mathematics

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Best Proximity Point Theorems Without Fuzzy P-Property for Several $(\psi - \phi)$-Weak Contractions in Non-Archimedean Fuzzy Metric Spaces

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Abstract: This paper addresses a problem of global optimization in a non-Archimedean fuzzy metric space context without fuzzy P-property. Specifically, it concerns the determination of the fuzzy distance between two subsets of a non-Archimedean fuzzy metric space. Our approach to solving this problem is to find an optimal approximate solution to a fixed point equation. This approach has been well studied within a category of problems called proximity point problems. We explore some new types of $(\psi - \phi)$-weak proximal contractions and investigate the existence of the unique best proximity point for such kinds of mappings. Subsequently, some fixed point results for corresponding contractions are proved, and some illustrative examples are presented to support the validity of the main results. Moreover, an interesting application in computer science, particularly in the domain of words has been provided. Our work is a fuzzy generalization of the proximity point problem by means of fuzzy fixed point method.

Keywords: best proximity point; global optimization; $(\psi - \phi)$-weak proximal contraction; fuzzy P-property; non-Archimedean fuzzy metric space; domain of words

MSC: 47H10; 47H09; 54H25

Citation: Zhou, M.; Saleem, N.; Roldán López de Hierro, A.F.; Liu, X. Best Proximity Point Theorems Without Fuzzy P-Property for Several $(\psi - \phi)$-Weak Contractions in Non-Archimedean Fuzzy Metric Spaces. *Mathematics* **2022**, *10*, 4031. https://doi.org/10.3390/math10214031

Academic Editors: Salvador Romaguera, Dimitrios Georgiou and Manuel Sanchis

Received: 3 October 2022
Accepted: 24 October 2022
Published: 30 October 2022

1. Introduction and Preliminaries

Let (X, d) be a metric space, and C, D be two nonempty subsets of X. Suppose that $T : C \to D$ is a non-self mapping. An element $x^* \in C$ is called a best approximate point of T if $d(x^*, Tx^*) = \inf\{d(T^*x, y) : y \in C\}$. One of the distinguished best approximation results attributed to Fan [1] assures the existence of a best approximation point of a continuous mapping of a nonempty compact convex subset of a Hausdorff locally convex topological vector space. As a generalization of the concept of a best approximated point, Basha and Veeramani [2] introduced the concept of best proximity point. We recall that an element $x' \in C$ is called a best proximity point of T if $d(x', Tx') = d(C, D) = \inf\{d(c, d) : (c, d) \in C \times D\}$. The best proximity point represents an optimal solution to the fixed point equation $Tx = x$, when the non-self mapping T has no fixed point, that is, to find an element x' such that the distance between x' and Tx' is minimum.

On the opposite hand, fuzzy set theory has been concerned in mathematics as a crucial tool to resolve the various uncertainties problems and ambiguities since Zadeh [3] introduced the idea of fuzzy set. Inspired by the idea of Zadeh, Kramosil and Michalek [4] introduced the concept of fuzzy metric space in 1975. Later on, George and Veermani [5] modified the definition in [4] to generate a Hausdorff topology. Such a topology is metrizable, and each metric will induce a fuzzy metric in Hausdorff topology. Fuzzy metric has some blessings over regular metric because of the flexibility and versatility that the fuzzy notions inherently possess. Fuzzy metrics are powerful tools for modeling various problems with uncertainties in reality. For instance, Gregori and Sapena [6] applied fuzzy metrics to the color image process to filter noisy images and to some other engineering problems of special interest. A fuzzy metric was applied to improve the color image filtering. Some filters were improved when some classical metrics were replaced with fuzzy metrics. For further details, readers are suggested to see [7,8]. Further to this, several fixed point results were established in fuzzy metric spaces. Some remarkable studies in this field are referred to [6,9–12]. More than that, the non-Archimedean property, an additional assumption, was added to the notion of fuzzy metric spaces to overcome some shortcomings in the study of fixed point theory. The non-Archimedean property weakens the criterion in the notion of fuzzy metric spaces; that is, the same real parameter can relate to the fuzzy distance between any three points of the underlying space. This property is very useful in practice because the main examples of fuzzy metric spaces that are handled in applications satisfy such a constraint. Many authors established several fixed point results which generalized fuzzy Banach contraction in fuzzy settings. For more details, readers are referred to [13–16].

Attempts to generalize the Banach contraction principle have been around for a long time. Nowadays, it remains an active branch of fixed point theory (for example, see [17–19]). Among these works, one such generalization is the concept of the weak contraction principle, which was first introduced by Alber et al. [20] in Hilbert spaces and later adapted to complete metric spaces by Rhoades [21]. A weak contraction mapping is intermediate between a contraction mapping and a non-expansive mapping. Later on, several authors created many results using weak contractions; see [22–25]. Saha et al. [26] introduced a weak contraction including two control functions in fuzzy metric spaces. In fuzzy metric spaces, control functions are used in similar ways to produce similar outcomes [27]. Compared with those results, in Saha's work, control functions are supposed to satisfy some other conditions suitable for fuzzy metric spaces. In 2019, Saha et al. [28] investigated the existence of the unique best proximity point for such weak contraction in fuzzy metric spaces by exploring P-property, which provided a way to obtain some proximity points after the unavailability of fixed points and approximate points for non-self mappings, extending and fuzzifying the existing results in metric spaces.

For the sake of completeness, we will recall and present some basic definitions, notations, lemmas and propositions used in the following.

Definition 1 ([5,29])**.** *A fuzzy metric space is a 3-tuple $(X, M, *)$, where X is a nonempty set, M is a fuzzy set on $X \times X \times (0, +\infty)$ such that for all $x, y, z \in X$ and $t, s > 0$:*

(F_1) $M(x, y, t) > 0$;
(F_2) $M(x, y, t) = 1$ *if and only if* $x = y$;
(F_3) $M(x, y, t) = M(y, x, t)$;
(F_4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(F_5) $M(x, y, \cdot) : (0, +\infty) \to [0, 1]$ *is continuous,*

where $*$ is a continuous t-norm. We recall that t-norm is a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ for which

(T_1) $*$ *is commutative and associative;*
(T_2) $*$ *is continuous;*
(T_3) $a * 1 = a$ *for all* $a \in [0, 1]$;
(T_4) $a * b \leq c * d$ *when* $a \leq c$ *and* $b \leq d$ *with* $a, b, c, d \in [0, 1]$.

If we replace (F_4) by (F_6): $M(x, y, t) * M(y, z, s) \leq M(x, z, \max\{t, s\})$ or $M(x, y, t) * M(y, z, t) \leq M(x, z, t)$ then the triple $(X, M, *)$ is called a non-Archimedean fuzzy metric space. Note that since (F_6) implies (F_4), each non-Archimedean fuzzy metric space is a fuzzy metric space.

Definition 2 ([5]). *Let $(X, M, *)$ be a fuzzy metric space (or a non-Archimedean fuzzy metric space). Then*

(a) *A sequence $\{x_n\}$ in X is said to be convergent to x if $\lim_{n\to+\infty} M(x_n, x, t) = 1$, for all $t > 0$;*

(b) *A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $M(x_n, x_m, t) > 1 - \epsilon$, for all $t > 0$ and $n, m \geq n_0$;*

(c) *A fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.*

The following lemmas were proved by Grabiec [11] for fuzzy metric spaces in the sense of Kramosil and Michálek. The conclusions are also valid for the fuzzy metric space given in Definition 1.

Lemma 1 ([11]). *Let $(X, M, *)$ be a fuzzy metric space. Then, $M(x, y, \cdot)$ is nondecreasing for all $x, y \in X$.*

Lemma 2 ([30]). *M is a continuous function on $X^2 \times (0, +\infty)$.*

For two given nonempty subsets A and B of a non-Archimedean fuzzy metric space $(X, M, *)$, the following notions are used throughout this paper:

$$M(A, B, t) = \sup\{M(x, y, t) : x \in A, y \in B\} \quad \text{for all} \quad t > 0.$$
$$A_0(t) = \{x \in A : M(x, y, t) = M(A, B, t), \quad \text{for some} \quad y \in B\},$$
$$B_0(t) = \{y \in B : M(x, y, t) = M(A, B, t), \quad \text{for some} \quad x \in A\}, \quad \text{for all} \quad t > 0.$$

Remark 1. *Similar to the above notations, there are notions of $A_0(t)$ and $B_0(t)$ that have been used in work [27,31,32]. The difference between them is that here, they are independent from the real parameter t.*

Most of the contractive conditions in fixed point theory make use of two metric terms: the distance between two distinct points, $d(x, y)$, and the distance between their images, $d(Tx, Ty)$, under the self-mapping T. However, for best proximity point theory, since $T : A \to B$ is not a self-mapping, for any x in A, one can not define $T^n(x)$, that is, for a fixed $x_0 \in A$, it is not possible to define the iterated sequence $x_{n+1} = Tx_n$, for each $n \in \mathbb{N}$, in a usual way. To overcome this shortcoming, recently, Sankar Raj [33,34] introduced a new property called P-property for a pair of disjoint nonempty subsets of a metric space. It is an essential geometrical property. The P-property is automatically valid for the pair (A, A). It has been proved in [34] that the P-property holds for any pair (A, B) of nonempty closed and convex subsets in a Hilbert space but not in an arbitrary Banach space. In metric fixed point theory, such a property for a pair of subsets is separately assumed for specific purposes. To solve the best proximity point problem in the setting of fuzzy metric spaces, Saha et al. [35] provided a fuzzy extension of P-property stated as follows.

Definition 3 ([35]). *Let (A, B) be a pair of nonempty disjoint subsets of a fuzzy metric space $(X, M, *)$. Then, the pair (A, B) is said to satisfy the fuzzy P-property if for all $t > 0$ and $x_1, x_2 \in A, y_1, y_2 \in B$, $M(x_1, y_1, t) = M(A, B, t)$ and $M(x_2, y_2, t) = M(A, B, t)$ jointly imply that $M(x_1, x_2, t) = M(y_1, y_2, t)$.*

Definition 4 ([27]). *Let A and B be two nonempty subsets of X where $(X, M, *)$ is a fuzzy metric space. An element $x^* \in A$ is defined as a best proximity point of the mapping $T : A \to B$ if it satisfies the condition that for all $t > 0$*

$$M(x^*, Tx^*, t) = M(A, B, t).$$

Definition 5 ([28])**.** *Let A and B be two subsets of X, where $(X, M, *)$ is a complete fuzzy metric space. Let $T : A \to B$ be a mapping that satisfies the following inequality*

$$\psi(M(Tx, Ty, t)) \leq \psi(M(x, y, t)) - \phi(M(x, y, t)),$$

where $x, y \in A, t > 0$ and $\psi, \phi : (0, 1] \to [0, \infty)$ are such that

(i) *ψ is monotone decreasing and continuous with $\psi(s) = 0 \Leftrightarrow s = 1$;*
(ii) *ϕ is lower semi continuous with $\phi(s) = 0 \Leftrightarrow s = 1$.*

Then, T is a weak contraction.

It is noted that in the case that $A = B$, the above definition reduces to that of a weak contraction introduced in [26] which is weaker than a fuzzy Banach contraction but stronger than a fuzzy non-expansive mapping.

Saha et al. in [28] established the existence and uniqueness of best proximity point for the weak contractions mentioned above in the frame of fuzzy metric spaces, as stated in the following theorem.

Theorem 1 ([28])**.** *Let $(X, M, *)$ be a complete fuzzy metric space. Let A and B be two closed subsets of X and $T : A \to B$ be a non-self weak contraction mapping such that the following conditions are satisfied:*

(i) *(A, B) satisfies the fuzzy P-property;*
(ii) *$T(A_0) \subseteq B_0$;*
(iii) *A_0 is nonempty.*

Then, there exists an element $x^* \in A$ which is a fuzzy best proximity point of T.

In this paper, we aim to enrich the study in the domain of fuzzy global optimization by the use of fuzzy fixed point methods. To this aim, we will provide four types of $(\psi - \phi)$-weak proximal contractions inspired by Saha's contraction and investigate the existence and uniqueness of the best proximity points for these types of weak proximal contractions without fuzzy P-property assumption in the frame of non-Archimedean fuzzy metric spaces (in the sense of George and Veermani). As a consequence, some fixed point results for corresponding contractions are proved, and some illustrative examples are presented to support the validity of the main results. Moreover, an interesting application in computer science, particularly in the domain of words, has been provided.

