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Multiplicity of Solutions for an Elliptic Kirchhoff Equation

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Abstract. In this paper we study the existence of positive solution to the Kirchhoff elliptic problem

$$\begin{cases} -\left(1+\gamma G'\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\right)\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$ $(N \geq 3)$, f is a locally Lipschitz continuous real function, $f(0) \geq 0$, $G' \in C(\mathbb{R}^+)$ and $G' \geq 0$. We prove the existence of at least two solutions with $L^\infty(\Omega)$ norm between two consecutive zeroes of f for large λ.

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Keywords. Elliptic Kirchhoff equation, Continua of solutions, Multiplicity of solutions.

1. Introduction

We consider the following Kirchhoff elliptic boundary value problem

$$\begin{cases} -\left(1+\gamma G'\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\right)\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P_{\lambda,\gamma})$$

where Ω is an open, bounded subset of $\mathbb{R}^N$ $(N \geq 3)$, $G \in C^1(\mathbb{R}^+)$ with $G, G' \geq 0$, $\gamma \geq 0$ and f is a locally Lipschitz continuous real function defined in $[0,\infty)$ with $f(0) \geq 0$. We extend f to $(-\infty, 0)$ by $f(s) = f(0) \geq 0$ for every $s < 0$ to guarantee that any solution of $(P_{\lambda,\gamma})$ is nonnegative. In the semilinear local case, $\gamma = 0$, it is well known (see [2]) that maximum principle implies the nonexistence of solution u with $\|u\|_{L^\infty(\Omega)} = \alpha$ provided that $f(\alpha) \leq 0$. K. J. Brown and H. Budin proved in [4], by using variational methods, how changes in the sign of f lead to multiple positive solutions of the equation for sufficiently large λ.

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In [6] P. Hess proved for λ big enough the existence of $w_{1,\lambda},\dots, w_{m,\lambda}$, $u_{1,\lambda},\dots,u_{m-1,\lambda}$, $2m-1$ positive solutions of the nonlinear elliptic eigenvalue problem $(P_{\lambda,0})$, if $f(0)>0$ and the graph of f has m positive humps and $m-1$ negative humps, each positive hump having greater area than the previous negative hump. More precisely, if α_k denotes the right end point of the k-th positive hump and $F(s) := \int_0^s f(t)dt$, then it is assumed that $f(0)>0$, $f(\alpha_1),\dots,f(\alpha_m)=0$ and

$$\max\{F(s) : 0 \le s \le \alpha_{k-1}\} < F(\alpha_k),\ k = 2,\dots,m.$$

He showed that $w_{1.\lambda} \le \alpha_1$ and $\|w_{k+1,\lambda}\|_{L^\infty(\Omega)}, \|u_{k,\lambda}\|_{L^\infty(\Omega)} \in (\alpha_k, \alpha_{k+1})$ for every $k = 1,\dots,m-1$. Moreover, $w_{1,\lambda} \le w_{2,\lambda} \le \dots \le w_{m,\lambda}$ and $u_{k,\lambda} \le w_{k+1,\lambda}$ for every $k=1,\dots,m-1$ since $w_{k,\lambda}$ is the maximal solution of $(P_{\lambda,0})$ in $[0,\alpha_k]$ with $k=1,\dots,m$.

Z. Liu proved, in [9], assuming that $f(0)>0$ and under the hypotheses in [6] that all those solutions are ordered, is that to say, $w_{1,\lambda} < u_{1,\lambda} < w_{2,\lambda} < u_{2,\lambda} < w_{3,\lambda} < \dots < u_{m-1,\lambda} < w_{m,\lambda}$.

