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Generalisation strategies and representations used by last-year elementary school students

Abstract: Recent research has highlighted the role of functional relationships in introducing elementary school students to algebraic thinking. This functional approach is here considered to study essential components of algebraic thinking such as generalization and its representation, and also the strategies used by students and their connection with generalization. This paper jointly describes the strategies and representations of generalisation used by a group of 33 sixth-year elementary school students, with no former algebraic training, in two generalisation tasks involving a functional relationship. The strategies applied by the students differed depending on whether they were working on specific or general cases. To answer questions on near specific cases they resorted to counting or additive operational strategies. As higher values or indeterminate quantities were considered, the strategies diversified. The correspondence strategy was the most used and the common approach when students generalised. Students were able to generalise verbally as well as symbolically and varied their strategies flexibly when changing from specific to general cases, showing a clear preference for a functional approach in the latter.

Keywords: algebraic thinking, early algebra, generalisation, strategies, functional relationships, representations.

Introduction

Research on algebraic reasoning at early ages is a fertile field of study with significant implications considering the breadth with which it has been addressed (Cañadas et al., 2019). Interest in the area has been further prompted by the new century's demands, which call for cultivating the ability to envisage the depth of the structures underlying mathematics (Blanton & Kaput, 2005). The early algebra curricular proposal promotes instruction in and the development of algebraic reasoning by affording an opportunity for deeper and more complex mathematical training from the time children are first schooled (Blanton & Kaput, 2005). The proposal has materialised with the inclusion of algebraic thinking in various countries' elementary school curricula (e.g. Common Core State Standards Initiative [CCSSI], 2010; Ministerio de Educación,Cultura yDeporte, 2014;Ministry of Education Singapore, 2012).). In the case of Spain, the elementary school curriculum indicates that throughout this stage it is intended that students can "describe and analyse situations of change, find patterns, regularities and mathematical laws in numerical, geometric and functional contexts, valuing their usefulness to make predictions" (Ministerio de Educación, Cultura y Deporte, 2014, p. 33).

We focus our interest in the functional approach to algebraic thinking that entails studying functions, relationships and change (Kaput, 2008). "Thinking in terms of and around relationships" (Rico, 2007, p. 56), i.e., functional thinking, is acknowledged to be one of the main components of algebraic reasoning (Warren & Cooper, 2005). Functional thinking favours the creation of a space for algebraic reasoning-linked experience, including reasoning, dealing with, generalising and representing the relationships existing between covarying quantities (Blanton et al., 2015; Blanton et al., 2011; Kaput, 2008; Stephens et al., 2017).

Within this context, generalization and its representation stand out as essential elements of algebraic thinking (e.g., Kaput, 2008; Radford, 2018; Warren et al., 2016). As Mason et al. (2005) explain "learners will only understand algebra as a language of expression if they perceive and express generalities for themselves" (p.23). The start point is existence of a multiplicity of representations that may be involved in the expression of the generalisation (e.g., natural language, algebraic symbolism, graphics) (e.g., Blanton y Kaput, 2005; Kaput, 2008; Carraher et al., 2008), either individually or combined, as well as other semiotic systems such as gestures (Radford, 2018). There are multiple studies in the early algebra framework, from the functional approach, exposing the student's algebraic potential to generalize and represent functional relationships, when they receive related instruction (e.g., Blanton y Kaput, 2004; Carraher et al., 2008; Warren y Cooper, 2008). In this study we are interested on students who have not received any algebraic education. We work with students who are ending elementary education and will start their learning of algebra in the following year. In this way we aim to provide information related to their skills of use in their later formal study of algebra in high school.

Current research reveals the interest of in-depth studying students' solving strategies in generalisation tasks (e.g., Amit & Neria, 2008; El Mouhayar & Jurdak, 2015), especially in functional contexts (e.g., e.g. Morales et al., 2018). In elementary education, even in the latter years, applying strategies that would lead to effective generalisation is a challenge for students (Barbosa et al.; Stacey, 1989; Zapatera Llinares, 2018). Some studies suggest that students find difficult to identify and justify the functional relationships underlying mathematical problems due in part to the strategies used (e.g., Moss & Beatty, 2006). From this insight, the strategies used by elementary school students to generalise in functional contexts stand out as a potential study area in the algebraic thinking framework to which we pretend to contribute.

Research interest

Within the context described, among the different strategies that students use when solving tasks, we pay attention to describing the strategies and representations that students use when generalizing as fundamental components of algebraic thinking. The joint study of strategies and representations of generalization is one of the contributions of this work. This paper aims to identify students' strategies in two generalization tasks and how they represent the generalization of the functional relationships they recognize within the variables involved.

Generalisation and representation

Generalisation plays an instrumental, even a core, part in mathematics (Mason et al., 1989). It lies at the heart of algebra (Mason et al., 2005). In their various approaches to generalisation, most authors stress recognition of regularity, generation of new cases and representation.

Pólya (1989) conceived generalisation to be the generation of new cases based on the regularity identified in a set of elements. Kaput (1999) referred to generalisation as extending reasoning beyond the cases considered by either explaining the similarity present or broadening reasoning by focusing on patterns, procedures and structures and their inter-relationships. Radford (2010) define algebraic generalisation as the ability to recognise regularity in a sequence of elements, realise its validity for all the elements of the same class and consequently formulate an expression to represent it. Stephens et al. (2017) distinguish between generalisation as a process and as a product, defining the latter as the result of any of these processes: identifying the regularity across cases, reasoning beyond the cases at issue or broadening the results beyond specific cases. In this study we assume Kaput's definition of generalisation (1999) applied to the functional context. It involves identifying, evidencing and representing the regularity underlying the task which connects the quantities involved.

One of the highlights of the generalisation process is progressively symbolic representation (Kaput, 2008). Representation is indisputably associated with generalisation and algebraic thinking (Kaput, 2008; Radford, 2018). Such thinking is not expressed exclusively through algebraic symbolism, however, but also in the form of natural language or gesturing, among other ways (Radford, 2018).