2. Best Proximity Point Results without Fuzzy *P*-Property

Let us begin with some definitions and propositions in the setting of the non-Archimedean fuzzy metric, which will be used to prove our main results.

Definition 6 ([36])**.** *A t-norm $*$ is said to be 1-boundary continuous if it is continuous at each point of the type $(1, s)$ where $s \in [0, 1]$ (that is, if $\{t_n\} \to 1$ and $\{s_n\} \to s$, then $\{t_n * s_n\} \to 1 * s = s$).*

Obviously, each continuous t-norm is 1-boundary continuous. Moreover, a t-norm called the Drastic norm $*_D$ defined by $t *_D s = \begin{cases} 0, & \text{if} \quad s, t < 0 \\ \min\{t, s\}, & \text{if} \quad t = 1 \quad \text{or} \quad s = 1 \end{cases}$ is the one which is not 1-boundary continuous.

Proposition 1 ([36])**.** *Let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\} \in [0, 1]$ be five sequences and let $L \in [0, 1]$ be a number such that $\{a_n\} \to L, \{b_n\} \to 1, \{d_n\} \to 1$ and $\{e_n\} \to L$. Suppose that $*$ is a 1-boundary continuous t-norm and that $a_n \geq b_n * c_n * d_n \geq e_n$ for all $n \in \mathbb{N}$. Then, $\{c_n\}$ converges to L.*

Proof. The conclusion can be drawn from the proof of Proposition 5 in [36]. □

Proposition 2. *The limit of a convergent sequence in a non-Archimedean fuzzy metric space (in the sense of George and Veeramani) whose t-norm is only* 1*-boundary continuous is unique.*

Proof. Let $\{x_n\}$ be a convergent sequence with $x_n \to x^*$ in a non-Archimedean fuzzy metric space $(X, M, *)$ whose t-norm $*$ is 1-boundary continuous.

Suppose that x' is another limit of $\{x_n\}$. From (F_6), we have that for all $t > 0$

$$M(x', x^*, t) \geq M(x', x_n, t) * M(x_n, x^*, t)$$

and

$$M(x', x_n, t) \geq M(x', x^*, t) * M(x^*, x_n, t).$$

Taking limit as $n \to +\infty$ in the above inequalities, we have for all $t > 0$

$$M(x', x^*, t) \geq 1 * 1 = 1 \quad \text{and} \quad 1 \geq M(x', x^*, t) * 1 = M(x', x^*, t),$$

which shows that $M(x', x^*, t) = 1$ for all $t > 0$, that is, $x' = x^*$. □

In 2011, Basha [37] introduced the concept of approximately compact sets in metric spaces. After that, Saleem et al. [38,39] provided the fuzzy extension of the concept illustrated in the following definition.

Definition 7 ([38])**.** *Let A and B be two nonempty subsets of a non-Archimedean fuzzy metric space. A is said to be approximately compact with respect to B if every sequence $\{x_n\}$ in A satisfying the condition that $M(y, x_n, t) \to M(y, A, t)$ for some $y \in B$ and all $t > 0$ has a convergent subsequence.*

It is noted that every set is approximately compact with respect to itself. If A intersects B, then $A \cap B$ is contained in both A_0 and B_0. Furthermore, it can be seen that if A is compact and B is approximately compact with respect to A, then the sets A_0 and B_0 are nonempty.

Let Ψ be the set of all functions $\psi : (0, 1] \to [0, \infty)$ satisfying the following conditions:

(i) ψ is monotone decreasing, that is, for all $t, s \in (0, 1]$, $t \leq s \Rightarrow \psi(t) \geq \psi(s)$;
(ii) ψ is continuous;
(iii) $\psi(t) = 0$ if and only if $t = 1$.

Here are some functions ψ belonging to Ψ:

(1) $\psi(t) = \frac{1-t}{t}$, for $0 < t \leq 1$;
(2) $\psi(t) = -\ln t$, for $0 < t \leq 1$;
(3) $\psi(t) = 1 - t$, for for $0 < t \leq 1$.

Let Φ be the set of all functions $\phi : (0, 1] \to [0, \infty)$ satisfying the following conditions:

(i) ϕ is lower semi continuous;
(ii) $\phi(t) = 0$ if and only if $t = 1$.

Here are some functions ϕ belonging to Φ:

(1) $\phi(t) = \frac{(1-k)(1-t)}{t}$, for $k \in (0, 1)$ and $0 < t \leq 1$;
(2) $\phi(t) = \frac{t-1}{t}$, for $0 < t \leq 1$;
(3) $\phi(t) = \frac{1}{t} - \frac{1}{\sqrt{t}}$, for $0 < t \leq 1$.

From now and onward, let A and B be nonempty subsets of X such that $A \cap B$ is nonempty. On the other hand, in the George and Veeramani's original definition, the t-norm should be continuous. In the following discussion, we will consider the more general case of a George and Veeramani's fuzzy metric space in which the t-norm is only 1-boundary continuous. Now, we are ready to introduce four new types of $(\psi - \phi)$-weak contractions in a fuzzy metric space $(X, M, *)$ and prove the corresponding best proximity point results in the setting of a non-Archimedean fuzzy metric space.

2.1. $(\psi - \phi)$-Weak Proximal Contraction of Type I and Its Best Proximity Point Results

Definition 8. *Let $(X, M, *)$ be a fuzzy metric space. A non-self mapping $T : A \to B$ is called a $(\psi - \phi)$-weak proximal contraction of type I, if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that for all $x, y, u, v \in A_0$ and $t > 0$*

$$\left.\begin{array}{l} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \text{ implies } \psi(M(u, v, t)) \leq \psi(M(x, y, t)) - \phi(M(x, y, t)).$$

Theorem 2. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space whose t-norm is 1-boundary continuous. Let A and B be two nonempty closed subsets of X such that A_0 is nonempty. Let $T : A \to B$ be a non-self mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$;
(ii) *T is a $(\psi - \phi)$-weak proximal contraction of type I;*
(iii) *For any sequence $\{y_n\} \subseteq B_0$ and $x \in A$ satisfying $M(x, y_n, t) \to M(A, B, t)$ as $n \to +\infty$, then $x \in A_0$.*

Then, T has a unique best proximity point x^* in A_0.

Proof. Let $x_0 \in A_0$ be an arbitrary element. Since $Tx_0 \in T(A_0) \subseteq B_0$, we can find $x_1 \in A_0$ such that

$$M(x_1, Tx_0, t) = M(A, B, t) \quad \text{for all} \quad t > 0.$$

Furthermore, since $Tx_1 \in T(A_0) \subseteq B_0$, it follows that there exists $x_2 \in A_0$ such that

$$M(x_2, Tx_1, t) = M(A, B, t) \quad \text{for all} \quad t > 0.$$

Recursively, we obtain a sequence $\{x_n\} \subseteq A_0$ satisfying

$$M(x_n, Tx_{n-1}, t) = M(A, B, t) \quad \text{for all} \quad n \in \mathbb{N}, \quad \text{and for all} \quad t > 0. \tag{1}$$

If there exists $n_0 \in \mathbb{N} \cup \{0\}$ for which $M(x_{n_0}, x_{n_0+1}, t) = 1$ for all $t > 0$, it follows that

$$\begin{aligned} M(A, B, t) &\geq M(x_{n_0}, Tx_{n_0}, t) \\ &\geq M(x_{n_0}, x_{n_0+1}, t) * M(x_{n_0+1}, Tx_{n_0}, t) \\ &= 1 * M(x_{n_0+1}, Tx_{n_0}, t) \\ &= M(A, B, t) \quad \text{for all} \quad t > 0. \end{aligned}$$

Hence, $M(x_{n_0}, Tx_{n_0}, t) = M(A, B, t)$ for all $t > 0$, so x_{n_0} is a best proximity point of T.

Without loss of generality, in the following, we may assume that $0 < M(x_n, x_{n+1}, t) < 1$ for all $n \in \mathbb{N} \cup \{0\}$, that is, $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Since T is a $(\psi - \phi)$-weak proximal contraction of type I, we obtain

$$\psi(M(x_{n+1}, x_{n+2}, t)) \leq \psi(M(x_n, x_{n+1}, t)) - \phi(M(x_n, x_{n+1}, t)) \leq \psi(M(x_n, x_{n+1}, t)). \tag{2}$$

Since ψ is monotone decreasing, then $M(x_{n+1}, x_{n+2}, t) \geq M(x_n, x_{n+1}, t)$, so the sequence $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence in $[0, 1]$ which is bounded above by 1 for all $t > 0$.

Taking $\lim_{n \to +\infty} M(x_n, x_{n+1}, t) = r(t)$, we shall claim that $r(t) = 1$ for all $t > 0$.

Indeed, taking limit as $n \to +\infty$ in (2), using the continuity of ψ and the lower semi continuity of ϕ, we obtain

$$\psi(r(t)) \leq \psi(r(t)) - \phi(r(t)),$$

which implies that $r(t) = 1$ for all $t > 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence.

Suppose on contrary that $\{x_n\}$ is not a Cauchy sequence. Then, there exist $\epsilon > 0$ and $0 < \lambda < 1$ such that for all positive integer k, there are $n_k, m_k \in \mathbb{N}$ with $n_k > m_k > k$ and

$$M(x_{m_k}, x_{n_k}, \epsilon) \leq 1 - \lambda. \tag{3}$$

Assume that n_k is the least integer exceeding m_k satisfying the above inequality, that is, equivalently,

$$M(x_{m_k}, x_{n_k-1}, \epsilon) > 1 - \lambda. \tag{4}$$

So, for all $k \in \mathbb{N}$, we obtain

$$\begin{aligned} 1 - \lambda &\geq M(x_{m_k}, x_{n_k}, \epsilon) \\ &\geq M(x_{m_k}, x_{n_k-1}, \epsilon) * M(x_{n_k-1}, x_{n_k}, \epsilon) \\ &> (1 - \lambda) * M(x_{n_k-1}, x_{n_k}, \epsilon). \end{aligned}$$

Since $\lim_{k \to +\infty} M(x_{n_k-1}, x_{n_k}, \epsilon) = 1$ and $*$ is continuous at 1-boundary, we deduce that

$$\lim_{k \to +\infty} M(x_{m_k}, x_{n_k}, \epsilon) = 1 - \lambda. \tag{5}$$

Taking into account that by (4),

$$1 - \lambda \geq M(x_{m_k}, x_{n_k-1}, \epsilon) * M(x_{n_k-1}, x_{n_k}, \epsilon) \geq (1 - \lambda) * M(x_{n_k-1}, x_{n_k}, \epsilon),$$

for all $k \in \mathbb{N}$. We can obtain by Proposition 1 that

$$\lim_{k \to +\infty} M(x_{m_k}, x_{n_k-1}, \epsilon) = 1 - \lambda,$$

which means that

$$\lim_{k \to +\infty} M(x_{m_k}, x_{n_k}, \epsilon) = \lim_{k \to +\infty} M(x_{m_k}, x_{n_k-1}, \epsilon) = 1 - \lambda. \tag{6}$$

Next, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} M(x_{m_k}, x_{n_k}, \epsilon) &\geq M(x_{m_k}, x_{m_k-1}, \epsilon) * M(x_{m_k-1}, x_{n_k-1}, \epsilon) * M(x_{n_k-1}, x_{n_k}, \epsilon) \\ &\geq M(x_{m_k}, x_{m_k-1}, \epsilon) * M(x_{m_k-1}, x_{m_k}, \epsilon) * M(x_{m_k}, x_{n_k}, \epsilon) * M(x_{n_k}, x_{n_k-1}, \epsilon) \\ &\quad * M(x_{n_k-1}, x_{n_k}, \epsilon). \end{aligned}$$

Clearly, $M(x_{m_k}, x_{n_k}, \epsilon) \to 1 - \lambda$, $M(x_{m_k}, x_{m_k-1}, \epsilon) \to 1$, $M(x_{n_k-1}, x_{n_k}, \epsilon) \to 1$ and $M(x_{m_k}, x_{m_k-1}, \epsilon) * M(x_{m_k-1}, x_{m_k}, \epsilon) * M(x_{m_k}, x_{n_k}, \epsilon) * M(x_{n_k}, x_{n_k-1}, \epsilon) * M(x_{n_k-1}, x_{n_k}, \epsilon) \to 1 - \lambda$.
Thus, from Proposition 1, we can obtain

$$\lim_{k \to +\infty} M(x_{m_k-1}, x_{n_k-1}, \epsilon) = 1 - \lambda.$$

As a consequence,

$$\lim_{k \to +\infty} M(x_{m_k}, x_{n_k}, \epsilon) = \lim_{k \to +\infty} M(x_{m_k-1}, x_{n_k-1}, \epsilon) = 1 - \lambda. \tag{7}$$

From (1), we know that

$$M(x_{m_k}, Tx_{m_k-1}, \epsilon) = M(A, B, \epsilon) \quad \text{and} \quad M(x_{n_k}, Tx_{n_k-1}, \epsilon) = M(A, B, \epsilon).$$

Next, by the contractive condition of T, we have

$$\psi(M(x_{m_k}, x_{n_k}, \epsilon) \leq \psi(M(x_{m_k-1}, x_{n_k-1}, \epsilon)) - \phi(M(x_{m_k-1}, x_{n_k-1}, \epsilon)).$$

Using (7), the continuity of ψ and lower semi continuity of ϕ, passing to the limit as $k \to +\infty$ in the above inequality, we obtain

$$\psi(1-\lambda) \leq \psi(1-\lambda) - \phi(1-\lambda),$$

which is a contradiction since $\phi(1-\lambda) \neq 0$.

