MAPS PRESERVING TWO-SIDED ZERO PRODUCTS ON BANACH ALGEBRAS

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ABSTRACT. Let A and B be Banach algebras with bounded approximate identities and let $\Phi : A \to B$ be a surjective continuous linear map which preserves two-sided zero products (i.e., $\Phi(a)\Phi(b) = \Phi(b)\Phi(a) = 0$ whenever $ab = ba = 0$). We show that Φ is a weighted Jordan homomorphism provided that A is zero product determined and weakly amenable. These conditions are in particular fulfilled when A is the group algebra $L^1(G)$ with G any locally compact group. We also study a more general type of continuous linear maps $\Phi : A \to B$ that satisfy $\Phi(a)\Phi(b) + \Phi(b)\Phi(a) = 0$ whenever $ab = ba = 0$. We show in particular that if Φ is surjective and A is a C^*-algebra, then Φ is a weighted Jordan homomorphism.

1. INTRODUCTION

Let A and B be Banach algebras. We will say that a linear map $\Phi : A \to B$ *preserves two-sided zero products* if for all $a, b \in A$,

$$ab = ba = 0 \implies \Phi(a)\Phi(b) = \Phi(b)\Phi(a) = 0. \tag{1.1}$$

Obvious examples of such maps are homomorphisms and antihomomorphisms. Their common generalizations are *Jordan homomorphisms*, i.e., linear maps $\Psi : A \to B$ satisfying

$$\Psi(a \circ b) = \Psi(a) \circ \Psi(b) \quad \forall a, b \in A,$$

where $a \circ b$ stands for the Jordan product $ab + ba$. Under the mild assumption that the centre of B does not contain nonzero nilpotent elements, every Jordan homomorphism from A onto B also preserves two-sided zero products [7, Lemma 7.20]. Next we recall that a linear map $W : B \to B$ is called a *centralizer* if

$$W(ab) = W(a)b = aW(b) \quad \forall a, b \in B.$$

We say that Φ is a *weighted Jordan homomorphism* if there exist an invertible centralizer W of B and a Jordan homomorphism Ψ from A to B such that $\Phi = W\Psi$. Observe that Φ preserves two-sided zero products if and only if Ψ does. We also remark that, by the closed graph theorem, every centralizer W is automatically continuous if B is a faithful algebra (i.e., $bB = Bb = \{0\}$ implies $b = 0$), and so, in this case, Φ is continuous if and only if Ψ is.

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Is every surjective continuous linear map $\Phi : A \to B$ which preserves two-sided zero products a weighted Jordan homomorphism? This question is similar to but, as it turns out, more difficult than a more thoroughly studied question of describing zero products preserving continuous linear maps (see the most recent publications [7, 11, 12, 14, 15] for historical remarks and references). It is known that the answer is positive if either A and B are C^*-algebras [3, Theorem 3.3] or if $A = L^1(G)$ and $B = L^1(H)$ where G and H are locally compact groups with $G \in$ [SIN] (i.e., G has a base of compact neighborhoods of the identity that is invariant under all inner automorphisms) [6, Theorem 3.1 (i)]. In fact, [3, Theorem 3.3] does not require that Φ satisfies (1.1) but only that for all $a,b \in A$,

$$ab = ba = 0 \implies \Phi(a) \circ \Phi(b) = 0. \tag{1.2}$$

This condition was also considered in the recent algebraic paper [8]. Observe also that it is more general than the condition that Φ preserves zero Jordan products ($a \circ b = 0$ implies $\Phi(a) \circ \Phi(b) = 0$) studied in [9].

The goal of this paper is to generalize and unify the aforementioned results from [3] and [6]. Our approach is based on the concept of a *zero product determined* Banach algebra. These are Banach algebras A with the property that every continuous bilinear functional $\varphi : A \times A \to \mathbb{C}$ satisfying $\varphi(a,b) = 0$ whenever $ab = 0$ is of the form $\varphi(a,b) = \tau(ab)$ for some continuous linear functional τ on A. We refer to the recent book [7] for a survey of these algebras. Let us for now only mention that they form a fairly large class of Banach algebras whose main representatives, C^*-algebras and group algebras of locally compact groups, are also weakly amenable Banach algebras having bounded approximate identities.

In Section 2, we show that the answer to our question is positive, i.e., a surjective continuous linear map $\Phi : A \to B$ which preserves two-sided zero products is a weighted Jordan homomorphism, provided that A is zero product determined and weakly amenable, and, additionally, both A and B have bounded approximate identities (Theorem 2.5). This in particular implies that that the restriction in [6, Theorem 3.1 (i)] that $G \in$ [SIN] is redundant (Corollary 2.7).

Section 3 is devoted to condition (1.2). We show that [3, Theorem 3.3] still holds if B is any Banach algebra with a bounded approximate identity, not only a C^*-algebra (Theorem 3.3). Our second main result regarding (1.2) considers the case where $A = \mathscr{A}(X)$ is the algebra of approximable operators (Theorem 3.4).

