

A repertoire of repulsive Keller–Segel models with logarithmic sensitivity: Derivation, traveling waves, and quasi-stationary dynamics

José Luis López 

Departamento de Matemática Aplicada and Excellence Research Unit “Modeling Nature” (MNat), Facultad de Ciencias, Universidad de Granada, Granada, Spain

Correspondence

José Luis López, Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, Granada 18071, Spain.
Email: jllopez@ugr.es

Communicated by: P. M. Mariano

Funding information

MINECO-Feder, Grant/Award Number: RTI2018-098850-B-I00; Junta de Andalucía, Grant/Award Number: PY18-RT-2422 and B-FQM-580-UGR

In this paper, we show how the chemotactic model

$$\begin{cases} \partial_t \rho = d_1 \Delta_x \rho - \nabla_x \cdot (\rho \nabla_x c) \\ \partial_t c = d_2 \Delta_x c + F(\rho, c, \nabla_x \rho, \nabla_x c, \Delta_x \rho) \end{cases}$$

introduced in Alejo and López (2021), which accounts for a chemical production–degradation operator of Hamilton–Jacobi type involving first- and second-order derivatives of the logarithm of the cell concentration, namely,

$$F = \mu + \tau c - \sigma \rho + A \frac{\Delta_x \rho}{\rho} + B \frac{|\nabla_x \rho|^2}{\rho^2} + C |\nabla_x c|^2,$$

with $\mu, \tau, \sigma, A, B, C \in \mathbb{R}$, can be formally reduced to a repulsive Keller–Segel model with logarithmic sensitivity

$$\begin{cases} \partial_t \rho = D_1 \Delta_x \rho + \chi \nabla_x \cdot (\rho \nabla_x \log(c)) \\ \partial_t c = D_2 \Delta_x c + \lambda \rho c - \beta c \end{cases}, \quad \chi, \lambda, \beta > 0,$$

whenever the chemotactic parameters are appropriately chosen and the cell concentration keeps strictly positive. In this way, some explicit solutions (namely, traveling waves and stationary cell density profiles) of the former system can be transferred to a number of variants of the latter by means of an adequate change of variables.

KEYWORDS

chemotaxis, logarithmic sensitivity, repulsive Keller–Segel model, Schrödinger–Doebner–Goldin equation, stationary solutions, traveling waves

MSC CLASSIFICATION

35C07, 35C08, 35K55, 35Q40, 35Q92

1 | INTRODUCTION AND MOTIVATION

The Keller–Segel model of chemotaxis^{1,2} is a system of partial differential equations of reaction–diffusion type describing the process through which a population of motile cells moves in the direction of the higher (attractive) or lower (repulsive) concentration of some chemical agent (chemoattractant/chemorepellent) in its environment. Both types of *taxis* are very present in life: On one hand, many bacteria (say, for instance, *Escherichia coli*) move towards the highest concentration of

This is an open access article under the terms of the Creative Commons Attribution-NonCommercial-NoDerivs License, which permits use and distribution in any medium, provided the original work is properly cited, the use is non-commercial and no modifications or adaptations are made.

© 2022 The Authors. Mathematical Methods in the Applied Sciences published by John Wiley & Sons, Ltd.

oxygen or glucose in their search for food (positive chemotaxis); on the other hand, many others swim away from alcohols, acids, and other unfavorable substances (negative chemotaxis).^{3,4}

The parabolic–parabolic Keller–Segel system takes the general form

$$\begin{cases} \partial_t \rho = d_1 \Delta_x \rho + \chi \nabla_x \cdot (\rho \nabla_x (\varphi(c))) \\ \partial_t c = d_2 \Delta_x c + f(\rho, c) \end{cases}, \quad (1)$$

where $\rho(t, x)$ and $c(t, x)$ stand for the density of motile cells and the concentration of chemical substance in time $t > 0$ and space $x \in \mathbb{R}^N$, respectively; $\varphi(c)$ is the so-called sensitivity function, which describes the sensitivity of cells with respect to the variation of chemical concentration and is typically given by $\varphi(c) = c$ or $\varphi(c) = \log(c)$; $\chi \neq 0$ is the chemotactic sensitivity parameter, whose absolute value accounts for the strength of the cell response to the chemical signal: $\chi < 0$ describes attractive chemotaxis, while $\chi > 0$ holds for repulsive chemotaxis; $d_1, d_2 \geq 0$ are the corresponding diffusion rates; and where the function $f(\rho, c)$ models the production–degradation mechanism of the chemical agent, being typically of the form $f(\rho, c) = \sigma\rho - \tau c$ or $f(\rho, c) = \lambda\rho c - \beta c$, where $\sigma, \tau, \lambda, \beta$ are constants.

Up to present, a great deal of work has been devoted to understand the behavior of solutions to a number of variants of the Keller–Segel system (1). We refer, for instance, to previous studies^{5–7} (and references therein) for a comprehensive approach to the state of art of the problem. Particularly, the global or blow-up character of solutions in different space dimensions⁸ (see previous studies^{9–27} among others) as well as the existence of some special shapes such as steady-state, soliton, or traveling wave solutions (see previous studies^{28–39} among others) are currently hot topics in this field. In a recent series of papers,^{40–42} the close connection between the parabolic–parabolic Keller–Segel system (1) (as well as other related models accounting for logistic growth of the cell population or nonstandard production–degradation mechanisms of the chemical substance) with

$$\chi < 0, \quad \varphi(c) = c, \quad f(\rho, c) = \sigma\rho - \tau c, \quad \sigma, \tau \geq 0,$$

and the so-called Doebner–Goldin family of Schrödinger (or modified complex Ginzburg–Landau) equations^{43,44}

$$i\partial_t \psi + \frac{1}{2} \Delta_x \psi + (\mu + \tau S - \sigma n) \psi = \mathcal{L}[n, \nabla_x n, \Delta_x n, J] \psi, \quad (2)$$

$$\mathcal{L} = -Q - \left(A - i \frac{d_1}{2} \right) \frac{\Delta_x n}{n} - B \frac{|\nabla_x n|^2}{n^2} - d_2 \nabla_x \cdot \left(\frac{J}{n} \right) + \left(C - \frac{1}{2} \right) \frac{|J|^2}{n^2}, \quad (3)$$

with $\mu = A = B = C = 0$, has been highlighted and subsequently exploited. In Equation (2), we used the following notation:

$$n = |\psi|^2 \quad (4)$$

is the quantum probability density,

$$J = \text{Im}(\bar{\psi} \nabla_x \psi) \quad (5)$$

is the electric current, S is an argument of the complex wavefunction ψ (see Guerrero et al⁴⁵ for a detailed discussion on its correct definition) that satisfies $\nabla_x S = \frac{J}{n}$, and

$$Q = -\frac{\Delta_x \sqrt{n}}{2\sqrt{n}} \quad (6)$$

is the quantum Bohm potential, where \bar{z} and $\text{Im}(z)$ stand for the complex conjugate and the imaginary part of the complex function z , respectively. In this context, the wavefunction can be decomposed in polar form as

$$\psi(t, x) = \sqrt{n(t, x)} e^{iS(t, x)},$$

in such a way that the modulus–argument couple (n, S) is straightforwardly shown to solve (1).

In spite of its profoundly different nature, the aforementioned link between both models has given rise to a number of interesting results. Particularly, Alejo and López⁴⁰ investigated various extended forms of the chemical production–degradation mechanism under which the existence of traveling wave solutions to the Keller–Segel system (1) is guaranteed. Indeed, they found a general balance term given by

$$F(\rho, c, \nabla_x \rho, \nabla_x c, \Delta_x \rho) = \mu + \tau c - \sigma \rho + A \frac{\Delta_x \rho}{\rho} + B \frac{|\nabla_x \rho|^2}{\rho^2} - C |\nabla_x c|^2, \quad (7)$$

such that the following generalization of the minimal Keller–Segel model

$$\begin{cases} \partial_t \rho = d_1 \Delta_x \rho - \nabla_x \cdot (\rho \nabla_x c) \\ \partial_t c = d_2 \Delta_x c + F \end{cases} \quad (8)$$

was shown to admit square-integrable cosh-type solitary wave solutions under the choice $\mu = \tau = \sigma = C = d_1 = 0$ provided that

$$A(A + B) < 0. \quad (9)$$

Notice that the relation (9) comes here to simplify that given in Alejo and López⁴⁰ (formula (12)), where the necessary condition $d_1 = 0$ was inadvertently omitted. In this case, (7) can be rewritten as

$$F(\nabla_x \rho, \Delta_x \rho) = A \Delta_x \log(\rho) + (A + B) |\nabla_x \log(\rho)|^2,$$

where the coefficients must have different sign to fulfill (9).

The purpose of this paper is to show that, despite its appearance, the generalized model (7)–(8) is not so far from the prototypical Keller–Segel system (1). As a matter of fact, we are intended to prove that when the diffusion rates and the chemotactic parameters are related to each other in an appropriate way, then the system (7)–(8) can be reformulated as a repulsive Keller–Segel model with logarithmic sensitivity

$$\begin{cases} \partial_t \rho = D_1 \Delta_x \rho + \chi \nabla_x \cdot (\rho \nabla_x \log(c)) \\ \partial_t c = D_2 \Delta_x c + \lambda \rho c - \beta c \end{cases}, \quad (10)$$

with $\chi, \lambda, \beta > 0$, via an adequate gauge transformation through which some special solutions of (7)–(8) (for instance, stationary and traveling wave solutions) can be straightforwardly transferred to the context of system (10).

Repulsive Keller–Segel models have not been dealt with very extensively in the mathematical literature. We review later on some of the known results concerning negative chemotaxis. Wang and Zhao⁴⁶ established the long-time dynamics of the unique global classical solution to the 1D initial-boundary value problem of Neumann type associated with (10), as well as its diffusion limit ($D_2 \rightarrow 0$). This model is reminiscent of those introduced in Othmer and Stevens⁴⁷ for the study of reinforced random walks and then examined in Levine and Sleeman⁴⁸ to deduce that the cell density may blow up in finite time or collapse to a spatially uniform distribution, depending on the choices of the chemotactic parameters. In both cases, there is no diffusion term present in the equation for the chemical ($D_2 = 0$). Cieřlak et al⁴⁹ consider the Keller–Segel chemorepulsion model with linear sensitivity $\varphi(c) = c$ and chemical production-decay law $f(\rho, c) = \sigma \rho - \tau c$, and prove the global existence and uniqueness of classical solutions to the associated initial-boundary value problem of Neumann type in 2D, as well as global existence of weak solutions in higher dimensions. Then, in Lin and Xiang,⁵⁰ these results are extended to account for power-like nonlinear sensitivity and chemical production functions. Indeed, it is shown that the strong damping effect due to chemotactic repulsion contributes to prevent blowup. The long-time convergence of this system to the (unique) constant solution is proved in Chen et al,⁵¹ which reinforces the idea of chemorepulsion as a mechanism that inhibits the formation of nontrivial patterns.

The paper is structured as follows. The next section is devoted to construct some families of solitary wave solutions to the nonlinear Schrödinger model (2)–(3), whose hydrodynamic components can be seen as special solutions of the Keller–Segel system (7)–(8). Particularly, traveling wave solutions (Section 2.1) and cell density stationary solutions (Section 2.2) are built up for the various eventual ranges of the involved parameters, and a summary of the results in each case is performed at the end of the corresponding sections (Sections 2.2.1, 2.2.2, and 2.2.3). In Section 3, we introduce a nonlinear Hopf–Cole-type transformation that converts the nonstandard aforesaid system into system (10), which is a prototypical Keller–Segel model with logarithmic sensitivity and positive mass action interaction (Section 3.1). Then, the explicit solutions previously found for Equations (7)–(8) are now transformed into solutions to a number of variants of system (10), which include terms proportional to either $\frac{\Delta_x n}{n}$ or $|\nabla_x \log(n)|^2$ in the production–degradation operator associated with the chemical substance. More precisely, several families of traveling wave solutions (Section 3.2.1) and stationary solutions (Sections 3.2.2, 3.2.3, and 3.2.4) are again constructed in the new context.