E. N. Dancer and K. Schmitt have shown in [5] that this relation between the area of the positive hump and the previous negative one is necessary for the existence of a positive solution with norm in the interval corresponding to the positive hump. Indeed, they showed that if $0<\alpha<\beta$ are two consecutive zeroes of f such that $f(s)>0$ for every $s\in(\alpha,\beta)$ then a necessary condition for the existence of solution u with $\|u\|_{L^\infty(\Omega)} = r \in (\alpha,\beta)$ is

$$\int_{s_0}^{r} f(s)ds > 0,\ \forall s_0 \in (0,\alpha). \tag{1.1}$$

In [7] J. García-Melián and L. Iturriaga have studied problem $(P_{\lambda,0})$ for a nonnegative function f with r positive zeros, $\alpha_1,\dots,\alpha_r$. They showed that, for large enough λ, there exist at least two solutions $w_{k,\lambda}, u_{k,\lambda}$ with $\|w_{k,\lambda}\|_{L^\infty(\Omega)} < \alpha_k < \|u_{k,\lambda}\|_{L^\infty(\Omega)}$ if f verifies a suitable non-integrability condition near each of its zeros. Specifically this condition is: there exists $\delta>0$ such that

$$\int_{\alpha_k}^{\alpha_k+\delta} \frac{f(t)}{(t-\alpha_k)^{\frac{2(N-1)}{N-2}}}dt = +\infty \tag{1.2}$$

for $k=1,...,r$. In addition, they obtained that

$$\lim_{\lambda\to+\infty} \|w_{k,\lambda}\|_{L^\infty(\Omega)} = \lim_{\lambda\to+\infty} \|u_{k,\lambda}\|_{L^\infty(\Omega)} = \alpha_k.$$

They used the sub- and supersolutions method and topological degree arguments combined with the use of a suitable Liouville theorem.

The above condition (1.2) was improved in [3] by B. Barrios, J. García-Melián and L. Iturriaga under the assumption that f has an isolated positive zero α such that

$$\frac{f(t)}{(t-\alpha)^{\frac{N+2}{N-2}}} \text{ is decreasing in } (\alpha,\alpha+\delta) \text{ for some small } \delta>0. \tag{1.3}$$

They proved, still for nonnegative f, that for large enough λ there exist at least two ordered positive solutions $w_{\alpha,\lambda} < u_{\alpha,\lambda}$ verifying the properties $\|w_{\alpha,\lambda}\|_{L^\infty(\Omega)} < \alpha < \|u_{\alpha,\lambda}\|_{L^\infty(\Omega)}$ and $w_{\alpha,\lambda}(x), u_{\alpha,\lambda}(x) \to \alpha$ uniformly on compact subsets of Ω as

$\lambda \to +\infty$. The existence of these solutions holds independently of the behavior of f near zero or infinity.

In this paper we are interested in the multiplicity results for the Kirchhoff problem $(P_{\lambda,\gamma})$ (in the case $\gamma > 0$) providing a description of the behavior of solutions for $\lambda \to +\infty$. Thus, in addition to $f(0) \geq 0$ we only assume

(H) There exists $0 < \alpha < \beta$ such that $f(\alpha) = f(\beta) = 0$, $f(s) > 0$ for every $s \in (\alpha, \beta)$ and

$$r_\alpha = \inf \left\{ r \in (\alpha, \beta) : \int_{s_0}^{r} f(s)ds > 0,\ \forall s_0 \in (0, \alpha) \right\} \text{ is finite.}$$

We observe that f may change sign and that hypothesis (H) implies that (1.1) is satisfied for every $r \in (r_\alpha, \beta)$.

In order to state the main result we use the following notation:

$$\mathcal{S} = \{(\mu, u) : u \in H_0^1(\Omega) \cap L^\infty(\Omega) \text{ solves } (P_{\mu,\gamma}), \mu > 0\}$$

and for any $B \subset \mathcal{S}$, $\mu > 0$ we define $B_\mu = \{u : (\mu, u) \in \mathcal{S}\}$.

Theorem 1.1. *Assume that f is a locally Lipschitz continuous real function with $f(0) \geq 0$ and satisfies (H). Assume also that $G \in C^1(\mathbb{R}^+)$ with $G, G' \geq 0$ and $\gamma \geq 0$. Then there exists $\overline{\lambda} > 0$ such that, for every $\lambda > \overline{\lambda}$, problem $(P_{\lambda,\gamma})$ admits at least two solutions $u_\lambda, w_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $u_\lambda \neq w_\lambda$ and $\|u_\lambda\|_{L^\infty(\Omega)}, \|w_\lambda\|_{L^\infty(\Omega)} \in (r_\alpha, \beta)$. Moreover, one of the following alternatives holds:*

1. *There exists an unbounded continuum $\Sigma \subset \mathcal{S}$ such that Σ_λ has at least two elements.*
2. *$\mathcal{S}_\lambda$ is infinite.*

Even more $\lim_{\lambda \to +\infty} \|w_\lambda\|_{L^\infty(\Omega)} = \beta$ and, if we assume that f is nonnegative, and it satisfies (1.3), then $\lim_{n \to \infty} \|u_{\lambda_n}\|_{L^\infty(\Omega)} = \alpha$ for some sequence $\lambda_n \to +\infty$.

