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Interacting with Indeterminate Quantities through Arithmetic Word Problems: Tasks to Promote Algebraic Thinking at Elementary School

Cristina Ayala-Altamirano [1], Eder Pinto [2,*], Marta Molina [3] and María C. Cañadas [4]

1 Department of Education, University of Almeria, Carretera San Urbano, La Cañada, 04120 Almería, Spain; altamira@ual.es
2 Faculty of Education, Universidad del Desarrollo, Av. Plaza 680, Santiago 7610658, Chile
3 Departamento de Didáctica de las Matemáticas y de las Ciencias Experimentales, Escuela de Educación y Turismo de Ávila, Universidad de Salamanca, Madrigal de las Altas Torres, 3, 05003 Ávila, Spain; martamolina@usal.es
4 Department of Mathematics Didactics, Faculty of Education Sciences, University of Granada, 18071 Granada, Spain; mconsu@ugr.es
* Correspondence: epinto@udd.cl

Abstract: In this study, we analyze how 9–10-year-old pupils work with equations, a central aspect of algebraic thinking in early grades and a cornerstone for more formal learning of algebra. Specifically, we seek: (a) to describe the main characteristics of the tasks that support algebraic thinking through a translation process from arithmetic word problems to algebraic language and vice versa, and (b) to identify how pupils refer to indeterminate quantities in these contexts and what meaning they give to them. The analysis focuses on the semantic congruence of the expressions proposed by them and on the dialogue they held during the translation process. We analyzed the oral discussion in the pools and the written responses to the problem that pupils posed. The results show that arithmetic word problems allow the indeterminate to become an object of thought for pupils, who represent it in multiple ways and refer to it when proposing equations that represent the structure of each problem. Another finding highlights that reflection on the interpretation of the equations supports the identification of two meanings associated with indeterminate quantities, namely, unknown and variable.

Keywords: arithmetic word problems; early algebra; indeterminate quantities; elementary education; problem posing; translation

MSC: 97H20

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1. Introduction

Currently, different curricular guidelines consider algebraic thinking as a transversal topic from the beginning of schooling [1–4]. These proposals recommend promoting in pupils the identification of general mathematical relationships and structures based on situations appropriate for their age, which are part of their daily experiences and natural intuitions. Despite the presence of algebraic thinking in the different curricula, there are still challenges on how to introduce this type of thinking in elementary education classrooms. Specifically, with this study, we seek to contribute with ways of approaching algebraic thinking from contents that have traditionally been seen as exclusively arithmetic. In this paper, we focus on identifying how elementary school pupils represent and refer to indeterminate quantities when they establish relationships between the resolution of arithmetic word problems (AWPs, hereafter) and their translation using algebraic language, and vice versa. This translation process will allow us to delve into the paths that pupils have to give meaning to the indeterminate.

Indeterminate quantities constitute a central aspect of algebraic thinking, which can be associated with different meanings depending on the context. They can be interpreted as a generalized number, an unknown quantity, a variable quantity, or a parameter. Getting elementary pupils to generate rich meanings from the indeterminate depends on the learning opportunities and diversity of learning experiences they are faced with [5]. In this study, we address how students interact with the indeterminate quantities through the process of problem posing and translating from algebraic to natural language, and vice versa.

A growing body of research has shown that pupils between the ages of 6 and 12 refer to and represent indeterminate quantities using multiple representations [6,7]. Regarding the use of algebraic language, and in particular the use of letters, elementary pupils accept its use and correctly represent variable quantities by generalizing relationships between quantities that covary [5,7–9]. It has been evidenced that teaching and learning environments that encourage children to utilize non-numerical symbols to represent indeterminate quantities, such as variable notation, can help them construct an understanding of variables [10]. However, in the transition to using this type of notation correctly, some errors and difficulties evidenced in higher grades are replicated [11–13]. For example, it is observed that pupils spontaneously assign values to literal symbols according to their position in the alphabet, or although they recognize that they can represent different values, they attribute specific values chosen at random [13]. On the other hand, the literature recommends giving concrete meaning to mathematical language through familiar contexts and recognizing familiarity as an important factor in the problem-solving process.

Considering the previous aspects, our proposal is to carry out tasks whose objective is to support the algebraic thinking of elementary school pupils and the correct representation of indeterminate quantities considering familiar contexts for them.

1.1. Algebraic Thinking

The conceptual framework that directs our study considers that algebraic thinking refers to indeterminate quantities, and these quantities are treated analytically, that is, even if the quantities are unknown, they are added, subtracted, multiplied, or divided [7]. More specifically, algebraic thinking can be understood as the four core practices of generalizing, representing, justifying, and reasoning with mathematical structure and relationships [14]. Specifically:

- *Generalize* can be interpreted, in a broad way, as the action of recognizing that some attributes of a mathematical situation can change, while others remain invariable [15]. Attending to generalization allows pupils to move away from the particularities associated with arithmetic calculation and, in turn, allows them to identify the structure and mathematical relationships involved in each situation [15].
- *Representing* general mathematical ideas can involve different semiotic means, some conventional and others not, such as gestures, the rhythm of speaking, and natural language [7]. The expression of generalization will have different degrees of sophistication depending on the means of representation used.
- *Justifying* generalizations requires pupils to determine and explain the truth of a conjecture or claim [7]. This supports a better understanding of the problem, its structure, and its relationships. Promoting justification in the classroom helps to: refine generalization [16]; for pupils to express themselves more clearly; and for teachers to make well-informed pedagogical decisions since they can understand what pupils think based on what they say or the use they make of signs [17].
- *Reasoning* involves treating generalizations as objects in themselves [18], which implies that pupils use the generalizations that they have found, represented, and justified in other types of mathematical situations.

The four core practices are embodied in the different approaches to early algebra: (a) *generalized arithmetic*, which involves generalizing, representing, justifying, and reasoning with arithmetic relationships, including fundamental properties of operations as well as other types of relationships on classes of numbers [16]; (b) *equivalence*, *expressions*, *equations*, and *inequalities*, which include developing a relational understanding of the equal sign and generalizing, representing, and reasoning with expressions, equations, and inequalities, including in their symbolic forms [14]; and (c) *functional thinking*, which includes generalizing relationships between co-varying quantities and representing, justifying, and reasoning with these generalizations through natural language, variable notation, drawings, tables, and graphs [18]. In this study, we focus on the equations, i.e., the second content area described previously.