2 | ON TRAVELING WAVES AND OTHER SPECIAL SOLUTIONS TO THE HAMILTON-JACOBI-KELLER-SEGEL SYSTEM (7)–(8)

In this section, we extend the results given in Alejo and López⁴⁰ to the case under consideration from now on: $\tau = 0$ and $\mu, \sigma, C \neq 0$. Particularly, choosing $C \neq 0$ in (7) puts now the second equation in (8) in the context of Hamilton–Jacobi models. In that spirit, we are intended to establish appropriate conditions on the chemotactic coefficients that guarantee the existence of solitary wave solutions to Equations (2)–(3) and thus to system (7)–(8) (see Alejo and López⁴⁰ and López⁴¹). More precisely, we shall show that the relation (9) is not necessary any more in the new scenario, but just $3A + 2B < 0$.

First, from the definitions (4) and (5), it is a simple matter to observe that $\nabla\psi = \left(\frac{\nabla n}{2n} + i\frac{J}{n}\right)\psi$. Then, after taking divergences, we achieve

$$\Delta_x\psi = \left(i\frac{\nabla_x \cdot J}{n} - \frac{|J|^2}{n^2} - 2Q\right)\psi,$$

with Q given as in (6). Then, the nonlinear Schrödinger model (2)–(3) can be rewritten as

$$i\partial_t\psi + (\mu - \sigma n)\psi = \left(\frac{i}{2}\left(d_1\frac{\Delta_x n}{n} - \frac{\nabla_x \cdot J}{n}\right) - A\frac{\Delta_x n}{n} - B\frac{|\nabla_x n|^2}{n^2} - d_2\nabla_x \cdot \left(\frac{J}{n}\right) + C\frac{|J|^2}{n^2}\right)\psi. \quad (11)$$

The construction is split into several steps.

Step 1: We first insert the profile $\psi = e^{r+is}$ into Equation (11) and separate into real and imaginary parts to obtain

$$\partial_t r = d_1\Delta_x r - \frac{1}{2}\Delta_x s + 2d_1|\nabla_x r|^2 - \nabla_x r \cdot \nabla_x s, \quad (12)$$

$$\partial_t s = 2A\Delta_x r + d_2\Delta_x s + 4(A+B)|\nabla_x r|^2 - C|\nabla_x s|^2 - \sigma e^{2r} + \mu. \quad (13)$$

Step 2: Now, the linear transformation

$$s' = \gamma r + \Lambda s \quad (14)$$

is introduced. Then,

$$\begin{aligned} \partial_t r &= \alpha_2\Delta_x r + \alpha_1\Delta_x s' + 2\alpha_2|\nabla_x r|^2 + 2\alpha_1\nabla_x r \cdot \nabla_x s', \\ \partial_t s' &= \beta_2\Delta_x r + \beta_1\Delta_x s' + \beta_3|\nabla_x r|^2 + \beta_4\nabla_x r \cdot \nabla_x s' + \beta_5|\nabla_x s'|^2 + \beta_6 e^{2r} + \beta_7, \end{aligned}$$

with

$$\begin{aligned} \alpha_1 &= -\frac{1}{2\Lambda}, \quad \alpha_2 = d_1 + \frac{\gamma}{2\Lambda}, \quad \beta_1 = d_2 - \frac{\gamma}{2\Lambda}, \\ \beta_2 &= 2\Lambda A + \gamma\left(d_1 - d_2 + \frac{\gamma}{2\Lambda}\right), \quad \beta_3 = 4\Lambda(A+B) + \gamma\left(2d_1 - \frac{\gamma}{\Lambda}(C-1)\right), \\ \beta_4 &= \frac{\gamma}{\Lambda}(2C-1), \quad \beta_5 = -\frac{C}{\Lambda}, \quad \beta_6 = -\sigma\Lambda, \quad \beta_7 = \mu\Lambda. \end{aligned}$$

As there are two degrees of freedom, we can set up $\alpha_1 = 1$ (by just choosing $\Lambda = -\frac{1}{2}$) and $\beta_1 = 0$ (if choosing $\gamma = -d_2$) to find

$$\partial_t r = (d_1 + d_2)\Delta_x r + \Delta_x s' + 2(d_1 + d_2)|\nabla_x r|^2 + 2\nabla_x r \cdot \nabla_x s', \quad (15)$$

$$\begin{aligned} \partial_t s' &= -(A + d_1 d_2)\Delta_x r - 2(A + B + d_2(d_1 - d_2(C-1)))|\nabla_x r|^2 \\ &\quad + 2d_2(2C-1)\nabla_x r \cdot \nabla_x s' + 2C|\nabla_x s'|^2 + \frac{\sigma}{2}e^{2r} - \frac{\mu}{2}. \end{aligned} \quad (16)$$

Step 3: By imposing stationarity of r , we get

$$\Delta_x((d_1 + d_2)r + s') + 2\nabla_x r \cdot \nabla_x((d_1 + d_2)r + s') = 0,$$

which invites us to consider the ansatz

$$s'(t, x) = -(d_1 + d_2)r(x) - \omega t, \quad \omega \in \mathbb{R}. \quad (17)$$

Finally, inserting this profile into Equation (16), we find

$$(A + d_1 d_2) \Delta_x r + 2(A + B - d_1^2 C) |\nabla_x r|^2 - \frac{\sigma}{2} e^{2r} = \omega - \frac{\mu}{2}. \quad (18)$$

As consequence, any couple (r, s) such that $r = r(x)$ solves Equation (18) and $s(t, x) = 2(d_1 r(x) + \omega t)$, is a (quasi-)stationary solution of the hydrodynamical system (12)–(13).

2.1 | Traveling wave solutions

It is a simple matter to notice that, for any solution (r, s') to (17)–(18), the couple

$$\begin{aligned} \tilde{r}(t, x) &= r(\xi), \\ \tilde{s}(t, x) &= s'(\xi) - \frac{1}{2}(x \cdot v) + \frac{C}{2}|v|^2 t = -d_2 r(\xi) - \omega t - \frac{1}{2}(x \cdot v) + \frac{C}{2}|v|^2 t, \end{aligned}$$

with arbitrary $\xi = x - vt$ and $v \in \mathbb{R}^N$, solves (15)–(16) when d_1 is set to zero. As consequence, given that $s = -2(d_2 r + \tilde{s})$ according to (14), we conclude that

$$\psi(t, x) = e^{r(\xi) + i(\xi \cdot v + (2\omega + (1-C)|v|^2)t)}$$

is a solution to Equation (11) provided that r solves Equation (18). In particular, solutions with the form

$$\psi(t, x) = \operatorname{sech} \left(\sqrt{-\frac{\sigma}{2N(3A+2B)}} \xi \right) e^{i \left(\xi \cdot v + \left(\mu - \frac{2\sigma(A+B)}{3A+2B} + (1-C)|v|^2 \right) t \right)} \quad (19)$$

are found provided that $3A + 2B < 0$. Indeed, inserting the profile $r(\xi) = \log \left(\operatorname{sech} \left(\sqrt{-\frac{\sigma}{2N(3A+2B)}} \xi \right) \right)$ into Equation (18) yields

$$\omega = \frac{\mu}{2} - \frac{\sigma(A+B)}{3A+2B}.$$

As consequence, the couple of functions

$$\begin{aligned} \rho(\xi) &= \operatorname{sech} \left(\sqrt{-\frac{\sigma}{2N(3A+2B)}} \xi \right)^2, \\ c(t, \xi) &= \xi \cdot v + \left(\mu - \frac{2\sigma(A+B)}{3A+2B} + (1-C)|v|^2 \right) t, \end{aligned}$$

solve the Keller–Segel-type system (7)–(8) for any given $v \in \mathbb{R}^N$.

2.2 | Cell density stationary solutions

We finally explore several scenarios in which Equation (18) can be explicitly solved. In this spirit, we set $N = 1$ and $\sigma, \mu > 0$ and consider the ansatz

$$r = \log(u),$$

so that Equation (18) can be rewritten as

$$(A + d_1 d_2) u u'' + \theta (u')^2 - \frac{\sigma}{2} u^4 - \nu u^2 = 0, \quad (20)$$

where we denoted

$$\theta = A + 2B - d_1(d_2 + 2d_1 C), \quad \nu = \omega - \frac{\mu}{2}. \quad (21)$$

2.2.1 | Case I: $\theta = A + d_1 d_2$

This relation holds when the chemotactic coefficients B and C are related through

$$B - d_1(d_2 + d_1 C) = 0, \quad (22)$$

in which case Equation (20) reads

$$\theta(u^2)'' = u^2(2\nu + \sigma u^2). \quad (23)$$

Then, there are two possible situations:

(I.I) If $\theta = 0$, then the only nontrivial solution to Equation (23) is given by $u \equiv \sqrt{\frac{\mu - 2\omega}{\sigma}}$ provided that $\omega \leq \frac{\mu}{2}$. Thus, the couple of functions

$$\rho \equiv \frac{\mu - 2\omega}{\sigma}, \quad c(t) = c_0 + 2\omega t, \quad (24)$$

solves the system (7)–(8) for all $c_0 \in \mathbb{R}$. As consequence, it suffices to consider

$$c_0 \geq 0, \quad 0 \leq \omega \leq \frac{\mu}{2},$$

so as to get $\rho, c \geq 0$ for all times.

(I.II) If $\theta \neq 0$, then $\rho = u^2$ satisfies

$$\rho'' = \rho(a + b\rho), \quad (25)$$

with

$$a = \frac{2\nu}{\theta}, \quad b = \frac{\sigma}{\theta}.$$

Multiplying Equation (25) by ρ' straightforwardly leads to

$$\left(\frac{1}{2}(\rho')^2 - \frac{a}{2}\rho^2 - \frac{b}{3}\rho^3 \right)' = 0,$$

from which it is easily deduced that

$$(\rho')^2 - \rho^2 \left(a + \frac{2b}{3}\rho \right) = K_0 \in \mathbb{R}. \quad (26)$$

In other terms, either $\rho \equiv \rho_0 \geq 0$ such that $\rho_0^2 \left(a + \frac{2b}{3}\rho_0 \right) + K_0 = 0$ or

$$\frac{\rho'}{\sqrt{\rho^2 \left(a + \frac{2b}{3}\rho \right) + K_0}} = \pm 1. \quad (27)$$

Note that the condition

$$F_{K_0}(\rho) := \rho^2 \left(a + \frac{2b}{3}\rho \right) + K_0 > 0$$

is required to make Equation (27) meaningful. Then, the particular choices $K_0 = 0$ and $K_0 = -\frac{a^3}{3b^2}$ are shown below to lead to explicit solutions. As a matter of fact, if a and b do not take negative values simultaneously, we have

$$\sqrt{F_0} = \frac{1}{\sqrt{3}}\rho\sqrt{2b\rho + 3a},$$

which makes sense provided that $\rho > -\frac{3a}{2b}$. Besides, if we disregard the case in which $a > 0$ and $b < 0$ simultaneously, then

$$\sqrt{F_{-\frac{a^3}{3b^2}}} = \frac{1}{\sqrt{3}}\left(\rho + \frac{a}{b}\right)\sqrt{2b\rho - a},$$

which makes sense provided that $\rho > \max \left\{ -\frac{a}{b}, \frac{a}{2b} \right\}$. We introduce the notation

$$\kappa_{\pm} = \sqrt{\pm \frac{2\nu}{\theta}} = \sqrt{\pm \frac{2\omega - \mu}{A + d_1 d_2}}, \quad (28)$$

for future convenience.