Hence, $\{x_n\}$ is a Cauchy sequence. The completeness of $(X, M, *)$ ensures that the sequence $\{x_n\}$ converges to some $x^* \in X$, that is, $\lim_{n\to+\infty} M(x_n, x^*, t) = 1$ for all $t > 0$. Moreover,

$$\begin{aligned} M(A,B,t) &= M(x_n, Tx_{n-1}, t) \\ &\geq M(x_n, x^*, t) * M(x^*, Tx_{n-1}, t) \\ &\geq M(x_n, x^*, t) * M(x^*, x_n, t) * M(x_n, Tx_{n-1}, t) \\ &= M(x_n, x^*, t) * M(x^*, x_n, t) * M(A,B,t). \end{aligned}$$

This implies that

$$\begin{aligned} M(A,B,t) &\geq M(x_n, x^*, t) * M(x^*, Tx_{n-1}, t) \\ &\geq M(x_n, x^*, t) * M(x^*, x_n, t) * M(A,B,t). \end{aligned}$$

Taking limit as $n \to +\infty$ in the above inequality, we obtain

$$M(A,B,t) \geq 1 * \lim_{n\to+\infty} M(x^*, Tx_{n-1}, t) \geq 1 * 1 * M(A,B,t),$$

that is

$$\lim_{n\to+\infty} M(x^*, Tx_{n-1}, t) = M(A,B,t),$$

and so by condition (iii), $x^* \in A_0$.

Since $x^* \in A_0$ and $Tx^* \in B_0$, then there exists $x' \in A_0$ such that

$$M(x', Tx^*, t) = M(A,B,t) \quad \text{for all} \quad t > 0. \tag{8}$$

From (1) and (8), using the contractive condition of T, we have

$$\psi(M(x', x_n, t)) \leq \psi(M(x^*, x_{n-1}, t)) - \phi(M(x^*, x_{n-1}, t)).$$

Passing to the limit as $n \to +\infty$ in the above inequality and using the continuity of ψ and M, the lower semi continuity of ϕ, we obtain

$$\lim_{n\to+\infty} \psi(M(x', x_n, t)) \leq \psi(1) - \phi(1) = 0,$$

which implies that $\lim_{n\to+\infty} M(x', x_n, t) = 1$ for all $t > 0$.

By Proposition 2, we conclude that $x' = x^*$, that is, $M(x^*, Tx^*, t) = M(x', x^*, t) = M(A,B,t)$, for all $t > 0$.

To prove the uniqueness of x^*, assume to the contrary that $0 < M(x^*, y^*, t) < 1$ for all $t > 0$ and y^* is another best proximity point of T, i.e., $M(y^*, Ty^*, t) = M(A,B,t)$ for all $t > 0$.

Then, from the contractive condition of T, we have

$$\psi(M(x^*, y^*, t)) \leq \psi(M(x^*, y^*, t)) - \phi(M(x^*, y^*, t)) < \psi(M(x^*, y^*, t)),$$

which is a contradiction and hence $M(x^*, y^*, t) = 1$ for all $t > 0$, that is, $x^* = y^*$. This completes the proof. □

Example 1. *Let* $X = [0,1] \times \mathbb{R}$ *be endowed with a standard fuzzy metric* $M : X \times X \times (0,+\infty) \to (0,1]$ *given by*

$$M(x,y,t) = \frac{t}{t + d(x,y)},$$

where $d : X \times X \to [0,+\infty)$ *is the metric defined by*

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|,$$

for all $x = (x_1, x_2)$ *and* $y = (y_1, y_2) \in X$. *It is obvious that the non-Archimedean property holds true provided by* $a * b = a \cdot b$ *for all* $a, b \in [0,1]$. *Thus,* $(X, M, *)$ *is a non-Archimedean fuzzy metric space. Moreover, the completeness of* $(X, M, *)$ *can be obtained from the the completeness of the metric space* (X, d) *(see Result 2.9 in [40]).*

Define $A = \{(0,x), 0 \le x \le 1\}$, *and* $B = \{(1,x), 0 \le x \le 1\}$.

So that, $d(A,B) = 1$ *and* $M(A,B,t) = \frac{t}{t+1}$, *for all* $t > 0$. *Obviously,* A, B *are nonempty closed subsets of* X.

Define a mapping $T : A \to B$ *by* $T(0,x) = (1, \frac{x}{2})$. *Clearly,* $A_0 = A, B_0 = B, T(A_0) \subseteq B_0$ *and the condition* (iii) *of Theorem 2 holds true.*

Let us consider $u = (0,u'), x = (0,x'), v = (0,v'), y = (0,y') \in A_0$ *such that*

$\begin{cases} M(u,Tx,t) = M(A,B,t) \\ M(v,Ty,t) = M(A,B,t) \end{cases}$ *are satisfied for all* $t > 0$.

Solving the above two equations, we have $u' = \frac{x'}{2}$, *and* $v' = \frac{y'}{2}$.

Define $\psi \in \Psi$ *and* $\phi \in \Phi$ *by* $\psi(s) = \frac{1-s}{2}$ *and* $\phi(s) = \frac{s-1}{2}$, *for all* $s \in (0,1]$.

Indeed, for all $t > 0$,

$$\psi(M(u,v,t)) = \frac{1 - M(u,v,t)}{2} = \frac{1}{2}(1 - \frac{t}{t + |u' - v'|}) = \frac{|x' - y'|}{4t + 2|x' - y'|},$$
$$\psi(M(x,y,t)) - \phi(M(x,y,t)) = 1 - M(x,y,t) = 1 - \frac{t}{t + |x' - y'|} = \frac{|x' - y'|}{t + |x' - y'|}.$$

So that, we have $\psi(M(u,v,t)) \le \psi(M(x,y,t)) - \phi(M(x,y,t))$.

Then, the non-self mapping $T : A \to B$ *is the* $(\psi - \phi)$*-weak proximal contraction of type I.*

All conditions of Theorem 2 are satisfied. After simple computation, we can conclude that $(0,0)$ *is the unique best proximity point of* T.

In view of Theorem 2, we can obtain the following theorem.

Theorem 3. *Let* $(X, M, *)$ *be a complete non-Archimedean fuzzy metric space whose t-norm is continuous at the 1-boundary. Let* A *and* B *be two nonempty closed subsets of* X *such that* A_0 *is nonempty and closed. Let* $T : A \to B$ *be a non-self mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$;

(ii) *T is a* $(\psi - \phi)$*-weak proximal contraction of type I.*

Then, T has a unique best proximity point $x^* \in A$, *which in fact belongs to* A_0.

Proof. Repeating the proof of Theorem 2, we can construct a Cauchy sequence $\{x_n\}$ in A_0. Since A_0 is closed, the completeness of $(X, M, *)$ guarantees that the sequence $\{x_n\}$ converges to some $x^* \in A_0$. Hence, the final conclusion can be drawn by running the same lines as the rest of the proof of Theorem 2. □

2.2. $(\psi - \phi)$-Weak Proximal Contraction of Type II and Its Best Proximity Point Results

Definition 9. *Let* $(X, M, *)$ *be a fuzzy metric space. A non-self mapping* $T : A \to B$ *is called a* $(\psi - \phi)$*-weak proximal contraction of type II, if there exist* $\psi \in \Psi$ *and* $\phi \in \Phi$ *such that for all* $x, y, u, v \in A_0$ *and* $t > 0$

$$\left.\begin{array}{r}M(u,Tx,t)=M(A,B,t)\\ M(v,Ty,t)=M(A,B,t)\end{array}\right\}\textit{ implies }\psi(M(u,v,t))\leq\psi(N(x,y,u,v,t))-\phi(N(x,y,u,v,t)),$$

where $N(x,y,u,v,t)=\max\{M(x,y,t),M(x,u,t)*M(y,v,t),M(x,v,t)*M(y,u,t)\}$.

Theorem 4. *Let $(X,M,*)$ be a complete non-Archimedean fuzzy metric space whose t-norm is continuous at the 1-boundary. Let A and B be two nonempty closed subsets of X such that A_0 is nonempty. Let $T:A\to B$ be a non-self mapping satisfying the following conditions:*

(i) $T(A_0)\subseteq B_0$;

(ii) *T is a $(\psi-\phi)$-weak proximal contraction of type II;*

(iii) *For any sequence $\{y_n\}\subseteq B_0$ and $x\in A$ satisfying $M(x,y_n,t)\to M(A,B,t)$ as $n\to+\infty$, then $x\in A_0$.*

Then, T has a unique best proximity point $x^*\in A$, which in fact belongs to A_0.

Proof. Let x_0 be an arbitrary element of A_0. Following the same routines in the proof of Theorem 2, we can construct a sequence $\{x_n\}$ in A_0 satisfying

$$M(x_n,Tx_{n-1},t)=M(A,B,t)\quad\text{for all}\quad n\in\mathbb{N}\cup\{0\},\text{for all}\quad t>0. \tag{9}$$

If there exists $n_0\in\mathbb{N}\cup\{0\}$ such that $M(x_{n_0},x_{n_0+1},t)=1$ for all $t>0$, it follows that

$$\begin{aligned}M(A,B,t)&\geq M(x_{n_0},Tx_{n_0},t)\\&\geq M(x_{n_0},x_{n_0+1},t)*M(x_{n_0+1},Tx_{n_0},t)\\&=1*M(x_{n_0+1},Tx_{n_0},t)\\&=M(A,B,t)\quad\text{for all}\quad t>0.\end{aligned}$$

Hence, $M(x_{n_0},Tx_{n_0},t)=M(A,B,t)$ for all $t>0$, so x_{n_0} is a best proximity point of T.

Without loss of generality, in the following, we may assume that $0<M(x_n,x_{n+1},t)<1$ for all $n\in\mathbb{N}\cup\{0\}$, that is, $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}\cup\{0\}$.