We refer here to (external) representations understood as "assertions in natural language, algebraic formulas, graphs or geometric figures, among others, [constituting] the medium whereby individuals exteriorize their mental images and representations to make them accessible to others" (Rico et al., 1997, p. 101). The term representations of generalisation refer to how generalisation is evidenced and externally expressed.

On the path to generalising functions, students can use and represent different relationships they identify between the variables. Smith (2008) distinguishes three types of relationships: (a) recursive, that consider the variation of a single variable relating its consecutive values, (b) correspondence, which addresses the relationship between pairs of corresponding values associated with the independent and dependent variable, and (c) covariation, which involves the analysis of how both variables covary, i.e., how the change in one variable affects the other. At the same time, the functional relationships used by students can be characterized in terms of their structure. The structure refers to how the regularity between the variables is organized and expressed (Pinto & Cañadas, 2017). That is, how indeterminate and/or numerical values are operated when the regularity is used or represented.

Generalisation in primary school continues to strengthen as a field of research (Cañadas et al., 2019; Hitt & González-Martín, 2016). Research focuses on how students represent generalisations and the strategies they use to generalise (Kaput, 2008; Morales et al., 2018; Ureña et al., 2019; Warren et al., 2016).

Studies with early elementary and pre-schoolers showed the students were able to identify variables and their relationships, use a variety of representations —including algebraic symbolism— and understand, represent, and progress in the expression of functional relationships, after receiving instruction (e.g., Blanton & Kaput, 2004; Blanton et al., 2015; Carraher et al., 2008; Warren & Cooper, 2005, 2008). Other authors have also focused on how generalisation is expressed. In a study with students from second (7 to 8 years old) to seventh year (12 to 13 years old), Radford (2018) observed both symbolic and non-symbolic representations of generalisation. He drew attention to the different semiotic systems (such as gesturing, language or symbolism) used to express generalisation, contending that each furnishes a different type of information on the treatment of and inter-relationships between variables and the algebraic structure of the sequences involved in the tasks. In the research by Amit and Neria (2008), talented students (11 to 13 years old) represented functional relationships linked to patterns through verbal representations, algebraic symbolism, and general verbal terms in algebraic expressions as a semi-symbolic representation.

In functional generalisation contexts, Torres et al. (2019) reported that second-year students (7-8 years old), without instruction, tended to use numerical and verbal representations in their answers, without generalising. Merino et al. (2013) also found that fifth graders (10 to 11 years old) mainly used verbal representation to present their reasoning. Pinto & Cañadas (2017, 2021) recognized that both third graders (8 to 9 years old) and mainly fifth graders (10-11 years old), represented verbally the generalisation of functional relationships. Ureña et al. (2019) determined that fourth year (10-11-year-old) students with no prior instruction in representation or functional tasks used a variety of systems (e.g., numerical, verbal, symbolic) to represent generalised functional relationships.

Strategies and generalisation

The procedures deployed to solve a problem, draw conclusions from a corpus of ideas and establish relationships are known as strategies (Rico, 1997). They inform about students' thinking processes when solving problems.

A variety of strategies applied in generalization contexts have been described in the literature. Stacey (1989) distinguished four strategies: (a) counting the elements on a figure, (b) direct proportionality, (c) difference between consecutive terms and (d) application of a linear functional model. Later studies such as Merino et al. (2013), Morales et al. (2018) and Zapatera Llinares (2018) define similar strategies used in generalization tasks. Two new strategies appeared in Merino et al. (2013)'s study: (a) the use of arithmetic operations unrelated to specific patterns (i.e. regularities) and (b) the repetition of the general statement of the task. It is interesting to notice the distinction made by Zapatera Llinares between a local (for a specific term) and global (for any term) application of the functional relationship connecting the two variables. These studies also report direct answers given without explanation of the process followed.

Stacey (1989) identified instability in elementary students' use of strategies for near (generalising with simple processes such as counting or drawings) and far (entailing more complex processes to determine a general rule or pattern) generalisation, along with a propensity to choose the simplest rather than the most precise option. Barbosa et al. (2012) observed sixth-year students (11 to 12 years old) to perform poorly in generalisation tasks with a visual component; even those earning the highest marks counted or used recursive patterning but did not generalise. Merino et al. (2013) pointed out that fifth year students (10-11 years old) changed strategies such as counting and direct answer in specific cases, to the use of arithmetic operations and, mainly, patterns (structures) in far and general cases. Other works highlight functional strategies for being linked to generalisation (e.g., Amit y Neria, 2008; El Mouhayar & Jurdak, 2015; Stacey, 1989). They consist of expressing, generalising or using implicitly or explicitly a functional relationship between two variables. In a similar study with 8- to 12-year-old students, Zapatera Llinares (2018) found that changing from additive strategies for near, to functional strategies (in which they identified and applied a function) for far generalisation, guaranteed successful generalisation. Amit and Neria (2008) determined that mathematically talented sixth- and seventh-year students (11 to 13 years old) used functional and recursive strategies to generalise linear and quadratic patterns. Especially, functional strategies stood out for their efficiency and scope to generalise. El Mouhayar and Jurdak (2015), also in a generalisation context of linear and quadratic figural patterns, focused on studying how the use of strategies from immediate-near cases to far-n case (understood as pattern generalisation types) varied in students' work across grades 4 to 11 (9 to 17 years old). They also highlighted recursive and functional strategies to be used in all tasks. However, unlike the first, the use of functional strategy tended to grow as the demand for generalisation towards the general case increased. Another common finding in various of the mentioned studies (e.g., Stacey, 1989; Zapatera Llinares, 2018) was the incorrect use of direct proportionality, primarily in general cases.

Methodology

This qualitative, descriptive and exploratory study was conducted with 33 sixth-year (11- to 12-years-old) elementary school students who volunteered to answer a questionnaire as a preliminary for participation in a project designed to stimulate mathematical talent ((Ramírez-Uclés & Cañadas, 2018). We intentionally work with these students because they were adequate to develop the objectives of the study in that we could assume a good attitude towards mathematics and that they would not have difficulties to generalise or to work with specific cases. Starting from this hypothesis, they could clearly display the strategies to generalise and the respective representations of generalisation.

