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A projection-less approach to Rickart Jordan structures



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To the memory of Professor
C.M. Edwards with admiration,
affection, and respect

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ABSTRACT

The main goal of this paper is to introduce and explore an appropriate notion of weakly Rickart JB^* -triples. We introduce weakly and weakly order Rickart JB^* -triples, and we show that a C^* -algebra A is a weakly (order) Rickart JB^* -triple precisely when it is a weakly Rickart C^* -algebra. We also prove that the Peirce-2 subspace associated with any tripotent in a weakly order Rickart JB^* -triple is a Rickart JB^* -algebra in the sense of Ayupov and Arzikulov. By extending a classical property of Rickart C^* -algebras, we prove that every weakly order Rickart JB^* -triple is generated by its tripotents.

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1. Introduction and preliminaries

The reference [59] is the founding work of the fruitful theory of Rickart and Baer C^* -algebras. C. E. Rickart [59] stated that “*Our general purpose is to study the structure of a B^* -algebra in terms of its projections. Such a study of course demands the existence of many projections a B^* -algebra is defined to be a B_p^* -algebra (now called a Rickart C^* -algebra) provided it contains, in a certain sense, “sufficiently many” projections.*” The chosen notion was built around left and right annihilators. For each nonempty subset S of an associative ring A , the *right-* and *left-annihilator* of S are defined by

$$R(S) = \{x \in A : sx = 0 \text{ for all } s \in S\}$$

and

$$L(S) = \{x \in A : xs = 0 \text{ for all } s \in S\},$$

respectively. If A is an associative $*$ -ring, a *projection* p in A will be a self-adjoint ($p^* = p$) idempotent ($p^2 = p$). A *Rickart $*$ -ring* is an associative $*$ -ring A such that, for each $a \in A$, $R(\{a\}) = pA$ for a (unique) projection p (see [12, §3, Definition 2]). In such a case we have $L(\{a\}) = (R(\{a^*\}))^* = (qA)^* = Aq$ for a suitable projection q . A *Rickart C^* -algebra* is a C^* -algebra which is also a Rickart $*$ -ring (cf. [12, §3, Definition 3] and the original work by Rickart [59]). Each Rickart $*$ -ring has a unity element and its involution is *proper*, i.e., $xx^* = 0 \Rightarrow x = 0$ (see [12, §3, Proposition 2]). The projections of a Rickart C^* -algebra form a lattice which is not necessarily complete (cf. [12, §3, Proposition 7 and Example 2]). A C^* -algebra A is called *weakly Rickart* if for each $x \in A$ there exists an *annihilating right projection* (briefly, ARP) of x , that is, a projection p satisfying $xp = x$, and $xy = 0$ implies $py = 0$. Let us observe that annihilating left projections (ALP) are similarly defined. The ARP and ALP of each element x are uniquely determined by x , and we shall denote them by $RP(x)$ and $LP(x)$, respectively. Every unital weakly Rickart C^* -algebra is a Rickart C^* -algebra, since for each $x \in A$ we have $R(\{x\}) = (\mathbf{1} - RP(x))A$. Rickart proved in [59, Theorem 2.10] that every Rickart C^* -algebra is generated by its projections.

As seen before, the definition of a Rickart $*$ -ring is given in terms of the annihilators of singletons. When singletons are replaced by general subsets we find the notion of Baer $*$ -ring. Concretely, a *Baer $*$ -ring* is an associative $*$ -ring A such that, for every nonempty subset $S \subset A$ we have $R(S) = pA$ for a suitable projection p in A (see [12, §4, Definition 1]). Baer $*$ -rings are precisely those Rickart $*$ -rings whose projections form a complete lattice, equivalently, every orthogonal family of projections has a supremum (cf. [12, §4, Proposition 1]). As introduced in the pioneering works of Kaplansky [46–48], an *AW $*$ -algebra* is a C^* -algebra that is a Baer $*$ -ring (see [12, §4, Definition 2]).

Since for each element a in a C^* -algebra A we have $R(\{a\}) = R(\{a^*a\})$, in the definition of Rickart C^* -algebra we can restrict our attention to the right-annihilators

of positive elements. Similarly, in the definition of AW*-algebras we can consider right-annihilators of sets of the form $\{a^*a : a \in S\}$, where S is any subset of the C*-algebra under study.

Each von Neumann algebra (i.e., a *-subalgebra of $B(H)$ whose bicommutant coincides with itself, or equivalently, by Sakai’s theorem [64], a C*-algebra which is also a dual Banach space) is an AW*-algebra [12, §4, Proposition 9]. After Sakai’s theorem, von Neumann algebras are also known as W*-algebras. Though AW*-algebras were actually introduced with the aim of finding an algebraic characterization of von Neumann or W*-algebras, it was soon shown by Dixmier that there exist commutative AW*-algebras which cannot be represented as von Neumann algebras (see [25] or [12, §7, Exercises 2, 3]). Wright found in [68] examples of monotone complete factors which are not von Neumann algebras. The reader has probably realized that we take the references [59,46,12,62] as the basic bibliography on Rickart and AW*-algebras.

In the list of problems and future directions in [61, page 144], A. Rodríguez-Palacios somehow anticipated and suggested the study of Rickart Jordan algebras as those Jordan algebras for which “the annihilator of every element in Zelmanov sense is generated by an idempotent” (see subsection 1.1 for the basic theory on Jordan algebras). However, we have to wait until 2016 to find the first study on Rickart and Baer Jordan algebras by Sh. A. Ayupov and F. N. Arzikulov (see [7]). The (outer) quadratic annihilator of a subset \mathcal{S} in a Jordan algebra M –with product \circ – is defined as the set

$$\text{Ann}(\mathcal{S}) = \mathcal{S}^{\perp_q} := \{a \in M : U_a(s) = 2(a \circ s) \circ a - (a \circ a) \circ s = 0, \quad \forall s \in \mathcal{S}\}.$$

A Jordan algebra M is called a Rickart Jordan algebra if for each element $a \in M^2$ there exists an idempotent $e \in M$ such that $\{a\}^{\perp_q} = U_e(M)$, where $U_e(x) := 2(e \circ x) \circ e - e^2 \circ x$. If in the definition of Rickart Jordan algebra, the sets given by a single element $a \in M^2$ are replaced by arbitrary subsets $\mathcal{S} \subset M^2$, we get the notion of Baer Jordan algebra (cf. [7]).

Rickart and Baer Jordan algebras are appropriate notions for JB-algebras, where we have projections and positive elements. It is shown by Ayupov and Arzikulov that for each C*-algebra A , its self-adjoint part, A_{sa} , is a Rickart (respectively, Baer) Jordan algebra if and only if A is a Rickart (respectively, Baer) C*-algebra [7,8]. A Rickart (respectively, Baer) JB*-algebra is a JB*-algebra M whose self-adjoint part is a Rickart (respectively, Baer) JB-algebra.

The original aim in Rickart’s studies was completed in the case of JB-algebras by F. N. Arzikulov who proved that a JB-algebra N is a Baer Jordan algebra if and only if N satisfies the following properties:

- (1) Every subset of pairwise orthogonal projections in the partially ordered set of projections has a least upper bound in this set;
- (2) Every maximal strongly associative subalgebra of N is generated by its projections (see [4, Theorem 2.1]).

The available notions of Rickart and Baer Jordan algebras have a strong dependence on quadratic annihilators, projections and positive elements. However, if we are interested in developing these notions in more general Jordan structures, like JB^* -triples, where projections and positive elements do not make any sense, we need an alternative approach. This is the main goal of this paper.

Section 2 is devoted to revisit the main results on Rickart and weakly Rickart C^* -algebras with the aim of finding a characterization which can be stated without appealing to projections and positive elements. We shall show (see Propositions 2.5 and 2.10) that by mixing and extending a characterization due to G. K. Pedersen in [56] with key contributions by P. Ara and D. Goldstein [2,3,35], the following characterizations hold for every C^* -algebra A :

- (a) A is a weakly Rickart C^* -algebra if, and only if, any of the following statements holds:
- (1) Given $x \in A$ and an inner ideal $J \subseteq A$ which is orthogonal to the inner ideal $I = A(x)$ of A generated by x , there exists a partial isometry e in A such that $I \subseteq A_2(e)$ and $J \subseteq A_0(e)$.
 - (2) Given $x \in A$ and an inner ideal $J \subseteq A$ with $I = A(x) \perp J$, there exists a partial isometry e in A such that $I \subseteq A_2(e)$, $e^*e = RP(x)$, $ee^* = LP(x)$, x is a positive element in the C^* -algebra $(A_2(e), \bullet_e, *_e)$, $A(x)$ is a C^* -subalgebra of the latter C^* -algebra and $J \subseteq A_0(e)$.
- (b) A is a Rickart C^* -algebra if, and only if, A is unital and for each $x \in A$ and each inner ideal $J \subseteq A$ which is orthogonal to $I = A(x)$, there exists a partial isometry e in A such that $I \subseteq A_2(e)$ and $J \subseteq A_0(e)$.

The advantage of the previous characterizations (especially the one in (a)(1)) relies on their independence of projections and positive elements, and can be therefore extended to wider settings. Before further extensions, in section 3 we explore the notions of weakly Rickart and SAJBW-algebras, both in terms of projections and positive elements. For example, a JB-algebra N is called a weakly Rickart JB-algebra if for each element $a \in N^+$ there exists a projection $p \in N$ such that $p \circ a = a$, and for each $z \in N$ with $U_z(a) = 0$ we have $p \circ z = 0$. In Proposition 3.14 we establish several characterizations of Baer or AJBW*-algebras, (weakly) Rickart JB^* -algebras and SAJBW*-algebras in terms of hereditary JB^* -subalgebras. After several technical conclusions in the line of classical results, we arrive to our main goal of section 3 in Theorem 3.16, where it is proved that every weakly Rickart JB^* -algebra is generated by its projections.

In section 4 we introduce several definitions of Rickart, weakly Rickart and weakly order Rickart JB^* -triples. We show that, thanks to the characterization of the corresponding notions for C^* -algebras presented in section 2, the new definitions coincide with the classical notions in the setting of C^* -algebras. Special interest is received by weakly order Rickart JB^* -triples. This new notion agrees with the concept of Rickart C^* -algebra in the C^* -setting. A weakly order-Rickart JB^* -triple E is a JB^* -triple satis-

fyng that for each $x \in E$, if we write $E(x)$ for the inner ideal of E generated by x , then for each inner ideal $J \subseteq E$ with $I = E(x) \perp J$, there exists a tripotent e in E such that x is positive in $E_2(e)$, and $J \subseteq E_0(e)$.

We prove in Proposition 4.4 that if E is a weakly order Rickart JB*-triple and $e \in E$ is a tripotent, then the Peirce-2 subspace $E_2(e)$ is a Rickart JB*-algebra. This allows us to conclude that every weakly order Rickart JB*-triple is generated by its tripotents (see Theorem 4.5).

Finally, in section 5 we explore the connections with von Neumann regularity, by showing that each inner ideal I of a weakly order Rickart JB*-triple E contains a dense subset of von Neumann regular elements (cf. Theorem 5.4).

1.1. Background and basic definitions

This subsection is aimed to provide a basic compendium on the Jordan structures studied in this note. The reader will find some brief historical introduction, definitions, notions and basic references. These contents are not really required to follow section 2, which has been written to be accessible with tools of C*-algebras.

The early contributions by Jordan, von Neumann and Wigner in the decade of 1930s led to the idea of employing non-associative structures, specially Jordan algebras, in quantum mechanics (see the interesting monograph [52] for a fantastic historical overview). A real or complex Jordan algebra is a non-necessarily associative algebra M whose product (denoted by \circ) is commutative and satisfies the *Jordan-identity*:

$$(a \circ b) \circ a^2 = a \circ (b \circ a^2) \quad (a, b \in M). \tag{1}$$

The Jordan algebra M is called unital if there exists a unit element $\mathbf{1}$ in M such that $\mathbf{1} \circ a = a$ for all $a \in M$. Jordan algebras are power associative, that is, a subalgebra generated by a single element is associative. In other words, for each $a \in M$ define $a^0 := \mathbf{1}$ if M is unital, $a^1 = a$ and $a^{n+1} = a \circ a^n$ ($n \geq 1$). Then $a^{n+m} = a^n \circ a^m$ for all natural numbers m and n [39, Lemma 2.4.5]. For each $a \in M$ we shall denote by T_a the Jordan multiplication operator by the element a , that is, $T_a(x) = a \circ x$ ($x \in M$).

An element a in a unital Jordan Banach algebra M is called *invertible* whenever there exists $b \in M$ satisfying $a \circ b = \mathbf{1}$ and $a^2 \circ b = a$. The element b is unique and it will be denoted by a^{-1} (cf. [39, 3.2.9] and [22, Definition 4.1.2]). We know from [22, Theorem 4.1.3] that an element $a \in M$ is invertible if and only if U_a is a bijective mapping, and in such a case $U_a^{-1} = U_{a^{-1}}$.

As in the associative case, an *involution* on a Jordan algebra M is a mapping $a \mapsto a^*$ satisfying $(a^*)^* = a$ and $(a \circ b)^* = a^* \circ b^*$ for all $a, b \in M$. The involution $*$ is called *proper* if $a \circ a^* = 0$ implies $a = 0$.

A very special source of examples is provided by associative algebras. Namely, suppose A is a real or complex associative algebra with product denoted by juxtaposition. Then the natural Jordan product $a \circ b := \frac{1}{2}(ab + ba)$ defines a structure of Jordan algebra on

A ; Jordan algebras of this type are called *special*, as they are isomorphic to subalgebras of associative algebras equipped with a new multiplication (a term coined by Jordan, von Neumann & Wigner [45]). There are Jordan algebras which are not special (cf. [39, Corollary 2.8.5]), these algebras are called *exceptional*.

Suppose that A is a C^* -algebra. The (associative) product of two self-adjoint elements in A need not be, in general, self-adjoint. Another good property of the natural Jordan product assures that the Jordan product of two self-adjoint elements in A also is in A_{sa} . Therefore, A_{sa} is a real Jordan subalgebra of A , but not an associative subalgebra.

A central notion in the study of Jordan algebras is the so-called U -mapping. Let a, b be two elements in a Jordan algebra M . The $U_{a,b}$ mapping is the linear map on M given by

$$U_{a,b}(x) = (a \circ x) \circ b + (b \circ x) \circ a - (a \circ b) \circ x,$$

for all $x \in M$. The mapping $U_{a,a}$ is usually denoted by U_a . The U -maps satisfy the following fundamental identity:

$$U_a U_b U_a = U_{U_a(b)}, \text{ for all } a, b \text{ in a Jordan algebra } M, \tag{2}$$

(see [39, 2.4.18]).

It is now the moment to introduce some analytic structures. A Jordan algebra M endowed with a complete norm satisfying $\|a \circ b\| \leq \|a\| \|b\|$, $a, b \in M$ is called a *Jordan Banach algebra*. A *JB-algebra* is a real Jordan Banach algebra N whose norm satisfies the following two geometric axioms:

- (i) $\|a^2\| = \|a\|^2$, for all $a \in N$;
- (ii) $\|a^2\| \leq \|a^2 + b^2\|$, for all $a, b \in N$,

(see [39, Definition 3.1.4]).

The Jordan mathematical model closest to C^* -algebras is given by the class of JB^* -algebras. A *JB^* -algebra* is a complex Jordan Banach algebra M together with an algebra involution $a \mapsto a^*$, whose norm satisfies the following generalization of the Gelfand-Naimark axiom:

$$\|U_a(a^*)\| = \|a\|^3, \text{ for every } a \in M.$$

Both of the just introduced Jordan structures are intrinsically related thanks to a result due to J. D. M. Wright proving that every JB -algebra corresponds to the self-adjoint part of a (unique) JB^* -algebra (see [69]).

If a C^* -algebra A is equipped with its original norm and involution and the Jordan product given by $a \circ b = \frac{1}{2}(ab + ba)$, then the resulting structure is a JB^* -algebra. Jordan $*$ -subalgebras of C^* -algebras are called JC^* -algebras, and their symmetric parts are known as JC -algebras. The class of JB^* -algebras is strictly bigger than the collection

of all associative C^* -algebras since, for example, the exceptional Jordan algebra $H_3(\mathbb{O})$ is a purely exceptional JB-algebra, that is, there is no nonzero homomorphism from $H_3(\mathbb{O})$ into a JC-algebra (cf. [39, §7.2]).