Since T is a $(\psi-\phi)$-weak proximal contraction of type II, we obtain that for all $t>0$

$$\psi(M(x_{n+1},x_{n+2},t))\leq\psi(N(x_n,x_{n+1},x_{n+1},x_{n+2},t))-\phi(N(x_n,x_{n+1},x_{n+1},x_{n+2},t)), \tag{10}$$

where

$$\begin{aligned}&N(x_n,x_{n+1},x_{n+1},x_{n+2},t)\\&=\max\{M(x_n,x_{n+1},t),M(x_n,x_{n+1},t)*M(x_{n+1},x_{n+2},t),M(x_n,x_{n+2},t)*M(x_{n+1},x_{n+1},t)\}\\&=\max\{M(x_n,x_{n+1},t),M(x_n,x_{n+1},t)*M(x_{n+1},x_{n+2},t),M(x_n,x_{n+2},t)\}\\&=\max\{M(x_n,x_{n+1},t),M(x_n,x_{n+2},t)\}.\end{aligned}$$

Case 1. Assume that $N(x_n,x_{n+1},x_{n+1},x_{n+2},t)=M(x_n,x_{n+2},t)$, that is,

$$M(x_n,x_{n+2},t)\geq M(x_n,x_{n+1},t)\quad\text{for all}\quad t>0. \tag{11}$$

It follows from (10) that

$$\begin{aligned}\psi(M(x_{n+1},x_{n+2},t))&\leq\psi(M(x_n,x_{n+2},t))-\phi(M(x_n,x_{n+2},t))\\&\leq\psi(M(x_n,x_{n+2},t)).\end{aligned} \tag{12}$$

From the monotone property of ψ, we have

$$M(x_{n+1},x_{n+2},t)\geq M(x_n,x_{n+2},t),\quad\text{for all}\quad t>0. \tag{13}$$

From (11) and (13), we obtain

$$M(x_{n+1},x_{n+2},t)\geq M(x_n,x_{n+2},t)\geq M(x_n,x_{n+1},t),\quad\text{for all}\quad t>0. \tag{14}$$

Case 2. Assume that $N(x_n, x_{n+1}, x_{n+1}, x_{n+2}, t) = M(x_n, x_{n+1}, t)$.
It follows from (10) that

$$\begin{aligned} \psi(M(x_{n+1}, x_{n+2}, t)) &\leq \psi(M(x_n, x_{n+1}, t)) - \phi(M(x_n, x_{n+1}, t)) \\ &\leq \psi(M(x_n, x_{n+1}, t)). \end{aligned} \tag{15}$$

In addition, from the monotone property of ψ, we have

$$M(x_{n+1}, x_{n+2}, t) \geq M(x_n, x_{n+1}, t) \quad \text{for all} \quad t > 0.$$

From Case 1 and Case 2, we obtain that $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence in $(0, 1]$ which is bounded above by 1. Then, there exists $r(t) \leq 1$ such that $\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = r(t)$ for all $t > 0$.

Meanwhile, again from (10) and Proposition 1, we have

$$\lim_{n\to+\infty} M(x_n, x_{n+2}, t) = r(t) \quad \text{for all} \quad t > 0. \tag{16}$$

Let us prove that $r(t) = 1$ for all $t > 0$.

Indeed, passing to the limit as $n \to +\infty$ in (12) and (15), using (16), and the continuity of ψ and M, and lower semi continuity of ϕ, we have

$$\psi(r(t)) \leq \psi(r(t)) - \phi(r(t)),$$

which implies that $r(t) = 1$ for all $t > 0$.

Therefore, $\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence.

Assuming that this is not true and proceeding as in the proof of Theorem 2, there exist $\epsilon > 0$ and $0 < \lambda < 1$ such that for all $k \in \mathbb{N}$, there are $n_k, m_k \in \mathbb{N}$ with $n_k > m_k > k$ such that

$$\lim_{k\to+\infty} M(x_{m_k}, x_{n_k}, \epsilon) = \lim_{k\to+\infty} M(x_{m_k-1}, x_{n_k-1}, \epsilon) = 1 - \lambda.$$

Applying the contractive condition of T, we obtain

$$\psi(M(x_{m_k}, x_{n_k}, \epsilon)) \leq \psi(N(x_{m_k-1}, x_{n_k-1}, x_{m_k}, x_{n_k}, \epsilon)) - \phi(N(x_{m_k-1}, x_{n_k-1}, x_{m_k}, x_{n_k}, \epsilon)), \tag{17}$$

where

$$\begin{aligned} N(x_{m_k-1}, x_{n_k-1}, x_{m_k}, x_{n_k}, \epsilon) = \max\{&M(x_{m_k-1}, x_{n_k-1}, \epsilon), M(x_{m_k-1}, x_{m_k}, \epsilon) * M(x_{n_k-1}, x_{n_k}, \epsilon), \\ &M(x_{m_k-1}, x_{n_k}, \epsilon) * M(x_{n_k-1}, x_{m_k}, \epsilon)\}. \end{aligned}$$

Using the continuity of ψ and M and the lower semi continuity of ϕ, passing to the limit as $k \to +\infty$ in the above inequality, we have

$$\psi(1 - \lambda) \leq \psi(1) - \phi(1) = 0,$$

and so $\psi(1 - \lambda) = 0$, contradictorily. Thus, $\{x_n\}$ is a Cauchy sequence.

The completeness of $(X, M, *)$ ensures that there exist $x^* \in X$ such that $\lim_{n\to+\infty} M(x_n, x^*, t) = 1$ for all $t > 0$.

Again, using condition (iii) and proceeding as in the proof of Theorem 2, we can find $x' \in A_0$ such that

$$M(x', Tx^*, t) = M(A, B, t) \quad \text{for all} \quad t > 0.$$

Consequently, from the contractive condition of T, we have

$$\psi(M(x', x_n, t)) \leq \psi(N(x^*, x_{n-1}, x', x_n, t)) - \phi(N(x^*, x_{n-1}, x', x_n, t)),$$

where

$$N(x^*, x_{n-1}, x', x_n, t) = \max\{M(x^*, x_{n-1}, t), M(x', x^*, t) * M(x_{n-1}, x_n, t), M(x^*, x_n, t) * M(x_{n-1}, x', t)\}.$$

Taking limit as $n \to +\infty$ in the above inequality and using the continuity of ψ and M, and lower semi continuity of ϕ, we have

$$\lim_{n\to+\infty} \psi(M(x', x_n, t)) \leq \psi(1) - \phi(1) = 0,$$

where implies that $\lim_{n\to+\infty} M(x', x_n, t)) = 1$ for all $t > 0$.

By Proposition 2, we conclude that $x' = x^*$, that is, $M(x', Tx^*, t) = M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$.

We proceed to prove that x^* is the unique best proximity point of T.

Assume to the contrary that $0 < M(x^*, y^*, t) < 1$ for all $t > 0$ and y^* is another best proximity point of T, i.e., $M(y^*, Ty^*, t) = M(A, B, t)$ for all $t > 0$.

Then, from the contractive condition of T, we have

$$\psi(M(x^*, y^*, t)) \leq \psi(N(x^*, y^*, x^*, y^*, t)) - \phi(N(x^*, y^*, x^*, y^*, t)),$$

where

$$\begin{aligned} N(x^*, y^*, x^*, y^*, t) &= \max\{M(x^*, y^*, t), M(x^*, x^*, t) * M(y^*, y^*, t), M(x^*, y^*, t) * M(y^*, x^*, t)\} \\ &= 1. \end{aligned}$$

Hence, $\psi(M(x^*, y^*, t)) \leq \psi(1) - \phi(1) = 0$ for all $t > 0$.

This implies that $M(x^*, y^*, t) = 1$ for all $t > 0$, that is, $x^* = y^*$. This completes the proof. □

Similar to Theorem 3, we can provide the following theorem.

Theorem 5. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space whose t-norm is continuous at the 1-boundary. Let A and B be two nonempty closed subsets of X such that A_0 is nonempty and closed. Let $T : A \to B$ be a non-self mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$;

(ii) *T is a $(\psi - \phi)$-weak proximal contraction of type II.*

Then, T has a unique best proximity point $x^* \in A$, which in fact belongs to A_0.

Proof. Repeating the proof of Theorem 4, we can construct a Cauchy sequence $\{x_n\}$ in A_0. Since A_0 is closed, the completeness of $(X, M, *)$ guarantees that there exists a point $x^* \in A_0$ such that $x_n \to x^*$. The rest of the proof is same as the one of Theorem 4. For brevity, we omit it. □

Example 2. *Let $X = \{re^{i\theta} : \frac{1}{4} \leq r, 0 \leq \theta \leq 2\pi\}$ be endowed with a fuzzy metric $M : X \times X \times (0, +\infty) \to (0, 1]$ given by*

$$M(x, y, t) = (\frac{t}{1+t})^{d(x,y)},$$

for all $t > 0$, where $d : X \times X \to [0, +\infty)$ is a metric defined by

$$d(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = |r_1 - r_2| + \min\{|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|\}.$$

Thus, the non-Archimedean property holds true provided by $a * b = a \cdot b$ for all $a, b \in [0, 1]$ and the triangle inequality property of d.

Moreover, let $\{x_n\}$ be a Cauchy sequence in $(X, M, *)$. Then, for $\epsilon, t > 0, 0 < \epsilon < 1$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $m, n \geq n_0$. Therefore, for each $t > 0$, $M(x_n, x_m, t)$ converges to 1 as $n, m \to +\infty$ which implies $d(x_n, x_m)$ converges to 0 as $n, m \to +\infty$. Thus, $\{x_n\}$ is also a Cauchy sequence in (X, d). Take $A = \{x_n, x_{n+1}, x_{n+2}, \cdots\}$ and $F_n = \bar{A_n}$. It is obvious that $\{F_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty closed sets with diameter $\delta(F_n)$ tending to 0. From the construct of X, we have that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Take $x \in \bigcap_{n=1}^{\infty} F_n$. If $\epsilon > 0$ is arbitrary, then there exists $N \in \mathbb{N}$ such that $\delta(F_n) < \epsilon$ and thus, $\delta(A_n) < \epsilon$, for $n \geq N$. This implies that $d(x_n, x) < \epsilon$ for $n \geq N$, so that x_n converges to x which implies that (X, d) is complete. Thus, the completeness of $(X, M, *)$ follows from the Theorem 3.3 in [40].

Define the following two closed subsets of X as follows:

$$A = \{re^{i\theta} : 2 \leq r \leq 4, 0 \leq \theta \leq \frac{\pi}{2}\}$$
$$B = \{re^{i\theta} : \frac{1}{4} \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}.$$

Define a mapping $T : A \to B$ by $T(re^{i\theta}) = \frac{1}{r-1} e^{i\frac{\theta}{2}}, 0 \leq \theta \leq \frac{\pi}{2}$ and $\psi, \phi : (0, 1] \to [0, +\infty)$ by $\psi(s) = -\ln s$ and $\phi(s) = -\frac{1}{4} \ln s$.

Then, $d(A, B) = 1$ and $M(A, B, t) = \frac{t}{1+t}$, for all $t > 0$.

Clearly, it is easy to check that

$$A_0 = \{2e^{i\theta} : 0 \leq \theta \leq \frac{\pi}{2}\} \quad \text{and} \quad B_0 = \{e^{i\theta} : 0 \leq \theta \leq \frac{\pi}{2}\},$$

and A_0 is closed with $T(A_0) \subset B_0$.

Now, let $u, x, v, y \in A_0$ such that $M(u, Tx, t) = M(v, Ty, t) = M(A, B, t) = \frac{t}{1+t}$ for all $t > 0$.

Then, we have $u = 2e^{i\theta_1}$, $x = 2e^{i2\theta_1}$, $v = 2e^{i\theta_2}$, $y = 2e^{i2\theta_2}$, with $\theta_1, \theta_2 \in [0, \frac{\pi}{4}]$. After simple calculation, we have

$$M(u, v, t) = (\frac{t}{1+t})^{d(u,v)} = (\frac{t}{1+t})^{|\theta_1 - \theta_2|};$$
$$M(x, y, t) = (\frac{t}{1+t})^{d(x,y)} = (\frac{t}{1+t})^{2|\theta_1 - \theta_2|};$$
$$M(x, u, t) = (\frac{t}{1+t})^{d(x,u)} = (\frac{t}{1+t})^{\theta_1};$$
$$M(y, v, t) = (\frac{t}{1+t})^{d(y,v)} = (\frac{t}{1+t})^{\theta_2};$$
$$M(x, v, t) = (\frac{t}{1+t})^{d(x,v)} = (\frac{t}{1+t})^{|2\theta_1 - \theta_2|};$$
$$M(y, u, t) = (\frac{t}{1+t})^{d(y,u)} = (\frac{t}{1+t})^{|2\theta_2 - \theta_1|}.$$

It follows from the above equations that

$$\begin{aligned} N(x, y, u, v, t) &= \max\{M(x, y, t), M(x, u, t) * M(y, v, t), M(x, v, t) * M(y, u, t)\} \\ &= \max\{(\frac{t}{1+t})^{2|\theta_1 - \theta_2|}, (\frac{t}{1+t})^{\theta_1 + \theta_2}, (\frac{t}{1+t})^{|2\theta_1 - \theta_2| \cdot |2\theta_2 - \theta_1|}\} \end{aligned}$$

and

$$\psi(M(u,v,t)) = -\ln(\frac{t}{1+t})^{|\theta_1-\theta_2|} = -|\theta_1-\theta_2|\ln(\frac{t}{1+t});$$
$$\psi(N(x,y,u,v,t)) - \phi(N(x,y,u,v,t)) = -\ln((N(x,y,u,v,t)) + \frac{1}{4}\ln(N(x,y,u,v,t))$$
$$= -\frac{3}{4}\ln(\max\{(\frac{t}{1+t})^{2|\theta_1-\theta_2|}, (\frac{t}{1+t})^{\theta_1+\theta_2}, (\frac{t}{1+t})^{|2\theta_1-\theta_2|\cdot|2\theta_2-\theta_1|}\}).$$

Since $|\theta_1-\theta_2| \leq 2|\theta_1-\theta_2| \leq \max\{2|\theta_1-\theta_2|, \theta_1+\theta_2, |2\theta_1-\theta_2|\cdot|2\theta_2-\theta_1|\}$, then T is a $(\psi-\phi)$-weak proximal contraction of type II. Thus, all the hypothesis of Theorem 5 are fulfilled. Therefore, we can deduce that the mapping T has a unique best proximity point in A_0 (which is the point 2).