The whole questionnaire consisted of five problems that address different contents (functions, operations with numbers, divisibility, plane measurement, and spatial constructions). According to our research interest here we analyse the answers to the first problem, the "potato seed" problem, as the only one that involved generalising functional relationships through two tasks. It was designed by Ramírez-Uclés & Cañadas, (2018) and was analyzed and validated by elementary and secondary teachers collaborating with the project. They assessed that the problem involved the ability to generalise, was appropriate to the students' age and mathematical knowledge, and was progressively complex. We designed this problem guided by the aim of the study. We reviewed the existing research literature and used the following criteria. The statement of the problem involves verbal and pictorial representation. In the first two specific cases the student is invited to make a pictorial representation of the situation. Two tasks that follows an inductive organization are derived from the problem and both accept different solving strategies. Each task implies a different linear functional relationship and requests justification of the answers, and there exists a dependency between them and the underlying functional relationships.

Task design

The potato seed problem consists in two tasks related to a same context (Figure 1). In the first task students had to determine the number of squares that could be drawn having seeds as vertices. The second task asks them to find the value of the sum of the orders of all the seeds (the order of a seed is the number of squares that has one of its vertices in that seed). The first functional relationship, which depends on the number of days, is $f(n) = 4n - 6$ (by obtaining that each day the number of squares increases by four except the first and second day when fewer are formed). The second functional relationship, which associates the sum of the orders with the number of squares, is $h(n) = 4 \cdot (4n - 6) = 16n - 24$.

[Figure 1]

The problem required solving both tasks for 3, 4, 100 and n days. This inductive approach was adopted to help students visualise the underlying regularity and identify and generalise the implicit functional relationships. In line with Amit and Neria (2008), this organization can promote a transition from a “warm up” case so that the student becomes familiar with the task, a tentative generalisation through the extension of the regularity to another specific case, and an informal generalisation through representation that the student prefers (e.g., verbally) until reaching a formal generalisation with algebraic symbolism.

Analysis

For data analysis we first defined the unit of analysis as each student's full answer to each task. Subsequently, we performed a three-phase analysis. In the initial phase, the first author of this paper elaborated two sets of categories using a content analysis approach informed by previous research on solving strategies (Amit & Neria, 2008; Barbosa et al., 2012; Merino et al., 2013;Morales et al., 2018; Stacey, 1989; Zapatera Llinares, 2018), and the representation of generalisation categories defined by Ureña et al. (2019).

To ensure the validity and reliability of the data analysis, in a second phase, three of the researchers performed a triangulation by experts. They analyzed a new random selection of written productions following the established categories. Once the members of the research team agreed on the coding of the results, the categories were written in its final form. Finally, after the categories have been established, in the third phase of the analysis, each of the students' written productions was exhaustively analyzed.

Below we defined he established categories to analyze the solving strategies.

- Counting: the result was obtained from the count of some elements in a pictorical representation.
- Additive operationality: the answer was found by explicit or implicit isolated additions not related to operations performed in previous or later responses to the task.
- Multiplicative operationality: the answer was found by explicit or implicit isolated multiplication or division not related to operations performed in previous responses to the task.

- Proportionality: proportional reasoning was used to obtain one of the terms as a product of others. This strategy is separated from the previous one to emphasize the specific reasoning and procedure involved.
- Correspondence: a correspondence functional relationship between the associated variables to describe the situation was established and used.
- Direct answer: answers were obtained with no specification of the procedure followed.
- Other: the procedure used could not be classified in any of the above.

Regarding to the representation of the generalisation, the students were deemed to express generalisation when they represented a general rule relating the variables according to a regularity recognised. Below we describe the categories used to classify how generalisation was represented by students.

- Student does not represent the generalisation.
- Student represents the generalisation. It is divided into three subcategories that are the types of representations of generalisation that we distinguish:
    - Verbal: the detected regularity is expressed through natural language.
    - Symbolic: the detected regularity is expressed by means of algebraic symbolism.
    - Multiple: the detected regularity is expressed using a combination of verbal and symbolic representations.

Three sections were distinguished for analysis: near cases-3 and 4, far case-100 and general case-n.

Results

Only three students (S4[1], S14 and S30) identified all the squares that could be drawn (Figure 2a). Twenty-seven students only identified the squares that rested on a horizontal row (Figure 2b). When considering only this latter type of square, the underlying functional relationships structures are: $3n - 4$ for the first task (by determining that each day the number of squares increases by 3 except the first and second day where fewer are formed) and $4s = 4(3n - 4) = 12n - 16$ for the second task. As the other three students (S9, S23 and S28) misinterpreted the geometric description given (Figure 2c) and furnished information irrelevant to the problem, so they were excluded from the analysis.

[Figure 2]

To present the main results we first describe for each of the tasks the strategies used by the students to solve the posed cases and obtain answers as a general context that will then allow us in next section to delve into those that were related to the generalisation and its respective representation.

Solving strategies used by the students

In both tasks the students evidenced the use of a diversity of strategies that varied depending on the case involved. Students used the same strategies in both tasks, except

[1] For reasons of confidentiality each student was assigned a number preceded by the letter S.

counting strategy and additive operationality which were inverted as the most and least frequent strategies between tasks. Correspondence and direct answer were also among the most used. At the same time, stands out a high number of students who do not answer the questions.

The most used and unused strategies are reversed between the first and second task, in 3- and 4-day cases. Counting is the most frequent strategy in first task and it is not evidenced in the second task, whereas the additive operationality strategy is the most used in the second one. This latter strategy is also indirectly based on counting to first determine the order of each seed before addition is performed. In general, counting did not occur in the final cases in both tasks; in these cases, students did not have neither draw any illustrations to support their answers.

Although in 100- and n-day cases more strategies are observed (e.g., proportionality, multiplicative operationality), the correspondence strategy is the most widely used in these cases.

Below we comment separately on the strategies applied by the students in each task.

Task 1. Number of squares

Counting, correspondence, multiplicative operationality, proportionality, and other strategies were used in the solution of this task. Some students also answered directly. Table 1 lists the number of students using each type of strategy in the first task, by case and globally.