We observe that in contrast with the semilinear case, $\gamma = 0$, due to the lack of a convenient comparison result for the Kirchhoff problem $(P_{\lambda,\gamma})$ we have no information on whether the solutions are ordered or not.

With respect to the plan of the paper, in the second section we show the main nonexistence results for $(P_{\lambda,\gamma})$, we establish the existence of $\overline{\lambda}$ and main properties of the solution w_λ as a minimum of a truncated functional. The third section is devoted to the existence of the second solution u_λ using some Leray-Schauder degree computations which leads to the continuum alternative. Finally, in the last section we collect all the results leading to the proof of the main result.

2. Qualitative Properties of Solutions

In this section we collect the main results concerning with the existence, nonexistence and main properties of solutions to the problem $(P_{\lambda,\gamma})$.

Let us remark that we are mainly interested in positive bounded solutions and the interaction between its $L^\infty(\Omega)$ norm and the positive zeroes of f. Thus, for every $k>0$, we may take

$$f_k(s)=f(\max\{0,\min\{s,k\}\}),\ s\in\mathbb{R}. \tag{2.1}$$

and consider the truncated problem

$$\begin{cases} -\left(1+\gamma G'\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\right)\Delta u=\lambda f_k(u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{P_k}$$

This truncated problem provides us with bounded solutions to ($P_{\lambda,\gamma}$) as it is stated in the following lemma.

Lemma 2.1. *Let $u\in H_0^1(\Omega)$ be a weak solution to (P_k), then $u\in L^\infty(\Omega)$ and u is a solution to ($P_{\lambda,\gamma}$) if $\|u\|_{L^\infty(\Omega)}\le k$.* □

The next result is related to the nonexistence of bounded solutions with a specific $L^\infty(\Omega)$ norm.

Lemma 2.2. *There is no bounded solution to ($P_{\lambda,\gamma}$) with $L^\infty(\Omega)$ norm equal to $r>0$ in the following cases:*

1. *$f(r)\le 0$.*
2. *$f(s)>0$ for every $s\in(\alpha,r]$, $f(\alpha)=0$ and $\int_{s_0}^r f(s)ds\le 0$ for some $s_0\in(0,\alpha)$.*

Proof. Observe that if $u\in H_0^1(\Omega)\cap L^\infty(\Omega)$ is a solution of ($P_{\lambda,\gamma}$) with $\|u\|_{L^\infty(\Omega)}=r$ and $\|u\|_{H_0^1(\Omega)}=r_0$ then

$$-\Delta u=\frac{\lambda}{1+\gamma G'(r_0)}f(u)\equiv\tilde f(u).$$

Thus, the first item is a direct consequence of the strong maximum principle ($\tilde f(r)\le 0$, see [2]) and the second is proved in [5] ($\int_{s_0}^r \tilde f(s)ds\le 0$).

Remark 2.3. Combining the results in the previous two lemmas, in the case $f(k)=0$, any solution u to (P_k) it is also a solution to ($P_{\lambda,\gamma}$) satisfying $\|u\|_{L^\infty(\Omega)}<k$. In addition, we may consider the functional $I_k:H_0^1(\Omega)\to\mathbb{R}$ given by

$$I_k(u)=\frac12\int_\Omega|\nabla u|^2+\frac12\gamma G\left(\int_\Omega|\nabla u|^2\right)-\lambda\int_\Omega\int_0^u f_k(s)ds,\ u\in H_0^1(\Omega),$$

and observe that $I_k\in C^1(H_0^1(\Omega))$ and critical points of I_k are solutions to (P_k).