1.2. Linear Equations in Elementary School

In this study, we focus on linear equations because these are deemed suitable for the age and their work is suggested in elementary school curricula [1–4]. We understand a linear equation is a mathematical sentence that involves an equal sign to show that two algebraic or numeric expressions are equivalent [14], with one or more unknowns. Radford [19] pointed out that using an equation to reason about the representation and communication of relationships between quantities is a cornerstone of algebra. In addition, many problems are better solved if the equation is first written to represent the problem statement. He highlighted that developing an understanding of how equations can be written to represent problems at elementary school can build a foundation for later learning of formal algebra.

1.3. Translation between Verbal Language and Algebraic Language

AWPs contain information that is presented exclusively through natural language, and to solve them and find the value of some unknown quantity it is necessary to apply one or more elementary mathematical operations. Within the framework of school algebra, AWPs encourage pupils to make sense of the indeterminate, which does not appear without support as it is a quantity of something that is not known [20]. In this context, problems can be represented using different representations. Their interpretation and solution can lead to several translations carried out by the solver.

Regarding the translation of natural language to algebraic language, most authors focus mainly on grades after elementary education. These studies have shown that to be successful in translating between natural language and algebraic language, elementary pupils must identify the variables involved, the relationships between them, and the syntax of the symbolic representation. Regarding the difficulties that they face, one of them is understanding the meaning of algebraic language since this type of representation is considered opaque to them. They tend to have difficulty visualizing the advantages of algebraic language [11], so elementary school pupils prefer to use arithmetic-type strategies and representations [21].

The reverse translation, from algebraic language to natural language, can be considered in the context of problem posing. This activity requires pupils to formulate mathematical problems from given situations that may include mathematical expressions or diagrams, or by reformulating existing problems [22]. Stoyanova [23] proposes three categories of problem-posing tasks: (a) free situations, (b) semi-structured situations, and (c) structured situations. In this study, we focus on the second category. These tasks are characterized by being based on an open situation, particularly an equation. From this point, we invite elementary school pupils to create a problem by applying mathematical procedures, concepts, and relationships from their own experiences. This type of task is associated with high cognitive demand; whoever invents the problems must reflect on the structure of the situation rather than on the procedures for solving the problem [24]. Previous studies have shown that posing problems from given mathematical equations or calculations requires understanding the meaning of the operations [25]. In addition, in this type of task, pupils usually follow an algorithmic process focused on the operational and non-semantic structure of the problems [26].

Problem posing, from the teaching perspective, is a means of evaluating the conceptions of pupils regarding a particular topic [22,27], and allows their abilities to use their mathematical knowledge to be recognized.

2. Research Objectives

In this study, we analyze the work completed by a group of 9–10-year-old pupils. Specifically, we seek to: (a) describe the main characteristics of the tasks that support algebraic thinking through a translation process from AWP to algebraic language and vice versa, and (b) identify how pupils refer to indeterminate quantities in these contexts and what meaning they give to them.

3. Materials and Methods

This study is part of a broader Classroom Teaching Experiment (CTE) [28], and is part of the research-design paradigm [29]. The general objective of the CTE was to guide pupils ages 9–10 in the expression and justification of general mathematical ideas by working on three approaches to school algebra.

3.1. Context

This study was conducted in the context of a summer school for pupils who had just finished 4th grade of the elementary school. The summer school is an activity organized every year by the Faculty of Education of the Universidad del Desarrollo (Santiago, Chile), with the aim of providing effective and fair opportunities to children through the development of thinking and innovation.

Both the design and the implementation of this activity contemplated a collaborative work that involved the participation of researchers and teachers from the area of mathematics education in Chile and Spain. Due to the health situation caused by the COVID-19 pandemic, the summer school was developed virtually. Specifically, the pupils accessed the activities from their homes, through different devices connected to the internet: mobile phones, tablets, or computers. Additionally, each pupil used physical materials that were provided by the university: a board with markers of different colors, a folder to record their findings on worksheets, and manipulatives. The students were encouraged to keep the cameras on, so that the teacher–researchers could observe their work. In turn, the students' families were asked to allow them to work on their own without the help of others.

3.2. Participants

This qualitative study involved 21 pupils who were between 9 and 10 years old and who had completed 4th grade online, given the health context. The pupils belonged to two schools that are part of the same Educational Foundation that serves children and young people from low-income sectors. Specifically, nine pupils from one school and 12 from another. Pupils' anonymity in this paper was ensured by assigning each a code: S_i, where $i = 1 \ldots 21$.

The pupils were selected with the help of their regular math teachers under the following three criteria: (a) gender parity (10 girls and 11 boys); (b) willingness to work during the summer; and (c) different paces of learning.

Regarding the previous knowledge of the pupils, it is important to point out that, although the Chilean curriculum contemplates a thematic topic of algebra and patterns [1], the learning objectives related to this topic corresponding to 4th grade were not fully addressed due to the country's health situation [30]. In the mathematics classes, the learning objectives that refer to the topic of numbers and operations were mainly addressed. Regarding the topic of patterns and algebra, we worked with numerical patterns that involve an operation, which was registered in tables. In the previous grade (3rd grade), pupils solved one-step equations involving addition and subtraction and a geometric symbol representing an unknown number. They employed strategies such as trial and error or the inverse operation. The mathematical representations that they used were numerical–symbolic ones and the emphasis of the classes was focused on promoting fluency in the calculation.

3.3. Design

The summer school was organized in 10 sessions, including a pre-test and a post-test, and following the approaches to algebraic thinking [14]: (a) generalized arithmetic; (b) equivalence, expressions, and equations; and (c) functional thinking (see Figure 1).