From a purely algebraic point of view, a *complex Jordan triple system* is a complex linear space E equipped with a triple product $\{x, y, z\}$ which is bilinear and symmetric in x, z and conjugate linear in y and satisfies the following ternary Jordan identity:

$$L(x, y)\{a, b, c\} = \{L(x, y)a, b, c\} - \{a, L(y, x)b, c\} + \{a, b, L(x, y)c\}, \tag{3}$$

for all $x, y, a, b, c \in E$, where $L(x, y) : E \rightarrow E$ is the linear mapping given by $L(x, y)z = \{x, y, z\}$.

The analytic structures known as *JB*-triples*, whose origins go back to the theory of holomorphic functions on infinite dimensional complex Banach spaces [49], are defined as those complex Jordan triple systems E which are Banach spaces satisfying the next “*geometric*” axioms:

- (a) For each $x \in E$, the operator $L(x, x)$ is hermitian with non-negative spectrum;
- (b) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in E$.

The triple product of each JB*-triple E is a non-expansive mapping, that is,

$$\|\{a, b, c\}\| \leq \|a\| \|b\| \|c\|, \tag{4}$$

for all $a, b, c \in E$ (cf. [37, Corollary 3]).

JBW*-triples (respectively, JBW*-algebras) are defined as those JB*-triples (respectively, JB*-algebras) which are also dual Banach spaces. The bidual of every JB*-triple is a JBW*-triple (see [24]). It is further known that each JBW*-triple admits a unique (isometric) predual and its product is separately weak* continuous [11] (see also [23, Theorems 5.7.20 and 5.7.38]).

Each C^* -algebra A carries a natural structure of JB*-triple with respect to the triple product given by

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a). \tag{5}$$

The same triple product equips the space $B(H, K)$, of all bounded linear operators between two complex Hilbert spaces, with structure of JB*-triple. In particular, there exist infinite-dimensional complex Hilbert spaces which are JB*-triples.

For each element a in a JB*-triple E , the symbol $Q(a)$ will denote the conjugate linear operator on E defined by $Q(a)(x) = \{a, x, a\}$. Every JB*-algebra M is a JB*-triple with triple product

$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*. \tag{6}$$

It follows that $Q(a)(x) = U_a(x^*)$ for all $a, x \in M$.

We refer to [39,22] and [23] for the basic background on JB*-triples and JB*-algebras.

An element $e \in E$ is called a *tripotent* if $\{e, e, e\} = e$. When a C*-algebra A is regarded as a JB*-triple with the triple product in (5), it is known that the tripotents in A are precisely the partial isometries in A . In the same way that each partial isometry in a C*-algebra A induces a Peirce decomposition, each tripotent e in a JB*-triple E produces a *Peirce decomposition* of E in the form $E = E_2(e) \oplus E_1(e) \oplus E_0(e)$, where $E_i(e)$ is the $\frac{i}{2}$ eigenspace of the operator $L(e, e)$, $i = 0, 1, 2$. This decomposition satisfies the following *Peirce rules*:

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0$$

and

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

when $i - j + k \in \{0, 1, 2\}$ and is zero otherwise. The *Peirce k -projection*, $P_k(e)$, is the natural projection of E onto $E_k(e)$. Peirce projections are non-expansive (cf. [33, Corollary 1.2]) and they can be expressed in the following terms:

$$P_2(e) = Q(e)^2, \quad P_1(e) = 2(L(e, e) - Q(e)^2),$$

and

$$P_0(e) = Id_E - 2L(e, e) + Q(e)^2.$$

It is known that the Peirce-2 subspace $E_2(e)$ is a JB*-algebra with unit e , Jordan product $x \circ_e y := \{x, e, y\}$ and involution $x^{*e} := \{e, x, e\}$, respectively. It is worth to note that a linear bijection between JB*-triples is an isometry if and only if it is a triple isomorphism (cf. [49, Proposition 5.5]). Consequently, the triple product in $E_2(e)$ is uniquely given by

$$\{x, y, z\} = (x \circ_e y^{*e}) \circ_e z + (z \circ_e y^{*e}) \circ_e x - (x \circ_e z) \circ_e y^{*e},$$

for all $x, y, z \in E_2(e)$.

A subspace B of a JB*-triple E is a JB*-subtriple of E if $\{B, B, B\} \subseteq B$. A JB*-subtriple I of E is called an *inner ideal* of E if $\{I, E, I\} \subseteq I$. A subspace I of a C*-algebra A is an *inner ideal* if $IAI \subseteq I$. Every hereditary σ -unital C*-subalgebra of a C*-algebra is an inner ideal. A complete study on inner ideals of JB*-triples is available in [28,29] and the references therein. It follows from Peirce rules that for each tripotent e in a JB*-triple E , the Peirce-2 subspace $E_2(e)$ is an inner ideal.

Let E be a JB*-triple. The JB*-subtriple, E_a , of E generated by a single element a is identified, via the Gelfand theory, with the commutative C*-algebra

$$C_0(\Omega_a) = \{f : \Omega_a \rightarrow \mathbb{C} \text{ continuous with } f(0) = 0 \text{ if } 0 \in \Omega_a\},$$

for a unique compact set Ω_a contained in $[0, \|a\|]$, such that $\|a\| \in \Omega_a$ and 0 cannot be isolated in Ω_a ; and under this identification a corresponds to the continuous function given by the embedding of Ω_a into \mathbb{C} (cf. [49, Corollary 1.15] and [50, Lemma 3.2]). A consequence of this representation affirms that every element in a JB^* -triple admits a cubic root and a $(2n - 1)$ th-root ($n \in \mathbb{N}$) belonging to the JB^* -subtriple that it generates. The sequence $(a^{[\frac{1}{2n-1}]})$ of all $(2n - 1)$ th-roots of a converges in the weak* (and also in the strong*) topology of E^{**} to a tripotent in E^{**} , denoted by $r_{E^{**}}(a)$, and called the *range tripotent* of a . The tripotent $r_{E^{**}}(a)$ is the smallest tripotent $e \in E^{**}$ satisfying that a is positive in the JBW^* -algebra $E_2^{**}(e)$. It is also known that, if $\|a\| = 1$, the sequence $(a^{[2n-1]})$, of all odd-powers of a , converges in the weak*- and strong*-topology of E^{**} to a tripotent (called the *support tripotent* of a , $u(a)$ in E^{**} , which satisfies $u(a) \leq a \leq r_{E^{**}}(a)$ in $E_2^{**}(r_{E^{**}}(a))$ (compare [27, Lemma 3.3]; beware that in [30], $r(a)$ is called the support tripotent of a). In case that a is a positive element in a JB^* -algebra M , the support and the range tripotents of a in M^{**} are projections, called the *support* and *range projections* of a in M^{**} .

For each element a in a JB^* -triple E (in which we generally do not have a cone of positive elements), the symbol $E(a)$ will stand for the norm-closure of $\{a, E, a\} = Q(a)(E)$ in E . It was proved by L. J. Bunce, C.-H. Chu and B. Zalar that $E(a)$ is precisely the norm-closed inner ideal of E generated by a . Clearly, $E_a \subset E(a)$. It is further shown in the just quoted reference that $E(a)$ is a JB^* -subalgebra of the JBW^* -algebra $E(a)^{**} = \overline{E(a)}^{w*} = E_2^{**}(r_{E^{**}}(a))$ and contains a as a positive element, where $r_{E^{**}}(a)$ is the range tripotent of a in E^{**} (cf. [16, Proposition 2.1]).

The reader will need some basic knowledge on the strong*-topology of a JB^* -triple. If we are given a norm-one functional φ in the predual, W_* , of a JBW^* -triple W , and a norm-one element z in W with $\varphi(z) = 1$, the mapping

$$(x, y) \mapsto \varphi \{x, y, z\}$$

defines a positive sesquilinear form on W . Moreover, the mapping does not depend on the chosen z , that is, if $w \in W$ satisfies $\varphi(w) = 1$, we have $\varphi \{x, y, z\} = \varphi \{x, y, w\}$, for all $x, y \in W$ (see [9, Proposition 1.2]). The mapping $x \mapsto \|x\|_\varphi := (\varphi \{x, x, z\})^{\frac{1}{2}}$, defines a prehilbertian seminorm on W . The strong*-topology (denoted by $S^*(W, W_*)$) is the topology on W generated by the family of all semi-norms $\|\cdot\|_\varphi$ with φ running in the unit sphere of the predual of W (cf. [10]). For the purposes of this note we recall that the triple product of every JBW^* -triple W is jointly strong* continuous on bounded sets. The first proof of this result appeared in [60], however the difficulties affecting Grothendieck’s inequalities in [9] also impacted the original proof and an alternative argument can be found in [57, Theorem 9]. The recent proof of the Barton-Friedman conjecture on Grothendieck’s inequalities for JB^* -triples in [38] reinstates the validity of the original proof.

The strong*-topology of a JB*-triple E is defined as the restriction to E of the strong*-topology of its bidual.

The notion of orthogonality for non-necessarily hermitian elements in a JB*-algebra actually requires to identify JB*-algebras inside the class of JB*-triples. The general notion reads as follows: elements a, b in a JB*-triple E are said to be *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. It is known that $a \perp b$ if and only if $b \perp a$ if and only if $E(a) \perp E(b)$ (see [17, Lemma 1] for additional details).

2. An orderless approach to Rickart C*-algebras

This section is devoted to explore some equivalent reformulations of the notions of (weakly) Rickart and Baer C*-algebras in which we do not need the natural partial order nor the cone of positive elements. Our departure point is a result by G. K. Pedersen from [56], where a reformulation in terms of hereditary subalgebras is established.

We begin by recalling the definition of another class of C*-algebras introduced by G. K. Pedersen in [56]. A *SAW*-algebra* is a C*-algebra A satisfying that for any two orthogonal positive elements x and y in A there is a positive element e in A (which is not assumed to be a projection) such that $ex = x$ and $ey = 0$. In the commutative setting these SAW*-algebras correspond to C*-algebras of the form $C_0(L)$ for some sub-Stonian (locally compact Hausdorff) space L . It should be remarked that sub-Stonian spaces, studied by K. Grove and G. K. Pedersen in [36], are defined as those locally compact Hausdorff spaces in which disjoint σ -compact open subspaces have disjoint compact closures.

A C*-subalgebra B of a C*-algebra A is said to be a *hereditary C*-subalgebra* of A if whenever $0 \leq a \leq b$ with $a \in A$ and $b \in B$, then $a \in B$, equivalently, B^+ is a face of A^+ . It is known that a hereditary C*-subalgebra B of C*-algebra A is σ -unital if and only if it has the form $B = \overline{xAx}$ for some positive $x \in A$.

Proposition 2.1 ([56, Proposition 1]). *Let A be a C*-algebra. Consider the following condition: Given two orthogonal hereditary C*-subalgebras B and C of A , there is an element e in A^+ which is a unit for B and annihilates C . Then the following statements hold:*

- (AW*) *If this condition holds for all pairs B, C , then A is an AW*-algebra;*
- (WRC*) *If this condition holds when B is σ -unital and C is arbitrary, then A is a weakly Rickart C*-algebra;*
- (SAW*) *If this condition is true when both B and C are σ -unital, then A is a SAW*-algebra.*

Remark 2.2. It should be noted that the implications in (AW*), (WRC*) and (SAW*) are actually equivalences and characterizations of AW*-algebras, weakly Rickart C*-algebra, and SAW*-algebras. Namely, if A is an AW*-algebra and B and C are two orthogonal,

hereditary C^* -subalgebras of A , by the hypothesis on A , there exists a projection p in A such that $R(C) = pA$. Clearly, $cp = 0$ for all $c \in C$, and since B and C are orthogonal, $B \subset R(C) = pA$. Having in mind that B and C are self-adjoint, we deduce that p is a unit for B and annihilates C . If A is a weakly Rickart C^* -algebra, B is the closure of xAx for some positive x , and C is arbitrary, by the assumptions on A , there exists a projection p in A such that $xp = x$ and $xy = 0$ implies $py = 0$. Therefore p is a unit for B and annihilates C . The remaining equivalence can be similarly obtained.

Although it is not explicit in [56, Proposition 1], the following equivalence also holds by the same arguments:

(RC*) The condition in Proposition 2.1 holds when C is σ -unital and B is arbitrary if, and only if, A is a Rickart C^* -algebra.

Let us briefly recall some basic facts on range projections. Suppose a is a positive element in a von Neumann algebra W . The *range projection* of a in W (denoted by $rp(a)$) is the smallest projection p in W satisfying $ap = a$. It is known that the sequence $((\frac{1}{n}\mathbf{1} + a)^{-1}a)_n$ is monotone increasing to $rp(a)$, and hence it converges to $rp(a)$ in the weak*-topology of W . If a is in the closed unit ball of W , the sequence $(a^{\frac{1}{n}})_n$ is monotone increasing and converges to $rp(a)$ in the weak*-topology of W . Actually, for any element x in W , the smallest projection l in W with $lx = x$ is called the *left range projection* of x and denoted by $s_l(x)$. The *right range projection* $s_r(x)$ is the smallest projection q in W with $xq = x$ (cf. [66, Definition 1.4] or [55, 2.2.7]). It is known that $r(x^*x) = s_r(x)$ and $r(xx^*) = s_l(x)$, while $r(xx^*) = s_l(x) = s_r(x)$ for any self-adjoint x . If x is an element in a C^* -algebra A , we shall usually employ the left and right range projections of x in A^{**} . If A is a Rickart C^* -algebra, for each $x \in A$ we have $s_r(x) \leq RP(x)$ and $s_l(x) \leq LP(x)$ in A^{**} .

An element e in a C^* -algebra A is a partial isometry if ee^* (equivalently, e^*e) is a projection in A . Each partial isometry $e \in A$ induces a Peirce decomposition of A in the form $A = A_2(e) \oplus A_1(e) \oplus A_0(e)$, where $A_2(e) = ee^*Ae^*e$, $A_1(e) = (\mathbf{1} - ee^*)Ae^*e \oplus ee^*A(\mathbf{1} - e^*e)$, and $A_0(e) = (\mathbf{1} - ee^*)A(\mathbf{1} - e^*e)$. The subspace $A_j(e)$ is called the Peirce- j subspace. The Peirce-2 subspace $A_2(e)$ is a unital C^* -algebra, with unit e , when equipped with the original norm, product $a \bullet_e b = ae^*b$ and involution $a^{*e} = ea^*e$ ($a, b \in A$).

A couple of projections p, q in a C^* -algebra A are said to be (Murray-von Neumann) *equivalent*, $p \sim q$, if $p = ee^*$ and $q = e^*e$ for some partial isometry $e \in A$. A unital C^* -algebra A is finite if $p \sim \mathbf{1}$ implies $p = \mathbf{1}$.

In our seeking of an order-free characterization of (weakly) Rickart C^* -algebras, which can be employed to define an appropriate notion in general JB*-triples, we shall need the following milestone result due to P. Ara: “*Left and right projections are (Murray-von Neumann) equivalent in Rickart C^* -algebras*” (see [2] where this famous conjecture by I. Kaplansky was proved). The same conclusion actually holds for weakly Rickart C^* -algebras. The result is included here for the lacking of an explicit reference.

Lemma 2.3. *Let A be a weakly Rickart C^* -algebra. Then the left and right projections of every element in A are Murray-von Neumann equivalent.*

Proof. Let x be an element in a weakly Rickart C^* -algebra A . If A is unital, the conclusion follows from Ara’s theorem [2, Theorem 2.5]. So, we shall assume that A is non-unital.

By [12, Theorem 5.1] (see also [63, Lemma 3.6]) we can find a unitization $A_1 = A \oplus \mathbb{C}\mathbf{1}$ of A which is a Rickart C^* -algebra. Fix $x \in A$. Let $RP(x)$ and $LP(x)$ denote the right and left projections of x (in A or in A_1). Let us observe that these symbols offer no ambiguity. More concretely, if $e = LP_A(x)$ is the ALP of x in A , Lemma 5.3 in [12] assures that e is the ALP of $x \in A$ in A_1 , that is, $LP_A(x) = LP_{A_1}(x)$. Similarly, $RP_A(x) = RP_{A_1}(x) \in A$.