The existence of solution to (P_k) when $f(k)=0$ is then standard as it is stated in the following lemma:

Lemma 2.4. *Assume that $f(k)=0$ for some $k>0$. Then there exists a solution u to ($P_{\lambda,\gamma}$) with $\|u\|_{L^\infty(\Omega)}<k$.*

Proof. Observe that, since f_k is bounded (which also implies that any solution of (P_k) belongs to $L^\infty(\Omega)$), the functional I_k is coercive. In addition, I_k is weak lower semicontinuous which implies that I_k has a global minimum w_λ which, taking into account Remark 2.3, is a solution of ($P_{\lambda,\gamma}$).

The solution given by Lemma 2.4 may be the trivial one and in the next result we give sufficient conditions to prove that it is not the trivial one. The proof follows closely that of [6] and we include it here for the reader convenience.

Lemma 2.5. *Assume that* $f(k) = 0$ *for some* $k > 0$ *and that for some* $0 < \varepsilon < k$ *there exists* $0 < s_0 < k - \varepsilon < s_1 < k$ *such that*

$$\max_{0 \le s \le k-\varepsilon} \left\{ \int_0^s f_k(t)dt \right\} = \int_0^{s_0} f_k(t)dt, \quad \int_{s_0}^{s_1} f_k(t)dt > 0.$$

Then there exists $\lambda_\varepsilon > 0$ *such that* $\|w_\lambda\|_{L^\infty(\Omega)} \ge k - \varepsilon$ *for every* $\lambda > \lambda_\varepsilon$*, where* w_λ *is the minimum of* I_k *established in Lemma 2.4.*

Proof. Assume that $\|w_\lambda\|_{L^\infty(\Omega)} < k - \varepsilon$ and observe that, in this case

$$\begin{aligned} \int_\Omega \int_0^{w_\lambda} f_k(t)dt &\le \int_\Omega \max_{0 \le s \le k-\varepsilon} \left\{ \int_0^s f_k(t)dt \right\} \\ &= |\Omega| \int_0^{s_1} f(t)dt - |\Omega| \int_{s_0}^{s_1} f(t)dt. \end{aligned} \tag{2.2}$$

Now we take $v \in C_0^\infty(\Omega)$ with $\|v\|_{L^\infty(\Omega)} \le k$ and $v(x) = s_1 > k - \varepsilon$ when $x \notin \Omega_\delta \equiv \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$ for some δ.

Therefore, using (2.2) we have:

$$\begin{aligned} \int_\Omega \int_0^v f_k(t)dt &= \int_{\Omega \setminus \Omega_\delta} \int_0^{s_1} f_k(t)dt + \int_{\Omega_\delta} \int_0^v f_k(t)dt \\ &= |\Omega| \int_0^{s_1} f(t)dt + \int_{\Omega_\delta} \int_{s_1}^v f_k(t)dt \\ &\ge \int_\Omega \int_0^{w_\lambda} f_k(t)dt + |\Omega| \int_{s_0}^{s_1} f(t)dt - C|\Omega_\delta|. \end{aligned}$$

Now we fix $\delta > 0$ such that $\eta \equiv |\Omega| \int_{s_0}^{s_1} f(t)dt - C|\Omega_\delta| > 0$ and we have that

$$\begin{aligned} I_k(v) - I_k(w_\lambda) &\le \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2}\gamma G\left(\int_\Omega |\nabla v|^2 \right) \\ &\quad - \lambda \int_\Omega \int_0^v f_k(s)ds + \lambda \int_\Omega \int_0^{w_\lambda} f_k(s)ds \\ &\le \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2}\gamma G\left(\int_\Omega |\nabla v|^2 \right) - \lambda\eta. \end{aligned}$$

On the other hand for $\lambda > \lambda_\varepsilon \equiv \frac{1}{\eta}\left(\frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\gamma G\left(\int_\Omega |\nabla v|^2\right)\right)$ (since v is fixed) we have that

$$I_k(v) < I_k(w_\lambda).$$

This implies that $\|w_\lambda\|_{L^\infty(\Omega)} > k - \varepsilon$ for every $\lambda > \lambda_\varepsilon$ and we finish the proof.

The following result is concerned with the existence of solution having $L^\infty(\Omega)$-norm between two consecutive zeros $0 < \alpha < \beta$ when (H) is satisfied.