[Table 1]

Overall, of the total of ninety productions we analyzed, eighteen times the students did not answer, this being more common in n-day case. In the other answers, most used strategies were the direct response followed by counting and correspondence. The least frequent strategy was additive operationality, that was not applied.

Counting strategy was used by most of the students (19) to reply to the questions with near cases (3 and 4- days). This strategy was used almost exclusively in these cases. The students represented all the squares comprising the answer (e.g., Figure 3) and sometimes organized them according to their size. This means that the students who applied counting based their answers on their pictorial representation of the squares. Student S20, for instance, replied that in four days there would be “8 squares, six with an area of 1 m^2 and two 4 m^2”.

[Figure 3]

Strategies were more diversified in 100- and n-day cases students’. Correspondence strategy, proportionality and multiplicative operationality were applied, with the first strategy being the one most widely used. Ten students (S1, S4, S5, S14, S15, S18, S19, S20, S26 and S30) switched from counting to correspondence strategy in the 100-day case. Students using the correspondence strategy established a relationship between the number of days lapsing and the number of squares. For instance, S1 applied the structure $3(n-4)+8$ (Figure 4). He used the eight squares that could be drawn in the first 4 days as a constant in the functional relationship and defined all other $n-4$ days as the variable term. Analogously, S18 took the five squares formed in the first 3 days as a constant, applying the structure $3(n-3)+5$. S19, S15 and S26 answered similarly.

[Figure 4]

Applying the correspondence strategy, three students (S5, S20 and S30) left out the constant term in the formulation of the structure of the functional relationships. S5 said that "you can draw 288 squares, because there are three squares per day" deriving that result by multiplying 3 times 96. That is, he used the structure $3(n-4)$. The student recognised the regularity but failed to take the number of day 4 squares into consideration. On the other hand, S30 represented a multiplicative structure with operations and words, using the latter as variables: "Solution=Nº of day × 3", after calculating the squares that could be drawn in the specific cases (Figure 5). Other students (S4 and S14) identified a total or partial regularity but used less clear structures. S14, for instance, replied that for 100 days "you can draw 200 squares because you can always draw two 1 m^2 squares plus the total number of days, or 102, and two squares less than the total number of days, or 98". That is, he referred to the structure $n+2\ +\ n-2$ wich is the same as $2n$.

[Figure 5]

The correspondence strategy was applied by five students (S1, S4, S5, S7 and S14) in n-day case. Four (S1, S4, S5 and S14) had applied the same strategy in the preceding case, where the fifth (S7) had used proportionality. Of the other six students who used the correspondence strategy in 100-day case, four (S19, S20, S26 and S30) failed to answer in the general case, despite their previous representation of structures. S18 replied directly with no clear connection to the previous data and S15 used counting.

To a lesser extent, proportionality and multiplicative operationality strategies were applied in 100- and n-day cases. Five students (S3, S6, S7, S11, S22) applied proportionality in 100-day case and three (S3, S22, S11) in the general case. These students applied a known formula unrelated to the data in the problem and to their prior results. By way of example, S7 used the eight squares she deemed could be drawn after 4 days (Figure 6). Other three students (S10, S12 and S33) used multiplicative operationality in 100-day case. S10 for instance divided "100:2=50 because you can draw two squares every day". In contrast, S12 doubled the 100 days to find 200 squares and S33 divided 300 by 4, in what we interpret to be dividing the total number of seeds by the number of vertices in a square. In n-day case S10 and S12 resorted to the same procedure while other student, S24, assigned the letter a value of 50 and divided by 4 to find the solution. Finally, in n-day case one student (S15) applied counting strategy, drawing and counting the squares based on the fifth day's data without evidence of why he selected that specific number of days to answer.

[Figure 6]

Task 2. Sum of orders

In the second task, correspondence, multiplicative operationality, proportionality, direct answer and other, were used as in the previous task. In this case additive operationality was also applied. The strategies used by the students to solve the second task are shown in Table 2.

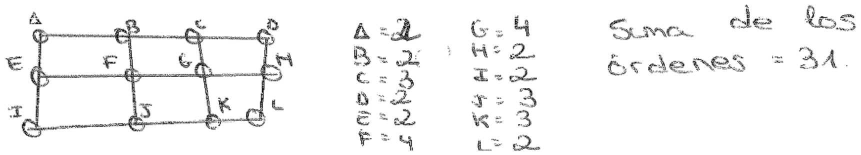
[Table 2]

In this task, almost a third of the students did not answer to the 100-day and n-day cases. In their answers they more frequently used the additive operationality, correspondence and direct answer strategies, respectively.

The additive operational strategy is the most used (24 students) in 3- and 4-day cases. This strategy as well as the counting strategy, and unlike the other strategies, depended on a visual component. The students who used the additive operational strategy added both implicitly and explicitly. For instance, S12 took the squares found in the 4-day case, numbered each seed and assigned an order to each of them as shown in Figure 7. He then summed all the quantities to find the respective solution.

[Figure 7]

Students who specified the order of each seed and then found the total were interpreted to add implicitly, given the arrangement of the data (see Figure 8).



[Handwritten note: sum of orders = 31]

Figure .8. S3's answer, 4-day case

As in task 1, the strategies used were more varied in 100- and n-day cases and differed from those deployed in 3- and 4-day cases, nonetheless, the no answer rate was high for the last two cases.

The correspondence strategy prevailed among the students who answered to the 100- and n-day cases. Seven (S1, S4, S7, S12, S14, S24 and S30) applied the correspondence strategy to 100-day case, changing from the additive operational strategy used in the preceding cases. By way of example, S1 expressed the relationship between the number of squares and the result of the sum of the orders by contending that "each square has four so (4·296) = 1184". According to his answers to the task, he followed the structure $4s$, where s is the number of squares obtained. S7 y S24 used the same correct structure. The rest of the students used other structures. S30, for example, represented the structure "$Day \times 9$", consistently with the result of the previous 4-day case. Of the seven students who applied this strategy in 100-day case, all except S30 who did not respond, used the correspondence in n-day case with the same structure.

