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ORIGINAL PAPER

# Extended Proinov $\mathfrak{X}$-contractions in metric spaces and fuzzy metric spaces satisfying the property $\mathcal{NC}$ by avoiding the monotone condition

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**Abstract**

In recent years, Fixed Point Theory has achieved great importance within Nonlinear Analysis especially due to its interesting applications in real-world contexts. Its methodology is based on the comparison between the distances between two points and their respective images through a nonlinear operator. This comparison is made through contractive conditions involving auxiliary functions whose role is increasingly decisive, and which are acquiring a prominent role in Functional Analysis. Very recently, Proinov introduced new fixed point results that have very much attracted the researchers' attention especially due to the extraordinarily weak conditions on the auxiliary functions considered. However, one of them, the nondecreasing character of the main function, has been used for many years without the chance of being replaced by another alternative property. In this way, several researchers have recently raised this question as an open problem in this field of study. In order to face this open problem, in this work we introduce a novel class of auxiliary functions that serve to define contractions, both in metric spaces and in fuzzy metric spaces, which, in addition to

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generalizing to Proinov contractions, avoid the nondecreasing character of the main auxiliary function. Furthermore, we present these new results in the setting of fuzzy metric spaces that satisfy the condition $\mathcal{NC}$, which open new possibilities in the metric theory compared to classic non-Archimedean fuzzy metric spaces. Finally, we include some illustrative examples to show how to apply the novel theorems to cases that are not covered by other previous results.

**Keywords** Fixed point · Fuzzy metric space · Contractivity condition · Property $\mathcal{NC}$ · Proinov theorem

**Mathematics Subject Classification** 47H10 · 47J26

## 1 Introduction

The most applicable characteristic of fixed point theory is its ability in order to solve an extensive class of functional equations. In fact, in his doctoral thesis, Banach introduced the celebrated contractivity condition

$$\mathsf{d}\,(\mathsf{Tu}, \mathsf{Tv}) \leq \lambda\,\mathsf{d}(\mathsf{u}, \mathsf{v}) \quad \text{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X}$$

(here $\mathsf{d}$ is a metric on $\mathsf{X}$, $\lambda$ is a scalar in the real interval $[0, 1)$ and $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ is a self-mapping) in order to guarantee the existence and the uniqueness of a fixed point of an operator $\mathsf{T}$. Such fixed point was actually the solution of a functional equation. This result went unnoticed at first, as it seemed like a simple detail within a much larger construction. However, it was the germ of a thriving field of study that is nowadays in a very flourishing stage.

The practise has left in evidence that two main components must be mixed to get a good result in the field of fixed point theory: a very general contractivity condition and an underlying abstract metric space. In order to correctly place the results that we are going to present here within their natural context and to better understand the main aims of this study, let us discuss in both directions.

On the one hand, real metric spaces were a breakthrough in the field of Mathematics. However, these spaces soon became obsolete for modeling natural phenomena. Many generalizations of the notion of "*metric*" were presented in an increasingly abstract way (see, for instance, modular spaces [1], partial metric spaces [2], Branciari spaces [3], $G$-metric spaces [4–6], $b$-metric spaces [7], etc.) In this line of research, it is worth highlighting the difficulty of experimenting with phenomena that cannot be measured in a precise manner, or in which randomness plays a predominant role. In these contexts the notion of "*fuzzy metric*", which was originally introduced by Menger [8] and later developed through different interpretations, took on great importance (see Kaleva and Seikkala [9], Heilpern [10], Kramosil and Michálek [11], Park [12], etc.; more examples and interrelationships between them can be found in Roldán et al [13]). Due to their generality and the interesting properties that they display, in this manuscript we will use the notion of "*fuzzy metric space*" introduced by George and Veeramani [14]. In fact, these spaces are so general that we will need to reduce our study to a wide class of spaces within this family that are characterized by the *property* $\mathcal{NC}$. This quality was very recently presented by Roldán López de Hierro et al [15] due mainly to the following two reasons: its good properties for working within the field of fixed point theory and the fact that this family encompasses the class of all *non-Archimedean* fuzzy metric spaces. Metric spaces can be seen as a very particular class of fuzzy metric spaces, so we will treat them separately: we will present our main results both in real metric spaces and in fuzzy metric spaces, since the former make it possible to assume weaker constraints.

On the other hand, the contractivity condition is the second basic ingredient of any advance in this area of knowledge. Since the 1950s, many researchers have presented remarkable extensions of Banach's theorem by introducing more and more generalized contractivity conditions. Among others, it must be cited the following contributions: Boyd and Wong [16], Caristi [17], Chatterjea [18], Hardy and Rogers [19], Kannan [20, 21], Mukherjea [22], Ćirić [23], Geragthy [24], Meir and Keeler [25], Gnana-Bhaskar and Lakshmikantham [26], Berinde and Borcut [27], Karapınar [28, 29], Samet, Vetro and Vetro [30], Khojasteh et al [31, 32], Roldán López de Hierro et al. [33, 40], Jleli and Samet [41], etc. Some equivalences were shown by Jachymski [42].

Very recently, one remarkable result has been introduced by Proinov [43]. His theorem has greatly and nicely surprised researchers in this area because the involved functions that this author used satisfied very weak conditions that became strong enough to be able to finally deduce a fixed-point existence and uniqueness theorem. Concretely, his contractivity condition is:

$$\psi(\mathsf{d}(\mathsf{Tu},\mathsf{Tv})) \leq \phi(\mathsf{d}(\mathsf{u},\mathsf{v})) \quad \text{for all } \mathsf{u},\mathsf{v} \in \mathsf{X} \text{ with } \mathsf{d}(\mathsf{Tu},\mathsf{Tv}) > 0, \tag{1}$$

where $\psi, \phi : (0,\infty) \to \mathbb{R}$ are auxiliary functions that, in the framework of Proinov theorem, have acquired the principal role. It is true that assumption (1) is not novel in the area of fixed point theory. It was firstly introduced by Dutta and Choudhury [44] under the expression:

$$\psi'(\mathsf{d}(\mathsf{Tu},\mathsf{Tv})) \leq \psi'(\mathsf{d}(\mathsf{u},\mathsf{v})) - \phi'(\mathsf{d}(\mathsf{u},\mathsf{v})),$$

(which is equivalent to (1) by identifying $\psi = \psi'$ and $\phi = \psi' - \phi'$), and it was successfully applied, for instance, by Lakzian and Samet [45], and by Harjani et al [46]. However, to be precise, before that, inspired by Rakotch [47], Boyd and Wong [16] had introduced a version in which $\psi$ is the identity mapping on $\mathsf{X}$ (see also Rhoades [48], and Alber and Guerre-Delabriere [49]).

Nevertheless, one of the main Proinov's contributions has been to highlight how such general functions $\psi$ and $\phi$ manage to develop all the necessary reasonings to reach the final objective. Specifically, such functions must only fulfill the following constraints:

$(a_1)$ $\psi$ is nondecreasing;
$(a_2)$ $\phi(\mathsf{s}) < \psi(\mathsf{s})$ for any $\mathsf{s} > 0$;
$(a_3)$ $\limsup_{\mathsf{s}\to\mathsf{e}^+} \phi(\mathsf{s}) < \lim_{\mathsf{s}\to\mathsf{e}^+} \psi(\mathsf{s})$ for any $\mathsf{e} > 0$.

Notice that there are not any condition about the continuity (or semi-continuity) of these functions, and they can take values on the whole set $\mathbb{R}$ (they are not restricted to a positive interval).

From our point of view, one of the conditions imposed on the function $\psi$ is very restrictive: by axiom $(a_1)$, it must be nondecreasing. Although this requirement has been often assumed among the hypotheses of many results in fixed point theory because it plays a key role on the proofs, this monotone condition can also be used in order to show some examples in which the involved operator has a unique fixed point and it satisfies a contractivity condition such as (1), but the function $\psi$ is not necessarily nondecreasing. In other words, it is relatively easy to find particular examples in which the function $\psi$ is strictly decreasing on some interval, so the main Proinov theorem is not applicable.

Having in mind all the above mentioned considerations, in this manuscript we introduce novel families of contractions through the control of the main properties of the involved auxiliary functions. The main aim is double:

- on the one hand, we describe a family of contractions that generalize the Proinov contractions but avoiding the monotone condition on the auxiliary function $\psi$. In fact, the constraints assumed on the auxiliary functions permit us to introduce new families of intermediate contractive mappings between the class of Proinov contractions and the novel extended class of contractions;
- on the other hand, the weak properties assumed on the family of auxiliary functions permit us to develop our main results in the setting both of metric spaces and also in fuzzy metric spaces by only doing very subtle changes. Taking into account that such kinds of abstract metric spaces are distinct in nature, we will show how the developed ideas can be carried out under the same theoretical scheme.

This paper is organized as follows. In Sect. 2, we introduce the necessary background and preliminaries to rightly understand the developments and ideas we will explain. In the third and the fourth sections, we present the main existence and uniqueness fixed point theorems associated to the novel class of contractions, initially in the setting of metric spaces and later in the framework of fuzzy metric spaces satisfying the property $\mathcal{NC}$. In Sect. 5 we illustrate the great versatility of the main results that can be applied in a wide variety of contexts and that generalize and extend well known results in the field of fixed point theory. Finally, we end this study by discussing some prospect work and by posing some open problems.

## 2 Preliminaries

For an optimal understanding of this paper, we introduce here some basic concepts and notations that could be found in Agarwal et al [6], and also in Shahzad et al [34]. Throughout this manuscript, let $\mathbb{R}$ be the family of all real numbers, let $\mathbb{I}$ be the real compact interval $[0, 1]$, let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of all positive integers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Henceforth, $\mathsf{X}$ will denote a non-empty set.

### 2.1 Preliminaries on fixed point theory

From now on, let $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ be a map from $\mathsf{X}$ into itself. If a point $\mathsf{u} \in \mathsf{X}$ verifies $\mathsf{Tu} = \mathsf{u}$, then $\mathsf{u}$ is a *fixed point of* $\mathsf{T}$. We denote by $\mathrm{Fix}(\mathsf{T})$ the set of all fixed points of $\mathsf{T}$.

A sequence $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}_0}$ is called a *Picard sequence of* $\mathsf{T}$ *based on* $\mathsf{u}_0 \in \mathsf{X}$ if $\mathsf{u}_{\ell+1} = \mathsf{Tu}_\ell$ for all $\ell \in \mathbb{N}_0$. Notice that, in such a case, $\mathsf{u}_\ell = \mathsf{T}^\ell \mathsf{u}_0$ for each $\ell \in \mathbb{N}_0$, where $\{\mathsf{T}^\ell : \mathsf{X} \to \mathsf{X}\}_{\ell\in\mathbb{N}_0}$ are the *iterates of* $\mathsf{T}$ defined by $\mathsf{T}^0 =$ identity, $\mathsf{T}^1 = \mathsf{T}$ and $\mathsf{T}^{\ell+1} = \mathsf{T} \circ \mathsf{T}^\ell$ for all $\ell \geq 2$.

Following Roldán and Shahzad [50], a sequence $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ in $\mathsf{X}$ is *infinite* if $\mathsf{u}_\ell \neq \mathsf{u}_k$ for all $\ell \neq k$, and $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ is *almost periodic* if there exist $n_0, N \in \mathbb{N}$ such that

$$\mathsf{u}_{n_0+\ell+Np} = \mathsf{u}_{n_0+\ell} \quad \text{for all } p \in \mathbb{N} \text{ and all } \ell \in \{0, 1, 2, \ldots, N-1\}.$$

**Proposition 1** (Roldán and Shahzad [50], Proposition 2.3) *Every Picard sequence is either infinite or almost periodic.*

**Proposition 2** (Roldán et al [51], Proposition 2) *Let* $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ *be a Picard sequence in a metric space* $(X, \mathsf{d})$ *such that* $\{\mathsf{d}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1})\} \to 0$. *If there are* $\ell_1, \ell_2 \in \mathbb{N}$ *such that* $\ell_1 < \ell_2$ *and* $\mathsf{u}_{\ell_1} = \mathsf{u}_{\ell_2}$, *then there is* $\ell_0 \in \mathbb{N}$ *and* $\mathsf{v} \in \mathsf{X}$ *such that* $\mathsf{u}_\ell = \mathsf{v}$ *for all* $\ell \geq \ell_0$ *(that is,* $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ *is constant from a term onwards). In such a case,* $\mathsf{v}$ *is a fixed point of the self-mapping for which* $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ *is a Picard sequence.*

**Lemma 3** (Agarwal et al. [6], Proinov [43]) *Let* $\{u_\ell\}_{\ell\in\mathbb{N}}$ *be a sequence in a metric space* $(X, d)$ *such that* $\{d(u_\ell, u_{\ell+1})\} \to 0$ *as* $\ell \to \infty$. *If the sequence* $\{u_\ell\}_{\ell\in\mathbb{N}}$ *is not* d*-Cauchy, then there exist* $e > 0$ *and two partial subsequences* $\{u_{p(\ell)}\}_{\ell\in\mathbb{N}}$ *and* $\{u_{q(\ell)}\}_{\ell\in\mathbb{N}}$ *of* $\{u_\ell\}_{\ell\in\mathbb{N}}$ *such that*

$$p(\ell) < q(\ell) < p(\ell+1) \quad \text{and} \quad e < d(u_{p(\ell)+1}, u_{q(\ell)+1}) \quad \text{for all } \ell \in \mathbb{N},$$
$$\lim_{\ell\to\infty} d(u_{p(\ell)}, u_{q(\ell)}) = \lim_{\ell\to\infty} d(u_{p(\ell)+1}, u_{q(\ell)}) = \lim_{\ell\to\infty} d(u_{p(\ell)}, u_{q(\ell)+1})$$
$$= \lim_{\ell\to\infty} d(u_{p(\ell)+1}, u_{q(\ell)+1}) = e.$$

## 2.2 Proinov contractions

Very recently, Proinov announced some results which unified many known results.

**Theorem 4** (Proinov [43], Theorem 3.6) *Let* $(X, d)$ *be a complete metric space and* $T : X \to X$ *be a mapping such that*

$$\psi(d(Tu, Tv)) \leq \phi(d(u, v)) \quad \text{for all } u, v \in X \text{ with } d(Tu, Tv) > 0, \tag{2}$$

*where the functions* $\psi, \phi : (0, \infty) \to \mathbb{R}$ *satisfy the following conditions:*

$(a_1)$ $\psi$ *is nondecreasing;*
$(a_2)$ $\phi(s) < \psi(s)$ *for any* $s > 0$*;*
$(a_3)$ $\limsup_{s\to e^+} \phi(s) < \lim_{s\to e^+} \psi(s)$ *for any* $e > 0$.

*Then* $T$ *has a unique fixed point* $v_0 \in X$ *and the iterative sequence* $\{T^\ell u\}_{\ell\in\mathbb{N}}$ *converges to* $v_0$ *for every* $u \in X$.

Taking into account the previous result, we will say that a self-mapping $T : X \to X$ is a *Proinov contraction* if there are two functions $\psi, \phi : (0, \infty) \to \mathbb{R}$, satisfying the above-mentioned properties $(a_1)$-$(a_3)$, such that the contractivity condition (2) holds.