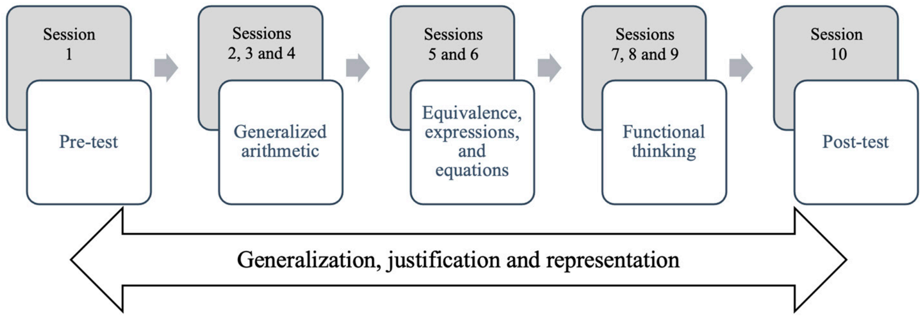

Figure 1. Organization of sessions.

In the first and last sessions, the pupils' responses to different algebraic tasks were assessed. Sessions 2–9 followed a similar structure, organized into three parts:

- *small groups* (4–5 pupils), in which the aim was for the pupils to dialogue and collaborate with each other in the search for regularities, conjectures, and solutions to the problems presented;
- *whole group*, where each group presented their findings and two teachers led the discussion so that the pupils synthesized their ideas; and
- *medium-sized groups* (10–11 pupils), in which the objective was to transfer what had been discussed to another similar situation or to delve into a finding from the previous parts on the problems presented. Each part supported the installation of spaces for cooperation, confrontation, and discussion of ideas.

3.4. Instruction Sequence: Sessions 2–5

In this study, we focus on session 6. However, we also describe, in general terms, what happened previously, without considering the initial assessment. In session 2, pupils expressed their general ideas through natural language by arguing what happens when odd and even numbers are added. Then, in sessions 3 and 4, they discussed the meaning of the equal sign. Here, for the first time, the letter is introduced as a representation for generalizing arithmetic properties (for example, the commutativity of addition). In this first encounter with letters, the pupils concluded that it could represent any number (generalized number). In the fifth and sixth sessions, we focused on the expression and resolution of equations. During these sessions we suggested to the pupils to (a) represent and solve equations using different strategies, (b) use the letter as a representation of an unknown, and (c) provide evidence to validate given explanations. We introduced the ideas of equality, equation, mathematical histories (an idea we use to refer to the translation from natural language to algebraic language), and letters as unknowns, among others. In session 5, they proposed an adaptation of the cards and envelopes problem described in [20] (Figure 2). The pupils were asked to express and solve the equation involved in the problem using manipulatives and pictorial representations. However, it became clear that it was necessary to deepen the approach to equations. Although they were able to represent them with manipulative material, drawings, and even letters, doubts remained about the meaning of the indeterminate.

Silvia and Carlos have some cards. Carlos has 3 cards and Silvia has 2 cards. Their mother prepares three envelopes with the same number of cards in each. She gives 1 envelope to Carlos and 2 to Silvia. Now both children have the same number of cards. How many letters are in an envelope?

Figure 2. Problem session 5.

3.5. Session 6

In the first part of the session, two problems were presented to the pupils:

- I bought a box of colored pencils. At home I had pencils, now I have 20 in total. How many pencils are in the box? and
- I have a basket of apples. Inside the basket, there are 20 green apples and other red ones. How many apples are in the basket?

These AWPs were represented with natural language and the pupils had to translate and represent with algebraic language. The first AWP involves an unknown, has a unique solution, and involves the structure $y + 15 = 20$. The other AWP implicates two unknowns whose values could not be determined due to the lack of data in the statement, and it involves the structure $y = 20 + b$.

The pupils had to represent the AWPs on their boards. Firstly, the pupils were asked to "tell the story" (i.e., to represent verbal sentences) with mathematical symbols. We were interested in pupils representing indeterminate quantities however they wanted. One by one they explained how they did it and discussed whether what their classmates did was the same as what each one represented. Then the possibility of representing them with letters was mentioned by the teacher and discussed within the group, as had happened in previous sessions. In Figure 3 the AWP and representations that the teacher raised for each situation are presented. The teacher presented two different representations for each problem: the drawing of the box and colored pencils or the basket of apples, and a symbolic–algebraic expression, with the use of letters. It was believed that familiar representations for the pupils, such as drawings, would allow them to understand the statement of the problem and the meaning of each element of the algebraic expression. After reaching an agreement on the representation of the problems in a group discussion, pupils solved each equation and discussed what the value of each unknown was.

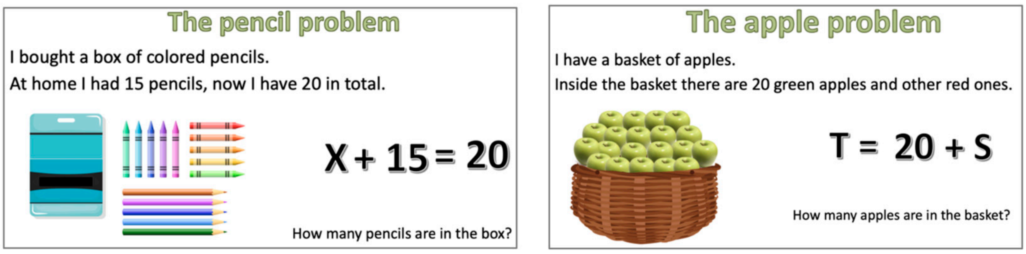

Figure 3. Problem session 6.

In the second part of the session, the teacher generated a space for discussion with the pupils about the discoveries obtained in the previous part. The discussion sought: (a) to investigate the use of the letter when expressing the equations, and (b) to collect evidence to determine whether the letter represents an unknown quantity or a variable quantity. The teacher asked questions such as: Does the expression $X + 15 = 20$ represent the pencil problem? Does the expression $T = 20 + S$ represent the apple problem? Do you know how many pencils are in the box? Can we know how many apples are in the basket? How do you know? What evidence do you have? Could you explain it in a different way to your partner?

At the end of this part, pupils were asked to answer a test via a virtual test to assess whether they could transfer what was discussed to other situations with similar characteristics. They were asked to choose the alternative that allowed them to tell each of the stories shown in Figure 4. In the second situation, there were two possible correct answers: $40 + r = j$ and $j + 40 = r$, the objective was to discuss the reasons for why both were correct.