The proportional or multiplicative operational strategies were used by a small number of students in this task (Table 2). In the 100 and n-day cases, S3 and S11 used proportionality, while in 100-day case S15 used multiplicative operationality, which both he and S5 adopted in n-day case. S15, for instance, associated the number of squares for 100 days with the number of vertices in a square answering '800 because there are 200 squares and each square has four vertices', drawing no connection between his answer and the previous variables and results.

Representations of generalisation and associated strategy

In all these cases students representing generalisation applied the correspondence strategy when solving the task. They established a relationship between variables which they generalised and represented. As shown in Table 3 they used different representations to express the generalisation. The representations of generalisation in both tasks took place in the answers to the questions about cases 100 and n-day.

[Table 3]

As evidenced in Table 3, although some students did not represent generalisation, they did show the recognition of a regularity consistent with the solutions they previously obtained. This is observed mainly in case 100 in both tasks. They numerically expressed the relationship between the quantities corresponding to each variable when indicating the operations performed to reach the answer. For example, in the answer to 100-day case in the second task, S24 wrote that the result of the sum of the orders was 244. This result was obtained from the multiplication 64×4, where 64 was the number of squares that they formed in 100 days. That is, he used the structure $4s$, where s is the number of squares. These students then represented the generalisation in n-day case. The exceptions are S15 who in the first task uses counting in the general case, and S24 who in the second task maintains the same form of numerical expression.

Below we comment on the use of each type of representations of generalisation exhibited in both tasks.

Verbal representation

Of the exhibited representations, the verbal representation of the generalisation stood out as the most frequent in both tasks and mainly in the 100-day case. In the first task in 100-day case seven students (Table 3) represented the generalisation verbally and three in n-day case. They verbalized the indeterminacy of the variables and the generalisation of a functional relationship between the number of days (independent variable) and the squares (dependent variable).

Most of the students that represented the generalisation verbally expressed that "each day increases by three squares" referring to the relationship between the number of squares (dependent variable identified in the problem) that could be drawn and the number of days (verbally expressed independent variable). Two students (S1 and S5) proposed this verbal expression and used structures such as $3(n-4)+8$ (S1, Figure 4) or $3(n-4)$ (S5). S20 wrote the equivalent verbal representation "because every day we get two 1 m^2 squares and one 4 m^2 square" and applied the structure $3(n-4)$. In n-day case S4, S5 and S7 represented generalisation verbally with very similar expressions to those exposed. The variety of structures under similar verbal expressions reflected a limited precision of verbal representations of generalisation. Nonetheless, all responses in 100-day case are accompanied with their calculations, a clue to understanding the structure of the functional relationship to which they referred (see Figure 4, for instance). We base our interpretation of their perception of generalisation on the grounds of the consistency of the answer with prior results.

In the second task, the verbal representation of the generalisation was manifested by S1 and S7 with expressions such as "the number of squares by 4" that were always related to the correct structure $4s$, both in 100- or n-day cases. In n-day case S4 used another verbal expression "every day increases by 16" that was associated with the structure $16n + 4$ applied in 100-day case.

Symbolic representation

The symbolic representation of the generalisation was used exclusively in n-day case and only by S1 in both tasks. By using algebraic symbolism, he represented the independent variable and the functional relationship that he identified. In the first task S1 described the regularity as “you can draw $(n \cdot 3)$ because three can be drawn every day”. That expression was consistent with the verbal representation he provided in the 100-day case answer, even though he failed to apply the structure $3(n-4)+8$ determined there (see Figure 4). In the second task he wrote the structure “$(n \cdot 3) \cdot 4$” which operates the number of squares $(n \cdot 3)$ found in 100-day case with the value of the sum of the orders by multiplying by 4.

Multiple representation

The multiple representation of generalisation is the second most used by students in both tasks with a slight majority in the n-day case of the second task (two students) (Table 3). This representation was characterized by involving the verbal representation of the variables as general terms and using numbers connected with arithmetic operations to indicate the relationship between the variables in a semi-symbolic expression. For example, S30 used this representation in 100-day case in both tasks. In first one he wrote the structure “$Day \times 9$” while in the second tasks expressed the structure “$Solution = Nº\ of\ day \times 3$”. In the first task S14, who represented verbally in 100-day case, switched to multiple representation in the n-day. In the following excerpt his expression of the regularity perceived lacked clarity to reveal the structure $n+2+n-2$.

> You can draw double the number of squares as on the day the seeds are planted. The number for day +2 is the number of squares that have an area of 2 m2. The number for day -2 is the number of squares with an area of 1 m2.

For the second task, S12 and S14 went from not representing generalisation, although they recognize a regularity in 100-day case, to multiple representation in n-day case. For example, S12 represented "$(The\ ‘n’\ on\ the\ day \times 2) \times 2 - 4 +$ $(the\ ‘n’\ on\ the\ day \times 4 - 4)$" to obtain the sum of the orders.

Discussion and conclusions

This article reports on last-year elementary school students' problem-solving strategies and their ways of representing generalisation in a context of functional thinking and within the early algebra frame. They were observed to deploy a variety of strategies, exhibiting flexibility to switch approaches between working on specific and general cases and to consistently use the same strategy in last cases. However, only a small number of students proved able to establish relationships between variables and represent the general rule governing the functional relationships underlying their solutions.

Solving strategies

This study supplements previous research on the strategies used by elementary school students in functional generalisation contexts, by describing the strategies deployed by sixth-year students without formal algebraic training. It characterizes the strategies used in specific and general cases and highlights the most often applied in representing generalisation of functional relationships.

In near cases the students used two different strategies depending on the task, both were related with a visual component. Counting was predominantly used in the first task while additive operational was applied in the second one, however, the latter also involved counting to obtain the order of each seed before adding. The use of such strategies might be related to a visual representation of the problem (e.g., Barbosa et al., 2012; El Mouhayar & Jurdak, 215; Stacey, 1989) and to the way the tasks were posed: in the second task students were asked to find a sum, which prompted them to use addition in light of the small number of data involved. In near cases the application of these strategies is consistent with results reported in other studies with elementary school students (e.g., Barbosa et al., 2012, Merino et al., 2013; Zapatera Llinares, 2018).