Lemma 2.6. *Assume* (H) *satisfied. Then there exist* $\underline{\lambda}, \overline{\lambda} > 0$ *such that*

$$\Lambda = \{\lambda \ge 0 : (P_{\lambda,\gamma}) \text{ admits solution } u \text{ with } \alpha \le r_\alpha < \|u\|_{L^\infty(\Omega)} < \beta\}$$

is a nonempty closed set with $[\overline{\lambda}, +\infty) \subset \Lambda \subset [\underline{\lambda}, +\infty)$.

Proof. Using Lemma 2.4 we have the existence of a global minimum w_λ of I_β which is a solution to problem ($P_{\lambda,\gamma}$). This solution may be the trivial one and we will prove now that, for large enough λ, $\|w_\lambda\|_{L^\infty(\Omega)} \in (r_\alpha, \beta)$. This is a consequence of Lemma 2.5 with $k = \beta$, $\varepsilon = \beta - \alpha$ and $s_1 > r_\alpha$. (Moreover, as in the proof of Lemma 2.5 we also have that $\lim_{\lambda\to+\infty} I_\beta(w_\lambda) = -\infty$).

As a consequence $[\overline{\lambda}, +\infty) \subset \Lambda$ for any $\overline{\lambda} > \lambda_\epsilon$. In particular $\Lambda \neq \emptyset$ and we can take $\underline{\lambda} = \inf \Lambda$. Observe that, since $(P_{0,\gamma})$ admits only the trivial solution, $0 \notin \Lambda$.

Now we prove that Λ is closed and as a consequence $\underline{\lambda} > 0$. Let λ_n be a convergent sequence in Λ and denote $\lambda = \lim_{n\to\infty} \lambda_n$.
Let take $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $r_\alpha < \|u_n\|_{L^\infty(\Omega)} < \beta$ satisfying

$$\begin{cases} -\left(1 + \gamma G'\left(\|\nabla u_n\|^2_{L^2(\Omega)}\right)\right) \Delta u_n = \lambda_n f(u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that the sequence $\lambda_n f(u_n) \left(1 + \gamma G'\left(\|\nabla u_n\|^2_{L^2(\Omega)}\right)\right)^{-1}$ is bounded in $L^\infty(\Omega)$. In addition, applying [8, Theorem 6.1] we deduce that the sequence u_n is bounded in $C^{0,\gamma}(\overline{\Omega})$. Consequently, Ascolí-Arzelá Theorem assures that u_n possesses a subsequence converging strongly in $C(\overline{\Omega})$ to $u \in L^\infty(\Omega)$ with $r_\alpha \le \|u\|_{L^\infty(\Omega)} \le \beta$.

Moreover, taking u_n as test function and using that f is continuous we get that u_n is bounded in $H_0^1(\Omega)$. This implies, in particular, that $u \in H_0^1(\Omega)$ and, up to a subsequence, $u_n \to u$ weakly in $H_0^1(\Omega)$. In addition, taking $u_n - u$ as test function, we can also assure that u_n strongly converges to u in $H_0^1(\Omega)$ and we can pass to the limit and it follows that

$$\begin{cases} -\left(1 + \gamma G'\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\right) \Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since u solves ($P_{\lambda,\gamma}$) the strong maximum principle allows us to assure that $r_\alpha \neq \|u\|_{L^\infty(\Omega)} \neq \beta$ (see Lemma 2.2). Therefore we have proved that $\lambda \in \Lambda$, i.e. Λ is closed and we conclude the proof.

Remark 2.7. Assume that (H) is satisfied and denote by w_λ the minimum of I_β established in Lemma 2.6 for every $\lambda \ge \overline{\lambda}$. Then, as a consequence of Lemma 2.5, for every $\varepsilon > 0$ there exists $\lambda_\varepsilon > 0$ such that $\|w_\lambda\|_{L^\infty(\Omega)} \ge \beta - \varepsilon$ for every $\lambda > \lambda_\varepsilon$.

Remark 2.8. Observe that when $f(k) \neq 0$, Lemma 2.4 also gives a bounded solution to (P_k) which is not necessarily a solution to ($P_{\lambda,\gamma}$) unless it is less than k. Moreover, Lemma 2.5 is also true for solutions to (P_k). Thus, if $\alpha < k < \beta$, with α, β as in Remark 2.7 we have that the norm of this bounded solution to (P_k) has to be greater than k.