1. There were 32 people in a room. Some people were sitting, and 12 others were standing.	2. In a cinema, there are many people. 40 watch Naruto and other people watch Batman.
$32 = m + 12$	$r + j = 40$
$32 + m = 12$	$40 + r = j$
$32 + 12 = m$	$j + 40 = r$

Figure 4. Test part 2.

Finally, in the third part, we wanted the pupils to extend what they had learned to other cases and they were asked to carry out the reverse process, that is, to create mathematical stories from the equation $25 + u = 45$. Each pupil wrote the problem that each one invented on the board and then discussed the relevance of each situation.

3.6. Analysis

We analyzed pupils' responses to (a) describe the characteristics of the tasks that support algebraic thinking, and (b) identify how they refer to indeterminate quantities. The authors of this study classified the responses concerning the first two parts of the session, and all the written and oral responses of the pupils to the AWPs presented and the invented problems in the third part. Discrepancies were then discussed until an agreement was reached.

To analyze the pupils' translations, we based our method on the ideas of Duval's [31] proposal. He pointed out that two representations are congruent when the following three conditions are met: (a) semantic correspondence between the significant units that constitute them; (b) semantic univocity, i.e., each initial significant unit of output corresponds to one and only one significant elementary unit of the input record; and (c) the order within the organization of the significant output units is maintained in the arrival representation. When one of these criteria is no longer met, the representations are not congruent with each other. However, this author added that two expressions can be referentially equivalent without being semantically congruent. Semantic congruence allows us to see the degree of transparency of the relationship between two representations.

If only correspondence and semantic uniqueness were observed, the translations were classified as equivalent.

Semantically consistent translations and their equivalents were considered correct translations. Regarding the incorrect translations, we identified four types of errors: (a) incomplete translation because some data of the statement are missing; (b) translation related to the problem but that does not refer to indeterminate quantities and includes the answer to the problem (e.g., $15 + 5 = 20$); (c) apparently correct translation, but the correspondence between the semantic units poorly relates the terms of the equation to the statement; and (d) translation with invented information or information unrelated to the problem.

When translating a given statement into natural language, the pupils' responses were classified into: (a) pictorial translation if drawings were used; (b) symbolic–arithmetic translation if using numbers and mathematical symbols; and (c) algebraic language translation if it used numbers, mathematical symbols, and some symbols to refer to indeterminate quantities (letters or "?").

To analyze the invented problems, we translated the problems posed into algebraic language, maintaining the structure of the given equation, in addition to a left–right congruence whenever possible. Then we compared the translation obtained with the proposed equation. We also identified if the pupils proposed a new context or adapted the problems proposed in the previous parts.

To identify how the pupils referred to indeterminate quantities, we analyzed the oral discussions in the pools and the written records of the problems they invented. We looked for linguistic expressions that used indefinite adjectives (e.g., little, a lot, one, another, too many, same, some, none, any), key phrases that conveyed that the pupils recognized an indeterminate amount in the analyzed contexts. For example: "it is a number that we do not know", and "it is the number you want", among others. In previous research, such as [5,15], elementary pupils referred to indeterminate quantities with the keywords "many" or "infinite" or the phrase "the number you want".

On the meaning given to indeterminate quantities, we analyzed the answers given to the questions Can we know the answer to the problem? Can we know what quantity the letter represents? In these cases, if the pupils answered that the problem only had one solution, we interpreted that the meaning associated with the indeterminate quantity was unknown. While if they answered that the answer to the problem could not be known because there was not enough information and that the answer depended on some data of the problem, we interpreted that the meaning given was that of a variable quantity.

4. Results

In this section we first present the answers of the pupils to the tasks in which they had to translate verbal statements into algebraic language. Then, the answers given in the inverse process, translation from algebraic language to natural language, will be discussed.

4.1. From Natural Language to Algebraic Language: The Pencil Problem

The first problem introduced involved the structure $y + 15 = 20$. In this instance, nineteen elementary school pupils participated in its resolution. Initially, the focus was on the translation of the statement into algebraic language. After this, as the last step, the unknown value was discussed. Table 1 shows a characterization of the translations proposed by the pupils. Thirteen pupils proposed a translation using algebraic language, one made a symbolic–arithmetic translation, four proposed a pictorial translation, and one pupil did not respond.

Table 1. Equations proposed by the pupils in the pencil problem (structure $y + 15 = 20$).

Translation	Pupil	Equation Structure	Equation Proposed by the Pupils	Correct Translation	Semantic Consistency		
					SC	**SU**	**OA**
Algebraic language	S_{15} and S_{16}	$y + 15 = 20$	$? + 15 = 20$	Yes	Yes	Yes	Yes
	S_5		$a + 15 = 20$	Yes	Yes	Yes	Yes
	S_3		$b + 15 = 20$	Yes	Yes	Yes	Yes
	S_{14}	$15 + y = 20$	$15 + ? = 20$	Yes	Yes	Yes	No
	S_1 and S_{13}		$15 + a = 20$	Yes	Yes	Yes	No
	S_7, S_8, S_{17} and S_{18}		$15 + x = 20$	Yes	Yes	Yes	No
	S_9		$15 + a = 20$ [1]	No	No	Yes	No
	S_6	$15 + y = 35$	$15 + a = 35$	No	No	Yes	No
Symbolic–arithmetic	S_{12}		$15 + 5 = 20$	No	No	Yes	No
Pictorial	S_2, S_4, S_{10} and S_{11}			No	No	Yes	No
Not responding	S_{19}						

SC = semantic correspondence; SU = semantic univocity; OA = order of apprehension. Note. [1] The equation is incorrect because the pupil says that he bought 15 pencils and "*a*" represents the ones he had at home.

Initially, pupils were asked to represent the problem statement freely on the board. Eleven of the pupils correctly translated the natural language expression into algebraic language. Four of them formulated an equation consistent with the pencil problem, while another seven proposed referentially equivalent equations, but these were not consistent since they did not meet the order of apprehension criterion. In this case, the structure represented by them was $15 + y = 20$, whose verbal statement would have to mention first the number of pencils they have (15) and then the number of pencils in the box (y).

In the design of the task, it was considered important that the pupils reflect on the equivalence between the expressions, that is why, in the discussion, the answers that involved the equations in the forms $y + 15 = 20$ and $15 + y = 20$ were contrasted. They were asked if they represented the same thing, even though the order of the addends was different. In this regard, S_8 pointed out: "if we change their order, the result does not change due to the commutative property". S_{15} also mentioned the commutative property to justify the equivalence between the expressions.