2. CONDITION (1.1)

Throughout, for a Banach space X, we write X^* for the dual of X and $\langle \cdot , \cdot \rangle$ for the duality between X and X^*. Let A be a Banach algebra. We turn A^* into a Banach A-bimodule by letting

$$\langle b, a \cdot \omega \rangle = \langle ba, \omega \rangle, \quad \langle b, \omega \cdot a \rangle = \langle ab, \omega \rangle \quad \forall a,b \in A,\ \forall \omega \in A^*.$$

The space of continuous derivations from A into A^* is denoted by $\mathscr{Z}^1(A,A^*)$. The main representatives of $\mathscr{Z}^1(A,A^*)$ are the so-called inner derivations. The inner derivation implemented by $\omega \in A^*$ is the map $\delta_\omega : A \to A^*$ defined by

$$\delta_\omega(a) = a \cdot \omega - \omega \cdot a \quad \forall a \in A.$$

The Banach algebra is called *weakly amenable* if every element of $\mathscr{Z}^1(A,A^*)$ is inner. For a thorough treatment of this property and an account of many interesting examples of weakly amenable Banach algebras we refer the reader to [10]. We should remark that the group algebra $L^1(G)$ of each locally compact group G and each C^*-algebra are weakly amenable [10, Theorems 5.6.48 and 5.6.77].

Our first result is a sharpening of [5, Theorem 2.7] (see also [7, Theorem 6.6]).

Theorem 2.1. *Let A be a Banach algebra, and suppose that:*

(a) *A is zero product determined;*
(b) *A has a bounded approximate identity;*
(c) *A is weakly amenable.*

Then there exists a constant $C \in \mathbb{R}^+$ such that for each continuous bilinear functional $\varphi\colon A \times A \to \mathbb{C}$ with the property that for all $a,b \in A$,

$$ab = ba = 0 \implies \varphi(a,b) = 0 \tag{2.1}$$

there exist $\sigma, \tau \in A^*$ such that

$$\|\sigma\| \le C\|\varphi\|, \quad \|\tau\| \le C\|\varphi\|,$$

and

$$\varphi(a,b) = \sigma(ab) + \tau(ba) \quad \forall a,b \in A.$$

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for A of bound M.

Since A is weakly amenable, the map $\omega \mapsto \delta_\omega$ from A^* to the Banach space $\mathscr{Z}^1(A,A^*)$ is a continuous linear surjection, and so, by the open mapping theorem, there exists a constant $N \in \mathbb{R}^+$ such that, for each $D \in \mathscr{Z}^1(A,A^*)$, there exists an $\omega \in A^*$ with

$$\|\omega\| \le N\|D\|$$

and

$$a\cdot\omega - \omega\cdot a = D(a) \quad \forall a \in A.$$

Towards the proof of the theorem, we proceed through a detailed inspection of the proof of [5, Theorem 2.7].

Define $\varphi_1, \varphi_2\colon A\times A \to \mathbb{C}$ by

$$\begin{aligned}\varphi_1(a,b) &= \tfrac{1}{2}\big[\varphi(a,b) + \varphi(b,a)\big],\\ \varphi_2(a,b) &= \tfrac{1}{2}\big[\varphi(a,b) - \varphi(b,a)\big] \quad \forall a,b \in A.\end{aligned}$$

It is clear that both φ_1 and φ_2 satisfy condition (2.1) and that

$$\|\varphi_1\| \le \|\varphi\|, \quad \|\varphi_2\| \le \|\varphi\|.$$

Since φ_1 is symmetric it follows from the last assertion of [5, Lemma 2.6] (or [7, Theorem 6.1]) that there exists a $\xi \in A^*$ such that

$$2\varphi_1(a,b) = \langle a\circ b, \xi\rangle \quad \forall a,b \in A. \tag{2.2}$$

For each $a \in A$, we observe that

$$\langle a\circ e_\lambda, \xi\rangle = 2\varphi_1(a,e_\lambda) \quad \forall \lambda \in \Lambda$$

and hence that

$$|\langle a\circ e_\lambda, \xi\rangle| \le 2M\|\varphi\|\|a\| \quad \forall \lambda \in \Lambda.$$

Taking limit and using $\lim_{\lambda\in\Lambda} a\circ e_\lambda = 2a$ we see that

$$2\langle a, \xi\rangle \le 2M\|\varphi\|\|a\|.$$

This shows that

$$\|\xi\| \le M\|\varphi\|. \tag{2.3}$$

Our next concern will be the behaviour of the skew-symmetric functional φ_2. By [4, Lemma 4.1] (or [7, Theorem 6.1]), there exists a $\psi \in A^*$ such that

$$\varphi_2(ab,c) - \varphi_2(b,ca) + \varphi_2(bc,a) = \langle abc, \psi \rangle \quad \forall a,b,c \in A.$$

The proof reveals that the functional ψ is defined by

$$\langle a, \psi \rangle = \lim_{\lambda \in \Lambda} \varphi_2(a, e_\lambda) \quad \forall a \in A,$$

so that

$$\|\psi\| \leq M\|\varphi\|. \tag{2.4}$$

By [5, Lemma 2.6] (or the proof of [7, Theorem 6.5]) , the map

$$D\colon A \to A^*, \quad \langle b, D(a) \rangle = \varphi_2(a,b) + \frac{1}{2}\langle a \circ b, \psi \rangle$$

is a continuous derivation, and clearly (using (2.4))

$$\|D\| \leq \|\varphi_2\| + \|\psi\| \leq (1+M)\|\varphi\|.$$

Consequently, there exists an $\omega \in A^*$ such that

$$\|\omega\| \leq N\|D\| \leq N(1+M)\|\varphi\| \tag{2.5}$$

and

$$a \cdot \omega - \omega \cdot a = D(a) \quad \forall a \in A,$$

and hence

$$\langle ba - ab, \omega \rangle - \varphi_2(a,b) = \tfrac{1}{2}\langle a \circ b, \psi \rangle \quad \forall a,b \in A.$$