## 2.3 Fuzzy metric spaces

A *triangular norm* [52] (for short, a *t-norm*) is a function $* : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ satisfying the following properties: associativity, commutativity, non-decreasing on each argument, has 1 as unity (that is, $t * 1 = t$ for all $t \in \mathbb{I}$). It is usual that authors consider continuous t-norms on their studies. Examples of t-norms are the following ones: $t *_m s = \min\{t, s\}$ (Minimum); $t *_P s = t\,s$ (product); $t *_L s = \max\{0, t + s - 1\}$ (Łukasiewicz). Next we introduce the notion of fuzzy metric space following the definition given by George and Veeramani.

**Definition 1** (*cf. George and Veeramani* [14]) A triplet $(X, M, *)$ is called a *fuzzy metric space* if $X$ is an arbitrary non-empty set, $*$ is a continuous t-norm and $M : X \times X \times (0, \infty) \to \mathbb{I}$ is a fuzzy set satisfying the following conditions, for each $u, v, w \in X$, and $t, s > 0$:

$(GV-1)$ $M(u, v, t) > 0$;
$(GV-2)$ $M(u, v, t) = 1$ for all $t > 0$ if, and only if, $u = v$;
$(GV-3)$ $M(u, v, t) = M(v, u, t)$;
$(GV-4)$ $M(u, w, t + s) \geq M(u, v, t) * M(v, w, s)$;
$(GV-5)$ $M(u, v, \cdot) : (0, \infty) \to [0, 1]$ is a continuous function.

The following one is the canonical way in which a metric space can be seen as a fuzzy metric space.

***Example 1*** Each metric space $(X, d)$ can be seen as a fuzzy metric space $(X, M^d, *)$, where $*$ is any t-norm, by defining $M^d : X \times X \times (0, \infty) \to \mathbb{I}$ as:

$$M^d(u, v, t) = \frac{t}{t + d(u, v)} \quad \text{for all } t > 0. \tag{3}$$

Notice that $0 < M^d(u, v, t) < 1$ for all $t > 0$ and all $u, v \in X$ such that $u \neq v$. Furthermore, $\lim_{t\to\infty} M^d(u, v, t) = 1$ for all $u, v \in X$.

***Remark 1*** There is a distinct, but remarkable, way to define a fuzzy metric space. It was introduced by Kramosil and Michálek [11]. Following this interpretation, a fuzzy metric on $X$ is a function $M : X \times X \times [0, \infty) \to \mathbb{I}$ satisfying similar properties to (GV-2)-(GV-5), but employing a subtle difference: the fuzzy metric $M$ is defined for $t = 0$ and it satisfies $M(u, v, 0) = 0$ for all $u, v \in X$. Metrically, this condition can be interpreted by saying that the fuzzy metric $M$ can take the value infinite, that is, it can generalize extended real metrics $d : X \times X \to [0, +\infty]$ for which it is possible the condition $d(u, v) = +\infty$. In this paper we do not analyze this case because the contractivity condition:

$$\psi(M(Tu, Tv, s)) \geq \phi(M(u, v, s)) \quad \text{for all } u, v \in X \text{ and all } s > 0$$

cannot accept the values $M(u, v, s) = 0$ or $M(Tu, Tv, s) = 0$ when the functions $\psi$ and $\phi$ are not defined for $t = 0$. As a consequence, we develop our study in fuzzy metric spaces in the sense of George and Veeramani.

**Lemma 5** (cf. Grabiec [53]) *If* $(X, M, *)$ *is a fuzzy metric space and* $u, v \in X$*, then each function* $M(u, v, \cdot)$ *is nondecreasing on* $(0, \infty)$.

**Definition 2** In a fuzzy metric space $(X, M, *)$, we say that a sequence $\{u_\ell\}_{\ell\in\mathbb{N}} \subseteq X$ is:

- $M$-*Cauchy* if for all $\varepsilon \in (0, 1)$ and all $t > 0$ there is $\ell_0 \in \mathbb{N}$ such that $M(u_\ell, u_k, t) > 1-\varepsilon$ for all $\ell, k \geq \ell_0$;
- $M$-*convergent to* $v \in X$ if for all $\varepsilon \in (0, 1)$ and all $t > 0$ there is $\ell_0 \in \mathbb{N}$ such that $M(u_\ell, v, t) > 1 - \varepsilon$ for all $\ell \geq \ell_0$ (in such a case, we write $\{u_\ell\} \to v$ and we will say the $v$ is an $M$-*limit* of $\{u_\ell\}$).

We say that the fuzzy metric space $(X, M, *)$ is *complete* (or $X$ is $M$-*complete*) if each $M$-Cauchy sequence in $X$ is $M$-convergent to a point of $X$.

**Proposition 6** *The limit of an* $M$*-convergent sequence in a fuzzy metric space is unique.*

**Definition 3** (*Istrăţescu* [54]) A fuzzy metric space $(X, M, *)$ is said to be *non-Archimedean* if

$$M(u, w, t) \geq M(u, v, t) * M(v, w, t) \quad \text{for all } u, v, w \in X \text{ and all } t > 0. \tag{4}$$

This property is equivalent to:

$$M(u, w, \max\{t, s\}) \geq M(u, v, t) * M(v, w, s) \quad \text{for all } u, v, w \in X \text{ and all } t, s > 0.$$

Altun and Miheţ [55] showed some examples of non-Archimedean fuzzy metric spaces.

**Proposition 7** (cf. Roldán et al [51], Proposition 2) *Let* $\{u_\ell\}_{\ell\in\mathbb{N}}$ *be a Picard sequence in a fuzzy metric space* $(X, M, *)$ *such that*

$$\lim_{\ell\to\infty} M(u_\ell, u_{\ell+1}, t) = 1 \quad \textit{for all } t > 0.$$

*If there are* $n_0, m_0 \in \mathbb{N}$ *such that* $n_0 < m_0$ *and* $u_{n_0} = u_{m_0}$*, then there is* $\ell_0 \in \mathbb{N}$ *and* $w \in X$ *such that* $u_\ell = w$ *for all* $\ell \geq \ell_0$ *(that is,* $\{u_\ell\}$ *is constant from a term onwards). In such a case,* $w$ *is a fixed point of the self-mapping for which* $\{u_\ell\}$ *is a Picard sequence.*

**Corollary 8** *Let* $\{u_\ell\}_{\ell\in\mathbb{N}}$ *be a Picard sequence in a fuzzy metric space* $(X, M, *)$ *such that* $u_\ell \neq u_{\ell+1}$ *for all* $\ell \in \mathbb{N}$ *and satisfying:*

$$\lim_{\ell\to\infty} M(u_\ell, u_{\ell+1}, t) = 1 \quad \textit{for all } t > 0.$$

*Then* $\{u_\ell\}_{\ell\in\mathbb{N}}$ *is an infinite sequence (that is,* $u_\ell \neq u_k$ *for all distinct* $\ell, k \in \mathbb{N}$*).*

***Proof*** If $\{u_\ell\}_{\ell\in\mathbb{N}}$ is not an infinite sequence, then there are $\ell_1, \ell_2 \in \mathbb{N}$ with $\ell_1 < \ell_2$ such that $u_{\ell_1} = u_{\ell_2}$. In such a case, the previous proposition guarantees that there is $\ell_0 \in \mathbb{N}$ and $w \in X$ such that $u_\ell = w$ for all $\ell \geq \ell_0$. In particular, $u_{\ell_0} = w = u_{\ell_0+1}$ which contradicts the fact that $u_\ell \neq u_{\ell+1}$ for all $\ell \in \mathbb{N}$. Then $\{u_\ell\}_{\ell\in\mathbb{N}}$ is an infinite sequence. □

## 3 Extended Proinov contractions in metric spaces

We start our study in the framework of metric spaces. Inspired by the above-mentioned Proinov theorem, we introduce here a new class of contractions in the setting of metric spaces that will be characterized by the following kind of auxiliary functions.

Let $\mathfrak{X}$ denote the family of pairs $(\psi, \phi)$ such that $\psi, \phi : (0, \infty) \to \mathbb{R}$ are functions satisfying the following conditions:

($\mathfrak{X}_1$) if $\{t_\ell\}_{\ell\in\mathbb{N}} \subset (0, \infty)$ is a sequence such that $\psi(t_{\ell+1}) \leq \phi(t_\ell)$ for all $\ell \in \mathbb{N}$, then $\{t_\ell\} \to 0$;

($\mathfrak{X}_2$) if $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0, \infty)$ are two sequences converging to the same limit $e \geq 0$ that satisfy $t_\ell > e$ and $\psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$, then $e = 0$;

($\mathfrak{X}_3$) if $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0, \infty)$ are two sequences such that $\{s_\ell\} \to 0$ and $\psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$, then $\{t_\ell\} \to 0$.

We show that the family $\mathfrak{X}$ is nonempty.

***Example 2*** If $\psi, \phi : (0, \infty) \to \mathbb{R}$ are defined by $\psi(s) = \lambda_1 s$ and $\phi(s) = \lambda_2 s$ for all $s > 0$, where $\lambda_1, \lambda_2 \in (0, \infty)$ are such that $\lambda_2 < \lambda_1$, then $(\psi, \phi) \in \mathfrak{X}$.

One of the main aims of this section is to prove that the pair $(\psi, \phi)$ of functions associated to a Proinov contraction belong to $\mathfrak{X}$, which is precisely the claim of the following statement.

**Lemma 9** *Let* $\psi, \phi : (0, \infty) \to \mathbb{R}$ *be functions verifying the following properties:*

*($a_1$)* $\psi$ *is nondecreasing;*

*($a_2$)* $\phi(s) < \psi(s)$ *for any* $s > 0$*;*

*($a_3$)* $\limsup_{s\to e^+} \phi(s) < \lim_{s\to e^+} \psi(s)$ *for any* $e > 0$.

*Then* $(\psi, \phi) \in \mathfrak{X}$.

***Proof*** On the one hand, let $\{t_\ell\}_{\ell\in\mathbb{N}} \subset (0,\infty)$ be a sequence such that $\psi(t_{\ell+1}) \le \phi(t_\ell)$ for all $\ell \in \mathbb{N}$. Since $t_\ell > 0$, property $(a_2)$ lead to

$$\psi(t_{\ell+1}) \le \phi(t_\ell) < \psi(t_\ell) \quad \text{for all } \ell \in \mathbb{N}. \tag{5}$$

As $\psi$ is nondecreasing, then $0 < t_{\ell+1} < t_\ell$ for all $\ell \in \mathbb{N}$. Let $e = \lim_{\ell\to\infty} t_\ell$. Clearly $e < t_\ell$ for all $\ell \in \mathbb{N}$. To prove that $e = 0$, suppose that $e > 0$. In such a case,

$$\psi(e) \le \lim_{s\to e^+} \psi(s) = \lim_{\ell\to\infty} \psi(t_\ell) = \lim_{\ell\to\infty} \psi(t_{\ell+1}).$$

Using (5), we deduce that

$$\lim_{\ell\to\infty} \phi(t_\ell) = \lim_{s\to e^+} \psi(s),$$

which contradicts $(a_3)$ because

$$\lim_{s\to e^+} \psi(s) = \lim_{\ell\to\infty} \phi(t_\ell) \le \limsup_{s\to e^+} \phi(s) < \lim_{s\to e^+} \psi(s).$$

Hence $e = 0$, which completes the proof of $(\mathfrak{X}_1)$.

On the other hand, let $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0,\infty)$ be two sequences such that $\{t_\ell\} \to e$, $\{s_\ell\} \to e$, and also satisfying $t_\ell > e$ and $\psi(t_\ell) \le \phi(s_\ell)$ for all $\ell \in \mathbb{N}$. Similarly, to prove that $e = 0$, suppose that $e > 0$. By property $(a_2)$,

$$\psi(t_\ell) \le \phi(s_\ell) < \psi(s_\ell) \quad \text{for all } \ell \in \mathbb{N}. \tag{6}$$

As $\psi$ is nondecreasing, then $e < t_\ell < s_\ell$ for all $\ell \in \mathbb{N}$. Therefore, as $\{t_\ell\} \to e$, $\{s_\ell\} \to e$ and their terms are strictly greater than $e$,

$$\psi(e) \le \lim_{s\to e^+} \psi(s) = \lim_{\ell\to\infty} \psi(t_\ell) = \lim_{\ell\to\infty} \psi(s_\ell).$$

Using (6), it follows that

$$\lim_{\ell\to\infty} \phi(s_\ell) = \lim_{s\to e^+} \psi(s),$$

which contradicts $(a_3)$ because

$$\lim_{s\to e^+} \psi(s) = \lim_{\ell\to\infty} \phi(s_\ell) \le \limsup_{s\to e^+} \phi(s) < \lim_{s\to e^+} \psi(s).$$

Hence $e = 0$, which proves $(\mathfrak{X}_2)$.

Finally, to check the property $(\mathfrak{X}_3)$, let $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0,\infty)$ be two sequences such that $\{s_\ell\} \to 0$ and $\psi(t_\ell) \le \phi(s_\ell)$ for all $\ell \in \mathbb{N}$. Since $s_\ell > 0$, then $\psi(t_\ell) \le \phi(s_\ell) < \psi(s_\ell)$, so $0 < t_\ell < s_\ell$ for all $\ell \in \mathbb{N}$. Hence $\{t_\ell\} \to 0$. □

Next we present the main result of this section.

**Theorem 10** *Let* $(X, d)$ *be a complete metric space and let* $T : X \to X$ *be a mapping for which there exists* $(\psi, \phi) \in \mathfrak{X}$ *such that*

$$\psi(d(Tu, Tv)) \le \phi(d(u, v)) \quad \textit{for all } u, v \in X \textit{ with } d(Tu, Tv) > 0. \tag{7}$$

*Then each iterative Picard sequence* $\{T^\ell u\}_{\ell\in\mathbb{N}}$ *converges to a fixed point of* $T$.

***Proof*** Let $\mathsf{u} \in \mathsf{X}$ be arbitrary and let define $\mathsf{u}_1 = \mathsf{u}$ and $\mathsf{u}_{\ell+1} = \mathsf{T}\mathsf{u}_\ell$ for all $\ell \in \mathbb{N}$. If there is $\ell_0 \in \mathbb{N}$ such that $\mathsf{u}_{\ell_0} = \mathsf{u}_{\ell_0+1}$, then $\mathsf{u}_{\ell_0}$ is a fixed point of $\mathsf{T}$. In such a case, $\{\mathsf{d}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1})\}_{\ell \geq \ell_0} = \{0\} \to 0$. On the contrary case, suppose that $\mathsf{u}_\ell \neq \mathsf{u}_{\ell+1}$ for all $\ell \in \mathbb{N}$. Then each $\mathsf{u}_\ell$ is not a fixed point of $\mathsf{T}$ and also

$$\mathsf{d}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}) > 0 \quad \text{and} \quad \mathsf{d}(\mathsf{T}\mathsf{u}_\ell, \mathsf{T}\mathsf{u}_{\ell+1}) > 0 \quad \text{for all} \quad \ell \in \mathbb{N}.$$

Applying the contractivity condition (7), we deduce that, for all $\ell \in \mathbb{N}$,

$$\psi(\mathsf{d}(\mathsf{u}_{\ell+1}, \mathsf{u}_{\ell+2})) = \psi(\mathsf{d}(\mathsf{T}\mathsf{u}_\ell, \mathsf{T}\mathsf{u}_{\ell+1})) \leq \phi\left(\mathsf{d}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1})\right).$$

If we define $\mathsf{s}_\ell = \mathsf{d}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1})$ for all $\ell \in \mathbb{N}$, the previous inequality means that the sequence $\{\mathsf{s}_\ell\}$ satisfies $\psi\left(\mathsf{s}_{\ell+1}\right) \leq \phi(\mathsf{s}_\ell)$ for all $\ell \in \mathbb{N}$. Under condition $(\mathfrak{X}_1)$, we deduce that $\{\mathsf{d}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1})\} = \{\mathsf{s}_\ell\} \to 0$.

