

Classification of zero mean curvature surfaces of separable type in Lorentz-Minkowski space

Seher Kaya
 Department of Mathematics
 Ankara University
 Ankara, Turkey
 seherkaya@ankara.edu.tr

Rafael López*
 Departamento de Geometría y Topología
 Instituto de Matemáticas (IEMath-GR)
 Universidad de Granada
 18071 Granada, Spain
 rcamino@ugr.es

Abstract

Consider the Lorentz-Minkowski 3-space \mathbb{L}^3 with the metric $dx^2 + dy^2 - dz^2$ in canonical coordinates (x, y, z) . A surface in \mathbb{L}^3 is said to be separable if satisfies an equation of the form $f(x) + g(y) + h(z) = 0$ for some smooth functions f , g and h defined in open intervals of the real line. In this article we classify all zero mean curvature surfaces of separable type, providing a method of construction of examples.

Keywords: Lorentz-Minkowski space, zero mean curvature, separable surface

AMS Subject Classification: 53A10, 53C42

1 Introduction

A zero mean curvature (ZMC, for short) surface in the Lorentz-Minkowski 3-space \mathbb{L}^3 is a non-degenerate surface whose mean curvature is zero at every point of the surface. There are two types of non-degenerate surfaces, namely, spacelike and timelike surfaces if the induced metric is Riemannian and Lorentzian respectively. A spacelike ZMC surface is called a maximal surface because it locally maximizes the area functional. A timelike ZMC surface is also called a timelike minimal surface. In general, we will allow ZMC surfaces of mixed type, that is, it may exist regions on the surface of spacelike type and timelike

*Partially supported by the grant no. MTM2017-89677-P, MINECO/AEI/FEDER, UE

type. On the other hand, the zero mean curvature equation in non-parametric form is a PDE which is of elliptic type (resp. hyperbolic type) in case the surface is spacelike (resp. timelike).

One of the problems in the theory of ZMC surfaces is the construction of examples. Among the different techniques commonly proposed in the literature, we point out the use of complex analysis by means of the Weierstrass representation [16, 24] and the Björling formula [3, 5] or the use of integrable systems in the method of loop groups [6, 14]. It is also of special interest to provide examples of surfaces with particular geometric properties. A clear example of this idea is the family of ZMC surfaces that are invariant by a uniparametric group of rotations of \mathbb{L}^3 . These surfaces, the so-called rotational ZMC surfaces, are classified: see for example [16, 18, 23]. More recently, there is a great activity in the search of explicit examples of ZMC surfaces of mixed type where the transition between the spacelike and timelike regions occurs along degenerate curves: here we refer [2, 9, 10, 11, 12, 13, 15, 22] without to be a complete list.

In this paper we present a new method for constructing ZMC surfaces in Lorentz-Minkowski space \mathbb{L}^3 by the technique of separable variables. The Lorentz-Minkowski \mathbb{L}^3 is the vector space \mathbb{R}^3 with canonical coordinates (x, y, z) and endowed with metric \langle, \rangle of signature $(+, +, -)$. Any surface of \mathbb{L}^3 is locally the zero level set $F(x, y, z) = 0$ of a smooth function F defined in an open set $O \subset \mathbb{R}^3$, being 0 a regular value of F . Our strategy in the construction method consists in to assume that the function F is a separable function of the three variables x , y and z . This transforms the zero mean curvature equation into an ODE which may be more manageable.

Definition 1.1. *A surface S in \mathbb{L}^3 is said to be separable if can be expressed as*

$$S = \{(x, y, z) \in \mathbb{L}^3 : f(x) + g(y) + h(z) = 0\}. \quad (1)$$

Here f , g and h are smooth functions defined in some intervals I_1 , I_2 and I_3 of \mathbb{R} , respectively. Also, $f'(x)^2 + g'(y)^2 + h'(z)^2 \neq 0$ for every $x \in I_1$, $y \in I_2$ and $z \in I_3$.

In this paper, we pose the following

Problem. *Classify all separable surfaces in \mathbb{L}^3 with zero mean curvature.*

A first example of separable surface occurs if one of the functions in (1) is linear in its variable. Without loss of generality, we suppose that h is the linear function $h(z) = az + b$, with $a, b \in \mathbb{R}$. If $a = 0$, then the implicit equation of the surface is $f(x) + g(y) + b = 0$, which says that the surface is a cylindrical surface whose base curve is a planar curve

contained in the xy -plane. If now we assume that the surface has zero mean curvature, then this planar curve is a straight-line and the surface is a plane. If $a \neq 0$, then the implicit equation of the surface is $z = -f(x)/a - g(y)/a - b/a$. In the literature, a surface expressed as $z = \phi(x) + \psi(y)$ is called a translation surface. The family of ZMC surfaces of translation type was classified in [16, 17, 23]. These surfaces are the (spacelike and timelike) analogous ones in the Lorentzian setting of the classical minimal surfaces in the Euclidean space discovered by Scherk in 1835 ([21]; see also [20]), together a flat B-scroll over a null curve which appears as an exceptional case when the surface is a timelike minimal surface [23]. In the rest of the paper we will assume that the surface is neither cylindrical nor translation type, or equivalently, none of the functions f , g and h are linear functions.

Other examples of separable surfaces are the rotational surfaces when the rotation axis is one of the coordinate axes. Indeed, if the rotational axis is the z -line, the implicit equation of the surface is $G(z) = x^2 + y^2$. In case that the rotational axis is the x -line (resp. the y -line), then the rotational surface is $G(x) = y^2 - z^2$ (resp. $G(y) = x^2 - z^2$). Rotational surfaces with zero mean curvature will appear as a particular case in our method of construction of separable ZMC surfaces: see the beginning of Section 3.

In the Euclidean space, the minimal surfaces of separable type were initially studied by Weingarten [24] in 1887 with the purpose to generalize the translation surfaces $z = \phi(x) + \psi(y)$ obtained by Scherk [21]. Later, in 1956-7 Fréchet gave a deep study of these surfaces obtaining explicit examples [7, 8]. Our paper is inspired of the calculations of Fréchet (see also [20, Sect. 5.2]). It deserves to point that Sergienko and Tkachev [22] gave an approach to the study of separable maximal surfaces in \mathbb{L}^3 because they were interested in the construction of doubly periodic maximal surfaces which satisfy an implicit equation of type $\zeta(z) = \phi(x)\psi(y)$, which is equivalent to (1).

The goal of our paper is to investigate the problem in all its generality, assuming mixed type causal character and providing an exhaustive method to construct all separable ZMC surfaces. This allows to present a plethora of new examples of ZMC surfaces, including also other known surfaces in the literature.

In Section 2, we compute the zero mean curvature equation of a separable surface obtaining for each one of the functions f , g and h a differential equation of fourth order. After successive integrations, we finally obtain three ODEs of first order on f , g and h respectively depending on a real constant K and a set of nine constants $a_i, b_i, c_i, 1 \leq i \leq 3$. These constants are not arbitrary because they are linked by six nonlinear equations. A first step in our strategy is to search the constants a_i, b_i and c_i satisfying the above six equations and once obtained these constants, to solve the ODEs. In general, this system can not be solved by quadratures and the solutions will be expressed in terms of elliptic

functions. We separate this discussion in the sections 3, 4 and 5 depending on the sign of the constant K . We will obtain explicit examples of separable ZMC surfaces and we will study the causal character of the surface and, if possible, its extension to regions of lightlike points in each case.

2 The method of construction of separable ZMC surfaces

The Lorentz-Minkowski space \mathbb{L}^3 is the vector space \mathbb{R}^3 with canonical coordinates (x, y, z) and endowed with the Lorentzian metric $\langle, \rangle = dx^2 + dy^2 - dz^2$. A vector $\vec{v} \in \mathbb{R}^3$ is said to be spacelike, timelike or lightlike if the inner product $\langle \vec{v}, \vec{v} \rangle$ is positive, negative or zero, respectively. The norm of \vec{v} is $|\vec{v}| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ if \vec{v} is spacelike and $|\vec{v}| = \sqrt{-\langle \vec{v}, \vec{v} \rangle}$ if \vec{v} is timelike. More generally, a surface (or a curve) S of \mathbb{L}^3 is called spacelike, timelike or lightlike if the induced metric on S is Riemannian, Lorentzian or degenerated, respectively. This property is called the causal character of S . We refer the reader to [19, 25] for some basics of \mathbb{L}^3 . In order to have no confusion, we denote by \mathbb{E}^3 for the Euclidean 3-space, that is, \mathbb{R}^3 endowed with the Euclidean metric $dx^2 + dy^2 + dz^2$.

Let S be a surface in \mathbb{L}^3 whose induced metric \langle, \rangle is non-degenerated. Recall that it is equivalent to say that S is spacelike (resp. timelike) if there is a unit normal timelike (resp. spacelike) vector field defined on S . Here we allow that the surface is of mixed type, so it may exist regions on the surface of spacelike and timelike type and, eventually, we will study if the surface can be extended to regions of \mathbb{L}^3 of lightlike type. In both types of surfaces, the mean curvature H is defined as the trace of the second fundamental form. If $H = 0$ everywhere, we say that S has zero mean curvature and we abbreviate by saying a ZMC surface.