Viewing this expression as a bilinear functional on $A \times A$, we see that the left-hand side is skew-symmetric and the right-hand side is symmetric. Therefore, both sides are zero. Thus

$$\varphi_2(a,b) = \langle ba - ab, \omega \rangle \quad \forall a,b \in A. \tag{2.6}$$

We then define

$$\sigma = \tfrac{1}{2}\xi - \omega, \quad \tau = \tfrac{1}{2}\xi + \omega.$$

From (2.3) and (2.5) we see that

$$\|\sigma\| \leq (\tfrac{1}{2}M + N + NM)\|\varphi\|,$$
$$\|\tau\| \leq (\tfrac{1}{2}M + N + NM)\|\varphi\|,$$

and from (2.2) and (2.6) we deduce that

$$\varphi(a,b) = \sigma(ab) + \tau(ba) \quad \forall a,b \in A. \qquad \square$$

We can now start our consideration of maps preserving two-sided zero products.

Lemma 2.2. *Let A be a Banach algebra, and suppose that:*

(a) *A is zero product determined;*
(b) *A has a bounded approximate identity;*
(c) *A is weakly amenable.*

Let B be a Banach algebra and let $\Phi\colon A \to B$ be a continuous linear map which preserves two-sided zero products. Then there exist:

- *a closed left ideal L of B containing $\Phi(A)$ and a continuous linear map $U\colon L \to B$,*
- *a closed right ideal R of B containing $\Phi(A)$ and a continuous linear map $V\colon R \to B$*

such that

$$U(xy) = xU(y), \quad V(zx) = V(z)x \quad \forall x \in B,\ \forall y \in L,\ \forall z \in R,$$
$$U(\Phi(a)) = V(\Phi(a)) \quad \forall a \in A,$$

and

$$U(\Phi(a \circ b)) = V(\Phi(a \circ b)) = \Phi(a) \circ \Phi(b) \quad \forall a, b \in A.$$

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for A of bound M, and let C be the constant given in Theorem 2.1.

We define

$$L = \left\{x \in B\colon \text{ the net } \big(x\Phi(e_\lambda)\big)_{\lambda\in\Lambda} \text{ is convergent}\right\},$$
$$U\colon L \to B, \quad U(x) = \lim_{\lambda\in\Lambda} x\Phi(e_\lambda) \quad \forall x \in L,$$

and

$$R = \left\{x \in B\colon \text{ the net } \big(\Phi(e_\lambda)x\big)_{\lambda\in\Lambda} \text{ is convergent}\right\},$$
$$V\colon R \to B, \quad V(x) = \lim_{\lambda\in\Lambda} \Phi(e_\lambda)x \quad \forall x \in R.$$

It is clear that L is a left ideal of B, R is a right ideal of B, and routine verifications, using that the net $(\Phi(e_\lambda))_{\lambda\in\Lambda}$ is bounded, show that both L and R are closed subspaces of B. It is also obvious that both U and V are continuous linear maps with $\|U\| \leq M\|\Phi\|$ and $\|V\| \leq M\|\Phi\|$ and that

$$U(xy) = xU(y), \quad V(zx) = V(z)x \quad \forall x \in B,\ \forall y \in L,\ \forall z \in R.$$

We claim that

$$\big(\Phi(a^2)\Phi(e_\lambda)\big)_{\lambda\in\Lambda} \to \Phi(a)^2 \quad \forall a \in A. \tag{2.7}$$

Fix an $a \in A$. By the Hahn-Banach theorem, for each $\lambda \in \Lambda$ there exists a $\xi_\lambda \in B^*$ with $\|\xi_\lambda\| = 1$ and

$$\left\langle \Phi(a^2)\Phi(e_\lambda) - \Phi(a)^2, \xi_\lambda \right\rangle = \left\|\Phi(a^2)\Phi(e_\lambda) - \Phi(a)^2\right\|. \tag{2.8}$$

For each $\lambda \in \Lambda$, we consider the continuous bilinear functional

$$\varphi_\lambda\colon A \times A \to \mathbb{C}, \quad \varphi_\lambda(u, v) = \langle \Phi(u)\Phi(v), \xi_\lambda\rangle \quad \forall u, v \in A,$$

which clearly satisfies (2.1) and

$$\|\varphi_\lambda\| \leq \|\Phi\|^2.$$

Hence Theorem 2.1 yields the existence of $\sigma_\lambda, \tau_\lambda \in A^*$ such that

$$\|\sigma_\lambda\| \leq C\|\Phi\|^2, \quad \|\tau_\lambda\| \leq C\|\Phi\|^2,$$

and

$$\langle \Phi(u)\Phi(v), \xi_\lambda\rangle = \sigma_\lambda(uv) + \tau_\lambda(vu) \quad \forall u, v \in A. \tag{2.9}$$