We distinguish three subcases:

Subcase a < 0: We first assume $K_0 = 0$, so that $F_0(\rho) > 0$ if and only if $\rho > -\frac{3a}{2b} > 0$ for $b > 0$ (in other words, the conditions $\theta > 0$ and $\nu < 0$ must be simultaneously satisfied). For instance, for an arbitrarily fixed $x_0 \in \mathbb{R}$, we may choose $\rho(x_0) = -\frac{3a}{2b} > 0$ and $\rho'(x_0) = 0$ as initial data, according to (26). Then, by integrating in Equation (27), we find

$$\rho_{\pm}(x) = \frac{3(\mu - 2\omega)}{\sigma} \left(\frac{1}{1 + \cos \left(C_0 \pm \sqrt{2}\kappa_{-}(x - x_0) \right)} \right) \quad (29)$$

with

$$C_0 = 2 \arctan \left(\frac{|\rho'(x_0)|}{\kappa_{-}\rho(x_0)} \right) > 0,$$

as follows easily from Equation (26) with $K_0 = 0$. The profiles (29) solve Equation (27) in $I_{\pm} = \left(x_0 \mp \frac{C_0}{\sqrt{2}\kappa_{-}} - \pi, x_0 \mp \frac{C_0}{\sqrt{2}\kappa_{-}} + \pi \right)$ given that

$$A > -d_1 d_2, \quad \omega < \frac{\mu}{2}. \quad (30)$$

Also

$$c_{\pm}(t, x) = d_1 \log(\rho_{\pm}(x)) + 2\omega t, \quad (31)$$

again by virtue of (14) and (17). It is a simple matter to check that for $\mu \geq \frac{2\sigma}{3}$, any value

$$0 \leq \omega \leq \frac{\mu}{2} - \frac{\sigma}{3}$$

yields the desired positivity of c_{\pm} .

Let us next take $K_0 = -\frac{a^3}{3b^2} > 0$ and $b > 0$. In this case, it suffices to have $\rho > -\frac{a}{b} = \frac{\mu - 2\omega}{\sigma} > 0$, as deduced previously. Integrating again in Equation (27) and after some calculus, we find

$$\rho_{\pm}(x) = \frac{\mu - 2\omega}{\sigma} \left(\frac{\cosh(C_0 \pm \kappa_{-}(x - x_0)) + 2}{\cosh(C_0 \pm \kappa_{-}(x - x_0)) - 1} \right), \quad (32)$$

with κ_{-} as in (28) and

$$C_0 = \log \left(\frac{\sigma |\rho'(x_0)| - \kappa_{-}(\sigma \rho(x_0) + 2\omega - \mu)}{\sigma |\rho'(x_0)| + \kappa_{-}(\sigma \rho(x_0) + 2\omega - \mu)} \right) < 0,$$

which solves Equation (27) in

$$I_{+} = \left(-\infty, x_0 - \frac{C_0}{\kappa_{-}} \right), \quad I_{-} = \left(x_0 + \frac{C_0}{\kappa_{-}}, \infty \right),$$

respectively, provided that the conditions (30) are fulfilled. Again, $c_{\pm}(t, x)$ are given by formula (31) and remain nonnegative by just choosing $0 \leq \omega \leq \frac{\mu - \sigma}{2}$.

Subcase a > 0: Let us first take $K_0 = 0$ and choose $b > 0$. Solving again Equation (27), we now arrive at

$$\rho_{\pm}(x) = \frac{3(2\omega - \mu)}{\sigma} \left(\frac{1}{\cosh(C_0 \pm \kappa_{+}(x - x_0)) - 1} \right), \quad (33)$$

with

$$C_0 = \log \left(\frac{|\rho'(x_0)| - \kappa_+ \rho(x_0)}{|\rho'(x_0)| + \kappa_+ \rho(x_0)} \right) < 0.$$

Besides, we have the already known expressions for c_{\pm} given by (31). The couple (33)–(31) is shown to solve (7)–(8) in

$$I_+ = \left(x_0 - \frac{\beta_0}{\kappa_+}, x_0 - \frac{C_0}{\kappa_+} \right), \quad I_- = \left(x_0 + \frac{C_0}{\kappa_+}, x_0 + \frac{\beta_0}{\kappa_+} \right),$$

respectively, with positivity of the profiles c_{\pm} . Here,

$$\beta_0 = C_0 + \operatorname{arccosh} \left(\frac{6\omega - 3\mu + \sigma}{\sigma} \right),$$

and the relations

$$A > -d_1 d_2, \quad \omega > \frac{\mu}{2},$$

have been required to hold.

We now tackle the case $K_0 = -\frac{a^3}{3b^2} < 0$ with $b > 0$. In this situation, we need $\rho > \frac{a}{2b} = \frac{2\omega - \mu}{2\sigma}$. Integrating in Equation (27) gives

$$\rho_{\pm}(x) = \frac{2\omega - \mu}{\sigma} \left(\frac{2 - \cos(C_0 \pm \kappa_+(x - x_0))}{1 + \cos(C_0 \pm \kappa_+(x - x_0))} \right), \quad (34)$$

with

$$C_0 = \frac{2}{\sqrt{3}} \arctan \left(\sqrt{\frac{2\sigma\rho(x_0) - 2\omega + \mu}{3(2\omega - \mu)}} \right) > 0,$$

which are well defined in $I_{\pm} = \left(x_0 \mp \frac{C_0}{\kappa_+} - \pi, x_0 \mp \frac{C_0}{\kappa_+} + \pi \right)$, respectively. Along with c_{\pm} as in (31), the functions ρ_{\pm} given in (34) solve (7)–(8) provided that

$$A > -d_1 d_2, \quad \omega > \frac{\mu}{2}.$$

Furthermore, positivity of c_{\pm} is guaranteed by just choosing $\omega \geq \frac{\mu}{2} + \sigma$, without any restriction on the intervals I_{\pm} . *Subcase $a = 0$:* As before, we integrate in Equation (27) to finally find

$$\rho_{\pm}(x) = \frac{\delta_0^2 \rho(x_0)}{(\delta_0 \mp (x - x_0))^2} \quad (35)$$

with $A > -d_1 d_2$, where we denoted

$$\delta_0 = \sqrt{\frac{6(A + d_1 d_2)}{\sigma \rho(x_0)}}. \quad (36)$$

Then, (ρ_{\pm}, c_{\pm}) are shown to solve (7)–(8) in

$$I_+ = (-\infty, x_0 + \delta_0), \quad I_- = (x_0 - \delta_0, \infty),$$

respectively. Furthermore, positivity of c_{\pm} (cf. 31) is guaranteed if the above intervals are restricted to

$$I_+ = \left(x_0 + \delta_0 \left(1 - \sqrt{\rho(x_0)} \right), x_0 + \delta_0 \right), \quad I_- = \left(x_0 - \delta_0, x_0 - \delta_0 \left(1 - \sqrt{\rho(x_0)} \right) \right).$$

Summary: Let the parameters B, C satisfy the relation (cf. 22)

$$B - d_1(d_2 + d_1 C) = 0.$$

We have:

- If $A = -d_1d_2$ and $0 \leq \omega \leq \frac{\mu}{2}$, then the constant states (24) are obtained.
- If $A > -d_1d_2$ and $\omega = \frac{\mu}{2}$, then profiles of the form (35)–(31) are obtained.
- If $A > -d_1d_2$ and $\omega > \frac{\mu}{2}$, then profiles of the form (33)–(31) are obtained.
- If $A > -d_1d_2$, $\mu \geq \frac{2\sigma}{3}$ and $0 \leq \omega \leq \frac{\mu}{2} - \frac{\sigma}{3}$, then profiles of the form (29)–(31) are obtained.
- If $A > -d_1d_2$ and $\omega \geq \frac{\mu}{2} + \sigma$, then profiles of the form (33)–(31) and (34)–(31) are obtained.
- If $A > -d_1d_2$, $\mu \geq \sigma$ and $0 \leq \omega \leq \frac{\mu - \sigma}{2}$, then profiles of the form (32)–(31) and (29)–(31) are obtained.

2.2.2 | Case II: $\theta = 0$, $A + d_1d_2 \neq 0$

This case holds when the chemotactic coefficients and the diffusion rates are related through

$$A + 2B - d_1(d_2 + 2d_1C) = 0. \quad (37)$$

Equation (20) then takes the form of an undamped Duffing equation

$$u'' = u(c + du^2), \quad (38)$$

where we denoted

$$c = \frac{\nu}{A + d_1d_2}, \quad d = \frac{\sigma}{2(A + d_1d_2)}.$$

Multiplying Equation (38) by u' leads to

$$\left(\frac{1}{2}(u')^2 - \frac{c}{2}u^2 - \frac{d}{4}u^4 \right)' = 0,$$

from which it is easily deduced that

$$(u')^2 - u^2 \left(c + \frac{d}{2}u^2 \right) = K_0 \in \mathbb{R}.$$

Then, either $u \equiv u_0 \geq 0$ such that $u_0^2 \left(c + \frac{d}{2}u_0^2 \right) + K_0 = 0$, or $\rho = u^2$ is easily seen to solve

$$\frac{\rho'}{\sqrt{\rho \left(\frac{d}{2}\rho^2 + c\rho + K_0 \right)}} = \pm 2, \quad (39)$$

given that

$$G_{K_0}(\rho) := \frac{d}{2}\rho^2 + c\rho + K_0 > 0.$$

In this framework, the particular choices $K_0 = 0$ and $K_0 = \frac{c^2}{2d}$ lead to explicit solutions. In the first case, it is required that $\rho > -\frac{2c}{d}$ in order to fulfill $G_0 > 0$, c and d not being simultaneously negative. The second case requires $d > 0$, which entails that $G_{\frac{c^2}{2d}}$ reaches its minimum value at $G_{\frac{c^2}{2d}} \left(-\frac{c}{d} \right) = 0$.

Then, we find:

(i) If $c < 0$ and $K_0 = 0$, then

$$\rho_{\pm}(x) = \frac{4(\mu - 2\omega)}{\sigma} \left(\frac{1}{1 + \cos \left(C_0 \pm \sqrt{2\kappa_-}(x - x_0) \right)} \right) \quad (40)$$

solves Equation (39) in $I_{\pm} = \left(x_0 - \frac{1}{\sqrt{2\kappa_-}}(\pi \pm C_0), x_0 + \frac{1}{\sqrt{2\kappa_-}}(\pi \mp C_0) \right)$, with

$$C_0 = 2 \arctan \left(\frac{|\rho'(x_0)|}{\sqrt{2\kappa_-}\rho(x_0)} \right) > 0,$$

provided that

$$A > -d_1 d_2, \quad \omega < \frac{\mu}{2}.$$

In these circumstances, it suffices to choose $\mu \geq \frac{\sigma}{2}$ and $0 \leq \omega \leq \frac{2\mu - \sigma}{4}$ in order to guarantee the nonnegativity of c_{\pm} , given as in (31).