We know that any surface S is locally given by an implicit equation $F(x, y, z) = 0$, where 0 is a regular value of F . The mean curvature H is calculated by means of the gradient and the Hessian matrix of F where the computations are similar as in \mathbb{E}^3 . The Lorentzian gradient of F is

$$\nabla^L F = (F_x, F_y, -F_z)$$

where, as it is usual, the subscripts indicate the partial derivatives with respect to the corresponding variable. The surface is spacelike (resp. timelike) if $\langle \nabla^L F, \nabla^L F \rangle < 0$ (resp. $\langle \nabla^L F, \nabla^L F \rangle > 0$) and consequently $N = \nabla^L F / |\nabla^L F|$ defines a unit normal vector field

on S . The mean curvature H is the Lorentzian divergence div^L of N ,

$$\operatorname{div}^L \left(\frac{\nabla^L F}{|\nabla^L F|} \right) = \left(\frac{F_x}{|\nabla^L F|} \right)_x + \left(\frac{F_y}{|\nabla^L F|} \right)_y - \left(\frac{F_z}{|\nabla^L F|} \right)_z = H.$$

Therefore the ZMC equation $H = 0$ is equivalent to

$$\frac{\Delta^L F}{|\nabla^L F|} + \frac{\epsilon}{|\nabla^L F|^3} (\nabla^L F)^t \cdot \operatorname{Hess} F \cdot \nabla^L F = 0,$$

where $\epsilon = 1$ (resp. $\epsilon = -1$) if S is spacelike (resp. timelike), $\Delta^L F = F_{xx} + F_{yy} - F_{zz}$ and

$$\operatorname{Hess} F = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix}.$$

Proposition 2.1. *If $S = F^{-1}(\{0\})$ is a non-degenerate surface in \mathbb{L}^3 , then $H = 0$ if and only if*

$$-\langle \nabla^L F, \nabla^L F \rangle \Delta^L F + (\nabla^L F)^t \cdot \operatorname{Hess} F \cdot \nabla^L F = 0. \quad (2)$$

Now suppose that the function $F = F(x, y, z)$ is of separable variables $F(x, y, z) = f(x) + g(y) + h(z)$. The causal character of S is determined by the sign of $\langle \nabla^L F, \nabla^L F \rangle = f'^2 + g'^2 - h'^2$, where the symbol $'$ indicates the derivative with respect to the corresponding variable. Thus if S is spacelike (resp. timelike) then $f'^2 + g'^2 - h'^2 < 0$ (resp. $f'^2 + g'^2 - h'^2 > 0$). The equation $H = 0$ in (2) is now

$$f''(g'^2 - h'^2) + g''(f'^2 - h'^2) - h''(f'^2 + g'^2) = 0. \quad (3)$$

Let us introduce the notation

$$u = f(x), \quad v = g(y), \quad w = h(z)$$

and

$$X(u) = f'^2, \quad Y(v) = g'^2, \quad Z(w) = h'^2.$$

In the new variables, the implicit equation of the surface (1) is now $u + v + w = 0$. In terms of the functions X , Y and Z , the causal character of the surface is determined by the sign of $X(u) + Y(v) - Z(w)$, with $u + v + w = 0$, which is spacelike (resp. timelike, lightlike) if the sign is negative (resp. positive, zero).

The ZMC equation (3) can be expressed as

$$A := (Y - Z)X' + (X - Z)Y' - (X + Y)Z' = 0, \quad (4)$$

for all values u , v and w with the condition $u + v + w = 0$.

Since we are assuming that the separable surface is not cylindrical or translation type, then none of the functions f , g and h are linear, in particular, none of the three functions X' , Y' or Z' can vanish identically in some open interval of the corresponding domain.

We need the following auxiliary result.

Lemma 2.2. *Let $Q = Q(u, v, w)$ be a smooth function defined in a domain $\Omega \subset \mathbb{R}^3$. If $Q(u, v, w) = 0$ for any triple of the section $\Omega \cap \Pi$, where Π is the plane of equation $u + v + w = 0$, then on the section we have*

$$Q_u = Q_v = Q_w.$$

Proof. If we write $w = -u - v$, then $Q(u, v, -u - v) = 0$. Differentiating with respect to u , we deduce $Q_u - Q_w = 0$. Changing the roles of u , v and w , we conclude the result. \square

Using this lemma, the identities $A_u - A_v = 0$, $A_v - A_w = 0$ and $A_u - A_w = 0$, are respectively

$$\begin{aligned} B_1 &:= (Y - Z)X'' - (X - Z)Y'' - (X' - Y')Z' = 0, \\ B_2 &:= (Y' + Z')X' + (X - Z)Y'' + Z''(X + Y) = 0, \\ B_3 &:= (Y - Z)X'' + (X' + Z')Y' + (X + Y)Z'' = 0. \end{aligned} \tag{5}$$

From (4) and the first two equations in (5), we have a system of linear equations on $Y - Z$, $X - Z$ and $X + Y$. The determinant of the coefficients of this system is $-M$, where

$$M = X''Y''Z' + X'Y''Z'' + X''Y'Z''.$$

We find directly that

$$\begin{aligned} M(Y - Z)X' &= -X'Y'Z'(Y''(X' + Z') + Z''(Y' - X')), \\ M(X - Z)Y' &= -X'Y'Z'(X''(Y' + Z') - Z''(Y' - X')), \\ M(X + Y)Z' &= -X'Y'Z'(X''(Y' + Z') + Y''(X' + Z')). \end{aligned}$$

Applying Lemma 2.2 again to the functions B_1 , B_2 and B_3 in (5), we have $(B_1)_v - (B_1)_w = 0$, $(B_2)_u - (B_2)_w = 0$ and $(B_3)_u - (B_3)_v = 0$. These three equations write, respectively,

$$\begin{aligned} X''(Y' + Z') - Y'''(X - Z) - Z''(Y' - X') &= 0, \\ X''(Y' + Z') + Y''(X' + Z') - Z'''(X + Y) &= 0, \end{aligned}$$

$$X'''(Y - Z) - Y''(X' + Z') - Z''(Y' - X') = 0,$$

and so

$$\begin{aligned} M(Y - Z)X' &= -X'Y'Z'(Y - Z)X''', \\ M(X - Z)Y' &= -X'Y'Z'(X - Z)Y''', \\ M(X + Y)Z' &= -X'Y'Z'(X + Y)Z'''. \end{aligned}$$

Since $X'Y'Z' \neq 0$, we deduce

$$\frac{X'''}{X'} = \frac{Y'''}{Y'} = \frac{Z'''}{Z'} = K, \quad (6)$$

where $K \in \mathbb{R}$ is a real constant. We solve these ODEs according to the sign of K .

1. Case $K > 0$. Let $K = k^2$, $k > 0$. The solutions of (6) are

$$\begin{aligned} X(u) &= a_1 + b_1 e^{ku} + c_1 e^{-ku}, \\ Y(v) &= a_2 + b_2 e^{kv} + c_2 e^{-kv}, \\ Z(w) &= a_3 + b_3 e^{kw} + c_3 e^{-kw}, \end{aligned} \quad (7)$$

where $a_i, b_i, c_i \in \mathbb{R}$, $1 \leq i \leq 3$. These nine constants are not arbitrary because there is a relation between them thanks to (4). We introduce the functions X , Y and Z in (4) and we replace w by $-u - v$ because $u + v + w = 0$. Then (4) is an equation of type

$$P_1 e^{-ku} + P_2 e^{-kv} + P_3 e^{ku} + P_3 e^{kv} + P_5 e^{-ku-kv} + P_6 e^{ku+kv} = 0.$$

Since the exponential functions are linearly independent, the coefficients P_i must be 0. A computation of P_i , $1 \leq i \leq 6$, and setting to be 0, yields

$$\begin{cases} (a_2 - a_3)c_1 + 2b_2b_3 = 0, \\ (a_1 - a_3)c_2 + 2b_1b_3 = 0, \\ (a_2 - a_3)b_1 + 2c_2c_3 = 0, \\ (a_1 - a_3)b_2 + 2c_1c_3 = 0, \\ (a_1 + a_2)b_3 + 2c_1c_2 = 0, \\ (a_1 + a_2)c_3 + 2b_1b_2 = 0. \end{cases} \quad (8)$$

2. Case $K < 0$. Let $K = -k^2$, $k > 0$. Then the solutions of (6) are

$$\begin{aligned} X(u) &= a_1 + b_1 \cos(ku) + c_1 \sin(ku), \\ Y(v) &= a_2 + b_2 \cos(kv) + c_2 \sin(kv), \\ Z(w) &= a_3 + b_3 \cos(kw) + c_3 \sin(kw). \end{aligned} \quad (9)$$

As in the previous case, and using that the trigonometric functions $\cos(ku)$, $\sin(ku)$, $\cos(kv)$ and $\cos(kv)$ are linearly independent, the relations between the constants a_i , b_i and c_i are now:

$$\begin{cases} (a_2 - a_3)c_1 - b_3c_2 - b_2c_3 = 0, \\ (a_1 - a_3)c_2 - b_3c_1 - b_1c_3 = 0, \\ (a_2 - a_3)b_1 + b_2b_3 - c_2c_3 = 0, \\ (a_1 - a_3)b_2 + b_1b_3 - c_1c_3 = 0, \\ (a_1 + a_2)b_3 + b_1b_2 - c_1c_2 = 0, \\ (a_1 + a_2)c_3 - b_2c_1 - b_1c_2 = 0. \end{cases} \quad (10)$$

3. Case $K = 0$. Now the solutions of (6) are

$$\begin{aligned} X(u) &= a_1 + b_1u + c_1u^2, \\ Y(v) &= a_2 + b_2v + c_2v^2, \\ Z(w) &= a_3 + b_3w + c_3w^2. \end{aligned} \quad (11)$$

As in the case $K \neq 0$, we insert the above solutions in (4) and replace w by $-u - v$, obtaining an equation of type

$$P_1 + P_2u + P_3v + P_4uv + P_5u^2 + P_6v^2 + P_7u^2v + P_8uv^2 = 0.$$

The functions on u and v are linearly independent, so all coefficients P_i must vanish, $1 \leq i \leq 8$. A computation of P_i yields $P_7 = P_8$ and P_4 is a linear combination of P_5 and P_6 . Once computed the coefficients P_i and setting to be 0, we deduce

$$\begin{cases} a_1(b_2 - b_3) + a_2(b_1 - b_3) - a_3(b_1 + b_2) = 0, \\ b_1b_2 + b_2b_3 + 2c_1(a_2 - a_3) + 2c_3(a_1 + a_2) = 0, \\ b_1b_2 + b_1b_3 + 2c_2(a_1 - a_3) + 2c_3(a_1 + a_2) = 0, \\ c_1(b_2 + b_3) + c_3(b_1 - b_2) = 0, \\ c_2(b_1 + b_3) - c_3(b_1 - b_2) = 0, \\ c_1c_2 - c_1c_3 - c_2c_3 = 0. \end{cases} \quad (12)$$

Definitively, the functions f , g and h in (1) are the solutions of the ODEs of first order (7), (9) and (11). We summarize in the following result the method of constructions of all separable ZMC surfaces of \mathbb{L}^3 .