Another important aspect considered in the design of the task was to guide the discussion so that the elementary pupils explicitly made the correspondence between the terms of the equation and the verbal statement. Focusing on indeterminate quantities and their representation, 10 pupils used letters (a, b, and x) and another four used the question mark "?" sign. In the discussion, the pupils referred to the letters or to the sign "?" as the number of pencils in the box they do not know. For example, S_3, who proposed the equation $b + 15 = 20$, said: "I thought. He says that he bought a box, but you don't know how many pencils that box has, and that represents "b". 15 at home. I used "b", but any letter can be any number." This last sentence highlights that he accepted that the representations of his classmates were also correct, even if different letters were used.

The reference to indeterminate quantities was not only made in the algebraic language translations, in this problem it was also evidenced in a pictorial translation (see Figure 5). S_2 represented the indeterminate quantity on her drawing using the symbol "?". She explained that she represented the 15 pencils and a box of mystery pencils. However, her translation is incomplete since she does not refer to or represent the total number of pencils (20). The pictorial representations of other pupils do not refer to indeterminate quantities either.

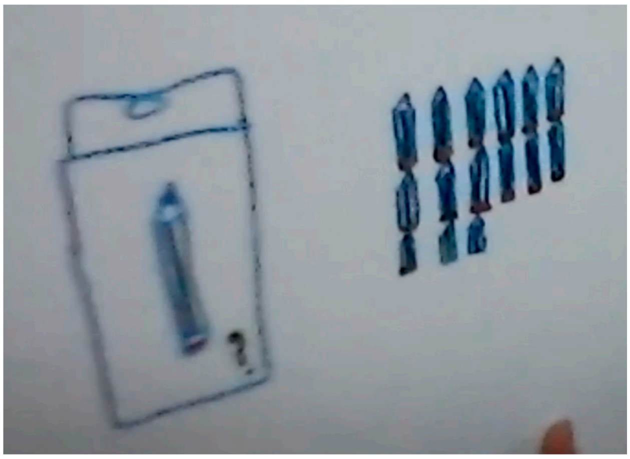

Figure 5. Translation of the verbal statement using drawings of S_2.

The last aspect considered in the design was the discussion about the value that the indeterminate quantity could take. In this problem, the pupils agreed that there could only be five pencils in the box. They complemented their argument by replacing the letter or the question mark "?" with that number and solving the sum to verify that the result was 20. In this way, they reaffirmed their idea that the indeterminate quantity can have only one value.

4.2. From Natural Language to Algebraic Language: The Apple Problem

The second problem discussed was that of apples, whose structure is $y = 20 + b$. As in the previous problem, the translation was carried out first and then the possible value of each of the indeterminate quantities was discussed. Table 2 shows the characterization of the pupils' translations. On this occasion, the number of correct translations with algebraic language was less than in the previous problem (9 out of 19); however, the number of pupils who made a translation of this type was the same. One pupil made a symbolic–arithmetic translation, three pupils a pictorial translation, and two pupils did not answer.

Table 2. Equations proposed by the pupils in the apple problem (structure $y = 20 + b$).

Translation	Pupil	Equation Structure	Equation Proposed by the Pupils	Correct Translation	Semantic Consistency		
					SC	US	OA
Algebraic language	S_5	$b + 20 = y$	$a + 20 = ?$	Yes	Yes	Yes	No
	S_{13}		$c + 20 = ?$	Yes	Yes	Yes	No
	S_{15}	$20 + b = y$	$20 + a = ?$	Yes	Yes	Yes	No
	S_2, S_6 and S_{12}		$20 + x = ?$	Yes	Yes	Yes	No
	S_3		$20 + c = ?$	Yes	Yes	Yes	No
	S_8		$20 + b = x$	Yes	Yes	Yes	No
	S_9		$20 + x = a$	Yes	Yes	Yes	No
	S_7 and S_{17}	$20 + b = b$	$20 + x = x$	No	Yes	No	No
	S_4	$20 + b = 40$	$20 + n = 40$	No	No	Yes	No
	S_{18}		$20 + x =$	No	No	Yes	No
Symbolic–arithmetic	S_{16}		$10 + 10 = 20$	No	No	No	No
Pictorial	S_1, and S_{11}			No	No	Yes	No
	S_{10}			No	No	No	No
Not responding	S_{14} and S_{19}						

SC = semantic correspondence; SU = semantic Univocity; OA = order of apprehension.

In the verbal statement of this problem, it first mentioned that there were apples in a basket (y), and then it described the type of apples inside (20 green and others that were red (b)). Pupils were expected to represent the total number of apples to the left of the equal sign and thus maintain the order of the statement. However, all of them represented it to their right, therefore, no expression was semantically consistent with the statement of the problem since the criterion of the same order of apprehension was not met.

Nine of the 19 elementary pupils correctly translated the verbal statement into algebraic language. Two of them proposed an equation of the form $b + 20 = y$, while seven pupils wrote an equation of the form $20 + b = y$. As in the previous problem, they recognized that these equations are equivalent and alluded to the commutative property of addition (explicitly or mentioned that the order of the addends does not change the result).

Including two indeterminate quantities in the problem statement also made it possible to discuss the univocity of one representation. S_7 and S_{17} represented the statement in the following way: $20 + x = x$. This equation is incorrect since it does not meet the univocity criterion, "x" represents the number of red apples and also the total number of apples in the basket. When discussing this translation, S_6 said that it could not be, and referring to indeterminate quantities as if they were known, he pointed out: "when you add the letter it will give you a result and it cannot be because they will give different numbers". S_9 added "it must be another symbol otherwise the result would be the same as the red [apples]. Instead of 'x' in the addition, you change it to 'a'". This discussion ended with S_8 suggesting that there should be five red apples to show that the result would be a quantity other than five.

S_{18} performed an incomplete translation ($20 + x$=). He pointed out that he could not complete the expression because he did not know yet how many red apples there were, and he could not write any number. These pupils recognized and represented the unknown number of apples; however, he was looking for a specific number to complete the expression next to the equal sign.