3. Leray-Schauder Degree Computations

In this section we perform the Leray Schauder degree computations to obtain the multiplicity result, again we follow closely [6].

Observe that u solves ($P_{\lambda,\gamma}$) and $\|u\|_{L^\infty(\Omega)} \le k$ if and only if

$$u = (-\Delta)^{-1}\left(\lambda f_k(u)\left(1+\gamma G'\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\right)^{-1}\right) \equiv \Phi_{\lambda,k}(u).$$

Moreover, $\Phi_{\lambda,k} : H^1_0(\Omega) \to \mathbb{R}$ is compact and we can use Leray-Schauder degree theory.

Lemma 3.1. *For every $\lambda, k > 0$, there exists $R > 0$ such that deg$(I - \Phi_{\lambda,k}, B_{R_\lambda}(0), 0) = 1$.*

Proof. Let us consider $H(t,u) = u - t\Phi_{\lambda,k}(u)$, for every $t \in [0,1]$ and $u \in H^1_0(\Omega)$. Since solutions to $H(t,u) = 0$ are bounded in $L^\infty(\Omega)$ we have that they are also bounded in $H^1_0(\Omega)$ and, for big enough $R > 0$, H is a valid homotopy and the result follows.

Observe that R depends on λ and k but it can be chosen constant in compact sets.

Now we denote by S_k to the set of solutions to ($P_{\lambda,\gamma}$) with $L^\infty(\Omega)$-norm less that k. Recall that, when $f(k) = 0$, S_k is the set of solutions to (P_k). Let us denote

$$U_{\varepsilon,k} = \left\{v \in H^1_0(\Omega) : \mathrm{dist}(v, S_k) < \varepsilon\right\}.$$

Lemma 3.2. *Let $0 < \alpha < \beta$ be two consecutive zeroes of f and $\lambda > 0$. Then there exists $\overline{\varepsilon} > 0$ such that deg$(I - \Phi_{\lambda,\beta}, U_{\varepsilon,\alpha}, 0) = 1$ for every $0 < \varepsilon < \overline{\varepsilon}$.*

Proof. It is clear that $\deg(I - \Phi_{\lambda,\alpha}, U_{\varepsilon,\alpha}, 0) = 1$ since there is no solution of $u = \Phi_{\lambda,\alpha}(u)$ in $B_R(0) \setminus U_{\varepsilon,\alpha}$.

Let us consider $H(t,u) = u - t\Phi_{\lambda,\alpha}(u) + (1-t)\Phi_{\lambda,\beta}(u)$, for every $t \in [0,1]$ and $u \in H^1_0(\Omega)$. We claim that H is a valid homotopy for small ε which implies the final result. Indeed, assume on the contrary that for some $\varepsilon_n \to 0$, $t_n \to t \in [0,1]$ and $u_n \in \partial U_{\varepsilon_n,\alpha}$ we have that

$$u_n = t_n\Phi_{\lambda,\alpha}(u_n) + (1-t_n)\Phi_{\lambda,\beta}(u_n),$$

or equivalently

$$-\left(1+\gamma G'\left(\|\nabla u_n\|^2_{L^2(\Omega)}\right)\right)\Delta u_n = \lambda(t_n f_\alpha(u_n) + (1-t_n)f_\beta(u_n)).$$

Whenever $\|u_n\|_{L^\infty(\Omega)} \le \alpha$, since $f_\beta(u_n) = f_\alpha(u_n)$ we have that $u_n \in S_\alpha$ which is a contradiction with the fact that $u_n \in \partial U_{\varepsilon_n,\alpha}$, in particular $\|u_n\|_{L^\infty(\Omega)} > \alpha$. Moreover, since $f_\beta(s) = f_\alpha(s) = 0$ for every $s \ge \beta$, using Lemma 2.2 we have that $\|u_n\|_{L^\infty(\Omega)} < \beta$.