Theorem 2.3. *Let S be a separable ZMC surface given by (1). Then the derivatives f' , g' and h' given in terms of the functions X , Y and Z are the following:*

1. Case $K \neq 0$. Then the functions X , Y and Z are given by expressions (7) or (9) and the constants a_i , b_i and c_i satisfy the relations (8) or (10) respectively.
2. Case $K = 0$. Then the functions X , Y and Z are given by the expressions (11) and the constants a_i , b_i and c_i satisfy the relations (12).

In order to integrate (7), (9) and (11), we remark that the functions in the right-hand sides of these equations must be positive because the functions X , Y and Z are positive. In addition, we need to consider a choice of the sign for the square roots of the first derivative, namely,

$$\int^u \frac{du}{\sqrt{X(u)}} = \pm x, \quad \int^v \frac{dv}{\sqrt{Y(v)}} = \pm y, \quad \int^w \frac{dw}{\sqrt{Z(w)}} = \pm w,$$

where the (+) or (-) sign is chosen depending on whether the derivatives $f'(x)$, $g'(y)$ and $h'(z)$ are positive or negative. For simplicity, we will usually choose the positive sign.

We now prove that varying the value of k in (7) or (9), the solution surface is the same up to a homothety. This will allow to fix the value of k in (7) or (9).

Proposition 2.4. *A change of k in the solutions of (7) or (9) produces a homothetical ZMC surface which is also of separable type.*

Proof. Let S be a separable ZMC surface given by $f(x) + g(y) + h(z) = 0$ and suppose that it is a solution of (7) or (9) with $\lambda \neq 0$. Define the functions $\tilde{f}(x) = \lambda f(x/\lambda)$, $\tilde{g}(y) = \lambda g(y/\lambda)$ and $\tilde{h}(z) = \lambda h(z/\lambda)$, defined in the intervals λI_1 , λI_2 and λI_3 respectively. Then the separable surface $\tilde{S} = \{(\tilde{x}, \tilde{y}, \tilde{z}) : \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{y}) + \tilde{h}(\tilde{z}) = 0\}$ is the dilation of the surface S by λ , in particular, \tilde{S} is also a ZMC surface.

With the utilized notation in this section, we have $\tilde{u} = \lambda u$, $\tilde{v} = \lambda v$ and $\tilde{w} = \lambda w$. Then $\tilde{X}(\tilde{u}) = \tilde{f}'(\tilde{x})^2 = f'(x/\lambda)^2 = X(u)$, and similarly, $\tilde{Y}(\tilde{v}) = Y(v)$ and $\tilde{Z}(\tilde{w}) = Z(w)$. It is immediate

$$\tilde{X}'(\tilde{u}) = \frac{d}{d\tilde{u}} \tilde{X}(\tilde{u}) = \frac{1}{\lambda} X'(u),$$

and similarly $\tilde{X}'''(\tilde{u}) = X'''(u)/\lambda^3$. From (6) we deduce

$$\tilde{K} = \frac{\tilde{X}'''(\tilde{u})}{\tilde{X}'(\tilde{u})} = \frac{1}{\lambda^2} \frac{X'''(u)}{X'(u)} = \frac{1}{\lambda^2} K,$$

obtaining the result. □

In the next sections we will show explicit examples of ZMC surfaces. In some particular cases, the ODEs can be solved by quadratures obtaining expressions in terms of trigonometric or hyperbolic functions. In general, we will see that the class of ZMC surfaces of

separable type can be expressed in terms of elliptic functions which are the inverses of certain elliptic integrals.

3 Separable ZMC surfaces: case $K = 0$.

In this section we will obtain particular examples of separable ZMC surfaces in case $K = 0$ of Theorem 2.3.

Firstly, we will prove that the rotational ZMC surfaces belong to the case $K = 0$, exactly when some of the constants c_i in (12) are 0. Indeed, and without loss of generality, we suppose $c_1 = 0$. From the last equation of (12), we have $c_2c_3 = 0$. Because the arguments are symmetric on c_2 and c_3 , we will assume in what follows that $c_2 = 0$. The fifth equation of (12) is $c_3(b_1 - b_2) = 0$. Now there are two possibilities.

1. Case $c_3 = 0$. Then the second and third equations of (12) are $b_2(b_1 + b_3) = 0$ and $b_1(b_2 + b_3) = 0$, respectively. If $b_1 = 0$ (resp. $b_2 = 0$), then $X' = 0$ (resp. $Y' = 0$), which it is not possible. Hence $b_1 = b_2 = -b_3$ with $b_i \neq 0$. Now the first equation of (12) yields $a_3 = a_1 + a_2$. Then $X(u) = a_1 + b_1u$, $Y(v) = a_2 + b_1v$ and $Z(w) = a_3 - b_1w$, in particular, $X + Y - Z = a_1 + a_2 - a_3 = 0$ proving that the surface is degenerate everywhere, which is not possible.
2. Case $c_3 \neq 0$. Then $b_2 = b_1 \neq 0$. If we set $b_1 = b$, then $X(u) = a_1 + bu$ and $Y(v) = a_2 + bv$ and the integration leads to

$$f(x) = \frac{b}{4}x^2 - \frac{a_1}{b}, \quad g(y) = \frac{b}{4}y^2 - \frac{a_2}{b}.$$

Thus the implicit equation of the surface is

$$x^2 + y^2 + \frac{4}{b}h(z) - 4(a_1 + a_2) = 0,$$

proving that the surface is a surface of revolution with respect to the z -axis.

As it was discussed in the Introduction, the rotational ZMC surfaces are classified, so we will discard this case, or equivalently, we will assume that $c_i \neq 0$ for all $1 \leq i \leq 3$.

3.1 Example where all a_i and b_i are 0

Let choose the constants as $a_i = b_i = 0$, $1 \leq i \leq 3$, in (12). Then

$$X(u) = c_1u^2, \quad Y(v) = c_2v^2, \quad Z(w) = c_3w^2,$$

where (12) reduces into the single equation $c_1c_2 - c_1c_3 - c_2c_3 = 0$. Note that $c_i > 0$, $1 \leq i \leq 3$, because X, Y and Z are positive functions. The solutions are

$$f(x) = \pm e^{\sqrt{c_1}x}, \quad g(y) = \pm e^{\sqrt{c_2}y}, \quad h(z) = \pm e^{\sqrt{c_3}z},$$

and the surface is

$$\{(x, y, z) \in \mathbb{L}^3 : \pm e^{\sqrt{c_1}x} \pm e^{\sqrt{c_2}y} \pm e^{\sqrt{c_3}z} = 0\}.$$

See Figure 2, left. For the causal character of the surface, we study the sign of $X + Y - Z$. By using $w = -u - v$,

$$X(u) + Y(v) - Z(w) = c_1u^2 + c_2v^2 - c_3w^2 = \frac{(c_1u - c_2v)^2}{c_1 + c_2}.$$

Since $c_i > 0$, the surface is timelike and it may be extended to regions of lightlike points if $c_1u - c_2v = 0$. If we choose opposite signs in the definition of the functions f and g , then $c_1u - c_2v \neq 0$ and the surface is always timelike. Now suppose that the functions have the same sign, that is, $f(x) = e^{\sqrt{c_1}x}$ and $g(y) = e^{\sqrt{c_2}y}$. Then the surface is

$$\{(x, y, z) \in \mathbb{L}^3 : e^{\sqrt{c_1}x} + e^{\sqrt{c_2}y} - e^{\sqrt{c_3}z} = 0\}$$

and

$$c_1u - c_2v = c_1e^{\sqrt{c_1}x} - c_2e^{\sqrt{c_2}y} = e^{\sqrt{c_1}x + \log(c_1)} - e^{\sqrt{c_2}y + \log(c_2)}.$$

Thus $c_1u - c_2v = 0$ is equivalent to $\sqrt{c_1}x + \log(c_1) = \sqrt{c_2}y + \log(c_2)$. Using $w = -u - v$, we deduce $\sqrt{c_3}z = \sqrt{c_1}x + \log(c_1 + c_2) - \log(c_2)$, obtaining a straight-line. We conclude that the surface can be extended to the set

$$\{(x, y, z) : \sqrt{c_1}x + \log(c_1) = \sqrt{c_2}y + \log(c_2), \sqrt{c_3}z = \sqrt{c_1}x + \log(c_1 + c_2) - \log(c_2)\}$$

which is a straight-line formed by lightlike points.