On how to represent indeterminate quantities, as in the previous problem, the pupils recognized that they could use different letters. On this occasion, since the problem involved two indeterminate quantities, the pupils used letters and the "?" sign in the same equation. It was common for them to use the "?" sign to refer to the total number of apples in the basket and a letter for the number of red apples.

In the pictorial translations, they also used letters and the "?" sign to represent the indeterminate quantities. An interesting aspect to highlight about these translations is that pupils such as S_2, S_3, and S_5 (whose answers are shown in Figure 6) used them in the first place to understand the problem and then, after being motivated by the teacher, proposed a translation using algebraic language. Pupil S_{11}, who also used drawings in the previous problem, represented the indeterminate quantity with the "?" symbol to refer to the unidentified number of apples. However, his representation was incomplete as he did not represent the total number of apples in the basket.

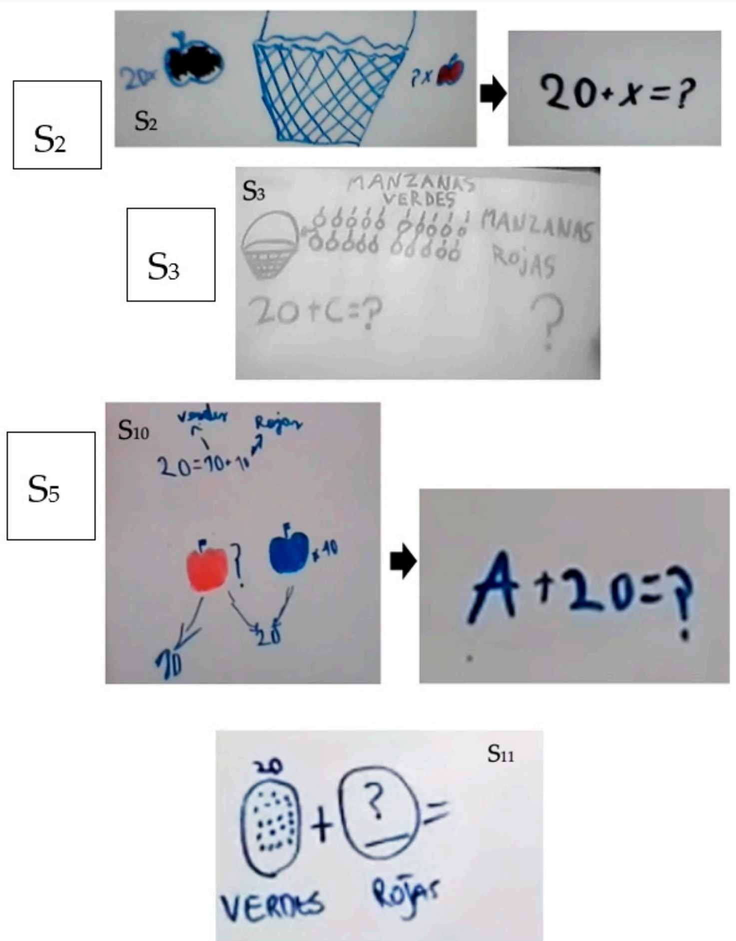

Figure 6. Translation of the verbal statement of pupils S_2, S_3, S_5 and S_{11}.

Finally, after representing the problem, the pupils agreed that they could not know how many apples there were in the basket, therefore they could not know the value of the indeterminate quantities. S_{18} said that there was not enough information. S_8 said "we are missing a clue. For instance, how much it would give or how much the result was." To which S_7 and S_6 responded by pointing out that one could also say the number of red apples, for example, 20 or 19. S_5, without referring to a certain quantity, argued: "it has to be greater than 21 for T. Because it says 'others', so it has to be more than 1 (referring to the number of red apples). In this case, the letter can have different values."

4.3. Transfer of What They Have Learned to Other Contexts

After solving the pencil/apple problems, described above, the pupils concluded that the indeterminate quantities can be represented in multiple ways, either with letters or with the "?" sign. Moreover, the same letter cannot represent two different data of the statement and those two expressions can be equivalent if it is observed that the commutative property of addition has been applied. Finally, regarding the solution of the equation, they pointed out that there are indeterminate quantities that can take a single value, while the value of others depends on the clues or information that are made explicit in the statement. To assess whether the pupils could transfer this knowledge to other contexts, they were asked about two new situations by means of a test. Pupils could choose more than one option and we made sure that everyone responded. Figure 7 shows the results of the test.

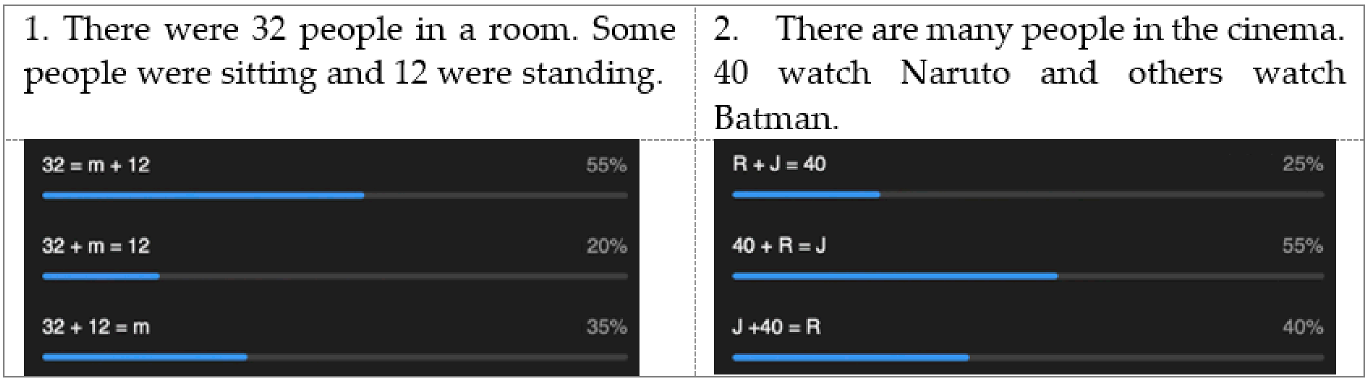

Figure 7. Test on translation from natural language to algebraic language.