From (2.8) and (2.9) we deduce that

$$\begin{aligned} \|\Phi(a^2)\Phi(e_\lambda) - \Phi(a)^2\| &= \langle \Phi(a^2)\Phi(e_\lambda), \xi_\lambda\rangle - \langle \Phi(a)^2, \xi_\lambda\rangle \\ &= \sigma_\lambda(a^2e_\lambda) + \tau_\lambda(e_\lambda a^2) - \sigma_\lambda(a^2) - \tau_\lambda(a^2) \\ &= \sigma_\lambda(a^2e_\lambda - a^2) + \tau_\lambda(e_\lambda a^2 - a^2) \quad \forall \lambda \in \Lambda. \end{aligned} \tag{2.10}$$

We now observe that

$$|\sigma_\lambda(a^2e_\lambda - a^2)| \leq C\|\Phi\|^2\|a^2e_\lambda - a^2\|,$$
$$|\tau_\lambda(e_\lambda a^2 - a^2)| \leq C\|\Phi\|^2\|e_\lambda a^2 - a^2\| \quad \forall \lambda \in \Lambda,$$

and so, taking limits and using that

$$\lim_{\lambda\in\Lambda}\|a^2e_\lambda - a^2\| = \lim_{\lambda\in\Lambda}\|e_\lambda a^2 - a^2\| = 0,$$

we see that

$$\lim_{\lambda\in\Lambda}\sigma_\lambda(a^2e_\lambda - a^2) = \lim_{\lambda\in\Lambda}\tau_\lambda(e_\lambda a^2 - a^2) = 0.$$

Taking limit in (2.10) we now deduce that

$$\lim_{\lambda\in\Lambda}\|\Phi(a^2)\Phi(e_\lambda) - \Phi(a)^2\| = 0,$$

which gives (2.7).

Of course, (2.7) gives

$$\Phi(a^2)\in L, \quad U(\Phi(a^2)) = \Phi(a)^2 \quad \forall a\in A. \tag{2.11}$$

In the same way as (2.7) one proves that

$$\big(\Phi(e_\lambda)\Phi(a^2)\big)_{\lambda\in\Lambda} \to \Phi(a)^2 \quad \forall a\in A,$$

which clearly yields

$$\Phi(a^2)\in R, \quad V(\Phi(a^2)) = \Phi(a)^2 \quad \forall a\in A. \tag{2.12}$$

It remains to prove that $\Phi(A)\subset L\cap R$. From (2.11) and (2.12) we deduce immediately that

$$\begin{aligned} &\Phi(a\circ b)\in L\cap R,\\ U(\Phi(a\circ b)) = &V(\Phi(a\circ b)) = \Phi(a)\circ\Phi(b) \quad \forall a,b\in A. \end{aligned} \tag{2.13}$$

For each $a\in A$, [1, Theorem II.16] gives $b,c\in A$ such that $a = bcb$, so that

$$a = \tfrac{1}{2}b\circ(b\circ c) - \tfrac{1}{2}b^2\circ c$$

and (2.13) then gives $\Phi(a)\in L\cap R$ and further $U(\Phi(a)) = V(\Phi(a))$. □

Lemma 2.3. *Let A and B be Banach algebras, and suppose that:*

(a) *A is zero product determined;*
(b) *A has a bounded approximate identity;*
(c) *A is weakly amenable;*
(d) *B is faithful.*

Let $\Phi\colon A\to B$ be a continuous linear map having dense range and preserving two-sided zero products. Then there exists an injective continuous centralizer $W\colon B\to B$ such that

$$W(\Phi(a\circ b)) = \Phi(a)\circ\Phi(b) \quad \forall a,b\in A.$$

Proof. We apply Lemma 2.2. Since Φ has dense range, it follows that $L = R = B$ and that $U = V$. Set $W = U\ (= V)$. Then W is a centralizer on B and

$$W(\Phi(a\circ b)) = \Phi(a)\circ\Phi(b) \quad \forall a,b\in A.$$

The only point remaining concerns the injectivity of W. We claim that

$$\ker W B^3 = B^3\ker W = \{0\}. \tag{2.14}$$

Let $x\in\ker W$. For each $a\in A$, we have

$$0 = W(x)\Phi(a^2) = W(x\Phi(a^2)) = xW(\Phi(a^2)) = x\Phi(a)^2,$$

and, since the range of Φ is dense, we arrive at

$$xy^2 = 0 \quad \forall y\in B.$$

We thus get

$$x(yz+zy)=0 \quad \forall x\in \ker W,\ \forall y,z\in B.$$

For all $x\in\ker W$ and $y,z,w\in B$ we have (using that $xz\in\ker W$)

$$(xyz)w=(-xzy)w=-(xz)yw=(xz)wy=x(zw)y=-xy(zw),$$

whence $xyzw=0$, and so $\ker W B^3=\{0\}$. Similarly we see that $B^3\ker W=\{0\}$. Thus, (2.14) holds.

It is an elementary exercise to show that an element b in a faithful algebra B satisfying $bB^3=B^3b=\{0\}$ must be 0. Indeed, one first observes that every $c\in B^2bB^2$ satisfies $cB=Bc=\{0\}$, which yields $B^2bB^2=\{0\}$. Similarly we see that this implies $B^2bB=BbB^2=\{0\}$, hence $BbB=B^2b=bB^2=\{0\}$, and finally $bB=Bb=\{0\}$. Therefore, $b=0$.