3.2 Example where all b_i are 0 and all a_i are not 0: case 1

In this subsection, we will show two examples where the difference will be that the constants a_i, b_i and c_i have opposite signs. For the first example, the constants a_i, b_i and c_i are

$$\begin{aligned} b_1 = b_2 = b_3 = 0, \quad c_1 = -\alpha^2\beta^2, c_2 = -\alpha^2, c_3 = -\beta^2 \\ a_1 = \alpha^2\beta^2, \quad a_2 = \alpha^2\beta^4, \quad a_3 = \alpha^4\beta^2, \end{aligned}$$

with $\alpha^2 - \beta^2 = 1$ and $\beta \neq 0$. Then it is immediate that they satisfy the equations (12) and the ODEs (11) are

$$X(u) = \alpha^2 \beta^2 - \alpha^2 \beta^2 u^2, \quad Y(v) = \alpha^2 \beta^4 - \alpha^2 v^2, \quad Z(w) = \alpha^4 \beta^2 - \beta^2 w^2.$$

After integrating, we obtain

$$\begin{aligned} x &= \int \frac{1}{\alpha\beta} \frac{du}{\sqrt{1-u^2}} = \frac{1}{\alpha\beta} \arcsin(u) \\ y &= \int \frac{1}{\alpha} \frac{dv}{\sqrt{\beta^4 - v^2}} = \frac{1}{\alpha} \arcsin\left(\frac{v}{\beta^2}\right) \\ z &= \int \frac{1}{\beta} \frac{dw}{\sqrt{\alpha^4 - w^2}} = \frac{1}{\beta} \arcsin\left(\frac{w}{\alpha^2}\right) \end{aligned}$$

Therefore

$$f(x) = \sin(\alpha\beta x), \quad g(y) = \beta^2 \sin(\alpha y), \quad h(z) = \alpha^2 \sin(\beta z)$$

and the surface is

$$\{(x, y, z) \in \mathbb{L}^3 : \sin(\alpha\beta x) + \beta^2 \sin(\alpha y) + \alpha^2 \sin(\beta z) = 0\}.$$

This surface is a triply periodic ZMC surface because the functions f , g and h are periodic. See Figure 1, left.

We study the causal character of the surface. Letting $w = -u - v$, we have

$$\begin{aligned} X(u) + Y(v) - Z(w) &= -\alpha^2 \beta^2 u^2 - \alpha^2 v^2 + \beta^2 w^2 = -\beta^2 (\beta^2 u^2 + \frac{v^2}{\beta^2} - 2uv) \\ &= -\beta^4 (\sin(\alpha\beta x) - \sin(\alpha y))^2 \leq 0. \end{aligned}$$

This implies that the surface is spacelike except satisfy the equation $\sin(\alpha\beta x) - \sin(\alpha y) = 0$, which are lightlike points. The region of these points is included in the set of straight-lines of equations

$$\{y = \beta x + 2\pi\mathbb{Z}, z = -\alpha x + 2\pi\mathbb{Z}\} \cup \{y = \pi - \beta x + 2\pi\mathbb{Z}, z = \alpha x + \pi + 2\pi\mathbb{Z}\}.$$

The second example appears when we reverse of sign all constant a_i and c_i , that is,

$$\begin{aligned} b_1 = b_2 = b_3 &= 0, \quad c_1 = \alpha^2 \beta^2, c_2 = \alpha^2, c_3 = \beta^2 \\ a_1 &= -\alpha^2 \beta^2, \quad a_2 = -\alpha^2 \beta^4, \quad a_3 = -\alpha^4 \beta^2. \end{aligned}$$

Then they satisfy (12) again and the integrations lead to

$$f(x) = \pm \cosh(\alpha\beta x), \quad g(y) = \pm \beta^2 \cosh(\alpha y), \quad h(z) = \pm \alpha^2 \cosh(\beta z)$$

and the surface is

$$\{(x, y, z) \in \mathbb{L}^3 : \pm \cosh(\alpha\beta x) \pm \beta^2 \cosh(\alpha y) \pm \alpha^2 \cosh(\beta z) = 0\}.$$

There must be two opposite signs in the implicit equation of the surface because on the contrary the above set would be empty. As in the above example, now the surface is a timelike minimal surface. In order to study when the surface can be extended by lightlike points, without loss of generality, we suppose that the implicit equation of the surface is $\cosh(\alpha\beta x) + \beta^2 \cosh(\alpha y) - \alpha^2 \cosh(\beta z) = 0$. Then

$$0 = \beta^2 u^2 + \frac{v^2}{\beta^2} - 2uv = \beta^2 (\cosh(\alpha\beta x) - \cosh(\alpha y))^2.$$

Hence $y = \pm\beta x$. By using $u + v + w = 0$, we deduce $\cosh(\alpha x) = \cosh(z)$, so $z = \pm\alpha x$. Thus the surface can be extended to four straight-lines of lightlike points which meet at the origin of coordinates. See Figure 1, right. Here we point out that a ZMC surface containing a non-degenerate straight-line can be extended by symmetry by the reflection principle [3, 15]. However, we can not expect symmetries along lightlike straight-lines. Recently, Akamine and Fujino have proved that if two lightlike line segments do intersect, a conelike singularity always appears at that intersection point, and hence there exists the reflection property [1]. This situation occurs for this surface: by the way, it did not appear in the surface of subsection 3.1 because there was only one straight-line.

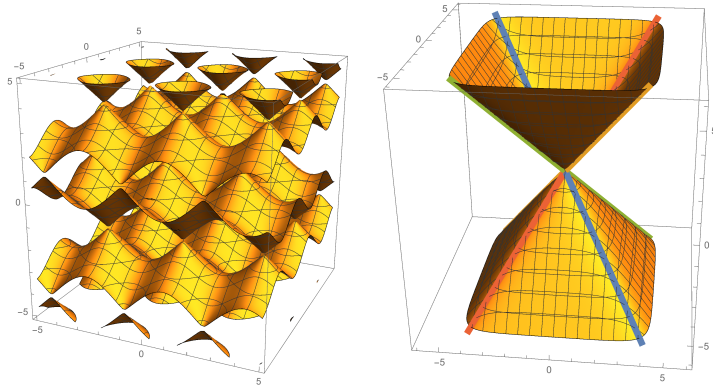


Figure 1: Left: the surface $\sin(\alpha\beta x) + \beta^2 \sin(\alpha y) + \alpha^2 \sin(\beta z) = 0$. Right: the surface $\cosh(\alpha\beta x) + \beta^2 \cosh(\alpha y) - \alpha^2 \cosh(\beta z) = 0$, where we have indicated the four straight-lines of lightlike points

3.3 Example where all b_i are 0 and all a_i are not 0: case 2

Let take the constant as $b_1 = b_2 = b_3 = 0$, $a_1 = a_2 = a_3/2 = a$ and $c_1 = c_2 = 2c_3 = 2c$, where a and c are non-zero real numbers. Then the functions are

$$X(u) = a + 2cu^2, \quad Y(v) = a + 2cv^2, \quad Z(w) = 2a + cw^2.$$

Since the functions X , Y and Z are positive, the case $a, c < 0$ is not possible. The computation of the integral depends on the sign of a and c , being

$$\int \frac{dt}{\sqrt{m + nt^2}} = \begin{cases} \frac{1}{\sqrt{n}} \operatorname{arcsinh}\left(\sqrt{\frac{n}{m}}t\right) & m > 0, n > 0 \\ \frac{1}{\sqrt{-n}} \arcsin\left(\sqrt{\frac{-n}{m}}t\right) & m > 0, n < 0 \\ \frac{1}{\sqrt{n}} \operatorname{arcosh}\left(\sqrt{\frac{-n}{m}}t\right) & m < 0, n > 0. \end{cases}$$

If $a, c > 0$, then the implicit equation of the surface is

$$\sqrt{\frac{a}{2c}} \sinh(\sqrt{2c}x) + \sqrt{\frac{a}{2c}} \sinh(\sqrt{2c}y) + \sqrt{\frac{2a}{c}} \sinh(\sqrt{c}z) = 0.$$

See Figure 2, right. If $a > 0$, $c < 0$, then

$$\sqrt{\frac{-a}{2c}} \sin(\sqrt{-2c}x) + \sqrt{\frac{-a}{2c}} \sin(\sqrt{-2c}y) + \sqrt{\frac{-2a}{c}} \sin(\sqrt{-c}z) = 0.$$

If $a < 0$, $c > 0$, then

$$\pm \sqrt{\frac{-a}{2c}} \cosh(\sqrt{2c}x) \pm \sqrt{\frac{-a}{2c}} \cosh(\sqrt{2c}y) \pm \sqrt{\frac{-2a}{c}} \cosh(\sqrt{c}z) = 0.$$

By using the equation

$$X(u) + Y(v) - Z(w) = c(u - v)^2,$$

we can say that the surface is spacelike or timelike if $c < 0$ or $c > 0$ respectively, except at the points $\{u = v, w = -2v\}$.

Let us observe that the solution of the case $a > 0$, $c < 0$ and the case $a < 0$, $c > 0$ are similar to the surfaces of subsection 3.2. In such a case, we know that the surface can be extended to lightlike points which are contained in a set of straight-lines.

Finally, in the case $a, c > 0$, the surface is a timelike minimal surface except in the set $u = v$, $w = -2v$. This set is now $f(x) = g(y)$, $h(z) = -2g(y)$, or equivalently, the straight-line $\{x = y, z = -\sqrt{2}y\}$.

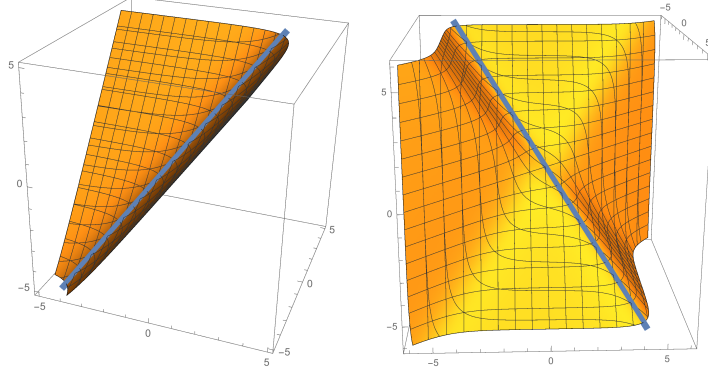


Figure 2: Left: the surface $e^x + e^y - e^{\sqrt{2}z}/2 = 0$ of subsection 3.1. Right: the surface $\sinh(\sqrt{2}x) + \sinh(\sqrt{2}y) + 2\sinh(z) = 0$ of subsection 3.3. We have indicated the straight-lines of lightlike points

4 Separable ZMC surfaces: case $K > 0$.

In this section we obtain particular examples of separable ZMC surfaces when $K > 0$ in Theorem 2.3. By Proposition 2.4, we can assume that $k = 2$ without loss of generality. To find explicit examples of separable ZMC surfaces, we follow the same procedure as in the previous section. Firstly we determine the real numbers a_i , b_i and c_i that satisfy (8), then we integrate the differential equations (7) and finally we will study the causal character of the surface.