In the same vein, generally the students did not resort to more specific cases than the two proposed. The use of pictorial representations is observed almost exclusively in these cases when the student is invited to make them. To answer the final cases and to represent the generalisation the main resources on which they base their reasoning were the numerical answers obtained in previous questions and the numerical expressions used to get the numerical computations. This result suggests a focus on the numerical rather the visual elements of the tasks and answers (Amit & Neria, 2008; Barbosa et al., 2012), which may be consequence of the type of learning experiences they have lived.

In 100-day case student strategies were more diverse. As the term was not a low number, neither consecutive or close to the previous ones, students could not continue using counting or operationality strategies and, as a consequence, the search of other procedures was promoted. Here as in n-day case, the correspondence strategy prevailed among the few students replying to the case. This strategy corresponds to the functional approach reported in other studies (e.g., Amit & Neria, 2008; El Mouhayar & Jurdak, 2015; Lannin et al., 2006; Stacey, 1989) but in our case only involved the correspondence relationship.

One of the primary contributions of this study is the relationship identified between the strategies used by last-year elementary school students and the ways they represented the generalisation. The correspondence strategy was the sole approach found to be associated with the representation of generalisation. The flexibility exhibited to switch strategies between specific and general cases was key in addressing the problem posed. The study stands out that switching to the correspondence strategy in the latter cases was an indication that other strategies were impractical in such cases, as students applied functions consistently in the last two cases. The study shows the scope of the correspondence strategy, allowing to represent the generalisation or show the recognition of a regularity in line with other studies that highlight functional strategies in generalisation tasks (e.g., Amit & Neria, 2008; El Mouhayar & Jurdak, 2015; Zapatera Llinares, 2018). That would suggest that the correspondence strategy is the one best suited to establishing relationships between data, ensuring coherence with prior answers and the problem posed. We agree that the use of this strategy was motivated by the proposal of distant cases and the cognitive demand in the transition from specific and familiar cases to cases that required more efficient and advanced strategies to solve and generalise (El Mouhayar & Jurdak, 2015; Lannin et al., 2006). In the first task fewer students used the correspondence strategy in the n-day case than in the 100-day case (Table 1), even after generalising in the latter, perhaps because they deemed they had completed the task or because they failed to understand the meaning of the alphanumerical symbolism.

We identified more diversity in the structures of functional relationships in the first task attributable to the fact that the functional relationship was more complex than in the second task (with multiplicative structure $4s$). We found that the diversity of structures, the modifications in the structures from one to another case, as well as the use of structures that do not fully correspond to the proposed tasks or the students' own results, may be due to errors in calculation or a trend focused on responding rather than refining responses. Another reason is the difficulty of the tasks that involved modelling non familiar situations (Lepak et al., 2018).

One finding likewise reported by other authors (Barbosa et al., 2012; Stacey, 1989; Zapatera Llinares, 2018) was the inappropriate use of proportionality, attributable to the desire to apply an efficient solving procedure (Lannin et al., 2006) or the incorrect over-generalisation of learned knowledge (Stacey, 1989). By contrast, some strategies identified in prior studies were not detected here. Despite being a common generalisation approach, recursive strategy (e.g., Amit & Neria, 2008; Carraher et al., 2008; Stacey, 1989) were not used by these sixth-year students, perhaps due to the near absence of consecutive cases and the potential of students in the search for efficient strategies.

Three groups of strategies were identified based on the coherence between the cases of each task as students progressed from case to case. The first group (direct answer and additive and multiplicative operational) consisted in procedures or reasoning applied to specific or isolated cases. In the second group (correspondence) students reasoned based on prior data, extending their reasoning to more general cases. In the third (proportionality and other) students applied reasoning associated with prior formulas or knowledge unrelated to the nature of the data in the problem posed or applied a strategy based exclusively on the data found in the immediately preceding case to find the answer to the case at hand.

Representations of generalisation

The representations of generalisation of functional relationships evidenced by sixth-grade students are another of the contributions of this research. We identified three representations considering the written nature of the task presented: verbal, symbolic and multiple representations. In this study, we reorganised the representations of generalisation proposed by Ureña et al. (2019) according to if the student evidences the recognition of a regularity and represents the generalisation (verbally, symbolically or multiple) or does not represent it.

An interesting result is that some of the students who used the correspondence strategy showed evidences of having detected and used a regularity although they did not represent the generalisation. It is an important finding that calls the attention towards cases in which students might be working with the functional relationships in an implicit way. This result could be due to the numerical nature of the case in question (100-day case) since most of these students represented generalisation in the n-day case (Table 3).

Generalisation was represented by more students in the first than in the second task. We consider that this finding could be due to the organization of the problem and the dependence that could be seen between the responses of the second task with those of the first task, in which the failure to answer or to correctly answer a case rendered it nearly impossible to identify a regularity. Eleven students represented generalisation in the first task, seven in the second and five in both (Table 3).

Generalization was most often represented verbally in both tasks, predominantly in the 100-day case. This result coincides with other investigations that recognize in primary school the use of verbal (and numeric) representations in general cases (e.g. Pinto & Cañadas, 2021; Torres et al., 2019). It is attributable both to the comfort and familiarity with these representations (Merino et al., 2013), their reduced use of pictorial representation and their inexperience with other types of representations.

As has been seen, the symbolic representation has not been an indispensable requirement to represent the generalisation. In fact, the symbolic representation was exhibited by one student in both tasks and only in n-day case. In the same line, students who shown multiple representation of generalisation also were able to identify, work with and represent indeterminate variables and hence express the functional relationship perceived. They used words as variables in algebraic expressions. We could recognize this last no conventional representation as a previous step to represent generalisation symbolically or even as semi symbolic representation (Amit y Neria, 2008). We reflect that this representation reveals the achievement of an algebraic maturity to express the generalisation in a general way, prior to the experience with formal algebra.