Observe that $\lambda(t_n f_\alpha(u_n) + (1-t_n)f_\beta(u_n))\left(1+\gamma G'\left(\|\nabla u_n\|^2_{L^2(\Omega)}\right)\right)^{-1}$ is bounded in $L^\infty(\Omega)$ and, applying [8, Theorem 6.1] we deduce that the sequence u_n is bounded in $C^{0,\gamma}(\overline{\Omega})$. Consequently, Ascolí-Arzelá Theorem assures that u_n possesses a subsequence converging strongly in $C(\overline{\Omega})$ to $u \in L^\infty(\Omega)$ with $\alpha \le \|u\|_{L^\infty(\Omega)} \le \beta$.

Moreover, taking u_n as test function we get that u_n is bounded in $H^1_0(\Omega)$. This implies, in particular, that $u \in H^1_0(\Omega)$ and, up to a subsequence, $u_n \to u$ weakly in $H_0^1(\Omega)$. In addition, taking $u_n - u$ as test function, we can also assure that u_n strongly converges to u in $H_0^1(\Omega)$ and we can pass to the limit and it follows that

$$\begin{cases} -\left(1+\gamma G'\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\right)\Delta u = \lambda(tf_\alpha(u)+(1-t)f_\beta(u)) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Even more, since $\varepsilon_n \to 0$ and $u_n \in U_{\varepsilon_n,\alpha}$ we have that $u \in S_\alpha$ which also implies that $\|u\|_{L^\infty(\Omega)} < \alpha$ and this is a contradiction.

With the notation of Lemma 2.6, let us take now $\lambda \in \Lambda$ and denote w_λ the solution of $(P_{\lambda,\gamma})$ with $r_\alpha < \|w_\lambda\|_{L^\infty(\Omega)} < \beta$. Let us denote $\Lambda_1 \subset \Lambda$ such that w_λ is the minimum of I_β and let us recall that $[\overline{\lambda}, +\infty) \subset \Lambda_1$. Observe that, by definition of Λ, $w_\lambda \notin S_\alpha$, in particular we can take $0 < \underline{\varepsilon} < \overline{\varepsilon}$ such that $U_{\varepsilon,\alpha} \cap B_\varepsilon(w_\lambda) = \emptyset$ for every $0 < \varepsilon < \underline{\varepsilon}$. In the following lemma we prove multiplicity of solution for every $\lambda \in \Lambda_1$.

Theorem 3.3. *Assume that (H) is satsified. For every $\lambda_0 \in \Lambda_1$ one of the following multiplicity results holds:*

1. *w_{λ_0} is not an isolated solution to $(P_{\lambda_0,\gamma})$.*
2. *There exists un unbounded continuum $\Sigma \subset \{(\lambda, u) : u$ solves $(P_{\lambda,\gamma})\}$ such that $(\lambda_0, w_{\lambda_0}) \in \Sigma$ and there exists a second solution $v_{\lambda_0} \neq w_{\lambda_0}$ with $(\lambda_0, v_{\lambda_0}) \in \Sigma$.*

Proof. Observe that whenever w_{λ_0} is isolated, taking into account that it is a minimum for I_β, it is proved in [10, Theorem 1.1] that

$$\deg(I - \Phi_{\lambda_0,\beta}, B_\varepsilon(w_{\lambda_0}), 0) = 1,\ \varepsilon < \underline{\varepsilon}.$$

Hence we may use [1, Theorem 4.4.1] to deduce the existence of the unbounded continuum Σ with $(\lambda_0, w_{\lambda_0}) \in \Sigma$. For the existence of the second solution v_{λ_0} with $(\lambda_0, v_{\lambda_0}) \in \Sigma$ we use the ideas of [1, Theorem 4.4.2] in the interval $[a, b]$ with $a < \underline{\lambda}$ and $b = \lambda_0$, $U_1 = B_\varepsilon(w_{\lambda_0})$ and taking into account that U may depend continuously on λ. Therefore we take

- $\Sigma_\alpha = \{(\lambda, u) :\ \lambda \in [a, b], u \text{ solves } (P_{\lambda,\gamma}), \|u\|_{L^\infty(\Omega)} \leq \alpha\}$,
- $V_{\varepsilon,\alpha} = \{(\lambda, v) \in \mathbb{R} \times H_0^1(\Omega) : \text{dist}((\lambda, v), \Sigma_\alpha) < \varepsilon\}$,
- $V_{\varepsilon,\alpha}(\lambda) = \{u \in H_0^1(\Omega) : (\lambda, u) \in V_{\varepsilon,\alpha}\}$ and
- $U(\lambda) = B_R(0) \setminus \overline{V}_{\varepsilon,\alpha}(\lambda)$, for every $\lambda \in [a, b]$.