If there are $\ell_1, \ell_2 \in \mathbb{N}$ such that $\ell_1 < \ell_2$ and $\mathsf{u}_{\ell_1} = \mathsf{u}_{\ell_2}$, then Proposition 2 ensures that there is $\ell_0 \in \mathbb{N}$ and $\mathsf{v} \in \mathsf{X}$ such that $\mathsf{u}_\ell = \mathsf{v}$ for all $\ell \geq \ell_0$. In such a case, $\mathsf{v}$ is a fixed point of $\mathsf{T}$, and the existence of fixed points is assured.

Next, suppose that $\mathsf{u}_{\ell_1} \neq \mathsf{u}_{\ell_2}$ for all $\ell_1, \ell_2 \in \mathbb{N}$ such that $\ell_1 \neq \ell_2$, that is, $\{\mathsf{u}_\ell\}_{\ell \in \mathbb{N}}$ is an infinite sequence. In particular, $\mathsf{d}\left(\mathsf{T}\mathsf{u}_{\ell_1}, \mathsf{T}\mathsf{u}_{\ell_2}\right) = \mathsf{d}\left(\mathsf{u}_{\ell_1+1}, \mathsf{u}_{\ell_2+1}\right) > 0$ for all $\ell_1, \ell_2 \in \mathbb{N}$ such that $\ell_1 \neq \ell_2$. To prove that $\{\mathsf{u}_\ell\}_{\ell \in \mathbb{N}}$ is a $\mathsf{d}$-Cauchy sequence, suppose that it is not. In such a case, Lemma 3 states that there exist $\mathsf{e} > 0$ and two partial subsequences $\{\mathsf{u}_{p(\ell)}\}_{\ell \in \mathbb{N}}$ and $\{\mathsf{u}_{q(\ell)}\}_{\ell \in \mathbb{N}}$ of $\{\mathsf{u}_\ell\}_{\ell \in \mathbb{N}}$ such that

$$p\left(\ell\right) < q\left(\ell\right) < p\left(\ell+1\right) \quad \text{and} \quad \mathsf{e} < \mathsf{d}(\mathsf{u}_{p(\ell)+1}, \mathsf{u}_{q(\ell)+1}) \quad \text{for all } \ell \in \mathbb{N}, \tag{8}$$

$$\begin{aligned} \lim_{\ell \to \infty} \mathsf{d}(\mathsf{u}_{p(\ell)}, \mathsf{u}_{q(\ell)}) &= \lim_{\ell \to \infty} \mathsf{d}(\mathsf{u}_{p(\ell)+1}, \mathsf{u}_{q(\ell)}) = \lim_{r \to \infty} \mathsf{d}(\mathsf{u}_{p(\ell)}, \mathsf{u}_{q(\ell)+1}) \\ &= \lim_{r \to \infty} \mathsf{d}(\mathsf{u}_{p(\ell)+1}, \mathsf{u}_{q(\ell)+1}) = \mathsf{e}. \end{aligned} \tag{9}$$

Applying the contractivity condition (7) we deduce that, for all $\ell \in \mathbb{N}$,

$$\psi(\mathsf{d}(\mathsf{u}_{p(\ell)+1}, \mathsf{u}_{q(\ell)+1})) = \psi(\mathsf{d}(\mathsf{T}\mathsf{u}_{p(\ell)}, \mathsf{T}\mathsf{u}_{q(\ell)})) \leq \phi(\mathsf{d}(\mathsf{u}_{p(\ell)}, \mathsf{u}_{q(\ell)})). \tag{10}$$

If we define $\mathsf{t}_\ell = \mathsf{d}(\mathsf{u}_{p(\ell)+1}, \mathsf{u}_{q(\ell)+1})$ and $\mathsf{s}_\ell = \mathsf{d}(\mathsf{u}_{p(\ell)}, \mathsf{u}_{q(\ell)})$ for all $\ell \in \mathbb{N}$, then (8), (9) and (10) guarantees that

$$\mathsf{t}_\ell > \mathsf{e} \quad \text{and} \quad \psi\left(\mathsf{t}_\ell\right) \leq \phi\left(\mathsf{s}_\ell\right) \quad \text{for all } \ell \in \mathbb{N}.$$

However, the fact that $\mathsf{e} > 0$ contradicts the property $(\mathfrak{X}_2)$. This contradiction comes from the assumption that $\{\mathsf{u}_\ell\}_{\ell \in \mathbb{N}}$ is not a $\mathsf{d}$-Cauchy sequence, which demonstrates that actually $\{\mathsf{u}_\ell\}_{\ell \in \mathbb{N}}$ is a $\mathsf{d}$-Cauchy sequence. As $(\mathsf{X}, \mathsf{d})$ is a complete metric space, there is $\mathsf{v} \in \mathsf{X}$ such that $\{\mathsf{u}_\ell\}_{\ell \in \mathbb{N}}$ $\mathsf{d}$-converges to $\mathsf{v}$. As the sequence $\{\mathsf{u}_\ell\}_{\ell \in \mathbb{N}}$ is infinite, there is $\ell_0 \in \mathbb{N}$ such that $\mathsf{u}_\ell \neq \mathsf{v}$ and $\mathsf{T}\mathsf{u}_\ell \neq \mathsf{T}\mathsf{v}$ for all $\ell \geq \ell_0$. The contractivity condition (7) leads to

$$\psi(\mathsf{d}(\mathsf{u}_{\ell+1}, \mathsf{T}\mathsf{v})) = \psi(\mathsf{d}(\mathsf{T}\mathsf{u}_\ell, \mathsf{T}\mathsf{v})) \leq \phi\left(\mathsf{d}(\mathsf{u}_\ell, \mathsf{v})\right).$$

It follows from property $(\mathfrak{X}_3^F)$ that $\{\mathsf{d}(\mathsf{u}_{\ell+1}, \mathsf{T}\mathsf{v})\} \to 0$, so $\mathsf{T}\mathsf{v} = \mathsf{v}$. This completes the proof. □

To deduce the uniqueness of the fixed point, it is necessary to add an additional condition.

**Theorem 11** *Under the hypotheses of Theorem 10, suppose that the pair $(\psi, \phi)$ satisfies the following property:*

$(\mathfrak{X}_4)$ *there is a subset* $\Omega \subseteq \mathsf{X}$ *such that* $\mathrm{Fix}(\mathsf{T}) \subseteq \Omega$ *and* $\psi(\mathsf{d}(\mathsf{u}, \mathsf{v})) > \phi(\mathsf{d}(\mathsf{u}, \mathsf{v}))$ *for all distinct* $\mathsf{u}, \mathsf{v} \in \Omega$.

*Then* $\mathsf{T}$ *has a unique fixed point* $\mathsf{v}_0 \in \mathsf{X}$ *and the iterative sequence* $\{\mathsf{T}^\ell \mathsf{u}\}_{\ell\in\mathbb{N}}$ *converges to* $\mathsf{v}_0$ *for every* $\mathsf{u} \in \mathsf{X}$.

***Proof*** To check the uniqueness of the fixed point of T, suppose that $\mathsf{v}_1, \mathsf{v}_2 \in \mathsf{X}$ are two distinct fixed points of T. Then $\mathsf{d}(\mathsf{Tv}_1, \mathsf{Tv}_2) = \mathsf{d}(\mathsf{v}_1, \mathsf{v}_2) > 0$. The contractive condition implies that

$$\psi(\mathsf{d}(\mathsf{v}_1, \mathsf{v}_2)) = \psi(\mathsf{d}(\mathsf{Tv}_1, \mathsf{Tv}_2)) \le \phi(\mathsf{d}(\mathsf{v}_1, \mathsf{v}_2)),$$

which contradicts the assumption $(\mathfrak{X}_4)$. □

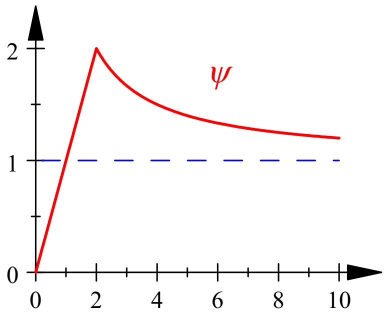
**Corollary 12** *Theorem 4 immediately follows from Theorems 10 and 11.*

***Proof*** Theorem 10 demonstrates the existence of fixed points and Theorem 11, its uniqueness. □

***Example 3*** Let $\mathsf{X} = [0, 1]$ endowed with the Euclidean metric and let $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ be defined by $\mathsf{Tu} = \mathsf{u}/2$ for all $\mathsf{u} \in [0, 1]$. Let also define $\psi, \phi : (0, \infty) \to \mathbb{R}$, for each $\mathsf{s} \in (0, \infty)$, by:

$$\psi(\mathsf{s}) = \begin{cases} \mathsf{s}, & \text{if } \mathsf{s} \in (0, 2), \\ 1 + \dfrac{2}{\mathsf{s}}, & \text{if } \mathsf{s} \in [2, \infty); \end{cases}$$

$$\phi(\mathsf{s}) = \psi(\mathsf{s})/2.$$




We claim that $(\psi, \phi) \in \mathfrak{X}$, so Theorems 10 and 11 can be employed in order to show that T has a unique fixed point. However, Theorem 4 is not applicable because $\psi$ is strictly decreasing in $(2, \infty)$.

To prove our claim, notice that, for all $\mathsf{u}, \mathsf{v} \in \mathsf{X}$,

$$\psi(\mathsf{d}(\mathsf{Tu}, \mathsf{Tv})) = \psi\left(\left|\frac{\mathsf{u}}{2} - \frac{\mathsf{v}}{2}\right|\right) = \left|\frac{\mathsf{u}}{2} - \frac{\mathsf{v}}{2}\right| = \frac{|\mathsf{u} - \mathsf{v}|}{2} = \phi(\mathsf{d}(\mathsf{u}, \mathsf{v})).$$

We check now $(\mathfrak{X}_1)$. Let $\{\mathsf{t}_\ell\} \subset (0, \infty)$ be such that $\psi(\mathsf{t}_{\ell+1}) \le \phi(\mathsf{t}_\ell)$ for all $\ell \in \mathbb{N}$, that is, $\psi(\mathsf{t}_{\ell+1}) \le \psi(\mathsf{t}_\ell)/2$ for all $\ell \in \mathbb{N}$. Therefore $\{\psi(\mathsf{t}_\ell)\} \to 0$, so $\{\mathsf{t}_\ell\} \to 0$. Next, let $\{\mathsf{t}_\ell\}, \{\mathsf{s}_\ell\} \subset (0, \infty)$ be two sequences converging to the same limit $\mathsf{e} \ge 0$ that satisfy $\mathsf{t}_\ell > \mathsf{e}$ and $\psi(\mathsf{t}_\ell) \le \phi(\mathsf{s}_\ell)$ for all $\ell \in \mathbb{N}$. If we suppose that $\mathsf{e} > 0$, taking into account that both functions $\psi$ and $\phi$ are continuous, then

$$\begin{aligned} 0 < \mathsf{e} = \psi(\mathsf{e}) &= \lim_{s\to \mathsf{e}} \psi(\mathsf{s}) = \lim_{\mathsf{s}\to \mathsf{e}^+} \psi(\mathsf{s}) = \lim_{\ell\to\infty} \psi(\mathsf{t}_\ell) \le \lim_{\ell\to\infty} \phi(\mathsf{s}_\ell) \\ &= \lim_{\mathsf{s}\to \mathsf{e}^+} \phi(\mathsf{s}) = \lim_{\mathsf{s}\to \mathsf{e}} \phi(\mathsf{s}) = \phi(\mathsf{e}) = \frac{\psi(\mathsf{e})}{2} = \frac{\mathsf{e}}{2}, \end{aligned}$$

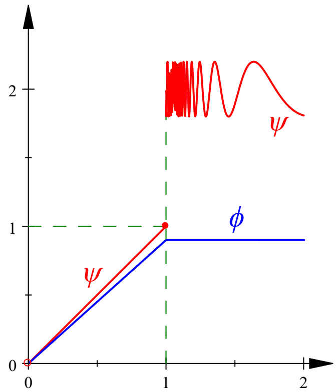
which is impossible. As a consequence, $\mathsf{e} = 0$ and this proves $(\mathfrak{X}_2)$. Finally, $(\mathfrak{X}_3)$ can be proved in a similar way.

***Example 4*** Let $\mathsf{X} = [0, 1] \cup \mathsf{Y}$ where $\mathsf{Y} = \{5\ell : \ell \in \mathbb{N}\} = \{5, 10, 15, \ldots\}$. Endowed with the Euclidean metric $\mathsf{d}(\mathsf{u}, \mathsf{v}) = |\mathsf{u} - \mathsf{v}|$, the set X is complete and unbounded. Let define $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ and $\phi, \psi : (0, \infty) \to \mathbb{R}$ as follows:

$$\mathsf{T}u = \begin{cases} \dfrac{u}{5}, & \text{if } u \in [0,1], \\ \dfrac{1}{u}, & \text{if } u \in Y; \end{cases}$$

$$\phi(s) = \begin{cases} 0.9\,s, & \text{if } 0 < s \leq 1, \\ 0.9, & \text{if } s > 1; \end{cases}$$

$$\psi(s) = \begin{cases} s, & \text{if } 0 < s \leq 1, \\ 2 + \dfrac{1}{5}\sin\left(\dfrac{5}{s-1}\right), & \text{if } s > 1. \end{cases}$$




Then $\mathsf{T}(X) = [0, 0.2]$, $\phi((0,\infty)) = (0, 0.9]$ and $\psi((0,\infty)) = (0,1] \cup [1.8, 2.2]$. Notice that if $u_0 = 0$ and $v_0 = 1$, then $e_0 = d(u_0, v_0) = 1$. However, $\psi$ is not monotone and the limit

$$\lim_{s \to e_0^+} \psi(s) = \lim_{s \to 1^+} \left[ 2 + \frac{1}{5}\sin\left(\frac{5}{s-1}\right)\right]$$

does not exist. As a consequence, the conditions $(a_1)$ and $(a_3)$ do not hold, so Theorem 4 is not applicable. However, we claim that $(\psi, \phi) \in \mathfrak{X}$.

($\mathfrak{X}_1$) Let $\{t_\ell\}_{\ell\in\mathbb{N}} \subset (0,\infty)$ be a sequence such that $\psi(t_{\ell+1}) \leq \phi(t_\ell)$ for all $\ell \in \mathbb{N}$. Then $\psi(t_{\ell+1}) \leq \phi(t_\ell) \leq 0.9$, so $\psi(t_{\ell+1}) \in (0, 0.9]$ for each $\ell \in \mathbb{N}$. This means that $t_{\ell+1} \in (0, 0.9]$ for each $\ell \in \mathbb{N}$. Hence the inequality $\psi(t_{\ell+1}) \leq \phi(t_\ell)$ means that $t_{\ell+1} \leq 0.9\,t_\ell$ for all $\ell \in \mathbb{N}$, and this condition guarantees that $\{t_\ell\} \to 0$.