By applying [2, Theorem 2.5] we deduce that $LP(x)$ and $RP(x)$ are equivalent projections in A_1 , that is, there exists a partial isometry $e \in A_1$ such that $e^*e = RP(x)$ and $ee^* = LP(x)$.

We shall finally show that $e \in A$. Let us write $e = e_1 + \lambda\mathbf{1}$ with e_1 in A and $\lambda \in \mathbb{C}$. Since $A \ni LP(x) = ee^* = e_1e_1^* + \lambda e_1^* + \bar{\lambda}e_1 + |\lambda|^2\mathbf{1}$, it follows that $\lambda = 0$, and thus $e = e_1 \in A$. \square

Remark 2.4. We have already commented that the idea behind Rickart’s original paper [59, Theorem 2.10] (see also [12, Proposition 8.1]) was to show that every Rickart C^* -algebra is generated by its projections. Actually, the same occurs for weakly Rickart C^* -algebras. Namely, let a be a positive element in a weakly Rickart C^* -algebra A . Let $p = RP(a)$ denote the ARP projection of a in A when the latter is regarded as a weakly Rickart C^* -algebra. It follows from [12, Proposition 5.6] that pAp is a Rickart C^* -algebra with unambiguous left and right projections for every element in pAp . It follows from the mentioned Theorem 2.10 in [59] that pAp is generated by its projections. In particular $a \in pAp$ can be approximated in norm by finite linear combinations of projections.

Given a positive element a in a C^* -algebra A , the hereditary C^* -subalgebra of A generated by a coincides with the norm closure, \overline{aAa} , of aAa and contains the C^* -subalgebra generated by a (see [54, Corollary 3.2.4]). This hereditary C^* -subalgebra is precisely the inner ideal of A generated by a , when A is regarded as a JB^* -triple (cf. [16, pages 19-20]). Therefore the symbol $A(a)$ will denote the hereditary C^* -subalgebra of A generated by a . It is further known, even in a more general setting, that $A(a)^{**}$ identifies with $(A^{**})_2(rp(a)) = rp(a)A^{**}rp(a)$ (because $rp(a)$ is a projection), and $A(a)$ is actually a C^* -subalgebra of this latter hereditary C^* -subalgebra of A^{**} (cf. [16, Proposition 2.1] whose proof is valid here too). It is worth mentioning that

$$A(a) = (A^{**})_2(rp(a)) \cap A. \tag{7}$$

Namely, the inclusion \subseteq is clear. We may clearly assume that $\|a\| \leq 1$. On the other hand, it is not hard to see that a is a strictly positive element in the hereditary C^* -subalgebra $I = (A^{**})_2(rp(a)) \cap A$, and hence $(a^{\frac{1}{n}})_n$ is an approximate identity in I (cf.

[66, Exercise 3 in page 31]). Given $x \in I$, the sequence $(a^{\frac{1}{n}}xa^{\frac{1}{n}})_n$ converges in norm to x and is contained in $A(a)$, therefore $x \in A(a)$.

It is well known that every σ -unital hereditary C^* -subalgebra of A is of the form $A(x)$, with x positive in A (cf. [54, Theorem 3.2.5], see also [55, §1.5] and [54, §3.2] for a detailed discussion on hereditary C^* -subalgebras and ideals). Moreover, as commented by G. K. Pedersen in [56, page 16], σ -unital hereditary C^* -subalgebras of A can be also represented in the form $\overline{(Ay)} \cap \overline{(y^*A)}$ with $y \in A$. Clearly, each hereditary C^* -subalgebra of the form $A(a)$ with $a \geq 0$ writes in the form $\overline{(Aa)} \cap \overline{(a^*A)}$ (just apply (7) in the non-trivial inclusion). On the other direction, for each $y \in A$, we shall show that $\overline{(Ay)} \cap \overline{(y^*A)} = A(y^*y)$. Indeed, since $\overline{(Ay)} \cap \overline{(y^*A)}$ is an inner ideal and contains y^*y , we deduce that $\overline{(Ay)} \cap \overline{(y^*A)} \supseteq A(y^*y)$. If we take $z \in \overline{(Ay)} \cap \overline{(y^*A)}$, we clearly have $r(y^*y)z = zr(y^*y) = z$, and thus $\overline{(Ay)} \cap \overline{(y^*A)} \subseteq A(y^*y)$ (cf. (7)).

For a general element x in a C^* -algebra A , the inner ideal of A generated by x can be described as the norm closure of xAx (cf. [16, pages 19-20]).

Let us recall that elements a, b in a C^* -algebra A are called orthogonal ($a \perp b$ in short) if $ab^* = b^*a = 0$. The orthogonal complement of a subset $\mathcal{S} \subset A$ is defined as $\mathcal{S}^\perp := \{a \in A : a \perp x \text{ for all } x \in \mathcal{S}\}$.

Proposition 2.5. *Let A be a C^* -algebra. Then the following statements hold:*

- (a) *A is a weakly Rickart C^* -algebra if, and only if, given $x \in A$ and an inner ideal $J \subseteq A$ with $I = A(x) \perp J$, there exists a partial isometry e in A such that $I \subseteq A_2(e)$ and $J \subseteq A_0(e)$;*
- (b) *A is a Rickart C^* -algebra if, and only if, A is unital and given $x \in A$ and an inner ideal $J \subseteq A$ with $I = A(x) \perp J$, there exists a partial isometry e in A such that $I \subseteq A_2(e)$ and $J \subseteq A_0(e)$.*

Proof. (a) (\Rightarrow) By Lemma 2.3 $LP(x)$ and $RP(x)$ are equivalent projections in A , that is, there exists a partial isometry $e \in A$ such that $e^*e = RP(x)$ and $ee^* = LP(x)$. Clearly, $x \in A_2(e)$, and hence $\{x, A, x\} \subseteq A_2(e)$. It follows that $A(x) \subseteq A_2(e)$.

On the other hand, for each $y \in J \perp A(x)$ we have $x^*y = 0 = yx^*$, and since $e^*e = RP(x) = LP(x^*)$ and $ee^* = LP(x) = RP(x^*)$ we deduce that $ee^*y = 0 = ye^*e$, witnessing that $y \in A_0(e)$.

(\Leftarrow) This implication follows from Proposition 2.1 and its proof in [56, Proposition 1], we shall revisit the argument for completeness. Fix $y \in A$ and consider the inner ideal $I = \overline{(Ay)} \cap \overline{(y^*A)} = A(y^*y)$. Let $R = R(y)$ denote the right annihilator of y in A . In this case $R \cap R^* = \{y^*y\}^\perp := J$. By the assumptions, there exists a partial isometry $e \in A$ such that

$$I \subseteq A_2(e) = ee^*Ae^*e \text{ and } J \subseteq A_0(e) = (1 - ee^*)A(1 - ee^*).$$

Therefore, $y^*y = ee^*y^*ye^*e$, and thus $s_r(y)e^*e = r_{A^{**}}(y^*y)e^*e = r_{A^{**}}(y^*y) = s_r(y)$ in A^{**} and $ye^*e = yr_{A^{**}}(y^*y)e^*e = yr_{A^{**}}(y^*y) = y$.

If $z \in R$, we have $zz^* \in R \cap R^* \subseteq A_0(e) = (\mathbf{1} - ee^*)A(\mathbf{1} - e^*e)$, which proves that $zz^* = (\mathbf{1} - ee^*)zz^*(\mathbf{1} - e^*e)$, and $zz^* = (\mathbf{1} - e^*e)zz^*(\mathbf{1} - ee^*)$. By repeating the arguments above we get $(\mathbf{1} - e^*e)r_{A^{**}}(zz^*) = r_{A^{**}}(zz^*)$ leading to $e^*e s_l(z) = 0$ in A^{**} , and to $e^*ez = e^*e s_l(z)z = 0$.

(b) This is clear from (a) and the fact that a C^* -algebra is a Rickart C^* -algebra if and only if it is weakly Rickart and unital (cf. [12, Proposition 5.2]). \square

The advantage of the previous proposition is that the equivalent reformulations do not depend on the natural partial order given by the cone of positive elements in a C^* -algebra.

Remark 2.6. Let A be a C^* -algebra. Clearly A is a SAW*-algebra if given $x, y \in A$ with $x \perp y$, there exists a partial isometry e in A such that $I = A(x) \subseteq A_2(e)$ and $J = A(y) \subseteq A_0(e)$ (cf. [56, Proposition 1]). We do not know if the reciprocal implication holds. The lacking of an analogue of Ara’s theorem in [2, Theorem 2.5] proving the equivalence of left and right projections in the setting of SAW*-algebras seems to be a major obstacle.

In the light of Pedersen’s result in Proposition 2.1 and the characterization in terms of inner ideals given in Proposition 2.5, it seems natural to ask if a characterization of Baer or AW*-algebras can be obtained in terms of inner ideals. If we assume some extra hypothesis the answer is yes.

Proposition 2.7. *Let A be a finite unital C^* -algebra. Then the following statements hold:*

- (a) *A is a Rickart C^* -algebra if, and only if, given $x \in A$ and an inner ideal $J \subseteq A$ with $I = A(x) \perp J$, there exists a partial isometry e in A such that $J \subseteq A_2(e)$ and $I \subseteq A_0(e)$;*
- (b) *A is an AW*-algebra if, and only if, for any family $\{x_i\}_i$ of mutually orthogonal elements in A and each inner ideal $J \subseteq A$ with $A(x_i) \perp J$ for all i , there exists a partial isometry $e \in A$ satisfying $J \subseteq A_2(e)$ and $A(x_i) \subseteq A_0(e)$ for all i .*

Proof. (a) (\Rightarrow) Let us fix $x \in A$. Since A is a finite Rickart C^* -algebra, $LP(x)$ and $RP(x)$ are unitarily equivalent [40, Theorem 4.1(c)], that is, there exists a unitary $u \in A$ such that $RP(x) = uLP(x)u^*$, and hence $\mathbf{1} - RP(x) = u(\mathbf{1} - LP(x))u^*$. Set $e = (\mathbf{1} - LP(x))u^*$. Clearly, e is a partial isometry with $ee^* = \mathbf{1} - LP(x)$ and $e^*e = \mathbf{1} - RP(x)$, and $x \in LP(x)$ A $RP(x) = (\mathbf{1} - ee^*)A(\mathbf{1} - e^*e) = A_0(e)$. This proves that $A(x) \subseteq A_0(e)$.

If we take $y \in J \perp I$, it follows that $yx^* = x^*y = 0$, which implies that $yRP(x) = LP(x)y = 0$, and thus $y \in (\mathbf{1} - LP(x))A(\mathbf{1} - RP(x)) = A_2(e)$.

(\Leftarrow) is a consequence of the equivalence in (RC*) in page 577.

(b) (\Rightarrow) Suppose A is an AW*-algebra (the projections in A form a complete lattice [12, Proposition 4.1]). Let us take a family $\{x_i\}_i$ of mutually orthogonal elements in A and an inner ideal $J \subseteq A$ with $A(x_i) \perp J$ for all i . It follows from the hypothesis that $RP(x_i) \perp RP(x_j)$ and $LP(x_i) \perp LP(x_j)$, for all $i \neq j$. By [40, Theorem 4.1(c)] $LP(x_i)$ and $RP(x_i)$ are unitarily equivalent, and hence equivalent via a partial isometry w_i for all i . Theorem 20.1(iii) in [12] proves the existence of a partial isometry w such that $ww^* = \bigvee_i LP(x_i)$, $w^*w = \bigvee_i RP(x_i)$ and $wRP(x_i) = w_i = LP(x_i)w$ for all i (i.e. orthogonal partial isometries in an AW*-algebra are addable). Applying once again that A is a finite Rickart C*-algebra we deduce that ww^* and w^*w are unitarily equivalent [40, Theorem 4.1(c)], and thus $\mathbf{1} - ww^*$ and $\mathbf{1} - w^*w$ are equivalent. Let us take a partial isometry e in A with $ee^* = \mathbf{1} - ww^*$ and $e^*e = \mathbf{1} - w^*w$. It is easy to check that, by construction,

$$\begin{aligned} A_0(e) &= (\mathbf{1} - ee^*)A(\mathbf{1} - e^*e) = ww^*Aw^*w \\ &= \left(\bigvee_i LP(x_i)\right)A\left(\bigvee_i RP(x_i)\right) \supseteq LP(x_{i_0})ARP(x_{i_0}) = A(x_{i_0}), \end{aligned}$$

for all i_0 . Given $y \in J$ and an index i_0 , it follows from the properties of the left and right projections of x_{i_0} and the fact that $J \perp x_{i_0}$, that

$$J \subseteq (\mathbf{1} - LP(x_{i_0}))A(\mathbf{1} - RP(x_{i_0})), \text{ for all } i_0,$$

and thus

$$\begin{aligned} J &\subseteq \left(\bigwedge_i (\mathbf{1} - LP(x_{i_0}))\right)A\left(\bigwedge_i (\mathbf{1} - RP(x_{i_0}))\right) \\ &= \left(\mathbf{1} - \bigvee_i LP(x_{i_0})\right)A\left(\mathbf{1} - \bigvee_i RP(x_{i_0})\right) = ee^*Ae^*e = A_2(e). \end{aligned}$$

(\Leftarrow) It follows from (a) that A is a Rickart C*-algebra, and thus A is unital. Let $\{p_i\}_{i \in \Gamma}$ be a family of mutually orthogonal projections in A . By applying the hypothesis to the inner ideal $J := \{x \in A : x \perp p_i \text{ for all } i \in \Gamma\}$, we deduce the existence of a partial isometry $e \in A$ such that $J \subseteq A_2(e)$ and $A(p_i) = A_2(p_i) \subseteq A_0(e)$ for all $i \in \Gamma$. The element $q = \mathbf{1} - ee^*$ is a projection in A satisfying $qp_i = p_i$ (equivalently, $q \geq p_i$) for all $i \in \Gamma$. Let r be any other projection in A with $r \geq p_i$ for all $i \in \Gamma$. The property $(\mathbf{1} - r)p_i = 0$ for all i , implies that $\mathbf{1} - r \in J \subseteq A_2(e)$, and thus $ee^*(\mathbf{1} - r) = \mathbf{1} - r$ and $(\mathbf{1} - ee^*)(\mathbf{1} - r) = 0$, witnessing that $q = \mathbf{1} - ee^* \leq r$, and therefore $q = \bigvee_i p_i$ in A . We have shown that every orthogonal family of projections in A has a supremum. Proposition 4.1(a) \Leftrightarrow (c) in [12] proves that A is an AW*-algebra. Actually, by applying the same argument with $\mathbf{1} - e^*e$ instead $\mathbf{1} - ee^*$ we get $\mathbf{1} - e^*e = \bigvee_i p_i = \mathbf{1} - ee^*$. \square

Remark 2.8. The characterization provided in Proposition 2.7 is not valid without the hypothesis of finiteness. Consider, for example, the Hilbert space $H = \ell_2$ with orthonormal basis $\{\xi_n : n \in \mathbb{N}\}$ and $A = B(H)$. Take a partial isometry v such that $\mathbf{1} - vv^* = \xi_1 \otimes \xi_1$ is a rank-one projection and $\mathbf{1} - v^*v = \xi_1 \otimes \xi_1 + \xi_2 \otimes \xi_2$ has rank 2. If for $J = \{v\}^\perp = A_0(v) \perp A(v) = A_2(v)$ there were a partial isometry e satisfying that $(\xi_1 \otimes \xi_1)A(\xi_1 \otimes \xi_1 + \xi_2 \otimes \xi_2) = A_0(v) = J \subseteq A_2(e)$ and $(\mathbf{1} - \xi_1 \otimes \xi_1)A(\mathbf{1} - \xi_1 \otimes \xi_1 - \xi_2 \otimes \xi_2) = A_2(v) \subseteq A_0(e)$ we would have $\xi_1 \otimes \xi_1 \leq ee^*$, $\xi_1 \otimes \xi_1 + \xi_2 \otimes \xi_2 \leq e^*e$, $\mathbf{1} - \xi_1 \otimes \xi_1 \leq \mathbf{1} - ee^*$ and $(\mathbf{1} - \xi_1 \otimes \xi_1 - \xi_2 \otimes \xi_2) \leq \mathbf{1} - e^*e$. Therefore $\xi_1 \otimes \xi_1 = ee^*$ and $\xi_1 \otimes \xi_1 + \xi_2 \otimes \xi_2 \leq e^*e$, which is impossible.