Thus, (2.14) shows that $\ker W=\{0\}$. □

Lemma 2.4. *Let A and B be Banach algebras, and suppose that B has a bounded approximate identity. Let $\Phi\colon A\to B$ be a surjective linear map, and let $W\colon B\to B$ be a linear map such that*

$$W(\Phi(a\circ b))=\Phi(a)\circ\Phi(b)\quad \forall a,b\in A.$$

Then W is surjective.

Proof. Set $x\in B$. By [1, Theorem II.16], there exist $y,z\in B$ such that $x=yzy$, so that

$$x=\tfrac{1}{2}y\circ(y\circ z)-\tfrac{1}{2}y^2\circ z.$$

Since Φ is surjective, we can choose $a,b,c,d\in A$ with

$$\Phi(a)=y,\quad \Phi(b)=y\circ z,\quad \Phi(c)=y^2,\quad \Phi(d)=z.$$

The condition on W now gives

$$\begin{aligned} W\big(\Phi(\tfrac{1}{2}a\circ b-\tfrac{1}{2}c\circ d)\big)&=\tfrac{1}{2}\Phi(a)\circ\Phi(b)-\tfrac{1}{2}\Phi(c)\circ\Phi(d)\\ &=\tfrac{1}{2}y\circ(y\circ z)-\tfrac{1}{2}y^2\circ z=x. \end{aligned}$$ □

We are now ready to establish our main result.

Theorem 2.5. *Let A and B be Banach algebras, and suppose that:*

(a) *A is zero product determined;*
(b) *A has a bounded approximate identity;*
(c) *A is weakly amenable;*
(d) *B has a bounded approximate identity.*

Let $\Phi\colon A\to B$ be a surjective continuous linear map which preserves two-sided zero products. Then Φ is a weighted Jordan homomorphism.

Proof. Since B has a bounded approximate identity, it follows that B is faithful. We conclude from Lemma 2.3 that there exists an injective continuous centralizer $W\colon B\to B$ such that

$$W(\Phi(a\circ b))=\Phi(a)\circ\Phi(b)\quad \forall a,b\in A. \tag{2.15}$$

Lemma 2.4 now shows that W is surjective.

Having proved that W is an invertible centralizer, we can define $\Psi=W^{-1}\Phi$ which is a surjective continuous linear map and, further, we deduce from (2.15) that Ψ is a Jordan homomorphism. Of course, $\Phi=W\Psi$. □

The crucial examples of zero product determined Banach algebras are the group algebras $L^1(G)$ for each locally compact group G and C^*-algebras [7, Theorems 5.19 and 5.21]. Furthermore, these Banach algebras are also weakly amenable and have bounded approximate identities. Therefore, it is legitimate to apply Theorem 2.5 in the case where A is a group algebra or a C^*-algebra.

Corollary 2.6. *Let G be a locally compact group, let B be a Banach algebra having a bounded approximate identity, and let $\Phi\colon L^1(G)\to B$ be a surjective continuous linear map which preserves two-sided zero products. Then Φ is a weighted Jordan homomorphism.*

Our final corollary generalizes [6, Theorem 3.1 (i)].

Corollary 2.7. *Let G and H be locally compact groups, and let $\Phi\colon L^1(G)\to L^1(H)$ be a surjective continuous linear map which preserves two-sided zero products. Then there exist a surjective continuous Jordan homomorphism $\Psi\colon L^1(G)\to L^1(H)$ and an invertible central measure $\mu\in M(H)$ such that $\Phi(f)=\mu*\Psi(f)$ for each $f\in L^1(G)$.*

Proof. By Corollary 2.6, there exist an invertible cetralizer W of $L^1(H)$ and a surjective continuous Jordan homomorphism $\Psi\colon L^1(G)\to L^1(H)$ such that $\Phi=W\Psi$. The centralizer W can be thought of as an element of the centre of the multiplier algebra of $L^1(H)$ which is, by Wendel's Theorem (see [10, Theorem 3.3.40]), isomorphic to the measure algebra $M(H)$. This gives a measure $\mu\in M(H)$ as required. $\square$

3. CONDITION (1.2)

We will not discuss Theorem 2.5 in the case where A is a C^*-algebra, because in this case condition (1.1) can be weakened to condition (1.2). Showing this is the main purpose of this section.

Lemma 3.1. *Let A be a Banach algebra, and suppose that:*

(a) *A is zero product determined;*
(b) *A has a bounded approximate identity.*

Let B be a Banach algebra and let $\Phi\colon A\to B$ be a continuous linear map satisfying condition (1.2)*. Then there exist a closed linear subspace J of B containing $\Phi(A)$ and a continuous linear map $W\colon J\to B$ such that*

$$W(\Phi(a\circ b))=\Phi(a)\circ\Phi(b)\quad\forall a,b\in A.$$

Moreover, if B has a bounded approximate identity and Φ is surjective, then W is a surjective map from B onto itself.

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for A of bound M.

We define

$$J=\big\{x\in B\colon\ \text{the net } \big(\Phi(e_\lambda)\circ x\big)_{\lambda\in\Lambda} \text{ is convergent}\big\}$$

and

$$W\colon J\to B,\quad W(x)=\lim_{\lambda\in\Lambda}\tfrac{1}{2}\Phi(e_\lambda)\circ x\quad\forall x\in J.$$

It is clear that J is a linear subspace of B and routine verifications, using that the net $(\Phi(e_\lambda))_{\lambda\in\Lambda}$ is bounded, show that J is a closed linear subspace of B. It is also obvious that W is a continuous linear map with $\|W\|\le M\|\Phi\|$.