If $c < 0$ and $K_0 = \frac{c^2}{2d}$, then $G_{\frac{c^2}{2d}}(\rho) = \left(\sqrt{\frac{d}{2}}\rho + \frac{c}{\sqrt{2d}} \right)^2$, and Equation (39) becomes

$$\frac{\rho'}{(d\rho + c)\sqrt{\rho}} = \pm \sqrt{\frac{2}{d}}. \quad (41)$$

As consequence, the functions

$$\rho_{\pm}(x) = \frac{\mu - 2\omega}{\sigma} \left(\frac{\cosh(C_0 \pm \kappa_{-}(x - x_0)) + 1}{\cosh(C_0 \pm \kappa_{-}(x - x_0)) - 1} \right) \quad (42)$$

are shown to solve Equation (41) in

$$I_+ = \left(-\infty, x_0 - \frac{C_0}{\kappa_+} \right), \quad I_- = \left(x_0 + \frac{C_0}{\kappa_+}, \infty \right),$$

respectively, provided that

$$A > -d_1 d_2, \quad \omega < \frac{\mu}{2}.$$

Here,

$$C_0 = \log \left(\frac{\sqrt{\sigma\rho(x_0)} - \sqrt{\mu - 2\omega}}{\sqrt{\sigma\rho(x_0)} + \sqrt{\mu - 2\omega}} \right) < 0.$$

Now, the choices $\mu \geq \sigma$ and $0 \leq \omega \leq \frac{\mu - \sigma}{2}$ are enough to ensure the positivity of c_{\pm} .

(ii) If $c > 0$ and $K_0 = 0$, then the functions

$$\rho_{\pm}(x) = \frac{4(2\omega - \mu)}{\sigma} \left(\frac{1}{\cosh(C_0 \pm \sqrt{2}\kappa_+(x - x_0)) - 1} \right) \quad (43)$$

solve Equation (39) in

$$I_+ = \left(-\infty, x_0 - \frac{C_0}{\sqrt{2}\kappa_+} \right), \quad I_- = \left(x_0 + \frac{C_0}{\sqrt{2}\kappa_+}, \infty \right),$$

respectively, provided that

$$A > -d_1 d_2, \quad \omega > \frac{\mu}{2}.$$

Here,

$$C_0 = \log \left(\frac{|\rho'(x_0)| - \sqrt{2}\kappa_+\rho(x_0)}{|\rho'(x_0)| + \sqrt{2}\kappa_+\rho(x_0)} \right) < 0.$$

Furthermore, c_{\pm} given as in (31) are shown to preserve positivity in

$$I_+ = \left(x_0 - \frac{\gamma_0}{\sqrt{2}\kappa_+}, x_0 - \frac{C_0}{\sqrt{2}\kappa_+} \right), \quad I_- = \left(x_0 + \frac{C_0}{\sqrt{2}\kappa_+}, x_0 + \frac{\gamma_0}{\sqrt{2}\kappa_+} \right),$$

respectively, where we denoted

$$\gamma_0 = C_0 + \operatorname{arccosh} \left(\frac{8\omega - 4\mu + \sigma}{8\omega - 4\mu} \right).$$

If $c > 0$ and $K_0 = \frac{c^2}{2d}$, then

$$\rho_{\pm}(x) = \frac{2\omega - \mu}{\sigma} \left(\frac{1 - \cos(C_0 \pm \kappa_+(x - x_0))}{1 + \cos(C_0 \pm \kappa_+(x - x_0))} \right) \quad (44)$$

solve Equation (39) in

$$I_{\pm} = \left(x_0 - \frac{\pi \pm C_0}{\kappa_+}, x_0 + \frac{\pi \mp C_0}{\kappa_+} \right),$$

respectively, provided that

$$A > -d_1 d_2, \quad \omega > \frac{\mu}{2}.$$

Here,

$$C_0 = 2 \arctan \left(\sqrt{\frac{\sigma \rho(x_0)}{2\omega - \mu}} \right) > 0.$$

In this case, the density profiles c_{\pm} (cf. 31) remain positive in the restricted intervals

$$I_+ = \left(x_0 - \frac{\varepsilon_0}{\kappa_+}, x_0 - \frac{C_0 - \pi}{\kappa_+} \right), \quad I_- = \left(x_0 + \frac{C_0 - \pi}{\kappa_+}, x_0 + \frac{\varepsilon_0}{\kappa_+} \right),$$

where we denoted

$$\varepsilon_0 = C_0 + \arccos \left(\frac{2\omega - \mu - \sigma}{2\omega - \mu + \sigma} \right).$$

(iii) If $c = 0$, then

$$\rho_{\pm}(x) = \frac{4\delta_0^2 \rho(x_0)}{\left(2\delta_0 \mp \sqrt{6}(x - x_0) \right)^2} \quad (45)$$

solve Equation (39) in

$$I_+ = \left(-\infty, x_0 + \frac{2\delta_0}{\sqrt{6}} \right), \quad I_- = \left(x_0 - \frac{2\delta_0}{\sqrt{6}}, \infty \right),$$

respectively, with δ_0 as defined in (36), provided that $A > -d_1 d_2$. Now, c_{\pm} are shown to remain positive in

$$I_+ = \left(x_0 + \frac{2\delta_0}{\sqrt{6}} \left(1 - \sqrt{\rho(x_0)} \right), x_0 + \frac{2\delta_0}{\sqrt{6}} \right),$$

$$I_- = \left(x_0 - \frac{2\delta_0}{\sqrt{6}}, x_0 + \frac{2\delta_0}{\sqrt{6}} \left(\sqrt{\rho(x_0)} - 1 \right) \right).$$

Summary: Let the parameters A, B, C satisfy the relation

$$A + 2B - 2d_1^2 C = d_1 d_2.$$

We have:

- If $A > -d_1 d_2$ and $\omega = \frac{\mu}{2}$, then profiles of the form (45)–(31) are obtained.
- If $A > -d_1 d_2$ and $\omega > \frac{\mu}{2}$, then profiles of the form (43)–(31) and (44)–(31) are obtained.
- If $A > -d_1 d_2$, $\mu \geq \frac{\sigma}{2}$ and $0 \leq \omega \leq \frac{\mu}{2} - \frac{\sigma}{4}$, then profiles of the form (40)–(31) are obtained.
- If $A > -d_1 d_2$, $\mu \geq \sigma$ and $0 \leq \omega \leq \frac{\mu - \sigma}{2}$, then profiles of the form (42)–(31) and (40)–(31) are obtained.

2.2.3 | Case III: $A + d_1 d_2 = 0$, $\theta \neq 0$

In this case, $B \neq d_1(d_2 + d_1 C)$, and Equation (20) reads

$$(u')^2 = u^2 (e + f u^2)$$

with

$$e = \frac{\nu}{\theta}, \quad f = \frac{\sigma}{2\theta}.$$

Hence, either $u \equiv \sqrt{\frac{-e}{f}} > 0$, or $\rho = u^2$ satisfies

$$\frac{\rho'}{\rho \sqrt{e + f \rho}} = \pm 2. \quad (46)$$

Denote

$$\phi_{\pm} = \sqrt{\frac{\pm(2\omega - \mu)}{B - d_1(d_2 + d_1 C)}}.$$

By proceeding as in the previous cases we find:

(i) If $e < 0$, then

$$\rho_{\pm}(x) = \frac{2(\mu - 2\omega)}{\sigma} \left(\frac{1}{1 + \cos(C_0 \pm \phi_{-}(x - x_0))} \right) \quad (47)$$

solve Equation (46) in $I_{\pm} = \left(x_0 - \frac{\pi \pm C_0}{\phi_{-}}, x_0 + \frac{\pi \mp C_0}{\phi_{-}} \right)$, provided that

$$\omega < \frac{\mu}{2}, \quad B > d_1(d_2 + d_1 C).$$

Here,

$$C_0 = 2 \arctan \left(\frac{|\rho'(x_0)|}{\phi_{-} \rho(x_0)} \right).$$

Also, the profiles c_{\pm} given by (31) are easily shown to remain positive for all $0 \leq \omega \leq \frac{\mu - \sigma}{2}$.

(ii) If $e > 0$, then

$$\rho_{\pm}(x) = \frac{2(2\omega - \mu)}{\sigma} \left(\frac{1}{\cosh(C_0 \pm \phi_{+}(x - x_0)) - 1} \right) \quad (48)$$

solve Equation (46) in

$$I_{+} = \left(-\infty, x_0 - \frac{C_0}{\phi_{+}} \right), \quad I_{-} = \left(x_0 + \frac{C_0}{\phi_{+}}, \infty \right),$$

respectively, provided that

$$\omega > \frac{\mu}{2}, \quad B > d_1(d_2 + d_1 C).$$

Here,

$$C_0 = \log \left(\frac{|\rho'(x_0)| - \phi_{+} \rho(x_0)}{|\rho'(x_0)| + \phi_{+} \rho(x_0)} \right).$$

Furthermore, the profiles c_{\pm} given by (31) remain positive when the above intervals are restricted to

$$I_{+} = \left(x_0 - \frac{\varphi_0}{\phi_{+}}, x_0 - \frac{C_0}{\phi_{+}} \right), \quad I_{-} = \left(x_0 + \frac{C_0}{\phi_{+}}, x_0 + \frac{\varphi_0}{\phi_{+}} \right),$$

respectively, where we denoted

$$\varphi_0 = C_0 - \operatorname{arccosh} \left(\frac{4\omega - 2\mu + \sigma}{\sigma} \right).$$

(iii) If $e = 0$, then

$$\rho_{\pm}(x) = \frac{4\psi_0^2 \rho(x_0)}{(2\psi_0 \mp (x - x_0))^2} \quad (49)$$

solve Equation (46) in

$$I_+ = (-\infty, x_0 + 2\psi_0), \quad I_- = (x_0 - 2\psi_0, \infty),$$

respectively, provided that $B > d_1(d_2 + d_1C)$. Here, we denoted

$$\psi_0 = \sqrt{\frac{B - d_1(d_2 + d_1C)}{\sigma\rho(x_0)}}.$$

Besides, the profiles c_{\pm} given by (31) remain positive in the restricted intervals

$$I_+ = \left(x_0 + 2\psi_0 \left(1 - \sqrt{\rho(x_0)} \right), x_0 + 2\psi_0 \right), \quad I_- = \left(x_0 - 2\psi_0, x_0 - 2\psi_0 \left(1 - \sqrt{\rho(x_0)} \right) \right).$$

Summary: Let $A = -d_1d_2$ and $B > d_1(d_2 + d_1C)$. We have:

- If $\omega = \frac{\mu}{2}$, then profiles of the form (49)–(31) are obtained.
- If $\omega > \frac{\mu}{2}$, then profiles of the form (48)–(31) are obtained.
- If $\mu \geq \sigma$ and $0 \leq \omega \leq \frac{\mu - \sigma}{2}$, then profiles of the form (47)–(31) are obtained.