Remark 2.9. The partial isometry e appearing in the statements of Proposition 2.5 need not be unique. Actually if e is a partial isometry satisfying the desired conclusion, then the partial isometry λe satisfies the same property for all λ in the unit sphere of \mathbb{C} .

The partial isometry e appearing in Proposition 2.5(a) induces a local order in the C^* -algebra $(A_2(e), \bullet_e, *_e)$ and we actually obtain a strengthened version of the statement.

Proposition 2.10. *Let A be a C^* -algebra. Then A is a weakly Rickart C^* -algebra if, and only if, given $x \in A$ and an inner ideal $J \subseteq A$ with $I = A(x) \perp J$, there exists a partial isometry e in A such that $I \subseteq A_2(e)$, $e^*e = RP(x)$, $ee^* = LP(x)$, x is a positive element in the C^* -algebra $(A_2(e), \bullet_e, *_e)$, $A(x)$ is a C^* -subalgebra of the latter C^* -algebra and $J \subseteq A_0(e)$.*

Proof. It suffices to prove the extra properties in the “only if” implication. Suppose A is a weakly Rickart C^* -algebra. We shall assume that A is non-unital, and its unitization $A_1 = A \oplus \mathbb{C}\mathbf{1}$ is a Rickart C^* -algebra [12, Theorem 5.1] (see also [63, Lemma 3.6]).

Fix $x \in A$. Another essential contribution by P. Ara and D. Goldstein (see [3, Corollary 3.5], [35, Corollary 7.4]) assures the existence of a polar decomposition for x , that is, there exists a partial isometry $e \in A_1$ such that $x = e|x|$, $ee^* = LP(x)$ and $e^*e = RP(x)$ (cf. also [12, Proposition 21.3]). If we write e in the form $e = e_1 + \lambda\mathbf{1}$ with $\lambda \in \mathbb{C}$, $e_1 \in A$, we infer from the fact $e_1e_1^* + \lambda e_1^* + \bar{\lambda}e_1 + |\lambda|^2\mathbf{1} = ee^* = LP(x) \in A$ that $e = e_1 \in A$, that is, weakly Rickart C^* -algebras satisfy the existence of polar decompositions.

Let $I = A(x)$ and let J be an inner ideal orthogonal to I . By considering the partial isometry e in the polar decomposition of x , we can easily check that x is a positive element in the C^* -algebra $(A_2(e), \bullet_e, *_e)$, namely, $ee^*(e|x|^{\frac{1}{2}})e^*e = e|x|^{\frac{1}{2}} = (e|x|^{\frac{1}{2}})^{*e}$, $(e|x|^{\frac{1}{2}}) \bullet_e (e|x|^{\frac{1}{2}}) = e|x| = x$, and hence x is positive in $(A_2(e), \bullet_e, *_e)$. Finally, given $y \in J$ the conditions $x \perp y$, $ee^* = LP(x)$ and $e^*e = RP(x)$ imply that $y \perp e$, and therefore $J \subseteq A_0(e)$. \square

Corollary 2.11. *Let A be a C^* -algebra. Then A is a weakly Rickart C^* -algebra if, and only if, given $x \in A$ there exists a partial isometry e in A such that $A(x) \subseteq A_2(e)$, x is a positive element in the C^* -algebra $(A_2(e), \bullet_e, *_e)$, and $A_0(e) = \{x\}^\perp$.*

Proof. (\Rightarrow) By applying Proposition 2.10 to $I = A(x)$ and $J = \{x\}^\perp$ we find a partial isometry $e \in A$ satisfying that $I \subseteq A_2(e)$, $e^*e = RP(x)$, $ee^* = LP(x)$, x is a positive element in the C^* -algebra $(A_2(e), \bullet_e, *_e)$, $A(x)$ is a C^* -subalgebra of $(A_2(e), \bullet_e, *_e)$, and $\{x\}^\perp = J \subseteq A_0(e)$.

We shall show that $\{x\}^\perp = A_0(e)$. To this end take $a \in A_0(e)$. The identities $xa^* = xRP(x)(\mathbf{1} - e^*e)a^* = x(e^*e)(\mathbf{1} - e^*e)a^* = 0$, and $a^*x = a^*(\mathbf{1} - ee^*)LP(x)x = a^*(\mathbf{1} - ee^*)(ee^*)x = 0$, show that $a \in \{x\}^\perp$.

(\Leftarrow) This is a clear consequence of Proposition 2.10, since for each $x \in A$ and each inner ideal $J \subseteq A$ with $I = A(x) \perp J$, by taking the partial isometry e given by the hypothesis we have $J \subseteq \{x\}^\perp = A_0(e)$ and $A(x) \subseteq A_2(e)$. \square

We have seen in the proof of Proposition 2.10 that, as a consequence of the result by P. Ara and D. Goldstein [3,35], weakly Rickart C^* -algebras satisfy polar decomposition. It is well known that the partial isometry appearing in the polar decomposition of an element a is uniquely determined by $|a|$ (cf. [12, Propositions 21.1 and 21.3]). We shall conclude this section by showing that the properties of the partial isometry e in Corollary 2.11 provide a characterization of the partial isometry in the polar decomposition.

Corollary 2.12. *Let x be an element in a weakly Rickart C^* -algebra A . Suppose e is a partial isometry in A . Then the following are equivalent:*

- (a) *e is the partial isometry in the polar decomposition of x ;*
- (b) *x is a positive element in the C^* -algebra $(A_2(e), \bullet_e, *_e)$, and $A_0(e) = \{x\}^\perp$.*

Proof. The implication (a) \Rightarrow (b) has been proved in the proof of Corollary 2.11.

(b) \Rightarrow (a) Since e is a partial isometry, the elements ee^* and e^*e are projections in A . It is known that ee^*Aee^* and e^*eAe^*e are Rickart C^* -algebras (cf. [12, Proposition 5.6]). Since the mapping $z \mapsto ze^*$ (respectively, $z \mapsto e^*z$) is a C^* -isomorphism from $(A_2(e), \bullet_e, *_e)$ onto ee^*Aee^* (respectively, e^*eAe^*e), we derive that $(A_2(e), \bullet_e, *_e)$ is a Rickart C^* -algebra.

We shall next show that the left and right projections of x in $A_2(e)$ both coincide with e . Since x is positive in $A_2(e)$, we have $RP_{A_2(e)}(x) = LP_{A_2(e)}(x) = q$. Clearly $q \leq e$ in $A_2(e)$. If $q < e$, the partial isometry (projection in $A_2(e)$) $e - q$ is orthogonal to q in $A_2(e)$ and also in A , because orthogonality in A can be given in terms of the triple product $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$ and $A_2(e)$ is closed for this triple product (see section 4 for additional details). When the triple product is computed with respect to the C^* -product of $A_2(e)$ and with respect to the one in A we have

$$x = \{q, x, q\}_{A_2(e)} = q \bullet_e x^{*e} \bullet_e q = qe^*ex^*ee^*q = qx^*q = \{q, x, q\}.$$

It follows that x belongs to $A_2(q)$, which combined with the fact $e - q \perp q$, implies that $x \perp e - q$. It follows from the hypotheses that $e - q \in A_0(e)$. Therefore $e - q = e \bullet_e (e - q) = ee^*(e - q) = 0$, leading to a contradiction.

Since x is positive in $A_2(e)$, $RP_{A_2(e)}(x) = LP_{A_2(e)}(x) = e$ in this C^* -algebra, and the mapping $z \mapsto e^*z$ is a C^* -isomorphism from $(A_2(e), \bullet_e, *_e)$ onto e^*eAe^*e , we deduce that e^*x is a positive element in A with $ee^* = LP_{A_2(e)}(x)e^* = LP(e^*x)$. Similarly, $e^*e = e^*RP_{A_2(e)}(x) = RP(e^*x)$ (have in mind that the left and right projections of xe^* and e^*x do not change when computed in A or in ee^*Aee^* or e^*eAe^*e , respectively [12, Proposition 5.6]). Furthermore, since

$$((e^*x)^*(e^*x))^n = (x^*ee^*x)^n = (x^*x)^n, \text{ for all natural } n,$$

it can be deduced, via functional calculus, that $|x| = e^*x$.

It clearly follows from the hypotheses that $x = ee^*x = e|x|$. We have seen above that $RP(e^*x) = e^*e$ and $ee^* = LP(e^*x)$. Therefore e is the partial isometry in the polar decomposition of x by uniqueness. \square

3. Jordan counterparts of Rickart and Baer $*$ -algebras in terms of projections

Sh. A. Ayupov and F. N. Arzikulov developed a deep study on the notions of Rickart and Baer $*$ -rings in the setting of real Jordan algebras in the papers [7,8,4,5]. Before entering into details, we introduce the required nomenclature.

Let M be a Jordan algebra. According to the standard notation (see [7,58]), the (*outer*) *quadratic annihilator* of a subset $S \subset M$ is the set

$$\text{Ann}(S) = S^{\perp_q} := \{a \in M : U_a(S) = \{0\}\}. \tag{8}$$

The *inner quadratic annihilator* of S is formed by the elements in the intersection of all kernels of all U -maps associated with elements in S defined by

$${}^{\perp_q}S := \{a \in M : U_s(a) = 0 \text{ for all } s \in S\}. \tag{9}$$

Let us denote $M^2 := \{a^2 : a \in M\}$ for the set of all elements in M which are the square of another element (do not confuse with the set of all elements of the form $a \circ b$ with $a, b \in M$). Clearly, each idempotent in M is inside M^2 . We consider the following two statements:

- (R1) For each element $a \in M^2$ there exists an idempotent $e \in M$ (i.e. $e^2 = e$) such that $\{a\}^{\perp_q} = U_e(M)$;
- (R2) For each element $x \in M$ there exists an idempotent $e \in M$ such that ${}^{\perp_q}\{x\} \cap M^2 = U_e(M) \cap M^2$.

In any Jordan algebra M , (R1) implies (R2) and both properties are equivalent when M is unital and lacks of nilpotent elements (cf. [7, Theorems 1.6 and 1.7]). According to [7,8], a Jordan algebra M satisfying condition (R1) (respectively, (R2)) is called a

Rickart Jordan algebra (respectively, an *inner Rickart Jordan algebra*). That is, each Rickart Jordan algebra is an inner Rickart Jordan algebra. It should be noted here that in [8] inner Rickart Jordan algebras are called *weak Rickart Jordan algebras*, however since the term weak Rickart algebra is employed in the associative setting with another meaning (for example, for an uncountable set Γ the commutative C^* -algebra $\ell_{\infty,c}(\Gamma)$ of all countably supported elements of the commutative von Neumann algebra $\ell_{\infty,c}(\Gamma)$ is weak Rickart but not an inner Jordan Rickart algebra see, for example, [12]), here we shall employ the term mentioned above.

The notion of (inner) Rickart is essentially addressed to real JB-algebras. For example, the exceptional JB-algebra $H_3(\mathbb{O})$ is a Rickart Jordan algebra (cf. [7, Proposition 3.4]). Moreover, for each associative Rickart $*$ -algebra A , its self-adjoint part A_{sa} is a Jordan algebra satisfying (R1) and (R2) (cf. [7, Proposition 1.1]). Reciprocally, if A is an associative $*$ -algebra with proper involution and A_{sa} is a Rickart Jordan algebra, then A is a Rickart $*$ -algebra in the usual sense ([7, Proposition 1.3]).

Every Rickart Jordan algebra possesses a unit element and lacks of nilpotent elements, it is further known that the set of idempotents of a Rickart Jordan algebra is a lattice, which is not, in general, complete (see [7, Lemma 1.4, Proposition 1.10]).

There exist examples of inner Rickart Jordan algebras without unit element (cf. [8, Remark 1 in page 32]). However the properties gathered in the next lemma hold:

Lemma 3.1 ([8, Lemma 2.3]). *Let M be an inner Rickart Jordan algebra. Then the following statements hold:*

- (a) *There exists an element $\mathbf{1}_2$ in M satisfying $a \circ \mathbf{1}_2 = a$ for every $a \in M^2$;*
- (b) *M^2 contains no non-trivial nilpotent elements.*

The element $\mathbf{1}_2$ given in the above statement (a) is a unit for those elements in M^2 . If M is generated by square elements (i.e., every element is a finite linear combination of elements in M^2), then the element $\mathbf{1}_2$ actually is a unit in M .

Corollary 3.2 ([7, Theorems 1.6 and 1.7]). *Suppose M is a Jordan algebra linearly generated by M^2 and containing no non-trivial nilpotent elements. Then M is a Rickart Jordan algebra if and only if it is an inner Rickart Jordan algebra.*

The lacking of associativity in Jordan algebras is somehow compensated with the celebrated *Macdonald’s theorem* asserting that if G is a multiplication operator in two variables x, y with $G(a, b) = 0$ for all a, b in all special Jordan algebras, then $G = 0$ in all Jordan algebras, equivalently, any polynomial identity in three variables, with degree at most 1 in the third variable, and which holds in all special Jordan algebras, holds in all Jordan algebras (cf. [39, Theorem 2.4.13]). The following identities, which hold true for any Jordan algebra M , can be directly deduced from Macdonald’s theorem:

$$2T_{a^l}U_{a^m,a^n} = 2U_{a^m,a^n}T_{a^l} = U_{a^{m+l},a^n} + U_{a^m,a^{n+l}}, \tag{10}$$

$$U_a^n = U_{a^n}, \tag{11}$$

for every natural numbers l, m, n (see [39, Lemma 2.4.21]).

In the set of all idempotents in a Jordan algebra M we can consider a partial order defined by $e \leq f$ if $e \circ f = e$. The following equivalences can be easily checked by applying (10) and (11):

$$e \leq f \Leftrightarrow e \in U_f(M) \Leftrightarrow U_e(M) \subseteq U_f(M). \tag{12}$$

A Jordan algebra M is called a *Baer Jordan algebra* if it satisfies the following property: For each subset $S \subset M^2$ there exists an idempotent $e \in M$ such that $S^{\perp_q} = U_e(M)$. We say that M is an *inner Baer Jordan algebra* if for each subset $S \subset M$ there exists an idempotent $e \in M$ such that ${}^{\perp_q}S \cap M^2 = U_e(M) \cap M^2$.

Let us observe that in [7,8,4–6] inner Baer Jordan algebras are called *weak Baer Jordan algebras*, which is a term not completely compatible with the notation in the associative setting.

Each Baer Jordan algebra is an inner Baer Jordan algebra [7, Theorem 2.6] or [8, Proposition 3.1]. If M is a Jordan algebra containing no nilpotent elements, then M is an inner Baer Jordan algebra if and only if it is a Baer Jordan algebra [7, Theorem 2.6]. As we have seen in the comments after Lemma 3.1, if a Jordan algebra M is linearly generated by elements in M^2 and M is an inner Baer Jordan algebra, then M is unital. A C*-algebra is a Baer C*-algebra if and only if A_{sa} is a Baer Jordan algebra (cf. [7, Propositions 2.1 and 2.3] or [8]).

To conclude our tour through the algebraic Jordan alter-egos of Rickart and Baer algebras, we appeal to a couple of results also proved by Sh. A. Ayupov and F. N. Arzikulov, where they establish that a Jordan algebra M is a Baer Jordan algebra if, and only if, it is a Rickart Jordan algebra and the set of all idempotents in M is a complete lattice (see [7, Theorem 2.7]); moreover, M is an inner Baer Jordan algebra if, and only if, it is an inner Rickart Jordan algebra and the set of all idempotents of M is a complete lattice (cf. [8, Theorem 3.5]).

Following [7,8,4,5], and in coherence with the terminology of C*-algebras, (*inner*) *Rickart JB-algebras* and (*inner*) *Baer JB-algebras* or *AJBW-algebras* are defined as those JB-algebras which are (*inner*) Rickart and (*inner*) Baer Jordan algebras, respectively. We shall also deal with the complex structures. A JB*-algebra M will be called a *Rickart JB*-algebra* (respectively, a *Baer JB*-algebra* or an *AJBW*-algebra*) if its self-adjoint part, M_{sa} , is a Rickart JB-algebra (respectively, a Baer JB-algebra or an AJBW-algebra). That is, M is a Rickart JB*-algebra if and only if for each $a \in M^+$ there exists a projection $p \in M$ such that

$$\{a\}^{\perp_q} \cap M_{sa} = U_p(M) \cap M_{sa} = Q_p(M) \cap M_{sa};$$

which by Corollary 3.2 is equivalent to prove that for each $x \in M_{sa}$ there exists a projection $p \in M$ such that

$$\perp^q \{x\} \cap M^+ = U_p(M) \cap M^+ = Q(p)(M) \cap M^+.$$

A similar restatement can be applied to the definition of Baer JB*-algebras. A JBW*-algebra (respectively, a JBW-algebra) is a JB*-algebra (respectively, a JB-algebra) which is a dual Banach space. It is known that a JB*-algebra M is a JBW*-algebra if, and only if, M_{sa} is a JBW-algebra (cf., for example, [53, Corollary 2.12]).