($\mathfrak{X}_2$) Let $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0,\infty)$ be two sequences converging to the same limit $e \geq 0$ that satisfy $t_\ell > e$ and $\psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$. Then $\psi(t_\ell) \leq \phi(s_\ell) \leq 0.9$, so $\psi(t_\ell) \in (0, 0.9]$ for each $\ell \in \mathbb{N}$. As a result, $t_\ell \in (0, 0.9]$ for each $\ell \in \mathbb{N}$, which means that $e < t_\ell \leq 0.9$ for each $\ell \in \mathbb{N}$. Since $\{t_\ell\}$ and $\{s_\ell\}$ converge to $e$, then there is $\ell_0 \in \mathbb{N}$ such that $t_\ell, s_\ell \in (0, 0.95]$ for each $\ell \geq \ell_0$. In particular, the inequality $\psi(t_\ell) \leq \phi(s_\ell)$ leads to $t_\ell \leq 0.9\,s_\ell$ for each $\ell \geq \ell_0$. Letting $\ell \to \infty$, we deduce that $e \leq 0.9\,e$, and as $e \geq 0$, then necessarily $e = 0$.

($\mathfrak{X}_3$) Let $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0,\infty)$ be two sequences such that $\{s_\ell\} \to 0$ and $\psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$. Since $\psi(t_\ell) \leq \phi(s_\ell) \leq 0.9$, then $\psi(t_\ell) \in (0, 0.9]$ and $t_\ell \in (0, 0.9]$ for each $\ell \in \mathbb{N}$. Furthermore, as $\{s_\ell\} \to 0$, then there is $\ell_0 \in \mathbb{N}$ such that $t_\ell, s_\ell \in (0, 0.9]$ for each $\ell \geq \ell_0$. In particular, the inequality $\psi(t_\ell) \leq \phi(s_\ell)$ leads to $t_\ell \leq 0.9\,s_\ell$ for each $\ell \geq \ell_0$. As a consequence, taking into account that $\{s_\ell\} \to 0$, we conclude that $\{t_\ell\} \to 0$.

Finally, we check that the contractivity condition (7) holds. Let $u, v \in X$ be two points such that $\mathsf{T}u \neq \mathsf{T}v$. In particular, $u \neq v$ and $\mathsf{T}u, \mathsf{T}v \in [0, 0.2]$, so $d(\mathsf{T}u, \mathsf{T}v) \leq 0.2$. We consider two cases.

- If $u, v \in [0,1]$, then $d(u,v) \leq 1$ and $d(\mathsf{T}u, \mathsf{T}v) \leq 0.2$, so

$$\psi(d(\mathsf{T}u,\mathsf{T}v)) = d(\mathsf{T}u,\mathsf{T}v) = \left|\frac{u}{5} - \frac{v}{5}\right| = 0.2\,|u - v| \leq 0.9\,|u - v|$$
$$= 0.9\,d(u,v) = \phi(d(u,v)).$$

- If $u \in Y$ or $v \in Y$, taking into account that $u \neq v$, then necessarily $d(u, v) \geq 4$, so $\phi(d(u, v)) = 0.9$, but as $d(Tu, Tv) \leq 0.2$, then

$$\psi(d(Tu, Tv)) = d(Tu, Tv) \leq 0.2 \leq 0.9 = \phi(d(u, v)).$$

In any case, the condition (7) holds, so Theorems 10 and 11 are applicable to conclude that T has a unique fixed point.

Theorems 10 and 11, together with Examples 3 and 4, show that assumptions $(\mathfrak{X}_1)$-$(\mathfrak{X}_4)$ are more general than Proinov's conditions $(a_1)$-$(a_3)$. In fact, the first set of assumptions help us to imagine new kind of intermediate families of functions for which Theorems 10 and 11 could be applied. This is the case of the following result.

**Lemma 13** *Let $\psi, \phi : (0, \infty) \to \mathbb{R}$ be functions verifying the following properties:*

*$(\mathfrak{a}_1)$ If $t, s \in (0, \infty)$ are such that $\psi(t) \leq \phi(s)$, then $t < s$.*
*$(\mathfrak{a}_2)$ for each $e > 0$ there exists the lateral limit $\lim_{s \to e^+} \psi(s) \in \mathbb{R}$;*
*$(a_3)$ $\limsup_{s \to e^+} \phi(s) < \lim_{s \to e^+} \psi(s)$ for any $e > 0$.*

*Then $(\psi, \phi) \in \mathfrak{X}$.*

***Proof*** We prove that the pair $(\psi, \phi)$ satisfies all properties $(\mathfrak{X}_1)$-$(\mathfrak{X}_3)$.

$(\mathfrak{X}_1)$ Let $\{t_\ell\}_{\ell \in \mathbb{N}} \subset (0, \infty)$ be a sequence such that $\psi(t_{\ell+1}) \leq \phi(t_\ell)$ for all $\ell \in \mathbb{N}$. Since $t_\ell > 0$, property $(\mathfrak{a}_1)$ leads to

$$0 < t_{\ell+1} < t_\ell \quad \text{for all } \ell \in \mathbb{N}.$$

Let $e = \lim_{\ell \to \infty} t_\ell$. Clearly $e < t_\ell$ for all $\ell \in \mathbb{N}$. To prove that $e = 0$, suppose that $e > 0$. In such a case,

$$\psi(t_{\ell+1}) \leq \phi(t_\ell) \quad \text{for all } \ell \in \mathbb{N} \quad \Rightarrow \quad \lim_{\ell \to \infty} \psi(t_{\ell+1}) \leq \limsup_{\ell \to \infty} \phi(t_\ell),$$

so

$$\lim_{s \to e^+} \psi(s) = \lim_{\ell \to \infty} \psi(t_{\ell+1}) \leq \limsup_{\ell \to \infty} \phi(t_\ell) \leq \limsup_{s \to e^+} \phi(s),$$

which contradicts the assumption $(a_3)$.

$(\mathfrak{X}_2)$ Let $\{t_\ell\}_{\ell \in \mathbb{N}}, \{s_\ell\}_{\ell \in \mathbb{N}} \subset (0, \infty)$ be two sequences converging to the same limit $e \geq 0$ that satisfy $t_\ell > e$ and $\psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$. In order to prove that $e = 0$ assume, by contradiction, that $e > 0$. In such a case, $0 < e < t_\ell$ for all $\ell \in \mathbb{N}$. Since $\psi(t_\ell) \leq \phi(s_\ell)$, then $t_\ell < s_\ell$, so $0 < e < t_\ell < s_\ell$ for all $\ell \in \mathbb{N}$. Furthermore,

$$\psi(t_\ell) \leq \phi(s_\ell) \quad \text{for all } \ell \in \mathbb{N} \quad \Rightarrow \quad \lim_{\ell \to \infty} \psi(t_\ell) \leq \limsup_{\ell \to \infty} \phi(s_\ell).$$

Hence, the following argument contradicts the assumption $(a_3)$ because:

$$\lim_{s \to e^+} \psi(s) = \lim_{\ell \to \infty} \psi(t_\ell) \leq \limsup_{\ell \to \infty} \phi(s_\ell) \leq \limsup_{s \to e^+} \phi(s).$$

This contradiction proves that $e = 0$.

$(\mathfrak{X}_3)$ Let $\{t_\ell\}_{\ell \in \mathbb{N}}, \{s_\ell\}_{\ell \in \mathbb{N}} \subset (0, \infty)$ be two sequences such that $\{s_\ell\} \to 0$ and $\psi(t_\ell) \leq \phi(s_\ell)$ for all $\ell \in \mathbb{N}$. By $(\mathfrak{a}_1)$, $0 < t_\ell < s_\ell$ for all $\ell \in \mathbb{N}$. Therefore, $\{t_\ell\} \to 0$. □

The previous result is the key tool of the following fixed point theorem, in which the function $\psi$ is not necessarily nondecreasing.

**Corollary 14** *Let* $(\mathsf{X}, \mathsf{d})$ *be a complete metric space and let* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *be a mapping for which there exists two functions* $\psi, \phi : (0, \infty) \to \mathbb{R}$ *satisfying the following properties:*

*(*$\mathfrak{a}_1$*) if* $\mathsf{t}, \mathsf{s} \in (0, \infty)$ *are such that* $\psi(\mathsf{t}) \leq \phi(\mathsf{s})$*, then* $\mathsf{t} < \mathsf{s}$*;*
*(*$\mathfrak{a}_2$*) for each* $\mathsf{e} > 0$ *there exists the lateral limit* $\lim_{\mathsf{s} \to \mathsf{e}^+} \psi(\mathsf{s}) \in \mathbb{R}$*;*
*(*$a_3$*)* $\limsup_{\mathsf{s} \to \mathsf{e}^+} \phi(\mathsf{s}) < \lim_{\mathsf{s} \to \mathsf{e}^+} \psi(\mathsf{s})$ *for any* $\mathsf{e} > 0$*.*

*Additionally, assume that*

$$\psi(\mathsf{d}(\mathsf{Tu}, \mathsf{Tv})) \leq \phi(\mathsf{d}(\mathsf{u}, \mathsf{v})) \quad \textit{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \textit{ with } \mathsf{Tu} \neq \mathsf{Tv}.$$

*Then the mapping* $\mathsf{T}$ *has a unique fixed point and the Picard sequence* $\{\mathsf{T}^\ell \mathsf{u}\}_{\ell \in \mathbb{N}}$ *converges to such fixed point whatever the initial point* $\mathsf{u} \in \mathsf{X}$*.*

***Proof*** Under the hypotheses of the present result, Lemma 13 ensures that $(\psi, \phi) \in \mathfrak{X}$. Therefore, Theorem 10 shows that each iterative Picard sequence $\{\mathsf{T}^\ell \mathsf{u}\}_{\ell \in \mathbb{N}}$ converges to a fixed point of $\mathsf{T}$. In particular, the mapping $\mathsf{T}$ has, at least, one fixed point. Furthermore, condition $(\mathfrak{a}_1)$ implies that $\psi(\mathsf{s}) > \phi(\mathsf{s})$ for all $\mathsf{s} \in (0, \infty)$. Hence assumption $(\mathfrak{X}_4)$ holds and Theorem 11 guarantees the uniqueness of such fixed point. □

## 4 Fixed point theory in fuzzy metric spaces satisfying the property $\mathcal{NC}$

The main aim of this section is to prove some new results about existence and uniqueness of fixed points of self-mappings in the setting of fuzzy metric spaces. Such new contractions will satisfy some assumptions that, as we will remark, are more general than supposed on Proinov theorem. In fact, we will show how the monotone condition on one auxiliary functions can be avoided in our main results. Before that, we start this section with a discussion about the algebraic tools that we will use to reach our objectives.

### 4.1 Some remarks about the assumptions on fuzzy contractive mappings

For the sake of completeness, we describe here some of the ideas that we will considered and that led us to the contents of the current manuscript, mainly in the framework of fuzzy metric spaces.

On the one hand, fuzzy metric spaces are distinct in nature than metric spaces. Their fuzzy metric takes values over the closed interval $\mathbb{I} = [0, 1]$ with a different meaning than real metric spaces: values near 1 mean that the points are *metrically* near, but values near to 0 have the contrary interpretation: the points are *metrically* far. However, the most important difference between such kind of abstract metric is the fact that a fuzzy metric $\mathsf{M} : \mathsf{X} \times \mathsf{X} \times (0, \infty) \to \mathbb{I}$ involves an argument $\mathsf{t} \in (0, \infty)$ that can be interpreted as follows: the value of $\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})$ is a performance of the probability of the event in which the distance between the points $\mathsf{u}$ and $\mathsf{v}$ of the underlying spaces is less than or equal to $\mathsf{t}$. Hence, the role is very significant, and it cannot be reduced to any intrinsic characteristic of a real metric.As a consequence, the hypotheses that usually are assumed when handling a fuzzy metric space are distinct than we can suppose on a real metric space. Having in mind this behavior and inspired by the content of Lemma 3, Roldán López de Hierro et al [15] introduced the *property* $\mathcal{NC}$ ($\mathcal{NC}$ means "*not Cauchy*") in a fuzzy space as follows: a fuzzy space $(\mathsf{X}, \mathsf{M})$ satisfies the *property* $\mathcal{NC}$ if for each sequence $\{\mathsf{u}_\ell\} \subseteq \mathsf{X}$ which is not $\mathsf{M}$-Cauchy and verifies $\lim_{\ell \to \infty} \mathsf{M}(\mathsf{u}_\ell, \mathsf{u}_{\ell+1}, \mathsf{t}) = 1$

for all $t > 0$, there are $e_0 \in (0, 1)$, $t_0 > 0$ and two partial subsequences $\{u_{r(\ell)}\}_{\ell\in\mathbb{N}}$ and $\{u_{s(\ell)}\}_{\ell\in\mathbb{N}}$ of $\{u_\ell\}$ such that, for all $\ell \in \mathbb{N}$,

$$\ell < r(\ell) < s(\ell) < r(\ell+1) \quad \text{and}$$
$$M\left(u_{r(\ell)}, u_{s(\ell)-1}, t_0\right) > 1 - e_0 \geq M\left(u_{r(\ell)}, u_{s(\ell)}, t_0\right),$$

and also

$$\lim_{\ell\to\infty} M\left(u_{r(\ell)}, u_{s(\ell)}, t_0\right) = \lim_{\ell\to\infty} M\left(u_{r(\ell)-1}, u_{s(\ell)-1}, t_0\right) = 1 - e_0.$$

This property is able to generalize the non-Archimedean property in the following sense: each non-Arquimedean fuzzy metric space $(X, M, *)$ whose t-norm $*$ is continuous satisfies the property $\mathcal{NC}$ (see [15]).

On the other hand, we are interested on a contractivity condition like:

$$\varphi(M(Tu, Tv, s)) \geq \eta(M(u, v, s)) \quad \text{for all } u, v \in X \text{ and all } s > 0,$$

for a mapping $T : X \to X$ from on a fuzzy metric space into itself. In this context, $\varphi, \eta : (0, 1] \to \mathbb{R}$ must be appropriate auxiliary functions. Inspired by the metric case, one can believe that it is not necessary that the functions $\varphi$ and $\eta$ are defined for $t = 1$; in such a case, the domain of $\varphi$ and $\eta$ could be the open interval $(0, 1)$. This can be considered as an interesting possibility if the fuzzy metric $M$ satisfies the additional condition

$$0 < M(u, v, s) < 1 \quad \text{for all } u, v \in X,\ u \neq v,\ \text{and all } s > 0.$$

This condition holds when $M$ is the fuzzy metric $M^d$ associated to a crisp metric $d$ on $X$ (see Example 1). However, there are fuzzy spaces in which there are two distinct points $u_0, v_0 \in X$ and a finite number $s_0 \in (0, \infty)$ such that $M(u_0, v_0, s_0) = 1$ and/or $M(Tu_0, Tv_0, s_0) = 1$. This is the case when $u = v$ because $M(u, u, s) = 1$ for all $s \in (0, \infty)$. To cover such case it is inevitable to have in mind that the functions $\varphi$ and $\eta$ must be defined for $t = 1$ and, although we will not use it, they must satisfy:

$$\varphi(1) \geq \eta(1).$$

Notice that this condition is compatible with the assumption

$$\eta(s) > \varphi(s) \quad \text{for any } s \in (0, 1)$$

that we will employ in some of the next results.