Applying [7, Theorem 6.1 and Remark 6.2] to the continuous bilinear map $\varphi\colon A\times A\to B$ defined by

$$\varphi(a,b)=\Phi(a)\circ\Phi(b)\quad\forall a,b\in A$$

we see that there exists a continuous linear map $S\colon A\to B$ such that

$$\Phi(a)\circ\Phi(b)=S(a\circ b)\quad\forall a,b\in A. \tag{3.1}$$

For each $a\in A$, we thus have

$$\Phi(e_\lambda)\circ\Phi(a)=S(e_\lambda\circ a)\quad\forall\lambda\in\Lambda.$$

Using $\lim_{\lambda\in\Lambda}e_\lambda\circ a=2a$ and the continuity of S, we see by taking limit that

$$\lim_{\lambda\in\Lambda}\Phi(e_\lambda)\circ\Phi(a)=2S(a).$$

This shows that $\Phi(a)\in J$ and that $W(\Phi(a))=S(a)$. On the other hand, using (3.1), we see that

$$W(\Phi(a\circ b))=S(a\circ b)=\Phi(a)\circ\Phi(b)\quad\forall a,b\in A. \tag{3.2}$$

Of course, if Φ is surjective, then $J=B$. Now suppose that, in addition, B has a bounded approximate identity. Then, on account of (3.2), Lemma 2.4 shows that W is surjective. □

Lemma 3.2. *Let A and B be a Banach algebras, let $\Phi\colon A\to B$ be a continuous linear map, and let $\omega\in B$. Suppose that:*

(a) *A is the closed linear span of its idempotents.*
(b) *$\Phi(a^2)\circ\omega=2\Phi(a)^2$ for each $a\in A$.*

Then $\omega^2\Phi(a)=\Phi(a)\omega^2$ for each $a\in A$.

Proof. Let $e\in A$ be an idempotent. From (b) we see that

$$\omega\Phi(e)+\Phi(e)\omega=\Phi(e)\circ\omega=\Phi(e^2)\circ\omega=2\Phi(e)^2. \tag{3.3}$$

By multiplying (3.3) by $\Phi(e)$ on the left we obtain

$$\Phi(e)\omega\Phi(e)+\Phi(e)^2\omega=2\Phi(e)^3 \tag{3.4}$$

and multiplying by $\Phi(e)$ on the right we get

$$\omega\Phi(e)^2+\Phi(e)\omega\Phi(e)=2\Phi(e)^3. \tag{3.5}$$

From (3.4) and (3.5) we arrive at $\omega\Phi(e)^2=\Phi(e)^2\omega$, which, on account of (3.3), yields

$$\omega^2\Phi(e)=\Phi(e)\omega^2.$$

Since A is the closed linear span of its idempotents, it follows that

$$\omega^2\Phi(a)=\Phi(a)\omega^2\quad\forall a\in A.$$ □

In the proof of the next results we will use the first Arens product on the second dual A^{**} of a Banach algebra A. We will denote this product by juxtaposition. Furthermore, we will use the following basic facts about the weak* continuity of the first Arens product which the reader can find in [10].

(A1) For each $a\in A$, the map $\xi\mapsto a\xi$ from A^{**} to itself is weak* continuous.
(A2) For each $\xi\in A^{**}$, the map $\zeta\mapsto\zeta\xi$ from A^{**} to itself is weak* continuous.
(A3) If A is a C^*-algebra, then the product in A^{**} is separately weak* continuous.

Theorem 3.3. *Let A be a C^*-algebra, let B be a Banach algebra having a bounded approximate identity, and let $\Phi\colon A\to B$ be a surjective continuous linear map such that for all $a,b\in A$,*

$$ab=ba=0 \implies \Phi(a)\circ\Phi(b)=0.$$

Then Φ is a weighted Jordan homomorphism.

Proof. By Lemma 3.1 there exists a surjective continuous linear map $W\colon B\to B$ such that

$$W(\Phi(a\circ b))=\Phi(a)\circ\Phi(b)\quad \forall a,b\in A. \tag{3.6}$$

We write $\Phi^{**}\colon A^{**}\to B^{**}$ and $W^{**}\colon B^{**}\to B^{**}$ for the second duals of the continuous linear maps $\Phi\colon A\to B$ and $W\colon B\to B$, respectively. We claim that

$$W^{**}(\Phi^{**}(x\circ y))=\Phi^{**}(x)\circ\Phi^{**}(y)\quad \forall x,y\in A^{**}. \tag{3.7}$$

Set $x,y\in A^{**}$, and take nets $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ in A such that

$$\begin{aligned}&(a_i)_{i\in I}\to x,\\ &(b_j)_{j\in J}\to y\quad \text{in } (A^{**},\sigma(A^{**},A^*)).\end{aligned}$$