Two elements a, b in a Jordan algebra A are said to operator commute if

$$a \circ (b \circ x) = (a \circ x) \circ b$$

for every $x \in A$. By the mentioned *Macdonald’s theorem* or by the *Shirshov-Cohn theorem* [39, Theorem 2.4.14], it can be easily checked that operator commutativity of a couple of elements in a Jordan algebra of self-adjoint operators can be equivalently verified in any Jordan subalgebra containing these elements (cf. [67, Proposition 1]).

A real Jordan algebra N is called *formally real* if for every $a_1, \dots, a_n \in N$ the condition $\sum_{i=1}^n a_i^2 = 0$ implies $a_1 = \dots = a_n = 0$ (see [39, §2.9]). Every JB-algebra is a formally real Jordan algebra. A Jordan subalgebra B of N is called *strongly associative* if the identity $(x \circ y) \circ a = x \circ (y \circ a)$ holds for all $x, a \in B$ and $y \in N$, equivalently, any pair of elements in B operator commute as elements in N . A family \mathcal{F} of elements of N is called *compatible* if the Jordan subalgebra $J(\mathcal{F})$ generated by \mathcal{F} is strongly associative.

The idea behind (weakly) Rickart and Baer C*-algebras is to find a subclass of C*-algebras, between general C*-algebras and von Neumann algebras, in which every element can be approximated in norm by finite linear combinations of projections. In the setting of AJBW*-algebras (i.e. Baer JB*-algebras) this goal is achieved by the following theorem, in which Arzikulov established a Jordan version of the original result proved by Kaplansky for AW*-algebras.

Theorem 3.3 ([4, Theorem 2.1]). *The following statements are equivalent for each JB-algebra N :*

- (a) N satisfies the following properties:
 - (1) Every subset of pairwise orthogonal projections in the partially ordered set of projections has a least upper bound in this set;
 - (2) Every maximal strongly associative subalgebra of N is generated by its projections (i.e., it coincides with the least closed subalgebra containing its projections);
- (b) N is an AJBW-algebra;
- (c) N is an inner AJBW-algebra.

Let M be a JB*-algebra. It is worth to notice that the JB*-subalgebra generated by a single self-adjoint element in M is strongly associative (cf. [22, Proposition 2.4.13 and Fact 3.3.34]). The set of all strongly associative subalgebras of M can be regarded as an inductive set when equipped with the order defined by inclusion. Therefore each

strongly associative JB^* -subalgebra of M is contained in a maximal strongly associative JB^* -subalgebra. It follows from Theorem 3.3 that every self-adjoint element in a $AJBW^*$ -algebra M can be approximated by finite linear combinations of projections in M –actually the same conclusion holds for any element in M . We shall see later that our notion of weakly Rickart JB^* -algebra also enjoys this property.

As in the case of C^* -algebras, a couple of projections p, q in a JB^* -algebra are called *orthogonal* if $p \circ q = 0$. Both notions are perfectly compatible in the case of a C^* -algebra regarded with its associative structure or as a JB^* -algebra.

One of the new contributions in this note is to explore the notions of weakly Rickart and SAW^* -algebras in the setting of JB^* -algebras. In order to develop our study, we shall follow a similar method to that introduced by Ayupov and Arzikulov focused on the self-adjoint part and the lattice of projections. In the setting of JB^* -algebras we cannot define properties in terms of the left or right multiplication operator by an element. We gather next some reinterpretations for latter purposes.

Lemma 3.4. *Let a and x be non-zero positive elements in a C^* -algebra. Then the following statements are equivalent:*

- (a) $ax = x$;
- (b) $a \circ x = x$;
- (c) $U_a(x) = x$.

Clearly, the elements a and x commute in case that any of the previous statements holds.

Proof. (a) \Rightarrow (b) and (c). This is clear because $xa = (ax)^* = x^* = x$, and thus $a \circ x = \frac{1}{2}(ax + xa) = x$.

Similarly, $U_a(x) = axa = xa = x$.

(b) \Rightarrow (a) We can clearly embed A inside its unitization, and thus assume that A is unital. Since $(1 - a) \circ x = 0$ with $1 - a \in A_{sa}$ and $x \geq 0$, [18, Lemma 4.1] implies that $x \perp (1 - a)$ in A (as JB^* - and as C^* -algebra), then $(1 - a)x = 0 = x(1 - a)$, which proves (a).

(c) \Rightarrow (a) If $\|a\| \leq 1$ the proof is much easier. First, the inequality $\|x\| = \|U_a(x)\| \leq \|a\|^2 \|x\|$ assures that $\|a\| = 1$. We can deduce from a simple induction argument that $U_{a^n}(x) = a^n x a^n = x$ for all natural n . Now, by applying that the sequence $(a^n)_n$ converges in the strong* topology of A^{**} to the support projection, $s(a)$, of a , together with the join strong* continuity of the product of A^{**} [65, Proposition 1.8.12], we obtain $s(a)x s(a) = x$. Finally, since $a = s(a) + (1 - s(a))a = s(a) + a(1 - s(a))$ in A^{**} , it follows that $ax = s(a)x + a(1 - s(a))x = x$.

For the general case we assume that $axa = x$. Since the same identity holds in A^{**} , it is easy to check that $aza = z$ for every z in the C^* -subalgebra of A generated by x (and also in the von Neumann subalgebra of A^{**} generated by x). Therefore, the identity $a r(x) a = r(x)$ holds in A^{**} . It is easy to deduce from the above that

$$(r(x) a r(x)) (r(x) a r(x)) = r(x).$$

Having in mind that $r(x) a r(x)$ is a positive element with $r(x) a r(x) \leq \|a\| r(x)$ whose square is $r(x)$, a simple application of the local Gelfand theory proves that $r(x) a r(x) = r(x)$.

Now by mixing the identities $a r(x) a = r(x)$ and $r(x) a r(x) = r(x)$ we get

$$a r(x) = (a r(x) a) r(x) = r(x), \text{ and } r(x) a = r(x) (a r(x) a) = r(x).$$

Finally, it is easy to see that $ax = ar(x)x = r(x)x = x = xr(x) = xr(x)a = xa$. \square

As in the associative setting of C^* -algebras, a JB^* -subalgebra B of a JB^* -algebra M is said to be a hereditary JB^* -subalgebra of M if whenever $0 \leq a \leq b$ with $a \in M$ and $b \in B$, then $a \in B$, equivalently, B^+ is a face of M^+ (cf. [26,15,1]).

It is known that a hereditary C^* -subalgebra B of a C^* -algebra A is σ -unital if and only if it has the form $B = \overline{xAx}$ for some positive $x \in A$. The same statement remains valid in the case of a JB^* -algebra M , where each σ -unital, hereditary JB^* -subalgebra is of the form $\overline{U_x(M)}$, for some positive $x \in M$.

Corollary 3.5. *Let a and x be positive elements in a JB^* -algebra M . Then the following statements are equivalent:*

- (a) $a \circ x = x$;
- (b) $U_a(x) = x$;
- (c) $a \circ z = z$ for all z in the inner ideal of M generated by x .

Furthermore, if any of the previous statements holds the elements a and x operator commute as elements of M , and $a \circ r(x) = r(x)$, where $r(x)$ denotes the range projection of x in M^{**} .

Proof. By Macdonald’s theorem (see also the Shirshov-Cohn theorem in [39] or [69, Corollary 2.2]), there exists a C^* -algebra A containing the JB^* -subalgebra of M generated by a and x as JB^* -subalgebra. Lemma 3.4 proves that (a) is equivalent to (b) in A , and hence in M . Since $ax = xa = x$ in A , [67, Proposition 1] assures that a and x operator commute in M .

The implication (c) \Rightarrow (a) is clear because $x \in M(x)$. To see the implication (a) \Rightarrow (c), we recall that (a) implies that a and x operator commute in M and $ax = xa = x$ in A (cf. Lemma 3.4). Then

$$\{a, x, z\} = (a \circ x) \circ z + (z \circ x) \circ a - (a \circ z) \circ x = x \circ z \quad (x \in M),$$

and thus, by the Jordan identity, we get

$$\begin{aligned}
 U_a U_x(y) &= \{a, \{x, y^*, x\}, a\} = -\{y^*, x, \{a, x, a\}\} + 2\{\{y^*, x, a\}, x, a\} \\
 &= -\{y^*, x, x\} + 2(x \circ y^*) \circ x \\
 &= -(x \circ y^*) \circ x - x^2 \circ y^* + (x \circ y^*) \circ x + 2(x \circ y^*) \circ x \\
 &= \{x, y^*, x\} = U_x(y),
 \end{aligned}$$

for all $y \in M$. This shows that $U_a(z) = z$ for every $z \in M(x)$. Now take $z \in M(x)$ positive, then, by the equivalence (a) \Leftrightarrow (b), $U_a(z) = z$ gives $a \circ z = z$. Since, each $z \in M(x)$ writes as a linear combination of four positive elements in $M(x)$, we have $a \circ z = z$. \square

The weak versions of Rickart and Baer Jordan algebras in the classical sense considered in Berberian’s book [12] have not been considered yet. The reader should be warned that, in order to work in the Jordan setting, the left and right multiplication operations do not make too much sense in a Jordan algebra.

Definition 3.6. Let N be a JB-algebra.

- ✓ We shall say that N is a weakly Rickart JB-algebra if for each element $a \in N^+$ there exists a projection $p \in N$ such that $p \circ a = a$, and for each $z \in N$ with $U_z(a) = 0$ we have $p \circ z = 0$.
- ✓ N is called a *weakly inner Rickart JB-algebra* if for each element $x \in N$ there exists a projection $p \in N$ such that $p \circ x = x$, and for each $z \in N^+$ with $U_x(z) = 0$ we have $p \circ z = 0$.

A JB*-algebra M will be called weakly Rickart or weakly inner Rickart if its self-adjoint part satisfies the same property.

Remark 3.7. Let N be a weakly Rickart JB-algebra. Then, for each $a \in N^+$, the projection p in Definition 3.6 is unique. This projection will be called the *range projection* of a in N ($RP_N(a) = RP(a)$ in short). Indeed, suppose that there exist projections p, p' in N such that

$$p \circ a = p' \circ a = a,$$

and for any $z \in N$ with $U_z(a) = 0$ we have $p \circ z = p' \circ z = 0$. It follows from the original assumptions that $(p - p') \circ a = 0$, and since $a \geq 0$, we deduce from (13) that $p - p' \perp a$. It then follows that $U_{(p-p')}(a) = \{p - p', a, p - p'\} = 0$. By applying the assumptions we get $p \circ (p - p') = 0 = p' \circ (p' - p)$, which implies that $p = p \circ p' = p'$.

It can be seen that $RP_N(a)$ is the smallest projection in N such that $a = p \circ a (= U_p(a))$. Namely, if q is any projection in N such that $q \circ a = a$, then $(RP_N(a) - q) \circ a = 0$, and thus $RP_N(a) \circ (RP_N(a) - q) = 0$, therefore $RP_N(a) \circ q = RP_N(a)$, which is equivalent to say that $RP_N(a) \leq q$.

Lemma 3.8. *Let N be a JB-algebra. Then N is weakly Rickart and unital if, and only if, it is a Rickart JB-algebra if, and only if, it is weakly inner Rickart and unital.*

Proof. Suppose N is a unital weakly Rickart JB-algebra with unit $\mathbf{1}$. Let us fix $a \in N^+$. By assumptions there exists a projection p in N such that $a \circ p = a$ and for each $z \in N$ with $U_z(a) = 0$ we have $p \circ z = 0$. Given $x \in \{a\}^{\perp q}$ we have $U_x(a) = 0$, and thus $p \circ x = 0$, in particular $(\mathbf{1} - p) \circ x = x$. We have shown that $\{a\}^{\perp q} \subseteq U_{\mathbf{1}-p}(N) = N_2(\mathbf{1} - p) = N_0(p)$. Conversely, if $x \in U_{\mathbf{1}-p}(N) = N_2(\mathbf{1} - p) = N_0(p)$, since $p \circ a = a$, we deduce that $a \in N_2(p)$, and consequently, $U_x(a) = 0$, by Peirce arithmetic.

Suppose now that N is a unital weakly inner Rickart JB-algebra with unit $\mathbf{1}$. So, given $x \in M$ there exists a projection $p \in N$ such that $p \circ x = x$ and for each $z \in N^+$ with $U_x(z) = 0$ we have $p \circ z = 0$. For each $z \in {}^{\perp q}\{x\} \cap N^2$ we have $U_x(z) = 0$, and hence $p \circ z = 0$. It follows that ${}^{\perp q}\{x\} \cap N^2 \subseteq U_{\mathbf{1}-p}(N) \cap N^2$. Reciprocally, each $z \in U_{\mathbf{1}-p}(N) \cap N^2$ is positive and must be orthogonal to $N_2(p)$ by Peirce arithmetic, then $z \in {}^{\perp q}\{x\} \cap N^2$, because $x \in N_2(p)$.

To conclude the proof we observe that every Rickart JB-algebra is unital and weakly (inner) Rickart. \square

Proposition 3.9. *Let p be a projection in a weakly Rickart JB^* -algebra M . Then the Peirce-2 subspace $M_2(p)$ is a Rickart JB^* -algebra with unambiguous range projections of positive elements in $M_2(p)$.*

Proof. Let us fix a positive element $a \in M_2(p)$. Clearly, a is positive in M . Let $q = RP(a)$ denote the range projection of a in M . Since $(p - q) \circ a = 0$, it follows from (13) that $(p - q) \perp a$, and thus $U_{(p-q)}(a) = 0$. Applying now that $q = RP(a)$ we get $q \circ (p - q) = 0$. Therefore, $p \circ q = q$ and thus $U_p(q) = q$, witnessing that $q \in M_2(p)$ and satisfies the properties of a range projection for a in $M_2(p)$. We have proved that $M_2(p)$ is a unital weakly Rickart JB^* -algebra, Lemma 3.8 gives the rest. \square

Let h and x be two elements in a JB^* -algebra M with h positive. We know from [18, Lemma 4.1] that

$$x \perp h \text{ if, and only if, } h \circ x = 0. \tag{13}$$

The *orthogonal annihilator* of a subset \mathcal{S} in a JB^* -triple E is defined as

$$\mathcal{S}_E^\perp = \mathcal{S}^\perp := \{y \in E : y \perp x, \forall x \in \mathcal{S}\}.$$

The next result with the basic properties of the orthogonal annihilator has been borrowed from [19, Lemma 3.1] and [30, Lemma 3.2].

Lemma 3.10 ([30, Lemma 3.2], [19, Lemma 3.1]). *Let \mathcal{S} be a nonempty subset of a JB^* -triple E . Then the following statements hold:*

- (a) \mathcal{S}^\perp is a norm closed inner ideal of E ;
- (b) $\mathcal{S} \cap \mathcal{S}^\perp = \{0\}$;
- (c) $\mathcal{S} \subseteq \mathcal{S}^{\perp\perp}$;
- (d) If $\mathcal{S}_1 \subseteq \mathcal{S}_2$ then $\mathcal{S}_2^\perp \subseteq \mathcal{S}_1^\perp$;
- (e) \mathcal{S}^\perp is weak* closed whenever E is a JBW*-triple.