### 4.2 The family of auxiliary functions

At a first sight, the following conditions could be considered as the corresponding properties for the functions $\varphi, \eta : (0, 1] \to \mathbb{R}$ to the fuzzy setting of hypotheses $(a_1)$-$(a_3)$:

($Þ_1$) $\varphi$ is nondecreasing;
($Þ_2$) $\eta(s) > \varphi(s)$ for any $s \in (0, 1)$;
($Þ_3$) $\liminf_{s\to L^-} \eta(s) > \lim_{s\to L^-} \varphi(s)$ for any $L \in (0, 1)$.

From our point of view, these conditions has three main drawbacks:

- on the one hand, as far as we know, they seems not to be strong enough in order to demonstrate an appropriate fixed point theorem in the framework of fuzzy metric spaces;

- on the other hand, the condition ($Þ_1$) is as much strict that it is relatively easy to find particular functions $\varphi$ and $\eta$ for which the main fixed point theorem is applicable but $\varphi$ is not necessarily nondecreasing (see Example 3);
- finally, condition ($Þ_3$) directly depends on assumption ($Þ_1$) because, in general, if $\varphi$ is not nondecreasing, it is not ensured the existence of the lateral limit $\lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s})$ for arbitrary values of $\mathsf{L} \in (0, 1)$.

Having in mind these disadvantages of the hypotheses of Proinov theorem 4, the main aim of the current section is to introduce a novel family of general auxiliary functions that permit us to prove (existence and uniqueness) fixed point results that can be particularized to the previous axioms ($Þ_1$)-($Þ_3$) by adding an additional fourth assumption. The reader can appreciate the importance of the role of value $\mathsf{t} = 1$ to the following development.

Next we introduce the announced family of auxiliary functions.

**Definition 4** Let $\mathfrak{X}^F$ denote the family of ordered pairs $(\varphi, \eta)$ such that $\varphi, \eta : (0, 1] \to \mathbb{R}$ are functions satisfying the following conditions:

($\mathfrak{X}_1^F$) if $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}} \subset (0, 1]$ is a sequence such that $\varphi(\mathsf{t}_{\ell+1}) \geq \eta(\mathsf{t}_\ell)$ for all $\ell \in \mathbb{N}$, then $\{\mathsf{t}_\ell\} \to 1$;

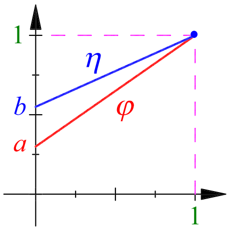
($\mathfrak{X}_2^F$) if $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}}, \{\mathsf{s}_\ell\}_{\ell\in\mathbb{N}} \subset (0, 1]$ are two sequences converging to the same limit $\mathsf{e} \in [0, 1]$ that satisfy $\mathsf{t}_\ell < \mathsf{e}$ and $\varphi(\mathsf{t}_\ell) \geq \eta(\mathsf{s}_\ell)$ for all $\ell \in \mathbb{N}$, then $\mathsf{e} = 1$;

($\mathfrak{X}_3^F$) if $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}}, \{\mathsf{s}_\ell\}_{\ell\in\mathbb{N}} \subset (0, 1]$ are two sequences such that $\{\mathsf{s}_\ell\} \to 1$ and $\varphi(\mathsf{t}_\ell) \geq \eta(\mathsf{s}_\ell)$ for all $\ell \in \mathbb{N}$, then $\{\mathsf{t}_\ell\} \to 1$.

We check that the family $\mathfrak{X}^F$ is nonempty by showing a great subfamily of pairs of functions that satisfy the previous assumptions.

***Example 5*** Given $a, b \in (0, 1)$ such that $a < b$, let $\varphi, \eta : (0, 1] \to \mathbb{R}$ be the functions defined, for each $\mathsf{s} \in (0, 1]$, by:

$$\varphi(\mathsf{s}) = a + (1 - a)\mathsf{s},$$
$$\eta(\mathsf{s}) = b + (1 - b)\mathsf{s}.$$



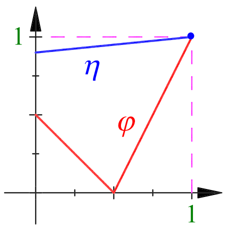

It is easy to directly check[1] that pair $(\varphi, \eta)$ satisfies all properties ($\mathfrak{X}_1^F$)-($\mathfrak{X}_3^F$), so it belongs to $\mathfrak{X}^F$.

In the previous example both functions $\varphi$ and $\eta$ are monotone increasing. Let show a pair $(\varphi, \eta) \in \mathfrak{X}^F$ in which the function $\varphi$ is not nondecreasing.

***Example 6*** Let $\varphi, \eta : (0, 1] \to \mathbb{R}$ given by:

$$\varphi(\mathsf{t}) = \begin{cases} 0.5 - \mathsf{t}, & \text{if } \mathsf{t} \in (0, 0.5], \\ 2\mathsf{t} - 1, & \text{if } \mathsf{t} \in (0.5, 1]; \end{cases}$$

$$\eta(\mathsf{t}) = \frac{9 + \mathsf{t}}{10}.$$




[1] A complete proof of this claim can be found on Appendix, page 25.

Then[2] $(\varphi, \eta) \in \mathfrak{X}^F$ but notice that the function $\varphi$ is strictly decreasing in the interval $(0, 0.5)$.

The reader can easily prove that the pairs given in Example 5 belong to $\mathfrak{X}^F$ by directly checking that they satisfy conditions $(\mathfrak{X}_1^F)$-$(\mathfrak{X}_3^F)$, but it is easier to apply the following result, whose assumptions were introduced in Roldán López de Hierro et al [56].

**Lemma 15** *Let $\varphi, \eta : (0, 1] \to \mathbb{R}$ be functions verifying the following properties:*

$(\text{Þ}_1)$ *$\varphi$ is nondecreasing;*
$(\text{Þ}_2)$ *$\eta(\mathsf{s}) > \varphi(\mathsf{s})$ for any $\mathsf{s} \in (0, 1)$;*
$(\text{Þ}_3)$ *$\liminf_{\mathsf{s}\to\mathsf{L}^-} \eta(\mathsf{s}) > \lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s})$ for any $\mathsf{L} \in (0, 1)$;*
$(\text{Þ}_4)$ *if $\mathsf{t} \in (0, 1]$ is such that $\varphi(\mathsf{t}) \geq \eta(1)$, then $\mathsf{t} = 1$.*

*Then $(\varphi, \eta) \in \mathfrak{X}^F$.*

***Proof*** Before proving the distinct properties $(\mathfrak{X}_1^F)$-$(\mathfrak{X}_3^F)$, we are going to demonstrate that

$$\begin{array}{c}\text{if } \{\mathsf{t}_\ell\}, \{\mathsf{s}_\ell\} \subset (0, 1] \text{ are such that } \varphi(\mathsf{t}_\ell) \geq \eta(\mathsf{s}_\ell) \text{ for all } \ell \in \mathbb{N}, \\ \text{then } \mathsf{t}_\ell \geq \mathsf{s}_\ell \text{ for all } \ell \in \mathbb{N}.\end{array} \tag{11}$$

We consider two cases. Let $\ell \in \mathbb{N}$. First, suppose that $\mathsf{s}_\ell = 1$. Then $\varphi(\mathsf{t}_\ell) \geq \eta(\mathsf{s}_\ell) = \eta(1)$, and property $(\text{Þ}_4)$ leads to $\mathsf{t}_\ell = 1$. Then $\mathsf{t}_\ell = 1 = \mathsf{s}_\ell$. On the contrary case, suppose that $\mathsf{s}_\ell < 1$. Then, using $(\text{Þ}_2)$, it follows that $\varphi(\mathsf{t}_\ell) \geq \eta(\mathsf{s}_\ell) > \varphi(\mathsf{s}_\ell)$ and, as $\varphi$ is nondecreasing, we deduce that $\mathsf{t}_\ell > \mathsf{s}_\ell$. In any case, (11) holds.

Next we study all properties one by one.

$(\mathfrak{X}_1^F)$ Let $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}} \subset (0, 1]$ be a sequence such that $\varphi(\mathsf{t}_{\ell+1}) \geq \eta(\mathsf{t}_\ell)$ for all $\ell \in \mathbb{N}$. Applying (11), we have that $\mathsf{t}_{\ell+1} \geq \mathsf{t}_\ell$ for all $\ell \in \mathbb{N}$. We now consider two cases. First, suppose that there is $\ell_0 \in \mathbb{N}$ such that $\mathsf{t}_{\ell_0} = 1$. As $\{\mathsf{t}_\ell\}$ is nondecreasing, then $\mathsf{t}_\ell = 1$ for all $\ell \geq \ell_0$. Hence, in this case, clearly $\{\mathsf{t}_\ell\} \to 1$.
Next suppose that $\mathsf{t}_\ell < 1$ for all $\ell \in \mathbb{N}$. Then property $(\text{Þ}_3)$ means that

$$\varphi(\mathsf{t}_{\ell+1}) \geq \eta(\mathsf{t}_\ell) > \varphi(\mathsf{t}_\ell), \tag{12}$$

and as $\varphi$ is nondecreasing, then $\mathsf{t}_{\ell+1} > \mathsf{t}_\ell$ for all $\ell \in \mathbb{N}$. Hence $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}}$ is an strictly increasing sequence such that $\mathsf{t}_\ell < \mathsf{t}_{\ell+1} < 1$ for all $\ell \in \mathbb{N}$. Let $\mathsf{L} = \lim_{\ell\to\infty} \mathsf{t}_\ell \in (0, 1]$ be its limit. To prove that $\mathsf{L} = 1$, we proceed by contradiction. Assume that $\mathsf{L} < 1$. As $\varphi$ is nondecreasing and $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}}$ is strictly increasing, then

$$\lim_{\ell\to\infty} \varphi(\mathsf{t}_\ell) = \lim_{\ell\to\infty} \varphi(\mathsf{t}_{\ell+1}) = \lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s}).$$

This limit exists because $\varphi$ is nondecreasing and $\varphi(1)$ is finite. Letting $\ell \to \infty$ in (12), we deduce that

$$\lim_{\ell\to\infty} \eta(\mathsf{t}_\ell) = \lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s}).$$

However, this fact contradicts the assumption $(\text{Þ}_3)$ because

$$\lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s}) = \lim_{\ell\to\infty} \eta(\mathsf{t}_\ell) \geq \liminf_{\mathsf{s}\to\mathsf{L}^-} \eta(\mathsf{s}) > \lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s}).$$

This contradiction leads us to $\mathsf{L} = 1$, so $\{\mathsf{t}_\ell\} \to 1$.

[2] A complete proof of this claim can be found on Appendix, page 26.

($\mathfrak{X}_2^F$) Let $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be two sequences converging to the same limit $e \in [0,1]$ that also satisfy $t_\ell < e$ and $\varphi(t_\ell) \geq \eta(s_\ell)$ for all $\ell \in \mathbb{N}$. Applying (11), we have that $t_\ell \geq s_\ell$ for all $\ell \in \mathbb{N}$, so $1 \geq e > t_\ell \geq s_\ell$ for all $\ell \in \mathbb{N}$. Then property ($\text{Þ}_2$) leads to

$$\varphi(t_\ell) \geq \eta(s_\ell) > \varphi(s_\ell) \quad \text{for all } \ell \in \mathbb{N}. \tag{13}$$

To prove that $e = 1$, we proceed by contradiction. Assume that $e < 1$. As $\varphi$ is nondecreasing and the sequences $\{t_\ell\}_{\ell\in\mathbb{N}}$ and $\{s_\ell\}_{\ell\in\mathbb{N}}$ are convergent to $e$ from the left, then

$$\lim_{\ell\to\infty} \varphi(t_\ell) = \lim_{\ell\to\infty} \varphi(s_\ell) = \lim_{s\to e^-} \varphi(s).$$

This limit exists because $\varphi$ is nondecreasing and $\varphi(1)$ is finite. Letting $\ell \to \infty$ in (13), we deduce that

$$\lim_{\ell\to\infty} \eta(s_\ell) = \lim_{s\to e^-} \varphi(s).$$

However, this fact contradicts the assumption ($\text{Þ}_3$) because

$$\lim_{s\to e^-} \varphi(s) = \lim_{\ell\to\infty} \eta(s_\ell) \geq \liminf_{s\to e^-} \eta(s) > \lim_{s\to e^-} \varphi(s).$$

This contradiction leads us to $e = 1$.

($\mathfrak{X}_3^F$) Let $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be two sequences such that $\{s_\ell\} \to 1$ and $\varphi(t_\ell) \geq \eta(s_\ell)$ for all $\ell \in \mathbb{N}$. Applying (11), we have that $t_\ell \geq s_\ell$ for all $\ell \in \mathbb{N}$. Therefore $s_\ell \leq t_\ell \leq 1$ for all $\ell \in \mathbb{N}$, so $\{t_\ell\} \to 1$. □

### 4.3 Fixed point theory in the setting of fuzzy metric spaces satisfying the property $\mathcal{NC}$

Inspired by the main theorems of Sect. 3 and once we have described the extended family of functions that we will use, next we present the main results in the context of complete fuzzy metric spaces satisfying the property $\mathcal{NC}$.