We divide this section in subsections according the numbers of the differential equations of (7) that can be solved by quadratures. The other solutions will be expressed in terms of elliptic integrals.

4.1 Case where the three differential equations are solved by quadratures

In this section we give four examples of ZMC surfaces of separable type where all integrals in (7) can be solved by quadratures.

4.1.1 Example 1

Consider the following constants:

$$\begin{aligned} a_1 &= 1, & b_1 &= 0, & c_1 &= 1 \\ a_2 &= 1, & b_2 &= 0, & c_2 &= m \\ a_3 &= 1, & b_3 &= -m, & c_3 &= 0, \end{aligned}$$

where $m \in \{-1, 1\}$. Then

$$X(u) = 1 + e^{-2u}, \quad Y(v) = 1 + me^{-2v}, \quad Z(w) = 1 - me^{2w}.$$

The integration of the first equation yields $f(x) = \log(\sinh(x))$. The integration of the other two differential equations depends on the sign of m :

$$g(y) = \begin{cases} \log(\sinh(y)) & m = 1 \\ \log(\cosh(y)) & m = -1, \end{cases} \quad h(z) = \begin{cases} -\log(\cosh(z)) & m = 1 \\ -\log(\sinh(z)) & m = -1. \end{cases}$$

In each case of m , the implicit equation of the surface is given by

$$\begin{aligned} \sinh(x) \sinh(y) &= \cosh(z) \quad (\text{case } m = 1) \\ \sinh(x) \cosh(y) &= \sinh(z) \quad (\text{case } m = -1). \end{aligned} \tag{13}$$

See Figure 3. The second surface in (13) is known as the timelike Scherk surface of second kind [10] which is an entire graph over the xy -plane.

For the causal character of the surface,

$$\begin{aligned} X(u) + Y(v) - Z(w) &= 1 + e^{-2u} + me^{-2v} + me^{-2u-2v} \\ &= (1 + e^{-2u})(1 + me^{-2v}) = X(u)Y(v) > 0. \end{aligned}$$

In case $m = 1$, the surface is timelike and it can not be extended to regions of lightlike points. In case $m = -1$, the surface is timelike again and also it can be extended to regions of lightlike points when $g'(y) = 0$, which is equivalent to $y = 0$. From the equation of the surface, we deduce $\sinh(x) = \sinh(z)$, so we have $x = z$. Thus the lightlike points consists in the straight-line $\{y = 0, x = z\}$. Therefore the surface is a timelike minimal surface that can be extended to one lightlike straight-line but the surface does not change type across this line.

4.1.2 Example: the Scherk surfaces

Consider the following constants:

$$\begin{aligned} a_1 &= 1, & b_1 &= 0, & c_1 &= -1 \\ a_2 &= 1, & b_2 &= 0, & c_2 &= m \\ a_3 &= 1, & b_3 &= m, & c_3 &= 0, \end{aligned}$$

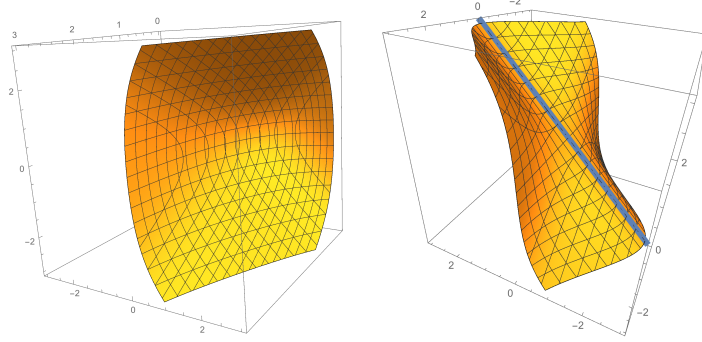


Figure 3: Left: the surface $\sinh(x) \sinh(y) = \cosh(z)$. Right: the surface $\sinh(x) \cosh(y) = \sinh(z)$

where $m \in \{-1, 1\}$. Then

$$X(u) = 1 - e^{-2u}, \quad Y(v) = 1 + me^{-2v}, \quad Z(w) = 1 + me^{2w}.$$

The integration yields $f(x) = \log(\cosh(x))$ and

$$g(y) = \begin{cases} \log(\sinh(y)) & m = 1 \\ \log(\cosh(y)) & m = -1 \end{cases} \quad h(z) = \begin{cases} -\log(\sinh(z)) & m = 1 \\ -\log(\cosh(z)) & m = -1. \end{cases}$$

For $m = 1$, the surface is $\cosh(x) \sinh(y) = \sinh(z)$, which appeared in the above subsection interchanging the roles of the variables x and y .

For $m = -1$, the implicit equation of surface is $\cosh(x) \cosh(y) = \cosh(z)$, see Figure 4, left. This surface is called the timelike Scherk surface of first kind [10]. The causal character of the surface is given by the sign of the function

$$X(u) + Y(v) - Z(w) = (1 - e^{-2u})(1 + me^{-2v}) = f'(x)^2 g'(y)^2,$$

and it says that the surface is timelike. We study the extension of the surface to regions of lightlike points. This occurs if $f'(x) = 0$ or $g'(y) = 0$, that is, $\sinh(x) = 0$ or $\sinh(y) = 0$. We conclude that this region is formed by four straight-lines of equations $\{x = 0, z = \pm y\}$ and $\{y = 0, z = \pm x\}$. Here we have a conelike point at the origin according [1].

If $m = -1$, it is possible to obtain new examples of separable ZMC surfaces by changing the signs of the constants a_i , b_i and c_i (this is not possible to do it for $m = 1$ because the functions Y and Z would be negative). Now

$$\begin{aligned} a_1 &= -1, & b_1 &= 0, & c_1 &= 1 \\ a_2 &= -1, & b_2 &= 0, & c_2 &= 1 \\ a_3 &= -1, & b_3 &= 1, & c_3 &= 0. \end{aligned}$$

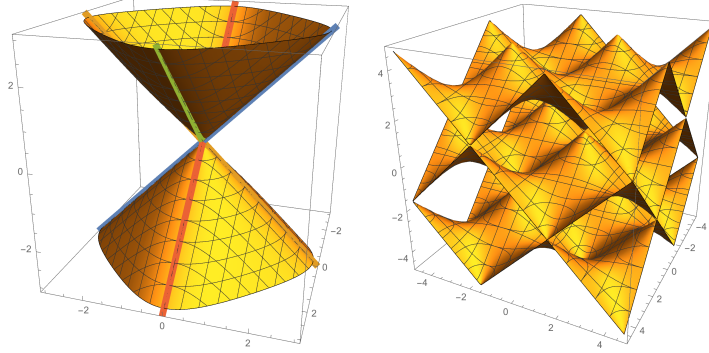


Figure 4: Left: the surface $\cosh(x) \cosh(y) = \cosh(z)$. Right: the surface $\sin(x) \sin(y) = \sin(z)$

The integration of the three differential equations gives the surface as

$$\{(x, y, z) \in \mathbb{L}^3 : \sin(x) \sin(y) = \sin(z)\}.$$

This surface is known in the literature as the spacelike Scherk surface [10], see Figure 4, right. Since the sign of the function $X + Y - Z$ has changed, the surface is now spacelike and can be extended to lightlike points when $f'(x)g'(y) = 0$. This occurs when $\cos(x) = 0$ or $\cos(y) = 0$. Thus the set of lightlike points is $\{x = \pi/2 + 2\pi\mathbb{Z}, y = z + 2\pi\mathbb{Z}\} \cup \{x = \pi - z + 2\pi\mathbb{Z}, y = \pi/2 + 2\pi\mathbb{Z}\}$. This maximal surface is a triply periodic surface and belongs to a family of triply periodic maximal surfaces which contains as particular examples, the H-type Schwarz surface and the D-type maximal surface [13]. Moreover, it is also the Scherk saddle tower under the motion of the Wick rotation of \mathbb{E}^3 [2].

4.1.3 Example 3: helicoids

A right helicoid in \mathbb{L}^3 is the surface obtained by moving a straight-line contained in a plane by means of a uniparametric group of skew motions of \mathbb{L}^3 around an axis contained in the plane ([4]). The right helicoids have zero mean curvature. In this subsection we will obtain the right helicoids whose axis is spacelike or timelike. These surfaces have regions of the three types of causal character. Firstly, consider the following constants

$$\begin{aligned} a_1 &= 0, & b_1 &= 1, & c_1 &= 0 \\ a_2 &= 0, & b_2 &= 0, & c_2 &= 1 \\ a_3 &= 2, & b_3 &= 1, & c_3 &= 1. \end{aligned}$$

Then the functions are

$$X(u) = e^{2u}, \quad Y(v) = e^{-2v}, \quad Z(w) = 2 + e^{2w} + e^{-2w}.$$

The integration yields

$$f(x) = -\log(-x), \quad g(y) = \log(y), \quad h(z) = \log(\tan(z)),$$

and the surface writes as $x = -y \tan(z)$. It is the right helicoid whose axis is the z -axis (timelike axis) and also known as the elliptic helicoid ([15]) or the helicoid of the first kind ([16]).