From the results we highlight the flexibility of sixth grade students to use different representations and change them between one case and another. The variety of representations corroborate students' algebraic thinking (e.g., Amit & Neria, 2008;; Kaput, 2008; Radford, 2018). Each expression of generalisation shows in a different way the variables, their relationships and the conceptual deepness with which they have been approached (Radford, 2018). Unlike symbolic or multiple representation, where the variables are explicit and the structure of the functional relationship is evident, in the verbal representation or in the responses of the students who do not represent but use the correspondence strategy, these are implicit.

As expected, students found it easier and were more prone to work with specific (where more students answered the questions) than general cases (e.g.Barbosa et al., 2012; Ureña et al., 2019; Zapatera Llinares, 2018). The explanation may be that the tasks involved demands associated with inter-cases dependency, their limited experience with generalisation problems such as the one proposed or the connection with the visual component from which students generalise information. An inadequate illustration or a poor visualization ability would influence the results they determine and the strategies they would use to solve them (Lannin et al., 2006). In n-day case one student even explicitly contended indeterminacy of the independent variable to be inconceivable, whilst others assigned the letter a fixed numerical value and solved from that perspective. Similar results have been reported for younger students (e.g., Molina et al., 2018; Ureña et al., 2019). Such evidence reinforces the idea to foster generalisation with tasks involving familiarisation with indeterminacy and its representations as a preamble to formal algebra instruction.

Unlike Ureña et al. (2019) fourth-year students, none of the sixth-year elementary schoolers used the generic representation generalisation. They found that representing the independent variable as "any" or "any number" prompted students to use generic examples. In their study outside mediation also induced participants to represent generalisation. The absence of such expressions in the wording of the cases used here may have contributed to students not representing generalisation in that way. However, we identified evidences of the multiple representation that were not evidenced by the fourth graders. We could conjecture that multiple representation is associated with more mathematical knowledge and experiences. Similar representations were highlighted by Amit and Neria in 11-13 years old mathematically talented students. Other authors (e.g., Blanton & Kaput, 2004; Blanton et al., 2015; Carraher et al., 2008; Warren & Cooper, 2005, 2008) reported younger students' ability to understand, represent functional relationships of varying complexity and even prove their reasoning with more specific cases. Those students had received instruction dealing with such content. The students in this study, further to the elementary school curriculum, were only expected to be able to identify patterns and regularities (Ministerio de Educación, Cultura y Deporte, 2014). That supports the idea that students of different ages are prone to functional thinking but need reinforcement and guidance to develop it fully. In the context of early algebra, then, this study is believed to furnish information on how students reaching the end of elementary school generalise and express functional relationships with no prior explicit training in those areas.

The instruction becomes essential to guiding students' experience, for it helps them organize their thoughts to find the right problem-solving strategies (Stacey, 1989). We highlight the need to provide spaces in which students learn and progressively develop increasingly advanced strategies to develop mathematical competences, including generalisation. This study also reports on the importance of analysing the procedures deployed and the inter-data relationships established, factors associated with the habit of checking one's answers. Consistently with other studies (Barbosa et al., 2012; Stacey, 1989), the students participating here did not verify the generalisation defined, a failure possibly attributable to their scant experience in this regard.

In another vein, while able to express generalisation, students failed to represent their reasoning clearly and methodically possibly due to lack of appropriate verbal skills. Other studies reported that students who identified variables and their inter-relationships were scantly able to express them clearly (Radford, 2018; Ureña et al., 2019). In our study that was attested to by the sparsity and even ambiguity of students' explanations. In this sense, the development and strengthening of students' communication skills for the adequate and accurate expression of their ideas and reasoning becomes also relevant (Barbosa et al., 2012).

In light of our findings and the importance of furthering algebraic reasoning beginning in elementary school, the study suggests that a number of issues (e.g., flexibility in the use of solving strategies, greater and better development of generalisation tasks in functional contexts and pre-algebra studies prior to formal instruction in secondary school) should be explored and attended in greater depth to promote spaces that motivate students to identify and represent generalisation in different ways.

Finally, we recognize the low number of participants as well as the consideration of only written answers as limitations of this study. Furthermore, as the findings are based on students who volunteered to participate, the research would benefit from supplementary information gathered from other sources involvingmore participants and students of other ages.

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Table 1. Strategies used by students (N=30) in task 1, by case

Strategies	3- and 4-day cases	100-day case	n-day case	Cumulative total
Counting	19	0	1	20
Additive operationality	0	0	0	0
Multiplicative operationality	0	3	3	6
Correspondence	0	10	5	15
Proportionality	0	5	3	8
Direct answer	10	7	4	21
Other	0	1	1	2
No answer	1	4	13	18
Total	30	30	30	90

Table 2. Strategies used by students (N=30) in task 2, by case

Strategies	3- and 4-day cases	100-day case	n-day case	Cumulative total
Counting	0	0	0	0
Additive operationality	24	3	0	27
Multiplicative operationality	0	1	2	3
Correspondence	0	7	6	13
Proportionality	0	2	2	4
Direct answer	3	4	5	12
Other	0	2	3	5
No answer	3	11	12	26
Total	30	30	30	90

Table 3. Students (N=30) representing generalisation

	Task 1		Task 2	
	100-day case	n-day case	100-day case	n-day case
Identifies a regularity but does not represent	2(S4, S15)		4(S4, S12, S14, S24)	1(S24[0])
Identifies a regularity and represents				
Verbal	7(S1, S5, S14, S18, S19, S20, S26)	3(S4[1], S5[0], S7)	2(S1, S7)	2(S4[1], S7[0])
Symbolic		1(S1[1])		1(S1[1])
Multiple	1(S30)	1(S14[1])	1(S30)	2(S12[1], S14[1])
Total	11 (S1, S4, S5, S7, S14, S15, S18, S19, S20, S26, S30)		7 (S1, S4, S7, S12, S14, S24, S30)	

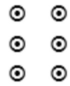
Note. [0]= The student used the same representation of the generalisation between 100-day case and n-day case. [1]= The student used a different representation of the generalisation in last both cases.

Figure 1

A farmer is going to plant potato seeds in his field.

The first day he plants three seeds in a straight line, spaced at 1 metre from one to the next (as in the figure on the right).