2.3. $(\psi-\phi)$-Weak Proximal Contraction of Type III and Its Best Proximity Point Results

Definition 10. *Let $(X, M, *)$ be a fuzzy metric space. A non-self mapping $T : A \to B$ is called a $(\psi-\phi)$-weak proximal contraction of type III if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that for all $x, y, u, v \in A_0$ and $t > 0$*

$$\left.\begin{array}{l} M(u,Tx,t) = M(A,B,t) \\ M(v,Ty,t) = M(A,B,t) \end{array}\right\} \text{implies } \psi(M(Tu,Tv,t)) \leq \psi(M(Tx,Ty,t)) - \phi(M(Tx,Ty,t)).$$

Theorem 6. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space whose t-norm is continuous at the 1-boundary. Let A and B be two nonempty closed subsets of X such that such that A_0 and B_0 are nonempty. Let $T : A \to B$ be a non-self mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$;*
(ii) *T is a $(\psi-\phi)$-weak proximal contraction of type III;*
(iii) *A is approximately compact with respect to B.*

Then, T has a best proximity point $x^* \in A$, which in fact belongs to A_0. In addition, if T is injective, then the best proximity point of T is unique.

Proof. As in the proof of Theorem 2, we can construct a sequence $\{x_n\}$ in A_0 such that

$$M(x_n, Tx_{n-1}, t) = M(A,B,t) \quad \text{for all} \quad t > 0.$$

Without loss of generality, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, that is, $0 < M(x_n, x_{n+1}, t) < 1$ for all $n \in \mathbb{N} \cup \{0\}, t > 0$.

Taking advantages from the contractive condition of T, we have

$$\begin{aligned} \psi(M(Tx_{n+1}, Tx_{n+2}, t)) &\leq \psi(M(Tx_n, Tx_{n+1}, t)) - \phi(M(Tx_n, Tx_{n+1}, t)) \\ &\leq \psi(M(Tx_n, Tx_{n+1}, t)). \end{aligned} \tag{18}$$

From the monotone property of ψ, we have

$$M(Tx_{n+1}, Tx_{n+2}, t) \geq M(Tx_n, Tx_{n+1}, t) \quad \text{for all} \quad t > 0.$$

Hence, the sequence $\{M(Tx_n, Tx_{n+1}, t)\}$ is an increasing sequence in $(0,1]$ which is bounded above by 1 and there exists $r(t) \leq 1$ such that $\lim_{n\to+\infty} M(Tx_n, Tx_{n+1}, t) = r(t)$ for all $t > 0$.

Using the continuity of ψ and M, the lower semi continuity of ϕ, passing to the limit as $n \to +\infty$ in (18), we obtain

$$\psi(r(t)) \leq \psi(r(t)) - \phi(r(t)),$$

which implies that $r(t) = 1$ for all $t > 0$.

Next, we shall prove that $\{Tx_n\}$ is a Cauchy sequence.

Assume that this is not true and applying the similar arguments as in the proof of Theorem 2, there exist $\epsilon > 0$ and $0 < \lambda < 1$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$ with $n_k > m_k > k$ such that

$$\lim_{k\to+\infty} M(Tx_{m_k}, Tx_{n_k}, t) = \lim_{k\to+\infty} M(Tx_{m_k-1}, Tx_{n_k-1}, t) = 1 - \lambda.$$

Using the contractive condition of T, we obtain that

$$\psi(M(Tx_{m_k}, Tx_{n_k}, t)) \leq \psi(M(Tx_{m_k-1}, Tx_{n_k-1}, t)) - \phi(M(Tx_{m_k-1}, Tx_{n_k-1}, t)) \quad \text{for all} \quad t > 0.$$

Taking limit as $k \to +\infty$ in the above inequality, from the continuity of ψ and M and the semi continuity of ϕ, we have

$$\psi(1-\lambda) \leq \psi(1-\lambda) - \phi(1-\lambda),$$

which is a contradiction since $\phi(1-\lambda) \neq 0$.

Hence, $\{Tx_n\}$ is a Cauchy sequence. The completeness of $(X, M, *)$ ensures that $\{Tx_n\}$ is convergent to some element y^* in B.

Furthermore,

$$\begin{aligned} M(A, y^*, t) &\geq M(x_n, y^*, t) \\ &\geq M(x_n, Tx_{n-1}, t) * M(y^*, Tx_{n-1}, t) \\ &= M(A, B, t) * M(y^*, Tx_{n-1}, t) \\ &\geq M(A, y^*, t) * M(y^*, Tx_{n-1}, t). \end{aligned}$$

Therefore, $M(x_n, y^*, t) \to M(A, y^*, t)$. In view of the fact that A is approximately compact with respect to B, $\{x_n\}$ has a subsequence $\{x_{l_k}\}$ convergent to some $x^* \in A$.

Therefore, it can be concluded from the continuity of M that

$$\lim_{k\to+\infty} M(x_{l_k}, Tx_{l_k-1}, t) = M(x^*, y^*, t) = M(A, B, t) \quad \text{for all} \quad t > 0.$$

As a consequence, x^* is an element of A_0, also, $y^* \in B_0$.

Since $T(A_0) \subseteq B_0$, then there exists $x' \in A_0$ such that $Tx' = y^*$.

From the contractive condition of T, we have

$$\psi(M(Tx_{l_k}, Tx^*, t)) \leq \psi(M(Tx_{l_k-1}, Tx', t)) - \phi(M(Tx_{l_k-1}, Tx', t)) \quad \text{for all} \quad t > 0.$$

Taking the limit as $k \to +\infty$ in the above inequality, we obtain

$$\begin{aligned} \psi(M(y^*, Tx^*, t)) &\leq \psi(M(y^*, y^*, t)) - \phi(M(y^*, y^*, t)) \\ &= \psi(1) - \phi(1) \\ &= 0. \end{aligned}$$

Hence, $M(y^*, Tx^*, t) = 1$ for all $t > 0$, that is, $y^* = Tx^*$.

Therefore, $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$, which implies that x^* is a best proximity point of T.

Suppose that T is injective and there is another element $z^* \in A_0$ such that $M(z^*, Tz^*, t) = M(A, B, t)$ for all $t > 0$.

Since T is a $(\psi - \phi)$-weak proximal contraction of type III, we have

$$\psi(M(Tx^*, Tz^*, t)) \leq \psi(M(Tx^*, Tz^*, t)) - \phi(M(Tx^*, Tz^*, t)),$$

which implies that $M(Tx^*, Tz^*, t) = 1$ for all $t > 0$, as mapping T is an injective mapping, hence $x^* = z^*$.

This completes the uniqueness of the best proximity point of T. □

Theorem 7. *Suppose that all the assumptions of Theorem 6 are satisfied except the assumption (iii) is replaced by $(iii)'$.*

$(iii)'$: For any sequence $\{x_n\} \in A_0$ and $y \in B$ satisfying $M(x_n, y, t) \to M(A, B, t)$ as $n \to +\infty$, then $y \in B_0$.

Then, the conclusion of Theorem 6 still holds.

Proof. In fact, proceeding the similar lines as in the proof of Theorem 6, we can obtain a Cauchy sequence $\{Tx_n\} \subseteq B_0$ with $x_n \in A_0$ for all $n \in \mathbb{N} \cup \{0\}$. The completeness of $(X, M, *)$ guarantees that there exists $y^* \in B$ such that $Tx_n \to y^*$.

Taking limit as $n \to +\infty$ in the following inequality

$$\begin{aligned} M(A,B,t) &\geq M(A,y^*,t) \\ &\geq M(x_n,y^*,t) \\ &\geq M(x_n,Tx_{n-1},t) * M(Tx_{n-1},y^*,t) \\ &= M(A,B,t) * M(Tx_{n-1},y^*,t). \end{aligned}$$

Therefore, we can have $M(x_n, y^*, t) \to M(A, B, t)$ for all $t > 0$.

In view of assumption $(iii)'$, $y^* \in B_0$. Since $T(A_0) \subseteq B_0$, then there exists $x^* \in A_0$ such that $Tx^* = y^*$. Since $y^* \in B_0$, there exists $x' \in A_0$ such that

$$M(x', y^*, t) = M(x', Tx^*, t) = M(A, B, t) \quad \text{for all} \quad t > 0.$$

Again, using the contractive condition of T, we have

$$\psi(M(Tx', Tx_n, t)) \leq \psi(M(Tx^*, Tx_{n-1}, t)) - \phi(M(Tx^*, Tx_{n-1}, t)) \quad \text{for all} \quad t > 0.$$

Using the continuity of ψ and M and the lower semi continuity of ϕ, passing to the limit as $n \to +\infty$ in the above inequality, we have

$$\psi(M(Tx', y^*, t)) \leq \psi(1) - \phi(1) = 0,$$

which implies that $M(Tx', y^*, t) = 1$ for all $t > 0$, that is, $Tx' = y^*$.

As mapping T is an injective mapping, hence $x' = x^*$, also $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$. This shows that x^* is a best proximity point of T.

For brevity, we omit the rest of the proof of the uniqueness of the desired point which is the same as the one in the proof of Theorem 6. □

Example 3. *Let X be $\mathbb{R}^2$ endowed with a fuzzy metric $M : X \times X \times (0, +\infty) \to (0, 1]$ given by*

$$M(x, y, t) = (\frac{t}{1+t})^{d(x,y)},$$

where $d : X \times X \to [0, +\infty)$ is the metric defined by

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|,$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in X$.

Thus, the non-Archimedean fuzzy property holds true provided by $a * b = a \cdot b$ for all $a, b \in [0, 1]$ and the triangle inequality property of metric d.

Using analysis similar to that in the proof of Example 2, we can show that $(X, M, *)$ is complete.

Define the sets as follows

$$A = \{(0, x) \in \mathbb{R}^2\}, \textit{ and } \quad B = \{(1, x) \in \mathbb{R}^2\}.$$

Thus, $d(A, B) = 1$ and $M(A, B, t) = \frac{t}{1+t}$ for all $t > 0$.

Define a mapping $T : A \to B$ by $T(0, x) = (1, \frac{x}{4})$ for all $(0, x) \in A$.

Notice that $A_0 = A$, $B_0 = B$, $T(A_0) \subset B_0$ and the condition $(iii)'$ of Theorem 7 holds true.

Define $\psi, \phi : (0, 1] \to [0, +\infty)$ by $\psi(s) = -\ln s$ and $\phi(s) = -\frac{1}{2}\ln s$.

Assume that there exist $u = (0, u'), x = (0, x'), v = (0, v'), y = (0, y') \in A_0$ such that $M(u, Tx, t) = M(A, B, t) = M(v, Ty, t)$ for all $t > 0$. Then, we have $4u' = x', 4v' = y'$.

Since

$$\begin{aligned}\psi(M(Tu, Tv, t)) &= -\ln M(Tu, Tv, t)\\ &= -d(Tu, Tv)\ln(\frac{t}{1+t})\\ &= -\frac{|u' - v'|}{4}\ln(\frac{t}{1+t})\\ &= -\frac{|x' - y'|}{16}\ln(\frac{t}{1+t}),\end{aligned}$$

$$\psi(M(Tx, Ty, t)) = -d(Tx, Ty)\ln(\frac{t}{1+t}) = -\frac{|x' - y'|}{4}\ln(\frac{t}{1+t}),$$

$$\phi(M(Tx, Ty, t)) = -\frac{1}{2}d(Tx, Ty)\ln(\frac{t}{1+t}) = -\frac{|x' - y'|}{8}\ln(\frac{t}{1+t}).$$

It is easy to check that T is a $(\psi - \phi)$-weak proximal contraction of type III. Hence, all the conditions of Theorem 7 are satisfied. Therefore, T has a unique best proximity point in A_0 (which is $(0, 0)$).

