answer to this problem, showing that in fact under the URED renorming every convex and compact subset K is a uniform retract, from any bounded set B containing K, by means of the nearest point map. Without loss of generality, it suffices to deal with the case when $B = B_X$ is the unit ball of X.

Definition 6.2. Given a subset K of a Banach space X, we will say that X is K-URED if it is uniformly rotund in the direction z for every $z \in span(K)$.

Definition 6.3. Given a subset K of a Banach space X, we will say that X is K-UR if for every pair of sequences $(x_n), (y_n) \in S_X$ such that $x_n - y_n \in K$ and $||x_n + y_n|| \to 2$ we have that $||x_n - y_n|| \to 0$.

Lemma 6.4. Let X be a Banach space that is uniformly rotund in the direction $z \in X \setminus \{0\}$. If there are $v_n, w_n \in S_X$ such that $v_n - w_n \to z$, then there exist $\widetilde{v_n}, \widetilde{w_n} \in S_X$ such that $v_n - \widetilde{v_n}, w_n - \widetilde{w_n} \to 0$ and $\widetilde{v_n} - \widetilde{w_n} = \lambda_n z$ for some $\lambda_n \in \mathbb{R}$.

Proof. First take for every $n \in \mathbb{N}$

$$t_n = \max\{t \ge 0 : w_n + tz \in S_X\}.$$

Let us prove that $t_n \to t \leq 1$. If we suppose that t > 1, then taking $\lambda_n = 1/t_n$ we have that

$$w_n + z = (1 - \lambda_n)w_n + \lambda_n(w_n + t_n z),$$

where $w_n, w_n + t_n z \in S_X$ and $w_n - (w_n + t_n z) = -t_n z$. In particular, $w_n + z \in B_X$ and we have that

$$1 \ge ||w_n + z|| \ge |||v_n|| - ||z - (v_n - w_n)||| \to 1,$$

so by the assumption that the norm is uniformly rotund in the direction z we get that $\lambda_n \to \lambda \in \{0,1\}$ which leads to a contradiction because $\lambda_n = 1/t_n \to 1/t \in (0,1)$. This means we can assume that $t \leq 1$.

If t = 1 then we define $\widetilde{w_n} = w_n$ and $\widetilde{v_n} = w_n + t_n z$ and we are done. If t < 1 we can assume that $t_n < 1$ for every $n \in \mathbb{N}$. In this case, we define $\widetilde{v_n} = \frac{w_n + z}{||w_n + z||}$ and

$$s_n = \max\{s \ge 0 : \widetilde{v_n} + s(-z) \in S_X\}.$$

Following the same argument as above, we know that $s = \lim s_n \leq 1$. As t < 1 we know that $||w_n + z|| > 1$, so $\frac{w_n}{||w_n + z||} = \widetilde{v_n} - \frac{z}{||w_n + z||} \in B_X$. Hence $s_n \geq \frac{1}{||w_n + z||}$. Just notice that $||w_n + z|| \to 1$ because

$$1 < ||w_n + z|| \le ||v_n|| + ||z - (v_n - w_n)|| \to 1,$$

 \mathbf{so}

$$1 \ge s = \lim s_n \ge \lim \frac{1}{||w_n + z||} = 1 \quad \Rightarrow \quad s = 1.$$

Finally, we are able to define $\widetilde{w_n} = \widetilde{v_n} + s_n(-z)$.

Proposition 6.5. Let X be a Banach space and $z \in S_X$, the following assertions are equivalent:

- X is uniformly rotund in the direction z.
- For every pair of sequences $x_n, y_n \in S_X$ such that $x_n y_n \to \lambda z$, for some $\lambda \in \mathbb{R}$, if $||x_n + y_n|| \to 2$ then $\lambda = 0$.

Proof. If X is uniformly rotund in the direction z and we suppose by contradiction that the second assertion does not hold, then there exist $x_n, y_n \in S_X$ such that $x_n - y_n \to \lambda z$ for some $\lambda \neq 0$ with $||x_n + y_n|| \to 2$. Then using Lemma 6.4 we get $\widetilde{x_n}, \widetilde{y_n} \in S_X$ such that $\widetilde{x_n} - x_n, \widetilde{y_n} - y_n \to 0$ and $\widetilde{x_n} - \widetilde{y_n} = \lambda_n z$. Now, we have

$$2 \ge ||\widetilde{x_n} + \widetilde{y_n}|| \ge ||x_n + y_n|| - (||x_n - \widetilde{x_n}|| + ||y_n - \widetilde{y_n}||) \to 2,$$

so $\lambda_n \to 0$ which is impossible because $\lambda_n z = \widetilde{x_n} - \widetilde{y_n} \to \lambda z \neq 0$. The other implication is straightforward.