**Theorem 16** *Let $(X, M, *)$ be an M-complete fuzzy metric space satisfying the property $\mathcal{NC}$ and let $T : X \to X$ be a mapping for which there exists $(\varphi, \eta) \in \mathfrak{X}^F$ such that*

$$\varphi(M(Tu, Tv, s)) \geq \eta(M(u, v, s)) \quad \textit{for all } u, v \in X \textit{ with } Tu \neq Tv \textit{ and all } s > 0. \tag{14}$$

*Then each iterative Picard sequence $\{T^\ell u\}_{\ell\in\mathbb{N}}$ converges to a fixed point $v_0 \in X$ of T for every $u \in X$. In particular, T has at least one fixed point.*

***Proof*** Let $\{u_\ell\}_{\ell\in\mathbb{N}}$ be the Picard sequence of T starting from an arbitrary initial point $u_1 \in X$. If there is $\ell_0 \in \mathbb{N}$ such that $u_{\ell_0} = u_{\ell_0+1}$, then $u_{\ell_0}$ is a fixed point of T. In this case, the first part of the proof is finished. Next suppose that $u_\ell \neq u_{\ell+1}$ for all $\ell \in \mathbb{N}$. Hence

$$M(u_\ell, u_{\ell+1}, s) > 0 \quad \text{for all } \ell \in \mathbb{N}.$$

Applying the contractivity condition (14), we deduce that, for all $\ell \in \mathbb{N}$ and all $s > 0$,

$$\varphi(M(u_{\ell+1}, u_{\ell+2}, s)) = \varphi(M(Tu_\ell, Tu_{\ell+1}, s)) \geq \eta(M(u_\ell, u_{\ell+1}, s)).$$

Given $s > 0$, if we define $s_\ell = M(u_\ell, u_{\ell+1}, s)$ for all $\ell \in \mathbb{N}$, the previous inequality means that the sequence $\{s_\ell\}$ satisfies $\varphi(s_{\ell+1}) \geq \eta(s_\ell)$ for all $\ell \in \mathbb{N}$. Property ($\mathfrak{X}_1^F$) guarantees that

$$\lim_{\ell\to\infty} M(u_\ell, u_{\ell+1}, s) = 1 \quad \text{for all } s > 0.$$

Furthermore, Corollary 8 guarantees that each two terms of the Picard sequence $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ are distinct, that is, $\mathsf{u}_{\ell_1} \neq \mathsf{u}_{\ell_2}$ for any $\ell_1, \ell_2 \in \mathbb{N}$ such that $\ell_1 \neq \ell_2$. In order to prove that $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ is a M-Cauchy sequence, we reason by contradiction. Suppose that $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ is not M-Cauchy. Since $(\mathsf{X}, \mathsf{M})$ satisfies the property $\mathcal{NC}$, there are $\mathsf{e}_0 \in (0, 1)$, $\mathsf{t}_0 > 0$ and two partial subsequences $\{\mathsf{u}_{p(\ell)}\}_{\ell\in\mathbb{N}}$ and $\{\mathsf{u}_{q(\ell)}\}_{\ell\in\mathbb{N}}$ of $\{\mathsf{u}_\ell\}$ such that, for all $\ell \in \mathbb{N}$,

$$\begin{aligned} &\ell < p(\ell) < q(\ell) < p(\ell+1) \qquad \text{and} \\ &\mathsf{M}(\mathsf{u}_{p(\ell)}, \mathsf{u}_{q(\ell)-1}, \mathsf{t}_0) > 1 - \mathsf{e}_0 \geq \mathsf{M}(\mathsf{u}_{p(\ell)}, \mathsf{u}_{q(\ell)}, \mathsf{t}_0), \end{aligned} \tag{15}$$

and also

$$\lim_{\ell\to\infty} \mathsf{M}(\mathsf{u}_{p(\ell)}, \mathsf{u}_{q(\ell)}, \mathsf{t}_0) = \lim_{\ell\to\infty} \mathsf{M}(\mathsf{u}_{p(\ell)-1}, \mathsf{u}_{q(\ell)-1}, \mathsf{t}_0) = 1 - \mathsf{e}_0. \tag{16}$$

Applying the contractivity condition (14), we deduce that, for all $\ell \in \mathbb{N}$,

$$\varphi(\mathsf{M}(\mathsf{u}_{p(\ell)}, \mathsf{u}_{q(\ell)}, \mathsf{t}_0)) = \varphi(\mathsf{M}(\mathsf{T}\mathsf{u}_{p(\ell)-1}, \mathsf{T}\mathsf{u}_{q(\ell)-1}, \mathsf{t}_0)) \geq \eta(\,\mathsf{M}(\mathsf{u}_{p(\ell)-1}, \mathsf{u}_{q(\ell)-1}, \mathsf{t}_0)\,).$$

If we define $\mathsf{e}_0' = 1 - \mathsf{e}_0 \in (0, 1)$, and also $\mathsf{t}_\ell = \mathsf{M}(\mathsf{u}_{p(\ell)}, \mathsf{u}_{q(\ell)}, \mathsf{t}_0)$ and $\mathsf{s}_\ell = \mathsf{M}(\mathsf{u}_{p(\ell)-1}, \mathsf{u}_{q(\ell)-1}, \mathsf{t}_0)$ for all $\ell \in \mathbb{N}$, then, by (16), the sequences $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}}, \{\mathsf{s}_\ell\}_{\ell\in\mathbb{N}} \subset (0, 1]$ converge, at the same time, to $\mathsf{e}_0' \in (0, 1)$ and they satisfy $\mathsf{t}_\ell > \mathsf{e}_0'$ and $\varphi(\mathsf{t}_\ell) \geq \eta(\mathsf{s}_\ell)$ for all $\ell \in \mathbb{N}$. As a consequence, assumption $(\mathfrak{X}_2^F)$ ensures that $\mathsf{e}_0' = 1$ which is a contradiction because $\mathsf{e}_0 > 0$ and $\mathsf{e}_0' = 1 - \mathsf{e}_0 < 1$.

This contradiction guarantees that $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ is a M-Cauchy sequence. As the fuzzy space $(\mathsf{X}, \mathsf{M})$ is M-complete, there is $\mathsf{v} \in \mathsf{X}$ such that $\{\mathsf{u}_\ell\}$ M-converges to $\mathsf{v}$, that is,

$$\lim_{\ell\to\infty} \mathsf{M}(\mathsf{u}_\ell, \mathsf{v}, \mathsf{t}) = 1 \quad \text{for all } \mathsf{t} > 0.$$

Next, we check that $\mathsf{v} \in \mathrm{Fix}(\mathsf{T})$. To prove it, assume, by contradiction, that $\mathsf{v} \in \mathsf{X} \setminus \mathrm{Fix}(\mathsf{T})$, that is, $\mathsf{v} \neq \mathsf{T}\mathsf{v}$. As the sequence $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ is infinite, then there is $\ell_0 \in \mathbb{N}$ such that $\mathsf{u}_\ell \neq \mathsf{v}$ and $\mathsf{u}_\ell \neq \mathsf{T}\mathsf{v}$ for all $\ell \geq \ell_0$. Using the contractivity condition (14), we deduce that, for all $\ell \geq \ell_0$ and all $\mathsf{t} > 0$,

$$\varphi(\mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{T}\mathsf{v}, \mathsf{t})) = \varphi(\mathsf{M}(\mathsf{T}\mathsf{u}_\ell, \mathsf{T}\mathsf{v}, \mathsf{t})) \geq \eta(\mathsf{M}(\mathsf{u}_\ell, \mathsf{v}, \mathsf{t}))\,.$$

Condition $(\mathfrak{X}_3^F)$ applied to the sequences

$$\{\,\mathsf{t}_\ell = \mathsf{M}(\mathsf{u}_{\ell+1}, \mathsf{T}\mathsf{v}, \mathsf{t})\,\}_{\ell\in\mathbb{N}} \quad \text{and} \quad \{\,\mathsf{s}_\ell = \mathsf{M}(\mathsf{u}_\ell, \mathsf{v}, \mathsf{t})\,\}_{\ell\in\mathbb{N}}$$

leads to $\{\mathsf{t}_\ell\} \to 1$, that is,

$$\lim_{\ell\to\infty} \mathsf{M}(\mathsf{u}_\ell, \mathsf{T}\mathsf{v}, \mathsf{t}) = 1 \quad \text{for all } \mathsf{t} > 0,$$

which means that $\{\mathsf{u}_\ell\}_{\ell\in\mathbb{N}}$ M-converges to $\mathsf{T}\mathsf{v}$, and the uniqueness of the limit of a convergent sequence in a fuzzy metric space concludes that $\mathsf{v}$ is a fixed point of $\mathsf{T}$. □

**Theorem 17** *Under the hypotheses of Theorem 16, assume that the pair* $(\varphi, \eta)$ *additionally satisfies:*

$(\mathfrak{X}_4^F)$ *for each two distinct fixed points* $\mathsf{u}_0, \mathsf{v}_0 \in \mathrm{Fix}(\mathsf{T})$ *there is* $\mathsf{s}_0 \in (0, \infty)$ *such that* $\eta(\mathsf{M}(\mathsf{u}_0, \mathsf{v}_0, \mathsf{s}_0)) > \varphi(\mathsf{M}(\mathsf{u}_0, \mathsf{v}_0, \mathsf{s}_0))$.

*Then the mapping* $\mathsf{T}$ *has a unique fixed point and the Picard sequence* $\{\mathsf{T}^\ell\mathsf{u}\}_{\ell\in\mathbb{N}}$ *converges to such fixed point whatever the initial point* $\mathsf{u} \in \mathsf{X}$.

***Proof*** If $u_0$ and $v_0$ are two distinct fixed points of T, then, for all $t > 0$,

$$\varphi(M(u_0, v_0, t)) = \varphi(M(Tu_0, Tv_0, t)) \geq \eta(M(u_0, v_0, t)),$$

which contradicts $(\mathfrak{X}_4^F)$. Hence, T can only have a unique fixed point. □

**Definition 5** In order to refer to the previous class of mappings, we will say that $T : X \to X$ is a *fuzzy $\mathfrak{X}^F$-contraction* on the fuzzy metric space $(X, M, *)$ if there exists $(\varphi, \eta) \in \mathfrak{X}^F$ such that

$$\varphi(M(Tu, Tv, s)) \geq \eta(M(u, v, s)) \quad \text{for all } u, v \in X \text{ with } Tu \neq Tv \text{ and all } s > 0.$$

Similarly, we will say that $T : X \to X$ is a $\mathfrak{X}$-*contraction* on the metric space $(X, d)$ if there exists $(\psi, \phi) \in \mathfrak{X}$ such that

$$\psi(d(Tu, Tv)) \leq \phi(d(u, v)) \quad \text{for all } u, v \in X \text{ with } d(Tu, Tv) > 0.$$

Under this nomenclature, Theorem 16 states that each fuzzy $\mathfrak{X}^F$-contraction defined on an M-complete fuzzy metric space satisfying the property $\mathcal{NC}$ has at least one fixed point, which is unique if the auxiliary pair $(\varphi, \eta)$ also satisfies the property $(\mathfrak{X}_4^F)$ (recall Theorem 17). In a similar way, each $\mathfrak{X}$-contraction defined on a complete metric space has at least one fixed point, which is unique when the property $(\mathfrak{X}_4)$ holds (see Theorems 10 and 11). In both cases, each iterative Picard sequence converges to a fixed point of the operator.

## 5 Consequences

This section is devoted to illustrate the powerful of the main results introduced in the previous two sections. We hope that the wide variety of consequences that we will present will definitively convince the reader of the enormous power and applicability of the results introduced throughout this manuscript.

### 5.1 A Proinov fuzzy theorem without a monotone condition

In order to avoid the condition about the nondecreasingness of the function $\varphi$, the following result can also be helpful. Notice that here the function $\varphi$ is not necessarily nondecreasing.

**Lemma 18** *Let $\varphi, \eta : (0, 1] \to \mathbb{R}$ be functions verifying the following properties:*

$(\beta_1)$ *if $t, s \in (0, 1]$ are such that $\varphi(t) \geq \eta(s)$, then either $t = s = 1$ or $t > s$;*
$(\beta_2)$ *for each $L \in (0, 1)$ there exists the lateral limit $\lim_{s \to L^-} \varphi(s) \in \mathbb{R}$;*
$(\beta_3)$ $\liminf_{s \to L^-} \eta(s) > \lim_{s \to L^-} \varphi(s)$ *for any $L \in (0, 1)$.*

Then $(\varphi, \eta) \in \mathfrak{X}^F$.

***Proof*** We check all properties.

$(\mathfrak{X}_1^F)$ Let $\{t_\ell\}_{\ell \in \mathbb{N}} \subset (0, 1]$ be a sequence such that $\varphi(t_{\ell+1}) \geq \eta(t_\ell)$ for all $\ell \in \mathbb{N}$. Applying $(\beta_1)$, we have that either $t_{\ell+1} = t_\ell = 1$ or $t_{\ell+1} > t_\ell$ for all $\ell \in \mathbb{N}$. In any case, $t_{\ell+1} \geq t_\ell$ for all $\ell \in \mathbb{N}$. If there is $\ell_0 \in \mathbb{N}$ such that $t_{\ell_0} = 1$, then necessarily $t_\ell = 1$ for all $\ell \geq \ell_0$, so $\{t_\ell\} \to 1$. Next suppose that $t_\ell < 1$ for all $\ell \in \mathbb{N}$. In such a case,

$0 < \mathsf{t}_\ell < \mathsf{t}_{\ell+1} < \mathsf{L} \leq 1$ for all $\ell \in \mathbb{N}$, where $\mathsf{L} = \lim_{\ell\to\infty} \mathsf{t}_\ell$. To prove that $\mathsf{L} = 1$, we proceed by contradiction. Assume that $\mathsf{L} < 1$. In such a case, using (ß$_2$) and (ß$_3$),

$$\lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s}) = \lim_{\ell\to\infty} \varphi(\mathsf{t}_{\ell+1}) \geq \liminf_{\ell\to\infty} \eta(\mathsf{t}_\ell) \geq \liminf_{\mathsf{s}\to\mathsf{L}^-} \eta(\mathsf{s}) > \lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s}),$$

which is a contradiction. Then $\mathsf{L} = 1$ and $\{\mathsf{t}_\ell\} \to 1$.

($\mathfrak{X}_2^F$) Let $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}}, \{\mathsf{s}_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be two sequences converging to the same limit $\mathsf{L} \in [0,1]$ that also satisfy $\mathsf{t}_\ell < \mathsf{L}$ and $\varphi(\mathsf{t}_\ell) \geq \eta(\mathsf{s}_\ell)$ for all $\ell \in \mathbb{N}$. Applying (ß$_1$), we deduce that either $\mathsf{t}_\ell = \mathsf{s}_\ell = 1$ or $\mathsf{t}_\ell > \mathsf{s}_\ell$ for all $\ell \in \mathbb{N}$. In any case, $\mathsf{t}_\ell \geq \mathsf{s}_\ell$ for all $\ell \in \mathbb{N}$, so $1 \geq \mathsf{L} > \mathsf{t}_\ell \geq \mathsf{s}_\ell$ for all $\ell \in \mathbb{N}$. To prove that $\mathsf{L} = 1$, we proceed by contradiction. Assume that $\mathsf{L} < 1$. In such a case, using (ß$_2$) and (ß$_3$),

$$\lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s}) = \lim_{\ell\to\infty} \varphi(\mathsf{t}_\ell) \geq \liminf_{\ell\to\infty} \eta(\mathsf{t}_\ell) \geq \liminf_{\mathsf{s}\to\mathsf{L}^-} \eta(\mathsf{s}) > \lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s}).$$

This contradiction leads us to $\mathsf{L} = 1$.

($\mathfrak{X}_3^F$) Let $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}}, \{\mathsf{s}_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be two sequences such that $\{\mathsf{s}_\ell\} \to 1$ and $\varphi(\mathsf{t}_\ell) \geq \eta(\mathsf{s}_\ell)$ for all $\ell \in \mathbb{N}$. Reasoning as before, we deduce that $\mathsf{t}_\ell \geq \mathsf{s}_\ell$ for all $\ell \in \mathbb{N}$. Therefore $\mathsf{s}_\ell \leq \mathsf{t}_\ell \leq 1$ for all $\ell \in \mathbb{N}$, so $\{\mathsf{t}_\ell\} \to 1$. □

***Remark 2*** Notice that condition (ß$_1$) leads to $\eta(\mathsf{s}) > \varphi(\mathsf{s})$ for any $\mathsf{s} \in (0,1)$, that is, (ß$_1$) implies (Þ$_2$). However, in Lemma 18, the function $\varphi$ is not necessarily nondecreasing. This can be a great advantage in many results, as the following one.

Notice that the following one is a Proinov-type fuzzy theorem without a monotone condition on $\varphi$.

**Corollary 19** *Let* $(\mathsf{X}, \mathsf{M}, *)$ *be an* $\mathsf{M}$*-complete fuzzy metric space satisfying the property* $\mathcal{NC}$ *and let* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *be a mapping for which there exists two functions* $\varphi, \eta : (0,1] \to \mathbb{R}$ *satisfying the following properties:*

($\beta_1$) *if* $\mathsf{t}, \mathsf{s} \in (0,1]$ *are such that* $\varphi(\mathsf{t}) \geq \eta(\mathsf{s})$*, then either* $\mathsf{t} = \mathsf{s} = 1$ *or* $\mathsf{t} > \mathsf{s}$*;*

($\beta_2$) *for each* $\mathsf{L} \in (0,1)$ *there exists the lateral limit* $\lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s}) \in \mathbb{R}$*;*

($\beta_3$) $\liminf_{\mathsf{s}\to\mathsf{L}^-} \eta(\mathsf{s}) > \lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s})$ *for any* $\mathsf{L} \in (0,1)$.