Theorem 8. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space whose t-norm is 1-boundary continuous. Let A and B be two nonempty closed subsets of X such that A_0 and B_0 are nonempty with B_0 is closed. Let $T : A \to B$ be a non-self injective mapping satisfying the assumptions (i)–(iii) (or $(iii)'$) presented in Theorem 6 or Theorem 7.*

Then, T has a unique best proximity point $x^* \in A$, which in fact belongs to A_0.

Proof. Analysis similar to the proof of Theorem 6 (or Theorem 7), we obtain a Cauchy sequence Tx_n in B_0. The completeness of $(X, M, *)$ and the closedness of B_0 ensures that there is $y^* \in B_0$ such that $Tx_n \to y^*$. The rest of the proof can run as the one of Theorem 6. □

2.4. $(\psi - \phi)$-Weak Proximal Contraction of Type IV and Its Best Proximity Point Results

Definition 11. *Let $(X, M, *)$ be a fuzzy metric space. A non-self mapping $T : A \to B$ is called a $(\psi - \phi)$-weak proximal contraction of type IV, if there exist $\psi \in \Psi, \phi \in \Phi$ such that for all $x, y, u, v \in A_0$ and $t > 0$*

$$\left.\begin{array}{l}M(u, Tx, t) = M(A, B, t)\\ M(v, Ty, t) = M(A, B, t)\end{array}\right\}\text{implies } \psi(M(u, v, t)) \le \psi(\tfrac{M(x,Tx,t)+M(y,Ty,t)}{2M(A,B,t)}) - \phi(\tfrac{M(x,Tx,t)+M(y,Ty,t)}{2M(A,B,t)}).$$

Theorem 9. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space whose Product norm $*_P$ is 1-boundary continuous. Let A and B be two nonempty closed subsets of X such that A_0 is nonempty. Let $T : A \to B$ be a non-self mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$;
(ii) *T is a $(\psi - \phi)$-weak proximal contraction of type IV;*
(iii) *A is approximately compact with respect to B.*

Then, T has a unique best proximity point $x^* \in A$, which in fact belongs to A_0.

Proof. As in the proof of Theorem 2, we can construct a sequence $\{x_n\} \subseteq A_0$ such that

$$M(x_n, Tx_{n-1}, t) = M(A, B, t) \quad \text{for all} \quad t > 0. \tag{19}$$

Without loss of generality, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ or equivalently, $0 < M(x_n, x_{n+1}, t) < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$.

Using the contractive condition of T and the monotone property of ψ, we have

$$\begin{aligned} &\psi(M(x_{n+1}, x_{n+2}, t)) \qquad (20)\\ &\leq \psi(\frac{M(x_n, Tx_n, t) + M(x_{n+1}, Tx_{n+1}, t)}{2M(A, B, t)}) - \phi(\frac{M(x_n, Tx_n, t) + M(x_{n+1}, Tx_{n+1}, t)}{2M(A, B, t)}) \\ &\leq \psi(\frac{M(x_n, Tx_n, t) + M(x_{n+1}, Tx_{n+1}, t)}{2M(A, B, t)}) \\ &\leq \psi(\frac{M(x_n, x_{n+1}, t) \cdot M(x_{n+1}, Tx_n, t) + M(x_{n+1}, n_{n+2}, t) \cdot M(x_{n+2}, Tx_{n+1}, t)}{2M(A, B, t)}) \\ &= \psi(\frac{M(x_n, x_{n+1}, t) \cdot M(A, B, t) + M(x_{n+1}, n_{n+2}, t) \cdot M(A, B, t)}{2M(A, B, t)}) \\ &= \psi(\frac{M(x_n, x_{n+1}, t) + M(x_{n+1}, n_{n+2}, t)}{2}), \end{aligned}$$

which implies that $M(x_{n+1}, x_{n+2}, t) \geq M(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N} \cup \{0\}, t > 0$.

Hence, the sequence $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence in $(0, 1]$ which is bounded above by 1 for all $t > 0$. Then, there exist $r(t) \leq 1$ such that $\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = r(t)$.

From (20), by taking limit inferior and denoting

$$\lim_{n\to+\infty} \inf(\frac{M(x_n, Tx_n, t) + M(x_{n+1}, Tx_{n+1}, t)}{2M(A, B, t)}) = l(t),$$

we obtain

$$\psi(r(t)) \leq \psi(l(t)) \leq \psi(r(t)), \qquad (21)$$

which implies that

$$r(t) = \lim_{n\to+\infty} M(x_n, x_{n+1}, t) \geq l(t) = \lim_{n\to+\infty} \inf(\frac{M(x_n, Tx_{n+1}, t) + M(x_{n+1}, Tx_{n+1}, t)}{2M(A, B, t)}) \geq r(t),$$

hence, $r(t) = l(t)$ for all $t > 0$.

Taking limit as $n \to +\infty$ in (20), we have

$$\psi(r(t)) \leq \psi(l(t)) - \phi(l(t)) = \psi(r(t)) - \phi(r(t)),$$

which implies that $r(t) = 1$ for all $t > 0$.

Taking into account the following inequalities

$$M(A, B, t) \geq M(x_n, Tx_n, t) \geq M(x_n, x_{n+1}, t) \cdot M(x_{n+1}, Tx_n, t),$$

and passing to the limit as $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} M(x_n, Tx_n, t) = M(A, B, t) \quad \text{for all} \quad t > 0. \qquad (22)$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence.

Taking advantages from the contractive condition, we obtain

$$\begin{aligned}\psi(M(x_{m+1},x_{n+1},t)) &\leq \psi(\frac{M(x_m,Tx_m,t)+M(x_n,Tx_n,t)}{2M(A,B,t)}) \\ &\quad - \phi(\frac{M(x_m,Tx_m,t)+M(x_n,Tx_n,t)}{2M(A,B,t)}) \\ &\leq \psi(\frac{M(x_m,Tx_m,t)+M(x_n,Tx_n,t)}{2M(A,B,t)}).\end{aligned}$$

Using (22) and taking the limit in the above inequality, we have

$$\lim_{n,m\to+\infty}\psi(M(x_{m+1},x_{n+1},t)) = \psi(1) = 0,$$

which together with the property of ψ, implies that $\lim_{n,m\to+\infty} M(x_{m+1},x_{n+1},t) = 1$ for all $t > 0$.

Hence, $\{x_n\}$ is a Cauchy sequence. From the completeness of $(X, M, *)$, we obtain $\{x_n\}$ converges to some point $x^* \in X$.

In the same manner as the proof of Theorem 2, we can prove that x^* is a unique best proximity point of T. For brevity, we omit the rest of the proof. □

Theorem 10. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space whose Product norm $*_P$ is 1-boundary continuous. Let A and B be two nonempty closed subsets of X such that A_0 is nonempty and closed. Let $T : A \to B$ be a non-self mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$;

(ii) T *is a $(\psi - \phi)$-weak proximal contraction of type IV.*

Then, T has a unique best proximity point $x^* \in A$, which in fact belongs to A_0.

Proof. Applying the same arguments as in the proof of Theorem 9, we can construct a Cauchy sequence $\{x_n\}$ in A_0. Moreover, since A_0 is closed, the completeness of $(X, M, *)$ guarantees that the sequence $\{x_n\}$ converges to some $x^* \in A_0$. The rest of the proof is same as the one of Theorem 2. □

Remark 2. *If we suppose that the non-self mapping T in Theorems 2, 4, 6, 7 and 9 is continuous, then we can omit the assumption (iii) (or $(iii)'$) in these theorems mentioned above.*

Example 4. *Let $X = \mathbb{R} \times \mathbb{R}$ and let $M : X \times X \times (0, +\infty) \to (0, 1]$ be a fuzzy metric given by*

$$M(x,y,t) = e^{-\frac{d(x,y)}{t}},$$

for all $t > 0$, where $d : X \times X \to [0, +\infty)$ is the standard metric

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

for all $x = (x_1, x_2)$ and $y = (y_1, y_2) \in X$.

The non-Archimedean property holds true provided by $a * b = a \cdot b$ for all $a, b \in [0, 1]$ and the triangle inequality property of metric d.

Analysis similar to that in the proof of Example 2, we can show that $(X, M, *)$ is complete.

Define the sets

$$A = \{(0,n) : n \in \mathbb{N}\} \cup \{(0,0)\} \quad and \quad B = \{(1,n) : n \in \mathbb{N}\} \cup \{(1,0)\}.$$

Hence, $d(A, B) = 1$ and $M(A, B, t) = e^{-\frac{1}{t}}$ for all $t > 0$. Clearly, A, B are nonempty closed subsets of X.

Define a mapping $T : A \to B$ *by* $T(x_1, x_2) = \begin{cases} (1, 2n), & (x_1, x_2) = (0, 2n). \\ (1, 0), & (x_1, x_2) = (0, 0). \end{cases}$

In addition, define $\psi \in \Psi$ *and* $\phi \in \Phi$ *by* $\psi(s) = -\frac{1}{4}\ln s$ *and* $\phi(s) = \frac{3}{4}\ln s$.

Notice that $A_0 = A, B_0 = B$ *and* $T(A_0) \subset B_0$. *So, all the conditions except condition* (ii) *of Theorem 10 are satisfied.*

Next, we will verify the condition (ii).

Assume that $M(u, Tx, t) = M(v, Ty, t) = M(A, B, t)$ *for some* $u, x, v, y \in A_0$. *Then,*

$$((u, x), (v, y)) \in \{((0, 0), (0, 0)), ((0, 2n), (0, n)) : n \in \mathbb{N}\}.$$

Now, we consider the following cases:

Case 1. If $(u, x) = ((0, 2n), (0, n))$ *and* $(v, y) = ((0, 2m), (0, m))$ *for* $n, m \in \mathbb{N}$, *we have*

$$\begin{aligned} M(u, v, t) &= e^{-\frac{d(u,v)}{t}} = e^{-\frac{2|n-m|}{t}}; \\ M(x, Tx, t) &= e^{-\frac{d(x,Tx)}{t}} = e^{-\frac{n+1}{t}}; \\ M(y, Ty, t) &= e^{-\frac{d(y,Ty)}{t}} = e^{-\frac{m+1}{t}}; \\ \frac{M(x, Tx, t) + M(y, Ty, t)}{2M(A, B, t)} &= \frac{e^{-\frac{n}{t}} + e^{-\frac{m}{t}}}{2}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\psi(M(u, v, t)) = -\frac{1}{4}\ln(e^{-\frac{2|n-m|}{t}}) = \frac{|n-m|}{2t}, \\ &\psi\left(\frac{M(x, Tx, t) + M(y, Ty, t)}{2M(A, B, t)}\right) - \phi\left(\frac{M(x, Tx, t) + M(y, Ty, t)}{2M(A, B, t)}\right) \\ &= -\ln\left(\frac{e^{-\frac{n}{t}} + e^{-\frac{m}{t}}}{2}\right) \\ &\geq -\ln(e^{-\frac{m+n}{2t}}) \\ &= \frac{m+n}{2t} \\ &> \frac{|n-m|}{2t} \\ &= \psi(M(u, v, t)). \end{aligned}$$

Case 2. If $(u, x) = ((0, 0), (0, 0))$ *and* $(v, y) = ((0, 2m), (0, m))$ *for all* $m \in \mathbb{N}$, *we have*

$$\begin{aligned} M(u, v, t) &= e^{-\frac{d(u,v)}{t}} = e^{-\frac{2m}{t}}; \\ M(x, Tx, t) &= e^{-\frac{d(x,Tx)}{t}} = e^{-\frac{1}{t}}; \\ M(y, Ty, t) &= e^{-\frac{d(y,Ty)}{t}} = e^{-\frac{2m+1}{t}}; \\ \frac{M(x, Tx, t) + M(y, Ty, t)}{2M(A, B, t)} &= \frac{1 + e^{-\frac{2m}{t}}}{2}. \end{aligned}$$