On account of (3.6), we have

$$W(\Phi(a_ib_j+b_ja_i))=\Phi(a_i)\Phi(b_j)+\Phi(b_j)\Phi(a_i)\quad \forall i\in I,\ \forall j\in J, \tag{3.8}$$

and the task is now to take the iterated limit $\lim_{j\in J}\lim_{i\in I}$ on both sides of the above equation. Throughout the proof, the limits $\lim_{i\in I}$ and $\lim_{j\in J}$ are taken with respect to the weak* topology. From (A3) and the weak* continuity of both Φ^{**} and W^{**} we deduce that

$$\begin{aligned}\lim_{j\in J}\lim_{i\in I}W^{**}\big(\Phi^{**}(a_ib_j+b_ja_i)\big)&=\lim_{j\in J}W^{**}\big(\Phi^{**}(xb_j+b_jx)\big)\\ &=W^{**}\big(\Phi^{**}(xy+yx)\big).\end{aligned} \tag{3.9}$$

From (A1)-(A2) (applied to the Arens product of B^{**}) and the weak* continuity of Φ^{**} we deduce that

$$\begin{aligned}\lim_{j\in J}\lim_{i\in I}\Phi(b_j)\Phi(a_i)&=\lim_{j\in J}\Phi(b_j)\Phi^{**}(x)\\ &=\Phi^{**}(y)\Phi^{**}(x).\end{aligned} \tag{3.10}$$

The remaining iterated limits

$$\lim_{j\in J}\lim_{i\in I}\Phi(a_i)\Phi(b_j)$$

must be treated with much more care than the previous ones. We regard A as a C^*-algebra acting on the Hilbert space of its universal representation, and we regard the continuous bilinear map

$$A\times A\to B,\quad (a,b)\to\Phi(a)\Phi(b)$$

as a continuous bilinear map with values in the Banach space B^{**} which is separately ultraweak-weak* continuous. By applying [13, Theorem 2.3], we obtain that the bilinear map above extends uniquely, without change of norm, to a continuous bilinear map $\phi : A^{**} \times A^{**} \to B^{**}$ which is separately weak* continuous. From this, and using (A1)-(A2) and the weak* continuity of Φ^{**}, we obtain

$$\begin{aligned}\lim_{j\in J}\lim_{i\in I}\Phi(a_i)\Phi(b_j) &= \lim_{j\in J}\lim_{i\in I}\phi(a_i,b_j)\\ &= \phi(x,y)\\ &= \lim_{i\in I}\lim_{j\in J}\phi(a_i,b_j)\\ &= \lim_{i\in I}\lim_{j\in J}\Phi(a_i)\Phi(b_j)\\ &= \lim_{i\in I}\Phi(a_i)\Phi^{**}(y)\\ &= \Phi^{**}(x)\Phi^{**}(y).\end{aligned} \tag{3.11}$$

From (3.8), (3.9), (3.10), and (3.11), it may be concluded that

$$\begin{aligned}W^{**}(\Phi^{**}(x\circ y)) =&\lim_{j\in J}\lim_{i\in I}W(\Phi(a_i\circ b_j))\\ =&\lim_{j\in J}\lim_{i\in I}\Phi(a_i)\Phi(b_j)+\lim_{j\in J}\lim_{i\in I}\Phi(b_j)\Phi(a_i)\\ =&\Phi^{**}(x)\circ\Phi^{**}(y),\end{aligned}$$

and (3.7) is proved.

Define $\omega = \Phi^{**}(1) \in B^{**}$, where 1 is the unit of the von Neumann algebra A^{**}. Setting $x = y$ in (3.7) we conclude that

$$W^{**}(\Phi^{**}(x^2)) = \Phi^{**}(x)^2 \quad \forall x \in A^{**}, \tag{3.12}$$

and setting $y = 1$ in (3.7) we see that

$$2W^{**}(\Phi^{**}(x)) = \omega\Phi^{**}(x) + \Phi^{**}(x)\omega \quad \forall x \in A^{**}. \tag{3.13}$$

From (3.12) and (3.13) we deduce that

$$\Phi^{**}(x^2)\circ\omega = 2\Phi^{**}(x)^2 \quad \forall x \in A^{**}.$$

Since A^{**} is a von Neumann algebra, it is the closed linear span of its projections and we are in a position to apply Lemma 3.2, which gives

$$\omega^2\Phi^{**}(x)) = \Phi^{**}(x)\omega^2 \quad \forall x \in A^{**}.$$

In particular,

$$\omega^2\Phi(a) = \Phi(a)\omega^2 \quad \forall a \in A.$$

Since Φ is surjective, it may be concluded that

$$\omega^2 u = u\omega^2 \quad \forall u \in B.$$

From (3.13) we see that, for each $a \in A$,

$$\begin{aligned}\omega W(\Phi(a)) =&\omega\tfrac{1}{2}\big(\omega\Phi(a)+\Phi(a)\omega\big)\\ =&\tfrac{1}{2}\big(\omega^2\Phi(a)+\omega\Phi(a)\omega\big)\\ =&\tfrac{1}{2}\big(\Phi(a)\omega^2+\omega\Phi(a)\omega\big)\\ =&\tfrac{1}{2}\big(\Phi(a)\omega+\omega\Phi(a)\big)\omega\\ =&W(\Phi(a))\omega.\end{aligned}$$

Since both Φ and W are surjective, it may be concluded that

$$\omega u = u\omega \quad \forall u \in B. \tag{3.14}$$

From (3.13) we now deduce that

$$W(\Phi(a)) = \tfrac{1}{2}\big(\omega\Phi(a) + \Phi(a)\omega\big) = \omega\Phi(a) \quad \forall a \in A,$$

and hence that

$$W(u) = \omega u \quad \forall u \in B.$$

Furthermore, for all $a, b \in B$, using (3.14) we obtain

$$W(ab) = \omega ab = W(a)b,$$

$$W(ab) = (\omega a)b = (a\omega)b = aW(b),$$

whence W is a centralizer on B. In order to prove that W is an invertible centralizer, it remains to show that W is injective. If $a \in \ker W$, then

$$aB = aW(B) = W(a)B = \{0\}$$

and therefore $a = 0$.