3 | FROM THE HYDRODYNAMIC SYSTEM (7)–(8) TO THE REPULSIVE KELLER–SEGEL SYSTEM (10): DERIVATION, COMPATIBILITY RELATIONS, AND EXPLICIT SOLUTIONS

3.1 | Derivation

In what follows, we set $\tau = 0$, $C = \frac{1}{2}$, and assume $\sigma, \mu > 0$ in (7). Then, following a similar strategy as in the previous section, we apply the biparametric change of variables

$$n = \rho, \quad S = \Lambda c + \gamma \log(\rho) \quad (50)$$

to system (8), with $\Lambda, \gamma \in \mathbb{R}$. On one hand, in the new variables the first equation keeps structurally unaltered:

$$\partial_t n = \left(d_1 + \frac{\gamma}{\Lambda} \right) \Delta_x n - \frac{1}{\Lambda} \nabla_x \cdot (n \nabla_x S). \quad (51)$$

On the other hand, the equation for the chemical agent is transformed into the following (deterministic) Kardar–Parisi–Zhang equation⁵²:

$$\partial_t S + \frac{1}{2\Lambda} |\nabla_x S|^2 = \left(d_2 - \frac{\gamma}{\Lambda} \right) \Delta_x S + \mu\Lambda - \sigma\Lambda n + \left((d_1 - d_2)\gamma + \frac{\gamma^2}{\Lambda} + A\Lambda \right) \frac{\Delta_x n}{n} + \left(d_2\gamma - \frac{\gamma^2}{2\Lambda} + B\Lambda \right) \frac{|\nabla_x n|^2}{n^2}. \quad (52)$$

The first idea consists in simultaneously killing the nonstandard terms depending upon $\frac{\Delta_x n}{n}$ and $\frac{|\nabla_x n|^2}{n^2}$. To this aim, the relation between the parameters γ, Λ appearing in (50) and the coefficients A, B must be as follows:

$$A = \frac{\gamma}{\Lambda^2} ((d_2 - d_1)\Lambda - \gamma), \quad B = \frac{\gamma}{\Lambda^2} \left(\frac{\gamma}{2} - d_2\Lambda \right). \quad (53)$$

Next, we are called to impose

$$d_2 - \frac{\gamma}{\Lambda} = \frac{1}{2\Lambda}, \quad (54)$$

in Equation (52) in order to combine the terms containing $\Delta_x S$ and $|\nabla_x S|^2$ conveniently, for which we distinguish the two following cases.

Case $d_2 = 0$: In this situation, Equation (54) straightforwardly leads to the choice $\gamma = -\frac{1}{2}$; thus, (53) reads

$$B = \frac{1}{8\Lambda^2} > 0, \quad A = \sqrt{2B} \left(d_1 - \sqrt{2B} \right). \quad (55)$$

Hence, on one hand, Equation (52) becomes

$$\partial_t S = \sqrt{2B} (\Delta_x S - |\nabla_x S|^2) + \frac{1}{2\sqrt{2B}} (\mu - \sigma n).$$

On the other hand, Equation (51) reads

$$\partial_t n = (d_1 - \sqrt{2B}) \Delta_x n - 2\sqrt{2B} \nabla_x \cdot (n \nabla_x S). \quad (56)$$

Notice that the condition

$$0 < B \leq \frac{d_1^2}{2},$$

or equivalently $A \geq 0$, is required in order that the diffusion rate in Equation (56) remains nonnegative. We finally introduce the Hopf–Cole transformation

$$u = e^{-S}, \quad (57)$$

which straightforwardly leads to

$$\begin{cases} \partial_t n = (d_1 - \sqrt{2B}) \Delta_x n + 2\sqrt{2B} \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = \sqrt{2B} \Delta_x u + \frac{\sigma}{2\sqrt{2B}} n u - \frac{\mu}{2\sqrt{2B}} u \end{cases}.$$

In conclusion, if $d_2 = 0$, then the extended Keller–Segel model

$$\begin{cases} \partial_t \rho = d_1 \Delta_x \rho - \nabla_x \cdot (\rho \nabla_x c) \\ \partial_t c = F \end{cases},$$

with production–degradation term given by

$$\begin{aligned} F(\rho, \nabla_x \rho, \Delta_x \rho, c, \nabla_x c) &= \mu - \sigma \rho + A \frac{\Delta_x \rho}{\rho} + B \frac{|\nabla_x \rho|^2}{\rho^2} - \frac{1}{2} |\nabla_x c|^2, \\ A &= \sqrt{2B} (d_1 - \sqrt{2B}) > 0, \\ B &= \frac{1}{8\Lambda^2} \leq \frac{d_1^2}{2}, \end{aligned}$$

is reduced to the repulsive Keller–Segel model

$$\begin{cases} \partial_t n = D_1 \Delta_x n + \chi \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = D_2 \Delta_x u + \lambda n u - \beta u \end{cases}, \quad (58)$$

via the transformation (cf. 50 and 57)

$$n = \rho, \quad u = \sqrt{\rho} e^{-\frac{c}{2\sqrt{2B}}}, \quad (59)$$

where

$$D_2 = \sqrt{2B}, \quad D_1 = d_1 - D_2, \quad \chi = 2D_2, \quad \lambda = \frac{\sigma}{2D_2}, \quad \beta = \frac{\mu}{2D_2}. \quad (60)$$

A distinctive feature of (58) is that the chemosensitivity must be twice the value of the chemical diffusivity, as evidenced in (60).

In this context, it may be emphasized that the derivation turns invalid if both diffusion rates vanish. Indeed, if in addition $d_1 = 0$, then $D_1 = -D_2 < 0$ according to (60), and thus, the cell diffusivity would become negative (see Remark below).

Case $d_2 > 0$: Now, Equation (54) yields

$$\Lambda = \frac{1 + 2\gamma}{2d_2}, \quad \gamma \neq -\frac{1}{2},$$

so that Equation (52) reads

$$\partial_t S = \frac{d_2}{1+2\gamma} (\Delta_x S - |\nabla_x S|^2) + \frac{1+2\gamma}{2d_2} (\mu - \sigma n). \quad (61)$$

This entails the restriction

$$\gamma > -\frac{1}{2}, \quad (62)$$

hence $\Lambda > 0$, so as to make it positive the diffusion coefficient in Equation (61). Also, inserting (54) into (53) yields

$$A = \frac{2d_2\gamma(d_2 - d_1(1+2\gamma))}{(1+2\gamma)^2}, \quad B = -\frac{2d_2^2\gamma(1+\gamma)}{(1+2\gamma)^2}. \quad (63)$$

Besides, Equation (51) becomes

$$\partial_t n = \left(d_1 + \frac{2d_2\gamma}{1+2\gamma} \right) \Delta_x n - \frac{2d_2}{1+2\gamma} \nabla_x \cdot (n \nabla_x S). \quad (64)$$

It is deduced from (63) that the original parameters A, B must be chosen such that

$$A \leq \frac{(d_2 - d_1)^2}{4}, \quad B > -\frac{d_2^2}{2}, \quad (65)$$

for consistency. Also, if $A = -d_1 d_2$, then $\gamma = -\frac{d_1}{2(d_1+d_2)}$ and $\Lambda = \frac{1}{2(d_1+d_2)}$; hence, the diffusion coefficient vanishes in Equation (64). Otherwise, γ must satisfy

$$\gamma \geq -\frac{d_1}{2(d_1+d_2)}, \quad (66)$$

which indeed is a stronger condition than (62). We perform again the change of variables (57) to obtain

$$\begin{cases} \partial_t n = \left(d_1 + \frac{2d_2\gamma}{1+2\gamma} \right) \Delta_x n + \frac{2d_2}{1+2\gamma} \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = \frac{d_2}{1+2\gamma} \Delta_x u + \frac{\sigma(1+2\gamma)}{2d_2} nu - \frac{\mu(1+2\gamma)}{2d_2} u \end{cases}.$$

In conclusion, the extended Keller–Segel model (7)–(8) with $\mu, \sigma > 0$, A and B satisfying (65) and chosen as in (63) with $\gamma > -\frac{d_1}{2(d_1+d_2)}$, is reduced to the repulsive Keller–Segel model (58) with

$$D_2 = \frac{d_2}{1+2\gamma}, \quad D_1 = d_1 + 2D_2\gamma, \quad \chi = 2D_2, \quad \lambda = \frac{\sigma}{2D_2}, \quad \beta = \frac{\mu}{2D_2}, \quad (67)$$

via the change of variables $(\rho, c) \mapsto (n, u)$, where

$$n = \rho, \quad u = \rho^{-\gamma} e^{-\frac{1+2\gamma}{2d_2} c}.$$

Again the chemosensitivity must be twice the value of the chemical diffusivity, as evidenced in (67).

Remark 1. It is noticeable the fact that the only Keller–Segel model of type (10) reachable from a Schrödinger–Doebner–Goldin system is the repulsive one. Indeed, starting from the attractive model

$$\begin{cases} \partial_t n = D_1 \Delta_x n - \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = D_2 \Delta_x u - \lambda nu \end{cases},$$

and going backwards in the transformations (50) and (57) with $\Lambda = 1$ (for simplicity), one is led to

$$\begin{cases} \partial_t \rho = (D_1 + \gamma) \Delta_x \rho + \nabla_x \cdot (\rho \nabla_x c) \\ \partial_t c = (D_2 - \gamma) \Delta_x c + \mathcal{F} \end{cases},$$

with

$$F = \lambda\rho + \gamma(D_2 - D_1 - \gamma)\frac{\Delta_x\rho}{\rho} - D_2\gamma(1 + \gamma)\frac{|\nabla_x\rho|^2}{\rho^2} - D_2|\nabla_x c|^2 - \gamma(2D_2 + 1)\frac{\nabla_x\rho}{\rho} \cdot \nabla_x c,$$

which shows the unavoidability of a term proportional to $\nabla_x \log(\rho) \cdot \nabla_x c$ unless the chemical diffusion rate is chosen $D_2 = -\frac{1}{2}$, which typically gives rise to ill-posed problems.

3.2 | Compatibility relations and explicit solutions

We are now concerned with the eventual translation of the special solutions constructed in Section 2 to the new framework described by the system (58).

3.2.1 | Traveling wave solutions

According to the results derived in Section 2.1, we must set $d_1 = 0$ and $3A + 2B < 0$ throughout Section 3.1 in order to recover the traveling wave solutions induced by the profiles (19). As a matter of fact, we have

$$3A + 2B = \frac{2d_2^2\gamma(1 - 2\gamma)}{(1 + 2\gamma)^2},$$

by virtue of (63). Then, taking into account that $\gamma \geq 0$ (cf. 66), it suffices to choose $d_2 > 0$ and $\gamma > \frac{1}{2}$. Under these conditions, the couple

$$n(\xi) = \operatorname{sech}\left(\frac{1 + 2\gamma}{2d_2} \sqrt{\frac{\sigma}{N\gamma(2\gamma - 1)}} \xi\right)^2, \quad (68)$$

$$u(t, \xi) = n(\xi)^{-\gamma} e^{-\frac{1+2\gamma}{2d_2} \left\{ v \cdot \xi + \left(\mu + \frac{2\sigma\gamma}{1-2\gamma} + \frac{1}{2}|v|^2 \right) t \right\}}, \quad (69)$$

with $\xi = x - vt$, is shown to solve the repulsive Keller–Segel model

$$\begin{cases} \partial_t n = \frac{2d_2\gamma}{1+2\gamma} \Delta_x n + \frac{2d_2}{1+2\gamma} \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = \frac{d_2}{1+2\gamma} \Delta_x u + \frac{\sigma(1+2\gamma)}{2d_2} n u - \frac{\mu(1+2\gamma)}{2d_2} u \end{cases}.$$