*Additionally, assume that*

$$\varphi(\mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s})) \geq \eta(\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})) \quad \textit{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \textit{ with } \mathsf{Tu} \neq \mathsf{Tv} \textit{ and all } \mathsf{s} > 0.$$

*Then the mapping* $\mathsf{T}$ *has a unique fixed point and the Picard sequence* $\{\mathsf{T}^\ell \mathsf{u}\}_{\ell\in\mathbb{N}}$ *converges to such fixed point whatever the initial point* $\mathsf{u} \in \mathsf{X}$.

***Proof*** It follows from Theorem 16 taking into account that $(\varphi, \eta) \in \mathfrak{X}$ by Lemma 18. Furthermore, condition ($\mathfrak{X}_4^F$) also holds because (ß$_1$) implies (Þ$_2$) (see Remark 2), so Theorem 17 is also applicable. □

## 5.2 Consequences in non-Archimedean fuzzy metric spaces

As we have commented in the introduction of Sect. 4, each non-Archimedean fuzzy metric space satisfies the property $\mathcal{NC}$. Therefore, the following result is a consequence of Theorems 16 and 17 applied to any non-Archimedean fuzzy metric space.

**Corollary 20** *Let* $(\mathsf{X}, \mathsf{M}, *)$ *be an* $\mathsf{M}$*-complete non-Archimedean fuzzy metric space and let* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *be a fuzzy* $\mathfrak{X}^F$*-contraction. Then each iterative Picard sequence* $\{\mathsf{T}^\ell \mathsf{u}\}_{\ell \in \mathbb{N}}$ *converges to a fixed point* $\mathsf{v}_0 \in \mathsf{X}$ *of* $\mathsf{T}$ *for every* $\mathsf{u} \in \mathsf{X}$*. In particular,* $\mathsf{T}$ *has at least one fixed point.*

*In addition to this, if for each two distinct fixed points* $\mathsf{u}_0, \mathsf{v}_0 \in \mathrm{Fix}(\mathsf{T})$ *there is* $\mathsf{s}_0 \in (0, \infty)$ *such that* $\eta(\mathsf{M}(\mathsf{u}_0, \mathsf{v}_0, \mathsf{s}_0)) > \varphi(\mathsf{M}(\mathsf{u}_0, \mathsf{v}_0, \mathsf{s}_0))$*, then* $\mathsf{T}$ *has a unique fixed point.*

Let us show how to apply the previous statement. It is known (see [15]) that if $(\mathsf{X}, \mathsf{d})$ is a metric space, then $(\mathsf{X}, \mathsf{M}^{\mathsf{d}})$, defined as in (3), is a non-Archimedean fuzzy metric space under any t-norm $*$ such that $\mathsf{t} * \mathsf{s} \leq \mathsf{t}\,\mathsf{s}$ for all $\mathsf{t}, \mathsf{s} \in \mathbb{I}$. then, the following result can be deduced.

**Corollary 21** *Let* $(\mathsf{X}, \mathsf{d})$ *be a complete metric space and let* $*$ *be a t-norm such that* $\mathsf{t} * \mathsf{s} \leq \mathsf{t}\,\mathsf{s}$ *for all* $\mathsf{t}, \mathsf{s} \in \mathbb{I}$*. Suppose that* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *is a mapping for which there exists* $(\varphi, \eta) \in \mathfrak{X}^F$ *such that*

$$\varphi\left(\frac{\mathsf{s}}{\mathsf{s} + \mathsf{d}(\mathsf{Tu}, \mathsf{Tv})}\right) \geq \eta\left(\frac{\mathsf{s}}{\mathsf{s} + \mathsf{d}(\mathsf{u}, \mathsf{v})}\right) \quad \textit{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \textit{ with } \mathsf{Tu} \neq \mathsf{Tv} \textit{ and all } \mathsf{s} > 0.$$

*Then each iterative Picard sequence* $\{\mathsf{T}^\ell \mathsf{u}\}_{\ell \in \mathbb{N}}$ *converges to a fixed point* $\mathsf{v}_0 \in \mathsf{X}$ *of* $\mathsf{T}$ *for every* $\mathsf{u} \in \mathsf{X}$*. In particular,* $\mathsf{T}$ *has at least one fixed point.*

*In addition to this, if for each two distinct fixed points* $\mathsf{u}_0, \mathsf{v}_0 \in \mathrm{Fix}(\mathsf{T})$ *there is* $\mathsf{s}_0 \in (0, \infty)$ *such that*

$$\eta\left(\frac{\mathsf{s}_0}{\mathsf{s}_0 + \mathsf{d}(\mathsf{u}_0, \mathsf{v}_0)}\right) > \varphi\left(\frac{\mathsf{s}_0}{\mathsf{s}_0 + \mathsf{d}(\mathsf{u}_0, \mathsf{v}_0)}\right),$$

*then* $\mathsf{T}$ *has a unique fixed point.*

The following result is a similar consequence.

**Corollary 22** *Let* $(\mathsf{X}, \mathsf{d})$ *be a complete metric space and let* $*$ *be a t-norm such that* $\mathsf{t} * \mathsf{s} \leq \mathsf{t}\,\mathsf{s}$ *for all* $\mathsf{t}, \mathsf{s} \in \mathbb{I}$*. Let* $\vartheta : (0, \infty) \to (0, 1)$ *be a nondecreasing and continuous function such that* $\lim_{t \to \infty} \vartheta(t) = 1$*. Suppose that* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *is a mapping for which there exists* $(\varphi, \eta) \in \mathfrak{X}^F$ *such that*

$$\varphi\left([\vartheta(\mathsf{s})]^{\mathsf{d}(\mathsf{Tu}, \mathsf{Tv})}\right) \geq \eta\left([\vartheta(\mathsf{s})]^{\mathsf{d}(\mathsf{u}, \mathsf{v})}\right) \quad \textit{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \textit{ with } \mathsf{Tu} \neq \mathsf{Tv} \textit{ and all } \mathsf{s} > 0.$$

*Then each iterative Picard sequence* $\{\mathsf{T}^\ell \mathsf{u}\}_{\ell \in \mathbb{N}}$ *converges to a fixed point* $\mathsf{v}_0 \in \mathsf{X}$ *of* $\mathsf{T}$ *for every* $\mathsf{u} \in \mathsf{X}$*. In particular,* $\mathsf{T}$ *has at least one fixed point.*

*In addition to this, if for each two distinct fixed points* $\mathsf{u}_0, \mathsf{v}_0 \in \mathrm{Fix}(\mathsf{T})$ *there is* $\mathsf{s}_0 \in (0, \infty)$ *such that*

$$\eta\left([\vartheta(\mathsf{s}_0)]^{\mathsf{d}(\mathsf{u}_0, \mathsf{v}_0)}\right) > \varphi\left([\vartheta(\mathsf{s}_0)]^{\mathsf{d}(\mathsf{Tu}_0, \mathsf{Tv}_0)}\right),$$

*then* $\mathsf{T}$ *has a unique fixed point.*

***Proof*** It follows from the fact that Altun and Miheţ [55, Example 1.3] proved that, under the previous hypotheses, if we define:

$$\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s}) = [\vartheta(\mathsf{s})]^{\mathsf{d}(\mathsf{u}, \mathsf{v})} \quad \text{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \text{ and all } \mathsf{s} > 0,$$

then $(\mathsf{X}, \mathsf{M}, *)$ is a non-Archimedean fuzzy metric space. □

### 5.3 Consequences for $\mathfrak{X}$-contractions and $\mathfrak{X}^F$-contractions under a kind of lateral continuity of the auxiliary functions

It is usual to consider auxiliary functions that are continuous or, at least, continuous from a side. In such a case, some constraints can be removed form the hypotheses of the main results because they can be deduced from the continuity. Under this additional assumption, we can deduce the following consequences in metric and fuzzy metric spaces. We start this subsection describing a result in metric spaces.

**Corollary 23** *Let* $(\mathsf{X}, \mathsf{d})$ *be a complete metric space and let* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *be a mapping for which there exists two continuous from the right functions* $\psi, \phi : (0, \infty) \to \mathbb{R}$ *satisfying the following property:*

$(\mathfrak{a}_1)$ *if* $\mathsf{t}, \mathsf{s} \in (0, \infty)$ *are such that* $\psi(\mathsf{t}) \leq \phi(\mathsf{s})$, *then* $\mathsf{t} < \mathsf{s}$.

*Additionally, assume that*

$$\psi(\mathsf{d}(\mathsf{Tu}, \mathsf{Tv})) \leq \phi(\mathsf{d}(\mathsf{u}, \mathsf{v})) \quad \textit{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \textit{ with } \mathsf{Tu} \neq \mathsf{Tv}.$$

*Then the mapping* $\mathsf{T}$ *has a unique fixed point and the Picard sequence* $\{\mathsf{T}^{\ell}\mathsf{u}\}_{\ell \in \mathbb{N}}$ *converges to such fixed point whatever the initial point* $\mathsf{u} \in \mathsf{X}$.

***Proof*** We show that, under the hypotheses of this result, conditions $(\mathfrak{a}_2)$ and $(a_3)$ in Corollary 14 holds. On the one hand, as $\psi$ is continuous from the right on $(0, \infty)$, then

$$\lim_{\mathsf{s} \to \mathsf{e}^+} \psi(\mathsf{s}) = \psi(\mathsf{e}) \in \mathbb{R} \quad \text{for all } \mathsf{e} > 0.$$

Hence, conditions $(\mathfrak{a}_2)$ holds. On the other hand, condition $(\mathfrak{a}_1)$ implies that $\psi(\mathsf{s}) > \phi(\mathsf{s})$ for all $\mathsf{s} \in (0, \infty)$. As a consequence, as the functions $\psi$ and $\phi$ are continuous from the right, then, for each $\mathsf{e} > 0$,

$$\limsup_{\mathsf{s} \to \mathsf{e}^+} \phi(\mathsf{s}) = \phi(\mathsf{e}) < \psi(\mathsf{e}) = \lim_{\mathsf{s} \to \mathsf{e}^+} \psi(\mathsf{s}).$$

This means that condition $(a_3)$ in Corollary 14 also holds. Therefore, such result is applicable. □

A similar reasoning can be developed in the setting of fuzzy metric spaces.

**Corollary 24** *Let* $(\mathsf{X}, \mathsf{M}, *)$ *be an* $\mathsf{M}$*-complete fuzzy metric space satisfying the property* $\mathcal{NC}$ *and let* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *be a mapping for which there exists two continuous from the left functions* $\varphi, \eta : (0, 1] \to \mathbb{R}$ *satisfying the following property:*

$(\beta_1)$ *if* $\mathsf{t}, \mathsf{s} \in (0, 1]$ *are such that* $\varphi(\mathsf{t}) \geq \eta(\mathsf{s})$, *then either* $\mathsf{t} = \mathsf{s} = 1$ *or* $\mathsf{t} > \mathsf{s}$.

*Additionally, assume that*

$$\varphi(\mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s})) \geq \eta(\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})) \quad \textit{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \textit{ with } \mathsf{Tu} \neq \mathsf{Tv} \textit{ and all } \mathsf{s} > 0.$$

*Then the mapping* $\mathsf{T}$ *has a unique fixed point and the Picard sequence* $\{\mathsf{T}^{\ell}\mathsf{u}\}_{\ell \in \mathbb{N}}$ *converges to such fixed point whatever the initial point* $\mathsf{u} \in \mathsf{X}$.

***Proof*** We prove that, under the hypotheses of this result, conditions $(\mathfrak{ß}_2)$ and $(\mathfrak{ß}_3)$ in Corollary 19 holds. On the one hand, as $\varphi$ is continuous from the left on $(0, 1)$, then

$$\lim_{\mathsf{s} \to \mathsf{L}^-} \varphi(\mathsf{s}) = \varphi(\mathsf{L}) \in \mathbb{R} \quad \text{for all } \mathsf{L} \in (0, 1).$$

Hence, conditions ($\beta_2$) holds. On the other hand, the assumption ($\beta_1$) implies that $\varphi(\mathsf{s}) < \eta(\mathsf{s})$ for all $\mathsf{s} \in (0,1)$. As a consequence, as the functions $\varphi$ and $\eta$ are continuous from the left, then, for each $\mathsf{L} > 0$,

$$\liminf_{\mathsf{s}\to\mathsf{L}^-} \eta(\mathsf{s}) = \eta(\mathsf{L}) > \varphi(\mathsf{L}) = \lim_{\mathsf{s}\to\mathsf{L}^-} \varphi(\mathsf{s}).$$

This means that condition ($\beta_3$) in Corollary 19 also holds. Therefore, such result is applicable. □

Obviously, Corollaries 23 and 24 also hold when the auxiliary functions $\psi$ and $\phi$ are continuous.

### 5.4 Miheţ-type fuzzy $\mathfrak{X}^F$-contractions

In the setting of fuzzy metric spaces in the sense of Kramosil and Michálek, Miheţ [57] studied the contractivity condition:

$$\mathsf{M}(\mathsf{u},\mathsf{v},\mathsf{t}) > 0 \quad \Rightarrow \quad \mathsf{M}(\mathsf{Tu},\mathsf{Tv},\mathsf{t}) \geq \psi\left(\mathsf{M}(\mathsf{u},\mathsf{v},\mathsf{t})\right), \tag{17}$$

where $\psi : [0,1] \to [0,1]$ belonged to the class $\Psi$ of all continuous and nondecreasing functions $\psi : [0,1] \to [0,1]$ satisfying $\psi(\mathsf{t}) > \mathsf{t}$ for all $\mathsf{t} \in (0,1)$. Notice that if $\psi \in \Psi$, then $\psi(0) \geq 0$ and $\psi(1) = 1$, so $\psi(\mathsf{t}) \geq \mathsf{t}$ for all $\mathsf{t} \in \mathbb{I}$. In particular, he proved that if $(\mathsf{X},\mathsf{M},*)$ is an M-complete non-Archimedean fuzzy metric spaces in the sense of Kramosil and Michálek (it is assumed that $*$ is continuous) and $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ is a fuzzy $\psi$-contractive mapping –which means that it satisfies the condition (17)–, then T has a fixed point provided that there exists $\mathsf{u} \in \mathsf{X}$ such that $\mathsf{M}(\mathsf{u},\mathsf{Tu},\mathsf{t}) > 0$ for all $\mathsf{t} > 0$. This kind of fuzzy contractions attracted much attention in the field of fixed point theory. Here, we do not want to compare his study with our main results because both of them are placed on distinct classes of fuzzy metric spaces. However, we are going to deduce a Miheţ-type result in the setting of fuzzy metric spaces in the sense of George and Veeramani.