By construction we have that $(P_{\lambda,\gamma})$ has no solution on $\partial V_{\varepsilon,\alpha}(\lambda)$ for every $\lambda \in [a, b]$ and $(P_{a,\gamma})$ has no solutions in $\overline{V}_{\varepsilon,\alpha}(a)$ and the result follows as in [1, Theorem 4.4.2].

Remark 3.4. Observe that $\deg(I - \Phi_{\lambda_0,\beta}, B_\varepsilon(w_{\lambda_0}), 0) = 1$ also implies, by the excision property combined with Lemma 3.1 and Lemma 3.2, that $\deg(I - \Phi_{\lambda_0,\beta}, B_R(0) \setminus (\overline{U}_{\varepsilon,\alpha} \cup \overline{B}_\varepsilon(w_{\lambda_0})), 0) = -1$ which also gives the multiplicity result.

4. Proof of Theorem 1.1 Completed

We can take $\overline{\lambda}$ given by Lemma 2.6. Thus, for every $\lambda > \overline{\lambda}$, Lemma 2.6 provides us with the solution w_λ with $\|w_\lambda\|_{L^\infty(\Omega)} \in (r_\alpha, \beta)$ as the minimum of I_β.

The existence of the second solution u_λ is consequence of Theorem 3.3 which also provides the multiplicity alternatives. Observe that, since no solution exist with $L^\infty(\Omega)$ norm in $[\alpha, r_\alpha]$ (see Lemma 2.2), then $u_\lambda \in \Sigma_\lambda$ implies $\|u_\lambda\|_{L^\infty(\Omega)} \in (r_\alpha, \beta)$.

As a consequence of Lemma 2.5 (see also Remark 2.7) we have that

$$\lim_{\lambda \to +\infty} \|w_\lambda\|_{L^\infty(\Omega)} = \beta.$$

Thus, only remains to prove that, if we assume that f is nonnegative and it satisfies (1.3) then $\lim \|u_{\mu_n}\|_{L^\infty(\Omega)} = \alpha$ for some divergent sequence μ_n. Indeed, in this case $r_\alpha = \alpha$ and this is a consequence of the results in [3] where it is proved the existence of v_λ solution to

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with $\lim_{\lambda \to +\infty} \|v_\lambda\|_{L^\infty(\Omega)} = \alpha$ and $\|v_\lambda\|_{L^\infty(\Omega)} > \alpha$. Thus, given $\lambda_n \to +\infty$ we can take $\mu_n = \lambda_n \left(1 + \gamma G'\left(\|\nabla v_{\lambda_n}\|^2_{L^2(\Omega)}\right)\right)$ and $u_{\mu_n} = v_{\lambda_n}$. Since $G' \geq 0$ we have that $\mu_n \to +\infty$ and the proof is completed. □

Remark 4.1. Let us observe that the multiplicity result cannot be deduced directly from the know results for the semilinear case. More precisely, given u a solution to the semilinear problem (4.1) then u is also a solution to

$$\begin{cases} -\left(1 + \gamma G'\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\right) \Delta u = \mu f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

with $\mu = \left(1 + \gamma G'\left(\|\nabla u\|^2_{L^2(\Omega)}\right)\right) \lambda$. This means that, for fixed λ, multiple solutions to (4.1) may lead to solutions to (4.2) for different values of the parameter μ.

This is not enough to describe the global behavior in the case $\gamma \neq 0$ since the reverse depend on G and $\|u\|_{H^1_0(\Omega)}$:

$$\lambda = \frac{\mu}{\left(1 + \gamma G'\left(\|\nabla u_\mu\|^2_{L^2(\Omega)}\right)\right)},$$

and we can assure that u_μ is a solution to (4.1) for this value of λ but we cannot assure that it is precisely v_λ.

Remark 4.2. As in the semilinear case, depending on the behavior of f below α and above β, the results proved in this paper may lead to the existence of solutions with $L^\infty(\Omega)$ norm less that α or greater than β. Even more, the existence of multiple zeroes of f may lead to a more general multiplicity result.

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