In the first situation, the majority chose the correct alternative, that is, 32 = m + 12. To justify that it was the correct option, the teacher (I) held the following dialogue with a pupil, in which he helped him to establish the semantic correspondence between the verbal statement and the terms of the equation.

I: S_3, what does 32 mean in the situation?
S_3: The people who were sitting. No, the people in the room.
I: And what does 12 mean?
S_3: The people who were sitting. Pretty sure.
I: And what does m mean?
S_3: The people who were standing.
I: And why with a letter?
S_3: Because we don't know how many people are standing . . . although we could know if we add 20 to 12, and there would be 32 people left in the room. The m can stand for 20.
S_{10}: I saw the voting and it's the other way around. Number 12 represents the people who were standing and the letter "m" the people who were sitting.
I: What about you? do you agree that this is the correct one?
S_{10}: Yes. In the test it was stated that there were a total of 32 people in the room. So, the final amount can't be twelve people, because those were the ones standing.
I: So, you discard the second option, because it says that the total is twelve and that cannot be.
S_{10}: Yes. And the result cannot be "m" either.
I: Why?
S_{10}: Because that represents the people who were sitting.

In the second situation, the discussion focused on the most voted option (*40* + r = j). S_{17} justified as follows.

S_{17}: Yes, it is correct. Because it says that 40 people watch Naruto and there are 40 there, plus "R" that would be the people who watch Batman, so we don't know actually how many people watch Batman. And just like J, that would be the result that we don't know about.

I: Why?

S_{17}: Because we don't have enough information.

I: What would we need for this information to be sufficient?

S_{17}: Either knowing the result of J, or how many people watch Batman.

The results of this test confirm that pupils accept representing indeterminate quantities with letters, referring to them as unknown quantities in their arguments. In addition, they manage to distinguish in which situations they can determine with certainty the value of the indeterminate quantity and in which they need more information.

4.4. From Algebraic Language to Natural Language: Inventing Problems

In the third part of the session, when pupils were in medium-sized groups, they were asked to create a story for the equation $25 + u = 45$. Twenty-one pupils participated in this part of the session and Table 3 presents the organization of the situations that they invented.

Table 3. Examples of invented situation by pupils.

Pupils	Structure	Examples of Invented Situations
S_3; S_7; S_8; S_9; S_5; S_{10}; S_{21}; S_{11}; S_{14}; S_{15}; S_2	$25 + u = 45$	There was a boy at the dentist. There were 25 people there. The boy did not know how many people had been treated, but he knew that there were 45 people scheduled. (S_{21}) We have a dinosaur that has 25 candies. Then another dinosaur arrives and gives him an indefinite number of sweets. After that, he counts them and finds out that he has 45 candies. Finally, another one arrives and asks him for 25 of them and the dinosaur ends up with 20 in total. (S_{14}) There were two children and each of them had a box of chocolates. The two had different amounts of it. One boy had 25 but the other did not know how many chocolates he had. They had 45 in total. (S_{10})
S_{20}; S_{12}	$45 = 25 + u$	There are 45 people sleeping, 25 of them in bed and some in the couch. How many people are sleeping in the couch? (S_{12})
S_{18}	$45 = u + 25$	I have a box with 45-pound cakes. Some vanilla and 25 chocolate. How many vanilla pound cakes do I have?
S_4	$u + t = 45$	There was a boy who went to a gumball machine and wanted to get the red ones. He didn't know how many of them were inside of it, but he did know that the total amount of gumballs in the machine was 45.
S_{17}	$25 + u$	In a supermarket there were 25 teddy bears and also some stuffed cats, but the amount of the latter is unknown. There are also 45 people who want to know how many stuffed cats are in the supermarket.
S_6	$25 + 20 = u$	Sofia has 25 books and Marta has 20. How many books are there in total?
S_{16}	$(45 + 25){:}20 = u$	Carla bought 45 candies and 25 lollipops and divided them among 20 friends, but she doesn't know how many candies she should give to each friend.
S_{19}	$(45 + 25){:}a = u$	Valentina bought 45 drinks and 25 sweets and wants to distribute them among her friends, but she does not know how many to distribute. How many drinks and sweets does she want to give to each of them?
S_1	$45 = 20 + 25$	There were 45 people in a mansion, 20 went on a trip and 25 stayed.
S_{13}	There is no record of their response.	

As we present in Table 3, eleven of the 21 pupils invented correct situations related to the given equation ($25 + u = 45$), and three pupils proposed equivalent translations ($45 = 25 + u$ or $45 = u + 25$). In addition, within this group of 14 pupils, six adapted the problem raised at the beginning of the session, and eight proposed a completely new context. They all referred to indeterminate quantities and used different expressions, such as: "he doesn't know how many he has, an indefinite number of, others, some, one and many". All these pupils made correct translations between the equation and the invented situation, which are semantically consistent. The problems invented by the rest of the pupils (S_4; S_{17}; S_6; S_1; S_{19}; S_{16}) did not have a correct translation. One pupil, S_{13}, did not present an invented situation.

In the discussions, as an example, we highlighted the arguments of S_{12} when evaluating whether the translations were correct or not. The pupil sought to establish the correspondence between the terms of the given equation and the data of the created problem; in addition, he referred to indeterminate quantities. The following dialogue shows his opinion on S_{18} problem.

I: S_{12}, does the story of S_{18} tell the same thing that the equation?
S_{12}: Yes.
I: Why?
S_{12}: It has the number we don't know, which is represented by the letter "u", and it has the numbers 25 and 45 which are the ones we know.

Later, the teacher asked them if the following story was correct: "I have a box with 45 pencils. 25 are red and 20 are blue". S_{12} said that he was not sure, but he thought so, because several had said that "that number that we did not know" was 20.

5. Conclusions and Discussion

In this study, we set two objectives: (a) to describe the main characteristics of the tasks that support algebraic thinking through a translation process from AWP to algebraic symbolism and vice versa; and (b) to identify how pupils refer to indeterminate quantities in these contexts and what meaning they attach to them. The results associated with each objective are discussed below.