In particular, the choice of a velocity field $v \in \mathbb{R}^N$ such that

$$|v|^2 = \frac{2(2\gamma(\sigma - \mu) + \mu)}{2\gamma - 1}$$

yields a “pure” traveling profile for the chemical substance, namely,

$$u(\xi) = n(\xi)^{-\gamma} e^{-\frac{1+2\gamma}{2d_2} v \cdot \xi}.$$

3.2.2 | Stationary solutions: Case I

Inserting the value $C = \frac{1}{2}$ into the formula (22) gives

$$B = d_1 \left(d_2 + \frac{d_1}{2} \right). \quad (70)$$

If $d_2 = 0$, then $B = \frac{d_1^2}{2}$ and $A = 0$, hence $\theta = 0$ (cf. 21). As a result, the only (quasi-)stationary profile inherited from Section 2.2.1 is that corresponding to the subcase (I.I), namely,

$$n \equiv \frac{\mu - 2\omega}{\sigma}, \quad u(t) = \left(\sqrt{\frac{\mu - 2\omega}{\sigma}} e^{-\frac{c_0}{2d_1}} \right) e^{-\frac{\omega}{d_1} t},$$

which solves the following repulsive Keller–Segel system without cell diffusion:

$$\begin{cases} \partial_t n = 2d_1 \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = d_1 \Delta_x u + \frac{\sigma}{2d_1} nu - \frac{\mu}{2d_1} u \end{cases} .$$

Our next purpose is to investigate how the production–degradation mechanism of the repulsive Keller–Segel system should be modified so as to inherit the richer stationary dynamics derived from the hydrodynamical Schrödinger–Doebner–Goldin model in Section 2.2.1. On one hand, it is clear from (54) that the condition $d_2 = 0$ leads to the only choice $\gamma = -\frac{1}{2}$. On the other hand, at least one of the relations stated in (53) must be removed in order to find nontrivial states. The price to pay in our search for richer steady-state dynamics is that the repulsive Keller–Segel model now becomes either

$$\begin{cases} \partial_t n = \left(d_1 - \frac{1}{2\Lambda}\right) \Delta_x n + \frac{1}{\Lambda} \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = \frac{1}{2\Lambda} \Delta_x u - \mu \Lambda u + \left(\sigma \Lambda n - \frac{4d_1^2 \Lambda^2 - 1}{8\Lambda} |\nabla_x \log(n)|^2\right) u \end{cases} ,$$

when

$$A = \frac{2d_1 \Lambda - 1}{4\Lambda^2}, \quad B = \frac{d_1^2}{2},$$

and where $\Lambda > \frac{1}{2d_1}$ for consistency, or

$$\begin{cases} \partial_t n = 2d_1 \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = d_1 \Delta_x u - \frac{\mu}{2d_1} u + \frac{1}{2d_1} \left(\sigma n - A \frac{\Delta_x n}{n}\right) u \end{cases} ,$$

when

$$B = \frac{1}{8\Lambda^2} = \frac{d_1^2}{2} \iff \Lambda = \frac{1}{2d_1},$$

and where $A > 0$ (for consistency with the general condition $A + d_1 d_2 > 0$ providing explicit solutions in Section 2.2.1). Under these conditions, the study carried out in Section 2.2.1 applies to provide nontrivial stationary states $n = \rho$ by just making $d_2 = 0$ in the expressions for κ_{\pm} (cf. 28) and δ_0 (cf. 36). Also, the chemical concentration is given by

$$u(t) = e^{-\frac{\omega t}{d_1}},$$

as follows from (59).

We now focus on the general case $d_2 > 0$. On one hand, by compatibility of (70) with the expression for B deduced in (63), γ only admits two possible values:

$$\gamma_+ = -\frac{d_1}{2(d_1 + d_2)}, \quad \gamma_- = -\frac{d_1 + 2d_2}{2(d_1 + d_2)}. \quad (71)$$

On the other hand, the requirement (66) restricts (71) to the only choice $\gamma = \gamma_+$, and thus, $\Lambda = \frac{1}{2(d_1 + d_2)}$. By (63), we now find that $A = -d_1 d_2$, and therefore, we can only recover the constant state from Section 2.2.1:

$$n \equiv \frac{\mu - 2\omega}{\sigma}, \quad u(t) = \left(\left(\frac{\mu - 2\omega}{\sigma} \right)^{\frac{d_1}{2(d_1 + d_2)}} e^{-\frac{c_0}{2(d_1 + d_2)}} \right) e^{-\frac{\omega}{d_1 + d_2} t},$$

as a solution to the following repulsive Keller–Segel system:

$$\begin{cases} \partial_t n = (d_1 + d_2) \Delta_x n + 2(d_1 + d_2) \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = (d_1 + d_2) \Delta_x u + \frac{\sigma}{2(d_1 + d_2)} nu - \frac{\mu}{2(d_1 + d_2)} u \end{cases} .$$

Again, a modification of the production–degradation mechanism in the Keller–Segel model is required if we are interested in importing the whole steady-state dynamics deduced in Section 2.2.1. As before, we are called to relax the condition (63)

in such a way that either the term $\frac{\Delta_x n}{n}$ or $\frac{|\nabla_x n|^2}{n^2}$ is retained in Equation (52). By first keeping the condition on B , we are led through (70) to $\gamma = \gamma_+$ as given in (71), so that the couples

$$n_{\pm} = \rho_{\pm}, \quad u(t) = e^{-\frac{\omega t}{d_1 + d_2}},$$

with ρ_{\pm} as in Section 2.2.1, solve the transformed Keller–Segel system

$$\begin{cases} \partial_t n = 2(d_1 + d_2) \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = (d_1 + d_2) \Delta_x u - \frac{\mu}{2(d_1 + d_2)} u + \frac{1}{2(d_1 + d_2)} \left(\sigma n - (A + d_1 d_2) \frac{\Delta_x n}{n} \right) u \end{cases}.$$

If on the contrary we skip the condition on B given in (63) and retain that on A , we find

$$A + d_1 d_2 = \frac{d_2}{(1 + 2\gamma)^2} (d_1 + 2\gamma(d_1 + d_2)) \geq 0$$

according to (66), and thus, the study carried out in Section 2.2.1 applies to provide nontrivial stationary states $n_{\pm} = \rho_{\pm}$ to the following nonstandard repulsive Keller–Segel model:

$$\begin{cases} \partial_t n = \left(d_1 + \frac{2d_2\gamma}{1+2\gamma} \right) \Delta_x n + \frac{2d_2}{1+2\gamma} \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = \frac{d_2}{1+2\gamma} \Delta_x u - \frac{\mu(1+2\gamma)}{2d_2} u + \frac{1}{4d_2} \left(2\sigma(1+2\gamma)n - \frac{(d_1+d_2)^2(1+2\gamma)^2 - d_2^2}{1+2\gamma} |\nabla_x \log(n)|^2 \right) u \end{cases},$$

with

$$u_{\pm}(t, x) = n_{\pm}(x) e^{-\frac{d_1 + 2(d_1 + d_2)\gamma}{2d_2} t - \frac{(1+2\gamma)\omega t}{d_2}}. \quad (72)$$

3.2.3 | Stationary solutions: Case II

We consider now the relation (37) with $C = \frac{1}{2}$, which gives

$$A + 2B = d_1(d_1 + d_2). \quad (73)$$

In this situation, the case $d_2 = 0$ can be disregarded as it leads to $B = \frac{d_1^2}{2}$ (by just combining 55 and 73), and consequently to $A = 0$, which contradicts the fact that $A + d_1 d_2 \neq 0$ in Section 2.2.2. As before, a modification of the production–degradation mechanism in the Keller–Segel system is required in order to incorporate to the model the stationary dynamics deduced in Section 2.2.2. Indeed, one of the two relations stated in (53) must be removed in order to give sense to the analysis there performed. Consequently, if only the condition on A is retained, we find

$$B = \frac{2d_1\Lambda(2d_1\Lambda - 1) + 1}{8\Lambda^2},$$

by virtue of (73) and the fact that $\gamma = -\frac{1}{2}$. Hence, the couple

$$n = \rho, \quad u(t, x) = n(x) e^{-\frac{2d_1\Lambda - \sqrt{2d_1\Lambda(2d_1\Lambda - 1)} + 1}{2\sqrt{2d_1\Lambda(2d_1\Lambda - 1)}} t - \frac{2\Lambda\omega t}{\sqrt{2d_1\Lambda(2d_1\Lambda - 1)} + 1}},$$

with ρ as given in Section 2.2.2, solves the generalized Keller–Segel model

$$\begin{cases} \partial_t n = \left(d_1 - \frac{1}{2\Lambda} \right) \Delta_x n + \frac{1}{\Lambda} \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = \frac{1}{2\Lambda} \Delta_x u - \mu \Lambda u + \left(\sigma \Lambda n - \frac{d_1(2d_1\Lambda - 1)}{4} |\nabla_x \log(n)|^2 \right) u \end{cases},$$

provided that $\Lambda > \frac{1}{2d_1}$ (see Equations (51)–(52)). On the contrary, if the condition retained in (53) is that on B , we then find from (73)

$$A = d_1^2 - \frac{1}{4\Lambda^2},$$

and now, the repulsive Keller–Segel system is given by

$$\begin{cases} \partial_t n = \left(d_1 - \frac{1}{2\Lambda}\right) \Delta_x n + \frac{1}{\Lambda} \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = \frac{1}{2\Lambda} \Delta_x u - \mu \Lambda u + \left(\sigma \Lambda n - \frac{d_1(2d_1\Lambda - 1)}{2} \frac{\Delta_x n}{n}\right) u \end{cases},$$

provided that $\Lambda > \frac{1}{2d_1}$. Under these conditions, the study carried out in Section 2.2.2 applies again to provide nontrivial stationary states $n = \rho$ with just $d_2 = 0$ in the expressions for κ_{\pm} (cf. 28) and δ_0 (cf. 36). Also,

$$u(t, x) = n(x)^{-\left(d_1\Lambda - \frac{1}{2}\right)} e^{-2\Lambda\omega t}. \quad (74)$$

Let now $d_2 > 0$. By imposing compatibility of (73) with the expressions for A and B found in (63), γ is shown to admit two possible values:

$$\gamma_+ = -\frac{d_1}{2(d_1 + d_2)}, \quad \gamma_- = -\frac{1}{2}.$$

Clearly, γ_- violates the necessary condition (62); hence, the only valid choice for γ is γ_+ (cf. 66). As consequence, the first relation in (63) yields $A = -d_1 d_2$, which again contradicts the hypothesis of Section 2.2.2.