**Lemma 25** *If $\psi \in \Psi$ is a Miheţ's auxiliary function (that is, $\psi : [0,1] \to [0,1]$ is continuous, nondecreasing and it satisfies $\psi(\mathsf{t}) > \mathsf{t}$ for all $\mathsf{t} \in (0,1)$) and we define $\varphi$ as the identity mapping on $(0,1]$ and $\eta$ as the restriction of $\psi$ to the interval $(0,1]$, then $(\varphi,\eta) \in \mathfrak{X}^F$.*

***Proof*** We study each property.

($\mathfrak{X}_1^F$) Let $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be a sequence such that $\varphi(\mathsf{t}_{\ell+1}) \geq \eta(\mathsf{t}_\ell)$ for all $\ell \in \mathbb{N}$. Notice that $\mathsf{t}_\ell > 0$ for all $\ell \in \mathbb{N}$. Furthermore,

$$\mathsf{t}_{\ell+1} = \varphi(\mathsf{t}_{\ell+1}) \geq \eta(\mathsf{t}_\ell) = \psi(\mathsf{t}_\ell) \geq \mathsf{t}_\ell \quad \text{for all } \ell \in \mathbb{N}.$$

Hence $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}}$ is a nondecreasing sequence. Let $\mathsf{e} \in (0,1]$ be its limit. Since $\mathsf{t}_{\ell+1} \geq \psi(\mathsf{t}_\ell)$ for all $\ell \in \mathbb{N}$ and $\psi$ is continuous, then $\mathsf{e} \geq \psi(\mathsf{e})$. If $\mathsf{e} < 1$, then $\psi(\mathsf{e}) > \mathsf{e}$. Hence, necessarily $\mathsf{e} = 1$ and $\{\mathsf{t}_\ell\} \to 1$.

($\mathfrak{X}_2^F$) Let $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}}, \{\mathsf{s}_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be two sequences converging to the same limit $\mathsf{e} \in [0,1]$ that satisfy $\mathsf{t}_\ell < \mathsf{e}$ and $\varphi(\mathsf{t}_\ell) \geq \eta(\mathsf{s}_\ell)$ for all $\ell \in \mathbb{N}$. Notice that $\mathsf{e} > 0$ because $\mathsf{e} > \mathsf{t}_\ell > 0$. Furthermore,

$$\mathsf{t}_\ell = \varphi(\mathsf{t}_\ell) \geq \eta(\mathsf{s}_\ell) = \psi(\mathsf{s}_\ell) \geq \mathsf{s}_\ell \quad \text{for all } \ell \in \mathbb{N}. \tag{18}$$

As $\psi$ is continuous, then $\mathsf{e} \geq \psi(\mathsf{e}) \geq \mathsf{e}$, so $\psi(\mathsf{e}) = \mathsf{e}$. However, this is only possible when $\mathsf{e} = 1$ because $\psi(\mathsf{t}) > \mathsf{t}$ for all $\mathsf{t} \in (0,1)$.

($\mathfrak{X}_3^F$) Let $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}}, \{\mathsf{s}_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be two sequences such that $\{\mathsf{s}_\ell\} \to 1$ and $\varphi(\mathsf{t}_\ell) \geq \eta(\mathsf{s}_\ell)$ for all $\ell \in \mathbb{N}$. Reasoning as in (18), we deduce that $1 \geq \mathsf{t}_\ell \geq \mathsf{s}_\ell$ for all $\ell \in \mathbb{N}$. Then $\{\mathsf{t}_\ell\} \to 1$. □

The previous lemma permit us to derive the following version of Miheţ's theorem.

**Corollary 26** *Let* $(\mathsf{X}, \mathsf{M}, *)$ *be an* $\mathsf{M}$*-complete fuzzy metric space (in the sense of George and Veeramani) satisfying the property* $\mathcal{NC}$ *and let* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *be a mapping for which there exists a Miheţ's auxiliary function* $\psi \in \Psi$ *(that is,* $\psi : [0,1] \to [0,1]$ *is continuous, nondecreasing and it satisfies* $\psi(\mathsf{t}) > \mathsf{t}$ *for all* $\mathsf{t} \in (0,1)$*) such that*

$$\mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) \geq \psi(\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})) \quad \textit{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \textit{ with } \mathsf{Tu} \neq \mathsf{Tv} \textit{ and all } \mathsf{s} > 0.$$

*Then the mapping* $\mathsf{T}$ *has a unique fixed point and the Picard sequence* $\{\mathsf{T}^\ell \mathsf{u}\}_{\ell\in\mathbb{N}}$ *converges to such fixed point whatever the initial point* $\mathsf{u} \in \mathsf{X}$.

***Proof*** Let define $\varphi$ as the identity mapping on $(0,1]$ and $\eta$ as the restriction of $\psi$ to the interval $(0,1]$. By Lemma 25, $(\varphi, \eta) \in \mathfrak{X}^F$ and the mapping $\mathsf{T}$ satisfies:

$$\varphi(\mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s})) = \mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s}) \geq \psi(\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})) = \eta(\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s}))$$

for all $\mathsf{u}, \mathsf{v} \in \mathsf{X}$ with $\mathsf{Tu} \neq \mathsf{Tv}$ and all $\mathsf{s} > 0$. Hence Theorem 16 guarantees that each iterative Picard sequence $\{\mathsf{T}^\ell \mathsf{u}\}_{\ell\in\mathbb{N}}$ converges to a fixed point $\mathsf{v}_0 \in \mathsf{X}$ of $\mathsf{T}$ for every $\mathsf{u} \in \mathsf{X}$. In particular, $\mathsf{T}$ has at least one fixed point. Furthermore, as $\eta(\mathsf{t}) = \psi(\mathsf{t}) > \mathsf{t} = \varphi(\mathsf{t})$ for all $\mathsf{t} \in (0,1)$, then condition ($\mathfrak{X}_4^F$) in Theorem 17 also holds, which demonstrates the uniqueness of the fixed point. □

## 6 Conclusions and prospect work

Having in mind the attractive results due to Proinov [43], in this paper we have wondered about the convenience of some of the very general hypotheses that such researcher handled in that paper. In particular, we have introduced two wide families of pairs of auxiliary functions ($\mathfrak{X}$ and $\mathfrak{X}^F$) in order to avoid the monotone condition on the main Proinov's statements. Such families, called $\mathfrak{X}$-*contractions*, have been used in the setting of metric spaces and fuzzy metric spaces to prove existence and uniqueness fixed point theorems that generalize and extend some well known results in this area of study.

However, much work must be done in this line of research in the future. For instance, we pose the following questions.

- *Open problem 1*: Do the presented results hold in the setting of more general fuzzy metric spaces? For instance, are they valid on fuzzy metric spaces in the sense of Kramosil and Michálek? A first but improvable approach was given in Sect. 5.4.
- *Open problem 2*: Inspired by the assumptions ($\mathfrak{X}_1^F$)-($\mathfrak{X}_3^F$) and the contractivity condition

$$\psi(\mathsf{M}(\mathsf{Tu}, \mathsf{Tv}, \mathsf{s})) \geq \phi(\mathsf{M}(\mathsf{u}, \mathsf{v}, \mathsf{s})) \quad \text{for all } \mathsf{u}, \mathsf{v} \in \mathsf{X} \text{ and all } \mathsf{s} > 0,$$

what other families of pairs of auxiliary functions $(\varphi, \eta)$ included on $\mathfrak{X}^F$ can be considered in order to develop fixed point theory on metric spaces and/or fuzzy metric spaces?

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## Declarations

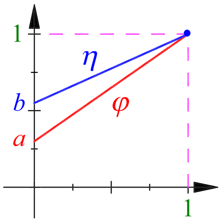
**Conflict of interest** The authors confirm that there are no conflicts of interest to this work.



## Appendix

***Proof*** **(Example 5, page 17)** Given $a, b \in (0,1)$ such that $a < b$, let $\varphi, \eta : (0,1] \to \mathbb{R}$ be the functions defined, for all $\mathsf{s} \in (0,1]$, by:

$$\varphi(\mathsf{t}) = a + (1-a)\mathsf{s},$$
$$\eta(\mathsf{t}) = b + (1-b)\mathsf{s}.$$




Notice that $\varphi$ and $\eta$ are continuous and

$$\varphi(\mathsf{t}) < \eta(\mathsf{t}) \quad \text{for all } \mathsf{t} \in (0,1). \tag{19}$$

We prove that $(\varphi, \eta) \in \mathfrak{X}^F$ by checking all properties.

($\mathfrak{X}_1^F$) Let $\{\mathsf{t}_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be a sequence such that $\varphi(\mathsf{t}_{\ell+1}) \geq \eta(\mathsf{t}_\ell)$ for all $\ell \in \mathbb{N}$. Therefore

$$a + (1-a)\mathsf{t}_{\ell+1} = \varphi(\mathsf{t}_{\ell+1}) \geq \eta(\mathsf{t}_\ell) = b + (1-b)\mathsf{t}_\ell.$$

Hence:

$$\begin{aligned} 0 < b - a &\leq (1-a)\mathsf{t}_{\ell+1} - (1-b)\mathsf{t}_\ell = (1-a)\mathsf{t}_{\ell+1} - (1-a+a-b)\mathsf{t}_\ell \\ &= (1-a)\mathsf{t}_{\ell+1} - (1-a)\mathsf{t}_\ell - (a-b)\mathsf{t}_\ell = (1-a)(\mathsf{t}_{\ell+1} - \mathsf{t}_\ell) + (b-a)\mathsf{t}_\ell. \end{aligned}$$

Therefore

$$0 \leq (b-a)(1-\mathsf{t}_\ell) \leq (1-a)(\mathsf{t}_{\ell+1} - \mathsf{t}_\ell) \quad \text{for all } \ell \in \mathbb{N}. \tag{20}$$

If there is some $\ell_0 \in \mathbb{N}$ such that $\mathsf{t}_{\ell_0} = 1$, then $0 \leq (1-a)(\mathsf{t}_{\ell_0+1} - 1)$, so $\mathsf{t}_{\ell_0+1} = 1$. In this case, $\mathsf{t}_\ell = 1$ for all $\ell \geq \ell_0$, so $\{\mathsf{t}_\ell\} \to 1$. On the contrary case, suppose that $\mathsf{t}_\ell < 1$ for all $\ell \in \mathbb{N}$. Then (20) implies that

$$0 < (b-a)(1-\mathsf{t}_\ell) \leq (1-a)(\mathsf{t}_{\ell+1} - \mathsf{t}_\ell) \quad \text{for all } \ell \in \mathbb{N}.$$

In particular, $t_\ell < t_{\ell+1} < 1$ for all $\ell \in \mathbb{N}$. Let $e = \lim_{\ell\to\infty} t_\ell \in (0,1]$. Since $\varphi$ and $\eta$ are continuous and $\varphi(t_{\ell+1}) \geq \eta(t_\ell)$ for all $\ell \in \mathbb{N}$, then $\varphi(e) \geq \eta(e)$. Taking into account (19), we deduce that $e = 1$, so $\{t_\ell\} \to 1$.

$(\mathfrak{X}_2^F)$ Let $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be two sequences converging to the same limit $e \in [0,1]$ that satisfy $t_\ell < e$ and $\varphi(t_\ell) \geq \eta(s_\ell)$ for all $\ell \in \mathbb{N}$. Clearly $e > 0$ because $e > t_\ell > 0$. Since $\varphi$ and $\eta$ are continuous and $\varphi(t_\ell) \geq \eta(s_\ell)$ for all $\ell \in \mathbb{N}$, then $\varphi(e) \geq \eta(e)$. Taking into account (19), we deduce that $e = 1$.

$(\mathfrak{X}_3^F)$ Let $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be two sequences such that $\{s_\ell\} \to 1$ and $\varphi(t_\ell) \geq \eta(s_\ell)$ for all $\ell \in \mathbb{N}$. Reasoning as in (20), we deduce that

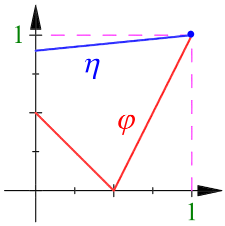
$$0 \leq (b-a)(1-s_\ell) \leq (1-a)(t_\ell - s_\ell) \quad \text{for all } \ell \in \mathbb{N}. \tag{21}$$

As a consequence, $s_\ell \leq t_\ell \leq 1$ for all $\ell \in \mathbb{N}$, so $\{t_\ell\} \to 1$. □

***Proof*** **(Example 6, page 18)** Let $\varphi, \eta : (0,1] \to \mathbb{R}$ given by:

$$\varphi(t) = \begin{cases} 0.5 - t, & \text{if } t \in (0, 0.5], \\ 2t - 1, & \text{if } t \in (0.5, 1]; \end{cases}$$

$$\eta(t) = \frac{9+t}{10}.$$




Notice that the functions $\varphi$ and $\eta$ are continuous and

$$\varphi(t) < \eta(t) \quad \text{for all } t \in (0,1). \tag{22}$$

We are going to prove that $(\varphi, \eta) \in \mathfrak{X}^F$. Before that, notice that

$$\text{if } t, s \in (0,1] \text{ are such that } \varphi(t) \geq \eta(s), \text{ then } t > 0.95. \tag{23}$$

To prove it, observe that $\eta(s) > 0.9$ for all $s \in (0,1]$, so $\varphi(t) \geq \eta(s) > 0.9$ implies that $2t - 1 > 0.9$, that is, $t > 0.95$.

We prove that $(\varphi, \eta) \in \mathfrak{X}^F$ by checking all properties.

$(\mathfrak{X}_1^F)$ Let $\{t_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be a sequence such that $\varphi(t_{\ell+1}) \geq \eta(t_\ell)$ for all $\ell \in \mathbb{N}$. By (23), $t_{\ell+1} > 0.95$ for all $\ell \in \mathbb{N}$, that is, $t_\ell > 0.95$ for all $\ell \geq 2$. Hence

$$\varphi(t_{\ell+1}) \geq \eta(t_\ell) \quad \Rightarrow \quad 2t_{\ell+1} - 1 \geq \frac{9+t_\ell}{10} \quad \Rightarrow \quad t_{\ell+1} - t_\ell \geq 19(1 - t_{\ell+1}) \geq 0. \tag{24}$$

Therefore $t_\ell \leq t_{\ell+1} \leq 1$ for all $\ell \in \mathbb{N}$. Let $e = \lim_{\ell\to\infty} t_\ell \in (0,1]$. Since $\varphi$ and $\eta$ are continuous and $\varphi(t_{\ell+1}) \geq \eta(t_\ell)$ for all $\ell \in \mathbb{N}$, then $\varphi(e) \geq \eta(e)$. Taking into account (22), we deduce that $e = 1$, so $\{t_\ell\} \to 1$.

$(\mathfrak{X}_2^F)$ Let $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be two sequences converging to the same limit $e \in [0,1]$ that satisfy $t_\ell < e$ and $\varphi(t_\ell) \geq \eta(s_\ell)$ for all $\ell \in \mathbb{N}$. Clearly $e > 0$ because $e > t_\ell > 0$. Since $\varphi$ and $\eta$ are continuous and $\varphi(t_\ell) \geq \eta(s_\ell)$ for all $\ell \in \mathbb{N}$, then $\varphi(e) \geq \eta(e)$. Taking into account (22), we deduce that $e = 1$.

$(\mathfrak{X}_3^F)$ Let $\{t_\ell\}_{\ell\in\mathbb{N}}, \{s_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ be two sequences such that $\{s_\ell\} \to 1$ and $\varphi(t_\ell) \geq \eta(s_\ell)$ for all $\ell \in \mathbb{N}$. Reasoning as in (24), we deduce that

$$0 \leq 19(1 - t_\ell) \leq t_\ell - s_\ell \quad \text{for all } \ell \in \mathbb{N}. \tag{25}$$

As a consequence, $s_\ell \leq t_\ell \leq 1$ for all $\ell \in \mathbb{N}$, so $\{t_\ell\} \to 1$. □

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