Proposition 6.6. Let X be a Banach space and $K \subset X$. If K is compact, then X is K-URED if and only if it is K-UR.

Proof. If X is K-URED, let us take $x_n, y_n \in S_X$ such that $x_n - y_n \in K$ and $||x_n + y_n|| \to 2$. Then, as K is compact, $x_n - y_n \to z \in K \subset span(K)$ so $||x_n - y_n|| \to 0$ just by Proposition 6.5. If X is K-UR then it is trivially K - URED.

Proposition 6.7. Let X be a Banach space and $K \subset B_X$ be a compact

convex subset. If X is K - URED then the nearest point map from B_X onto K is uniformly continuous.

Proof. We are going to argue by contradiction. Assuming that the nearest point map $R: B_X \to K$ is not uniformly continuous, then there exists an $\varepsilon > 0$ and a pair of sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subset B_X$ such that $||x_n - y_n|| \to 0$ and $||Rx_n - Ry_n|| > \varepsilon$ for every $n \in \mathbb{N}$. By compactness, we may assume that the sequences $(||Rx_n - x_n||), (||Ry_n - y_n||), (Rx_n)$ and (Ry_n) are all convergent. In particular, we claim that $\lim ||Rx_n - x_n|| = \lim ||Ry_n - y_n|| = d \in \mathbb{R}^+$. Let us first prove that $\lim ||Rx_n - x_n|| = \lim ||Ry_n - y_n||$. Otherwise, we may assume that $\lim ||Rx_n - x_n|| > \lim ||Ry_n - y_n|| + \rho$ for some $\rho > 0$. Then there is an $n \in \mathbb{N}$ such that $||Rx_n - x_n|| > ||Ry_n - y_n|| + \rho/2$ and $||x_n - y_n|| \le \rho/2$, so

$$||Ry_n - x_n|| \le ||Ry_n - y_n|| + ||x_n - y_n|| \le ||Ry_n - y_n|| + \rho/2 < ||Rx_n - x_n||$$

This contradicts the definition of R. Now, if d = 0 then $Rx_n, x_n \to p \in K$ and $Ry_n, y_n \to q \in K$ so p = q because $x_n - y_n \to 0$. This means that $\lim ||Rx_n - Ry_n|| \to 0$ which is impossible.

Assuming that d > 0, we are going to use the previous Lemma 6.4 with $v_n = \frac{Rx_n - x_n}{||Rx_n - x_n||}$, $w_n = \frac{Ry_n - y_n}{||Ry_n - y_n||}$ and $z = \frac{p-q}{d}$ where $p = \lim Rx_n$ and $q = \lim Ry_n$. Indeed,

$$v_n - w_n = \frac{Rx_n}{||Rx_n - x_n||} - \frac{Ry_n}{||Ry_n - y_n||} + \frac{y_n||Rx_n - x_n|| - x_n||Ry_n - y_n||}{||Rx_n - x_n|| \, ||Ry_n - y_n||}$$

so $v_n - w_n \to z = \frac{p-q}{d}$. Let us take $\widetilde{v_n}, \widetilde{w_n} \in S_X$ given by Lemma 6.4. As

$$||\widetilde{v_n} + \widetilde{w_n}|| \ge ||v_n + w_n|| - (||\widetilde{v_n} - v_n|| + ||\widetilde{w_n} - w_n||),$$

we have that $2 \ge \lim ||\widetilde{v_n} + \widetilde{w_n}|| \ge \lim ||v_n + w_n||$. For this reason, in order to prove that $||\widetilde{v_n} + \widetilde{w_n}|| \to 2$, it is enough to prove that $\left|\left|\frac{v_n + w_n}{2}\right|\right| \to 1$.