Other choice of the constants is the following:

$$\begin{aligned} a_1 &= 0, & b_1 &= 1, & c_1 &= 0 \\ a_2 &= -2, & b_2 &= 1, & c_2 &= 1 \\ a_3 &= 0, & b_3 &= 0, & c_3 &= 1. \end{aligned}$$

Then

$$X(u) = e^{2u}, \quad Y(v) = -2 + e^{2v} + e^{-2v}, \quad Z(w) = e^{-2w}.$$

The integration of the three differential equations yields

$$f(x) = -\log(-x), \quad g(y) = \log(-\tanh(y)), \quad h(z) = \log(z),$$

and the implicit equation of the surface is $x = z \tanh(y)$. This surface is the right helicoid whose axis is the y -axis (spacelike axis) and known as the hyperbolic helicoid ([15]) or the helicoid of the second kind ([16]). Each helicoid is transformed into the other one by means of the Wick rotation [2].

4.1.4 Example 4

Consider the following constants:

$$\begin{aligned} a_1 &= -2, & b_1 &= 1, & c_1 &= 1 \\ a_2 &= -2, & b_2 &= 1, & c_2 &= 1 \\ a_3 &= -1, & b_3 &= 1/2, & c_3 &= \frac{1}{2}. \end{aligned}$$

Then

$$X(u) = -2 + e^{2u} + e^{-2u}, \quad Y(v) = -2 + e^{2v} + e^{-2v}, \quad Z(w) = -1 + \frac{1}{2}e^{2w} + \frac{1}{2}e^{-2w}.$$

The integration of the first differential equation is

$$x = \int \frac{df}{\sqrt{-2 + e^{2f} + e^{-2f}}} = \int^{e^f} \frac{d\tau}{\sqrt{(1 - \tau^2)^2}},$$

and similarly for the second one. For the third equation, we have

$$z = \int \frac{dg}{\sqrt{-1 + e^{2f}/2 + e^{-2f}/2}} = \int^{e^h} \frac{\sqrt{2}d\tau}{\sqrt{(1 - \tau^2)^2}}.$$

The integration by quadratures depends on the case of $\tau^2 < 1$ or $\tau^2 > 1$. If $\tau^2 < 1$ in the first equation, that is, if $f < 0$, then $f(x) = \log \tanh(x)$ and if $\tau^2 > 1$ ($f > 0$), then $f(x) = -\log \tanh(x)$. Since the functions f , g and h can not have the same sign, without loss of generality, we suppose that $f, g < 0$ and $h > 0$. In such a case, $g(y) = \log(\tanh(y))$ and $h(z) = -\log(\tanh(z/\sqrt{2}))$. Thus the surface is

$$\{(x, y, z) \in \mathbb{L}^3 : \tanh(x) \tanh(y) = \tanh(z/\sqrt{2})\}.$$

This surface appeared in [22] and it is of mixed type: see Figure 5, left.

4.2 Case where two differential equations are solved by quadratures

We will give three examples where two differential equations in (7) can be solved by quadratures and the solution of the other differential equation is given in terms of elliptic integrals.

4.2.1 Example 1

Consider the following constants that satisfy (8):

$$\begin{aligned} a_1 &= 1, & b_1 &= 0, & c_1 &= 1 \\ a_2 &= -1, & b_2 &= 1, & c_2 &= 0 \\ a_3 &= 2m - 1, & b_3 &= m, & c_3 &= m - 1, \end{aligned}$$

where m is a real parameter. The functions X , Y and Z are

$$X(u) = 1 + e^{-2u}, \quad Y(v) = -1 + e^{2v}, \quad Z(w) = 2m - 1 + me^{2w} + (m - 1)e^{-2w}.$$

Notice that $Z(w)$ is a positive function, so the value of the parameter m is not arbitrary: for example, if $m = 0$, then $Z(w) = -1 - e^{-2w} < 0$, which is not possible. The solutions of the first two differential equations are

$$f(x) = \log(\sinh(x)), \quad g(y) = -\log(\sin(y)).$$

For the third equation,

$$\int^{e^h} \frac{d\tau}{\sqrt{m\tau^4 + (2m-1)\tau^2 + m-1}} = \pm z,$$

or

$$\int^{e^h} \frac{d\tau}{\sqrt{(\tau^2+1)(m\tau^2-1+m)}} = \pm z. \quad (14)$$

This integral is elliptic and can not be solved by quadratures in general. We show two examples by taking particular values of m .

1. Case $m = 1/2$. Then $Z(w) = \sinh(2w)$ and (14) is

$$\xi := \sqrt{2} \int^{e^h} \frac{1}{\sqrt{\tau^4-1}} d\tau = z. \quad (15)$$

Let V the inverse of the function ξ . Then

$$h(z) = \log\left(V\left(\frac{z}{\sqrt{2}}\right)\right)$$

and the surface is

$$\{(x, y, z) \in \mathbb{L}^3 : \sinh(x)V\left(\frac{z}{\sqrt{2}}\right) = \sin(y)\}.$$

On the other hand,

$$\begin{aligned} X(u) + Y(v) - Z(w) &= e^{-2u} + e^{2v} - \frac{e^{2w} - e^{-2w}}{2} \\ &= \frac{1}{2} e^{-2u-2v} \frac{\cosh(x)^4 - \cos(y)^4}{\sin(y)^4} > 0 \end{aligned}$$

and the surface is timelike at every point, except when $\cosh(x) = \cos^2(y) = 1$. However, there are not points with $\cosh(x) = 1$ and $\sin(y) = 0$ because of the definitions of the functions f and g . Thus the surface is timelike and can not be extended to lightlike points.

2. Case $m = 1$. Now the integral (14) can be solved by quadratures, exactly, $h(z) = -\log(\sinh(z))$. The surface is

$$\{(x, y, z) \in \mathbb{L}^3 : \sinh(x) = \sin(y) \sinh(z)\}.$$

This surface is singly periodic along the y -direction by the periodicity of $\sin(y)$, see Figure 5, middle. We study the causal character of the surface. Since $w = -u - v$, we have

$$X(u) + Y(v) - Z(w) = e^{-2u} + e^{2v} - e^{2w} - 1 = (1 + e^{-2u-2v})(e^{2v} - 1) > 0$$

because $e^{2v} - 1 = g'(y)^2 > 0$. Then the surface is timelike. At the set of points where $g'(y) = 0$, that is $\cos(y) = 0$, the surface can be extended into a region of lightlike points. By the equation of the surface, $\sinh(x) = \sinh(z)$ or $\sinh(x) = -\sinh(z)$. This set is formed by the straight-lines $\{x = z, y = \pi/2 + 2\pi\mathbb{Z}\} \cup \{x = -z, y = -\pi/2 + 2\pi\mathbb{Z}\}$. Let us observe that this surface is a singly periodic surface along the y -axis and it is the Wick rotation of the timelike Scherk surface of second kind that appeared in (13): see details in [2].

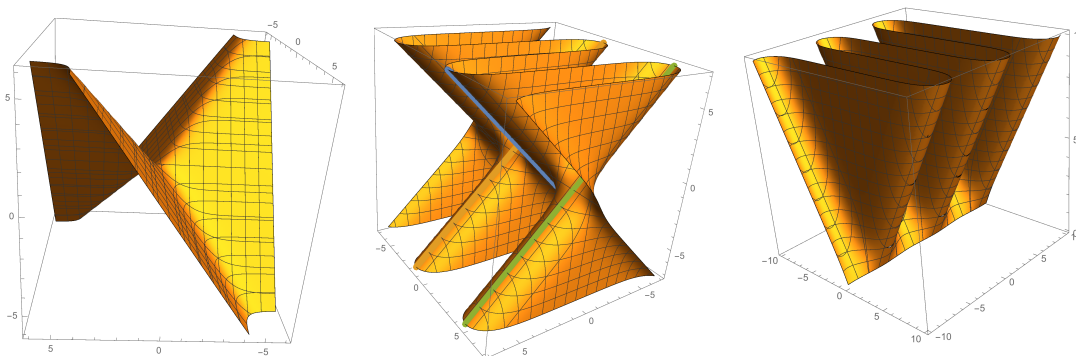


Figure 5: Left: the surface $\tanh(x) \tanh(y) = \tanh(z/\sqrt{2})$. Middle: the surface $\sinh(x) = \sin(y) \sinh(z)$. Right: the surface $M(y/\sqrt{2}) = \sinh(x)/\sinh(z)$

4.2.2 Example 2

Consider the following constants:

$$\begin{aligned} a_1 &= 1, & b_1 &= 1, & c_1 &= 0 \\ a_2 &= 1 - 2m, & b_2 &= m - 1, & c_2 &= m \\ a_3 &= 1, & b_3 &= 0, & c_3 &= 1, \end{aligned}$$

where m is a real parameter. The functions X , Y and Z are

$$X(u) = 1 + e^{2u}, \quad Y(v) = 1 - 2m + (m - 1)e^{2v} + me^{-2v}, \quad Z(w) = 1 + e^{-2w}.$$

The integrations of the first and third differential equation yield

$$f(x) = -\log(\sinh(x)), \quad h(z) = \log(\sinh(z)).$$

For the function g , we have the elliptic integral

$$\pm y = \int^{e^g} \frac{d\tau}{\sqrt{(m-1)\tau^4 + (1-2m)\tau^2 + m}} = \int^{e^g} \frac{d\tau}{\sqrt{(\tau^2-1)((m-1)\tau^2-m)}}. \quad (16)$$

1. Case $m = 0$. Then $g(y) = -\log(\cosh(y))$ and the implicit equation of the surface is $\sinh(x) \cosh(y) = \sinh(z)$ which appeared in the subsection 4.1.
2. Case $m = 1$. Then the surface is $\sinh(x) = \sin(y) \sinh(z)$, which appeared again in the above subsection.
3. Case $m = 1/2$. The integral (16) is now

$$\psi := \int^{e^g} \frac{d\tau}{\sqrt{1-\tau^4}} = \frac{y}{\sqrt{2}}. \quad (17)$$

Let $M(\psi)$ be the inverse of the function ψ . Then

$$g(y) = \log\left(M\left(\frac{y}{\sqrt{2}}\right)\right),$$

and the surface is

$$\{(x, y, z) \in \mathbb{L}^3 : \sinh(z)M\left(\frac{y}{\sqrt{2}}\right) = \sinh(x)\}.$$

This surface is singly periodic along the y -axis, see Figure 5, right. For the causal character, we determine the sign of the function

$$\begin{aligned} X(u) + Y(v) - Z(w) &= \frac{1}{2}e^{-2u-2v} (e^{4u}(e^{2w} + 1)^2 - (e^{2u} + 1)^2) \\ &= \frac{1}{2}e^{-2u-2v} \frac{\cosh(z)^4 - \cosh(x)^4}{\sinh(x)^4}, \end{aligned}$$

hence the surface is of mixed type. The lightlike points is the set of points such that $\cosh(x) = \cosh(z)$, that is, $\{(x = z, M(y/2) = 1\} \cup \{(x = -z, M(y/2) = 1\}$ up to periodicity. By the function M , this set of points is formed by straight-lines.