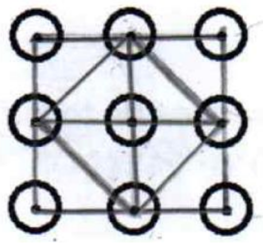
On the second day he plants three more seeds in a line parallel to and 1 metre away from the one he planted the day before, with the seeds again at 1 metre from one another.



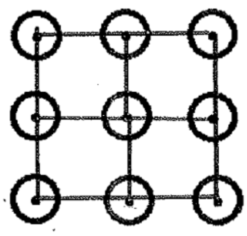
. - After the third day, the field looks like this:

Figure 2

2a



2b



2c

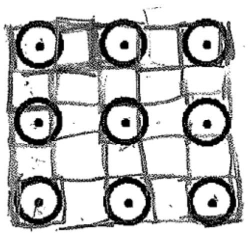


Figure 3

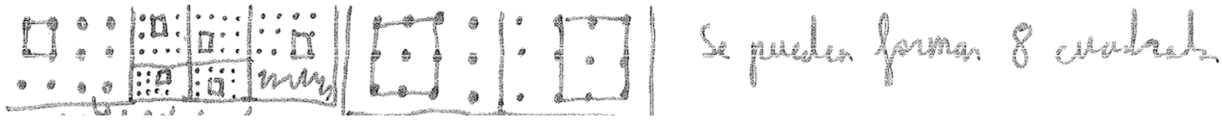


Figure 4

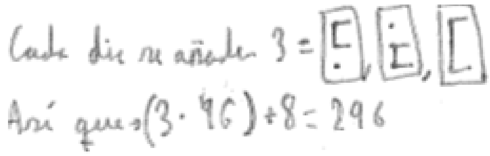
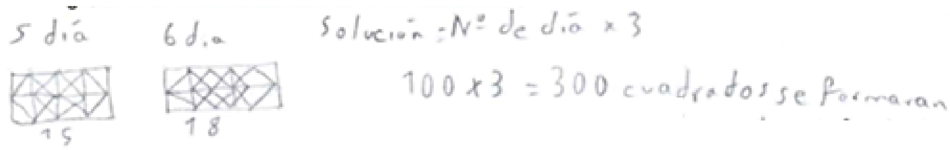


Figure 5



5 día
6 día
Solución : Nº de día x 3
100 x 3 = 300 cuadrados se formaran
15
18

Figure 6

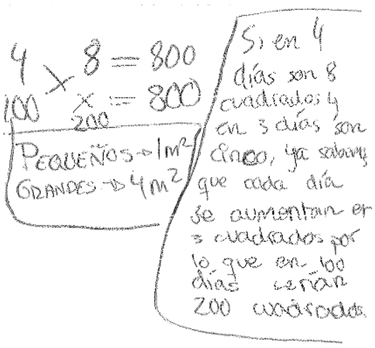

4 × 8 = 800
100 x = 800
200
PEQUEÑOS → 1m²
GRANDES → 4m²
Si en 4 días son 8 cuadrados y en 3 días son cinco, ya sabemos que cada día se aumentan en 3 cuadrados por lo que en 100 días serían 200 cuadrados.

Figure 7

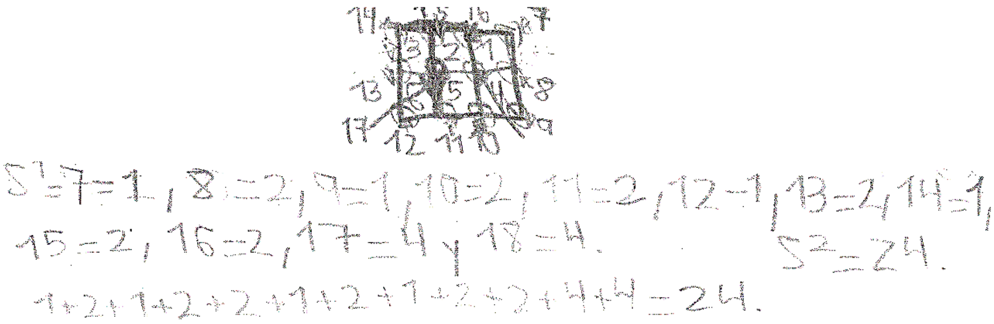

S¹=7=1, 8=2, 9=1, 10=2, 11=2, 12=1, 13=2, 14=1,
15=2, 16=2, 17=4 y 18=4. S²=24.
1+2+1+2+2+1+2+1+2+2+4+4=24.

Figure 8

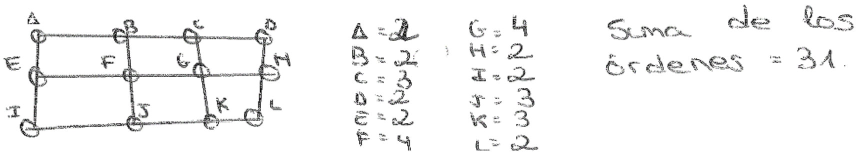

A
B
C
D
E
F
G
H
I
J
K
L
A = 2
B = 2
C = 3
D = 2
E = 2
F = 4
G = 4
H = 2
I = 2
J = 3
K = 3
L = 2
Suma de los órdenes = 31.

Figure captions

Figure 1. The potato seed problem

Figure 2. Geometric representation of the squares: a. All squares, b. Squares resting on rows, c. Different interpretation

Figure 3. S1's answer, n-*day* case. [Handwritten note: "You can draw 8 squares"]

Figure 4. S1's answer, 100-day case. [Handwritten note: "Three are added each day. So (3·96)+8 = 296"]

Figure 5. S30's answer, 100-day case. [Handwritten note: day 5 (15); day 6 (18). Solution = No. of days x 3. 100 x 3 = 300 squares]

Figure 6. S7's answer, 100-day case. [Handwritten note: Small = 1 m^2; large=4 m^2. If in 4 days there are 8 squares and in 3 days there are 5, we know that with each day there are 3 more, so after 100 days we'd have 200 squares]

Figure 7. S12's answer, 4-day case

Figure 8. S3's answer, 4-day case