Hence, we have

$$
\begin{aligned}
&\psi(M(u,v,t)) = -\frac{1}{4}\ln(e^{-\frac{2m}{t}}) = \frac{m}{2t},\\
&\psi(\frac{M(x,Tx,t)+M(y,Ty,t)}{2M(A,B,t)}) - \phi(\frac{M(x,Tx,t)+M(y,Ty,t)}{2M(A,B,t)})\\
&= -\ln(\frac{1+e^{-\frac{m}{t}}}{2})\\
&\geq -\ln(e^{-\frac{m}{t}})\\
&= \frac{m}{t}\\
&> \frac{m}{2t}\\
&= \psi(M(u,v,t)).
\end{aligned}
$$

Case 3. If If $(u,x) = (v,y) = ((0,0),(0,0))$, *we have*

$$
\begin{aligned}
M(u,v,t) &= e^{-\frac{d(u,v)}{t}} = 1;\\
M(x,Tx,t) &= e^{-\frac{d(x,Tx)}{t}} = e^{-\frac{1}{t}};\\
M(y,Ty,t) &= e^{-\frac{d(y,Ty)}{t}} = e^{-\frac{1}{t}};\\
\frac{M(x,Tx,t)+M(y,Ty,t)}{2M(A,B,t)} &= \frac{1}{2}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&|\psi(M(u,v,t)) = -\frac{1}{4}\ln 1 = 0,\\
&\psi(\frac{M(x,Tx,t)+M(y,Ty,t)}{2M(A,B,t)}) - \phi(\frac{M(x,Tx,t)+M(y,Ty,t)}{2M(A,B,t)})\\
&= -\ln(\frac{1}{2})\\
&> 0\\
&= \psi(M(u,v,t)).
\end{aligned}
$$

Now, we can conclude that T *is a* $(\psi-\phi)$*-weak proximal contraction of type* IV. *Therefore, all the conditions of Theorem 10 are satisfied. Therefore,* T *has a unique best proximity point in* A_0 *(which is* $(0,0)$*).*

2.5. Consequences in Fixed Point Theory

In this section, we will discuss results which ensure the existence of fixed point for some $(\psi-\phi)$-weak contractions $T : X \to X$ in the setting of a non-Archimedean fuzzy metric space $(X, M, *)$. These properties can be considered as applications of the above stated results in fixed point theory. They are obtained by considering $A = B = X$ in the second section.

We start with providing four new types of $(\psi-\phi)$-weak contractions for self mappings in a non-Archimedean fuzzy metric space $(X, M, *)$.

Definition 12. *Let* $(X, M, *)$ *be a fuzzy metric space. A mapping* $T : X \to X$ *is called a* $(\psi-\phi)$*-contraction of type I if there exist* $\psi \in \Psi$ *and* $\phi \in \Phi$ *such that for all* $x, y \in X$ *and* $t > 0$

$$\psi(M(Tx,Ty,t)) \leq \psi(M(x,y,t)) - \phi(M(x,y,t)). \tag{23}$$

Definition 13. *Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is called a $(\psi - \phi)$-contraction of type II if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that for all $x, y, \in X$ and $t > 0$*

$$\psi(M(Tx, Ty, t)) \leq \psi(N(x, y, Tx, Ty, t)) - \phi(N(x, y, Tx, Ty, t)), \tag{24}$$

where $N(x, y, Tx, Ty, t) = \max\{M(x, y, t), M(x, Tx, t) * M(y, Ty, t), M(x, Ty, t) * M(y, Tx, t)\}$.

Definition 14. *Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is called a $(\psi - \phi)$-contraction of type III if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that for all $x, y \in X$ and $t > 0$*

$$\psi(M(T^2x, T^2y, t)) \leq \psi(M(Tx, Ty, t)) - \phi(M(Tx, Ty, t)). \tag{25}$$

Definition 15. *Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is called a $(\psi - \phi)$-contraction of type IV if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that for all $x, y \in X$ and $t > 0$*

$$\psi(M(Tx, Ty, t)) \leq \psi(\frac{M(x, Tx, t) + M(y, Ty, t)}{2}) - \phi(\frac{M(x, Tx, t) + M(y, Ty, t)}{2}). \tag{26}$$

Theorem 11. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space whose t-norm is 1-boundary continuous. Let $T : X \to X$ be a $(\psi - \phi)$-contraction of type I. Then, T has a unique fixed point x^* in X.*

Proof. Consider a self mapping $T : X \to X$ which is a $(\psi - \phi)$-contraction of type I. Proceeding the similar argument as in the proof of Theorem 2, we can generate a fuzzy Picard sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ which converges to $x^* \in X$, that is, $\lim_{n \to +\infty} M(x_n, x^*, t) = 1$ for all $t > 0$.

Taking $x = x_n, y = x^*$ in (23), we have

$$\psi(M(Tx_n, Tx^*, t)) \leq \psi(M(x_n, x^*, t)) - \phi(M(x_n, x^*, t)).$$

Passing to the limit as $n \to +\infty$ in the above inequality and using the continuity of ψ and M, the lower semi continuity of ϕ, we obtain

$$\lim_{n \to +\infty} \psi(M(Tx_n, Tx^*, t)) \leq \psi(1) - \phi(1) = 0,$$

which implies $\lim_{n \to +\infty} M(Tx_n, Tx^*, t) = M(x^*, Tx^*, t) = 1$ for all $t > 0$. Analysis similar to that in the proof of Theorem 2 shows x^* is a unique fixed point of mapping T. □

Remark 3. *It is noted that the consequence of the best proximity result stated as Theorem 2 can be reduced to Theorem 11 which coincides with Theorem 2,2 in [26].*

Theorem 12. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space whose t-norm is 1-boundary continuous. Let $T : X \to X$ be a $(\psi - \phi)$-contraction of type II. Then, T has a unique fixed point x^* in X.*

Proof. Consider a self mapping $T : X \to X$ which is a $(\psi - \phi)$-contraction of type II. Proceeding the similar argument as in the proof of Theorem 4, we can generate a fuzzy Picard sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ which converges to $x^* \in X$, that is $\lim_{n \to +\infty} M(x_n, x^*, t) = 1$ for all $t > 0$. Following the rest of the proof of Theorem 4, we can prove that x^* is a unique fixed point of T. □

Theorem 13. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space whose t-norm is 1-boundary continuous. Let $T : X \to X$ be a $(\psi - \phi)$-contraction of type III. Then, T has a fixed point x^* in X. Moreover, if T is injective, the fixed point is unique.*

Proof. Consider a self mapping $T : X \to X$ which is a $(\psi - \phi)$-contraction of type III. Proceeding the similar argument as in the proof of Theorem 6, we can generate a fuzzy Picard sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ which converges to $x^* \in X$, that is $\lim_{n\to+\infty} M(x_n, x^*, t) = 1$ for all $t > 0$. Along with the same route in the proof of Theorem 6 we can prove that x^* is a unique fixed point of T. □

Theorem 14. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space whose Product t-norm is 1-boundary continuous. Let $T : X \to X$ be a $(\psi - \phi)$-contraction of type IV. Then, T has a unique fixed point x^* in X.*

Proof. Consider a self mapping $T : X \to X$ which is a $(\psi - \phi)$-contraction of type IV. Proceeding the similar argument as in the proof of Theorem 9, we can generate a fuzzy Picard sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ which converges to $x^* \in X$, that is $\lim_{n\to+\infty} M(x_n, x^*, t) = 1$ for all $t > 0$. Along with the same route in the proof of Theorem 9, we can prove that x^* is a unique fixed point of T. □

3. Application in Domain of Words

As fuzzy metric spaces are linked in a very natural way with applications in the computer sciences, in this section, we will show the existence of fixed point for $(\psi - \phi)$-contraction of type I mappings on the domain of words when it is endowed with a non-Archimedean fuzzy metric.

Suppose that a nonempty set of alphabets is denoted by σ and σ_∞ represents the set of all finite and infinite words over σ. Note that ϕ represents the empty words/sequence contained in σ_∞. Suppose a prefix order on σ_∞, denoted by $\bigvee$ and defined as

$$a \bigvee b \text{ if and only if } a \text{ is a prefix of } b.$$

For every nonempty (word) $a \in \sigma_\infty$, and the length of a is $l(a) \in [1, +\infty)$ and $l(\phi) = 0$. Furthermore, if $a \in \sigma_\infty$ is finite, then $l(a) < +\infty$ and we write

$$a = a_1a_2, \dots, a_n,$$

otherwise we write

$$a = a_1a_2, \dots .$$

Now, for $a, b \in \sigma_\infty$, the common prefix of a and b is represented by $a \bigcap b$. It is to be noted that $a = b$ if and only if $a \bigvee b$ and $b \bigvee a$ and $l(a) = l(b)$. Define $R_\bigvee : \sigma_\infty \times \sigma_\infty \to [0, +\infty)$ by

$$R_\bigvee(a, b) = \begin{cases} 0, & \text{iff } a = b; \\ 2^{-l(a)}, & \text{iff } a \bigvee b; \\ 2^{-l(b)}, & \text{iff } b \bigvee a; \\ 2^{-l(a \bigcap b)}, & \text{otherwise.} \end{cases}$$

If $a \bigvee b$, then $a \bigcap b = a$ and if $b \bigvee a$, then $b \bigcap a = b$. Therefore, for all $a, b \in \sigma_\infty$, we can write

$$R_\bigvee(a, b) = \begin{cases} 0, & \text{iff } a = b; \\ 2^{-l(a \bigcap b)}, & \text{otherwise.} \end{cases}$$

$R_\bigvee$ is a complete Baire metric [41] on σ_∞. Define a fuzzy metric over σ_∞ as:

$$M_{R_\bigvee}(a, b, t) = e^{-\frac{R_\bigvee(a,b)}{t}}.$$

Then, $(\sigma_\infty, M, *)$ represents a complete non-Archimedean fuzzy metric space, where the t-norm is $a * b = ab$. The Quicksort algorithm gives the recurrence relation

$$a_m = 0, \text{ for } m = 1,$$
$$a_m = \frac{2(m-1)}{m} + \frac{m+1}{m} a_{m-1}, \text{ for } m \geq 2.$$

For more details, we refer the readers to [42,43]. For $\sigma = [0, +\infty)$, in correspondence to the above sequence, we define the function $f : \sigma_\infty \to \sigma_\infty$ that assigns $f(a) := f((a))_1 f((a))_2, \ldots$ to $a := a_1 a_2, \ldots$ and is defined by

$$\begin{cases} f((a))_m = 0, & \text{for } m = 1; \\ f((a))_m = \frac{2(m-1)}{m} + \frac{m+1}{m} f(a_{m-1}), & \text{for } m \geq 2. \end{cases}$$

Note that

$$l(f((a))) = l(a) + 1,$$

for all $a \in \sigma_\infty$ and in particular

$$l(f((a)) = +\infty,$$

whenever $l(a) = +\infty$. By definition of f, we have

$$a \bigvee b \quad \text{if and only if} \quad f(a) \bigvee f(b)$$

and this implies that

$$f(a \bigcap b) \bigvee f(a) \bigcap f(b)$$

for all $a, b \in \sigma_\infty$. Hence, $l(f(a \bigcap b)) \leq l(f(a) \bigcap f(b))$, for all $a, b \in \sigma_\infty$. We apply Theorem 11 and prove that the functional f has a fixed point. Also, define $\psi \in \Psi$ and $\phi \in \Phi$ by $\psi(s) = \frac{1-s}{4s}$ and $\phi(s) = \frac{s-1}{2s}$. Then, consider the following two cases:

Case 1: If $a = b$, then we have

$$M_{R_\vee}(f(a), f(b), t) = 1 = M_{R_\vee}(a, a, t).$$

and

$$\psi(M_{R_\vee}(f(a), f(b), t)) = \psi(M_{R_\vee}(a, a, t)) - \phi(M_{R_\vee}(a, a, t)) = 0.$$

Case 2: If $a \neq b$, then for all $t > 0$, we have

$$l(f(a) \bigcap f(b)) \geq l(f(a \bigcap b)),$$

that is, $-\frac{2^{-l(f(a) \bigcap f(b))}}{t} \geq -\frac{2^{-l(a \bigcap b)}}{t}$, further

$$e^{-\frac{2^{-l(f(a) \bigcap f(b))}}{t}} \geq e^{-\frac{2^{-l(a \bigcap b)}}{t}}.$$

Now,

$$
\begin{aligned}
M_{R_{\vee}}(f(a), f(b), t) &= e^{-\frac{2^{-l(f(a)\cap f(b))}}{t}} \\
&\geq e^{-\frac{2^{-l(f(a\cap b))}}{t}} \\
&= e^{-\frac{2^{-(l(a\cap b)+1)}}{t}} \\
&= e^{-\frac{2^{-l(a\cap b)}.2^{-1}}{t}} \\
&= (e^{-\frac{2^{-l(a\cap b)}}{t}})^{2^{-1}} \\
&\geq e^{-\frac{2^{-l(a\cap b)}}{t}} \\
&= M_{R_{\vee}}(a, b, t),
\end{aligned}
$$

for all $a, b \in \sigma_\infty$, and all $t > 0$.