Since W is an invertible centralizer on B, (3.6) shows that $W^{-1}\Phi$ is a Jordan homomorphism, and hence Φ is a weighted Jordan homomorphism. □

Our final concern will be the algebra $\mathscr{A}(X)$ of approximable operators on a Banach space X. It is shown in [2] that $\mathscr{A}(X)$ has the so-called property $\mathbb{B}$ for each Banach space X (see also [7, Example 5.15]). Further, it is known that $\mathscr{A}(X)$ has a bounded left approximate identity if and only if the Banach space X has the bounded approximation property (see [10, Theorem 2.9.37]). In this case, $\mathscr{A}(X)$ is actually a zero product determined Banach algebra (see [2, Lemma 2.3] or, alternatively, [7, Proposition 5.5]). Another remarkable feature of $\mathscr{A}(X)$ is that it has a bounded approximate identity if and only if X^* has the bounded approximation property (see [10, Theorem 2.9.37]).

Theorem 3.4. *Let X be a Banach space such that X^* has the bounded approximation property, let B be a Banach algebra having a bounded approximate identity, and let $\Phi\colon \mathscr{A}(X) \to B$ be a surjective continuous linear map such that for all $S, T \in \mathscr{A}(X)$,*

$$ST = TS = 0 \implies \Phi(S) \circ \Phi(T) = 0.$$

Then Φ is a weighted Jordan homomorphism.

Proof. We begin by applying Lemma 3.1 to obtain a surjective continuous linear map $W : B \to B$ such that

$$W(\Phi(S \circ T)) = \Phi(S) \circ \Phi(T) \quad \forall S, T \in \mathscr{A}(X). \tag{3.15}$$

Let $(E_\lambda)_{\lambda\in\Lambda}$ be a bounded approximate identity for $\mathscr{A}(X)$. Then we regard $\big(\Phi(E_\lambda)\big)_{\lambda\in\Lambda}$ as a bounded net in the second dual B^{**} of B. It follows from the Banach-Alaoglu theorem that this net has a $\sigma(B^{**}, B^*)$-convergent subnet. Hence, by passing to a subnet, we may supose that $(E_\lambda)_{\lambda\in\Lambda}$ is a bounded approximate identity for $\mathscr{A}(X)$ such that

$$\lim_{\lambda\in\Lambda} \Phi(E_\lambda) = \omega \quad \text{in } (B^{**}, \sigma(B^{**}, B^*))$$

for some $\omega \in B^{**}$.

Set $T \in \mathscr{A}(X)$. Writting E_λ for S in (3.15), we obtain

$$W(\Phi(E_\lambda T + TE_\lambda)) = \Phi(E_\lambda)\Phi(T) + \Phi(T)\Phi(E_\lambda) \quad \forall \lambda \in \Lambda, \tag{3.16}$$

and our next goal is to take limits on both sides of (3.16). Since

$$\lim_{\lambda\in\Lambda}(E_\lambda T + TE_\lambda) = 2T \quad \text{in } (\mathscr{A}(X), \|\cdot\|),$$

the continuity of $W\Phi$ gives

$$\lim_{\lambda\in\Lambda} W(\Phi(E_\lambda T + TE_\lambda)) = 2W(\Phi(T)) \quad \text{in } (B, \|\cdot\|). \tag{3.17}$$

On the other hand, since

$$\lim_{\lambda\in\Lambda} \Phi(E_\lambda) = \omega \quad \text{in } (B^{**}, \sigma(B^{**}, B^*))$$

and $\Phi(T) \in B$, we can appeal to (A1)-(A2) to deduce that

$$\begin{aligned} &\lim_{\lambda\in\Lambda} \Phi(E_\lambda)\Phi(T) = \omega\Phi(T), \\ &\lim_{\lambda\in\Lambda} \Phi(T)\Phi(E_\lambda) = \Phi(T)\omega \quad \text{in } (B^{**}, \sigma(B^{**}, B^*)). \end{aligned} \tag{3.18}$$

Hence, taking limits in (3.16) and using (3.17) and (3.18), we obtain

$$2W(\Phi(T)) = \omega\Phi(T) + \Phi(T)\omega. \tag{3.19}$$

Having (3.15) and (3.19) and using that $\mathscr{A}(X)$ is the closed linear span of its idempotents, we can now apply Lemma 3.2 to obtain

$$\omega^2\Phi(T) = \Phi(T)\omega^2 \quad \forall T \in \mathscr{A}(X).$$

From the surjectivity of Φ we deduce that

$$\omega^2 a = a\omega^2 \quad \forall a \in B.$$

By using the same method as in the proof of Theorem 3.3 we verify that $\omega a = a\omega$ for each $a \in B$, that $W(a) = \omega a$ for each $a \in A$, and that W is an invertible centralizer on B such that $W^{-1}\Phi$ is a Jordan homomorphism. □

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