5.1. Characteristics of the Tasks

In the experience described above, two types of tasks are presented: (a) those referring to the process of translating statements from natural language (AWP) to algebraic language; and (b) its reverse process, focused on the invention of problems from an equation expressed in algebraic notation. In general, asking the pupils to first translate the problem before looking for the answer allowed them to explain and visualize the structure of the problems and helped them identify what the unknown information was. Another study has shown that elementary pupils of similar ages solve arithmetic–algebraic problems; however, they fail to express them with algebraic symbols [32]. In this study, we show an experience in which pupils think in terms of indeterminate quantities, they also express them, and, even later, solve them through intuitive strategies such as trial and error or substitution. We suggest that this was supported given that our main objective was not to look for solutions but to "tell the story" of the statement and discuss the semantic consistency of the translations. We complement this with the findings of another similar research [19] in which it is highlighted that problems are better solved if the equation is first written to represent the problem statement. Additionally, the results obtained in the research carried out with high school pupils are ratified, in which it is pointed out that in order to be successful in translating the problem, the pupil is required to identify the quantities involved in the problem (known or not) and the relationships that the statement establishes between them [19].

Another important feature to highlight in the proposed tasks is the number of indeterminate quantities involved in the equations. In current curricula [1,2], it is proposed that pupils in these grades solve equations with one indeterminate quantity. In this study, it is evident that proposing a situation with two unknowns in the second place helped to develop a new meaning of the indeterminate quantities, as a variable number that depends on the information that we know about the problem. This meaning is associated with the study of functions, but, in this case, it emerges naturally within the framework of typical tasks of the algebra approach in terms of equivalence, expressions, and equations, highlighting the potential of the context of problem solving with the purpose of making possible the distinction between different meanings of the indeterminate quantities.

Problem posing was used as a tool to evaluate the possibility of transferring what was learned to other contexts. In this task, the pupils were able to reflect on the structure of the situation rather than on the procedures for solving the problem. We assume that this is a consequence of what was promoted in the previous tasks. According to the findings of other investigations, pupils usually follow an algorithmic process focused on the operative structure of the equation [22,26]; in our case, although some pupils also did this, they were in the minority. The vast majority of elementary pupils in this study were able to set out contextualized and self-interested problems in response to task demands. In particular, they presented relatively few non-mathematical problems and statements.

Finally, another contribution of this study is to show how tasks that seem to be very simple for pupils at this level, favor discussion and allow pupils to refer to indeterminate quantities in their arguments. Verbal arithmetic problems, which pupils usually solve by focusing only on arithmetic calculations, in translation tasks managed to have an algebraic character, while the indeterminate became an object of thinking of the pupils at the same time. Family contexts and less linguistic complexity helped them to visualize the relationship between each of the significant units expressed in both natural language and symbolic language (semantic correspondence between semantic units). Encouraging them to think about whether the expressions "told the story" mentioned in the problem helped them discuss the relevance of using certain symbols in their equations. For example, they agreed that the same letter could not be used to represent different things in the problem (semantic unambiguity between representations) or that the order in which the addends are written does not tell the story in the same order (same order of apprehension), but mathematically it is correct because of the commutativity of addition. Contrasting an equation of the $a + b = c$ form with an equation of the same form, but with only one known value (a) (in which a and c are known natural numbers), allowed us to discuss the possibility that the indeterminate quantities had more than one possible value. In previous research, these types of equations were presented to pupils [33]; however, there was no comparison or discussion of the meaning of each one, nor a relationship with everyday situations. This study extends the findings of previous research in which the importance of everyday contexts to develop algebraic skills in secondary school is highlighted; in our case, we show evidence in elementary education. An open line of research is to investigate the interpretation of equations whose structures are more complex, which supports the development of relational thinking that encourages pupils to identify regularities between expressions. In future works, it will also be relevant to test the results obtained by replicating the study described here with pupils of similar ages, in order to corroborate the effect of the characteristics of the tasks considered here.

5.2. Ways of Referring to Indeterminate Quantities and Meanings Given

The results show that the translation and invention of AWP along with the analysis of its semantic congruence allowed the indeterminate to become an object of thought for the pupils. This was evidenced in the multiple representations used by elementary pupils when solving and creating problems.

On the one hand, those who were part of this study identified the indeterminate quantities involved in the different contexts and represented them, in most cases, with letters or with a question mark "?". This is similar to that shown in previous studies in which the tasks involved patterns or functions [5,7,8]. On the other hand, they referred to the indeterminate in everyday contexts using multiple linguistic expressions when inventing problems; for example, they used indeterminate adjectives such as "others, some, many" or expressions such as "I didn't know how many" or "an indefinite number of . . . " Previous research that analyzes pupils' algebraic thinking when solving tasks that involve patterns [7] or functions [15], highlights the importance of analyzing natural language to identify how pupils express their algebraic ideas without necessarily using algebraic notation. The expressions that we identified in this study could be used in other contexts to highlight the indeterminate quantity and pupils can recognize it in different contexts.

In addition, the characteristics of the tasks proposed in this study allowed the elementary pupils to associate the indeterminate quantities with two different meanings. As an unknown quantity with a fixed value when they had enough information in the statement about it, or as a variable number that depends on the values given to some of the unknown quantities. This distinction is of great relevance given the complexity of the polysemy of indeterminate quantities in algebraic contexts.

As we have mentioned in our conceptual framework, in previous research with elementary pupils who needed to refer to or give meaning to indeterminate quantities [5,8,9], pupils showed a tendency to assign specific values to them and make mistakes that are also evident in higher grades [11,12]. In this case, it is important to highlight that this did not occur; we assume that it was because the proposed situations were associated with contexts that were close to the pupils and involved numbers that allowed calculations to be carried out easily. The pupils gave meaning to the indeterminate considering that it was something that is not known in history, an element that they also highlighted in their study [18]. This helped them focus on the information given in the statement and avoid choosing numbers at random or resorting to the alphabet. It is worth mentioning that this type of error was observed in session 4, in which the letter was presented as a representation for indeterminate quantities for the first time and in a mathematical context when representing some properties of the addition. One limitation of our study is that it did not investigate how the previous difficulties were overcome until reaching the session that we discuss in this work. An open line of work is to analyze the relationship between the different areas of algebra and carry out a global analysis of all the sessions, which will allow the information presented here to be completed, describing how the different tasks proposed cause different or similar ways of referring to indeterminate quantities and give meaning to them.

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