After a simple computation, we have

$$
\begin{aligned}
\psi(M_{R_{\vee}}(f(a), f(b), t)) &= \frac{1 - M_{R_{\vee}}(f(a), f(b), t)}{4M_{R_{\vee}}(f(a), f(b), t)} \\
&= \frac{1}{4}e^{\frac{2^{-l(f(a)\cap f(b))}}{t}} - \frac{1}{4} \\
&\leq \frac{1}{4}e^{\frac{2^{-l(a\cap b)}}{t}} - \frac{1}{4} \\
&\leq \frac{1}{2}e^{\frac{2^{-l(a\cap b)}}{t}} - \frac{1}{2} \\
&= \psi(M_{R_{\vee}}(a, b, t)) - \phi(M_{R_{\vee}}(a, b, t)).
\end{aligned}
$$

Thus, all conditions of Theorem 11 are satisfied and f has a fixed point $a = a_1a_2\ldots$, which is the required fixed point of f. Hence, we have

$$
\begin{aligned}
a_1 &= 0, \\
a_n &= \frac{2(n-1)}{n} + \frac{n+1}{n}, \text{for } n \geq 2.
\end{aligned}
$$

Remark 4. *The prefix order $\bigvee$ on σ_∞ defined as above is a partial order on σ_∞ (domain of words) which is associated with the graph via the relation*

$$a \bigvee b \text{ if and only if } (a, b) \in E(G),$$

where $E(G)$ is the set of edges of graph G and the graph $G = (V(G), E(G))$ with $V(G) = \sigma_\infty$. Some problems related to the domain of words can be considered in connection with the graphs as well to solve some problems related to networks.

4. Conclusions

To find an optimal distance between two objects is considered an important problem in mathematics, which is at the root of several studies related to fields such as geometry, mechanics, computer sciences, optimization, etc. Here, our candidates are sets and the framework of study is the non-Archimedean fuzzy metric spaces. In this paper, we defined some new classes of $(\psi - \phi)$-weak proximal contractions and proved the existence of unique best proximity point/fixed point for such contractions in complete non-Archimedean fuzzy metric spaces without fuzzy P-property. Moreover, we also present some examples to illustrate the main results and an application in computer science, particularly in the domain of words as well. The prospects of the present study lie in the possibilities of further generalizations as well as in its possible applications in the domains of fuzzy geometry, fuzzy optimization, etc. The present paper is a fuzzy generalization of the proximity point problem by means of fuzzy fixed point methodologies. Meanwhile, we put forward some directions worthy further consider in the future:

(1) To generalize the existing proximal contractions and to prove the existence of the unique best proximity point in non-Archimedean fuzzy spaces;
(2) Under the assumption that $A_0(t)$ and $B_0(t)$ are dependent from the real parameter t, what proximal contractions and other properties can guarantee the existence of the unique best proximity point;
(3) To study the best proximity point results and related applications for the p-cyclic proximal contractions in the fuzzy settings;
(4) As fuzzy quasi metric spaces are linked in a very natural way with applications in computer sciences (see [41]), so the results in this paper can be investigated in connection with fuzzy quasi metric spaces with some applications.

Author Contributions: Conceptualization, M.Z. and N.S.; formal analysis, M.Z., N.S.; investigation, M.Z., N.S., A.F.R.L.d.H. and X.L.; writing—original draft preparation, M.Z. and N.S.; writing—review and editing, M.Z., N.S., A.F.R.L.d.H. and X.L. All authors have read and agreed to the published version of the manuscript.

Funding: There is no funding to support this work.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Fan, K. Extension of two fixed point theorems of F.E Browder. *Math. Z.* **1969**, *112*, 234–240. [CrossRef]
2. Basha, S.S.; Veeramani, P. Best approximations and best proximity pairs. *Acta Sci. Math.* **1997**, *63*, 289–300.
3. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
4. Kramosil, I.; Michalek, J. Fuzzy metrics and statistical metric spaces. *Kybernetika* **1975**, *11*, 336–344.
5. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* **1994**, *64*, 395–399. [CrossRef]
6. Gregori, V.; Sapena, A. On fixed-point theorems in fuzzy metric spaces. *Fuzzy Sets Syst.* **2002**, *125*, 245–252. [CrossRef]
7. Samuel, M.; Gregori, V.; Guillermo, P.F.; Latorre, P. A fast impulsive noise color image filter using fuzzy metrics. *Real Time Imaging* **2005**, *11*, 417–428.
8. Samuel, M.; Gregori, V.; Guillermo, P.F.; Sapena, A. Local self adaptive fuzzy filter for imulsive noise removal in color images. *Signal Process.* **2008**, *88*, 390–398.
9. Choudhury, B.S.; Das, K.; Das, P. Coupled coincidence point results for compatible mappings in partially ordered fuzzy metric spaces. *Fuzzy Sets Syst.* **2013**, *222*, 84–97. [CrossRef]
10. Roldán López de Hierro, A.F.; Sintunavarat, W. Common fixed point theorems in fuzzy metric spaces using the CLR_g property. *Fuzzy Sets Syst.* **2016**, *282*, 131–142. [CrossRef]
11. Grabiec, M. Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **1998**, *27*, 385–389. [CrossRef]
12. Wardowski, D. Fuzzy contractive mappings and fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **2013**, *222*, 108–114. [CrossRef]
13. Mihęt, C.D. Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces. *Fuzzy Sets Syst.* **2008**, *159* 739–744. [CrossRef]
14. Vetro, C. Fixed points in weak non-Archimedean fuzzy metric spaces. *Fuzzy Sets Syst.* **2011**, *162*, 84–90. [CrossRef]
15. Samanta, T.K.; Mohinta, S. Common fixed point theorems for single and set-valued maps in Non-Archimedean fuzzy metric spaces. *Acta Univ. Sapientiae Math.* **2012**, *4*, 132–144.
16. Altun, I.; Miheţ, C.D. Ordered non-Archimedean fuzzy metric spaces and some fixed-point results. *Fixed Point Theory Appl.* **2010**, *2010*, 782680. [CrossRef]
17. Mustafa, Z.; Parvaneh, V.; Roshan, J.R. b_2-Metric spaces and some fixed point theorems. *Fixed Point Theory Appl.* 2014, *2014*, 144. [CrossRef]
18. Roshan, J.R.; Shobkolaei, N.; Sedghi, S.; Parvaneh, V.; Radenović, S. Common fixed point theorems for three maps in discontinuous G_b-metric spaces. *Acta Math. Sci. Ser. B Chin. Ed.* **2014**, *34B*, 1643–1654. [CrossRef]
19. Shobkolaei, N.; Sedghi, S.; Roshan, J.R.; Altun, I. Common fixed point of mappings satisfying almost generalized $(S, T)-$contractive condition in partially ordered partial metric spaces. *Appl. Math. Comput.* **2012**, *219*, 443–452. [CrossRef]
20. Alber, Y.I.; Guerre-Delabriere, S. Principle of weakly contractive maps in Hilbert spaces. *New Results Oper. Theory Appl.* **1997**, *98*, 7–22.
21. Rhoades, B.E. Some theorems on weakly contractive maps. *Nonlinear Anal. TMA* **2001**, *47*, 2683–2693. [CrossRef]
22. Chidume, C.E.; Zegeye, H.; Aneke, S.J. Approximation of fixed points of weakly contractive non self maps in Banach spaces. *J. Math. Anal. Appl.* **2002**, *270*, 189–199. [CrossRef]

23. Roshan, J.R.; Parvaneh, V.; Radenović, S. Some coincidence point results for generalized $(\psi,\varphi)-$weakly contractions in ordered $b-$metric spaces. *Fixed Point Theory Appl.* **2015**, *2015*, 68. [CrossRef]
24. Dorić, D. Common fixed point for generalized (φ,ψ)-weak contractions. *Appl. Math. Lett.* **2009**, *22*, 75–78. [CrossRef]
25. Harjani, J.; Sadarangani, K. Fixed point theorems for weakly contractive mappings in partially ordered sets. *Nonlinear Anal. TMA* **2009**, *71*, 3403–3410. [CrossRef]
26. Saha, P.; Choudhury, B.S.; Das, P. A weak contraction in a fuzzy metric spaces. *J. Uncertain Syst.* **2019**, *13*, 151–160.
27. Vetro, C.; Salimi, P. Best Proximity Point Results in Non-Archimedean Fuzzy Metric Spaces. *Fuzzy Inf. Eng.* **2013**, *4*, 417–429. [CrossRef]
28. Saha, P.; Guria, S.; Choudhury, B.S.; De la Sen, M. Determining Fuzzy Distance between Sets by Application of Fixed Point Technique Using Weak Contractions and Fuzzy Geometric Notions. *Symmetry* **2019**, *11*, 812. [CrossRef]
29. Schweizer, B.; Sklar, A. Statistical metric spaces. *Pac. J. Math.* **1960**, *10*, 314–334. [CrossRef]
30. Rodríguez-López, J.; Romaguera, S. The Hausdorff fuzzy metric on compact sets. *Fuzzy Sets Syst.* **2004**, *147*, 273–283. [CrossRef]
31. Murthy, P.P.; Kenvat, R. Best proximity points in non-Archimedean fuzzy metric spaces. *Ser. Math. Inform.* **2015**, *30*, 479–488.
32. Paknazar, M. Non-Archimedean fuzzy metric spaces and best proximity point theorems. *Sahand Commun. Math. Anal. (SCMA)* **2018**, *9*, 85–112.
33. Sankar Raj, V. A best proximity point theorem for weakly contractive non-self mappings. *Nonlinear Anal. TMA* **2011**, *74*, 4804–4808. [CrossRef]
34. Sankar Raj, V. Best proximity point theorems for non-self mappings. *Fixed Point Theory* **2013**, *14*, 447–454.
35. Saha, P.; Guria, S.; Choudhury, B.S. Determing fuzzy distance through non-self fuzzy contractions. *Yugosl. J. Oper. Res.* **2019**, *29*, 325–335. [CrossRef]
36. Roldán López de Hierro, A.F.; Karapınar, E.; Shahzad, N. Fuzzy ample spectrum contractions in (more general than) non-Archimedean fuzzy metric spaces. *arXiv* **2021**, arXiv:2104.09155.
37. Basha, S.S. Best proximity point theorems. *J. Approx. Theory* **2011**, *163*, 1772–1781. [CrossRef]
38. Saleem, N.; Abbas, M.; Raza, Z. Optimal coincidence best approximation solution in non-Archimedean Fuzzy Metric Spaces. *Iran. J. Fuzzy Syst.* **2016**, *13*, 113–124.
39. De la Sen, M.; Abbas, M.; Saleem, N. On optimal fuzzy best proximity coincidence points of proximal contractions involving cyclic mappings in non-Archimedean fuzzy metric spaces. *Mathematics* **2017**, *5*, 22.
40. George, A.; Veeramani, P. On some results of analysis for fuzzy metric spaces. *Fuzzy Sets Syst.* **1997**, *90*, 365–368. [CrossRef]
41. Romaguera, S.; Sapena, A.; Tirado, P. The Banach fixed point theorem in fuzzy quasi-metric spaces with application to the domain of words. *Topol. Appl.* **2007**, *154*, 2196–2203. [CrossRef]
42. Flajolet, P. Analytic analysis of algorithms. In *Lecture Notes in Computer Science*; Springer: Berlin/Heidelberg, Germany, 1992.
43. Kruse, R. *Data Structures and Program Design*; Prentice-Hall: New York, NY, USA, 1999.