Let us then correct the production–degradation Keller–Segel operator in order to recover the explicit solutions found in 2.2.2. If only the condition on B is retained in (63), then we arrive at

$$A = \frac{d_1(d_1 + d_2)(1 + 2\gamma)^2 + 4d_2^2\gamma(1 + \gamma)}{(1 + 2\gamma)^2},$$

by virtue of (73). As consequence, the profiles $n_{\pm} = \rho_{\pm}$ given in Section 2.2.2 are shown to solve the modified Keller–Segel system

$$\begin{cases} \partial_t n = \left(d_1 + \frac{2d_2\gamma}{1+2\gamma}\right) \Delta_x n + \frac{2d_2}{1+2\gamma} \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = \frac{d_2}{1+2\gamma} \Delta_x u - \frac{\mu(1+2\gamma)}{2d_2} u + \frac{1}{2d_2} \left(\sigma(1 + 2\gamma)n - (d_1 + d_2)(d_1 + 2(d_1 + d_2)\gamma) \frac{\Delta_x n}{n}\right) u \end{cases},$$

along with the profiles (72) for u_{\pm} . On the contrary, if the condition retained in (63) is that on A , then

$$B = \frac{d_1(d_1 + d_2)(1 + 2\gamma)^2 - 2d_2\gamma(d_2 - d_1(1 + 2\gamma))}{2(1 + 2\gamma)^2},$$

and the repulsive Keller–Segel system now becomes

$$\begin{cases} \partial_t n = \left(d_1 + \frac{2d_2\gamma}{1+2\gamma}\right) \Delta_x n + \frac{2d_2}{1+2\gamma} \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = \frac{d_2}{1+2\gamma} \Delta_x u - \frac{\mu(1+2\gamma)}{2d_2} u + \frac{1}{4d_2} \left(2\sigma(1 + 2\gamma)n - (d_1 + d_2)(d_1 + 2(d_1 + d_2)\gamma) |\nabla_x \log(n)|^2\right) u \end{cases}.$$

3.2.4 | Stationary solutions: Case III

In this case, the following relations must be fulfilled:

$$A = -d_1 d_2, \quad B \neq \frac{d_1(d_1 + 2d_2)}{2}. \quad (75)$$

By assuming $d_2 = 0$, we find $0 = A = \sqrt{2B} (d_1 - \sqrt{2B})$ and $B = \frac{1}{8\Lambda^2} > 0$, as deduced in (55). As consequence B must take the value $B = \frac{d_1^2}{2}$, which contradicts (75). Proceeding as in the previous cases, if we retain the condition on A stated in (53) and release that on B , then we are led to the choice $\Lambda = \frac{1}{2d_1}$. The profiles $n = \rho$ given in Section 2.2.3, along with the chemical densities

$$u(t, x) = n(x) e^{-\frac{d_1 - \sqrt{2B}}{2\sqrt{2B}} - \frac{\omega t}{\sqrt{2B}}},$$

are thus shown to solve the modified Keller–Segel system

$$\begin{cases} \partial_t n = 2d_1 \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = d_1 \Delta_x u - \frac{\mu}{2d_1} u + \frac{1}{2d_1} \left(\sigma n + \left(\frac{d_1^2}{2} - B \right) |\nabla_x \log(n)|^2 \right) u \end{cases},$$

provided that $B < \frac{d_1^2}{2}$ (see Equations 51–52). If on the contrary we just retain the condition on B in (53), then the couple (n, u) solves the generalized Keller–Segel model

$$\begin{cases} \partial_t n = \left(d_1 - \frac{1}{2\Lambda} \right) \Delta_x n + \frac{1}{\Lambda} \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = \frac{1}{2\Lambda} \Delta_x u - \mu \Lambda u + \frac{1}{4\Lambda} \left(4\sigma \Lambda^2 n + (2d_1 \Lambda - 1) \frac{\Delta_x n}{n} \right) u \end{cases},$$

provided that $\Lambda \neq \frac{1}{2d_1}$, with n as in Section 2.2.3 and u given by (74).

Let us now consider $d_2 > 0$. On one hand, the first identity in (63) leads to

$$-d_1 d_2 = A = \frac{2d_2 \gamma (d_2 - d_1(1 + 2\gamma))}{(1 + 2\gamma)^2},$$

or equivalently

$$\gamma = -\frac{d_1}{2(d_1 + d_2)}. \quad (76)$$

On the other hand, the second identity in (63) straightforwardly leads to

$$B = \frac{d_1(d_1 + 2d_2)}{2},$$

which again contradicts (75). In conclusion, the production–degradation term of the Keller–Segel system must be rectified again so as to reproduce the steady-state dynamics exhibited in Section 2.2.3. By just retaining the condition on A stated in (63), we arrive at the expression for γ given in (76), which in turn leads to the diffusionless system

$$\begin{cases} \partial_t n = 2(d_1 + d_2) \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = (d_1 + d_2) \Delta_x u - \frac{\mu}{2(d_1 + d_2)} u + \frac{1}{2(d_1 + d_2)} \left(\sigma n + \left(\frac{d_1(d_1 + d_2)}{2} - B \right) |\nabla_x \log(n)|^2 \right) u \end{cases}, \quad (77)$$

provided that $B \neq \frac{d_1(d_1 + d_2)}{2}$. Here, the couples $(n_{\pm} = \rho_{\pm}, u(t) = e^{-\frac{\omega t}{d_1 + d_2}})$ solve (77), with ρ_{\pm} as given in Section 2.2.3. Finally, if only the condition on B is retained in (63), then the couples (n_{\pm}, u_{\pm}) solve

$$\begin{cases} \partial_t n = \left(d_1 + \frac{2d_2 \gamma}{1 + 2\gamma} \right) \Delta_x n + \frac{2d_2}{1 + 2\gamma} \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = \frac{d_2}{1 + 2\gamma} \Delta_x u - \frac{\mu(1 + 2\gamma)}{2d_2} u + \frac{1}{2} \left(\frac{\sigma(1 + 2\gamma)}{d_2} n + \frac{d_1 + 2(d_1 + d_2)\gamma}{1 + 2\gamma} \frac{\Delta_x n}{n} \right) u \end{cases},$$

provided that $\gamma \neq -\frac{d_1}{2(d_1 + d_2)}$. Here, ρ_{\pm} are as in Section 2.2.3 and u_{\pm} are those given by (72).

Remark 2. When neither the constraint on A nor that on B in (53) or (63) is retained, a peculiar situation arises that deserves to be highlighted. Indeed, when the coefficients affecting $\frac{\Delta_x n}{n}$ and $|\nabla_x \log(n)|^2$ in (52) have opposite sign, namely,

$$d_1 \gamma + \frac{\gamma^2}{\Lambda} + A \Lambda = \frac{\gamma^2}{2\Lambda} - B \Lambda, \quad (78)$$

then both terms combine to give $\Delta_x \log(n)$. On one hand, if $d_2 = 0$, then $\gamma = -\frac{1}{2}$, and Equation (78) is translated into

$$\Lambda_{\pm} = \frac{d_1 \pm \sqrt{d_1^2 - 2(A+B)}}{4(A+B)}. \quad (79)$$

On the other hand, if $d_2 > 0$, then $\Lambda = \frac{1+2\gamma}{2d_2}$, and Equation (78) becomes

$$\gamma_{\pm} = \frac{-d_1 d_2 - 2(A+B) \pm d_2 \sqrt{d_1^2 - 2(A+B)}}{2(2(A+B) + 2d_1 d_2 + d_2^2)}. \quad (80)$$

Later on, we analyze the only admissible situations in the perspective of Section 2.

Case I: If $d_2 = 0$, then $B = \frac{d_1^2}{2}$ and (79) reads

$$\Lambda_{\pm} = \frac{d_1 \pm \sqrt{-2A}}{2(d_1^2 + 2A)}, \quad A \leq 0.$$

Thus, the corresponding Keller–Segel model is written as

$$\begin{cases} \partial_t n = D_1 \Delta_x n + \chi \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = D_2 \Delta_x u - \beta u + (\lambda n + \phi \Delta_x \log(n)) u \end{cases},$$

with

$$D_1 = d_1 - D_2, \quad D_2 = \frac{d_1^2 + 2A}{d_1 + \sqrt{-2A}}, \quad \chi = 2D_2, \quad \beta = \frac{\mu}{2D_2}, \quad \lambda = \frac{\sigma}{2D_2}, \quad \phi = \frac{(A + d_1^2)\sqrt{-2A} - d_1 A}{2(d_1^2 + 2A)},$$

provided that $-\frac{d_1^2}{2} < A \leq 0$.

If $d_2 > 0$, then $B = \frac{d_1(d_1 + 2d_2)}{2}$, and (80) becomes

$$\gamma_{\pm} = \frac{-2(A + d_1 d_2) - d_1(d_1 + d_2) \pm d_2 \sqrt{-2(A + d_1 d_2)}}{2(2(A + d_1 d_2) + (d_1 + d_2)^2)},$$

which is only meaningful for values of A such that $A + d_1 d_2 \leq 0$. Consequently, the solutions constructed in Section 2 are not transferable to this situation.

Case II: If $d_2 = 0$, then $A + 2B = d_1^2$ and (79) reads

$$\Lambda_{\pm} = \frac{d_1 \pm \sqrt{-A}}{2(d_1^2 + A)}, \quad A \leq 0.$$

Thus, the corresponding Keller–Segel models is written as

$$\begin{cases} \partial_t n = D_1 \Delta_x n + \chi \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = D_2 \Delta_x u - \beta u + (\lambda n + \phi \Delta_x \log(n)) u \end{cases},$$

with

$$D_1 = d_1 - D_2, D_2 = \frac{d_1^2 + A}{d_1 + \sqrt{-A}}, \chi = 2D_2, \beta = \frac{\mu}{2D_2}, \lambda = \frac{\sigma}{2D_2}, \phi = \frac{d_1(d_1\sqrt{-A} - A)}{2(d_1^2 + A)},$$

provided that $-d_1^2 < A \leq 0$.

If $d_2 > 0$, then $A + 2B = d_1(d_1 + d_2)$, and (80) becomes

$$\gamma_{\pm} = \frac{-(A + d_1d_2) - d_1(d_1 + d_2) \pm d_2\sqrt{-(A + d_1d_2)}}{2(A + d_1d_2 + (d_1 + d_2)^2)}.$$

As in the preceding case, these expressions are only meaningful for values of A such that $A + d_1d_2 \leq 0$. Consequently, the solutions constructed in Section 2 are not transferable to this situation.

Case III: If $d_2 = 0$, then $A = 0$ and (79) reads

$$\Lambda_{\pm} = \frac{d_1 \pm \sqrt{d_1^2 - 2B}}{4B}, B < \frac{d_1^2}{2}.$$

Thus, the corresponding Keller–Segel model is written as

$$\begin{cases} \partial_t n = D_1 \Delta_x n + \chi \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = D_2 \Delta_x u - \beta u + (\lambda n + \phi \Delta_x \log(n)) u \end{cases},$$

with

$$D_1 = d_1 - D_2, D_2 = \frac{2B}{d_1 + \sqrt{d_1^2 - 2B}}, \chi = 2D_2, \beta = \frac{\mu}{2D_2}, \lambda = \frac{\sigma}{2D_2}, \phi = \frac{\sqrt{d_1^2 - 2B}}{2},$$

provided that $0 \leq B < \frac{d_1^2}{2}$.

If $d_2 > 0$, then $A = -d_1d_2$, and (80) becomes

$$\gamma_{\pm} = \frac{d_1d_2 - 2B \pm d_2\sqrt{d_1^2 + 2d_1d_2 - 2B}}{2(d_2^2 + 2B)}, B < \frac{d_1(d_1 + 2d_2)}{2}.$$