4.2.3 Example 3

We show two new examples of separable ZMC surfaces with similar choices of the constants. The first example corresponds with the choice of constants

$$\begin{aligned} a_1 &= -1, & b_1 &= 0, & c_1 &= 1 \\ a_2 &= 1, & b_2 &= -1, & c_2 &= 0 \\ a_3 &= m, & b_3 &= \frac{1-m}{2}, & c_3 &= \frac{-1-m}{2}, \end{aligned}$$

where m is a real parameter. The functions X , Y and Z are

$$X(u) = -1 + e^{-2u}, \quad Y(v) = 1 - e^{2v}, \quad Z(w) = m + \frac{1-m}{2}e^{2w} - \frac{1+m}{2}e^{-2w}.$$

The integration of the first two differential equations yields

$$f(x) = \log(\sin(x)), \quad g(y) = -\log(\cosh(y)).$$

For the function h , we have

$$\int^{e^h} \frac{d\tau}{\sqrt{(1-m)\tau^4 + 2m\tau^2 - 1 - m}} = \int^{e^h} \frac{d\tau}{\sqrt{(\tau^2 - 1)((m-1)\tau^2 - m - 1)}} = \frac{z}{\sqrt{2}}. \quad (18)$$

In general the integral (18) is elliptic. We show two particular examples.

1. Case $m = 1$. Then the integration of (18) yields $h(z) = \log(\cosh(z))$ and the surface is

$$\{(x, y, z) \in \mathbb{L}^3 : \sin(x) \cosh(z) = \cosh(y)\}.$$

Notice that the values of x such that $\sin(x) = 0$ are not in the domain of the surface, so the surface is not singly periodic along the x -axis: see Figure 6, left. In this case,

$$X(u) + Y(v) - Z(w) = (1 - e^{-2u})(e^{-2w} - 1) > 0$$

and the surface is timelike. The surface extends to lightlike points in the set $f'(x)h'(z) = 0$, that is, $\cos(x) = 0$ or $\sinh(z) = 0$. Without loss of generality, we assume that the domain of f is $(0, \pi)$, then $\cos(x) = 0$ yields $x = \pi/2$, so the set of lightlike points is formed by two straight-lines, namely, $\{x = \pi/2, z = \pm y\}$. If $\sinh(z) = 0$, then $z = 0$ and from the equation of the surface, $\sin(x) = \cosh(y)$, hence $x = \pi/2$, $y = 0$, showing that the point $(\pi/2, 0, 0)$ is a singularity of the surface. Furthermore the surface reflects along this singularity ([1]).

2. Case $m = 0$. Then the integral (18) is

$$\int^{e^h} \frac{d\tau}{\sqrt{\tau^4 - 1}} = \frac{z}{\sqrt{2}}.$$

This integral has appeared in (15). Then the surface is

$$\{(x, y, z) \in \mathbb{L}^3 : \sin(x)V\left(\frac{z}{\sqrt{2}}\right) = \cosh(y)\}.$$

This surface also appeared in [22] and it is a doubly periodic surface by the periodicity of the functions $V(z/\sqrt{2})$ and $\sin(x)$.

The second example corresponds with the constants

$$\begin{aligned} a_1 &= -1, & b_1 &= 0, & c_1 &= 1 \\ a_2 &= m, & b_2 &= -\frac{m+1}{2}, & c_2 &= \frac{1-m}{2} \\ a_3 &= -1, & b_3 &= 1, & c_3 &= 0, \end{aligned}$$

where $m \in \mathbb{R}$. The integration of f and h is

$$f(x) = \log(\sin(x)), \quad h(z) = -\log(\sin(z)).$$

For the function g , we have

$$\int^{e^g} \frac{d\tau}{\sqrt{-(m+1)\tau^4 + 2m\tau^2 + 1 - m}} = \int^{e^g} \frac{d\tau}{\sqrt{(\tau^2 - 1)(-(m+1)\tau^2 + 1 - m)}} = \frac{y}{\sqrt{2}}.$$

We discuss two cases:

1. Case $m = 1$. Then $g(y) = -\log(\cosh(y))$ and the surface is

$$\{(x, y, z) \in \mathbb{L}^3 : \sin(x) = \cosh(y) \sin(z)\}.$$

Let us observe that the surface is doubly-periodic in the xz -plane, see Figure 6, middle. Moreover,

$$\begin{aligned} X(u) + Y(v) - Z(w) &= 1 + e^{-2u} - e^{2v} - e^{-2u-2v} = (e^{2v} - e^{-2u})(e^{-2v} - 1) \\ &= \frac{\sin(x)^2 - \cosh(y)^2}{\sin(x)^2 \cosh(y)^2} (\cosh(y)^2 - 1) \leq 0. \end{aligned}$$

Thus the surface is spacelike except in those points where $y = 0$. By the equation of the surface, $\sin(x) = \sin(z)$ and this set of points are the straight-lines with equation $\{y = 0, z = x + 2\pi\mathbb{Z}\} \cup \{y = 0, z = \pi - x + 2\pi\mathbb{Z}\}$.

2. Case $m = 0$. Now the elliptic integral coincides with (17) and the equation of the surface is

$$\{(x, y, z) \in \mathbb{L}^3 : \sin(x)M(y/\sqrt{2}) = \sin(z)\}.$$

This surface is doubly periodic because the functions $M(y/\sqrt{2})$ and $\sin(z)$ are periodic. In Figure 6, right we show a piece of the surface. In fact, and up periodic translations, the surface contains the pair of orthogonal spacelike lines $\{x = 0, z = 0\} \cup \{z = 0, y = y_o\}$, with $M(y_o/\sqrt{2}) = 0$, $y_o \neq 0$, where the surface can be repeated by symmetry reflections thanks to the reflection principle for maximal surfaces [3]. Also the surface contains singularities at the points $(\pi/2, 0, \pi/2)$ and $(-\pi/2, 0, \pi/2)$ and their translations by periodicity.

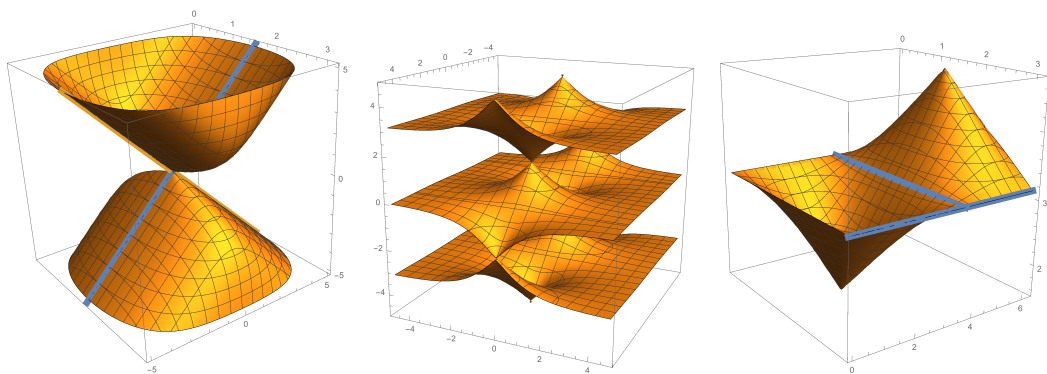


Figure 6: Left: the surface $\sin(x)\cosh(z) = \cosh(y)$. Middle: the surface $\sin(x) = \cosh(y)\sin(z)$. Right. the surface $\sin(x)M(y/\sqrt{2}) = \sin(z)$ where we show the pair of straight-lines through which the surface can be repeated by symmetries

4.3 Case that none of the three integrals can be solved by quadratures

Consider the constants in (8) given by

$$\begin{aligned} a_1 &= -2m^2 + 1, & b_1 &= m^4, & c_1 &= 1, \\ a_2 &= -1, & b_2 &= 1, & c_2 &= m^2, \\ a_3 &= 1, & b_3 &= 1, & c_3 &= m^2, \end{aligned}$$

where $m \in \mathbb{R}$. Then

$$X(u) = -2m^2 + 1 + m^4 e^{2u} + e^{-2u}, \quad Y(v) = -1 + e^{2v} + m^2 e^{-2v}, \quad Z(w) = 1 + e^{2w} + m^2 e^{-2w}.$$

The case $m = 0$ appeared in the subsection 4.2.1.