We should note that the orthogonal annihilator of a subset \mathcal{S} in a JB*-algebra M need not coincide with the quadratic annihilators defined in (8) and (9). In general we have

$$(\mathcal{S}^\perp)^* \subseteq {}^{\perp q}\mathcal{S}, \text{ and } (\mathcal{S}^*)^\perp \subseteq \mathcal{S}^{\perp q}, \text{ for all } \mathcal{S} \subset M, \tag{14}$$

where $\mathcal{S}^* = \{x^* : x \in \mathcal{S}\}$. The equalities do not necessarily hold. For example, let e be a complete tripotent in $M = B(H)$ which is not unitary (for example a partial isometry satisfying $ee^* = \mathbf{1}$ and $p = e^*e \neq \mathbf{1}$). Clearly, $\{e\}^\perp = M_0(e) = \{0\}$ and $\{e^*\}^\perp = M_0(e^*) = \{0\}$. It is easy to check that ${}^{\perp q}\{e\} = M_1(e) = M(\mathbf{1} - p)$ and $(\mathbf{1} - p)M \subseteq \{e\}^{\perp q}$.

Lemma 3.11. *Let \mathcal{S} be a set of positive elements in a JB*-algebra M . Then*

$$\mathcal{S}^{\perp q} \cap M_{sa} = \mathcal{S}^\perp \cap M_{sa}, \text{ and } {}^{\perp q}\mathcal{S} \cap M^+ = \mathcal{S}^\perp \cap M^+.$$

Proof. The inclusion \supseteq is clear from (14). Fix $s \in \mathcal{S}$ and $h \in \mathcal{S}^{\perp q} \cap M_{sa}$. We can find, via Macdonald’s or Shirshov-Cohn theorem, a C^* -algebra B containing s and h as positive and hermitian elements, respectively. Since $s = b^2$ for some $b \in M$ and also in B , and $0 = U_h(s) = U_h(b^2) = (hb)(bh)^*$, we deduce that $hb = bh = 0$, and hence $hs = hb^2 = 0$ and $h \circ s = 0$ in B and in M . This is enough to guarantee that $h \perp s$ (cf. [18, Lemma 4.1]). The other equality can be proved similarly. \square

The following lemma is probably known, but it is included here for the lacking of an explicit source.

Lemma 3.12. *Let \mathcal{S} be a subset of positive elements in a JB*-algebra M . Then the orthogonal annihilator of \mathcal{S} , \mathcal{S}^\perp , is a triple inner ideal and a hereditary JB*-subalgebra of M .*

Proof. Clearly \mathcal{S}^\perp is a closed subspace and an inner ideal (see Lemma 3.10). Let us take $x \in \mathcal{S}^\perp$ and $s \in \mathcal{S}$. Since $s \perp x$ with $s \geq 0$, we deduce from [18, Lemma 4.1] that $x \circ s = 0$, and hence $x^* \circ s = 0$, which is equivalent to $s \perp x^*$, and consequently $x^* \in \mathcal{S}^\perp$.

The elements $h = \frac{x+x^*}{2}$ and $k = \frac{x-x^*}{2i}$ lie in $M_{sa} \cap \mathcal{S}^\perp$. Since $h^2 \circ s = \{h, h, s\} = 0$, a new application of [18, Lemma 4.1] proves that $h^2 \in \mathcal{S}^\perp$. Similarly, $k^2 \in \mathcal{S}^\perp$. Actually, since $h + k \in \mathcal{S}^\perp$, we can similarly deduce that $(h + k)^2 \in$

\mathcal{S}^\perp . It follows from this that $h \circ k \in \mathcal{S}^\perp$, and $x^2 = h^2 - k^2 + 2ih \circ k \in \mathcal{S}^\perp$ too.

Finally, let us take $0 \leq a \leq b$ with $b \in \mathcal{S}^\perp$ and any $s \in \mathcal{S}^\perp$ then $0 \leq U_s(a) \leq U_s(b) = 0$. It follows from Lemma 3.11 that $a \in \mathcal{S}^\perp$. \square

Our next definition is now fully justified by the previous results.

Definition 3.13. Let N be a JB-algebra. We shall say that N is a SAJBW-algebra if for any $x, y \in N^+$ with $x \circ y = 0$ there exists $e \in N^+$ (not necessarily a projection) such that $e \circ x = x$ and $e \circ y = 0$. A JB^* -algebra M will be called a SAJBW*-algebra if its self-adjoint part is a SAJBW-algebra.

The next proposition is a generalization of Pedersen’s result in Proposition 2.1 to the setting of JB^* -algebras. Our new notions of weakly Rickart and SAJBW*-algebras are the missing ingredients to complete the whole picture.

Proposition 3.14. *Let M be a JB^* -algebra. Consider the following property: Given two orthogonal, hereditary JB^* -subalgebras B and C of M , there is a positive e in M which is a unit for B and annihilates C .*

- (a) *The previous property holds for all pairs of hereditary JB^* -subalgebras B, C if, and only if, M is an $AJBW^*$ -algebra;*
- (b) *The property holds when B is the inner ideal generated by a positive element and C is arbitrary if, and only if, M is a weakly Rickart JB^* -algebra;*
- (c) *The property holds when C is the inner ideal generated by a positive element and B is arbitrary if, and only if, M is a Rickart JB^* -algebra;*
- (d) *The property holds when both B and C are inner ideals generated, each one of them by a single positive element if, and only if, M is a SAJBW*-algebra.*

Proof. (d) (\Rightarrow) By considering two positive elements x, y in M with $x \circ y = 0$, the inner ideals $M(x)$ and $M(y)$ are orthogonal, and hence by hypothesis, there exists a positive $e \in M$ which is a unit for $M(x)$ and annihilates $M(y)$. Clearly, $e \circ x = x$ and $e \circ y = 0$.

(\Leftarrow) If M is a SAJBW*-algebra, given positive elements x, y in M with $x \circ y = 0$, there exists a positive e in M such that $e \circ x = x$ and $e \circ y = 0$ –the latter being equivalent to $y \perp e$ by (13). Corollary 3.5 implies that e is a unit for $B = \overline{U_x(M)}$. Furthermore, for each a in M the elements e and $U_y(a)$ are orthogonal since $y \perp e$ and $\{e\}^\perp$ is an inner ideal of M and hence contains all elements in $U_y(M) = Q(y)(M)$. It follows that e annihilates $C = \overline{U_y(M)}$.

(b) (\Rightarrow) Fix a positive $a \in M$, by applying the hypothesis to $B = M(a)$ and $C = \{a\}^\perp$ we find a positive $e \in M$ which is a unit for $M(a)$ and annihilates $\{a\}^\perp$. We know from Corollary 3.5 that e and a operator commute, and thus $(e^n - e) \circ a = 0$ for all natural n . Having in mind (13), the properties of e assure that $0 = e \circ (e^n - e) = e^{n+1} - e^2$ for

all natural n . A simple application of the local Gelfand theory on the commutative and associative JB*-algebra generated by e proves that e is a projection.

Now we take any $z \in \{a\}^{\perp q} \cap M_{sa}$. By Lemma 3.11, $\{a\}^{\perp q} \cap M_{sa} = \{a\}^{\perp} \cap M_{sa}$, and thus the properties of e imply that $e \circ z = 0$. Therefore, M is a weak Rickart JB*-algebra.

(\Leftarrow) Suppose now that M is a weak Rickart JB*-algebra. Take $B = \overline{U_x(M)}$ and C as in the statement, with x positive in M . It follows from the hypothesis that there exists a projection $p \in M$ satisfying $p \circ x = x$ and $p \circ z = 0$ for all $z \in M_{sa}$ with $U_z(x) = 0$. Clearly, each $c \in C_{sa}$ satisfies $U_c(x) = 0$, and thus $p \circ c = 0$ for all $c \in C$.

(c) (\Rightarrow) For $C = M(0) = \{0\}$ and $B = M$, the hypothesis implies the existence of a unit element $\mathbf{1} \in M$. Pick $a \in M^+$. Since $B = \{a\}^{\perp}$ and $C = M(a)$ are two orthogonal hereditary JB*-subalgebras, by hypothesis, there exists a positive $e \in M$ which is a unit for B and annihilates C . That is $e \in \{a\}^{\perp}$, and hence $e \circ e = e$, witnessing that e is a projection in M .

As before, Lemma 3.11 proves that $\{a\}^{\perp q} \cap M_{sa} = \{a\}^{\perp} \cap M_{sa}$. It follows from the properties of e that $\{a\}^{\perp q} \cap M_{sa} \subseteq \{\mathbf{1} - e\}^{\perp} = U_e(M)$. Reciprocally, if $z \in U_e(M) \cap M_{sa}$, since $e \circ a = 0$, and hence $a \in U_{\mathbf{1}-e}(M)$, it follows that $z \in \{a\}^{\perp} \cap M_{sa} = \{a\}^{\perp q} \cap M_{sa}$.

(\Leftarrow) We assume now that M is a Rickart JB*-algebra. Take $C = \overline{U_x(M)}$ and B as in the statement, with x positive in M . Under these circumstances there exists a projection $p \in M$ satisfying $\{x\}^{\perp q} \cap M_{sa} = U_p(M) \cap M_{sa}$. Clearly, each $b \in B$ lies in $\{x\}^{\perp q}$, and thus $p \circ b = b$ for all $b \in B$. Since $p \in U_p(M)$, we have $U_p(x) = 0$. Having in mind that x is positive, we deduce, via Shirshov-Cohn theorem, that p and x are orthogonal. Consequently, by Peirce arithmetic, p annihilates $C = \overline{U_x(M)}$.

(a) (\Rightarrow) Taking $B = \{0\}$ and $C = A$, we find a unit $\mathbf{1} \in M$. Fix a subset $\mathcal{S} \subseteq M^+$. The inner ideal $C = S^{\perp}$ is a hereditary JB*-subalgebra of M , and the same happens to $B = (C \cap M^+)^{\perp}$. Clearly, $B \perp C$. By assumptions, there exists a positive e in M which is a unit for B and annihilates C . In particular $\mathbf{1} - e$ lies in C and hence $e \circ (\mathbf{1} - e) = 0$. Thus, e is a projection.

Lemma 3.11 implies that $S^{\perp q} \cap M_{sa} = S^{\perp} \cap M_{sa} = U_{\mathbf{1}-e}(M) \cap M_{sa}$, where the last equality follows from the same arguments given in the proof of (c).

(\Leftarrow) We assume finally that M is an AJBW*-algebra (it is, in particular, unital). Taking B and C as in the statement, for C^+ , there exists a projection p in M such that $(C^+)^{\perp q} = U_p(M_{sa})$. Since $B^+ \subseteq (C^+)^{\perp q}$, p is the unit element in $U_p(M_{sa})$, and every element in B is a linear combination of four positive elements in B , p must be a unit for B . On the other hand, each positive $c \in C$ satisfies that $U_p(c) = 0$, and thus p is orthogonal to each positive element in C . Therefore, p is orthogonal to C , as desired. \square

Let A be a C*-algebra. It follows from the previous proposition and from Proposition 2.1 that A is an AJBW*-algebra (respectively, a Rickart, a weakly Rickart or a SAJBW*-algebra) if and only if it is an AW*-algebra (respectively, a Rickart, a weakly Rickart or a SAW*-algebra). The statement concerning AJBW*-algebras and AW*-algebras (respectively, Rickart JB*-algebras and Rickart C*-algebras) can be derived from the results by Ayupov and Arzikulov in [7, Propositions 1.1, 1.3, 2.1 and 2.3].

The next technical lemma will be required in the main result of this section. Before presenting the result, we recall some facts on operator commutativity. By the Shirshov–Cohn theorem [39, Theorem 2.4.14] any two self-adjoint elements a and b in a JB^* -algebra M generate a JB^* -subalgebra that can be realized as a JC^* -subalgebra of some $B(H)$ (see also [69, Corollary 2.2]). Furthermore, under this identification, a and b commute in the usual sense whenever they operator commute in M (compare Proposition 1 in [67]). By the same arguments, for any pair of self-adjoint elements a and b in M we have

$$a \text{ and } b \text{ operator commute if and only if } a^2 \circ b = 2(a \circ b) \circ a - a^2 \circ b \tag{15}$$

Lemma 3.15. *Let M be a weakly Rickart JB^* -algebra. Let a, b be two elements in M with a positive. Suppose that a and b operator commute. Then $RP(a)$ and b operator commute.*

Proof. Let $p = RP(a) \in M$. Let us write, $b = b_1 + ib_2$, where each b_j is self-adjoint for every $j = 1, 2$ and a operator commutes with b_1 and b_2 . Let us consider the element $c_j = p \circ b_j - b_j$. Having in mind that a and b_j operator commute and $p = RP(a)$ we obtain

$$c_j \circ a = (p \circ b_j - b_j) \circ a = (p \circ a) \circ b_j - b_j \circ a = a \circ b_j - b_j \circ a = 0.$$

Since a is positive, the above identity proves that $a \perp c_j$ (cf. (13)). It follows from the properties of the range projection that $p \circ c_j = 0$, that is, $p \circ (p \circ b_j - b_j) = 0$, or equivalently, $p \circ (p \circ b_j) = p \circ b_j = p^2 \circ b_j$, which is equivalent to say that p and b_j operator commute (cf. (15)). It follows that p and $b = b_1 + ib_2$ operator commute too. \square

We can now establish a generalization of the result proved by Arzikulov in Theorem 3.3 in the line of Rickart’s original result.

Theorem 3.16. *Every weakly Rickart JB^* -algebra is generated by its projections.*

Proof. We can clearly reduce our argument to positive elements. Let a be a positive element in a weakly Rickart JB^* -algebra M . Let $p = RP(a)$ denote the range projection of a in M (cf. Remark 3.7). It follows from Proposition 3.9 that $M_2(p)$ is a Rickart JB^* -algebra with unambiguous range projections of positive elements.

Let B be a maximal strongly associative JB^* -subalgebra of M containing the element a . It follows from Lemma 3.15 that B contains the range projection of every positive element $c \in B$. Therefore B is a weakly Rickart associative JB^* -algebra, or equivalently, a commutative weakly Rickart C^* -algebra (cf. Propositions 3.14 and 2.1). Finally, it follows from Remark 2.4 that B (and hence M) is generated by its projections. We can also consider a maximal strongly associative JB^* -subalgebra C of $M_2(p)$ containing a and p . In this case C is a Rickart associative JB^* -algebra, or equivalently, a commutative Rickart C^* -algebra (cf. Lemma 3.15, Propositions 3.14 and 2.1). \square

4. Rickart JB*-triples

The definitions of Baer and Rickart JB*-algebras introduced by Ayupov and Arzikulov and the notions of weakly Rickart and SAJBW*-algebras developed in the previous section depend extremely on the existence of a cone of positive elements. This is a handicap if we want to work on the wider setting of JB*-triples, where the existence of a cone of positive elements is, in general, impossible.

Furthermore, projections make no sense in the wider setting of JB*-triples; and the role of projections is in general played by tripotents. As in the original study by Rickart, our aim is to find an appropriate notion, in terms of orthogonal annihilators, local order and range tripotents, to assure that a JB*-triple satisfying this property contains sufficiently many tripotents.

The characterizations of (weakly) Rickart C*-algebras established in section 2 (see Propositions 2.5 and 2.10) offer a perspective which allows us to consider these notions in the wider setting of JB*-triples.

Definition 4.1. Let E be a JB*-triple.

- ✓ E is called a SAJBW*-triple if for any $x, y \in E$ with $x \perp y$, there exists a tripotent $e \in E$ satisfying $x \in E_2(e)$ and $y \in E_0(e)$.
- ✓ E is a weakly Rickart (wR) JB*-triple if given $x \in E$ and an inner ideal $J \subseteq E$ with $I = E(x) \perp J$, there exists a tripotent e in E such that $I \subseteq E_2(e)$ and $J \subseteq E_0(e)$.
- ✓ E is a weakly order-Rickart (woR) JB*-triple if given $x \in E$ and an inner ideal $J \subseteq E$ with $I = E(x) \perp J$, there exists a tripotent e in E such that x is positive in $E_2(e)$, and $J \subseteq E_0(e)$.
- ✓ E is called a Rickart JB*-triple if it is weakly Rickart and admits a unitary element.

For a JB*-triple E , the following implications hold: E is a Rickart JB*-triple $\Rightarrow E$ is a wR JB*-triple, and E is a woR JB*-triples $\Rightarrow E$ is a wR JB*-triple.