Equivalently, we are going to prove that $\left\| \frac{(Rx_n - x_n) + (Ry_n - y_n)}{2} \right\| \to d$. If that does not hold, then there exists $\rho > 0$ such that

$$d = \lim ||Rx_n - x_n|| > \lim \left| \left| \frac{(Rx_n - x_n) + (Ry_n - y_n)}{2} \right| \right| + \rho$$

Now, there exists an $n \in \mathbb{N}$ such that

$$||Rx_n - x_n|| > \left| \left| \frac{(Rx_n - x_n) + (Ry_n - y_n)}{2} \right| \right| + \rho/2 \quad , \quad ||x_n - y_n|| < \rho.$$

So taking into account the convexity of K, the next inequality is a contradiction with the definition of R:

$$\left\| \frac{Rx_n + Ry_n}{2} - x_n \right\| \le \left\| \frac{(Rx_n - x_n) + (Ry_n - y_n)}{2} \right\| + \left\| \frac{x_n + y_n}{2} - x_n \right\| < \left\| \frac{(Rx_n - x_n) + (Ry_n - y_n)}{2} \right\| + \rho/2 < \|Rx_n - x_n\|.$$

Finally, we know that $||\widetilde{v_n} + \widetilde{w_n}|| \to 2$ and $\widetilde{v_n} - \widetilde{w_n} = \lambda_n(p-q)$ with $\lambda_n \in \mathbb{R}$. As the norm of X is K-URED, we get that $\lambda_n \to 0$. This is in fact impossible because

$$||\widetilde{v_n} - \widetilde{w_n}|| \ge ||v_n - w_n|| - (||\widetilde{v_n} - v_n|| + ||\widetilde{w_n} - w_n||) \to ||z|| > 0.$$

Recall that thanks to Zizler's result ([Ziz71]) every separable Banach space has an equivalent URED renorming.

Corollary 6.8. If X is a separable Banach space, then it can be renormed so that the nearest point map from B_X onto any compact and convex subset of B_X is uniformly continuous.

The next proof is a slight modification of the proof given by V. Zizler to renorm separable spaces with URED norm ([Ziz71]).

Proposition 6.9. For every compact subset K of a Banach space X, there is an equivalent K-URED norm on X.

Proof. Let us take V = span(K) which has to be a separable Banach space. Now, there is a countable set $\{f_n\}_{n \in \mathbb{N}} \subset S_{X^*}$ separating the points of V. We define the linear operator $T: X \to \ell_2$ as $Tx = \left(\frac{f_n(x)}{2^n}\right)_{n \in \mathbb{N}}$. It is easily seen that T is bounded and its restriction to V is injective. Our new equivalent norm is going to be

$$|||x||| = \sqrt{||x||_X^2 + ||Tx||_2^2},$$

where $|| \cdot ||_X$ is the initial norm on X and $|| \cdot ||_2$ is the usual norm on ℓ_2 . If we suppose that there exists a bounded sequence $(x_n) \subset X$ such that

(6.1)
$$2(|||x_n + z|||^2 + |||x_n|||^2) - |||2x_n + z|||^2 \to 0,$$

for some $z \in V$, then

 $\begin{array}{l} 2(||x_n+z||^2+||x_n||^2)-||2x_n+z||^2+2(||Tx_n+Tz||_2^2+||Tx_n||_2^2)-||2Tx_n+Tz||_2^2\rightarrow 0. \\ \text{In particular, } 2(||Tx_n+Tz||_2^2+||Tx_n||_2^2)-||2Tx_n+Tz||_2^2\rightarrow 0. \\ \text{Making use of Proposition 1 of [Ziz71], the equivalence } 1 \Leftrightarrow 7 \text{ means that } \ell_2 \text{ is not uniformly rotund in the direction } Tz, \text{ leading to a contradiction because } \ell_2 \text{ is in fact UR. Then, (6.1) does not hold for any bounded sequence } (x_n) \subset X \\ \text{which again by the equivalence } 1 \Leftrightarrow 7 \text{ means that } X \text{ is uniformly rotund in the direction } z. \\ \blacksquare \end{array}$

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Corollary 6.10. For every compact and convex subset $K \subset B_X$ of a Banach space X, there is an equivalent norm on X such that the nearest point map from B_X onto K is well defined and uniformly continuous.

The following corollaries were recently established in [CCW21].

Corollary 6.11. For every compact and convex subset K of a Banach space X there is a uniformly continuous retraction from X onto K.

Proof. We may assume without loss of generality that $K \subset B_X$. It suffices to compose the 2-Lipschitz retraction of X onto B_X with the uniformly continuous retraction from B_X onto K obtained earlier.

Corollary 6.12. Compact convex subsets of Banach spaces are absolute uniform retracts.

Proof. Just embed the metric space into $\ell_{\infty}(\Gamma)$ for some set Γ and use Corollary 6.11.

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