Thus, the corresponding Keller–Segel model is written as

$$\begin{cases} \partial_t n = D_1 \Delta_x n + \chi \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = D_2 \Delta_x u - \beta u + (\lambda n + \phi \Delta_x \log(n)) u \end{cases},$$

with

$$D_1 = d_1 + 2\gamma_+ D_2, D_2 = \frac{d_2^2 + 2B}{d_1 + d_2 + \sqrt{\Theta}}, \chi = 2D_2, \beta = \frac{\mu}{2D_2}, \lambda = \frac{\sigma}{2D_2}, \phi = \frac{\Theta + (d_1 + d_2)\sqrt{\Theta}}{2(d_1 + d_2 + \sqrt{\Theta})},$$

provided that $-\frac{d_2^2}{2} \leq B < \frac{d_1(d_1 + 2d_2)}{2}$, where we denoted

$$\Theta = d_1^2 + 2d_1d_2 - 2B.$$

4 | CONCLUDING REMARKS

Starting from the most general form of a Keller–Segel-type model,

$$\begin{cases} \partial_t \rho = d_1 \Delta_x \rho - \nabla_x \cdot (\rho \nabla_x c) \\ \partial_t c = d_2 \Delta_x c + \mu - \sigma \rho + A \frac{\Delta_x \rho}{\rho} + B \frac{|\nabla_x \rho|^2}{\rho^2} + C |\nabla_x c|^2 \end{cases}$$

obtained as the hydrodynamic system associated with the cubic Schrödinger–Doebner–Goldin equation

$$i \partial_t \psi + \frac{1}{2} \Delta_x \psi + (\sigma n - \mu) \psi = \left(a \frac{\Delta_x n}{n} - b \frac{|\nabla_x n|^2}{n^2} - d_2 \nabla_x \cdot \left(\frac{J}{n} \right) + c \frac{|J|^2}{n^2} \right) \psi,$$

with

$$a = -A + \frac{1}{4} + i \frac{d_1}{2}, \quad b = B + \frac{1}{8}, \quad c = C - \frac{1}{2}$$

and

$$\psi = \sqrt{\rho} e^{ic},$$

we derived the standard form of the Keller–Segel model with singular sensitivity accounting for negative chemotaxis:

$$\begin{cases} \partial_t n = D_1 \Delta_x n + 2D_2 \nabla_x \cdot (n \nabla_x \log(u)) \\ \partial_t u = D_2 \Delta_x u + \frac{\sigma}{2D_2} n u - \frac{\mu}{2D_2} u \end{cases},$$

via an adequate change of variables, provided that some specific relations are fulfilled among the original chemotactic parameters A, B, C and diffusion rates $d_1, d_2 \geq 0$. Here,

$$D_2 = \begin{cases} \frac{d_2}{1+2\gamma} & \text{if } d_2 > 0, \gamma \geq -\frac{d_1}{2(d_1+d_2)} \\ \sqrt{2B} & \text{if } d_2 = 0, 0 \leq B \leq \frac{d_1^2}{2} \end{cases}, \quad D_1 = \begin{cases} d_1 + 2d_2\gamma & \text{if } d_2 > 0, \gamma \geq -\frac{d_1}{2(d_1+d_2)} \\ d_1 - D_2 & \text{if } d_2 = 0, 0 \leq B \leq \frac{d_1^2}{2} \end{cases}.$$

Then, several families of explicit solutions to a number of variants of the latter system (namely, those including terms proportional to either $\frac{\Delta_x n}{n}$ or $|\nabla_x \log(n)|^2$ in the chemical production–degradation operator) have been constructed from their counterparts on the former. More precisely, sech-shaped traveling wave solutions were found (cf. 68–69) as well as a range of stationary cell densities involving trigonometric and hyperbolic functions, along with other stationary profiles with second-degree polynomial decay. Besides, the corresponding chemical concentrations are shown to decay exponentially in time.

ACKNOWLEDGEMENTS

The author is partially supported by MINECO-Feder (Spain), Research Grant RTI2018-098850-B-I00, as well as by Junta de Andalucía (Spain), Project PY18-RT-2422 and B-FQM-580-UGR. Funding for open access charge: Universidad de Granada/CBUA.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

ORCID

José Luis López  <https://orcid.org/0000-0002-2878-6557>

REFERENCES

1. Keller EF, Segel LA. Model for chemotaxis. *J Theor Biol.* 1971;30(2):225-234.
2. Keller EF, Segel LA. Traveling bands of chemotactic bacteria: A theoretical analysis. *J Theor Biol.* 1971;30:235-248.
3. Adler J. Chemotaxis in bacteria. *Annu Rev Biochem.* 1975;44:341-356.
4. Tso W-W, Adler J. Negative chemotaxis in *Escherichia coli*. *J Bacteriology.* 1974;118(2):560-576.

5. Bellomo N, Bellouquid A, Tao Y, Winkler M. Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues. *Math Models Meth Appl Sci*. 2015;25(9):1663-1763.
6. Horstmann D. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences I. *Jahresberichte der DMV*. 2004;105(3):103-165.
7. Horstmann D. From 1970 until present: The Keller-Segel model in chemotaxis and its consequences II. *Jahresberichte der DMV*. 2004;106(2):51-69.
8. Tao Y, Wang L, Wang Z-A. Large-time behavior of a parabolic-parabolic chemotaxis model with logarithmic sensitivity in one dimension. *Discrete Contin Dyn Syst - Ser B*. 2013;18(3):821-845.
9. Ahn J, Kang K, Lee J. Global well-posedness of logarithmic Keller-Segel type systems. *J Differ Equ*. 2021;287:185-211.
10. Blanchet A. On the parabolic-elliptic Patlak-Keller-Segel system in dimension 2 and higher. *Séminaire Laurent Schwartz—EDP et Applications*. 2011-2012;8:1-26.
11. Blanchet A, Dolbeault J, Perthame B. Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. *Electronic J Differ Equ*. 2006;2006(44):1-33.
12. Calvez V, Corrias L, Ebde MA. Blow-up, concentration phenomenon and global existence for the Keller-Segel model in high dimension. *Comm PDE*. 2012;37(4):561-584.
13. Del Pino M, Wei J. Collapsing steady states of the Keller-Segel system. *Nonlinearity*. 2006;19:661-684.
14. Gajewski H, Zacharias K. Global behaviour of a reaction-diffusion system modelling chemotaxis. *Math Nachr*. 1998;195:77-114.
15. Hillen T, Painter K. Global existence for a parabolic chemotaxis model with prevention of overcrowding. *Adv Appl Math*. 2001;26(4):280-301.
16. Huang Y, Bertozzi A. Self-similar blowup solutions to an aggregation equation in \mathbb{R}^N . *SIAM J Appl Math*. 2010;70(7):2582-2603.
17. Liu D. Global solutions in a fully parabolic chemotaxis system with singular sensitivity and nonlinear signal production. *J Math Phys*. 2020;61:21503.
18. Liu D, Tao Y. Boundedness in a chemotaxis system with nonlinear signal production. *Appl Math J Chinese Univ*. 2016;31(4):379-388.
19. Souplet P, Winkler M. Blow-up profiles for the parabolic-elliptic Keller-Segel system in dimensions $n \geq 3$. *Comm Mth Phys*. 2019;367:665-681.
20. Winkler M. Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source. *Comm Partial Differ Equ*. 2020;35:1516-1537.
21. Winkler M. The two-dimensional Keller-Segel system with singular sensitivity and signal absorption: global large-data solutions and their relaxation properties. *Math Models Meth Appl Sci*. 2016;26(05):987-1024.
22. Winkler M. Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system. *J Math Pures et Appliquées*. 2013;100(5):748-767.
23. Winkler M. Global solution in a fully parabolic chemotaxis system with singular sensitivity. *J Math Meth Appl Sci*. 2011;34:176-190.
24. Yang Y, Chen H, Liu W. On existence of global solutions and blow-up to a system of reaction-diffusion equation modelling chemotaxis. *SIAM J Math Anal*. 2001;33(4):763-785.
25. Wang G, Wei J. Steady state solutions of a reaction-diffusion system modeling chemotaxis. *Math Nachr*. 2002;233-234:221-236.
26. Zhang Q. Global bounded solutions to a Keller-Segel system with singular sensitivity. *Appl Math Lett*. 2020;107:106397.
27. Zhuang M, Wang W, Zheng S. Boundedness in a fully parabolic chemotaxis system with logistic-type source and nonlinear production. *Nonlinear Anal RWA*. 2019;47:473-483.
28. Arias M, Campos J, Soler J. Cross-diffusion and traveling waves in porous-media flux-saturated Keller-Segel models. *Math Models Meth Appl Sci*. 2018;28:2103-2129.
29. Bramburger JJ. Exact minimum speed of traveling waves in a Keller-Segel model. *Appl Math Lett*. 2021;111:106594.
30. Calvez V, Perthame B, Yasuda S. Traveling wave and aggregation in a flux-limited Keller-Segel model. *Kinet Relat Models*. 2018;11:891-909.
31. Campos J, García C, Soler J. Kinks and solitons in linear and nonlinear-diffusion Keller-Segel type models with logarithmic sensitivity; 2021.
32. Li T, Park J. Traveling waves in a chemotaxis model with logistic growth. *Discrete Contin Dyn Syst - Ser B*. 2019;24:6465-6480.
33. Lin T, Ni W, Takagi I. Large amplitude stationary solutions to a chemotaxis system. *J Differ Equ*. 1988;72:1-27.
34. Nadine G, Perthame B, Ryzhik L. Traveling waves for the Keller-Segel system with Fisher birth terms. *Interf Free Bound*. 2008;10:517-538.
35. Salako RB, Shen W. Existence of traveling wave solutions of parabolic-parabolic chemotaxis systems. *Nonlinear Anal RWA*. 2018;42:93-119.
36. Schwetlick H. Traveling waves for chemotaxis-systems. *PAMM*. 2003;3:476-478.
37. Wang Z-A. Mathematics of traveling waves in chemotaxis -review paper. *Discrete Contin Dyn Syst - Ser B*. 2013;18:601-641.
38. Li T, Wang Z-A. Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis. *SIAM J Appl Math*. 2009;70(5):1522-1541.
39. Wang Z-A. Wavefront of an angiogenesis model. *Discrete Contin Dyn Syst - Ser B*. 2012;17(8):2849-2860.
40. Alejo MA, López JL. Modeling chemotaxis with nonstandard production/degradation mechanisms from Doebner-Goldin theory: existence of solitary waves. *Physica D*. 2021;426:132989.
41. López JLA. Quantum approach to Keller-Segel dynamics via a dissipative nonlinear Schrödinger equation. *Discrete Contin Dyn Syst - Ser A*. 2021;41:2601-2617.
42. López JL. On nonstandard chemotactic dynamics with logistic growth induced by a modified complex Ginzburg-Landau equation. *Studies Appl Math*. 2022;148(1):248-269.

43. Doebner HD, Goldin G. On a general nonlinear Schrödinger equation admitting diffusion currents. *Phys Lett A*. 1992;162:397-401.
44. Doebner HD, Goldin G. Properties of nonlinear Schrödinger equations associated with diffeomorphism group representations. *J Phys A: Math Gen*. 1994;7:1771-1780.
45. Guerrero P, López JL, Montejo-Gámez J, Nieto J. Wellposedness of a nonlinear, logarithmic Schrödinger equation of Doebner–Goldin type modeling quantum dissipation. *J Nonlinear Sci*. 2012;22:631-663.
46. Wang Z-A, Zhao K. Global dynamics and diffusion limit of a one-dimensional repulsive chemotaxis model. *Comm Pure Appl Analysis*. 2013;12(6):3027-3046.
47. Othmer H, Stevens A. Aggregation, blowup and collapse: The ABC's of taxis in reinforced random walks. *SIAM J Appl Math*. 1997;57:1044-1081.
48. Levine HA, Sleeman BD. A system of reaction diffusion equations arising in the theory of reinforced random walks. *SIAM J Appl Math*. 1997;57:683-730.
49. Cieślak T, Laurençot P, Morales-Rodrigo C. Global existence and convergence to steady states in a chemorepulsion system. *Parabolic and Navier-Stokes Equations*, Vol. 81: Banach Center Publications; 2008:105-117.
50. Lin K, Xiang T. Strong damping of chemo-repulsion prevents blow-up. *J Math Phys*. 2021;62:41508.
51. Chen L, Kong F, Wang Q. Global and exponential attractor of the repulsive Keller–Segel model with logarithmic sensitivity. *European J Appl Math*. 2021;32:599-617.
52. Kardar M, Parisi G, Zhang Y-C. Dynamic scaling of growing interfaces. *Phys Rev Lett*. 1986;56:889-892.

How to cite this article: López JL. A repertoire of repulsive Keller–Segel models with logarithmic sensitivity: Derivation, traveling waves, and quasi-stationary dynamics. *Math Meth Appl Sci*. 2022;1-25. doi:10.1002/mma.8638