Let A be a C*-algebra. It follows from Proposition 2.5 that A is a Rickart or a weakly Rickart C*-algebra if and only if it is a Rickart or a weakly Rickart JB*-triple, respectively. Furthermore, Propositions 2.10 and 2.5 prove that a C*-algebra is a wR JB*-triple if and only if it is a woR JB*-triple. So, our definition is consistent with the previous notions. We do not know if A being a SAW*-algebra implies that A is a SAJBW*-triple. For the reciprocal, suppose that A is a SAJBW*-triple. Fix two positive elements $x, y \in A$ with $xy = 0$. By hypothesis there exists a partial isometry e with $x \in A_2(e)$ and $y \in A_0(e)$. Since $x = ee^*xe^*e$ and $x \geq 0$, it can be shown that $x = ee^*x = xee^* = e^*ex = xe^*e$. Similarly, $ee^*y = yee^* = ye^*e = ee^*y = 0$. Therefore A is a SAW*-algebra.

The examples provided in [59,12,56] show that, even in the category of abelian C*-algebras, the classes of SAJBW*-triples, weakly Rickart JB*-triples and Rickart JB*-triples are mutually different.

In the setting of JB^* -algebras we do not know if there is a relation between being a Rickart or a weakly Rickart JB^* -algebra as seen in section 3 and the corresponding notion as JB^* -triple. The lacking of polar decompositions makes invalid the natural arguments. What we can prove is the following connection between JB^* -algebras which are woR JB^* -triples and weakly Rickart JB^* -algebras.

Proposition 4.2. *Let M be a JB^* -algebra which is a woR JB^* -triple, then M is a weakly Rickart JB^* -algebra. Actually, it suffices to assume that every positive element a in M admits a range tripotent $R(a)$ in M , and in such a case the range tripotent of a in M is precisely the range projection of a in M as weakly Rickart JB^* -algebra.*

Before presenting the proof, we establish a result proving the existence of range tripotents for elements in woR JB^* -triples.

Lemma 4.3. *Let E be a JB^* -triple. Then the following statements hold:*

- (a) *If a is an element in E and e, v are two tripotents in E such that a is positive in $E_2(e)$ and in $E_2(v)$ with $\{a\}^\perp = E_0(e)$, then $e \leq v$;*
- (b) *Let us assume that E is a woR JB^* -triple. Then for each element a in E there exists a unique tripotent $e \in E$ satisfying that a is positive in $E_2(e)$ and $\{a\}^\perp = E_0(e)$.*

Proof. (a) Let e and v be tripotents in E satisfying the properties in the statement. Let $r_{E^{**}}(a)$ denote the range tripotent of a in E^{**} . Since a is positive in $E_2(e) \subseteq E_2^{**}(e)$, it follows that a is positive in the JBW^* -algebra $E_2^{**}(e)$, and hence $r_{E^{**}}(a) \leq e$ as tripotents in E^{**} . Therefore $e = r_{E^{**}}(a) + (e - r_{E^{**}}(a))$ with $r_{E^{**}}(a) \perp (e - r_{E^{**}}(a))$, and hence $\{a, a, e\} = \{a, a, r_{E^{**}}(a)\}$. Similarly, $\{a, a, v\} = \{a, a, r_{E^{**}}(a)\}$. It then follows that the triple product $\{a, a, e - v\} = 0$ in E , or equivalently, $a \perp (e - v)$, that is, $e - v \in \{a\}^\perp$. The assumptions on e imply that $e - \{e, e, v\} = \{e, e, e - v\} = 0$, or equivalently, $\{e, e, v\} = e$. Lemma 1.6 or Corollary 1.7 in [33] implies that $v \geq e$.

(b) Let e and v satisfying the hypotheses in (b) (both exist by the assumptions on E). It follows from (a) that $e \leq v$ and $v \leq e$. Therefore $e = v$ as claimed. \square

Let a be an element in a woR JB^* -triple E . The unique tripotent e given by Lemma 4.3 is called the *range tripotent* of a in E , and will be denoted by $R_E(a)$. It follows from Lemma 4.3(a) that $R_E(a)$ is the smallest tripotent e in E satisfying that a is positive in the unital JB^* -algebra $E_2(e)$.

Let us briefly recall that for each self-adjoint element h in a JB^* -algebra M , the mapping U_h is positive on M , that is, it maps positive elements to positive elements [39, Proposition 3.3.6].

Proof of Proposition 4.2. Let us fix a positive element a in M . Let $e = R_E(a)$ denote the range tripotent of a in E . Since the involution on M is a conjugate linear triple automorphism on M we have $0 \leq a = a^*$ in $M_2(e^*)$ and

$$\{a\}^\perp = \{a^*\}^\perp = (\{a\}^\perp)^* = (M_0(e))^* = M_0(e^*),$$

witnessing that e^* satisfies the properties of the range tripotent of a in M , and by the uniqueness of this element $e = e^*$. That is, e is a self-adjoint tripotent in M , and thus, by the local Gelfand theory, $e = p - q$, where p and q are two orthogonal projections in M .

It follows from the properties of the range tripotent $e = p - q$ that $0 \leq a$ in $M_2(e)$. Since $0 \leq -q \leq e$ in $M_2(e)$, the element $-q$ is a projection in $M_2(e)$. Therefore, having in mind that, by Kaup’s theorem, the triple product on $M_2(e)$ is uniquely given by the restriction of the triple product of M and by the JB*-structure of $M_2(e)$, the element

$$U_{-q}^{M_2(e)}(a) = \{-q, a^{*e}, -q\} = \{-q, a, -q\} = \{q, a, q\} = U_q(a)$$

is positive in $M_2(e)$ (cf. [39, Proposition 3.3.6]), and in $M_2(-q)$. Since, $M_2(-q) = M_2(q)$ with $(M_2(-q))_{sa} = (M_2(q))_{sa}$ we deduce the existence of $y \in (M_2(-q))_{sa} = (M_2(q))_{sa} \subseteq M_{sa}$ such that

$$U_q(a) = y \circ_{-q} y = \{y, -q, y\} = -\{y, q, y\} = -U_y(q),$$

which implies that $U_q(a)$ is a negative element in M .

On the other hand, since a is positive in M and q is a projection, the element $U_q(a)$ must be positive in M [39, Proposition 3.3.6], which combined with the previous conclusion leads to $U_q(a) = 0$. It follows from the first statement in Lemma 3.11 that $q \in \{a\}^{\perp q} \cap M_{sa} = \{a\}^\perp \cap M_{sa}$, that is, $q \perp a$. The properties of the range tripotent imply that $q \in M_0(e) = M_0(p - q)$, and thus $q \perp (p - q)$, and so $q = 0$.

We have therefore shown that the range tripotent $e = R_M(a)$ of a in M is a projection in this JB*-algebra. It can be easily checked that $e \circ a = \{e, e, a\} = a$ and for each $z \in M_{sa}$ with $U_z(a) = 0$ we have $p \circ z = 0$ (cf. Lemma 3.11), that is M is a weakly Rickart JB*-algebra. \square

An element u in a unital JB*-algebra M is called *unitary* if it is invertible with inverse u^* . In the setting of JB*-triples, the word *unitary* is applied to those elements u such that $L(u, u)$ is the identity mapping. Clearly, every unitary u in a JB*-triple E is a tripotent with $E_2(u) = E$ —this is actually a characterization. There is no ambiguity in case that a unital JB*-algebra M is regarded as a JB*-triple because both notions are equivalent [13, Proposition 4.3].

Our next result is a strengthened version of Proposition 4.3. We recall first that for each tripotent e in a JB*-triple E and each unitary complex number λ , the mapping

$$S_\lambda(e) = \lambda^2 P_2(e) + \lambda P_1(e) + P_0(e) \tag{16}$$

is a triple automorphism on E [33, Lemma 1.1]. It can be easily deduced from this fact that the mapping

$$R_\lambda(e) = P_2(e) + \lambda P_1(e) + \lambda^2 P_0(e) \tag{17}$$

also is a triple automorphism on E .

Proposition 4.4. *Let E be a woR JB^* -triple. Then for each tripotent $e \in E$, the Peirce-2 subspace $E_2(e)$ is a Rickart JB^* -algebra.*

Proof. Having in mind Proposition 4.2 and Lemma 3.8(a), it suffices to show that each positive element a in $E_2(e)$ admits a range tripotent in $E_2(e)$. Let $v = R_E(a)$ be the range tripotent of a in E . Let $S_{-1} = S_{-1}(e) = P_2(e) - P_1(e) + P_0(e)$ denote the triple automorphism on E given in (16). Let us observe that $S_{-1}(a) = a$ because $a \in E_2(e)$.

Since a is positive in $E_2(v)$ with $\{a\}_E^\perp = E_0(v)$, we deduce that $a = S_{-1}(a)$ is positive in $E_2(S_{-1}(v))$ with

$$\{a\}_E^\perp = \{S_{-1}(a)\}_E^\perp = S_{-1}(\{a\}_E^\perp) = S_{-1}(E_0(v)) = E_0(S_{-1}(v)).$$

That is, $S_{-1}(v)$ satisfies the properties of the range tripotent for a , and hence it follows from its uniqueness that $v = S_{-1}(v) = P_2(e)(v) - P_1(e)(v) + P_0(e)(v)$. This equality proves that $v = P_2(e)(v) + P_0(e)(v)$, where $P_2(e)(v)$ and $P_0(e)(v)$ are two orthogonal tripotents in E .

If in the previous argument we replace $S_{-1}(e)$ with $R_i(e)$, and we apply it to $v = P_2(e)(v) + P_0(e)(v)$, we derive that $v = R_i(e)(v) = P_2(e)(v) - P_0(e)(v)$, witnessing that $v = P_2(e)(v)$. Now, it can be easily seen that $v = P_2(e)(v) \in E_2(e)$ satisfies the properties of the range tripotent for a in $E_2(e)$ (and in E). This concludes the proof. \square

We can now establish the result which has motivated our study. We shall see that every woR JB^* -triple contains an abundant collection of tripotents.

Theorem 4.5. *Every weakly order Rickart JB^* -triple is generated by its tripotents.*

Proof. Let a be an element in a woR JB^* -triple E . Let $e = R_E(a)$ be the range tripotent of a in E . Proposition 3.9 assures that $E_2(e)$ is a Rickart JB^* -algebra. By construction, a is a positive element in $E_2(e)$, and hence Theorem 3.16 implies that a can be approximated in norm by finite linear combinations of projections in $E_2(e)$. The proof concludes by just observing that, since $E_2(e)$ is a JB^* -subtriple of E , every projection in $E_2(e)$ is a tripotent in E . \square

5. Von Neumann regularity

Regular elements in the sense of von Neumann have been intensively studied in the associative setting of C^* -algebras (cf. [41,42,14] and [59, §3]) as well as in the wider setting of JB^* -triples (see [31,32,50,20,21] and [44]).

Motivated by the study conducted by Rickart on von Neumann regular elements in B_p^* -algebras (now called Rickart C^* -algebras) in [59, §3], we devote this section to explore von Neumann regular elements in woR JB^* -triples.

An element a in a JB^* -triple E is called *von Neumann regular* if and only if there exists $b \in E$ such that $Q(a)b = a$, $Q(b)a = b$ and $[Q(a), Q(b)] := Q(a)Q(b) - Q(b)Q(a) = 0$ (cf. [50, Lemma 4.1] or [31,32,20]). The element $b \in E$ satisfying the previous properties is unique and is called the *generalized inverse* of a in E (denoted by a^\dagger). However, there exist von Neumann regular elements $a \in E$, for which we can find many elements $c \in E$ such that $Q(a)c = a$.

Several useful characterizations of von Neumann regular elements in JB^* -triples can be found in [31,32,50,20]. For our purposes here, we recall that an element a in a JB^* -triple E , whose range tripotent in E^{**} is denoted by $r_{E^{**}}(a) = r(a)$, is von Neumann regular if, and only if, $r(a) \in E$ and a is positive and invertible in the unital JB^* -algebra $E_2(r(a))$, and in such a case a^\dagger is precisely the inverse of a in $E_2(r(a))$ (cf. [20, §2, pages 191 and 192]). It is further known that in this case $L(a, a^\dagger) = L(a^\dagger, a) = L(r(a), r(a))$ (see [20, §2, page 192] and [51, Lemma 3.2]).

The next lemma goes in the line of [43, Lemma 2.2] and [59, Theorem 3.2].

Lemma 5.1. *Let e be a tripotent in a JB^* -triple E . The following statements hold:*

- (a) *Every invertible element a in the unital JB^* -algebra $E_2(e)$ is von Neumann regular in E with $r_{E^{**}}(a)$ being a unitary element in $E_2(e)$.*
- (b) *Suppose that x is an element in E with $\|e - x\| < 1$. Then $Q(e)(x)$ and $P_2(e)(x)$ are von Neumann regular elements whose range tripotents (i.e. $r(Q(e)(x))$ and $r(P_2(e)(x))$, respectively) in E^{**} belong to $E_2(e)$ and are unitaries in the latter JB^* -algebra. Moreover, $r(Q(e)(x))$ and $r(P_2(e)(x))$ satisfy the properties of the range tripotent in a woR JB^* -triple for the elements $Q(e)(x)$ and $P_2(e)(x)$, respectively. The latter conclusion holds for the range tripotent in E^{**} of any invertible element $a \in E_2(e)$.*

Proof. (a) The statement is essentially proved in [43, Remark 2.3]. Namely, if a is invertible in $E_2(e)$, the just quoted remark assures that the range tripotent $r = r_{E_2^*(e)}(a)$ of a in the bidual of $E_2(e)$ is a unitary element in $E_2(e)$. It is clear that r must be also the range tripotent of a in E^{**} and belongs to E . It follows from the characterization of von Neumann regular elements from [20], seen before this lemma, that a is von Neumann regular in E .

(b) Since $\|e - x\| < 1$ and $Q(e)$ and $P_2(e)$ are non-expansive mappings fixing the element e , we get $\|e - Q(e)(x)\|, \|e - P_2(e)(x)\| < 1$. Having in mind that $E_2(e)$ is a unital JB^* -algebra with unit e and $Q(e)(x), P_2(e)(x) \in E_2(e)$, we deduce that these two elements are invertible in $E_2(e)$. The first part of the statement now follows from (a).

We shall only prove the last statement for $P_2(e)(x)$. To simplify the notation, let $r = r(P_2(e)(x)) \in E_2(e)$ denote the range tripotent of $P_2(e)(x)$. Clearly, $P_2(e)(x)$ is

positive in $E_2(r)$ (let us note that $E_2(r) = E_2(e)$ as sets because r is a unitary in $E_2(e)$). Finally, it follows from Lemma 3.2 in [19] that $\{P_2(e)(x)\}^\perp = E_0(r)$, which concludes the argument. \square

The next result is a triple version of [59, Theorem 3.3].

Proposition 5.2. *Let E be a woR JB^* -triple. Suppose that a is a von Neumann regular element in E . Then the range tripotent of a in E as woR JB^* -triple coincides with the range tripotent of a in E^{**} (and in E), that is $R(a) = r_{E^{**}}(a)$. Furthermore $a^\dagger \in E_2(R(a))$ is the inverse of a in $E_2(R(a))$ and $R(a^\dagger) = R(a)$.*

Proof. We know from Lemma 5.1(b) that the range tripotent $r(a)$ satisfies the properties of the range tripotent of a in the definition of woR JB^* -triple. Then the uniqueness of $R(a)$ (see Lemma 4.3(b)) implies that $R(a) = r(a)$.

It is known that $r = r(a) = R(a)$ and a^\dagger both belong to the JB^* -subtriple of E generated by a (cf. [51, Lemma 3.2]), and hence $a^\dagger \in E_2(R(a))$. Finally, we know from the properties of the generalized inverse that a^\dagger is the inverse of a in $E_2(r)$. \square

As we have seen in subsection 1.1, for each element a in a JB^* -triple E , its triple spectrum $\Omega_a \subseteq [0, \|a\|]$ can be employed to identify the JB^* -subtriple, E_a , of E generated by a with the commutative C^* -algebra $C_0(\Omega_a)$, and under this identification a corresponds to the continuous function given by the embedding of Ω_a into \mathbb{C} (cf. [49, Corollary 1.15] and [50, Lemma 3.2]). The triple spectrum Ω_a does not change when computed with respect to any JB^* -subtriple F of E containing the element a [50, Proposition 3.5(vi)]. It is further known that a is von Neumann regular if and only if $0 \notin \Omega_a$ (cf. [50, Lemma 4.1]). In particular if F is a JB^* -subtriple of a JB^* -triple E , then an element $a \in F$ is von Neumann regular in F if and only if it is von Neumann regular in E . Furthermore, if $a \in E$ is von Neumann regular, then a^\dagger and $r(a)$ both belong to the JB^* -subtriple of E generated by a .