Consider the interesting case $m = 1$. Now

$$X(u) = -1 + e^{2u} + e^{-2u}, \quad Y(v) = -1 + e^{2v} + e^{-2v}, \quad Z(w) = 1 + e^{2w} + e^{-2w}.$$

The solution of the first differential equation is

$$\int \frac{df}{\sqrt{-1 + e^{2f} + e^{-2f}}} = x.$$

Then

$$x = \int \frac{df}{\sqrt{-1 + e^{2f} + e^{-2f}}} = \int^{e^f} \frac{d\tau}{\sqrt{1 - \tau^2 + \tau^4}}.$$

Denote $t = \mathcal{F}(\xi)$ the inverse of the elliptic integral $\int (1 - \tau^2 + \tau^4)^{-1/2} d\tau$. Similarly, let $\mathcal{G}(\xi)$ the inverse of the elliptic integral $\int (1 + \tau^2 + \tau^4)^{-1/2} d\tau$. Then the surface takes the form $\log \mathcal{F}(x) + \log \mathcal{F}(y) + \log \mathcal{G}(z) = 0$, or equivalently

$$\mathcal{F}(x)\mathcal{F}(y)\mathcal{G}(z) = 1.$$

5 Separable ZMC surfaces: case $K < 0$.

In this section we obtain examples of separable ZMC surfaces when $K < 0$ in Theorem 2.3. Using Proposition 2.4 and without loss of generality, we suppose that $K = -1$ and $k = 1$. We know that the constants a_i, b_i and c_i satisfy (10) and the functions f, g and h of 1) are the solutions of the differential equations (11). In this section, we will only show some examples and pictures, see Figure 7.

5.1 Example 1

Consider the constants

$$\begin{aligned} a_1 &= 1, & b_1 &= 0, & c_1 &= 1 \\ a_2 &= 1, & b_2 &= 0, & c_2 &= 1 \\ a_3 &= \frac{1}{2}, & b_3 &= \frac{1}{2}, & c_3 &= 0. \end{aligned}$$

Then

$$X(u) = 1 + \sin(u), \quad Y(v) = 1 + \sin(v), \quad Z(w) = \frac{1}{2} + \frac{1}{2} \cos(w).$$

The integrations of the three equations yield

$$f(x) = 2 \arctan \left(\frac{1 + \sinh(x/\sqrt{2})}{1 - \sinh(x/\sqrt{2})} \right),$$

$$g(y) = 2 \arctan \left(\frac{1 + \sinh(y/\sqrt{2})}{1 - \sinh(y/\sqrt{2})} \right),$$

$$h(z) = 2 \arctan (\sinh(z/2)).$$

After some manipulations, the implicit equation of the surface is

$$\frac{\sinh(x/\sqrt{2}) \sinh(y/\sqrt{2}) - 1}{\sinh(x/\sqrt{2}) + \sinh(y/\sqrt{2})} = -\sinh(z/2).$$

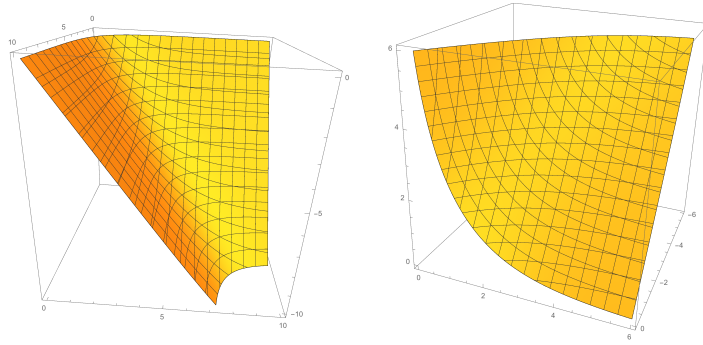


Figure 7: The surfaces of Example 1 of Section 5

5.2 Example 2

Consider the following constants:

$$\begin{aligned} a_1 &= \frac{1}{2}, & b_1 &= \frac{1}{2}, & c_1 &= 0 \\ a_2 &= 1, & b_2 &= 0, & c_2 &= 1 \\ a_3 &= \frac{1}{3}, & b_3 &= 0, & c_3 &= \frac{1}{3}. \end{aligned}$$

With a similar argument as the previous example, the integrations of the three differential equations by quadratures leads to

$$f(x) = 2 \arctan (\sinh(x/2)),$$

$$g(y) = 2 \arctan \left(\frac{1 + \sinh(y/\sqrt{2})}{1 - \sinh(y/\sqrt{2})} \right),$$

$$h(z) = 2 \arctan \left(\frac{1 + \sinh(z/\sqrt{6})}{1 - \sinh(z/\sqrt{6})} \right).$$

Then the implicit equation of the surface is

$$\frac{\sinh(y/\sqrt{2}) \sinh(z/\sqrt{6}) - 1}{\sinh(y/\sqrt{2}) + \sinh(z/\sqrt{6})} = -\sinh(x/2).$$

References

- [1] S. Akamine, H. Fujino, Duality of boundary value problems for minimal and maximal surfaces, arXiv: 1909.00975 [math.DG] (2019).
- [2] S. Akamine, R. K. Singh, Wick rotations of solutions to the minimal surface equation, the zero mean curvature equation and the Born-Infeld equation. Proc. Indian Acad. Sci. Math. Sci. 129 (2019), Art. 35, 18 pp.
- [3] L. J. Alías, R. M. B. Chaves, P. Mira, Björling problem for maximal surfaces in Lorentz-Minkowski space. Math. Proc. Cambridge Philos. Soc. 134 (2003), 289–316.
- [4] Chr. C. Beneki, G. Kaimakamis, B. J. Papantoniou, Helicoidal surfaces in three-dimensional Minkowski space. J. Math. Anal. Appl. 275 (2002), 586–614.
- [5] R. M. B. Chaves, M. P. Dussan, M. Magid, Björling problem for timelike surfaces in the Lorentz-Minkowski space, J. Math. Anal. Appl. 337 (2011), 481–494.
- [6] J. Dorfmeister, J. Inoguchi, M. Toda, Weierstrass-type representation of timelike surfaces with constant mean curvature. In: Contemp. Math., vol. 308, pp. 77–99. Am. Math. Soc., Providence (2002).
- [7] M. Fréchet, Détermination des surfaces minima du type $a(x) + b(y) = c(z)$. Rend. Circ. Mat. Palermo 5 (1956), 238–259.
- [8] M. Fréchet, Détermination des surfaces minima du type $a(x) + b(y) = c(z)$. Rend. Circ. Mat. Palermo 6 (1957), 5–32.
- [9] S. Fujimori, Y. Kawakami, M. Kokubu, W. Rossman, M. Umehara, K. Yamada, Entire zero-mean curvature graphs of mixed type in Lorentz-Minkowski 3-space. Q. J. Math. 67 (2016), 801–837.
- [10] S. Fujimori, Y. W. Kim, S-E. Koh, W. Rossman, H. Shin, H. Takahashi, M. Umehara. K. Yamada, S-D. Yang, Zero mean curvature surfaces in L^3 containing a light-like line. C.R. Acad. Sci. Paris. Ser. I 350 (2012), 975–978.
- [11] S. Fujimori, Y. W. Kim, S-E. Koh, W. Rossman, H. Shin, M. Umehara. K. Yamada. S-D. Yang, Zero mean curvature surfaces in Lorentz-Minkowski 3-space and 2-dimensional fluid mechanics. Math. J. Okayama Univ. 57 (2015) 173–200.

- [12] S. Fujimori, Y. W. Kim, S.-E. Koh, W. Rossman, H. Shin, M. Umehara, K. Yamada, S.-D. Yang, Zero mean curvature surfaces in Lorentz-Minkowski 3-space which change type across a light-like line. *Osaka J. Math.* 52 (2015), 285–297.
- [13] S. Fujimori, W. Rossman, M. Umehara, K. Yamada, S.-D. Yang, Embedded triply periodic zero mean curvature surfaces of mixed type in Lorentz-Minkowski 3-space. *Michigan Math. J.* 63 (2014), 189–207.
- [14] J. Inoguchi and M. Toda, Timelike minimal surfaces via loop groups. *Acta Appl. Math.* 83 (2004), 313–355.
- [15] Y. W. Kim, S.-E. Koh, H. Shin, S.-D. Yang, Spacelike maximal surfaces, timelike minimal surfaces, and Björling representation formulae. *J. Korean Math. Soc.* 48 (2011), 1083–1100.
- [16] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space L^3 . *Tokyo J. Math.* 6 (1983), 297–309.
- [17] H. Liu, Translation surfaces with constant mean curvature in 3-dimensional spaces. *J. Geom.* 64 (1999), 141–149.
- [18] R. López, Timelike surfaces with constant mean curvature in Lorentz three-space. *Tohoku Math. J. (2)* 52 (2000), 515–532.
- [19] R. López, Differential geometry of curves and surfaces in Lorentz-Minkowski space. *Int. Electron. J. Geom.* 7 (2014), 44–107.
- [20] J. C. C. Nitsche, *Lectures on Minimal Surfaces*. Cambridge University Press, Cambridge, 1989.
- [21] H. F. Scherk, Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen. *J. Reine Angew. Math.* 13 (1835), 185–208.
- [22] V. Sergienko, V. G. Tkachev, Doubly periodic maximal surfaces with singularities. *Proceedings on analysis and geometry (Russian) (Novosibirsk Akademgorodok, 1999)*, pp. 571–584, Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000.
- [23] I. Van de Woestijne, *Minimal surfaces of the 3-dimensional Minkowski space. Geometry and topology of submanifolds, II (Avignon, 1988)*, World Sci. Publ., Teaneck, NJ, 1990, pp. 344–369.
- [24] J. Weingarten, Ueber die durch eine Gleichung der Form $\mathfrak{X} + \mathfrak{Y} + \mathfrak{Z} = 0$ darstellbaren Minimalflächen. *Nachr. Königl. Ges. d. Wissensch. Univ. Göttingen* (1887), 272–275.
- [25] T. Weinstein, *An Introduction to Lorentz Surfaces*. New York: De Gruyter expositions in Mathematics, 1996.