Our next goal is a triple version of [59, Theorem 3.13] and a refinement of Theorem 4.5.

Proposition 5.3. *Let E be a woR JB^* -triple. Suppose a is an element in E whose range tripotent is $R(a)$. Then for each $\varepsilon > 0$ there exists a tripotent $e_\varepsilon \in E$ and an element b in the JB^* -subtriple of E generated by a satisfying $e_\varepsilon \leq R(a)$, $\{b, R(a), b\} = a$, $\{b, e_\varepsilon, b\}$ is von Neumann regular and $\|a - \{b, e_\varepsilon, b\}\| < \varepsilon$.*

Proof. Proposition 4.4 assures that $E_2(R(a))$ is a Rickart JB^* -algebra. By definition, a is positive in $E_2(R(a))$. Let C be a maximal strongly associative JB^* -subalgebra of $E_2(R(a))$ containing a . Lemma 3.15 implies that C is a Rickart JB^* -algebra. Therefore C is a commutative Rickart C^* -algebra whose product and involution will be denoted by \cdot and $*$, respectively –observe that $*$ coincides with $*_{R(a)}$.

Given $\varepsilon > 0$, having in mind that C is a commutative C^* -algebra, Theorem 3.13 in [59] proves the existence of a projection $e_\varepsilon \in C$ satisfying $e_\varepsilon \leq R(a)$, $e_\varepsilon \cdot a = \{e_\varepsilon, a, e_\varepsilon\} = P_2(e_\varepsilon)(a)$ is von Neumann regular in C and $\|a - P_2(e_\varepsilon)(a)\| < \varepsilon$.

As observed in [34, comments after Theorem 2.1], since a is a positive in $E_2(R(a))$ (and in C), the JB^* -subtriple E_a of $E_2(R(a))$ (and of C) generated by a coincides with the JB^* -subalgebra that a generates. Therefore the square root of a in C lies in E_a . Let $b \in E_a$ denote the square root of a in C . By applying that C is a commutative C^* -algebra, it can be deduced that $\{b, e_\varepsilon, b\} = (b \cdot b) \cdot e_\varepsilon = a \cdot e_\varepsilon$ is von Neumann regular in C . Clearly, $\{b, R(a), b\} = a$.

Finally, since C is a JB^* -subtriple of E , the element e_ε is a tripotent in E with $e_\varepsilon \leq R(a)$, $\{b, e_\varepsilon, b\}$ is von Neumann regular in E and $\|a - \{b, e_\varepsilon, b\}\| < \varepsilon$. \square

We can now prove that every inner ideal in a woR JB^* -triple E contains an abundant collection of von Neumann regular elements.

Theorem 5.4. *Let I be an inner ideal of a woR JB^* -triple E . Then the von Neumann regular elements of I are dense in I . Each von Neumann regular element x in I is contained in $E_2(R(x)) = I_2(R(x))$, where $R(x) \in I$ and $E_2(R(x))$ is a Rickart JB^* -algebra. Furthermore, if $I \neq \{0\}$, then I contains a non-zero tripotent, actually I contains the generalized inverse and the range tripotent of each non-zero element in I .*

Proof. Let us fix $a \in I$. Proposition 5.3 proves that we can approximate a in norm by von Neumann regular elements of the form $\{b, e, b\}$, where $e \in E$ is a tripotent satisfying $e \leq R(a)$ and $b \in E_a$. Having in mind that I is an inner ideal we deduce that $E_a \subseteq I$, and $\{b, e, b\} \in I$, which concludes the proof of the first statement. The second statement is a consequence of Propositions 5.2 and 4.4.

Take now $a \in I \setminus \{0\}$. In this case $E_a \subseteq E(a)$. By the conclusion in the first paragraph, we can approximate a in norm by a sequence $(a_n)_n$ of non-zero von Neumann regular elements in I . It follows from Proposition 5.2 that the range tripotent of each a_n in E , $R(a_n)$, coincides with its range tripotent in E^{**} and by the theory on von Neuman regular elements $a_n^\dagger, R(a_n) \in E_{a_n} \subseteq E(a_n) \subseteq I$, which concludes the proof. \square

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References

- [1] E.M. Alfsen, F.W. Shultz, *Geometry of State Spaces of Operator Algebras*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 2003.
- [2] P. Ara, Left and right projections are equivalent in Rickart C^* -algebras, *J. Algebra* 120 (1989) 433–448.
- [3] P. Ara, D. Goldstein, A solution of the matrix problem for Rickart C^* -algebras, *Math. Nachr.* 164 (1993) 259–270.
- [4] F.N. Arzikulov, On abstract JW-algebras, *Sib. Math. J.* 39 (1998) 18–24.
- [5] F.N. Arzikulov, AJW-algebras of type I and their classification, *Sib. Adv. Math.* 8 (1998) 30–48.
- [6] F.N. Arzikulov, On an analogue of the Peirce decomposition, *Sib. Math. J.* 40 (1999) 413–418.
- [7] Sh.A. Ayupov, F.N. Arzikulov, Jordan counterparts of Rickart and Baer $*$ -algebras, *Uzbek. Mat. Zh.* (1) (2016) 13–33.
- [8] Sh.A. Ayupov, F.N. Arzikulov, Jordan counterparts of Rickart and Baer $*$ -algebras II, *São Paulo J. Math. Sci.* 13 (2019) 27–38.
- [9] T. Barton, Y. Friedman, Grothendieck’s inequality for JB^* -triples and applications, *J. Lond. Math. Soc.* (2) 36 (1987) 513–523.
- [10] T. Barton, Y. Friedman, Bounded derivations of JB^* -triples, *Q. J. Math. Oxf. Ser.* (2) 41 (1990) 255–268.
- [11] T. Barton, R.M. Timoney, Weak $*$ -continuity of Jordan triple products and its applications, *Math. Scand.* 59 (1986) 177–191.
- [12] S.K. Berberian, Baer $*$ -Rings, *Die Grundlehren der mathematischen Wissenschaften*, vol. 195, Springer-Verlag, New York-Berlin, 1972.
- [13] R. Braun, W. Kaup, H. Upmeyer, A holomorphic characterisation of Jordan- C^* -algebras, *Math. Z.* 161 (1978) 277–290.
- [14] L.G. Brown, G.K. Pedersen, On the geometry of the unit ball of a C^* -algebra, *J. Reine Angew. Math.* 469 (1995) 113–147.
- [15] L.J. Bunce, On compact action in JB -algebras, *Proc. Edinb. Math. Soc.* (2) 26 (1983) 353–360.
- [16] L.J. Bunce, C.-H. Chu, B. Zalar, Structure spaces and decomposition in JB^* -triples, *Math. Scand.* 86 (2000) 17–35.
- [17] M. Burgos, F.J. Fernández-Polo, J.J. Garcés, J. Martínez Moreno, A.M. Peralta, Orthogonality preservers in C^* -algebras, JB^* -algebras and JB^* -triples, *J. Math. Anal. Appl.* 348 (2008) 220–233.
- [18] M. Burgos, F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, Orthogonality preservers revisited, *Asian-Eur. J. Math.* 2 (2009) 387–405.
- [19] M. Burgos, J.J. Garcés, A.M. Peralta, Automatic continuity of biorthogonality preservers between weakly compact JB^* -triples and atomic JBW^* -triples, *Stud. Math.* 204 (2011) 97–121.
- [20] M. Burgos, A. Kaidi, A. Morales, A.M. Peralta, M.I. Ramírez, von Neumann regularity and quadratic conorms in JB^* -triples and C^* -algebras, *Acta Math. Sin. Engl. Ser.* 24 (2) (2008) 185–200.
- [21] M.J. Burgos, A.C. Márquez-García, A. Morales-Campoy, A.M. Peralta, Linear maps between C^* -algebras preserving extreme points and strongly linear preservers, *Banach J. Math. Anal.* 10 (3) (2016) 547–565.
- [22] M. Cabrera García, Á. Rodríguez Palacios, *Non-Associative Normed Algebras*, vol. 1, The Vidav-Palmer and Gelfand-Naimark theorems, *Encyclopedia of Mathematics and Its Applications*, vol. 154, Cambridge University Press, Cambridge, 2014.
- [23] M. Cabrera García, Á. Rodríguez Palacios, *Non-Associative Normed Algebras*, vol. 2, Representation theory and the Zel’manov approach, *Encyclopedia of Mathematics and Its Applications*, vol. 167, Cambridge University Press, Cambridge, 2018.
- [24] S. Dineen, The second dual of a JB^* -triple system, in: J. Múgica (Ed.), *Complex Analysis, Functional Analysis and Approximation Theory*, in: North-Holland Math. Stud., vol. 125, North-Holland, Amsterdam-New York, 1986, pp. 67–69.
- [25] J. Dixmier, Sur certains espaces consideres par M. H. Stone, *Summa Bras. Math.* 2 (1951) 151–182.
- [26] C.M. Edwards, Ideal theory in JB -algebras, *J. Lond. Math. Soc.* (2) 16 (1977) 507–513.

- [27] C.M. Edwards, G.T. Rüttimann, On the facial structure of the unit balls in a JBW*-triple and its predual, *J. Lond. Math. Soc.* (2) 38 (1988) 317–322.
- [28] C.M. Edwards, G.T. Rüttimann, Inner ideals in C*-algebras, *Math. Ann.* 290 (1991) 621–628.
- [29] C.M. Edwards, G.T. Rüttimann, A characterization of inner ideals in JB*-triples, *Proc. Am. Math. Soc.* 116 (1992) 1049–1057.
- [30] C.M. Edwards, G.T. Rüttimann, Peirce inner ideals in Jordan*-triples, *J. Algebra* 180 (1996) 41–66.
- [31] A. Fernández López, E. García Rus, E. Sánchez Campos, M. Siles Molina, Strong regularity and generalized inverses in Jordan systems, *Commun. Algebra* 20 (7) (1992) 1917–1936.
- [32] A. Fernández López, E. García Rus, E. Sánchez Campos, M. Siles Molina, Strong regularity and Hermitian Hilbert triples, *Q. J. Math. Oxf. Ser.* (2) 45 (177) (1994) 43–55.
- [33] Y. Friedman, B. Russo, Structure of the predual of a JBW*-triple, *J. Reine Angew. Math.* 356 (1985) 67–89.
- [34] J.J. Garcés, A.M. Peralta, One-parameter groups of orthogonality preservers on JB*-algebras, *Adv. Oper. Theory* 6 (2021), Paper No. 43, 25 pp.
- [35] D. Goldstein, Polar decomposition in Rickart C*-algebras, *Publ. Mat.* 39 (1995) 5–21.
- [36] K. Grove, G.K. Pedersen, Sub-Stonean spaces and corona sets, *J. Funct. Anal.* 56 (1984) 124–143.
- [37] Y. Friedman, B. Russo, The Gelfand-Naimark theorem for JB*-triples, *Duke Math. J.* 53 (1986) 139–148.
- [38] J. Hamhalter, O. Kalenda, A.M. Peralta, H. Pfitzner, Grothendieck’s inequalities for JB*-triples: proof of the Barton–Friedman conjecture, *Trans. Am. Math. Soc.* 374 (2021) 1327–1350.
- [39] H. Hanche-Olsen, E. Størmer, *Jordan Operator Algebras*, Pitman, London, 1984.
- [40] D. Handelman, *Finite Rickart C*-Algebras and Their Properties*. Studies in Analysis, *Adv. in Math. Suppl. Stud.*, vol. 4, Academic Press, New York-London, 1979, pp. 171–196.
- [41] R. Harte, M. Mbekhta, On generalized inverses in C*-algebras, *Stud. Math.* 103 (1) (1992) 71–77.
- [42] R. Harte, M. Mbekhta, Generalized inverses in C***-algebras. II, *Stud. Math.* 106 (2) (1993) 129–138.
- [43] F.B. Jamjoom, A.M. Peralta, A.A. Siddiqui, H.M. Tahlawi, Approximation and convex decomposition by extremals and the λ -function in JBW*-triples, *Q. J. Math.* 66 (2) (2015) 583–603.
- [44] F.B. Jamjoom, A.M. Peralta, A.A. Siddiqui, H.M. Tahlawi, Extremally rich JB*-triples, *Ann. Funct. Anal.* 7 (4) (2016) 578–592.
- [45] P. Jordan, J. von Neumann, E.P. Wigner, On an algebraic generalization of the quantum mechanical formalism, *Ann. Math.* (2) 35 (1934) 29–64.
- [46] I. Kaplansky, Projections in Banach algebras, *Ann. Math.* (2) 53 (1951) 235–249.
- [47] I. Kaplansky, Algebras of type I, *Ann. Math.* (2) 56 (1952) 460–472.
- [48] I. Kaplansky, Modules over operator algebras, *Am. J. Math.* 75 (1953) 839–858.
- [49] W. Kaup, A Riemann Mapping Theorem for bounded symmetric domains in complex Banach spaces, *Math. Z.* 183 (1983) 503–529.
- [50] W. Kaup, On spectral and singular values in JB*-triples, *Proc. R. Ir. Acad., A Math. Phys. Sci.* 96 (1) (1996) 95–103.
- [51] W. Kaup, On Grassmanian associated with JB*-triples, *Math. Z.* 236 (2001) 567–584.
- [52] M. Liebmann, H. Rühak, B. Henschenmacher, Non-associative algebras and quantum physics. A historical perspective, arXiv:1909.04027.
- [53] J. Martínez, A.M. Peralta, Separate weak*-continuity of the triple product in dual real JB*-triples, *Math. Z.* 234 (2000) 635–646.
- [54] G.J. Murphy, *C*-Algebras and Operator Theory*, Academic Press, Inc., Boston, MA, 1990.
- [55] G.K. Pedersen, *C*-Algebras and Their Automorphism Groups*, London Mathematical Society Monographs, vol. 14, Academic Press, London, 1979.
- [56] G.K. Pedersen, SAW*-algebras and Corona C*-algebras, contributions to non-commutative topology, *J. Oper. Theory* 15 (1986) 15–32.
- [57] A.M. Peralta, A. Rodríguez Palacios, Grothendieck’s inequalities for real and complex JBW*-triples, *Proc. Lond. Math. Soc.* (3) 83 (2001) 605–625.
- [58] A.M. Peralta, B. Russo, Automatic continuity of triple derivations on C*-algebras and JB*-triples, *J. Algebra* 399 (2014) 960–977.
- [59] C.E. Rickart, Banach algebras with an adjoint operation, *Ann. Math.* (2) 47 (1946) 528–550.
- [60] Á. Rodríguez Palacios, On the strong* topology of a JBW*-triple, *Q. J. Math. Oxf. Ser.* (2) 42 (165) (1991) 99–103.
- [61] A. Rodríguez-Palacios, Jordan structures in analysis, in: *Jordan Algebras*, Oberwolfach, 1992, de Gruyter, Berlin, 1994, pp. 97–186.
- [62] K. Saitô, J.D.M. Wright, *Monotone Complete C*-Algebras and Generic Dynamics*, Springer Monographs in Mathematics, Springer, London, 2015.

- [63] K. Saitô, J.D.M. Wright, On defining AW*-algebras and Rickart C*-algebras, *Q. J. Math.* 66 (2015) 979–989.
- [64] S. Sakai, A characterization of W*-algebras, *Pac. J. Math.* 6 (1956) 763–773.
- [65] S. Sakai, C*- and W*-Algebras, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 60, Springer-Verlag, New York-Heidelberg, 1971.
- [66] M. Takesaki, Theory of Operator Algebras I. *Encyclopaedia of Mathematical Sciences*, vol. 124, Reprint of the first (1979) edition, *Operator Algebras and Non-commutative Geometry*, vol. 5, Springer-Verlag, Berlin, 2002.
- [67] D. Topping, Jordan algebras of self-adjoint operators, *Mem. Am. Math. Soc.* 53 (1965), 48 pp.
- [68] J.D.M. Wright, Wild AW*-factors and Kaplansky-Rickart algebras, *J. Lond. Math. Soc.* (2) 13 (1976) 83–89.
- [69] J.D.M. Wright, Jordan C*-algebras, *Mich. Math. J.* 24 (1977) 291–302.